Abstracts

Beyond Endoscopy and elliptic terms

James Arthur

Beyond Endoscopy is the proposal [4] of Langlands for applying the stable trace formula to the general principle of functoriality. It will at best be a long term enterprise, calling for the ideas and efforts of many mathematicians. However, there seems also to be a good deal of hidden structure in the problem, which should lead to a number of well defined and perhaps accessible questions. In the lecture, we reviewed the main premises of Beyond Endoscopy. We then discussed the elliptic regular terms in the trace formula for $\mathrm{GL}(n+1)$. Our aim was to give some sense of the hidden structure within these often impenetrable objects.

We restrict our attention to the group

$$G = G(n) = \mathrm{GL}(n+1)$$

over $\mathbb{Q}$, with Langlands dual group equal to

$$\hat{G} = \hat{G}(n) = \mathrm{GL}(n+1, \mathbb{C}).$$

The stable trace formula then reduces to the ordinary trace formula. This amounts to an identity

$$I_{\text{geom}}(f) = I_{\text{spec}}(f), \qquad f \in C_c^\infty(G(\mathbb{A})), \tag{1}$$

between a geometric expansion on the left and a spectral expansion on the right.

While many of the more exotic terms on each side of (1) are likely to have important roles in Beyond Endoscopy, we consider only the approximate identity

$$I_{\text{ell,reg}}(f) \sim I_2(f), \qquad f \in C_c^\infty(G(\mathbb{A})/Z_+), \tag{2}$$

of "primary terms." These are the regular elliptic orbital integrals

$$\begin{aligned} I_{\text{ell,reg}}(f) &= \sum_\gamma \mathrm{vol}(\gamma)\,\mathrm{Orb}(\gamma, f) \\ &= \sum_{\gamma \in \Gamma_{\text{ell,reg}}(G)} \mathrm{vol}(Z_+ G_\gamma(F) \setminus G_\gamma(\mathbb{A}) \int_{G_\gamma(\mathbb{A}) \setminus G(\mathbb{A})} f(x^{-1}\gamma x)\,dx) \end{aligned}$$

from the geometric side, and the automorphic characters

$$\begin{aligned} I_2(f) &= \sum_\pi \mathrm{tr}(\pi(f)) \\ &= \sum_{\pi \in \Pi_2(G, Z_+)} \mathrm{tr}(\int_{G(\mathbb{A})/Z_+} f(x)\pi(x)\,dx) \end{aligned}$$

from the spectral side that occur in the discrete spectrum, with Z_+ being the central subgroup

$$Z_+ = Z_G(\mathbb{R})^0 \subset G^+(\mathbb{R}) = \{x_\infty \in G(\mathbb{R}) \;:\; \det(x_\infty) > 0\} \subset G(\mathbb{A})$$

of $G(\mathbb{R})$. These terms are quite challenging enough for now!

For simplicity, we also restrict the test function f. We set

$$f = f_\infty f^\infty = f_\infty f_p^k f^{\infty,p},$$

for a fixed prime p and nonnegative integer k. The archimedean factor f_∞ is in $C_c^\infty(G^+(\mathbb{R})/Z_+)$, while $f^{\infty,p}$ is the characteristic function

$$\chi(\prod_{q\neq p} G(\mathbb{Z}_p))$$

on $G(\mathbb{A}^{\infty,p})$, and f_p^k is the function on $G(\mathbb{Z}_p)$ given by the product

$$p^{-k/(n+1)} \cdot \chi(\{x_p \in \mathfrak{g}(n+1, \mathbb{Q}_p) \ : \ |\det x_p|_p = p^{-k}\}),$$

for the Lie algebra $\mathfrak{g} = \mathrm{gl}(n+1)$ of G. With this function, it is easy to see that the orbital integral $\mathrm{Orb}(\gamma, f)$ in (2) vanishes unless the characteristic polynomial of γ has integral coefficients. More precisely, we can assume that γ is such that

$$\det(\lambda I - \gamma) = \lambda^{n+1} - a_1\lambda + \cdots + (-1)^n a_n \lambda + (-1)^{n+1} a_{n+1} = p_b(\lambda),$$

for $b = (a_1, \dots, a_n)$ in $\mathbb{Z}^n$, and $a_{n+1} = \det\gamma = p^k$. We then have a bijection $\gamma \to b$ onto the set of $b \in \mathbb{Z}^n$ such that $p_b(\lambda)$ is irreducible.

The mapping $\gamma \to b$ gives a parametrization

$$I_{\mathrm{ell,reg}}(b, f) = \mathrm{vol}(\gamma)\,\mathrm{Orb}(\gamma, f)$$

of the elliptic regular terms by (certain) points $b \in \mathbb{Z}^n$. This parametrization, which was a foundation of Ngo's proof of the fundamental lemma, represents a real change of outlook. It imposes a new structure on the elliptic regular terms in the trace formula. With its focus on irreducible, monic, integral polynomials rather than the roots of these polynomials, it hints at a new role for the theory of equations, which emphasizes Galois resolvents and the explicit determination of Galois groups.

In general, there is typically a parallel structure between geometric and spectral terms in the trace formula. The Galois theoretic properties of the elliptic regular terms on the geometric side would now be analogous to the functorial properties of the discrete, L^2-terms on the spectral side. Whether this is more than just an analogy might depend on the answer to the following question, posed in [3].

Is it possible to apply the Poisson summation formula to the sum

$$I_{\mathrm{ell,reg}}(f) = \sum_{b\in\mathbb{Z}^n} I_{\mathrm{ell,reg}}(b, f) = \sum_{\gamma} \mathrm{vol}(\gamma)\,\mathrm{Orb}(\gamma, f), \quad \gamma \to b,$$

over b in (a subset of) $\mathbb{Z}^n$? In other words, can we write

$$I_{\mathrm{ell,reg}}(f) = \sum_{\xi\in\mathbb{Z}^n} \hat{I}_{\mathrm{ell,reg}}(\xi, f)? \tag{3}$$

The question is very subtle, and clearly fails if it is interpreted literally. For example, the sum over b is taken only over a proper subset of $\mathbb{Z}^n$. Nevertheless, Altug was able to solve the problem in the case $n = 2$ of GL(2). He overcame the analytic difficulties by expanding each $I_{\mathrm{ell,reg}}(b, f)$ into a double sum over integers

l and f, and then taking the original sum over b inside these two supplementary sums. (See [1, Theorem 4.2].) In the lecture, we did not have the time to discuss the difficulties that remain in the case $n > 1$ of higher rank. There has been some progress, but there are also problems that are still to be solved.

In any case, if (3) can be established in general, the summation index ξ could be regarded as an additive spectral variable. It would also index its own monic, integral polynomial $p_\xi(\lambda)$ of degree n. Might there be some concrete relations between the Galois theoretic properties of the polynomials $p_\xi(\lambda)$ on the geometric side and the functorial properties of the representations π on the spectral side. In particular is there any relation between those ξ such that the Galois group of $p_\xi(\lambda)$ is a *proper* subgroup of the symmetric group $S = S_{n+1}$ and the cuspidal automorphic representations π that are *proper* functorial transfers to the general linear group $G = \mathrm{GL}(n+1)$? I have no evidence at all for this, and it is safe to say that any sort of answer lies well in the future. Nevertheless, I find the question intriguing, and worthy of consideration.

The more immediate reason for seeking a Poisson formula (3) is that it seems to be amenable to a geometric description of the spectral contribution of the noncuspidal (residual) representations on the spectral side of (2). In [3] it was conjectured that the trivial one-dimensional representation π of $G(\mathbb{A})$ should correspond to the additive spectral parameter $\xi = 0$ in (3). This problem was solved, again for the case $n = 1$ of GL(2), by Altug [1, Theorem 6.1]. At the end of the lecture, we included a few words on a conjectured extension [2, Section 3] of this result to higher rank that would account for all residual representations π.

References

[1] A. Altuğ. Beyond Endoscopy via the trace formula – I: Poisson summation and isolation of special representations. *Compos. Math.*, 151(10):1791–1820, 2015.

[2] J. Arthur. Functoriality and the Trace Formula. To appear in *Geometric aspects of the Trace Formula*, Simons Symposia Series, Springer.

[3] E. Frenkel, R. Langlands, and B.C. Ngo. Formule des traces et fonctorialité: Le début d'un programme. *Ann. Sci. Math. Québec*, 34:199–243, 2010.

[4] R. Langlands. Beyond Endoscopy. In *Contributions to Automorphic Forms, Geometry, and Number Theory*, pages 611–697. Johns Hopkins Univ. Press, Baltimore, MD, 2004.

Reporter: Chao Li and Chen Wan