Functoriality and the Trace Formula

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Abstract. We shall summarize two different lectures that were presented on Beyond Endoscopy, the proposal of Langlands to apply the trace formula to the principle of functoriality. We also include an elementary description of functoriality, and in the last section, some general reflections on where the study of Beyond Endoscopy might be leading.

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Contents

Foreword

This note is a summary of the Simons Symposium lecture from April 2016, and a lecture a month later at Luminy. We have added some further reflections in Section 4, and an elementary description of functoriality in Section 1. The topic is Beyond Endoscopy, the proposal of Langlands for using the trace formula to attack the general principle of functoriality. Our discussion here will be brief and largely expository. We refer the reader to the original papers [L4], [FLN] and [L5] of Langlands (partly in collaboration with Frenkel and Ngo) for details, and to the expository parts of the articles [Ar4] and [Ar5] for more expansive discussion.

1. The principle of functoriality

The principle of functoriality was introduced by Langlands as a series of conjectures in his original article [L1]. Despite the fact it is now almost fifty years old, and that it has been the topic of various expository articles, functoriality is still not widely known among mathematicians. In our attempt to give an elementary introduction, we shall describe the central core of functoriality, its assertion for the unramified components of automorphic representations. One could in fact argue that the other assertions of functoriality, both local and global, should be treated as postulates for the separate theory of endoscopy.

Because our goal is only to give some sense of the basic ideas, we shall not aim for complete generality. In particular, we shall work until Section 4 over the ground field of rational numbers $\mathbb{Q}$, rather than an arbitrary number field. We take G to be a connected, quasisplit reductive group over $\mathbb{Q}$. Then G comes with its L-group

$$^{L}G = \hat{G} \rtimes \mathrm{Gal}(E/\mathbb{Q}),$$

where $\hat{G}$ is the complex connected dual group of G, and $E/\mathbb{Q}$ is any suitable finite Galois extension through which the canonical action of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\hat{G}$ factors. For example, we could take $G = G(n) = \mathrm{GL}(n+1)$, the general linear group of semisimple rank n over $\mathbb{Q}$. Since G is split in this case, the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\hat{G} = \mathrm{GL}(n+1, \mathbb{Q})$ is trivial. We are therefore free to take $E = \mathbb{Q}$, and

$$^{L}G = \hat{G} = \mathrm{GL}(n+1, \mathbb{C}).$$

We define an automorphic representation π of G to be an irreducible, unitary representation π of $G(\mathbb{A})$ that "occurs in" the decomposition of the unitary representation of $G(\mathbb{A})$, the group of adelic points in G, by right translation on $L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}))$. This is an informal definition, which is not completely precise (because $L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}))$ generally has a continuous spectrum), and somewhat restrictive (since the general definition allows for nonunitary extensions of the relevant parameters to the complex domain). (See [L2].) We recall that π is a (restricted) tensor product

$$\pi = \bigotimes_{v} \pi_v,$$

where $v \in \{v_\infty = v_{\mathbb{R}},\ v = v_p\ (p \text{ prime})\}$ ranges over the completions of $\mathbb{Q}$, and π_v is an irreducible unitary representation of $G(\mathbb{Q}_v)$. Because π comes with an implicit condition of weak continuity, π_v is unramified for almost all v. This means that $\pi_v = \pi_{v_p} = \pi_p$ is determined by a concrete character of induction, represented by a semisimple conjugacy class $c(\pi_p)$ in ^{L}G whose image in $\mathrm{Gal}(E/\mathbb{Q})$ equals the Frobenius class F_p. The automorphic representation thus comes with a family

$$c(\pi) = c^S(\pi) = \{c_p(\pi) = c(\pi_p)\ :\ p \notin S\}$$

of semisimple conjugacy classes in ^{L}G, where S is a finite set of valuations that contains the archimedean place ∞.

The family $c(\pi)$ of semisimple conjugacy classes attached to π is a concrete set of data that is in large part the reason why we are interested in automorphic representations. It is believed to govern some of the fundamental workings of the arithmetic world. In recognition of this possibility, and in analogy with the L-functions Artin had earlier attached to the finite dimensional representations of a finite Galois group, Langlands was led in [L1] to define an automorphic L-function. To do so, one would want to have a family of conjugacy classes in a general linear group $\mathrm{GL}(N, \mathbb{C})$, rather than the complex (disconnected) group ${}^LG$. An automorphic L-function therefore requires the datum of a finite dimensional representation

$$r : {}^LG \to \mathrm{GL}(N, \mathbb{C}),$$

as well as an automorphic representation π of G. It is defined in terms of the characteristic polynomials of the semisimple conjugacy classes $r(c_v(\pi))$ by an Euler product

$$L^S(s, \pi, r) = \prod_{p \notin S} L_p(s, \pi, r) = \prod_{p \notin S} \det(1 - r(c_p(\pi))p^{-s})^{-1},$$

which converges for the real part $\Re(s)$ of $s \in \mathbb{C}$ in some right half plane.

Langlands conjectured that for any π and r, the L-function $L^S(s, \pi, r)$ has analytic continuation to a meromorphic function of s in the complex plane (with an implicit understanding that the poles and residues could be determined explicitly), and a functional equation that relates its values at s and $(1 - s)$. His definitions actually presupposed supplementary local factors $L_v(s, \pi, r) = L(s, \pi_v, r)$ at the places $v \notin S$, with the expectation that the Euler product over all v would satisfy a particularly simple functional equation. However, the unramified L-function $L^S(s, \pi, r)$ remains the most important component, since it is built out of the family $c^S(\pi)$ of conjugacy classes that contains the fundamental arithmetic data.

The principle of functoriality can be regarded as an identity between automorphic L-functions for two groups. Suppose that G' is a second connected quasisplit group over $\mathbb{Q}$, and that ρ is an L-homomorphism from ${}^LG'$ to ${}^LG$, which is to say a commutative diagram

$$\begin{tikzcd}
{}^LG' \arrow[rr, "\rho"] \arrow[dr] & & {}^LG \arrow[dl] \\
& \mathrm{Gal}(E/\mathbb{Q}) &
\end{tikzcd}$$

.

Functoriality asserts that for every automorphic representation π' of G', there is an automorphic representation π of G such that

$$L^S(s, \pi, r) = L^S(s, \pi', r \circ \rho),$$

for every r. This is essentially the condition

$$c_p(\pi) = \rho(c_p(\pi')), \tag{1.1}$$

on the two families of conjugacy classes. Langlands actually introduced functoriality more generally for inner twists of quasisplit groups (or in other words, for arbitrary connected reductive groups), and also for the ramified places $v \in S$. However, these supplementary assertions are more complex, and are now seen as part of the theory of endoscopy. The assertion (1.1) can therefore be regarded as the essence of functoriality.

In addition to defining automorphic L-functions and introducing the principle of functoriality (before it was so named), Langlands sketched the following four applications in his seminal paper [L1].

(i) *Analytic continuation and functional equation*: Langlands pointed out that the analytic continuation and functional equation for a general automorphic L-function would follow from functoriality and the special case that $G = \mathrm{GL}(N)$ and $r = St_N$, the standard N-dimensional representation of $\mathrm{GL}(N)$. This special case (at least for cuspidal π) was established soon afterwards by Godement and Jacquet [GJ].

(ii) *Artin L-functions*: We have noted that quasisplit groups are the natural setting for functoriality. The Galois factor $\mathrm{Gal}(E/\mathbb{Q})$ is then an essential part of the L-group ${}^LG$. In particular, the construction naturally includes the seemingly trivial case that G is the 1-element group $\{1\}$. Its L-group will then be an arbitrary finite Galois group $\mathrm{Gal}(E/\mathbb{Q})$, while r becomes simply an N-dimensional representation of $\mathrm{Gal}(E/\mathbb{Q})$. The associated automorphic L-function $L(s, \pi, r)$ (with π being of course the trivial 1-dimensional automorphic representation of G) is then just the general Artin L-function $L^S(s, r)$. The principle of functoriality can thus be interpreted as an identity

$$L^S(s, r) = L^S(s, \pi, St_N) \tag{1.2}$$

between a general Artin L-function and a standard automorphic L-function for $\mathrm{GL}(N)$. This represents a general and completely unexpected formulation of nonabelian class field theory. It identifies purely arithmetic objects, Artin L-functions, with objects associated with harmonic analysis, automorphic L-functions, thereby proving that the arithmetic L-functions have analytic continuation and functional equation (and with control over their poles). Abelian class field theory amounts to the special case that the dimension N of r equals 1. Its original aim was to establish that abelian Artin L-functions are the Hecke-Tate L-functions attached to the automorphic representations of $\mathrm{GL}(1)$, and thereby have analytic continuation and functional equation.

(iii) *Generalized Ramanujan conjecture*: The generalized Ramanujan conjecture asserts that a cuspidal automorphic representation $\pi = \bigotimes_v \pi_v$ of $\mathrm{GL}(N)$ is *tempered*. This means that the character

$$f_v \to \mathrm{tr}(\pi(f_v)), \qquad f_v \in C_c^\infty(\mathrm{GL}(N, F_v)),$$

of each local constituent π_v of π is tempered, in the sense that it extends to a continuous linear form on the Schwartz space $\mathcal{C}(\mathrm{GL}(N, F_v))$ on $\mathrm{GL}(N, F_v)$ defined by Harish-Chandra. We recall that the classical Ramanujan conjecture applies to the case $N = 2$, and π comes from the cusp form of weight 12 and level 1. It was proved by Deligne [D], who established more generally (for $N = 2$) that the conjecture holds if π is attached to any holomorphic cusp form. (The case that π comes from a Maass form remains an important open problem.) Langlands observed that functoriality, combined with expected properties of the correspondence $\pi' \to \pi$, would imply the generalized Ramanujan conjecture for $\mathrm{GL}(N)$. His representation theoretic argument is strikingly similar to Deligne's geometric proof.

(iv) *Sato-Tate conjecture*: The Sato-Tate conjecture for the distribution of the numbers $N_p(E)$ of solutions (mod p) of an elliptic curve E over $\mathbb{Q}$ has a general analogue for automorphic representations. Suppose for example that π is a cuspidal automorphic representation of $\mathrm{GL}(N)$. The generalized Ramanujan conjecture of (iii) asserts that the conjugacy classes

$$c_p(\pi) = \begin{pmatrix} c_{p,1}(\pi) & & 0 \\ & \ddots & \\ 0 & & c_{p,N}(\pi) \end{pmatrix} \Big/ S_N,$$

have eigenvalues of absolute value 1. The generalized Sato-Tate conjecture describes their distribution in the maximal torus $U(1)^N$ of the maximal compact subgroup $U(N)$ of the dual group $\mathrm{GL}(N, \mathbb{C})$. If π is *primitive* (a notion that requires functoriality even to define, as we will describe in Section 4), the distribution of these classes should be given by the weight function in the Weyl integration formula for the unitary group $U(N)$. Langlands sketched a rough argument for establishing such a result from general functoriality. Clozel, Harris, Shepherd-Barron and Taylor followed this argument in their proof of the original Sato-Tate conjecture, but using base change for $\mathrm{GL}(N)$ and deformation results in place of functoriality. (See [T].)

2. The trace formula

We take G again to be a connected quasisplit group over $\mathbb{Q}$. As above, the group $G(\mathbb{Q})$ then embeds diagonally in the locally compact group $G(\mathbb{A})$ of points in G with values in the adele ring $\mathbb{A}$ of $\mathbb{Q}$. It is convenient to write

$$Z_+ = A_G(\mathbb{R})^0,$$

where A_G is the split component of the centre of G over $\mathbb{Q}$. The quotient $Z_+G(\mathbb{Q})\backslash G(\mathbb{A})$ then has finite volume with respect to the right $G(\mathbb{A})$-invariant measure. This implies that the discrete spectrum

$$L^2_{\mathrm{disc}} = L^2_{\mathrm{disc}}(Z_+G(\mathbb{Q}) \setminus G(\mathbb{A})) \subseteq L^2(Z_+G(\mathbb{Q}) \setminus G(\mathbb{A}))$$

in the corresponding Hilbert space of square integrable functions (that is, the subspace that decomposes discretely under the unitary action of $G(\mathbb{A})$ by right translation) is nonzero.

The trace formula for G is an identity

$$I_{\mathrm{geom}}(f) = I_{\mathrm{spec}}(f) \tag{2.1}$$

between a geometric expansion and a spectral expansion. The terms in the expansions are distributions in a test function f, which we take to be in the space

$$\mathcal{D}(G) = C_c^\infty(Z_+ \setminus G(\mathbb{A})).$$

To study Beyond Endoscopy, one will have to work with the stable trace formula, a refinement of the basic trace formula whose terms are stable distributions. For general linear groups, however, the stable trace formula reduces to the standard trace formula. In the interest of simplicity, we assume until further notice that G is the general linear group $G(n) = \mathrm{GL}(n+1)$ of semisimple rank n, in which

$$Z_+ = \left\{ \begin{pmatrix} r & & 0 \\ & \ddots & \\ 0 & & r \end{pmatrix} \ : \ r > 0 \right\} \subset G(\mathbb{R}).$$

We will then be able to work with the standard trace formula.

The primary terms in the trace formula are those in the elliptic regular part

$$I_{\mathrm{ell,reg}}(f) = \sum_{\gamma \in \Gamma_{\mathrm{ell,reg}}(G)} \mathrm{vol}(Z_+ G_\gamma(\mathbb{Q}) \setminus G_\gamma(\mathbb{A})) \int_{G_\gamma(\mathbb{A}) \setminus G(\mathbb{A})} f(x^{-1}\gamma x)\, dx \tag{2.2}$$

of the geometric side, and include those in the "square integrable" part

$$I_2(f) = \sum_{\pi \in \Pi_2(G)} \mathrm{tr}(\pi(f)) \tag{2.3}$$

of the spectral side. These terms were reviewed in the papers [Ar4] and [Ar5], so we can be brief here. We note that $\Gamma_{\mathrm{ell,reg}}(G)$ is the set of conjugacy classes γ in $G(\mathbb{Q})$ such that centralizer G_γ of γ in G is an anisotropic torus modulo A_G over $\mathbb{Q}$, while $\Pi_2(G)$ is the set of irreducible representations π of $G(\mathbb{A})$ that occur in L^2_{disc}. It is known [MW] for the general linear group here that any such representation occurs with multiplicity 1, so there are no coefficients in the sum of characters on the right hand side of (2.3).

The core of functoriality concerns the subset $\Pi_{\mathrm{cusp}}(G)$ of cuspidal representations in $\Pi_2(G)$. These are the representations of $G(\mathbb{A}) = \mathrm{GL}(n+1, \mathbb{A})$ in $\Pi_2(G)$ that should be tempered, according to the generalized Ramanujan conjecture discussed in Section 1. In the other direction, we have the representations that give the remaining primary spectral terms. They lie in the complement of $\Pi_2(G)$ in the set of $\Pi_{\mathrm{disc}}(G)$ of representations that support the "discrete part" $I_{\mathrm{disc}}(f)$ of the spectral side of the trace formula. (See [Ar5, Section 4] for a comprehensive review of $I_{\mathrm{disc}}(f)$.) The remaining terms in the trace formula lie in the complements of $I_{\mathrm{ell,reg}}(f)$ and $I_{\mathrm{disc}}(f)$ in $I_{\mathrm{geom}}(f)$

and $I_{\text{spec}}(f)$ respectively. These supplementary terms were reviewed in [Ar4]. They also seem to be important for Beyond Endoscopy, more so perhaps than has been the case in the theory of endoscopy itself. However, the implications of the supplementary terms are also more subtle. They are best left for the future.

One of the fundamental goals of Beyond Endoscopy is to isolate the contribution of the cuspidal terms

$$I_{\text{cusp}}(f) = \sum_{\pi \in \Pi_{\text{cusp}}(G)} \operatorname{tr}(\pi(f)) \tag{2.4}$$

to the geometric side. This would entail a study of the difference

$$I_{\text{geom},-}(f) = I_{\text{geom}}(f) - I^{+}_{\text{spec}}(f) \tag{2.5}$$

between the geometric side and the noncuspidal part

$$I^{+}_{\text{spec}}(f) = I_{\text{spec}}(f) - I_{\text{cusp}}(f)$$

of the spectral side. Ideally, one would like a supplementary geometric expansion for this difference. The identity

$$I_{\text{geom},-}(f) = I_{\text{cusp}}(f) \tag{2.6}$$

would then become a more direct formula for the trace of f on the cuspidal discrete spectrum.

There is an implicit premise of functoriality, which concerns what we might call the "functorial source" of any cuspidal representation $\pi \in \Pi_{\text{cusp}}(G)$. By this we mean a minimal pair

$$(G', \pi'), \qquad\qquad \pi' \in \Pi_{\text{cusp}}(G'),$$

such that π is a functorial image of π' under some L-embedding

$$\rho' : {}^{L}G' \to {}^{L}G.$$

The premise is that the functorial source of π should be closely related to the poles at $s = 1$ of L-functions $L^S(s, \pi, r)$, as r varies over finite dimensional representations of ${}^{L}G$. Beyond Endoscopy is a strategy for expanding the trace formula so as to include information about the poles of automorphic L-functions.

To motivate the proposed constructions, we assume for a moment that each $\pi \in \Pi_{\text{cusp}}(G)$ does satisfy the principle of functoriality. In particular, we suppose that the consequences of functoriality described in Section 1 are valid. Since these include the meromorphic continuation of L-functions, we then can define an enhanced cuspidal expansion

$$I^{r}_{\text{cusp}}(f) = \sum_{\pi \in \Pi_{\text{cusp}}(G)} m_\pi(r) \operatorname{tr}(\pi(f)) \tag{2.7}$$

for any r that is weighted with coefficients equal to the orders

$$m_\pi(r) = \operatorname{res}_{s=1}\left(-\frac{d}{ds} \log L^S(s, \pi, r)\right) = -\operatorname{ord}_{s=1} L^S(s, \pi, r) \tag{2.8}$$

of poles at $s = 1$ of the relevant L-functions. If r equals the trivial 1-dimensional representation $1 = 1_G$ of ${}^L G$, $L^S(s, \pi, r)$ is just the (incomplete) Riemann zeta function $\zeta^S(s)$ for any π. It of course has a pole of order 1 at $s = 1$. In this case, $m_\pi(r)$ equals 1, and $I^r_{\text{cusp}}(f)$ reduces to the trace $I_{\text{cusp}}(f)$ on the cuspidal discrete spectrum. Thus, $I^1_{\text{cusp}}(f)$ satisfies the trace formula (2.6), which we are hoping will eventually reduce to something approaching a reasonable geometric expansion. In the general case, we can ask whether the enhanced cuspidal expansion $I^r_{\text{cusp}}(f)$ might also have reasonable geometric expansion. This would then be a more general trace formula, the "r-trace formula" attached to any finite dimensional representation r of ${}^L G$.

Langlands' idea is to construct the distribution $I^r_{\text{cusp}}(f)$ from the special case that $r = 1$. We can write the test function $f \in \mathcal{D}(G)$ as the product of a smooth, compactly supported function on the group

$$Z_+ \setminus G(\mathbb{Q}_S) = Z_+ \setminus G(\mathbb{R}) \times \left(\prod_{v \in S - \{v_{\mathbb{R}}\}} G(\mathbb{Q}_v) \right)$$

with the characteristic function of the compact group

$$K^S = \prod_{v_p \notin S} G(\mathbb{Z}_p),$$

for a finite set of valuations S on $\mathbb{Q}$ that contains the archimedean place $v_{\mathbb{R}}$. Given r, and any valuation $v_p \notin S$, we define a new function $f^r_p \in \mathcal{D}(G)$ as in Section 2 of [Ar4], namely as a product

$$f^r_p(x) = f(x) h^r_p(x_p), \qquad x \in G(\mathbb{A}), \tag{2.9}$$

where x_p is the component of x in $G(\mathbb{Q}_p)$, and h^r_p is the unramified spherical function on $G(\mathbb{Q}_p)$ whose Satake transform equals

$$\hat{h}^r_p(c_p) = \operatorname{tr}(r(c_p)),$$

for any Frobenius-Hecke class c_p in ${}^L G_p$. We are assuming that each $\pi \in \Pi_{\text{cusp}}(G)$ satisfies the generalized Ramanujan conjecture, as one of the consequences of functoriality. The Euler product

$$L^S(s, \pi, r) = \prod_{p \notin S} \det(1 - r(c(\pi_p)) p^{-s})^{-1}$$

of the associated (incomplete) L-function will then converge for $\Re(s) > 1$. In fact, the L-function will satisfy all the conditions of the Tauberian theorem proved in the appendix of Section 2.1 of [Se]. The order of the pole of $L^S(s, \pi, r)$ should therefore be equal to

$$m_\pi(r) = \lim_{N \to \infty} |S_N|^{-1} \sum_{p \in S_N} \log(p) \operatorname{tr}(r(c(\pi_P))),$$

if

$$S_N = \{p \notin S \ : \ p \leq N\}.$$

It will then follow from the definition of $I^r_{\mathrm{cusp}}(f)$ and the function f^r_p that

$$I^r_{\mathrm{cusp}}(f) = \lim_{N\to\infty} |S_N|^{-1} \sum_{p\notin S_N} \log(p) I_{\mathrm{cusp}}(f^r_p). \tag{2.10}$$

(See for example the derivation of the formula (A.2) on p. 253 of [Ar2], from which the factor $\log(p)$ was inadvertently omitted, or the original discussion from [L4, Section 1.5], which leads to an equivalent limit.)

The limit formula (2.10) should thus be a consequence of the properties of L-functions implied by functoriality. However, functoriality is the ultimate goal of Beyond Endoscopy, not something we can assume in trying to carry it out. We are in no position to assume the meromorphic continuation of L-functions, or even a definition of orders $m_\pi(r)$ with which we defined $I^r_{\mathrm{cusp}}(f)$. All we can say is that we expect a limit formula (2.10) to be valid. Langlands' proposal is to try to establish such a formula from the putative geometric expansion (2.6) of $I_{\mathrm{cusp}}(f)$. For if the limit (2.10) were valid, it would also apply to (2.6). It would then give rise to an enhanced trace formula

$$I^r_{\mathrm{geom},-}(f) = I^r_{\mathrm{cusp}}(f), \tag{2.11}$$

with the left hand side defined as a limit

$$I^r_{\mathrm{geom},-}(f) = \lim_{N\to\infty} |S_N|^{-1} \sum_{p\notin S_N} \log(p) I_{\mathrm{geom},-}(f^r_p), \tag{2.12}$$

for any finite dimensional representation r of ${}^LG$.

We therefore return to our basic setting, with G still being the group $\mathrm{GL}(n+1)$, but with no a priori assumption on functoriality. The idea of Langlands is to establish a formula (2.11) directly. We are hoping the distribution $I_{\mathrm{geom},-}(f)$ in (2.6) can be expressed by some approximation of a geometric expansion. One would try to establish (2.11) by applying the limit to each of the terms in the expansion of $I_{\mathrm{geom},-}(f^r_p)$. This would establish the existence of the spectral limit (2.10). One could then try to use the resulting formula (2.11) to study it as a spectral expansion in the original function f.

3. A stratification

The strategy for constructing an r-trace formula (2.12) is predicated on the existence of a geometric-like expansion of $I_{\mathrm{geom},-}(f)$, the left hand side of the trace formula (2.6) for $I_{\mathrm{cusp}}(f)$. This is a serious matter. The individual terms in $I_{\mathrm{geom},-}(f)$ include the nontempered characters from the complement of $\Pi_{\mathrm{cusp}}(G)$ in $\Pi_2(G)$. For these terms, the analogue of the limit (2.11) will not exist. As emphasized in [L4], there will have to be some striking cancellations of terms in the difference (2.5) before one can even consider the possibility of a limit (2.11). We shall review the main construction from the paper [Ar5], which represents a conjectural geometric expansion that appears to be closely related to $I_{\mathrm{geom},-}(f)$.

The cancellation problem was posed by Langlands in [L4], and made more explicit in the joint paper [FLN]. A. Altug solved the problem for the group GL(2) in his thesis [Al1]. He then published his solution in the later paper [Al2]. In this section here, we shall give a brief summary of Section 5 of the paper [Ar5], the aim of which was to describe a conjectural analogue for $\mathrm{GL}(n+1)$ of Altug's solution for GL(2).

The supplementary terms in the trace formula are undoubtedly relevant to the problem. Some of them were examined in [L4], and were found to have some interesting new properties. However, a systematic analysis of the supplementary terms in the context of Beyond Endoscopy has not been undertaken. We shall follow [Ar5] in ignoring them. That is, we replace $I_{\mathrm{geom}}(f)$ and $I_{\mathrm{spec}}(f)$, the geometric and spectral sides of the initial trace formula (2.1), by their primary parts $I_{\mathrm{ell,reg}}(f)$ and $I_{\mathrm{disc}}(f)$. We shall then write

$$I_{\mathrm{ell,reg}}(f) \sim I_{\mathrm{disc}}(f) \tag{3.1}$$

in place of (2.1), without any attempt to describe what the approximation means. The symbol $\sim$ is to be taken heuristically, and maybe interpreted loosely as, "pretend they are equal"!

The formula (2.6) becomes the approximation formula

$$I_{\mathrm{ell,reg},-}(f) \sim I_{\mathrm{cusp}}(f) \tag{3.2}$$

for $I_{\mathrm{cusp}}(f)$ in terms of the difference

$$I_{\mathrm{ell,reg},-}(f) = I_{\mathrm{ell,reg}}(f) - I^{+}_{\mathrm{disc}}(f) \tag{3.3}$$

between the primary geometric expansion and the noncuspidal part

$$I^{+}_{\mathrm{disc}}(f) = I_{\mathrm{disc}}(f) - I_{\mathrm{cusp}}(f) \tag{3.4}$$

of the primary spectral expansion. The notation (3.4), incidentally, differs from that of Section 4 of [Ar5], where we reviewed the representations $\Pi_{\mathrm{disc}}(G)$ whose characters support $I_{\mathrm{disc}}(G)$. In [Ar5], we wrote

$$I_{\mathrm{disc}}(f) = \sum_{m} I^{0}_{\mathrm{disc}}(m, f), \qquad (m+1)|(n+1), \tag{3.5}$$

for the decomposition of $I_{\mathrm{disc}}(f)$ into components supported on characters of $G = \mathrm{GL}(n+1)$ whose cuspidal source ranges over the smaller general linear groups $\mathrm{GL}(m+1)$ (embedded diagonally in $\mathrm{GL}(n+1)$) [Ar5, (4.10)]. In particular, we wrote

$$I_{\mathrm{cusp}}(f) = I^{0}_{\mathrm{disc}}(n, f) = I^{0}_{\mathrm{disc}}(f).$$

The noncuspidal part $I^{+}_{\mathrm{disc}}(f)$ of $I_{\mathrm{disc}}(f)$ here then represents the sum in (3.5) over *proper* divisors $(m+1)$ of $(n+1)$. In any case, the essential point is that the left hand side of (3.2) is the difference (3.3) between a geometric expansion and a spectral expansion. One would like to absorb the spectral part in the geometric part, leaving what one would hope to be some modified geometric expansion.

An important change of perspective was introduced in the paper [FLN]. The authors there parametrized the semisimple conjugacy classes that index terms in geometric expansions by points in the base of the Steinberg-Hitchin fibration. In the case $G = G(n) = \mathrm{GL}(n+1)$ we are considering here, the base of the Steinberg-Hitchin fibration is a product

$$\mathcal{A}(n) = \mathcal{B}(n) \times \mathbb{G}_m$$

of affine n-space $\mathcal{B}(n)$ with the multiplicative group $\mathbb{G}_m = \mathrm{GL}(1)$. The proposal in [FLN] in this case is to identify points $\gamma \in \Gamma_{\mathrm{ell,reg}}(G)$ with their characteristic polynomials $p_\gamma(\lambda)$. For there is a bijection $\gamma \to a$ from $\Gamma_{\mathrm{ell,reg}}(G)$ onto the subset of $\mathcal{A}_{\mathrm{irred}}(n, \mathbb{Q})$ of elements

$$a = (a_1, \ldots, a_n, a_{n+1})$$

in $\mathcal{A}(n, \mathbb{Q})$ such that characteristic polynomial

$$p_a(\lambda) = p_\gamma(\lambda) = \lambda^{n+1} - a_1\lambda^n + \cdots + (-1)^n a_n \lambda + (-1)^{n+1} a_{n+1}$$

is irreducible over $\mathbb{Q}$.

We follow [L4], [Al2] and [Ar5] in restricting the test function f. For simplicity, we take it to be of the form

$$f = f_\infty \cdot f^\infty = f_\infty \cdot f^{\infty,p} \cdot f_p^k$$

specified at the beginning of Section 3 of [Ar5]. For this choice, the summand of γ in (2.2) vanishes unless the irreducible monic polynomial $p_a(\lambda)$ has integral coefficients, with constant term equal to p^k or $-p^k$. We can therefore write

$$I_{\mathrm{ell,reg}}(f) = \sum_{b \in \mathcal{B}(n,\mathbb{Z})} \left\{ \sum_{\gamma \in \Gamma_{\mathrm{ell,reg}}(b)} \mathrm{vol}(\gamma)\, \mathrm{Orb}(\gamma, f^\infty)\, \mathrm{Orb}(\gamma, f_\infty) \right\}, \tag{3.6}$$

where $\mathrm{vol}(\gamma)$ is the volume term in (2.2), $\mathrm{Orb}(\gamma, f^\infty)$ and $\mathrm{Orb}(\gamma, f_\infty)$ are the local factors of the global orbital integral in (2.2), and $\Gamma_{\mathrm{ell,reg}}(b)$ is the preimage of

$$\{a = (b, \varepsilon p^k) \ : \ \varepsilon \in \{\pm 1\},\ p_a(\lambda) \text{ irreducible}\}$$

in $\Gamma_{\mathrm{ell,reg}}(G)$, a set of order 0, 1 or 2. The primary geometric expansion in thus given by a sum over the lattice $\mathcal{B}(n, \mathbb{Z})$ in the real vector space $\mathcal{B}(n, \mathbb{R})$. A key question posed in [FLN] is whether one can apply the Poisson summation formula to this sum. The question cannot be taken literally, since the summands contain arithmetic factors that do not extend to functions $\mathcal{B}(n, \mathbb{R})$. The problem is to transform (3.6) into a different expression to which Poisson summation can be applied.

There are a number of difficulties. In addition to the two arithmetic factors $\mathrm{vol}(\gamma)$ and $\mathrm{Orb}(\gamma, f^\infty)$ of the summands in (3.6), there is a purely arithmetic constraint in the sum itself. It is taken only over elements $b \in \mathcal{B}(n, \mathbb{Z})$ such that at least one of the two characteristic polynomials

$$p_a(\lambda) = p_{(b, \varepsilon p^k)}(\lambda), \qquad \varepsilon \in \{\pm 1\}, \tag{3.7}$$

is irreducible over $\mathbb{Q}$. The various difficulties were discussed in Section 3 of [Ar4], as well as in Section 4.1 of [Al2]. In the case of GL(2), Altug was able to overcome them all. In particular, he enlarged $I_{\mathrm{ell,reg}}(f)$ to an extended geometric expansion $\bar{I}_{\mathrm{ell,reg}}(f)$ by adding terms for the characteristic polynomials (3.7) that are reducible. This could only be accomplished after the original expansion had been manipulated to accommodate various problems of convergence. Altug then rearranged the terms in his expression for $\bar{I}_{\mathrm{ell,reg}}(f)$ so as to be able to apply the one variable Poisson summation formula for the lattice $\mathcal{B}(1,\mathbb{Z}) = \mathbb{Z}$ in $\mathcal{B}(1,\mathbb{R}) = \mathbb{R}$. The result is an expansion

$$\bar{I}_{\mathrm{ell,reg}}(f) = \sum_{\xi \in \mathbb{Z}} \hat{\bar{I}}_{\mathrm{ell,reg}}(\xi, f). \tag{3.8}$$

(See [Al2, Theorem 4.2] and the discussion in Section 3 of [Ar5].)

Having established (3.8), Altug then examined the contribution of the noncuspidal representations to the right hand side. The noncuspidal part

$$I^+_{\mathrm{disc}}(f) = I_{\mathrm{disc}}(f) - I_{\mathrm{cusp}}(f)$$

of $I_{\mathrm{disc}}(f)$ can be written as a sum

$$(I_2(f) - I_{\mathrm{cusp}}(f)) + (I_{\mathrm{disc}}(f) - I_2(f)).$$

Each of the two summands is a scalar multiple of an irreducible character in f. The first is the character of the trivial 1-dimensional representation, while the second is a multiple of a singular induced character (the term (vi) on p. 517 of [JL]). Altug showed that each of the summands contributes only to the term with $\xi = 0$ in (3.8). In fact, he was able to decompose $\hat{\bar{I}}(0, f)$ into a term that equals the first summand and a term that equals the second summand, together with an explicit integral that for at least some purposes represents a manageable error term [Al2, Theorem 6.1, Lemma 6.2]. This is a striking confirmation (and extension) for GL(2) of the conjecture in [FLN] that for any G, the trivial 1-dimensional representation should contribute only to the term with $\xi = 0$ in the conjectural Poisson summation formula.

What should be the analogue for $\mathrm{GL}(n+1)$, where there are many more singular automorphic representations, of the singular term with $\xi = 0$ in the expansion (3.8) for GL(2)? We would of course first require an analogue of the expansion (3.8) itself. This does not yet exist. What is lacking is a suitable interpretation of the nonarchimedean orbital integrals $\mathrm{Orb}(\gamma, f^\infty)$ in (3.6), as discussed in Section 2 of [Ar5]. We shall just assume that we have obtained an extension

$$\bar{I}_{\mathrm{ell,reg}}(f) = \sum_{\xi \in \Xi(n,\mathbb{Z})} \hat{\bar{I}}_{\mathrm{ell,reg}}(\xi, f) \tag{3.9}$$

for $G = \mathrm{GL}(n+1)$ of Altug's Poisson formula (3.8). This would include an approximation $\bar{I}_{\mathrm{ell,reg}}(f)$ of $I_{\mathrm{ell,reg}}(f)$, with terms indexed by arbitrary characteristic polynomials. (We have chosen different notation $\Xi(n)$ for affine n-space here, to suggest that its elements are to regarded as spectral variables.) Does the noncuspidal part $I^+_{\mathrm{disc}}(f)$ of $I_{\mathrm{disc}}(f)$ then have a transparent contribution to the right hand side of (3.9)? The answer conjectured in [Ar5, Section 5] is yes. It takes the form of a stratification of $\Xi(n)$, with strata parametrized by divisors $(m+1)$ of $(n+1)$.

For every proper divisor $(m+1)$ of $(n+1)$, we assume inductively that we have defined an open subset $\Xi^0(m)$ of affine m-space $\Xi(m)$. We use this to define a locally closed subset

$$\Xi^0(m,n) = \{(\xi_m, 0, \xi_m, 0, \dots, 0, \xi_m) \ : \ \xi_m \in \Xi^0(m)\} \tag{3.10}$$

of $\Xi(n)$, where if

$$(n+1) = (m+1)(d+1),$$

the vector in the brackets contains $(d+1)$-copies of the smaller vector ξ_m, and d-copies of the component 0. The number of components of this vector therefore equals

$$m(d+1) + d = md + m + d = (n+1) - 1 = n,$$

so that $\Xi^0(m,n)$ is indeed a subset (obviously locally closed) of $\Xi(n)$. We complete the inductive definition by requiring that $\Xi(n)$ be the disjoint union

$$\Xi(n) = \coprod_m \Xi^0(m,n), \qquad (m+1)|(n+1), \tag{3.11}$$

over *all* divisors $(m+1)$ of $(n+1)$ of the subsets $\Xi^0(m,n)$. For it follows from (3.10) that the remaining ingredient, the open subset $\Xi^0(n)$ of $\Xi(n)$, equals $\Xi^0(n,n)$. It is therefore defined by

$$\Xi^0(n) = \Xi^0(n,n) = \Xi(n) \setminus \coprod_{m \neq n} \Xi^0(m,n).$$

The stratification is obviously compatible with the $\mathbb{Z}$-structure on $\Xi(n)$. That is

$$\Xi(n,\mathbb{Z}) = \coprod_m \Xi^0(m,n,\mathbb{Z}), \tag{3.12}$$

where

$$\begin{aligned} \Xi^0(m,n,\mathbb{Z}) &= \Xi^0(m,n) \cap \Xi(n,\mathbb{Z}) \\ &= \{(\xi_m, 0, \xi_m, 0, \dots, 0, \xi_m) \ : \ \xi_m \in \Xi^0(m,\mathbb{Z})\}. \end{aligned}$$

We can therefore apply it to the distribution-valued function

$$\hat{\hat{I}}_{\mathrm{ell,reg}}(f) : \xi \to \hat{\hat{I}}_{\mathrm{ell,reg}}(\xi, f), \qquad \xi \in \Xi(n,\mathbb{Z}),$$

in the putative Poisson expansion (3.9). We obtain a decomposition

$$\hat{\hat{I}}_{\mathrm{ell,reg}}(f) = \sum_m \hat{\hat{I}}^0_{\mathrm{ell,reg}}(m,f), \qquad (m+1)|(n+1), \tag{3.13}$$

where

$$\hat{\hat{I}}^0_{\mathrm{ell,reg}}(m,f) = \sum_{\xi \in \Xi^0(m,n,\mathbb{Z})} \hat{\hat{I}}^0_{\mathrm{ell,reg}}(\xi, f). \tag{3.14}$$

Observe that (3.13) is completely parallel to the decomposition (3.5) of $I_{\mathrm{disc}}(f)$ taken from [Ar5].

It is clear that (3.12) is a generalization from 1 to n of Altug's decomposition of $\Xi(1,\mathbb{Z}) = \mathbb{Z}$ in the case of GL(2) into the two subsets $\Xi^0(0,1,\mathbb{Z}) = \{0\}$ and

$$\Xi^0(1,1,\mathbb{Z}) = \Xi^0(1,\mathbb{Z}) = \{\xi \in \mathbb{R} \ : \ \xi \neq 0\}.$$

As we have noted, he established that the noncuspidal representations in $I_{\mathrm{disc}}(f)$ contribute entirely to the term in (3.14) for GL(2) with $\xi = 0$. The question is, to what degree does this phenomenon persist in the case of $\mathrm{GL}(n+1)$. The answer will have to wait until we have a corresponding Poisson expansion (3.9), the explicit form of which we could study in detail.

4. Further thoughts

I would like to conclude with a few general observations. We have been discussing the proposal of Langlands for applying the trace formula to the principle of functoriality. It appears that the general problem breaks rather cleanly into four subproblems. These are cumulative in that each depends on the solution of its predecessors. It goes without saying that they are all difficult! We shall say a few words on each of them in turn.

For this section, we take G to be a general connected, quasisplit group over a number field F. The discussion for $\mathrm{GL}(n+1)$ of the last two sections remains essentially the same for G here, with one significant proviso: the trace formula for G must be replaced by the stable trace formula. In particular, the cuspidal trace $I_{\mathrm{cusp}}(f)$ has to be replaced by the stable cuspidal trace $S_{\mathrm{cusp}}(f)$, in which multiplicities become stable multiplicities, and cuspidal automorphic representations become cuspidal automorphic L-packets. This makes no difference in the case $G = \mathrm{GL}(n+1)$ above, since the trace formula reduces to the ordinary trace formula.

The first subproblem would be to find a geometric-like expansion for $S_{\mathrm{cusp}}(f)$, which is to say, for the stable analogue of the expression in (2.6) for $I_{\mathrm{cusp}}(f)$. Our notation $I_{\mathrm{geom},-}(f)$ for this expression reflects our hope for a geometric solution, rather than just the definition (2.5) of the expression as a difference of a geometric expression and a spectral expression. This is the problem we discussed in Section 3, with both Altug's solution for $\mathrm{GL}(2)$ and its conjectural extension to $\mathrm{GL}(n+1)$. Note however that the real problem demands the stable analogue of the full expression (2.5), and not just of the primary part (3.3) that we discussed in Section 3. A full solution would require a comprehensive "Beyond Endoscopic" analysis of the supplementary terms in the stable trace formula. The scattered remarks in Section 6 of [Ar5] hint at the seriousness of any such undertaking.

The second subproblem would be to establish a stable r-trace formula for $S^r_{\mathrm{cusp}}(f)$, the stable cuspidal trace, weighted for any finite dimensional representation r of ${}^LG$ according to (2.7). This question was described for $\mathrm{GL}(n+1)$ in Section 2. It would require a solution to the first subproblem in order to study the stable analogue for the limit in (2.12). In fact, one would need to know specific details of a solution even to think about the general question. For this reason, no doubt, little is known about the second subproblem. The papers [Al3] and [Al4] of Altug represent progress in the case of $\mathrm{GL}(2)$.

Trace formulas have been most powerful when they could be compared with other trace formulas. Beyond Endoscopy will be no exception to this rule. The third subproblem, which we have not discussed here, would be to construct a further trace formula for comparison with the r-trace formula. It would be a formula for what we called the *primitive* (stable, cuspidal) trace $P_{\mathrm{cusp}}(f)$. By this, we mean the contribution to $S_{\mathrm{cusp}}(f)$ of those cuspidal automorphic L-packets whose "functorial source" is G itself, which is to say that they do not represent proper functorial images. In principle, $P_{\mathrm{cusp}}(f)$ cannot even be defined without functoriality. In practice, we would try to establish a "primitization" of the stable trace formula, and more generally of the r-trace formula, for any r. This would be a decomposition

$$S^r_{\mathrm{cusp}}(f) = \sum_{G'} \iota(r, G')\ \hat{P}^{\tilde{G}'}_{\mathrm{cusp}}(f') \tag{4.1}$$

of the r-cuspidal trace into components parametrized by quasisplit groups G' (which are actually supposed to represent "elliptic, beyond endoscopic data $(G', \mathcal{G}', \xi')$ with auxiliary datum $(\tilde{G}', \tilde{\xi}')$"), where f' is a function for $\tilde{G}'$ attached to f by stable transfer. This speculative formula is described in Section 2 of [Ar4]. Its statement no doubt calls for further thought and possible revision. For example, our suggestion in [Ar4] that

$$\iota(r, G') = m'(r)\iota(G, G') \tag{4.2}$$

where $m'(r)$ equals the multiplicity of the trivial representation of $\mathcal{G}'$ in $r \circ \xi'$ and $\iota(G, G')$ is independent of r, is just an uninformed guess. In any case, with the specialization of (4.1) to $r = 1$, we would be able to complete the inductive definition by setting

$$P_{\mathrm{cusp}}(f) = S_{\mathrm{cusp}}(f) - \sum_{G' \neq G} \iota(1, G')\hat{P}^{\tilde{G}'}_{\mathrm{cusp}}(f'). \tag{4.3}$$

The fourth subproblem would be to deduce functoriality itself from the primitization (4.1) of the r-trace formula. It is related to the problem that as r-varies, the dimension data

$$m'(r) = m_{G'}(r) \tag{4.4}$$

do not determine G' uniquely [AYY], [Y]. Something of this question seems reminiscent of techniques from the global theory of endoscopy in [Ar3]. I made a couple of remarks to this effect in [Ar4, Section 3, Question VII], but I have not thought seriously about them. Perhaps this fourth subproblem should wait until we know more about the earlier three.

One of the aims of the paper [Ar4] was to draw comparisons between the ideas in Beyond Endoscopy implicit in [L4], [FLN] and [L5] and techniques from the theory of endoscopy. These analogies work particularly well in the context of the four subproblems I have described. I shall recall them very briefly, if for no other reason to try to clarify my own thoughts!

The first subproblem was to establish a geometric formula for the stable trace $S_{\mathrm{cusp}}(f)$. This would be analogous to the original (invariant) trace formula for G. The second was to establish a geometric formula for the (stable) r-trace $S^r_{\mathrm{cusp}}(f)$, for any finite dimensional representation r of ${}^L G$. This would be parallel to the twisted trace formula for any automorphism and (abelian) automorphic character for G. The third was to establish a primitization of the trace formula for $S_{\mathrm{cusp}}(f)$, and more generally, for $S^r_{\mathrm{cusp}}(f)$. This is parallel to the stabilization of the ordinary and twisted trace formulas for G. And finally, we have the fourth subproblem of trying to deduce functoriality from the primitization of $S_{\mathrm{cusp}}(f)$ and $S^r_{\mathrm{cusp}}(f)$. This would seem to be parallel to establishing the endoscopic classification of automorphic representations for (quasisplit) classical groups G from the stabilization of their trace formulas, and of the twisted trace formulas for general linear groups.

We have completed our description of four subproblems that make up Langlands' proposal of Beyond Endoscopy. The trace formula is clearly at the centre of each of them. There are also other approaches to functoriality that are not primarily based on the trace formula. One is to develop ideas of Braverman and Kazhdan [BK1], [BK2] that are based on Vinberg's theory of monoids (See [N1], [N2].) Its goal is to study automorphic L-functions directly. If one could establish their analytic continuation and functional equation directly, one might be able to use a converse theorem to establish functoriality. Another approach is based on relative trace formulas [Sak], [V]. One aim here is to study *generic* global L-packets through general analogues of the Kuznetsov trace formula. This would sidestep the nontempered automorphic representations, whose removal represents the serious problem discussed in Section 3 (and described as the first general subproblem above).

It is possible that ideas from several different points of view might ultimately have to be used together. But it seems to me that the use of the trace formula will be indispensable if we are to fully understand functoriality. For our ultimate goal should be a classification of automorphic representations for any G that goes beyond the principle of functoriality. I am not thinking of the endoscopic classification of representations, such as was established for quasisplit classical groups in [Ar3] and [Mok]. It was in terms of global L-packets, and the functorial transfers from endoscopic groups that govern the packets. What we would like now is a refined classification of cuspidal automorphic packets for G in terms of primitive cuspidal automorphic packets for smaller groups, as in the primitization (4.1) (with $r = 1$) of the stable trace formula for $S_{\mathrm{cusp}}(f)$. As far as I can see, this would not appear to be accessible, even in principle, without extensive use of the trace formula.

The primitization of $S_{\mathrm{cusp}}(f)$ would also not be enough as it stands. There are a number of questions to be answered before we could treat (4.1) (with $r = 1$) as a well defined decomposition of $\Pi_{\mathrm{cusp}}(G)$ (or rather of the associated family $\Phi_{\mathrm{cusp}}(G)$ of global, cuspidal automorphic L-packets for G). They include questions about the orders of poles of L-functions $L(s, \pi, r)$, where π represents a packet in $I_{\mathrm{cusp}}(G)$. For example, given π, can we find a datum G' indexing the sum on the right hand side of (4.1) such that the nonnegative integers (2.8) and (4.4) satisfy

$$m_{G'}(r) = m_\pi(r), \tag{4.5}$$

for every representation r of ${}^LG$? This question was raised at the very beginning of the foundational article [L4]. We had better hope for an affirmative answer, since any alternative would seem to lead to chaos. A second question concerned the uniqueness of G'. However, the later examples in [AYY] and [Y] tell us that G' need not be uniquely determined up to $\hat{G}$-conjugacy of ${}^LG'$ in ${}^LG$, or possibly more precisely, up to isomorphism of G' as a "beyond endoscopic datum." (See [Ar4, Section 2]. This notion of isomorphism is presumably similar to its analogue [KS, p. 18] for endoscopic data.)

We can try to refine the two questions together, as follows.

Refined question: Given $\pi \in \Pi_{\mathrm{cusp}}(G)$ as above, can we find a pair

$$(G', c'), \tag{4.6}$$

such that

(i) the pair (G', c') is a functorial source of π,
(ii) G' and π satisfy (4.5), and
(iii) the pair (G', c') is uniquely determined up to isomorphism?

We have used the term "functorial source" only informally up until now. To say more precisely what we mean in (i) here, G' represents an "elliptic beyond endoscopic datum $(G', \mathcal{G}', \xi')$ with auxiliary datum $(\tilde{G}', \tilde{\xi}')$" (as in (4.1), and as described following [Ar4, (2.3)]), while

$$c' = \{c'_v \ : \ v \notin S\}$$

is a family of semisimple conjugacy classes in $\mathcal{G}'$ whose image $\xi'(c)$ in ${}^LG$ equals $c(\pi)$, and whose image $\tilde{c}' = \tilde{\xi}'(c')$ in ${}^L\tilde{G}'$ equals $c(\tilde{\pi}')$, for a representation $\tilde{\pi}' \in \Pi_{\mathrm{cusp}}(\tilde{G}')$ that occurs in the decomposition of the representation $P^{\tilde{G}'}_{\mathrm{cusp}}$. It is understood that both π and $\tilde{\pi}'$ are of Ramanujan type, in the sense that they represent cuspidal global L-packets for G and $\tilde{G}'$.

It would be very nice if the question has an affirmative answer as stated. I would be content to think that it works in principle, even if my formulation might not be quite correct. The question does represent a classification of cuspidal automorphic representations of G, or rather, cuspidal global L-packets for G. However, it is quite ungainly. There is undoubtedly a better way to formulate it.

I am thinking of Langlands' automorphic Galois group L_F. According to [L3], [K] and [Ar1], it is a hypothetical locally compact extension

$$1 \to K_F \to L_F \to W_F \to 1,$$

where W_F is the global Weil group of F, and K_F is a compact connected group that was in fact conjectured to be simply connected in [Ar1]. We expect the (equivalence classes of) irreducible, unitary N-dimensional representations of L_F to be in canonical bijection with the unitary, cuspidal automorphic representations of $\mathrm{GL}(N)$. More generally, for the quasisplit group G, the set of (isomorphism classes of) bounded, L-homomorphisms of L_F into ${}^LG$ that are discrete, in the sense that their image is not contained in any proper parabolic subgroup ${}^LP$ of ${}^LG$, should be in canonical bijection with the set of (isomorphism classes of) global L-packets of unitary, cuspidal automorphic representations of G. It follows from this property, and the expected compatibility of L_F with the local Langlands groups [Ar1, (1.1)], that the existence of L_F implies the principle of functoriality.

The construction of the hypothetical group L_F in [Ar1] is related to the classification suggested above, but it is simpler. It would be interesting to try to compare them. In particular, are the hypotheses on cuspidal automorphic L-packets that support the construction in Section 4 of [Ar1] essentially the same as those of the refined question above? If so, Langlands' program for Beyond Endoscopy, which is based on the trace formula, could be regarded as a proposal to construct L_F as well as to establish the principle of functoriality. And indeed, it is not unreasonable to expect that the undiscovered mathematical path to functoriality, whatever its technical foundation, must lead also to the automorphic Galois group L_F.

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