L-functions and Automorphic Representations

James Arthur*

Abstract. Our goal is to formulate a theorem that is part of a recent classification of automorphic representations of orthogonal and symplectic groups. To place it in perspective, we devote much of the paper to a historical introduction to the Langlands program. In our attempt to make the article accessible to a general mathematical audience, we have centred it around the theory of L-functions, and its implicit foundation, Langlands' principle of functoriality.

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Preface

Suppose that $f(x)$ is a monic polynomial of degree n with integral coefficients. For any prime number p, we can then write $f(x)$ as a product

$$f(x) \equiv f_1(x) \cdots f_r(x) \qquad (\operatorname{mod} p)$$

of irreducible factors modulo p. It is customary to leave aside the finite set S of primes for which these factors are not all distinct. For those that remain, we consider the mapping

$$p \longrightarrow \Pi_p = \{n_1, \dots, n_r\}, \qquad n_i = \deg\big(f_i(x)\big),$$

from primes $p \notin S$ to partitions Π_p of n. Here are two basic questions:

(I) Is there some independent way to characterize the preimage

$$\mathcal{P}(\Pi) = \{p \notin S : \ \Pi_p = \Pi\}$$

of any partition Π of n?

(II) What is the density of $\mathcal{P}(\Pi)$ in the set of all primes, or for that matter, the set of all positive integers.

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Suppose for example that $f(x) = x^2 + 1$. Then S consists of the single prime 2, while

$$\mathcal{P}(1,1) = \{p : \ p \equiv 1 \pmod 4\}$$

and

$$\mathcal{P}(2) = \{p : \ p \equiv 3 \pmod 4\}.$$

This well known supplement of the law of quadratic reciprocity gives us a striking answer to (I). As for (II), $\mathcal{P}(1,1)$ and $\mathcal{P}(2)$ are each known to have density $\frac{1}{2}$ in the set of all primes. Combined with the prime number theorem, this gives an asymptotic formula

$$\lim_{x\to\infty} |\mathcal{P}(\Pi, x)|/\Big(\frac{x}{\log x}\Big) = 1/2, \qquad \Pi = \{1,1\},\, \{2\},$$

for the set

$$\mathcal{P}(\Pi, x) = \{p \in \mathcal{P}(\Pi) : \ p \leq x\}$$

of primes $p \in \mathcal{P}(\Pi)$ with $p \leq x$. If the generalized Riemann hypothesis can be proved, there will be much sharper asymptotic estimates.

In general, the two questions have significance that goes well beyond their obvious initial interest. The first could perhaps be regarded as the fundamental problem in algebraic number theory. The second has similar standing in the area of analytic number theory. Both questions are central to the Langlands program.

In this article we combine an introduction to the theory of automorphic forms with a brief description of a recent development in the area. I would like to make the discussion as comprehensible as I can to a general mathematical audience. The theory of automorphic forms is often seen as impenetrable . Although the situation may be changing, the aims and techniques of the subject are still some distance from the common "mathematical canon". At the suggestion of Bill Casselman, I have tried to present the subject from the perspective of the theory of L-functions. These are concrete, appealing objects, whose behaviour reflects the fundamental questions in the subject. I will use them to illustrate the basic tenets of the Langlands program. As we shall see in §4, L-functions are particularly relevant to the principle of functoriality, which can be regarded as a foundation of the Langlands program.

The new development is a classification [A4] of automorphic representations of classical groups G, specifically orthogonal and symplectic groups, in terms of those of general linear groups $GL(N)$. It was established by a multifaceted comparison of trace formulas. These are the trace formula for G [A1] and its stabilization [A2], which is now unconditional thanks to the proof of the fundamental lemma [N], and the twisted trace formula for $GL(N)$ [LW] and its stabilization, which is still under construction [W1], [W2], [MW2]. We refer the reader to the surveys [A3], [A5], and [A6], each written from a different perspective, for a detailed description of the classification. We shall be content here to formulate a consequence of the classification in terms of our two themes, L-functions and the principle of functoriality.

Upon reflection after its completion, I observe that the article is not typical of plenary reports for an ICM. It represents a broader, and perhaps denser, introduction than is customary. I hope that the nonspecialist for whom the article is intended will find the details comprehensive enough without being overwhelming.

1. Classical introduction

Recall that a Dirichlet series is an infinite series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}, \tag{1.1}$$

for a complex variable s and complex coefficients a_n. If the coefficients have moderate growth, the series converges when s lies in some right half plane in $\mathbb{C}$. If they are bounded, for example, the series converges absolutely when the real part $\mathrm{Re}(s)$ of s is greater than 1. More generally, if the coefficients satisfy a bound

$$|a_n| \leq cn^k,$$

for positive real numbers c and k, the series converges absolutely in the right half plane $\mathrm{Re}(s) > k+1$. Since the convergence is uniform in any smaller right half plane, the sum of the series is an analytic function of s on the open set $\mathrm{Re}(s) > k+1$.

The most famous example is the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1.2}$$

Since the coefficients are all equal to 1, this converges as an analytic function of s for $\mathrm{Re}(s) > 1$. Euler had studied the series earlier for real values of s. He discovered a remarkable formula

$$\zeta(s) = \prod_p \left(\frac{1}{1-p^{-s}}\right) \tag{1.3}$$

for $\zeta(s)$ as a product over all prime numbers p, which he proved using only the fundamental theorem of arithmetic and the formula for the sum of a geometric series. With the later theory of complex analysis, Riemann was able to extend the domain. He showed that the function has analytic continuation to a meromorphic function of s in the entire complex plane, which satisfies a functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin(\pi s/2)\Gamma(1-s)\zeta(1-s) \tag{1.4}$$

in terms of its values at points s and $1-s$. He observed further that if

$$L_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2),$$

where $\Gamma(\cdot)$ is the gamma function (here and in (1.4)), the product

$$L(s) = L_{\mathbb{R}}(s)\zeta(s) = L_{\mathbb{R}}(s) \cdot \prod_{p} \Big(\frac{1}{1-p^{-s}}\Big) \tag{1.5}$$

satisfies the symmetric functional equation

$$L(s) = L(1-s). \tag{1.6}$$

The functions $\zeta(s)$ and $L(s)$ are both analytic in the complex plane, except for a simple pole at $s = 1$. Riemann conjectured that the only zeros of $L(s)$ lie on the vertical line $\mathrm{Re}(s) = 1/2$. This is the famous Riemann hypothesis, regarded by many as the most important unsolved problem in mathematics. Its interest stems from the fact that the zeros $\{\rho = 1/2 + it\}$ of $L(s)$ on this line are in some sense dual to prime numbers, or more accurately, to logarithms $\{\gamma = \log p^n\}$ of prime powers. We can think of the former as a set of spectral data and the latter as a set of geometric data, which are related to each other by a Fourier transform. The Riemann hypothesis implies a very sharp asymptotic estimate for the number

$$\pi(x) = |\mathcal{P}(x)| = |\{p \le x\}|$$

of primes less than or equal to x. One particularly explicit form [Schoen] of the estimate is

$$|\pi(x) - \ell i(x)| \le \frac{1}{8\pi}\sqrt{x}\log x, \qquad x \ge 2658, \tag{1.7}$$

for the principle value integral

$$li(x) = \int_0^x \frac{1}{\log t}\,\mathrm{d}t.$$

Because the function $li(x)$ is easy to approximate for large values of x, and is asymptotic to a function $\dfrac{x}{\log x}$ that strongly dominates the error term, this is a striking estimate indeed. For example, if $x = 10^{100}$, one sees that $\pi(x)$ and $li(x)$ are positive integers with 97 digits each, the first 47 of which match!

Dirichlet later generalized Riemann's construction to series of the form

$$L^N(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \tag{1.8}$$

where $\chi(n)$ is what later became known as a Dirichlet character. We recall that χ is a complex valued function on $\mathbb{N}$ such that

$$\chi(nm) = \chi(n)\chi(m),$$

$$\chi(n+N) = \chi(n),$$

and

$$\chi(n) = 0, \qquad \text{if } gcd(n,N) > 1,$$

where N is a positive integer called the *modulus* of χ. One says that χ is *primitive* if its nonzero values are not given by restriction of a Dirichlet character with modulus a proper divisor of N (in which case N is called the proper divisor of χ). This means that x generates the cyclic group of characters on the multiplicative group $(\mathbb{Z}/\mathbb{Z}N)^*$. The series (1.8) behaves very much like the Riemann zeta function. It converges if $\mathrm{Re}(s) > 1$. It has an Euler product

$$L^N(s,\chi) = \prod_p \Big(\frac{1}{1-\chi(p)p^{-s}}\Big). \tag{1.9}$$

It also has analytic continuation to a meromorphic function on the complex plane. However, its functional equation is a little more interesting.

To state it, we can suppose without loss of generality that χ is primitive. We account for the gamma function and powers of π in the analogue of (1.4) by setting

$$L_{\mathbb{R}}(s,\chi) = (\pi)^{-(s+a)/2}\Gamma\big((s+a)/2\big),$$

where

$$a = a(\chi) = \begin{cases} 0, & \text{if } \chi(-1) = 1 \\ 1, & \text{if } \chi(-1) = -1. \end{cases}$$

The product

$$L(s,\chi) = L_{\mathbb{R}}(s,\chi)L^N(s,\chi) = \Gamma_{\mathbb{R}}(s,\chi)\cdot\prod_p \frac{1}{1-\chi(p)p^{-s}} \tag{1.10}$$

then satisfies the functional equation

$$L(s,\chi) = \varepsilon(s,\chi)L(1-s,\overline{\chi}), \tag{1.11}$$

where $\overline{\chi}$ is the complex conjugate of χ, and

$$\varepsilon(s,\chi) = N^{-1/2-\varepsilon}\varepsilon(\chi), \tag{1.12}$$

for

$$\varepsilon(\chi) = \varepsilon(1/2,\chi) = i^{-a}N^{-1/2}\Big(\sum_{n=1}^{N-1}\chi(n)e^{2\pi i n/N}\Big), \qquad i = \sqrt{-1}.$$

The expression in the brackets is a Gauss sum. It is the analogue for a finite field (or ring) of the classical gamma function for the field $\mathbb{R}$. Its generalizations are an important part of the functional equations of nonabelian L-functions.

Dirichlet introduced his L-series to study the prime numbers in an arithmetic progression. Suppose that χ is primitive and nontrivial. Dirichlet showed that the L-function (1.8) (or its normalization (1.10)) is actually an entire function of $s \in \mathbb{C}$, and in addition, that its value $L(1,\chi)$ at 1 is nonzero. He then used this to show that for any integer a with $gcd(a,N) = 1$, the number

$$\pi(a,N) = |\mathcal{P}(a,N)| = |\{p \equiv a (\operatorname{mod} N)\}|$$

of primes p in the arithmetic sequence

$$a, a+N, a+2N, \ldots.$$

is infinite. The generalized Riemann hypothesis is the assertion that the only zeros of the entire function $L(s,\chi)$ again lie on the line $\mathrm{Re}(s) = \frac{1}{2}$. It implies an analogue

$$|\pi(x,a,N) - \phi(N)^{-1} li(x)| < C\sqrt{x}\log x, \qquad x \geq 2,$$

of the asymptotic estimate (1.7), for the number

$$\pi(x,a,N) = |\{p \in \mathcal{P}(a,N) : \ p \leq x\}|$$

of primes in the arithmetic sequence less than or equal to x. The familiar Euler function

$$\phi(N) = |\{a : \ 1 \leq a \leq N, \ gcd(a,N) = 1\}|$$

equals the number of such arithmetic progressions.

These remarks illustrate the power of L-functions. They are directed at some of the deepest analytic questions on the distribution of prime numbers. The L-functions we have described are just the beginning. They are the simplest among an enormous but unified collection of L-functions, which have the potential to resolve fundamental arithmetic questions about prime numbers, as well as refinements of the analytic questions we have looked at.

2. Artin L-functions and class field theory

There seems not to be complete agreement on what kind of Dirichlet series (1.1) should be called an L-function. To qualify, it should certainly converge to an analytic function in some right half plane $\mathrm{Re}(s) > k+1$. It should also have an Euler product

$$\prod_p (1 + c_{p,1}p^{-s} + c_{p,2}p^{-2s} + \cdots), \qquad \mathrm{Re}(s) > k+1, \tag{2.1}$$

for a family of coefficients $C = \{c_{p,n}\}$. Finally, it ought to have (or at least be expected to have) analytic continuation and functional equation. We shall take these conditions as our working definition.

The Euler product often arises naturally as an incomplete product, taken over the primes outside a finite set S. However, the expected functional equation is generally best stated for the completed product, in which one adds factors $L_p(s,C)$ for the primes $p \in S$, as well as a factor $L_{\mathbb{R}}(s,C)$ for the "archimedean prime" $\mathbb{R}$ as above. If we also write $L_p(s,C)$ for the factors with $p \notin S$, we then have the given incomplete L-function

$$L^S(s,C) = \prod_{p \notin S} L_p(s,C),$$

and its better behaved completion

$$L(s,C) = L_{\mathbb{R}}(s,C)\Big(\prod_{p\in S} L_p(s,C)\Big)L^S(s,C). \tag{2.2}$$

Treating Dirichlet L-functions as a guide, we would be looking for a functional equation

$$L(s,C) = \varepsilon(s,C)L(1-s,C^\vee), \tag{2.3}$$

where $C^\vee$ is some "dual" family of coefficients attached naturally to C, and

$$\varepsilon(s,C) = b_C^{(1/2-s)}\varepsilon(\tfrac{1}{2},C),$$

for a positive real number b_C, and a complex number $\varepsilon(\frac{1}{2},C)$ that is independent of s.

The higher L-functions that will be our topic are of two kinds, automorphic and arithmetic. The former are primarily analytic objects, while the latter are algebraic. The Riemann zeta function is the common ancestor of them all. It is a mainstay of analytic number theory. However, it can also be regarded as the "trivial" case of the arithmetic L-functions we shall describe in this section. Dirichlet L-functions $L(s,\chi)$ are automorphic. As in the case of the zeta function, it is the application of analysis (real, complex and harmonic) to $L(s,\chi)$ that leads to its analytic continuation and functional equation, and to the location of any poles.

An important family of arithmetic L-functions was introduced by Emile Artin. These are attached to N-dimensional representations

$$r:\ \Gamma_{E/\mathbb{Q}} \longrightarrow GL(N,\mathbb{C})$$

of the Galois group $\Gamma_{E/\mathbb{Q}} = \mathrm{Gal}(E/\mathbb{Q})$ of a finite Galois extension E of $\mathbb{Q}$. We shall describe them here from the perspective of the questions (I) and (II) raised in the preface.

For any E, we have the finite set $S = S_E$ of prime numbers p that ramify in E, and for any unramified $p \notin S_E$, a canonical conjugacy class Frob_p (the *Frobenius* class) in $\Gamma_{E/\mathbb{Q}}$. This is one of the first constructions encountered in algebraic number theory. To be concrete, we can take E to be the splitting field of a monic, integral polynomial $f(x)$ of degree n, as in the preface. For any prime number p, we then have the corresponding factorization of $f(x)$ into irreducible factors $f_i(x)$ modulo p, with degrees n_i. This embeds S_E in the finite set $S = S_f$ of primes p for which these factors are distinct. The choice of $f(x)$ also identifies $\Gamma_{E/\mathbb{Q}}$ with a conjugacy class of subgroups of the symmetric group S_n. For any unramified prime $p \notin S_E$, Frob_p is then mapped to the conjugacy class in S_n defined by the partition $\Pi_p = \{n_1,\ldots,n_r\}$ of n. In particular, the set

$$\mathrm{Spl}(E/\mathbb{Q}) = \{p \notin S:\ \mathrm{Frob}_p = 1\}$$

of prime numbers that *split completely* in E, is the set of p such that $f(x)$ breaks into linear factors modulo p. It is known [T2, p. 165] that $\mathrm{Spl}(E/Q)$ characterizes

E. In other words, the mapping

$$E \longrightarrow \mathrm{Spl}(E/\mathbb{Q}),$$

from Galois extensions of $\mathbb{Q}$ to subsets of prime numbers, is *injective*. A variant of the question (I) would be to characterize its image. This would amount to a classification of Galois extensions E of $\mathbb{Q}$.

For any $p \notin S$, the image $r(\mathrm{Frob}_p)$ under the representation r of $\Gamma_{E/\mathbb{Q}}$ gives a semisimple conjugacy class in the complex general linear group $GL(N, \mathbb{C})$. Artin defined its local L-factor

$$L_p(s, r) = \det\bigl(1 - r(\mathrm{Frob}_p)p^{-s}\bigr)^{-1}, \qquad p \notin S, \tag{2.4}$$

in terms of the associated characteristic polynomial. It clearly has an expansion in terms of powers of p^{-s}, and therefore has the general form of the factor of p in (2.1). Artin then defined an incomplete L-function as the Euler product

$$L^S(s, r) = \prod_{p \notin S} L_p(s, r), \tag{2.5}$$

which he conjectured had analytic continuation with functional equation of the general form (2.3). The question (II) will be reflected in the analytic properties of the resulting function of s. However, the analytic continuation and functional equation of (2.5), let alone the relevant analogue of the Riemann hypothesis, is a more serious proposition. Since $L^S(s, r)$ is a fundamentally arithmetic object, it cannot be studied in terms of the kind of analysis that Dirichlet applied to the L-functions $L(s, \chi)$. Artin treated it indirectly.

Suppose that the Galois group $\mathrm{Gal}(E/\mathbb{Q})$ is abelian, and that r is irreducible, and therefore one-dimensional. The classes $r(\mathrm{Frob}_p)$ are then just nonzero complex numbers. The (incomplete) Artin L-function becomes a product

$$L^S(s, r) = \prod_{p \notin S} \frac{1}{1 - r(\mathrm{Frob}_p)p^{-s}}$$

that resembles the Euler product (1.9) of a Dirchlet L-function. Indeed, if p divides the modulus of χ (written $p|N$), $\chi(p)$ vanishes, and the corresponding product (1.9) can then be written

$$L^S(s, \chi) = \prod_{p \notin S} \frac{1}{1 - \chi(p)p^{-s}},$$

where

$$S = \{p : \ p|N\}.$$

This formal similarity between the products $L^S(s, r)$ and $L^S(s, \chi)$ turns out in fact to be an identity. More precisely, for any one-dimensional Galois representation r, there is a Dirichlet character χ such that the function $L^S(s, r)$ equals $L^S(s, \chi)$. The new L-function therefore has the analytic behaviour of the Dirichlet L-function $L^S(s, \chi)$. In particular, it has analytic continuation, and its completed L-function

(2.2) (with r in place of C) satisfies the functional equation (2.3) (with $r^\vee = \overline{r}$ in place of $C^\vee$).

The assertion that for every r there is a χ is a rather deep classical result, known as the Kronecker-Weber theorem. It finesses the question of the behaviour of abelian Artin L-functions by imposing limits on the set of abelian extensions of $\mathbb{Q}$. From an elementary calculation in algebraic number theory, one obtains the converse theorem that for every χ there is an r. More precisely, if χ has modulus N, there is a one-dimensional representation r of the abelian Galois group of the cyclotomic Galois extension $\mathbb{Q}(e^{2\pi i/N})$ of $\mathbb{Q}$ such that $\chi(p)$ equals $r(\mathrm{Frob}_p)$, for any p that does not divide N. The Kronecker-Weber theorem can then be interpreted as the assertion that any finite abelian extension of $\mathbb{Q}$ is contained in the cyclotomic extension of Nth roots of 1, for some N.

The Kronecker-Weber theorem predated Artin by many years. However, Artin was working over an arbitrary number field F, a finite field extension of $\mathbb{Q}$, rather than $\mathbb{Q}$ itself. The definitions we have made so far are easily extended from $\mathbb{Q}$ to F. The ring $\mathfrak{o} = \mathfrak{o}_F$ of algebraic integers of F does not generally have unique factorization, so one must replace prime numbers p in $\mathbb{Z}$ with prime ideals $\mathfrak{p}$ in $\mathfrak{o}$, and integers n in $\mathbb{Z}$ with general (integral) ideals $\mathfrak{a}$ in $\mathfrak{o}$. Any ideal then has a norm

$$N\mathfrak{a} = |\mathfrak{o}/\mathfrak{a}| = (N\mathfrak{p}_1)^{a_1} \cdots (N\mathfrak{p}_r)^{a_r} = |\mathfrak{o}/\mathfrak{p}_1|^{a_1} \cdots |\mathfrak{o}/\mathfrak{p}_r|^{a_r}, \tag{2.6}$$

where

$$\mathfrak{a} = \mathfrak{p}_1^{a_1} \cdots \mathfrak{p}_r^{a_r}$$

is its unique factorization into prime ideals. The definition of a Dirichlet series (1.1) for $\mathbb{Q}$ can then be extended to F by replacing the sum over n by a sum over $\mathfrak{a}$, and the corresponding variable n^{-s} by $(N\mathfrak{a})^{-s}$. The same goes for an Euler product (2.1). A Dirichlet series for F does reduce to a Dirichlet series for $\mathbb{Q}$, since the norm of a prime ideal $\mathfrak{p}$ is a power of a prime number p. However, a Dirichlet or Artin L-function over F represents something different, even though it can be regarded as a Dirichlet series (1.1) with Euler product (2.1).

Artin worked from the beginning over F. He defined the L-function $L^S(s, r)$ for any N-dimensional representation

$$r : \ \Gamma_{E/F} \longrightarrow GL(N, \mathbb{C}) \tag{2.7}$$

of the Galois group of a finite Galois extension E/F. Algebraic number theory again tells us that for any $\mathfrak{p}$ outside the finite set $S = S_E$ of prime ideals for F that ramify in E, there is a canonical conjugacy class $\mathrm{Frob}_\mathfrak{p} = \mathrm{Frob}_{E/F,\mathfrak{p}}$ in $\Gamma_{E/F}$. The definition (2.4) therefore does extend to F if we replace p^{-s} by $N\mathfrak{p}^{-s}$. Artin also introduced factors for the ramified primes $\mathfrak{p} \in S$, and for the archimedean "primes" v for F (now a finite set S_∞ rather than just the one completion $\mathbb{R}$ of $\mathbb{Q}$). He conjectured that the resulting product $L(s, r)$ had functional equation (2.3), with an ε-factor $\varepsilon(s, r)$ he formulated in terms of r.

It was in this context that Artin obtained the analytic continuation and functional equation for the abelian L-functions $L(s, r)$. Dirichlet characters χ can still be defined by a variant of the prescription for $\mathbb{Q}$ above, and the same analysis that works for $\mathbb{Q}$ gives the analytic continuation and functional equation of a general Dirichlet L-function $L(s,\chi)$. With this in mind, Artin established an F-analogue of the Kronecker-Weber theorem, known now as the Artin reciprocity law. It again asserts that for any one-dimensional Galois representation r over F, there is a Dirichlet character χ over F such that $\chi(\mathfrak{p})$ equals $r(\mathrm{Frob}_{\mathfrak{p}})$, for every unramified prime ideal $\mathfrak{p}$ for F. This leads to the identity $L(s,r) = L(s,\chi)$ of L-functions, and therefore the desired analytic properties of the arithmetic L-function $L(s,r)$.

The Artin reciprocity law is one of the central assertions of class field theory. Unlike the general constructions above, it becomes much deeper in the passage from $\mathbb{Q}$ to F, even though the assertion remains similar. As in the case $F = \mathbb{Q}$, the general law asserts that the abelian field extensions over F are limited to those attached to Dirichlet characters χ. These may then be classified by the "reciprocity law"

$$r(\mathrm{Frob}_{\mathfrak{p}}) = \chi(\mathfrak{p}), \qquad \mathfrak{p} \notin S, \tag{2.8}$$

according again to the remark on [T2, p. 165].

For completeness, we note that Artin L-functions are but the simplest in the general family of arithmetic L-functions, called motivic L-functions. A $\mathbb{Q}$-*motive* M *over* F (which we will not try to define!) also comes with a finite dimensional representation $r_{M,\ell}$ of the absolute Galois group $\Gamma_F = \Gamma_{\overline{F}/F}$. In this general case, however, it takes values in an ℓ-adic general linear group $GL(N,\mathbb{Q}_\ell)$, for a variable prime number $\ell \notin S$. It therefore gives rise to a (continuous) homomorphism

$$r_M = \bigotimes_{\ell} r_{M,\ell} : \ \Gamma_F \longrightarrow \widetilde{\prod_{\ell}} GL(N,\mathbb{Q}_\ell) \tag{2.9}$$

from Γ_F to a large, totally disconnected group. It therefore represents a much larger quotient of Γ_F than does a complex valued representation (2.7). The motive should also come with a finite set S of prime ideals in F such that $r_{M,\ell}$ is unramified in any prime $\mathfrak{p} \notin S \cup S_\ell$, where S_ℓ is the set of primes in F that divide the prime ℓ of $\mathbb{Q}$. The family $\{r_{M,\ell}\}$ of ℓ-adic representations is conjectured to be *compatible*, in the sense that the image $r_{M,\ell}(\mathrm{Frob}_{\mathfrak{p}})$ of the associated Frobenius class in $GL(N,\mathbb{Q}_\ell)$ is the image of a semisimple conjugacy class $r_M(\mathrm{Frob}_{\mathfrak{p}})$ in $GL(N,\mathbb{Q})$ that is independent of ℓ. The incomplete L-function of M is then given by an Euler product

$$L^S(s,M) = \prod_{\mathfrak{p}\notin S} \det\bigl(1 - r_M(\mathrm{Frob}_{\mathfrak{p}})(N\mathfrak{p})^{-s}\bigr)^{-1}, \tag{2.10}$$

which converges in some right half plane, and which is again expected to have analytic continuation with functional equation. Notice that any complex representation (2.7) of $\Gamma_{E/F}$ that is defined over $\mathbb{Q}$ gives a compatible family (2.9) of ℓ-adic representations. It represents a $\mathbb{Q}$-motive over F *of dimension* 0.

3. Automorphic L-functions

Dirichlet L-functions are the analytic counterparts of abelian Artin L-functions. Class field theory, the culmination of many decades of effort by number theorists past, represents a classification of the finite abelian field extensions of any number field. It tells us that any abelian Artin L-function is a Dirichlet L-function. An L-function of the former sort therefore inherits all of the rich properties that can be made available for the latter through analysis. What are the analytic counterparts of nonabelian Artin L-functions? They are the automorphic L-functions introduced by Robert Langlands [L1] in 1970.

Automorphic representations are the nonabelian generalizations of Dirichlet characters, and their abelian generalizations introduced later by Erich Hecke. They are defined for any connected, reductive algebraic group G over the number field F. Algebraic groups represent a conceptual hurdle for many, but a reader is invited to take G to be the general linear group $GL(N)$ of invertible $(N \times N)$-matrices. For any ring A (abelian, with 1), $G(A)$ is then equal to the group $GL(N, A)$ of $(N \times N)$-matrices with entries in A and determinant equal to a unit in A. We want to take A to be the ring $A = \mathbb{A}_F$ of adeles of F. This is a locally compact topological ring, in which F embeds as a discrete subring. It is often a second hurdle, but avoiding it would make matters considerably more complicated. The idea is really quite simple and natural.

By definition, the group of adelic points of G is a restricted direct product

$$G(\mathbb{A}) = \widetilde{\prod_v} G(F_v), \tag{3.1}$$

taken over the valuations v on F. For any v, F_v is the locally compact field obtained by completing F with respect to v. It is modeled on the standard case of the completion $F_v = \mathbb{R}$ of $F = \mathbb{Q}$ with respect to the usual absolute value $|\cdot|_v = |\cdot|$. We recall that the complementary valuations for $F = \mathbb{Q}$ are the nonnegative functions

$$|u|_p = \begin{cases} p^{-r}, & \text{if } u = (a/b)p^r, \ \text{ for } a, b, r \in \mathbb{Z}, \ (a,p) = (b,p) = 1, \\ 0, & \text{if } u = 0, \end{cases}$$

on $\mathbb{Q}$, parametrized by prime numbers p. In general, the restricted direct product is the subgroup of elements

$$x = \prod_v x_v, \qquad x_v \in G(F_v),$$

in the direct product such that for almost all valuations v, x_v lies in the maximal compact subgroup $G(\mathfrak{o}_v)$ of points in $G(F_v)$ with values in the compact subring

$$\mathfrak{o}_v = \{u_v \in F_v : \ |u_v|_v \leq 1\}$$

of integers in F_v. It becomes a locally compact group under the appropriate direct limit topology. The group $G(F)$ embeds in $G(F_v)$ (as a dense subgroup). The diagonal embedding of $G(F)$ into $G(\mathbb{A})$ then exists (because an element in $G(F)$ is integral at almost all valuations v), and is easily seen to have discrete image.

Since $G(F)$ is discrete in $G(\mathbb{A})$, the quotient $G(F)\backslash G(\mathbb{A})$ is a reasonable object. It comes with a right invariant measure, which is determined up to a positive multiplicative constant. One can therefore form the associated space $L^2\big(G(F)\backslash G(\mathbb{A})\big)$ of square-integrable functions. It is a Hilbert space, equipped with the unitary representation

$$\big(R(y)\phi\big)(x) = \phi(xy), \qquad x, y \in G(\mathbb{A}),\ \phi \in L^2\big(G(F)\backslash G(\mathbb{A})\big),$$

of $G(\mathbb{A})$ by right translation. One could describe an *automorphic representation* of G to be an irreducible representation of $G(\mathbb{A})$ that occurs in the spectral decomposition of R. This description is actually more of an informal characterization than a definition. It is also more restrictive than the formal definition in [BJ] and [L4]. We shall take the broader definition, without recalling its two equivalent formulations established in [L4]. We will then call automorphic representations that satisfy the narrower spectral condition above *globally tempered*.

Suppose for example that G is the abelian algebraic group $GL(1)$. Then $G(\mathbb{A})$ is the multiplicative group $\mathbb{A}^*$ of elements x in $\mathbb{A}$ whose components $x_v \in F_v$ are all nonzero and of valuation 1 for almost all v. This is the group of ideles, introduced earlier by Chevalley. A (globally tempered) automorphic representation of $G = GL(1)$ is a character χ on the idele class group $F^*\backslash\mathbb{A}^*$, or in other words, a continuous F^*-invariant homomorphism from $\mathbb{A}^*$ to the group $U(1)$ of complex numbers of absolute value 1. It is the generalization of a Dirichlet character introduced by Hecke, which he called a Grossencharakter, and which is now generally referred to as a Hecke character. Hecke worked in the classical context of ideals $\mathfrak{a}$, but his constructions are a little easier to formulate now in the language of ideles. It is an early illustration of the convenience of the language of adeles. In this regard, a Dirichlet character is just a Hecke character of finite order.

One of the remarkable discoveries of Langlands has been the fundamental role played by a certain dual group of G. The dual group is a complex connected reductive group $\widehat{G}$, whose Coxeter-Dynkin diagram is the dual of the diagram of G. It comes with an action of the absolute Galois group $\Gamma_F = \Gamma_{\overline{F}/F}$, a compact totally disconnected group, which factors through the finite quotient $\Gamma_{E/F}$ of Γ_F attached to some finite Galois extension E of F. Langlands built this action into the dual group as the semidirect product

$${}^L G_E = \widehat{G} \rtimes \Gamma_{E/F},$$

or more canonically

$${}^L G = \widehat{G} \rtimes \Gamma_F,$$

that is now known as the L-group. If G equals $GL(N)$ for example, $\widehat{G}$ is just the complex general linear group $GL(N, \mathbb{C})$. In this case, the action of Γ_F on $\widehat{G}$ is trivial, so we can take $E = F$. For the case that G is orthogonal or symplectic, the families that will be our ultimate interest, we refer the reader to the beginning of §5.

Automorphic L-functions $L(s,\pi,r)$ were defined by Langlands for any G. They depend on an automorphic representation π of G and a finite dimensional representation

$$r: \ {}^LG \longrightarrow GL(N,\mathbb{C}) \tag{3.2}$$

of ${}^LG$. It is understood that r is continuous on Γ_F and analytic on $\widehat{G}$, and in particular, that it factors through a finite quotient $\Gamma_{E/F}$ of Γ_F. Its image is therefore a complex group with finitely many connected components. We will review the definition in the rest of this section.

We first recall [F] that any π can be written as a restricted tensor product

$$\pi = \widetilde{\bigotimes_v} \pi_v, \qquad \pi_v \in \Pi(G_v), \tag{3.3}$$

where $\Pi(G_v)$ is the set of irreducible representations of the locally compact completion $G(F_v)$ of $G(F)$. The interest is not so much in the individual constituents π_v of π, as in the relations they need to satisfy among themselves in order that the product be automorphic. Much of the data that characterize these representations is quite explicit. For example, it is a consequence of what it means for (3.3) to be a restricted direct product that the irreducible representations π_v of $G(F_v)$ will be unramified[1] for almost all v. A well known integral transform, introduced into p-adic harmonic analysis by Satake, leads to a canonical mapping[2]

$$\pi_v \longrightarrow c(\pi_v)$$

from the set of unramified representations π_v of $G(F_v)$ to the set of semisimple conjugacy classes in ${}^LG$. The automorphic representation π thus gives rise to a family

$$c^S(\pi) = \{c_v(\pi) = c(\pi_v) : \ v \notin S\}$$

of semisimple classes in ${}^LG$.

If we are given r as well as π, we obtain a family

$$\big\{r\big(c_v(\pi)\big) : \ v \notin S\big\}$$

of semisimple conjugacy classes in $GL(N,\mathbb{C})$. The incomplete automorphic L-function of π and r is then defined in terms of the characteristic polynomials of these classes. It equals the product

$$L^S(s,\pi,r) = \prod_{v\notin S} L(s,\pi_v,r_v), \tag{3.4}$$

[1]This means that F_v is nonarchimedean, that $G_v = G \times_F F_v$ is quasisplit and split over an unramified extension of F_v, and that the restriction of π_v to a hyperspecial maximal compact subgroup K_v of $G(F_v)$ contains the trivial 1-dimensional representation.

[2]The mapping becomes a bijection if one takes the restricted form ${}^LG_E$ of the L-group, and then takes its range to be the set of semisimple conjugacy classes in ${}^LG_E$ whose image in $\Gamma_{E/F}$ equals the Frobenius class $\mathrm{Frob}_v = \mathrm{Frob}_{E/F,v}$, if v is unramified in E. This basic condition on the Satake transform was observed by Langlands in [L1].

where

$$L(s,\pi_v,r_v) = \det\bigl(1 - r\bigl(c_v(\pi)\bigr)q_v^{-s}\bigr)^{-1}, \tag{3.5}$$

and

$$q_v = p_v^{f_v} = |\mathfrak{o}_v/\mathfrak{p}_v|$$

is the order of the residue class field of F_v. The product is easily seen to converge for s in some right half plane, and is clearly a Dirichlet series (1.1) with Euler product (2.1). The definition can be compared with that of the incomplete Artin L-function (2.4) and (2.5), or rather its generalization from $\mathbb{Q}$ to F. The analogy is clear, even though the earlier definition was in terms of ideals rather than the formulation here in terms of valuations.

Langlands introduced automorphic L-functions in [L1]. He conjectured that they have analytic continuation, with a very precise functional equation

$$L(s,\pi,r) = \varepsilon(s,\pi,r)\,L(1-s,\pi,r^\vee), \tag{3.6}$$

where

$$L(s,\pi,r) = L_S(s,\pi,r)\,L^S(s,\pi,r) = \prod_v L(s,\pi_v,r_v) \tag{3.7}$$

is a completed L-function obtained by appending a finite product

$$L_S(s,\pi,r) = \prod_{v\in S} L(s,\pi_v,r_v)$$

of suitable factors at the ramified valuations $v \in S$ (including the archimedean valuations $v \in S_\infty$ of F), and

$$\varepsilon(s,\pi,r) = \prod_{v\in S} \varepsilon(s,\pi_v,r_v,\psi_v) \tag{3.8}$$

is a finite product of local monomials in q_v^{-s}. The local ε-factors on the right would depend on the local components ψ_v of a fixed, nontrivial additive character ψ on the group $F\backslash\mathbb{A}$, while the global product on the left hand side of (3.8) would be independent of ψ. Langlands did not define the ramified local L- and ε-factors in [L1]. Nevertheless, his introduction of the general automorphic L-function in [L1], with its proposed functional equation (3.6), was an enormous step beyond the abelian automorphic L-functions of Hecke.[3] It depends above all on the L-group ${}^LG$ he introduced at the same time.

We have now described two fundamental families of L-functions. They are the arithmetic L-functions of the last section and the automorphic L-functions of this section. Langlands later conjectured that the former family is a subset of the latter. In other words, for any motive M (which for present purposes we can take to be an N-dimensional representation of a finite Galois group $\Gamma_{E/F}$), there should be a pair (π, r) such that the completed L-functions satisfy

$$L(s,M) = L(s,\pi,r). \tag{3.9}$$

[3] Hecke also introduced some nonabelian L-functions for the group $G = GL(2)$ and the standard two dimensional representation r.

In particular, the analytic continuation and functional equation for $L(s,M)$ would follow from the same properties for $L(s,\pi,r)$. This would be a striking and far reaching generalization of the method used by Artin to establish the analytic continuation and functional equation of abelian Artin L-functions.

There is actually a theory that applies directly to nonabelian Artin L-functions. Richard Brauer established a general property of the representations of a finite group (the Brauer induction theorem), which he used to express any nonabelian Artin L-function $L(s,r)$ as a quotient of finite products of abelian Artin L-functions $L(s,r_1)$ (over finite extensions F_1 of F). Combined with the results of Artin described in §2, this shows that $L(s,r)$ has analytic continuation, with a functional equation of the desired sort. However, it does not give much control over the analytic behaviour of $L(s,r)$. In particular, it gives no information on a fundamental conjecture of Artin, which asserts that an irreducible, nonabelian Artin L-function is entire.

Brauer's theorem has, however, led to important results on local arithmetic L- and ε-factors. These apply more generally to the variant of the Galois group that Weil introduced as a consequence of class field theory. Like the absolute Galois group Γ_F, the Weil group W_F is defined if F is a local or a global field. It is a locally compact group, equipped with a continuous homomorphism $W_F \to \Gamma_F$ with dense image and connected kernel, whose maximal abelian quotient is given by

$$W_F^{ab} = W_F/W_F^c \cong \begin{cases} F^*, & \text{if } F \text{ is local,} \\ F^*\backslash\mathbb{A}^*, & \text{if } F \text{ is global.} \end{cases} \tag{3.10}$$

If F is the global field we are discussing here, W_F comes with a conjugacy class of embeddings

$$\begin{array}{ccc} W_{F_v} & \longrightarrow & \Gamma_{F_v} \\ \big\downarrow & & \big\downarrow \\ W_F & \longrightarrow & \Gamma_F \end{array}$$

for any v, that are compatible with the abelianization (3.10). (See [T3, §1].) These properties imply that the Artin reciprocity law described in §2 extends to a canonical isomorphism from the group of 1-dimensional representations of W_F to the group of (1-dimensional) automorphic representations of $G = GL(1)$. Moreover, the Brauer induction theorem extends to N-dimensional representations r of W_F. It leads to a global L-function

$$L(s,r) = \prod_v L(s,r_v) = \prod_{v\in S} L(s,r_v) \prod_{v\notin S} \det\bigl(1 - r(\mathrm{Frob}_v)q_v^{-s}\bigr)^{-1} \tag{3.11}$$

that has analytic continuation and functional equation

$$L(s,r) = \varepsilon(s,r)\,L(s,r^\vee), \tag{3.12}$$

for a global ε-factor

$$\varepsilon(s,r) = \prod_{v\in S} \varepsilon(s,r_v,\psi_v). \tag{3.13}$$

As in the special case of Artin L-functions, we obtain little control over the analytic behaviour of the global L-functions $L(s,r)$ in the full complex domain. The global interest in these results is therefore limited. However, Deligne used them to establish important local results [T3, §2]. He showed that the local L-functions $L(s,r_v)$ in (3.11) and ε-factors $\varepsilon(s,r_v,\psi_v)$ in (3.13) have a canonical local definition. In particular, they can be constructed independently of the global representation r.

The global L-functions $L(s,r)$ attached to representations of W_F are not all motivic, unlike Artin L-functions. We cannot therefore really regard them as arithmetic. On the other hand, they are not automorphic, since they are not generally defined in terms of automorphic representations. Perhaps they should be regarded as objects that lie between the two classes. In any case, $L(s,r)$ should still be equal to an automorphic L-function. This is again among the original conjectures of Langlands in [L1]. Deligne's constructions then become important for the local classification of representations, and in particular, for comparison with the local L- and ε-factors in (3.7) and (3.8).

4. The principle of functoriality

Langlands' conjectural functional equation (3.6) for a general automorphic L-function is very deep, and far from known. However, among the cases that are known, there is one that deserves special mention. It is the standard automorphic L-function, in which G equals $GL(N)$, and r equals the standard N-dimensional representation St of ${}^LG_F = GL(N,\mathbb{C})$.

Abelian Hecke L-functions are given by the further special case that $G = GL(1)$. Hecke established their analytic continuation and functional equation, using the classical language of ideals. Tate later simplified Hecke's proof by introducing the ring of adeles $\mathbb{A}$. In his famous thesis [T1], he established the results through the interplay of multiplicative harmonic analysis on the idele class group

$$F^*\backslash\mathbb{A}^* = GL(1,F)\backslash GL(1,\mathbb{A})$$

with additive harmonic analysis on the group $\mathbb{A}$.

Following Langlands' paper [L1], Godement and Jacquet [GJ] extended the method of Tate to $GL(N)$, with the additive group of adelic matrices $M_{N\times N}(\mathbb{A})$ in place of $\mathbb{A}$. It follows from their results and later refinements [J] that the standard (completed) L-function

$$L(s,\pi) = L(s,\pi,St) \tag{4.1}$$

for any automorphic representation π of $GL(N)$ is well defined, and has analytic continuation with functional equation

$$L(s,\pi) = \varepsilon(s,\pi)\, L(1-s,\pi^\vee).$$

This method also gives further information about the analytic behaviour of standard L-functions. For example, if the automorphic representation π is cuspidal,

$L(s,\pi)$ is an entire function of s unless $N=1$ and $\pi(x)=|x|^\lambda$ for some $\lambda\in\mathbb{C}$, in which case $L(s,\pi)$ is analytic apart from a simple pole at $s=1-\lambda$.

For any G, we write $\Pi_{\text{aut}}(G)$ for the set of automorphic representations of G (in the broad sense of [L4] we have agreed upon). We then write

$$\mathcal{C}_{\text{aut}}(G)=\{c(\pi):\ \pi\in\Pi_{\text{aut}}(G)\} \tag{4.2}$$

for the set of families $c^S(\pi)$ of automorphic conjugacy classes, taken up to the equivalence relation defined by $c^S\sim c_1^S$ if $c_v=c_{1,v}$ for almost all v. We emphasize that these are concrete objects. They represent the fundamental data encompassed in the seemingly abstract notion of an automorphic representation. As we have noted, the arithmetic significance of these data is not so much in the value of any one class $c_v(\pi)$ as in the relationships among the classes as v varies.

In his original paper [L1], Langlands made a profound conjecture that later became known as the principle of functoriality. We shall state it in the restricted form that applies to the concrete families $\mathcal{C}_{\text{aut}}(G)$.

Principle of Functoriality (Langlands). *Suppose that G and G' are quasisplit*[4] *groups over the number field F. Suppose also that*

$$\rho:\ {}^LG'\longrightarrow {}^LG$$

is an L-homomorphism (that is, a continuous, analytic homomorphism that commutes with the two projections onto Γ_F) between their L-groups. Then if $c'=\{c'_v\}$ lies in $\mathcal{C}_{\text{aut}}(G')$, the family

$$c=\rho(c')=\{\rho(c'_v)\}$$

lies in $\mathcal{C}_{\text{aut}}(G)$. In other words, if $c'=c(\pi')$ for some $\pi'\in\Pi_{\text{aut}}(G')$, then $c=c(\pi)$ for some $\pi\in\Pi_{\text{aut}}(G)$.

The principle of functoriality is the central problem in the theory of automorphic forms. It asserts that the internal relations in an automorphic family $c'=c(\pi')$ for G', whatever they might be, are reflected in the internal relations in some automorphic family $c=c(\pi)$ for G. The principle of functoriality has been established in a significant number of cases. But as challenging as these have been, they pale in comparison with the cases that have not been established.

In the same paper [L1], Langlands pointed out some fundamental applications of functoriality. The first concerns the automorphic L-functions he had just introduced.

Suppose that $G',G,\rho,c'=c(\pi')$ and $c=c(\pi)$ are as in the assertion of functoriality. If r is an N-dimensional representation of ${}^LG$, the composition $r\circ\rho$ is an N-dimensional representation of ${}^LG'$. We then obtain an identity

$$L^S(s,\pi',r\circ\rho)=L^S(s,\pi,r) \tag{4.3}$$

of incomplete automorphic L-functions from the definitions, and of course, the principle of functoriality. This relation might seem almost routine at first glance,

[4]See the brief description of this property at the beginning of the next section.

certainly not the sweeping observation it actually is. But consider the special case with G, $GL(N)$, $c = c(\pi)$ and $c_N = c(\pi_N)$ in place of G', G, c' and c respectively. Then ρ can be identified with an N-dimensional representation r of ${}^LG$, and (4.3) specializes to an identity

$$L^S(s, \pi, r) = L^S(s, \pi_N, St) = L^S(s, \pi_N).$$

The general incomplete automorphic L-function on left thus equals a standard incomplete L-function, the function for which we already have analytic continuation and functional equation. If we set[5]

$$L_S(s, \pi, r) = L_S(s, \pi_N)$$

and

$$\varepsilon(s, \pi, r) = \varepsilon(s, \pi_N),$$

for the supplementary terms, the completed L-function satisfies

$$L(s, \pi, r) = L_S(s, \pi, r)L^S(s, \pi, r) = L_S(s, \pi_N)L^S(s, \pi_N) = L(s, \pi_N), \tag{4.4}$$

and the general functional equation (3.6) then follows from its analogue for standard L-functions.

A second immediate application of functoriality pointed out by Langlands in [L1] is to nonabelian class field theory. It concerns the seemingly trivial case of functoriality with $G' = \{1\}$. Despite its apparent simplicity, however, this case comes with answers to the two general questions (I) and (II) from the preface.

If G' equals $\{1\}$, the dual group $\widehat{G}'$ also equals $\{1\}$, but the L-group ${}^LG'$ is still the absolute Galois group Γ_F. We again take the second group G to be $GL(N)$. An L-homomorphism from ${}^LG'$ to ${}^LG$ will be continuous (by definition) on its totally disconnected domain Γ_F. It can therefore be identified with an N-dimensional representation

$$r : \Gamma_{E/F} \longrightarrow GL(N, \mathbb{C}) \tag{4.5}$$

of the Galois group of some finite Galois extension E of F. The only automorphic representation of G' is the trivial representation 1. However, the associated automorphic family $c(1)$ is still interesting. It is represented by the set

$$c^S(1) = \{c_v(1) = \mathrm{Frob}_v : v \notin S\}$$

of Frobenius conjugacy classes in $\Gamma_{E/F}$ of primes v of F that are unramified in E, according to footnote 2 from the last section. The principle of functoriality in this

[5]In order that the two left hand sides here depend only on π, we assume implicitly that π_N is *isobaric*, in the sense of [L5, §2]. It is then the *unique* automorphic representation of $GL(N)$ with the given eigenfamily $c(\pi_N) = c(\pi)$, according to [JS]. Notice that we do not obtain a local construction for the factors in these supplementary terms, unlike in their analogues (3.11), (3.13) for representations of Weil groups. This requires a stronger (and more complex) assertion of functoriality as in [L1], and is predicated on a local classification of representations, such as that obtained for special orthogonal and symplectic groups in [A4].

case asserts that its r-image $r\bigl(c(1)\bigr)$ is automorphic for $GL(N)$. In other words, there is an automorphic representation π of $GL(N)$ such that

$$c_v(\pi) = r\bigl(c_v(1)\bigr) = r(\mathrm{Frob}_v), \qquad v \notin S.$$

This can be regarded as a general answer to the question (I). Since it includes an analytic characterization of the set $\mathrm{Spl}(E/F)$ of primes of F that split completely in E, it also amounts to a classification theory for general Galois extensions of F, the long standing dream of earlier number theorists. The arithmetic data $\{\mathrm{Frob}_v\}$ that characterize finite Galois extensions E of F ([T2, p. 165]), and that are conveniently packaged by the characters of continuous, finite dimensional representations of Γ_F, can be represented by the analytic data $\{c_v(\pi)\}$ of automorphic representations of general linear groups.

Langlands' formulation of nonabelian class field theory has implications for L-functions. It follows from the definitions (2.4), (2.5), (3.5) and (3.4) that for $G' = \{1\}$, the automorphic L-function $L(s, 1, r)$ on the left hand side of (4.4) is equal to the completed Artin L-function $L(s, r)$. It therefore equals an automorphic L-function $L(s, \pi_N) = L(s, \pi)$ for $GL(N)$. We should note here that the general principle of functoriality implicitly includes some common spectral properties of the two automorphic representations π' and π. In particular, if the representation r in (4.4) is irreducible, the automorphic representation π_N of $GL(N)$ should be cuspidal. If $N \geq 2$, this implies that the automorphic L-function $L(s, \pi_N)$ is entire, as we noted at the beginning of the section. On the other hand, the Artin conjecture mentioned near the end of §3 asserts that the irreducible L-function $L(s, r)$ is entire. The principle of functoriality, in the case $G' = \{1\}$ and $G = GL(N)$, therefore implies this well known conjecture of almost one hundred years. By relating $L(s, r)$ to a standard automorphic L-function $L(s, \pi_N)$, we would obtain what could be considered an answer to the question (II). For since we now have an understanding of the poles of $L(s, \pi_N)$, and can perhaps hope someday to have also an understanding of its zeros, we would have the means to estimate the distribution of the classes $\{\mathrm{Frob}_v\}$. Notice that this is a beautiful generalization of the indirect method in Section 2 used by Artin to study abelian L-functions. In both cases, an analytic problem for arithmetic L-functions is solved by showing that these functions are also automorphic L-functions, for which the analytic behaviour is better understood.

In addition to the two striking consequences of functoriality in [L1] we have just described, Langlands proposed two further applications. One is to the generalized Ramanujan conjecture, the other to a generalization of the conjecture of Sato-Tate. Both have implications for L-functions. For the sake of completeness, we shall say a word on each.

The generalized Ramanujan conjecture can be formulated for any G. It asserts that the local constituents π_v of certain natural automorphic representations π in the discrete spectrum of G are (locally) tempered, in the extension to the local groups $G(F_v)$ by Harish-Chandra of the definition from classical Fourier analysis. If $G = GL(N)$, for example, it is the unitary cuspidal automorphic representations to which the conjecture applies. The connection with L-functions, suggested by Langlands and reinforced by later local harmonic analysis, is that the local components π_v of π will be tempered if and only if the L-functions $L(s,\pi,r)$ are analytic in the right half plane $\mathrm{Re}(s) > 1$. Given this property, Langlands deduced the generalized Ramanujan conjecture from the principle of functoriality, and the fact that a representation π in the discrete spectrum is automatically unitary [L1, p. 43–49]. For more recent progress for the group $G = GL(2)$, we refer the reader to [KiS] and [Ki].

A generalized Sato-Tate conjecture is only hinted at in [L1], and then just in the last paragraph on p. 49. It would apply to any automorphic representation π of G that satisfies the generalized Ramanujan conjecture. We assume the principle of functoriality. The Ramanujan condition is then valid. It implies that for any unramified place $v \notin S$, the conjugacy class $c_v(\pi)$ in ${}^LG$ intersects a fixed maximal compact subgroup ${}^LK = \widehat{K} \rtimes \Gamma_F$ in ${}^LG = \widehat{G} \rtimes \Gamma_F$, and can therefore be identified with a conjugacy class in ${}^LK$. The problem is to determine the distribution of these classes as v varies. If the family $c(\pi)$ is a proper functorial image of a family $c(\pi')$ for some group G', one could determine the distribution of $c(\pi)$ from that of $c(\pi')$. One can therefore assume that π is *primitive*, in the sense that it is not a proper functorial image from some G'. We would then expect[6] that

$$-\operatorname{ord}_{s=1}\big(L(s,\pi,r)\big) = [r:1]$$

for any finite dimensional representation ρ of ${}^LG$. That is, the order of the pole of $L(s,\pi,r)$ at $s=1$ equals the multiplicity of the trivial representation of ${}^LG$ in r. It would then follow from the Wiener-Ikehara Tauberian theorem that the distribution of the classes $\{c_v(\pi)\}$ in ${}^LK$ is given by the Haar measure on ${}^LK$. In concrete terms, one would be able to express the distribution of classes

$$\{c_v(\pi) \cap {}^LT : \ v \notin S\}$$

in a maximal torus ${}^LT = \widehat{T} \rtimes \Gamma_F$ in ${}^LK$ in terms of the explicit density function on ${}^LT$ obtained from the Haar measure that occurs in the Weyl integration formula. (See [Se, §2, Appendix].)

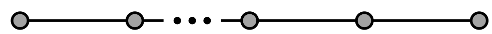
I have attempted to introduce the subject through Langlands' original paper [L1] without discussing subsequent developments. The most famous of these is undoubtedly Wiles' work on the Shimura-Taniyama-Weil conjecture [Wi], which he used to prove Fermat's Last Theorem. An important foundation for Wiles' work was the Langlands-Tunnell theorem that two-dimensional representations r of solvable Galois groups $\Gamma_{E/F}$ satisfy Artin's conjecture. This followed from base change for $GL(2)$ [L6], established by Langlands as an early application of the trace formula. (See [A1, §26], for example.) I mention also that R. Taylor has established the classical Sato-Tate conjecture, which applies to the group $G = GL(2)$, by using base change for $GL(N)$ [AC] and other means to extract what is needed from the unproven principle of functoriality. (See [Tay] and the references there.)

[6] This is actually a little stronger than the principle of functoriality, of which it represents a converse of sorts. However, Langlands' recent ideas [L7] for attacking the principle of functoriality, speculative as they may be, would treat this question as well.

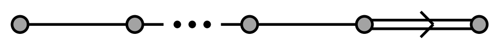
5. Orthogonal and symplectic groups

The monograph [A4] contains a classification of automorphic representations of quasisplit orthogonal and symplectic groups over the number field F. The groups of interest are attached to the four infinite families of complex simple Lie algebras. These in turn are represented by the following four infinite families of Coxeter-Dynkin diagrams, for which I am indebted to W. Casselman.

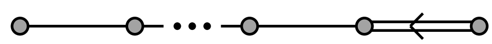
Type $\mathbf{A_n}$



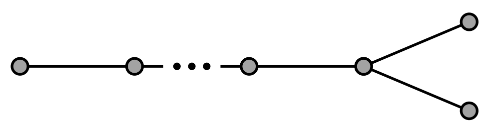
Type $\mathbf{B_n}$



Type $\mathbf{C_n}$



Type $\mathbf{D_n}$



For corresponding complex groups, we could take the special linear groups $SL(n+1,\mathbb{C})$, the odd orthogonal groups $SO(2n+1,\mathbb{C})$, the symplectic groups $Sp(2n,\mathbb{C})$ and the even orthogonal groups $SO(2n,\mathbb{C})$. The family $\mathbf{A}_n$ is the starting point for the classification. Since the representation theory is simplest for general linear groups [JS], [MW1], we take the reductive groups $GL(N,\mathbb{C})$, $N=n+1$, as the complex representatives for this family.

We want to take these groups over the number field F, which is not algebraically closed. But according to a fundamental theorem of Chevalley, any one of these complex groups corresponds naturally to a canonical group over F. It is the *split* group attached to the given diagram (and centre). Our interest is actually in *quasisplit* groups. These are obtained by twisting the Galois action on any given split group by a supplementary Galois action on the diagram. The symmetry group of a diagram is the group of bijections of the set of vertices that preserve all edges and directions. It equals $\mathbb{Z}/2\mathbb{Z}$ in type $\mathbf{A}_n$, is trivial in types $\mathbf{B}_n$ and $\mathbf{C}_n$, and equals[7] $\mathbb{Z}/2\mathbb{Z}$ in type $\mathbf{D}_n$. A quasisplit group is determined by a homomorphism from the Galois group Γ_F to the symmetry group of the diagram. Following [A5], we will not treat nonsplit, quasisplit groups of type $\mathbf{A}_n$. These are unitary groups, for which we refer the reader to [Mok]. Since a quasisplit group of type $\mathbf{B}_n$ or $\mathbf{C}_n$ is split, we have only then to consider type $\mathbf{D}_n$. In this case, a quasisplit group is determined by a quotient of Γ_F of order 1 or 2, or in other words, a Galois extension E/F of degree 1 or 2.

[7] If $n=4$, this group is actually isomorphic to S_3, but we agree to consider only the standard symmetries that interchange the two right hand vertices in the diagram.

From now on, G will stand exclusively for one of our quasisplit groups of type $\mathbf{B}_n$, $\mathbf{C}_n$ or $\mathbf{D}_n$. Its construction above relies on the identification of the supplementary Galois action on the diagram with the Galois action on the underlying split group (determined by a fixed splitting). The transfer of this action to the dual group $\widehat{G}$ is what is used to define the semidirect product ${}^LG = \widehat{G} \rtimes \Gamma_F$. (See [K, §1].). We list the three families of objects $(G, \widehat{G}, {}^LG_E)$ explicitly, where E/F is the minimal Galois extension through which the action of Γ_F factors.

Type $\mathbf{B_n}$: $G = SO(2n+1)$ is split, $\widehat{G} = Sp(2n, \mathbb{C}) = {}^LG_E$, $E = F$.

Type $\mathbf{C_n}$: $G = Sp(2n)$ is split, $\widehat{G} = SO(2n+1, \mathbb{C}) = {}^LG_E$, $E = F$.

Type $\mathbf{D_n}$: $G = SO(2n)$ is quasisplit, $\widehat{G} = SO(2n, \mathbb{C})$, ${}^LG_E = SO(2n, \mathbb{C}) \rtimes \Gamma_{E/F}$, $\deg(E/F) \in \{1, 2\}$.

The other family corresponds to diagrams of type $\mathbf{A}_n$. We are taking the underlying group in this case to be the split group $GL(N)$, with dual group $GL(N, \mathbb{C})$, and minimal L-group ${}^L\big(GL(N)\big)_E$ also equal to $GL(N, \mathbb{C})$, for $N = n + 1$ and $E = F$.

The monograph [A4] is devoted to a classification of automorphic representations of any of our groups G in terms of those of general linear groups. In the rest of this section, we shall discuss how it relates to functoriality and L-functions, the central themes of this article. The classification is based on two general cases of the principle of functoriality. We shall describe them each in turn, following the remarks at the end of §1 of [A5].

Cases of Functoriality: 1. This case arises from the natural embedding of a complex classical group into a complex general linear group. According to our understanding, G is any one of our quasisplit classical groups of type $\mathbf{B}_n$, $\mathbf{C}_n$ or $\mathbf{D}_n$ over F. There is then a canonical embedding of the dual group $\widehat{G}$ into a general linear group $GL(N, \mathbb{C})$, for N equal to $2n$, $2n+1$ and $2n$ respectively. If G is split over F, this extends trivially to a canonical L-embedding

$$ {}^LG = \widehat{G} \times \Gamma_F \longrightarrow {}^L\big(GL(N)\big) = GL(N, \mathbb{C}) \times \Gamma_F $$

of the full L-group of G to that of $GL(N)$. In the special case of type $\mathbf{C}_n$, we also obtain a nonstandard L-embedding of ${}^LG$ into ${}^L\big(GL(N)\big)$ for any quadratic extension E/F, by mapping the quotient $\Gamma_{E/F} = \Gamma_F/\Gamma_E$ isomorphically into the central subgroup $\{\pm 1\}$ of the image of $O(2n+1, \mathbb{C})$ in $GL(N, \mathbb{C})$. If G is not split over F, it is of type $\mathbf{D}_n$. The associated quadratic quotient $\Gamma_{E/F}$ then acts on $\widehat{G} = SO(2n, \mathbb{C})$ through the nonidentity connected component of the complex group $O(2n, \mathbb{C})$. This leads again to a canonical L-embedding of L-groups

$$ {}^LG = \widehat{G} \rtimes \Gamma_F \longrightarrow {}^L\big(GL(N)\big) = GL(N, \mathbb{C}) \times \Gamma_F. $$

2. In the second general case, G is as in the first. This time, however, we take a product

$$ G' = G'_1 \times G'_2 $$

of smaller such groups. We require that the dual group

$$\widehat{G}' = \widehat{G}'_1 \times \widehat{G}'_2$$

come with a natural embedding into $\widehat{G}$. This means that

$$\widehat{G}' = Sp(2m, \mathbb{C}) \times Sp(2n-2m, \mathbb{C}) \subset Sp(2n, \mathbb{C}) = \widehat{G},$$

$$\widehat{G}' = SO(2m, \mathbb{C}) \times SO(2n+1-2m, \mathbb{C}) \subset SO(2n+1, \mathbb{C}) = \widehat{G},$$

and

$$\widehat{G}' = SO(2m, \mathbb{C}) \times SO(2n-2m, \mathbb{C}) \subset SO(2n, \mathbb{C}) = \widehat{G},$$

for integers $0 \leq m \leq n$, when G is of type $\mathbf{B}_n$, $\mathbf{C}_n$ and $\mathbf{D}_n$ respectively. If G is of type $\mathbf{B}_n$, G' is split, and the L-embedding of ${}^L G'$ into ${}^L G$ extends trivially to an L-embedding of ${}^L G'$ into ${}^L G$. If G is of type $\mathbf{C}_n$, G and G'_2 are split, but G'_1 can be a quasisplit group defined by an extension E_1 of F of degree 1 or 2. In this case, we obtain an L-embedding

$${}^L G' = (\widehat{G}'_1 \times \widehat{G}'_2) \rtimes \Gamma_F \longrightarrow {}^L G = \widehat{G} \times \Gamma_F$$

from the nonstandard embedding of the second factor ${}^L G'_2$ attached to the quadratic extension E_1/F. Finally, if G is of type $\mathbf{D}_n$, it is the quasisplit group defined by an extension $E = F(\sqrt{d})$ of degree 1 or 2. We can then take G'_1 and G'_2 to be quasisplit groups of types D_m and D_{n-m} defined by any extensions $E_1 = F(\sqrt{d_1})$ and $E_2 = F(\sqrt{d_2})$ such that $d_1 d_2$ equals d. It is then easy to see that there is a natural L-embedding of L-groups

$${}^L G' = (\widehat{G}'_1 \times \widehat{G}'_2) \rtimes \Gamma_F \longrightarrow {}^L G = \widehat{G} \rtimes \Gamma_F.$$

We thus obtain two basic cases of the principle of functoriality by taking the L-homomorphism ρ to be any one of the L-embeddings we have just described. The first is at the heart of the classification of representations of G (both local and global) in terms of those of $GL(N)$. The second provides the foundation for an understanding of the precise functorial correspondence from G to $GL(N)$.

Theorem 5.1. *The principle of functoriality stated in §4 is valid if ρ is any one of the L-embeddings in the two general cases described above.*

This theorem is a consequence of the classification of representations of G in [A4]. It has a significant application to Rankin-Selberg products. These are the automorphic L-functions whose arithmetic analogues correspond to tensor products of finite dimensional representations of Γ_F (or W_F).

We first review the standard theory of Rankin-Selberg products for general linear groups. In this case, the underlying quasisplit group is a product $GL(N_1) \times GL(N_2)$, while the underlying representation $r = r_N$ of its L-group is given by the standard representation

$$g = g_{N_1} \times g_{N_2} : \ X \longrightarrow g_{N_1} \times {}^t g_{N_2}, \qquad g_{N_i} \in GL(N_i, \mathbb{C}),$$

of the dual group $GL(N_1, \mathbb{C}) \times GL(N_2, \mathbb{C})$ on the $N = N_1 N_2$-dimensional vector space of complex $(N_1 \times N_2)$-matrices X. For any automorphic representation $\pi_N = \pi_{N_1} \otimes \pi_{N_2}$ of this group, we can form the incomplete L-function

$$L^S(s, \pi_{N_1} \times \pi_{N_2}) = L^S(s, \pi_N, r_N)$$

of (3.4). In this case, it is known how to define the local L-functions

$$L(s, \pi_{N_1,v} \times \pi_{N_2,v}) = L(s, \pi_{N,v}, r_{N,v}) \tag{5.1}$$

and ε-factors

$$\varepsilon(s, \pi_{N_1,v} \times \pi_{N_2,v}, \psi_v) = \varepsilon(s, \pi_{N,v}, r_{N,v}, \psi_v) \tag{5.2}$$

in a purely local manner for all valuations v, in such a way that the completed L-function

$$L(s, \pi_{N_1} \times \pi_{N_2}) = L_S(s, \pi_{N_1} \times \pi_{N_2})\, L^S(s, \pi_{N_1} \times \pi_{N_2}) = L_S(s, \pi_N, r_N)\, L^S(s, \pi_N, r_N)$$

has analytic continuation and functional equation

$$L(s, \pi_{N_1} \times \pi_{N_2}) = \varepsilon(s, \pi_{N_1} \times \pi_{N_2})\, L(1 - s, \pi_{N_1}^\vee \times \pi_{N_2}^\vee),$$

for the associated global ε-factor

$$\varepsilon(s, \pi_{N_1} \times \pi_{N_2}) = \varepsilon(s, \pi_N, r_N).$$

The general principle of functoriality applies to the mapping r_N from $GL(N_1, \mathbb{C}) \times GL(N_2, \mathbb{C})$ to $GL(N, \mathbb{C})$. However, it is far from known in this case. On the other hand, and in contrast to Langlands' first application of functoriality described in §4, the analytic continuation and functional equation of Rankin-Selberg products has been established directly. There have been two different approaches to the theory, both of which lead to the same results. The original method [JS], [JPS], [MW1, Appendix], [CP] combines certain integrals with the Poisson summation formula, in a way that is reminiscent of Tate's thesis [T1] (which applies to the special case that $(N_1, N_2) = (N, 1)$). The other approach, known as the Langlands-Shahidi method [L2], [Sha], [CPS], combines Whittaker models and intertwining operators with the analytic continuation and functional equations for Eisenstein series established by Langlands in his study [L3] of continuous automorphic spectra. It is capable of considerably broader application.

Our application of Theorem 5.1 is to Rankin-Selberg products for classical groups, specifically a product $G_1 \times G_2$ of any two groups from our general family of quasisplit special orthogonal and symplectic groups. From the standard L-embeddings

$$\rho_i : \ {}^L G_i \longrightarrow {}^L\big(GL(N_i)\big), \qquad i = 1, 2,$$

of Case 1 above, we obtain a homomorphism

$$\rho_1 \times \rho_2 : \ {}^L(G_1 \times G_2) \longrightarrow GL(N_1, \mathbb{C}) \times GL(N_2, \mathbb{C}).$$

If $\pi = \pi_1 \otimes \pi_2$ is an automorphic representation of $G_1 \times G_2$, and r is the composition $r_N \circ (\rho_1 \times \rho_2)$, we can form the partial L-function

$$L^S(s, \pi_1 \times \pi_2) = L^S(s, \pi, r)$$

for the group $G_1 \times G_2$. We apply Theorem 5.1 to the two L-embeddings ρ_i. It attaches to the two automorphic representations $\pi_i \in \Pi_{\mathrm{aut}}(G_i)$ two (self-dual, isobaric) automorphic representations $\pi_{N_i} \in \Pi_{\mathrm{aut}}(N_i)$ for the general linear groups $GL(N_i)$, such that

$$L^S(s, \pi_1 \times \pi_2) = L^S(s, \pi_{N_1} \times \pi_{N_2}).$$

In other words, the partial L-function for $G_1 \times G_2$ on the left equals its analogue for $GL(N_1) \times GL(N_2)$ on the right. It follows from the theory we have just described for general linear groups that we can define the supplementary L-factor

$$L_S(s, \pi_1 \times \pi_2) = L_S(s, \pi_{N_1} \times \pi_{N_2}) \tag{5.3}$$

and the global ε-factor

$$\varepsilon(s, \pi_1 \times \pi_2) = \varepsilon(s, \pi_{N_1} \times \pi_{N_2}) \tag{5.4}$$

for $G_1 \times G_2$ so that the completed L-function

$$L(s, \pi_1 \times \pi_2) = L_S(s, \pi_1 \times \pi_2) L^S(s, \pi_1 \times \pi_2) \tag{5.5}$$

has analytic continuation, with the functional equation

$$L(s, \pi_1 \times \pi_2) = \varepsilon(s, \pi_1 \times \pi_2)\, L(1 - s, \pi_1 \times \pi_2). \tag{5.6}$$

Our discussion for $G_1 \times G_2$ does not to this point include a local theory. That is, it does not give a local construction of the factors implicit in the left hand sides of (5.3) and (5.4). This is in contrast to the theory for $GL(N)$, which not only gives a local construction for the factors (5.1) and (5.2) for the right hand side, but also relates them (according to the local Langlands correspondence for $GL(N)$) to their arithmetic analogues in (3.11) and (3.13), for representations $r_{N_1,v} \otimes r_{N_2,v}$ of the local Weil groups W_{F_v}. The stronger results for $G_1 \times G_2$ follow, at least for representations π_i that are globally tempered, from the local and global classifications in [A4].

In summary, Theorem 5.1 establishes two cases of functoriality for quasisplit orthogonal and symplectic groups. As a corollary of the first case, we also obtain the analytic continuation of the corresponding Rankin-Selberg L-functions (5.5), with functional equation (5.6). This last result is very much in the spirit of our earlier discussion. Like Artin's proof of analytic continuation and functional equation for the abelian L-functions that bear his name, and Langlands' reduction of the analytic properties of general Artin L-functions and general automorphic L-functions to the principle of functoriality, the approach is indirect. Rather than deal with the unknown L-functions directly, we establish classification theorems that limit their scope. That is, contrary perhaps to what might have been expected, the unknown L-functions are in fact included among a class of L-functions whose analytic behaviour *is* understood.

6. Remarks on the classification

We have not described the classification [A4] that might have been the natural topic of this article, having chosen instead to focus on its simpler implications for our historical introduction to the Langlands program. The classification is given by Theorems 1.5.1, 1.5.2 and 1.5.3 of [A4]. It is also summarized from different points of view in the three surveys [A3], [A5] and [A6]. Partial results were established earlier for generic representations in [CKPS] and [GRS], by quite different methods. It is now possible to see where the generic representations of these papers fit into the general classification [A4, Proposition 8.3.2].

We conclude with a few very general remarks on the structure of the classification. The first of the two cases of functoriality in Theorem 5.1 gives a canonical mapping

$$\mathcal{C}_{\mathrm{aut}}(G) \longrightarrow \mathcal{C}_{\mathrm{aut}}(N) = \mathcal{C}_{\mathrm{aut}}\big(GL(N)\big), \tag{6.1}$$

from automorphic eigenfamilies for our classical group G to automorphic eigenfamilies for $GL(N)$. The methods of [A4] are designed for the representations that occur in the spectral decomposition of $L^2\big(G(F)\backslash G(\mathbb{A})\big)$, namely the subset $\Pi(G) \subset \Pi_{\mathrm{aut}}(G)$ of automorphic representations we are calling globally tempered. The version of Theorem 5.1 that arises[8] most directly from [A4] actually applies to automorphic eigenfamilies that are globally tempered, namely the image $\mathcal{C}(G) \subset \mathcal{C}_{\mathrm{aut}}(G)$ of $\Pi(G)$ under the mapping $\pi \to c(\pi)$. The restriction of (6.1) can be seen from [A4] to give a canonical mapping

$$\mathcal{C}(G) \longrightarrow \mathcal{C}(N) \tag{6.2}$$

from $\mathcal{C}(G)$ to the image $\mathcal{C}(N) \subset \mathcal{C}_{\mathrm{aut}}(N)$ of the set $\Pi(N)$ of globally tempered automorphic representations of $GL(N)$. We shall comment briefly on the general steps needed to obtain a classification[9] of $\Pi(G)$ from (6.2).

The general linear group $GL(N)$ has the remarkable property that the mapping $\pi_N \to c(\pi_N)$ from $\Pi(N)$ to $\mathcal{C}(N)$ is a bijection. This follows from fundamental theorems of Jacquet-Shalika [JS] and Moeglin-Waldspurger [MW1]. (See [A4, §1.3] and [A6, §4].) The composition

$$\Pi(G) \longrightarrow \mathcal{C}(G) \longrightarrow \mathcal{C}(N) \xleftarrow{\sim} \Pi(N)$$

then gives a mapping $\pi \to \pi_N$ from $\Pi(G)$ to $\Pi(N)$. Langlands' theory of Eisenstein series [L3] constructs the automorphic spectrum of any group in terms of automorphic discrete spectra. For our group G, it is therefore enough to classify the subset $\Pi_2(G)$ of representations in $\Pi(G)$ that occur in the discrete spectrum, the subspace $L^2_{\mathrm{disc}}\big(G(F)\backslash G(\mathbb{A})\big)$ of $L^2\big(G(F)\backslash G(\mathbb{A})\big)$ that decomposes under right translation by $G(\mathbb{A})$ into a direct *sum* of irreducible representations. To classify

[8] It is not stated explicitly in [A4]. In fact, the analogue of the second case of Theorem 5.1 is not quite true for $\mathcal{C}(G)$, thanks to an interesting pathology discovered by Cogdell and Piatetskii-Shapiro. (See [A5, §3] and [A6, §8].)

[9] In principle, one can obtain a classification of the larger set $\Pi_{\mathrm{aut}}(G)$ from that of $\Pi(G)$ in [A4] and the criterion for automorphy in [L4]. (See [A6, §8].)

automorphic representations of G in terms of those of $GL(N)$, we would need to give an explicit description of the restricted mapping

$$\pi \longrightarrow \pi_N, \qquad \pi \in \Pi_2(G), \tag{6.3}$$

from $\Pi_2(G)$ to $\Pi(N)$. Specifically, we would need to characterize the image and the kernel of this mapping.

To characterize the image of (6.3), it is necessary to analyze the (globally tempered) automorphic representations of $GL(N)$ that are self-dual. This is not difficult to do, using the general structure of the set of self-dual, N-dimensional representations of an arbitrary Galois group $\Gamma_{E/F}$ for guidance, and the classification in [MW1] of the automorphic, relatively discrete spectrum of $GL(N)$ [A4, §1.2, 1.4]. The problem, it then turns out, is to establish two necessary and sufficient conditions for a self-dual *cuspidal* automorphic representation π_N of $GL(N)$ to lie in the image of (6.3). One is a familiar condition [CKPS], [GRS] on the existence of a pole at $s = 1$ of a certain automorphic L-function of π_N. The other is a more technical condition in harmonic analysis, which is harder to state, but which is at the centre of the argument. The two conditions are among the last things to be established in the classification, but they lead in the end to a clear description of the image of (6.3).

The fibres of (6.3) are often large. They occur in packets

$$\Pi_\psi, \qquad \psi \in \Psi_2(G),$$

parametrized by a family $\Psi_2(G)$ of objects ψ that is in canonical bijection with the image of (6.3), the subset of $\Pi(N)$ we have just discussed. These global "parameters" have localizations ψ_v at valuations v, which are parameters in the more familiar sense. They belong to the set $\Psi(G_v)$ of local L-homomorphisms[10]

$$\psi_v : \; L_{F_v} \times SU(2) \longrightarrow {}^L G_v,$$

taken up to $\widehat{G}$-conjugacy in the local L-group ${}^L G_v$, such that the image of ψ_v in $\widehat{G}$ is relatively compact. A significant part of the global classification in [A4] is purely local. To every local parameter $\psi_v \in \Psi(G_v)$, one has to attach a canonical, *finite* set $\Pi_{\psi_v} \subset \Pi_{\text{unit}}(G_v)$ of irreducible unitary representations of $G(F_v)$. A global packet Π_ψ is then defined as the set of restricted tensor products

$$\Pi_\psi = \Big\{ \pi = \widetilde{\bigotimes_v} \, \pi_v : \; \pi_v \in \Pi_{\psi_v} \Big\} \tag{6.4}$$

of representations from the corresponding local packets. The subset $\Phi_{\text{bdd}}(G_v)$ of parameters $\phi_v \in \Psi(G_v)$ that are constant on the factor $SU(2)$ are known as (bounded) Langlands parameters. A prerequisite for the study of the general packets Π_{ψ_v} in Chapter 7 of [A4] is the proof in Chapter 6 of [A4] of the local Langlands

[10] In the domain, L_{F_v} is the local Langlands group. It is defined as the local Weil group W_{F_v} if v is archimedean, and the product of W_{F_v} with a separate copy of $SU(2)$ if v is nonarchimedean

correspondence for G_v. This asserts[11] that the set $\Pi_{\mathrm{temp}}(G_v)$ of (locally) tempered, irreducible representations of $G(F_v)$ is a disjoint union over $\phi_v \in \Phi_{\mathrm{bdd}}(G_v)$ of the local Langlands packets Π_{ϕ_v}.

The global classification thus depends on a local description of representations in order to define the global packets (6.4). The main global result of [A5] is Theorem 1.5.2. It gives a multiplicity formula for any irreducible representation of $G(\mathbb{A})$ in the discrete spectrum. More precisely, the theorem asserts that any representation in $\Pi_2(G)$ lies in a unique global packet Π_ψ. For any $\pi \in \Pi_\psi$, it then gives an explicit multiplicity formula for π in $L^2_{\mathrm{disc}}\big(G(F)\backslash G(\mathbb{A})\big)$ in terms of invariants attached to its local constituents. Its proof is a multifaceted induction, which includes most of the other results in [A4], and takes up much of the volume.

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Department of Mathematics, University of Toronto, Toronto, Canada, M5S 2E4
E-mail: arthur@math.toronto.edu