EIGENFAMILIES, CHARACTERS AND MULTIPLICITIES

JAMES ARTHUR

Contents

Foreword

This article is expository. It consists of a short description of the main results of [A2], namely a characterization of the automorphic discrete spectrum of a quasisplit orthogonal or symplectic group G. The article [A3] also contains a summary of the results of [A2]. However, we simplified the discussion there by defining global parameters in terms of the hypothetical global Langlands group L_F. Our focus here will be somewhat different. In particular, we shall formulate the global parameters we need as in the original monograph, simplified somewhat, but still without recourse to the undefined group L_F.

We are assuming for the moment that the field F is global (of characteristic 0). We recall that the global Langlands group L_F is a hypothetical, locally compact extension of the global Weil group W_F by a subgroup K_F that is compact, connected and (if we are prepared to be optimistic) even simply connected. It would be characterized by the property that its irreducible, unitary, N-dimensional representations parametrize unitary cuspidal automorphic representations of the general linear group $GL(N)$ over F. However, its existence is far deeper than any theorems now available. The present role of L_F is therefore confined to one of motivation and guidance.

The global parameters ψ in [A2] were in fact defined crudely in terms of cuspidal automorphic representations of general linear groups (rather than irreducible finite dimensional representations of the hypothetical group L_F). This leads to a workable substitute $\mathcal{L}_\psi$ for L_F. But as the notation suggests, it has the unfortunate property of being dependent on ψ. We would be better off having a group that at the very least is independent of ψ. I had originally planned to include the construction of such a group in this paper. It is a locally compact group $\widetilde{L}_F^*$ over W_F that is indeed independent of ψ, and which for the purposes of [A2] should serve as a substitute for the universal group L_F. It amounts to an extension of the group $\widetilde{L}^*_{F,\mathrm{reg}}$ introduced in [A2, §8.5]. However, the construction of $\widetilde{L}^*_F$ is related to questions in base change and automorphic induction that, for me at least, require some further thought. Rather than take the time here, I shall leave it for another paper.

This article will therefore be restricted to our brief survey of results from [A2]. It consists of three sections, each devoted to its own general theme. We have chosen the title to reflect these themes, and to draw attention to another difference from the survey [A3]. We have tried here to motivate the results from a more elementary and explicit point of view. Each theme leads naturally to the next, until we end in §3 with the global multiplicity formula for G. I hope that the two surveys will be complementary, despite inevitably having much in common. In this article we have emphasized the underlying context of the results (including the role of L_F and its possible substitutes), while [A3] was designed more as a guide to their proofs. In particular, there will be no discussion here of the trace formula for G and its stabilization, or the twisted trace formula for $GL(N)$, and its conditional stabilization on which the results still depend.

In §1, we describe automorphic families

$$c = \{c_v : \ v \notin S\}$$

of Hecke eigenvalues for G. The general transfer of these objects is perhaps the most concrete and fundamental manifestation of Langlands's principle of functoriality. However, the endoscopic transfer of Hecke eigenfamilies leads immediately to the more complex question of how automorphic spectra behave under transfer. This question cannot be framed in the absence of further local information. It forces us to provide a corresponding local theory of endoscopic transfer.

In §2, we describe the classification of irreducible representations of a localization $G(F_v)$ of G. These results will be formulated explicitly in terms of irreducible characters, and the transfer factors of Kottwitz, Langlands and Shelstad. We will then be able to state the main global theorem in §3. It gives a decomposition of the automorphic discrete spectrum

$$L^2_{\mathrm{disc}}\big(G(F)\backslash G(\mathbb{A})\big)$$

of G in terms of global, "square integrable" parameters $\psi \in \widetilde{\Psi}_2(G)$. The data ψ are the global objects that would be defined naturally in terms of the hypothetical group L_F, but which must in practice be constructed in a more prosaic manner.

The results described in §1–§3 are special cases of Langlands's conjectural theory of endoscopy. They also give special cases of the broader principle of functoriality. However, they occupy a special niche within the general theory. This is because a global parameter $\psi \in \widetilde{\Psi}_2(G)$ is uniquely determined by its associated Hecke eigenfamily

$$c(\psi) = \big\{c_v(\psi) = c(\psi_v) : \ v \notin S\big\},$$

regarded in fact as a family of conjugacy classes in a complex general linear group $GL(N, \mathbb{C})$. In other words, the automorphic representation theory of G is governed by the concrete objects introduced early in §1. This circumstance is also behind the construction of the group $\widetilde{L}^*_F$, which we have postponed for now.

We conclude the introduction with a review of the relevant groups. We take F to be a local or global field of characteristic 0, and G to be a quasisplit, special orthogonal or symplectic group over F. (We assume always that G is "classical", in the sense that it is not an outer twist of the split group $SO(8)$ by a triality automorphism.) For the first three sections of this paper, we follow the conventions from the beginning of [A3]. Then G has a complex dual group $\widehat{G}$, and a corresponding L-group

$$^{L}G = \widehat{G} \rtimes \Gamma_{E/F}.$$

We are taking $\Gamma_{E/F} = \mathrm{Gal}(E/F)$ to be the Galois group of a suitable finite extension E/F. If G is split, for example, the absolute Galois group $\Gamma = \Gamma_F = \Gamma_{\overline{F}/F}$ acts trivially on $\widehat{G}$, and we often take $E = F$.

There are three general possibilities for G, whose description we take from page 2 of [A3]. They correspond to the three infinite families of simple groups $\mathbf{B}_n$, $\mathbf{C}_n$ and $\mathbf{D}_n$, and are as follows.

Type $\mathbf{B}_n$: $G = SO(2n+1)$ is split, and $\widehat{G} = Sp(2n, \mathbb{C}) = {}^{L}G$.

Type $\mathbf{C}_n$: $G = Sp(2n)$ is split, and $\widehat{G} = SO(2n+1, \mathbb{C}) = {}^{L}G$.

Type $\mathbf{D}_n$: $G = SO(2n)$ is quasisplit, and $\widehat{G} = SO(2n, \mathbb{C})$. In this case, we can take ${}^{L}G$ to be the semidirect product of $SO(2n, \mathbb{C})$ with $\Gamma_{E/F}$, where E/F is an arbitrary extension of degree 1 or 2 whose Galois group acts by outer automorphisms on $SO(2n, \mathbb{C})$ (which is to say, by automorphisms that preserve a fixed splitting of $SL(2n, \mathbb{C})$). The nontrivial outer autmorphism of $SO(2n, \mathbb{C})$ is induced by conjugation by some element in its complement in $O(2n, \mathbb{C})$.

The other infinite family of simple groups is of course $\mathbf{A}_n$. We regard the split (reductive) group $GL(N)$, with $N = n+1$, as our representative from this family. Its role is different. For we are treating the representations of $GL(N)$ as known objects, in terms of which we want to classify the representations of G. We write

$$\widetilde{\theta}(N):\ x \longrightarrow \widetilde{J}(N)\ {}^{t}x^{-1}\widetilde{J}(N)^{-1}, \qquad \widetilde{J}(N) = \begin{pmatrix} & 0 & & 1 \\ & & -1 & \\ & & \iddots & \\ (-1)^{N+1} & & & 0 \end{pmatrix},$$

for the outer automorphism of $GL(N)$ that stabilizes the standard splitting, and

$$\widetilde{G}(N)^{+} = GL(N) \rtimes \langle \widetilde{\theta}(N) \rangle$$

for the semidirect product of $GL(N)$ with the group (of order 2) generated by $\widetilde{\theta}(N)$. It is a union of the connected component

$$\widetilde{G}(N) = GL(N) \rtimes \widetilde{\theta}(N)$$

and the identity component $\widetilde{G}(N)^0 = GL(N)$. Of special interest are the irreducible representations of $GL(N)$ that are $\widetilde{\theta}(N)$-stable, which is to say that they extend to the group $\widetilde{G}(N)^{+}$.

We have introduced the "minimal" L-group ${}^LG = \widehat{G} \rtimes \Gamma_{E/F}$ above for simplicity. It suffices for many purposes. However, one is sometimes forced to take the Galois extension E/F to be large. For this, it is easiest just to take the "maximal" L-group, either its Galois form

$$ {}^LG = \widehat{G} \rtimes \Gamma_F, $$

or its Weil form

$$ {}^LG = \widehat{G} \rtimes W_F. $$

The former will be used at some point in §3, while the latter is used for the Langlands group L_F and its approximation $\widetilde{L}_F^*$.

The integers n of course refer to the number of vertices in the relevant Coxeter-Dynkin diagrams. In the expository interests of this article, we will generally focus on a given orthogonal or symplectic group G, rather than the set of G attached as twisted endoscopic data to a given general linear group. In other words, we will usually fix G, and then take the general linear group $GL(N)$ attached to the standard representation of ${}^LG$. It will thus be understood implicitly that N equals $2n$, $2n+1$ and $2n$ in the three cases $\mathbf{B}_n$, $\mathbf{C}_n$ and $\mathbf{D}_n$. Note that if G is of type $\mathbf{C}_n$, and we happen to be working with the maximal, Galois form

$$ {}^LG = \widehat{G} \rtimes \Gamma_F = SO(2n+1, \mathbb{C}) \rtimes \Gamma_F $$

of the L-group, the standard representation is understood to be trivial on the Galois factor Γ_F. This represents the *canonical* twisted endoscopic datum for $GL(N)$, whose complement would be given by the set of embeddings parametrized by characters of Γ_F of order 2. (See [A2, §1.2].)

1. Hecke eigenfamilies

In this section, we take the field F to be global. Our theme will be the families of Hecke eigenvalues, Hecke eigenfamilies, at the heart of automorphic representations. They are conjectured to carry information that would characterize much of the arithmetic word, according to a basic premise of the Langlands program.

We begin with the general linear group $GL(N)$. We shall recall two fundamental theorems for this group. These are the global foundation for the study of automorphic representations of the other three families of groups G.

Given N, we consider the set $\Psi_{\mathrm{sim}}(N) = \Psi_{\mathrm{sim}}\big(GL(N)\big)$ of triplets consisting of:

(i) a decomposition $N = mn$, for positive integers m and n;

(ii) an irreducible, unitary, cuspidal automorphic representation μ of the group $GL(m)$;

(iii) the unique irreducible representation ν of the group $SU(2)$ of dimension n.

Theorem 1.1 (Moeglin-Waldspurger [MW]). *There is a canonical bijection*

$$ \psi \longrightarrow \pi_\psi, \qquad \psi \in \Psi_{\mathrm{sim}}(N), \tag{1.1} $$

from $\Psi_{\mathrm{sim}}(N)$ onto the set of irreducible unitary representations of $GL(N,\mathbb{A})$ that occur in the automorphic, relative discrete spectrum $L^2_{\mathrm{disc}}\big(GL(N,F)\backslash GL(N,\mathbb{A})\big)$ of $GL(N)$. Moreover, for any ψ, π_ψ occurs in the relative discrete spectrum with multiplicity one.

Moeglin and Waldspurger construct π_ψ explicitly as a multi-residue of a cuspidal Eisenstein series attached to μ. More precisely, a certain Eisenstein multi-residue provides an intertwining operator from a global Langlands quotient, the global Speh representation π_ψ obtained by parabolic induction from the nonunitary representation

$$x \longrightarrow \mu(x_1)|\det x_1|^{\frac{n-1}{2}} \otimes \mu(x_2)|\det x_2|^{\frac{n-3}{2}} \otimes \cdots \otimes \mu(x_n)|\det x_n|^{-\frac{n-1}{2}} \tag{1.2}$$

of the standard Levi subgroup

$$M_P(\mathbb{A}) = \big\{x = (x_1, \dots, x_n) : \; x_i \in GL(m, \mathbb{A})\big\}$$

of $GL(N, \mathbb{A})$, to a constituent of the relative discrete spectrum. The deepest part of the theorem is to show that there is nothing further in the relative discrete spectrum. This entails a sustained analysis of Chapter 7 of Langlands's monograph [L1], and the various supplementary residues that can arise from it.

Corollary 1.2. *Let $\Psi(N) = \Psi\big(GL(N)\big)$ be the set of pairs consisting of*
(i) *a partition $N = N_1 + \cdots + N_r$ of N;*
(ii) *a formal unordered sum*

$$\psi = \psi_1 \boxplus \cdots \boxplus \psi_r, \qquad \psi_i \in \Psi_{\mathrm{sim}}(N_i).$$

There is then a bijection

$$\psi \longrightarrow \pi_\psi, \qquad \psi \in \Psi(N),$$

from $\Psi(N)$ onto the set of irreducible constituents of the full automorphic spectrum $L^2\big(GL(N, F)\backslash GL(N, \mathbb{A})\big)$ of $GL(N)$.

The corollary is a consequence of Langlands's general construction of automorphic spectra from relative discrete spectra of Levi subgroups. For the given element $\psi \in \Psi(N)$, π_ψ is given by parabolic induction of the unitary representation

$$\pi_{\psi_1}(x_1) \otimes \cdots \otimes \pi_{\psi_r}(x_r)$$

of the standard Levi subgroup

$$M_P(\mathbb{A}) = \big\{x = (x_1, \dots, x_r) : \; x_i \in GL(N_i, \mathbb{A})\big\}$$

of $GL(N, \mathbb{A})$.

If we had the hypothetical Langlands group L_F at our disposal, $\Psi(N)$ could be identified with the set of unitary, N-dimensional representations

$$\psi : \; L_F \times SU(2) \longrightarrow GL(N, \mathbb{C}) \tag{1.3}$$

of the product of L_F with $SU(2)$. The subset $\Psi_{\mathrm{sim}}(N)$ would then be identified with the set of irreducible representations in $\Psi(N)$. As matters stand here, the irreducible representation of ν of $SU(2)$ attached to an element $\psi \in \Psi_{\mathrm{sim}}(N)$ is not explicit in the construction. One sees only its weights, which are represented by the quasicharacters

$$|\cdot|^{\frac{n-1}{2}}, |\cdot|^{\frac{n-3}{2}}, \dots, |\cdot|^{-\frac{n-1}{2}}$$

in (1.2). We often write $\Phi_{\mathrm{bdd}}(N)$ for the subset of elements $\phi = \psi$ in $\Psi(N)$ whose simple factors ψ_i come with the trivial representation $\nu_i = 1$ of $SU(2)$. They would of course correspond to representations (1.3) that are trivial on the factor $SU(2)$.

Suppose that π is an irreducible (admissible) representation of $GL(N, \mathbb{A})$. Then π is unramified at almost all valuations v of F. We recall that for any v, the Satake transform gives a canonical bijection

$$\pi_v \longrightarrow c(\pi_v)$$

from the set of unramified irreducible representations π_v of $GL(N, F_v)$ to the set of semisimple conjugacy classes c_v in the dual group $GL(N, \mathbb{C})$ of $GL(N)$. The given global representation π thus gives rise to a family

$$c(\pi) = \big\{c_v(\pi) = c(\pi_v) : \ v \notin S\big\}$$

of semisimple conjugacy classes in $GL(N, \mathbb{C})$, parametrized by a cofinite set of valuations v, and taken up to the equivalence relation obtained by setting $c \sim c'$ if $c_v = c'_v$ for almost all v. We will call $c(\pi)$ a *Hecke eigenfamily*. It represents a set of simultaneous eigenvalues for the action of the factors of the restricted tensor product

$$\mathcal{H}^S_{\mathrm{un}}(N) = \widetilde{\bigotimes_{v \notin S}} \mathcal{H}_{v,\mathrm{un}}(N)$$

of local unramified Hecke algebras

$$\mathcal{H}_{v,\mathrm{un}}(N) = C_c^\infty\big(GL(N, \mathcal{O}_v) \backslash GL(N, F_v) / GL(N, \mathcal{O}_v)\big),$$

relative to the hyperspecial maximal compact subgroup

$$K^S(N) = \prod_{v \notin S} K_v(N) = \prod_{v \notin S} GL(N, \mathcal{O}_v)$$

of $GL(N, \mathbb{A}^S)$, on the space of $K^S(N)$-invariant vectors of π.

Suppose that ψ belongs to the set $\Psi(N)$ defined in Corollary 1.2. We then obtain a Hecke eigenfamily

$$c(\psi) = c(\pi_\psi) = \big\{c_v(\psi) = c(\pi_{\psi,v}) : \ v \notin S\big\} \tag{1.4}$$

from the irreducible representation π_ψ of $GL(N, \mathbb{A})$. This is to be regarded as a concrete datum, which is attached to the formal object ψ through the automorphic representation π_ψ. According to the remarks following the statements of Theorem 1.1 and Corollary 1.2, $c(\psi)$ is given explicitly in terms of the Hecke eigenfamilies

$$c(\mu_i) = \big\{c_v(\mu_i) : \ v \notin S\big\}, \qquad 1 \le i \le r,$$

of the cuspidal components of the constituents ψ_i of ψ. More precisely, if $\psi \in \Psi_{\mathrm{sim}}(N)$ is as in Theorem 1.1, then

$$c_v(\psi) = c_v(\mu) \otimes c_v(\nu) = c_v(\mu) q_v^{\frac{n-1}{2}} \oplus \cdots \oplus c_v(\mu) q_v^{-\frac{n-1}{2}}, \tag{1.5}$$

while if $\psi \in \Psi(N)$ is a general element as in Corollary 1.2, we have

$$c_v(\psi) = c_v(\pi_i) \oplus \cdots \oplus c_v(\pi_r). \tag{1.6}$$

These objects represent explicit conjugacy classes in $GL(N,\mathbb{C})$.

We write

$$\mathcal{C}(N) = \big\{ c(\psi) : \ \psi \in \Psi(N) \big\} \tag{1.7}$$

for the set of Hecke eigenfamilies attached to elements in $\Psi(N)$.

Theorem 1.3 (Jacquet-Shalika [JS]). *The mapping*

$$\psi \ \longrightarrow \ c(\psi), \qquad \psi \in \Psi(N),$$

is a bijection from $\Psi(N)$ to $\mathcal{C}(N)$.

Historically, Theorem 1.3 predated Theorem 1.1. It applied to a class of automorphic representations of $GL(N)$ Langlands introduced in [L2], and called *isobaric*. At the time, it was not known whether these included the constituents of the automorphic discrete spectrum. Theorem 1.1 implies that these constituents are distinct and isobaric. It therefore yields the interpretation above of the original theorem of Jacquet and Shalika.

The injectivity of the mapping is of course the point of Theorem 1.3. It implies that any information that might be contained in a constituent π_ψ of the automorphic spectrum of $GL(N)$ ought to be reflected somehow in the corresponding Hecke eigenfamily $c(\psi)$. Since $c(\psi)$ appears to contain less information, the ramified local constituents of π_ψ being an obvious gap, and since it is itself just a concrete set of complex parameters, the assertion is quite remarkable. What about the other half of the problem? Can one characterize the image $\mathcal{C}(N)$ of the mapping within the set of all Hecke eigenfamilies? The question is too broad as stated, and would not be expected to have a reasonable answer. Langlands's point of view was to look instead for reciprocity laws between Hecke eigenfamilies in $\mathcal{C}(N)$ and data obtained from other sources. It is in this context that we can frame the classification of automorphic representations of the groups G.

The transition from general linear groups $GL(N)$ to our classical groups G begins with the contragredient involution

$$\pi \ \longrightarrow \ \pi^\vee(x) = \pi({}^t x^{-1}) \cong \big(\pi \circ \widetilde{\theta}(N)\big)(x), \qquad x \in GL(N,\mathbb{A}),$$

on irreducible representations π of $GL(N,\mathbb{A})$. This operation also defines a natural involution $\psi \to \psi^\vee$ on $\Psi(N)$ such that

$$\pi_{\psi^\vee} = \pi_\psi^\vee.$$

It follows from the definitions that

$$c^\vee(\psi) \stackrel{\text{def}}{=\!=} c(\psi^\vee) = \big\{ c_v^\vee(\psi) = c_v(\psi)^{-1} : v \notin S \big\}.$$

We write

$$\widetilde{\Psi}(N) = \big\{ \psi \in \Psi(N) : \ \psi^\vee = \psi \big\}$$

and

$$\widetilde{\mathcal{C}}(N) = \big\{ c \in \mathcal{C}(N) : c^\vee = c \big\}$$

for the subsets of self-dual elements in $\Psi(N)$ and $\mathcal{C}(N)$. They are in bijection under the mapping of the last theorem. As we will see, the automorphic representation theory of the groups G is governed by these sets.

Suppose that G is a quasisplit special orthogonal or symplectic group over F, as at the end of the foreword. Satake transforms and unramified local Hecke algebras are again defined for G, as they are for any connected reductive group over F. An irreducible (admissible) representation π of $G(\mathbb{A})$ then yields a Hecke eigenfamily

$$c(\pi) = \big\{c_v(\pi) = c(\pi_v):\ v \notin S\big\}.$$

Its components $c_v(\pi)$ are semisimple classes in the L-group ${}^LG$, which we define as usual by $\widehat{G}$-conjugacy in the case $\mathbf{B}_n$ and $\mathbf{C}_n$. If G is of type $\mathbf{D}_n$, however, we agree to define the classes in ${}^LG$ by $O(2n,\mathbb{C})$ conjugacy (rather than conjugacy by the subgroup $\widehat{G} = SO(2n,\mathbb{C})$ of index 2). We continue to regard the family as an equivalence class under the relation $c \sim c'$ defined as for $GL(N)$ above.

Given G, we write $\widetilde{\mathcal{C}}(G)$ for the set of Hecke eigenfamilies $c(\pi)$, where π ranges over irreducible representation of $G(\mathbb{A})$ that occur in the automorphic spectrum of $L^2\big(G(F)\backslash G(\mathbb{A})\big)$.

Theorem 1.4. *The embedding of* ${}^LG$ *into* $GL(N,\mathbb{C})$ *gives a canonical mapping*

$$\widetilde{\mathcal{C}}(G) \ \longrightarrow\ \widetilde{\mathcal{C}}(N). \tag{1.8}$$

The theorem asserts that the Hecke eigenfamily for $GL(N)$ attached to an automorphic Hecke eigenfamily for G is automorphic for $GL(N)$. This is essentially Proposition 3.4.1 of [A2], particularly the ensuing Corollary 3.4.3. The corollary actually applies to the discrete spectrum of G, but an easy comparison of Levi subgroups of G and $GL(N)$, together with Langlands's construction of continuous spectra by Eisenstein series, leads to the general result.

It is easy to see that the mapping (1.8) is injective. (See the elementary analysis of [A2, §1.2], with the group Λ_F there taken to be infinite cyclic.) We can therefore regard $\widetilde{\mathcal{C}}(G)$ as a subset of $\widetilde{\mathcal{C}}(N)$. One can actually characterize this subset. To do so, however, would require some of the deeper results of [A2], so we shall put the matter aside for the moment.

Our main focus is the automorphic representation theory of G. We have just seen that the Hecke eigenfamilies attached to automorphic representations of G are among the automorphic Hecke eigenfamilies for $GL(N)$. This is a reciprocity law of the sort mentioned earlier. It represents a proof of a small part of Langlands's principle of functoriality (so called "weak functoriality" for the pair G and $GL(N)$, and the standard embedding of ${}^LG$ into $GL(N,\mathbb{C})$).

To understand the automorphic representation theory of G, we need to supplement the reciprocity law. We would like to make it the foundation for a broader description of the contribution of any element c in $\widetilde{\mathcal{C}}(N)$ (a set whose objects we are regarding as known) to the automorphic spectrum of G. It is enough just to consider the discrete spectrum, by the theory of Eisenstein series. We can therefore pose the problem more precisely as follows. Given any element $\psi \in \widetilde{\Psi}(N)$, and any irreducible representation π in the set

$$\big\{\pi \in \Pi\big(G(\mathbb{A})\big):\ c(\pi) = c(\psi)\big\},$$

find an explicit formula for the multiplicity

$$m_\psi(\pi) = m_{G,\psi}(\pi)$$

of π in the automorphic discrete spectrum of G. This of course would give information about the subset $\widetilde{\mathcal{C}}(G)$ of $\widetilde{\Psi}(N)$. For if $m_\psi(\pi)$ is nonzero for any such π, the Hecke eigenfamily $c(\psi)$ lies in $\widetilde{\mathcal{C}}(G)$. However, the most significant implication of the problem is that it demands an understanding of local representation theory.

In the next section we will describe the local theory of endoscopy for the completions $G(F_v)$ of G. We will formulate results for irreducible representations π_v of $G(F_v)$ explicitly in terms of their characters. This will allow us to describe the answer of the multiplicity question in §3.

2. Local character relations

Throughout this section, we take the field F to be local. We fix a quasisplit special orthogonal or symplectic group G, as at the end of the foreword. The local Langlands group L_F is given by a simple prescription, unlike its hypothetical global counterpart. By definition, we have

$$L_F = \begin{cases} W_F, & \text{if } F \text{ is archimedean,} \\ W_F \times SU(2), & \text{if } F \text{ is } p\text{-adic,} \end{cases}$$

where W_F is the (local) Weil group of F. We are therefore free to define local parameters as L-homomorphisms from L_F to the L-group ${}^LG$.

In [A3, §1], we introduced four families of local parameters for G, and four families of irreducible representations of $G(F)$. These give four pairs $\big(\Phi(G), \Pi(G)\big)$, $\big(\widetilde{\Phi}(G), \widetilde{\Pi}(G)\big)$, $\big(\widetilde{\Phi}_{\mathrm{bdd}}(G), \widetilde{\Pi}_{\mathrm{temp}}(G)\big)$ and $\big(\widetilde{\Psi}(G), \widetilde{\Pi}_{\mathrm{unit}}(G)\big)$ of loosely associated objects. In the first pair, $\Phi(G)$ is the set of L-homomorphisms

$$\phi : \ L_F \longrightarrow \ {}^LG,$$

taken up to $\widehat{G}$-conjugacy, and $\Pi(G)$ is the set of irreducible representations of $G(F)$, taken up to the usual notion of equivalence. The second pair is a quotient

$$\big(\widetilde{\Phi}(G), \widetilde{\Pi}(G)\big) = \big(\Phi(G)/\sim, \, \Pi(G)/\sim\big)$$

of the first. The equivalence relation $\sim$ is trivial in case G is of type $\mathbf{B}_n$ or $\mathbf{C}_n$, and is defined by conjugation of ${}^LG$ by $O(2n, \mathbb{C})$ and $G(F)$ by $O(2n, F)$ (rather than by $SO(2n, \mathbb{C})$ and $SO(2n, F)$) in case G is of type $\mathbf{D}_n$. In the third pair, $\widetilde{\Phi}_{\mathrm{bdd}}(G)$ is the set of (equivalence classes of) parameters in $\widetilde{\Phi}(G)$ of bounded image, and $\widetilde{\Pi}_{\mathrm{temp}}(G)$ is the set of (equivalence classes of) tempered representations in $\widetilde{\Pi}(G)$. In the fourth pair, $\widetilde{\Psi}(G)$ is the set of equivalence classes of L-homomorphisms

$$\psi : \ L_F \times SU(2) \ \longrightarrow \ {}^LG \tag{2.1}$$

such that the restriction of ψ to L_F lies in $\widetilde{\Phi}_{\mathrm{bdd}}(G)$, and $\widetilde{\Pi}_{\mathrm{unit}}(G)$ is the subset of unitary representations in $\widetilde{\Pi}(G)$.

Parameters ψ in the last set $\widetilde{\Psi}(G)$ can be extended analytically to the larger domain $L_F \times SL(2,\mathbb{C})$. For any such ψ, we write

$$\phi_\psi(u) = \psi\left(u, \begin{pmatrix} |u|^{\frac{1}{2}} & 0 \\ 0 & |u|^{-\frac{1}{2}} \end{pmatrix}\right), \qquad u \in L_F,$$

where $|u|$ is the pullback to L_F of the canonical absolute value on W_F. We obtain a mapping

$$\psi \longrightarrow \phi_\psi, \qquad \psi \in \widetilde{\Psi}(G),$$

from $\widetilde{\Psi}(G)$ to $\widetilde{\Phi}(G)$, which is easily seen to be injective. Since we can regard $\widetilde{\Phi}_{\mathrm{bdd}}(G)$ as the subset of parameters in $\widetilde{\Psi}(G)$ that are trivial on the factor $SU(2)$, we obtain canonical embeddings

$$\widetilde{\Phi}_{\mathrm{bdd}}(G) \subset \widetilde{\Phi}(G) \subset \widetilde{\Phi}(G).$$

Similar definitions apply to the group $GL(N)$. We write $\Phi(N) = \Phi\big(GL(N)\big)$, $\Phi_{\mathrm{bdd}}(N) = \Phi_{\mathrm{bdd}}\big(GL(N)\big)$, $\Pi(N) = \Pi\big(GL(N)\big)$, $\Pi_{\mathrm{temp}}(N) = \Pi_{\mathrm{temp}}\big(GL(N)\big)$, and so on. The quotient sets (for groups of type $\mathbf{D}_n$) denoted by a tilde are not relevant to general linear groups. We shall instead use the notation as in §1 to denote subsets of self-dual objects for $GL(N)$.

Theorem 2.1 (Langlands [L3], Harris-Taylor [HT], Henniart [He], Scholze [Sch])**.** *There is a unique bijection*

$$\phi \longrightarrow \pi_\phi, \qquad \phi \in \Phi(N),$$

from $\Phi(N)$ *onto* $\Pi(N)$ *that is compatible with Rankin-Selberg L-functions and* ε*-factors, with the automorphism* $\widetilde{\theta}(N)$ *of* $GL(N)$*, and with tensor products by* 1*-dimensional representations, and that transforms determinants to central characters. Furthermore, the mapping restricts to a bijection between the subsets* $\Phi_{\mathrm{bdd}}(N)$ *and* $\Pi_{\mathrm{temp}}(N)$ *of* $\Phi(N)$ *and* $\Pi(N)$*, and restricts further to a bijection between the subsets* $\widetilde{\Phi}_{\mathrm{bdd}}(N)$ *and* $\widetilde{\Pi}_{\mathrm{temp}}(N)$ *of self-dual elements in* $\Phi_{\mathrm{bdd}}(N)$ *and* $\Pi_{\mathrm{temp}}(N)$*.*

Theorem 2.1 establishes a strong form of the local Langlands correspondence for the group $GL(N)$. For us, it will be the starting point of a local theory of endoscopy for the group G. In this regard, its role amounts to a local analogue of that played by the two global Theorems 1.1 and 1.3.

We return to our group G over F. For any parameter ψ in the subset $\widetilde{\Psi}(G)$ of $\widetilde{\Psi}(N)$, we can define the centralizer

$$S_\psi = \mathrm{Cent}\big(\mathrm{im}(\psi), \widehat{G}\big) \tag{2.2}$$

in $\widehat{G}$ of its image, a complex reductive subgroup of $\widehat{G}$. We can then form the quotient

$$\mathcal{S}_\psi = S_\psi / S_\psi^0 Z(\widehat{G})^{\Gamma_{E/F}} \tag{2.3}$$

where $Z(\widehat{G})^{\Gamma_{E/F}}$ is the subgroup of $\Gamma_{E/F}$-invariants in the centre of $\widehat{G}$. For our group G here, $\mathcal{S}_\psi$ is a finite, abelian 2-group.

Theorem 2.2. (a) *For any $\psi \in \widetilde{\Psi}(G)$, there is a finite "multi-set" $\widetilde{\Pi}_\psi$ in $\widetilde{\Pi}_{\mathrm{unit}}(G)$ (or more precisely, a finite set over $\widetilde{\Pi}_{\mathrm{unit}}(G)$), together with a canonical mapping*

$$\pi \longrightarrow \langle \cdot, \pi \rangle, \qquad \pi \in \widetilde{\Pi}_\psi,$$

from $\widetilde{\Pi}_\psi$ to the group $\widehat{\mathcal{S}}_\psi$ of linear characters on $\mathcal{S}_\psi$, both determined by twisted character relations from $GL(N)$.

(b) *Suppose that $\psi = \phi$ lies in the subset $\widetilde{\Phi}_{\mathrm{bdd}}(G)$ of $\widetilde{\Psi}(G)$. Then the elements in $\widetilde{\Pi}_\phi$ are tempered and multiplicity free (so that $\widetilde{\Pi}_\phi$ is a subset of $\widetilde{\Pi}_{\mathrm{temp}}(G)$). Moreover, the mapping from $\widetilde{\Pi}_\phi$ to $\widehat{\mathcal{S}}_\phi$ is injective, and bijective if F is p-adic. Finally, the set $\widetilde{\Pi}_{\mathrm{temp}}(G)$ is a disjoint union over $\phi \in \widetilde{\Phi}_{\mathrm{bdd}}(G)$ of the packets $\widetilde{\Pi}_\phi$.*

The theorem is stated in [A2, §1.5] as Theorem 1.5.1. It is proved together with its quantitive analogue, which we will state here as Theorem 2.3, in Chapters 6 and 7 of [A2]. The methods are global, specifically, a multifaceted comparison of global trace formulas. However, Theorem 2.1 for $GL(N)$ is an indispensable local ingredient. It allows us to attach representations of $GL(N,F)$ to parameters $\psi \in \widetilde{\Psi}(G)$ through the mapping

$$\widetilde{\Psi}(G) \longrightarrow \widetilde{\Psi}(N),$$

which is defined by the embedding of ${}^LG$ into $GL(N,\mathbb{C})$. Since the mapping is injective (see [A2, §1.2]), we can identify $\widetilde{\Psi}(G)$ with a subset of $\widetilde{\Psi}(N)$, and hence with a set of self-dual unitary representations of $GL(N,F)$. This transforms the proof of the theorem to a series of questions in harmonic analysis, which centre around the problem of attaching packets of representations of $G(F)$ to certain self-dual representations of $GL(N,F)$.

Part (b) of Theorem 2.2 is essentially the local Langlands correspondence for G, while part (a) is a weaker assertion for the more general parameters $\psi \in \widetilde{\Psi}(G)$. Taken as a whole, the theorem is to be regarded as a qualitative theory of local endoscopy for G. To have an explicit form of the theory, however, we need to specify the endoscopic character relations of (a). These will be formulated as the quantitative supplement Theorem 2.3 mentioned above.

Characters are remarkable objects, which are at the heart of local harmonic analysis. Their importance is of course tied to the fact that they determine the representations from which they are derived. As functions that are complex valued rather than matrix valued, they are more explicit, and more amenable to techniques in harmonic analysis.

Character theory for groups over local fields is a centrepiece of the work of Harish-Chandra. Suppose that π is an irreducible (admissible) representation of $G(F)$. Harish-Chandra proved first that the mapping

$$f \longrightarrow f_G(\pi) = \mathrm{tr}\big(\pi(f)\big), \qquad f \in C_c^\infty\big(G(F)\big),$$

is defined, and is a distribution on $G(F)$. He then established the much deeper theorem that is a function [Ha1], [Ha4]. More precisely,

$$f_G(\pi) = \int_{G(F)} \Theta_G(\pi, x) f(x) dx,$$

for a locally integrable function

$$\Theta_G(\pi, x), \qquad x \in G(F),$$

whose restriction to the open dense subset $G_{\mathrm{reg}}(F)$ of (strongly) regular points in $G(F)$ is analytic. It is this function that is the *character* of π. Its integral against any f depends only on its restriction to $G_{\mathrm{reg}}(F)$, which is in turn invariant under conjugation. We can therefore write

$$f_G(\pi) = \int_{\Gamma_{\mathrm{reg}}(G)} I_G(\pi, \gamma) f_G(\gamma) d\gamma, \qquad f \in C_c^\infty\big(G(F)\big), \tag{2.4}$$

where $\Gamma_{\mathrm{reg}}(G)$ is the set of $G(F)$-conjugacy classes in $G_{\mathrm{reg}}(F)$, equipped with the measure $d\gamma$ defined by a set of Haar measures on the maximal tori

$$G_\gamma(F) = \mathrm{Cent}\big(\gamma, G(F)\big),$$

while

$$I_G(\pi, \gamma) = |D(\gamma)|^{\frac{1}{2}} \Theta_G(\pi, \gamma), \tag{2.5}$$

for the Weyl discriminant $D(\gamma)$ of G, and

$$f_G(\gamma) = |D(\gamma)|^{\frac{1}{2}} \int_{G_\gamma(F) \backslash G(F)} f(x^{-1}\gamma x) dx$$

is the *orbital integral* of f at γ, defined by the quotient dx of a fixed Haar measure on $G(F)$ and the chosen measure on $G_\gamma(F)$.

The function $I_G(\pi, \gamma)$ is known as the *normalized character* of π. We have included it in the discussion in order to make a point. We are trying to demonstrate that the theorems we quote describe interesting, concrete objects, which can sometimes be quite explicit. This is particularly so for normalized characters. Suppose for example that F is archimedean and that π is tempered. Then Harish-Chandra shows that if γ is restricted to a connected component in the intersection of $G_{\mathrm{reg}}(F)$ with a maximal torus in G over F, then $I_G(\pi, \gamma)$ is a linear combination of exponential functions of γ, with complex coefficients that can be described explicitly [Ha2], [Ha3]. This may be regarded as an analogue of the Weyl character formula for compact connected groups, which is particularly striking if we replace the irreducible character by a stable character (2.11). If F is p-adic, the normalized character $I_G(\pi, \gamma)$ is deeper. It seems to be some combination of a finite germ expansion near the singular set (with coefficients and germs of functions concrete but highly complex objects), modulated by some unknown Gauss sums at intermediate distance from the singular set, followed by a function that in some cases is again like an analogue of the Weyl character formula. All of this is very interesting, but unlike the archimedean case, far from known. Our view of normalized p-adic characters will sometimes be more like that of global Hecke eigenfamilies. Rather than trying to calculate them explicitly, we would search for reciprocity laws among normalized characters on different groups.

There are three variants of these definitions we need to mention. The first is the normalized character

$$\widetilde{I}_G(\pi,\gamma)=\sum_{\pi_*} I_G(\pi_*,\gamma),\qquad \pi\in\widetilde{\Pi}(G),\ \gamma\in\widetilde{\Gamma}_{\mathrm{reg}}(G), \tag{2.6}$$

of an element $\pi\in\widetilde{\Pi}(G)$. It is a sum of irreducible characters, taken over the set $\Pi(\pi)$ (of order 1 or 2) of irreducible representations π_* in the equivalence class π. The summands depend on γ as an element in $\Gamma_{\mathrm{reg}}(G)$, but the sum itself can be regarded as a function of γ in the obvious geometric analogue

$$\widetilde{\Gamma}_{\mathrm{reg}}(G)=\Gamma_{\mathrm{reg}}(G)/\sim$$

of the spectral quotient $\widetilde{\Pi}(G)=\Pi(G)/\sim$. Once again, $\widetilde{I}_G(\pi,\gamma)$ equals $I_G(\pi,\gamma)$ unless G is of type $\mathbf{D}_n$.

The second variant is a twisted character on $GL(N)$. Suppose that ψ belongs to $\widetilde{\Psi}(N)$. Then Theorem 2.1 gives rise to a representation π_ψ of $GL(N,F)$ that is self-dual, and that therefore has an extension $\widetilde{\pi}_\psi$ to the group $\widetilde{G}(N,F)^+$ generated by $\widetilde{G}(N,F)$. There is in fact a canonical extension determined by the theory of Whittaker models. (If $\psi=\phi$ lies in the subset $\widetilde{\Phi}_{\mathrm{bdd}}(N)$ of $\widetilde{\Psi}(N)$, for example, one takes $\widetilde{\Pi}_\psi$ to be the extension that stabilizes a Whittaker vector for π_ψ. In general, π_ψ does not have a Whittaker model, but one can still work with the standard induced representation of which π_ψ is the Langlands quotient. See [A2, §2.2].) Clozel has extended the Harish-Chandra character theorem to nonconnected reductive groups. One can therefore write the distribution

$$\widetilde{f}_N(\psi)=\mathrm{tr}\big(\widetilde{\pi}_\psi(\widetilde{f})\big),\qquad \widetilde{f}\in C_c^\infty\big(\widetilde{G}(N,F)\big),$$

as

$$\widetilde{f}_N(\psi)=\int_{\widetilde{\Gamma}_{\mathrm{reg}}(N)}\widetilde{I}_N(\widetilde{\pi}_\psi,\widetilde{\gamma})\widetilde{f}_N(\widetilde{\gamma})d\widetilde{\gamma}, \tag{2.7}$$

for a smooth function $\widetilde{I}_N(\widetilde{\pi}_\psi,\widetilde{\gamma})$ of $\widetilde{\gamma}$ in the set $\widetilde{\Gamma}_{\mathrm{reg}}(N)$ of strongly regular, $GL(N,F)$-orbits in $\widetilde{G}(N,F)$. This function is the normalized twisted character of $\widetilde{\pi}_\psi$.

The third variant is a stable character for G. Suppose that ψ belongs to the subset $\widetilde{\Psi}(G)$ of $\widetilde{\Psi}(N)$. We then define a smooth function

$$\widetilde{S}^G(\psi,\delta)=\sum_{\widetilde{\gamma}\in\widetilde{\Gamma}_{\mathrm{reg}}(N)}\widetilde{I}_N(\widetilde{\pi}_\psi,\widetilde{\gamma})\,\overline{\Delta(\delta,\widetilde{\gamma})} \tag{2.8}$$

of δ in the stable version

$$\widetilde{\Delta}_{\mathrm{reg}}(G)=\Delta_{\mathrm{reg}}(G)/\sim$$

of the set $\widetilde{\Gamma}_{\mathrm{reg}}(G)$. The elements in $\Delta_{\mathrm{reg}}(G)$ are thus stable conjugacy classes in $G(F)$. In other words, they are the equivalence classes under the relation on $\Gamma_{\mathrm{reg}}(G)$ defined by $G(\overline{F})$-conjugacy (rather than the relation of $G(F)$-conjugacy that defines $\Gamma_{\mathrm{reg}}(G)$). The coefficients

$$\Delta(\delta,\widetilde{\gamma}),\qquad \delta\in\widetilde{\Delta}_{\mathrm{reg}}(G),\ \widetilde{\gamma}\in\widetilde{\Gamma}_{\mathrm{reg}}(N),$$

in the sum are Kottwitz-Shelstad twisted transfer factors [KS], for the automorphism $\widetilde{\theta}(N)$ of $GL(N)$ and the twisted endoscopic group G. They are functions that are simple enough to be quite explicit, yet deep enough to be very interesting.

There is one other point, which for us pertains to ordinary endoscopy for G (rather than twisted endoscopy for the group $GL(N)$). It concerns a bijective correspondence

$$(G', \psi') \leftrightarrow (\psi, s), \qquad \psi \in \widetilde{\Psi}(G), \ s \in S_{\psi,\mathrm{ss}}, \tag{2.9}$$

where G' is an endoscopic group for G, and ψ' belongs to the corresponding set $\widetilde{\Psi}(G')$. This entirely elementary construction can be regarded as an implicit foundation for the theory. If s belongs to the set $S_{\psi,\mathrm{ss}}$ of semisimple elements in the centralizer S_ψ, G' has the property that

$$\widehat{G}' = \mathrm{Cent}(s, \widehat{G})^0.$$

The Galois action on $\widehat{G}'$ that suffices to define G' as a quasisplit group over F is then determined in a natural way by the parameter ψ. Once we have G', the corresponding parameter ψ' is defined as the natural preimage of ψ. Now the connected centralizer $\widehat{G}'$ is a product of general linear groups with a pair of complex special orthogonal or symplectic groups. The quasisplit group G' is therefore given by a similar product. The stable character

$$\widetilde{S}'(\phi', \delta') = \widetilde{S}^{G'}(\phi', \delta'), \qquad \delta' \in \widetilde{\Delta}_{\text{G-reg}}(G'),$$

on $G'(F)$ attached to ϕ' is consequently a product of functions of the kind we have defined. Indeed, the factors for the orthogonal or symplectic components of $\widehat{G}'$ are given by analogues of (2.8), while the factor for any general linear group, in which stable conjugacy reduces to ordinary conjugacy, is just an irreducible character.

Ordinary endoscopy of course also comes with transfer factors

$$\Delta(\delta', \gamma), \qquad \delta' \in \widetilde{\Delta}_{\text{G-reg}}(G'), \ \gamma \in \widetilde{\Gamma}_{\mathrm{reg}}(G).$$

These are the original factors of Langlands and Shelstad [LS]. They were suggested by Shelstad's earlier work for real groups, which was in turn motivated by Harish-Chandra's work [Ha2], [Ha3] on characters and orbital integrals. Like their twisted variants above, they are also defined by very interesting, explicit formulas.

For simplicity, we shall state our refined supplement of Theorem 2.2 for parameters $\psi = \phi$ in the subset $\widetilde{\Phi}_{\mathrm{bdd}}(G)$ of $\widetilde{\Psi}(G)$.

Theorem 2.3. *Suppose that ϕ is a local parameter in the set $\widetilde{\Phi}_{\mathrm{bdd}}(G)$, that ξ is a character on the abelian 2-group $\mathcal{S}_\phi$, and that π is the element in the packet $\widetilde{\Pi}_\phi$ such that*

$$\xi(x) = \langle x, \pi \rangle, \qquad x \in \mathcal{S}_\phi.$$

Then the character of π is given by the formula

$$\widetilde{\Phi}_G(\pi, \gamma) = |\mathcal{S}_\phi|^{-1} \sum_{x \in \mathcal{S}_\phi} \sum_{\delta' \in \widetilde{\Delta}_{\mathrm{reg}}(G')} \xi(x)^{-1} \widetilde{S}'(\phi', \delta') \Delta(\delta', \gamma), \tag{2.10}$$

for any $\gamma \in \widetilde{\Gamma}_{\mathrm{reg}}(G)$. On the right hand side, (G', ϕ') is the preimage of (ϕ, s), for any $s \in S_{\phi,\mathrm{ss}}$ that maps to the given index of summation $x \in \mathcal{S}_\phi$, while $\widetilde{S}'(\phi', \delta')$ is the corresponding stable character, and $\Delta(\delta', \gamma)$ is the Langlands-Shelstad transfer factor for G and G'.

It is clear that Theorem 2.3 characterizes the objects of Theorem 2.2 uniquely in terms of the characters $\widetilde{\Phi}_G(\pi, \gamma)$. If F equals $\mathbb{R}$, the result was established for general groups by Shelstad. (See [S2].) In this case, the mapping from $\widetilde{\Pi}_\phi$ to $\widehat{\mathcal{S}}_\phi$ is only injective. If ξ lies in the complement of its image, π is to be interpreted simply as 0, and the assertion of the lemma becomes a vanishing formula.

It is also clear that the theorem gives reciprocity laws among local characters on different groups. It relates characters on G with twisted characters on general linear groups. In fact, it does more. If we sum each side of (2.10) over $\xi \in \widehat{\mathcal{S}}_\phi$, we observe that the summand of any $x \neq 1$ on the right vanishes. Since the transfer factor for the endoscopic group $G' = G$ can be taken to be 1, this gives the familiar formula

$$\widetilde{S}^G(\phi, \delta) = \sum_{\pi \in \widetilde{\Pi}_\phi} \widetilde{I}_G(\pi, \delta) \tag{2.11}$$

for a stable character. If we substitute its analogue for G' back into (2.10), and apply Fourier inversion for the group $\mathcal{S}_\phi$, we obtain reciprocity laws among characters on G and its endoscopic groups G'.

Theorem 2.3 is essentially Theorem 2.2.1 of [A2], with its interpretation [A2, §8.3] in terms of normalized characters. It is actually the special case for elements $\psi = \phi$ in the subset $\widetilde{\Phi}_{\mathrm{bdd}}(G)$ of $\widetilde{\Psi}(G)$. However, one can easily state the general result, again in terms of normalized characters. The character formula (2.10) will remain valid for an arbitrary element $\psi \in \widetilde{\Psi}(G)$ provided that we make two small changes. We must replace the irreducible character π on the left hand side by the reducible sum

$$\sigma = \bigoplus_\pi \pi$$

over the preimage in $\widetilde{\Pi}_\psi$ of the given character $\xi \in \widehat{\mathcal{S}}_\psi$; we also must replace the factor $\xi(x)^{-1}$ on the right hand side with its translate $\xi(s_\psi x)^{-1}$ by the point

$$s_\psi = \psi\left(1, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$

in $\mathcal{S}_\psi$. The general form of Theorem 2.3 is then just the amended version

$$\widetilde{\Phi}_G(\sigma, \gamma) = |\mathcal{S}_\psi|^{-1} \sum_{x \in \mathcal{S}_\psi} \sum_{\delta' \in \widetilde{\Delta}_{\mathrm{reg}}(G')} \xi(s_\psi x)^{-1} \widetilde{S}'(\psi', \delta') \Delta(\delta', \gamma) \tag{2.12}$$

of (2.10). We observe that the analogue for the stable character of ψ of the sum (2.11), whose value at ψ' appears on the right hand side of the general form (2.12) of (2.10), becomes

$$\widetilde{S}^G(\psi, \delta) = \sum_\sigma \langle s_\psi, \sigma \rangle \widetilde{I}_G(\sigma, \gamma) = \sum_{\pi \in \widetilde{\Pi}_\psi} \langle s_\psi, \pi \rangle \widetilde{I}_G(\pi, \delta).$$

We also note that the representations σ above are indeed often reducible. However, in the p-adic case, Moeglin [M] has shown that the packet $\widetilde{\Pi}_\psi$ is a *subset* of $\widetilde{\Pi}(G)$, so the reducible representations σ are at least multiplicity free.

We make one other observation on Theorem 2.3, in preparation for the global discussion of the next section. It represents a straightforward extension of the theorem, needed to account for the possible failure of the generalized Ramanujan conjecture for $GL(N)$.

Let $\widetilde{\Psi}^+(G)$ be the set of equivalence classes of all L-homomorphisms (2.1). It is thus composed of mappings ψ whose restriction to L_F need not lie in the subset $\widetilde{\Phi}_{\mathrm{bdd}}(G)$. Using complex parameters in $\widehat{G}$, one sees that $\widetilde{\Psi}^+(G)$ is a complex manifold, of which $\widetilde{\Psi}(G)$ is a real submanifold. One observes also that the preimage $\widetilde{\Psi}^+_S(G)$ in $\widetilde{\Psi}^+(G)$ of any complex reductive subgroup S of $\widehat{G}$ (taken up to conjugacy), under the mapping

$$\psi \longrightarrow S_\psi, \qquad \psi \in \widetilde{\Psi}^+(G),$$

is a locally closed submanifold of $\widetilde{\Psi}^+(G)$. If $\widetilde{\Psi}^+_S(G)$ is nonempty, its subset

$$\widetilde{\Psi}_S(G) = \widetilde{\Psi}^+_S(G) \cap \widetilde{\Psi}(G)$$

is a nonempty, real analytic submanifold of $\widetilde{\Psi}^+_S(G)$. Suppose that ξ is a character on the abelian 2-group

$$\mathcal{S} = S/S^0 Z(\widehat{G})^{\Gamma_{E/F}}.$$

Each side of (2.12) is then defined as a real analytic function of $\psi \in \widetilde{\Psi}_S(G)$, which can be analytically continued to the larger space $\widetilde{\Psi}^+_S(G)$. The formula (2.12) therefore holds for any $\psi \in \widetilde{\Psi}^+_S(G)$, and hence for any parameter in the general set $\widetilde{\Psi}^+(G)$. The price we pay for this extension is that the constituents of a more general packet $\widetilde{\Pi}_\psi$ become representations induced from a nonunitary parameter, which no longer need to be irreducible or unitary. (See the more explicit description in [A2, p. 45–46].) We will use this extended form of Theorem 2.3 to construct global packets in the next section.

3. Global multiplicities

In this section we return to the case that the field F is global. We shall state the global multiplicity formula in terms of objects formulated in the first two sections. The set $\Psi(N)$ is again the family of global objects attached to $GL(N)$ in the statement of Corollary 1.2. For each valuation v of F, we write $\Psi_v(N)$, $\Pi_v(N)$, $\Psi_v^+(N)$, etc. for the sets of local objects attached to F_v in the last section.

For any v, there is a localization mapping

$$\psi \longrightarrow \psi_v, \qquad \psi \in \Psi(N),$$

from $\Psi(N)$ to the local set $\Psi_v^+(N)$. It is given by the composition

$$\psi \longrightarrow \pi_\psi \longrightarrow \pi_{\psi,v} \longrightarrow \psi_v,$$

where the left hand arrow is the bijection of Corollary 1.2, the middle arrow is given by the local F_v-constituent of the representation π_ψ, and the right hand arrow is the inverse of the bijection of Theorem 2.1 (or rather, its extension to the larger domain $\Psi_v^+(N)$). We are interested in the analogue of this mapping for our quasisplit special orthogonal or symplectic group G over F.

We have not yet attached a global subset $\widetilde{\Psi}(G)$ of $\widetilde{\Psi}(N)$ to G. For the moment, we introduce only the smaller set

$$\widetilde{\Psi}_{\mathrm{sim}}(G) = \big\{\psi \in \widetilde{\Psi}_{\mathrm{sim}}(N) : \ c(\psi) \in \widetilde{\mathcal{C}}(G)\big\}. \tag{3.1}$$

This is the essential case. We did define the local set $\widetilde{\Psi}(G_v)$ in §2. We also noted that the mapping of $\widetilde{\Psi}(G_v)$ into $\widetilde{\Psi}_v(N)$ given by the embedding of ${}^LG_v$ into $GL(N,\mathbb{C})$ is itself an embedding. The same also being true of the larger sets $\widetilde{\Psi}^+(G_v)$ and $\widetilde{\Psi}_v^+(N)$, we can regard the local set $\widetilde{\Psi}^+(G_v)$ as a subset $\widetilde{\Psi}_v^+(N)$.

Proposition 3.1. *If ψ belongs to the subset $\widetilde{\Psi}_{\mathrm{sim}}(G)$ of $\widetilde{\Psi}(N)$, its localization ψ_v lies in the subset $\widetilde{\Psi}^+(G_v)$ of $\widetilde{\Psi}_v^+(N)$. In other words, ψ_v maps the group $L_{F_v} \times SU(2)$ into the subgroup ${}^LG_v$ of $GL(N,\mathbb{C})$.*

This is essentially Theorem 1.4.2 of [A2]. Along with its purely global companion Theorem 1.4.1, it is the starting point for many of the constructions of [A2], including that of the sets $\widetilde{\Psi}(G)$. Theorems 1.4.1 and 1.4.2 are carried as induction hypotheses throughout [A2]. These induction assumptions are not completely resolved until §8.2 of [A2], at which point one would finally be able to see how the set $\widetilde{\Psi}(G)$ constructed in [A2, §1.4] is related to the set $\widetilde{\mathcal{C}}(G)$ of Hecke eigenfamilies we have defined in §1 here.

To describe the automorphic discrete spectrum of G, we need to introduce a global subset $\widetilde{\Psi}_2(G)$ of $\widetilde{\Psi}(N)$ that contains $\widetilde{\Psi}_{\mathrm{sim}}(G)$. It consists of the set of formal, unordered direct sums

$$\psi = \psi_1 \boxplus \cdots \boxplus \psi_r, \qquad \psi_i \in \Psi_{\mathrm{sim}}(N_i),$$

as in the statement of Corollary 1.2, but which satisfy the following three supplementary conditions:

(i) the constituents ψ_i of ψ are *self-dual* and *distinct*;

(ii) for each i, ψ_i lies in the subset $\widetilde{\Psi}_{\mathrm{sim}}(G_i)$ of $\widetilde{\Psi}_{\mathrm{sim}}(N_i)$ attached to a special orthogonal or symplectic group G_i over F, such that $\widehat{G}_i$ and $\widehat{G}$ are of the same type, either both orthogonal or both symplectic;

(iii) The central character η_ψ of the automorphic representation π_ψ of $GL(N)$ equals the product
$$\eta_{\psi_1} \cdots \eta_{\psi_r}$$
of the central character of the representations π_{ψ_i} of $GL(N_i)$.

If we were using the inductive definition of the subset $\widetilde{\Psi}(G)$ of $\widetilde{\Psi}(N)$ from [A2, §1.4], it would follow immediately that $\widetilde{\Psi}_2(G)$ is contained in this subset. We will return to the question after stating the next theorem.

Suppose that ψ belongs to $\widetilde{\Psi}_2(G)$. There is then a canonical embedding

$$\widehat{G}_1 \times \cdots \times \widehat{G}_r \ \hookrightarrow \ \widehat{G},$$

up to conjugation by $\widehat{G}$ in the cases $\mathbf{B}_n$ and $\mathbf{C}_n$ and by the group $O(2n,\mathbb{C})$ in case $\mathbf{D}_n$. This follows from condition (ii). The condition (iii) leads to an embedding of L-groups. However, we must formulate it in terms of the broader form

$$^LG = \widehat{G} \rtimes \Gamma_F, \qquad \Gamma = \Gamma_F = \Gamma_{\overline{F}/F},$$

of the L-group, rather than the abbreviated version introduced for simplicity at the end of the introduction. (The action on $\widehat{G}$ of the absolute Galois group Γ_F factors through the quotient $\Gamma_{E/F}$ of Γ_F of order 1 or 2, which is why the simpler version of the L-group suffices for many purposes.) The L-group of the product $G_1 \times \cdots \times G_r$ becomes a fibre product

$$^L(G_1 \times \cdots \times G_r) = \prod_{i=1}^{r} ({}^LG_i \to \Gamma_F)$$

of L-groups over Γ_F. The condition (iii) then implies that the embedding of dual groups above extends to an L-embedding

$$^L(G_1 \times \cdots \times G_r) \hookrightarrow {}^LG \tag{3.2}$$

of L-groups.

The embedding (3.2) of L-groups leads directly to an object that governs the global multiplicity formula. It is the centralizer

$$S_\psi = \mathrm{Cent}\big({}^L(G_1 \times \cdots \times G_r), \widehat{G}\big) \tag{3.3}$$

in $\widehat{G}$ of the image of the embedding. This is a finite abelian 2-group, as is the quotient

$$\mathcal{S}_\psi = S_\psi / Z(\widehat{G})^\Gamma = S_\psi / Z(\widehat{G})^{\Gamma_{E/F}}. \tag{3.4}$$

If v is any valuation, we can apply Proposition 3.1 to any of the groups G_i. We see that the localization ψ_v of ψ maps the product $L_{F_v} \times SU(2)$ into the subgroup $^L(G_1 \times \cdots \times G_r)$ of ^LG. In particular, ψ_v belongs to the subset $\widetilde{\Psi}^+(G_v)$ of $\widetilde{\Psi}_v^+(N)$. We thus obtain a mapping

$$x \longrightarrow x_v, \qquad x \in \mathcal{S}_\psi,$$

from $\mathcal{S}_\psi$ to the centralizer quotient attached in the last section to the localization ψ_v. Letting v vary, we form a global packet

$$\widetilde{\Pi}_\psi = \left\{ \pi = \widetilde{\bigotimes_v} \pi_v : \ \pi_v \in \widetilde{\Pi}_{\psi_v} \right\}, \tag{3.5}$$

where the restricted tensor product is over products π such that the character $\langle \cdot, \pi_v \rangle$ on $\mathcal{S}_{\psi_v}$ equals 1 for almost all v. Any $\pi \in \widetilde{\Pi}_\psi$ then restricts to a character

$$\langle x, \pi \rangle = \prod_v \langle x_v, \pi_v \rangle, \qquad x \in \mathcal{S}_\psi, \tag{3.6}$$

on $\mathcal{S}_\psi$.

The global packet $\widetilde{\Pi}_\psi$ is a set of irreducible representations of $G(\mathbb{A})$ if G is of type $\mathbf{B}_n$ or $\mathbf{C}_n$. If G is of type $\mathbf{D}_n$, however, the global packet is a set of global objects whose local constituents are elements in the set $\widetilde{\Pi}_{\psi_v}$ over $\widetilde{\Pi}(G_v)$, which means that they are to be regarded as orbits of irreducible representations of $G(F_v)$ under the group

$$O(N, F_v)/SO(N, F_v) \cong \mathbb{Z}/2\mathbb{Z}, \qquad N = 2n.$$

The underlying reason for this (and the other variants we have already encountered) is the comparison with $GL(N)$, which we will not discuss in this paper, but which is nonetheless at the heart of the proofs. It leads naturally to representations of the group $O(N, F_v)$ rather than $SO(N, F_v)$, which amount to orbits of representation of $SO(N, F)$. To describe the decomposition of the discrete spectrum, we have consequently to introduce the locally symmetric Hecke algebra

$$\widetilde{\mathcal{H}}(G) = \widetilde{\bigotimes_v} \widetilde{\mathcal{H}}(G_v).$$

It consists of functions on $G(\mathbb{A})$ in the ordinary Hecke algebra

$$\mathcal{H}(G) = \widetilde{\bigotimes_v} \mathcal{H}(G_v)$$

that on each subgroup $G(F_v)$ are unrestricted in the cases $\mathbf{B}_n$ and $\mathbf{C}_n$, but that are symmetric under the automorphism $\widetilde{\theta}(N)$ in case G is of type $\mathbf{D}_n$. We recall that $\mathcal{H}(G_v)$ equals the algebra $C_c^\infty\big(G(F_v)\big)$ of smooth (which is to say, locally constant) functions of compact support if F_v is nonarchimedean, but is the subalgebra of K_v-finite functions in $C_c^\infty\big(G(F_v)\big)$ if F_v is archimedean. Our use of the Hecke algebra rather than $C_c^\infty\big(G(\mathbb{A})\big)$ is a minor matter in this context, which need not concern us.

Theorem 3.2. *There is an $\widetilde{\mathcal{H}}(G)$-module isomorphism*

$$L^2_{\mathrm{disc}}\big(G(F)\backslash G(\mathbb{A})\big) \cong \bigoplus_{\psi \in \widetilde{\Psi}_2(G)} \ \bigoplus_{\pi \in \widetilde{\Pi}_\psi(\varepsilon_\psi)} m_\psi \pi, \tag{3.7}$$

where m_ψ equals 1 or 2, while

$$\varepsilon_\psi : \ \mathcal{S}_\psi \ \longrightarrow \ \{\pm 1\}$$

is a linear character defined explicitly in terms of symplectic ε-factors, and

$$\widetilde{\Pi}_\psi(\varepsilon_\psi) = \big\{\pi \in \widetilde{\Pi}_\psi : \ \langle \cdot, \pi\rangle = \varepsilon_\psi\big\} \tag{3.8}$$

is the subset of the global packet $\widetilde{\Pi}_\psi$ attached to ε_ψ.

This is Theorem 1.5.2 of [A2], which was not established completely until near the end [A2, §8.2] of the volume. It asserts that any constituent of the automorphic discrete spectrum of G must lie in a global packet of the form

$$\widetilde{\Pi}_\psi, \qquad \psi \in \widetilde{\Psi}_2(G).$$

It also asserts that for any such packet, an element $\pi \in \widetilde{\Pi}_\psi$ occurs in the discrete spectrum if and only if the associated character $\langle \cdot, \pi\rangle$ on $\mathcal{S}_\psi$ equals ε_ψ, in which case π occurs with multiplicity 1 or 2. The objects ε_ψ and m_ψ have explicit formulas, which we shall discuss presently.

At the suggestion of the referee, let me add a further comment on the case of $\mathbf{D}_n$. If G belongs to the complementary cases of type $\mathbf{B}_n$ and $\mathbf{C}_n$, the assertion of the theorem is clear. It is a precise formula for the multiplicity of a given irreducible representation π in the automorphic discrete spectrum of G. But if G is of type $\mathbf{D}_n$, the formula is slightly weaker. In this case, it gives only a sum of multiplicities, taken over all irreducible representations π' of $G(\mathbb{A})$ in the equivalence class

$$\pi = \bigotimes_v \pi_v, \qquad \pi_v \in \widetilde{\Pi}(G_v),$$

defined by products of orbits (of order 1 or 2) in $G(\mathbb{A})$. The equivalence class could contain infinitely many irreducible representations π', but only finitely many of them will occur with nonzero multiplicity. The question is related to the integer m_ψ, on which we will comment at the end of the paper.

We have not been emphasizing proofs in this article. In fact, we have sometimes left out critical remarks on a given proof in our attempt to state the result as vividly as possible. The multiplicity formula (3.7) is a case in point. It is closely related to another fundamental global result, which we call the stable multiplicity formula [A2, Theorem 4.1.2], and which we apply to the preimage (G', ψ') of a global pair

$$(\psi, s), \qquad \psi \in \widetilde{\Psi}_2(G),\ s \in S_{\psi,\mathrm{ss}},$$

under the global analogue of the bijective correspondence discussed briefly in the last section. Combined with the global transfer of functions from G to G', this leads to a formula [A2, Corollary 4.1.3] that includes reciprocity laws among Hecke eigenfamilies for G and its elliptic endoscopic groups G'. These complement the reciprocity laws between Hecke eigenfamilies for G and $GL(N)$ given by Theorem 1.4.

The global arguments are complex. But very roughly speaking, the multiplicity formula (3.7) follows from the stable multiplicity formula (as expressed in Corollary 4.1.3 of [A2]), and the ψ-component of the stabilization of the trace formula of G, for *any* element $\psi \in \Psi(N)$ [A2, (4.1.2)]. As we have said, they are resolved only in §8.2 of [A2].

Theorem 1.5.2 could perhaps be regarded as the central result of [A2], especially considering that it requires the local results even to state. Formulated as Theorem 3.2, it is certainly the culmination of the discussion in this paper. It is the third and last step in our attempt to present the classification of automorphic representations of G. We recall that the first step was the reciprocity law of Theorem 1.4. It tells us that the Hecke eigenfamily attached to any automorphic representation of G is among the automorphic Hecke eigenfamilies for $GL(N)$, objects we are taking to be understood. This raised the question we have just answered with Theorem 3.2, given its interpretation as an explicit description of the contribution of a Hecke eigenfamily to the discrete spectrum of G. The theorem was in turn founded on the results of §2. As we recall, they consist of the explicit local transfer of characters provided by Theorems 2.2 and 2.3.

In §1, we raised the question of describing the set $\widetilde{\mathcal{C}}(G)$, as defined prior to Theorem 1.4, explicitly as a subset of $\widetilde{\mathcal{C}}(N)$. We can now give an answer. Let us first define the subset $\widetilde{\Psi}(G)$ of global objects in $\widetilde{\Psi}(N)$ that are attached to G.

A general (standard) Levi subgroup of G takes the form

$$M \cong GL(N_1') \times \cdots \times GL(N_{r'}') \times G_-,$$

for positive integers $N_1', \ldots, N_{r'}'$ and N_- such that

$$2N_1' + \cdots + 2N_{r'}' + N_- = N.$$

The factor G_- is a special orthogonal or symplectic group relative to $GL(N_-)$ such that $\widehat{G}_-$ and $\widehat{G}$ are of the same type, either both orthogonal or both symplectic, and such that the quadratic character η_{G_-} that defines G_- as a quasisplit outer twist equals its analogue η_G for G. Given M, we write $\widetilde{\Psi}_M(G)$ for the set of elements

$$\psi = (\psi_1 \boxplus \cdots \boxplus \psi_{r'} \boxplus \psi_{r'}^\vee \boxplus \cdots \boxplus \psi_1^\vee) \boxplus \psi_-, \tag{3.9}$$

where $\psi_i \in \Psi_{\mathrm{sim}}(N_i)$ and $\psi_- \in \widetilde{\Psi}_2(G_-)$. We then define $\widetilde{\Psi}(G)$ to be the union over M of the subsets $\widetilde{\Psi}_M(G)$ of $\widetilde{\Psi}(N)$. This becomes quite explicit if we take account Theorem 1.5.3 of [A2], an important global result we have not yet mentioned. It characterizes the subset $\widetilde{\Psi}_{\mathrm{sim}}(G)$ of simple objects $\psi \in \widetilde{\Psi}_{\mathrm{sim}}(N)$ in terms of their self-dual cuspidal components μ, according to whether it is the symmetric square L-function or the skew-symmetric L-function of μ that has a pole at $s = 1$. (See the remarks on p. 33–34 of [A2] as well as the statement of Theorem 1.5.3.) Applied to the simple summands of ψ_-, this gives an explicit description of the subset $\widetilde{\Psi}_2(G)$ of $\widetilde{\Psi}(G)$. The general definition (3.9) then leads to an explicit characterization of the subset $\widetilde{\Psi}(G)$ of $\widetilde{\Psi}(N)$.

The set $\widetilde{\Psi}(G)$ is obviously closely related to the subset $\widetilde{\mathcal{C}}(G)$ of $\widetilde{\mathcal{C}}(N)$. We might expect that $\widetilde{\mathcal{C}}(G)$ is just the set

$$\big\{c(\psi) : \ \psi \in \widetilde{\Psi}(G)\big\}, \tag{3.10}$$

but this is not quite the case. For it is conceivable that there could be elements ψ in $\widetilde{\Psi}_2(G)$ such that the set $\widetilde{\Pi}_\psi(\varepsilon_\psi)$ of Theorem 3.2 is empty. There would then be no contribution of ψ to the discrete spectrum of G, and by application of Theorem 1.3 to the definition (3.9), no contribution of ψ to any part of the spectrum. Examples of this phenomenon were found some years ago by Cogdell and Piatetskii-Shapiro [CP], by different methods. The general question depends of course on the definition of the sign character, which we have not yet discussed. In any case, the function (3.6) represents a mapping

$$\widetilde{\Pi}_\psi \ \longrightarrow \ \widehat{\mathcal{S}}_\psi \tag{3.11}$$

of the global packet of ψ to the finite group of linear characters on $\mathcal{S}_\psi$. We write $\widetilde{\Psi}_{2,\mathrm{aut}}(G)$ for the subset of elements $\psi \in \widetilde{\Psi}_2(G)$ such that the sign character ε_ψ lies in the image of this mapping. It is then clear that the collection

$$\widetilde{\mathcal{C}}_2(G) = \big\{c(\psi) : \ \psi \in \widetilde{\Psi}_{2,\mathrm{aut}}(G)\big\}$$

is the subset of Hecke eigenfamilies in $\widetilde{\Psi}(N)$ of the form $c(\pi)$, where π ranges over irreducible representations of $G(\mathbb{A})$ that occur in the automorphic *discrete* spectrum of G. More generally, the original set from Theorem 1.4 is given by

$$\widetilde{\mathcal{C}}(G) = \big\{c(\psi) : \ \psi \in \widetilde{\Psi}(G), \ \psi_- \in \widetilde{\Psi}_{2,\mathrm{aut}}(G_-)\big\}. \tag{3.12}$$

It can be characterized explicitly as a subset of $\widetilde{\Psi}(N)$ according to the remarks above.

The slightly ungainly description (3.12) is forced on us by the definition of $\widetilde{\mathcal{C}}(G)$ prior to the statement of Theorem 1.4. We could instead have defined $\widetilde{\mathcal{C}}(G)$ simply as the larger set (3.10). This would make sense from the perspective of the volume [A2], where the family $\widetilde{\Psi}(G)$ was defined [A2, §1.4] early in the process. The understanding would then be that for some elements $c = c(\psi)$ in $\widetilde{\mathcal{C}}_2(G)$ say, *every* element π in the corresponding global packet $\widetilde{\Pi}_\psi$ could have multiplicity 0 in the automorphic discrete spectrum of G. However, such a convention would not be in keeping with this article, and our emphasis on the reciprocity laws satisfied by Hecke eigenfamilies. The point does not arise if $\psi = \phi$ lies in the subset $\widetilde{\Phi}_{\mathrm{bdd}}(G)$ of $\widetilde{\Psi}(G)$. For there is always an element $\pi_v \in \widetilde{\Pi}_{\psi_v}$ such that $\langle \cdot, \pi_v\rangle = 1$, for any v, and it is easy to see that $\varepsilon_\psi = \varepsilon_\phi = 1$ in this case. The discrepancy, which is relatively rare in any case, can only occur then if the global parameter ψ is among those for which Ramanujan's conjecture is known to fail.

Incidentally, the image of the mapping (3.11) is related to a completely different question from the volume [A2]. It concerns the refinements for groups of type $\mathbf{D}_n$ studied in §8.4 of [A2]. The problem is to characterize the irreducible representations π' of $G(\mathbb{A})$ in an orbit π from a global packet $\widetilde{\Pi}_\psi$ that occur in the automorphic discrete spectrum of G. The problem was solved in the special case that $\psi = \phi$ lies in the subset $\widetilde{\Phi}_{\mathrm{bdd}}(G)$, and the mapping (3.11) is surjective, and in fact, under the weaker condition that the mapping

$$\mathcal{S}_\phi \ \longrightarrow \ \mathcal{S}_{\phi_{\mathbb{A}}} = \prod_v \mathcal{S}_{\phi_v}$$

is injective.

It remains to say something about m_ψ and ε_ψ, the essential numerical ingredients of the theorem. The integer m_ψ is easily defined. It equals 1 unless G equals $SO(2n)$ and the integers N_i attached to the constituents ψ_i of ψ are all even, in which case $m_\psi = 2$. This integer obviously bears on the question of the multiplicity with which an irreducible representation π' occurs in the automorphic discrete spectrum, but one also needs information about the local packets $\widetilde{\Pi}_{\psi_v}$ attached to ψ. For a full statement, once again in the case that $\psi = \phi$ lies in the subset $\widetilde{\Phi}_{\mathrm{bdd}}(G)$ of $\widetilde{\Psi}(G)$, see [A3, §3 (vii)].

The sign character ε_ψ would also be straightforward to define, except that we would first have to describe some internal structure of the group ${}^L(G_1 \times \cdots \times G_r)$ we used to define S_ψ. In [A2, §1.4], we attached a complex group $\mathcal{L}_\psi$ over Γ_F to the cuspidal factors μ_i of the constituents ψ_i. There is then an L-embedding

$$\mathcal{L}_\psi \times SL(2,\mathbb{C}) \ \longrightarrow \ {}^L(G_1 \times \cdots \times G_r),$$

the centralizer in $\widehat{G}$ of whose image in ${}^LG$ equals that of ${}^L(G_1 \times \cdots \times G_r)$, namely the group S_ψ. The character ε_ψ is defined [A2, §1.5 and §4.6] in terms of global Rankin-Selberg L-functions $L(s, \mu_i \times \mu_j)$ that are symplectic.

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