Representation Theory, Automorphic Forms, and Complex Geometry
A Tribute to Wilfried Schmid

# Characters and modules

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There are two different ways to classify representations of compact connected Lie groups. One is the construction by harmonic analysis of irreducible characters. It is due to H. Weyl, in his original work that culminated in the famous Weyl character formula. (See [**W**, p. 377–385] for an elementary description for the case of compact unitary groups.) The other is the algebraic construction of irreducible modules by highest weights. This is best known as part of the theory of complex semisimple Lie algebras, but it is easily transformed to a classification for compact groups. The two theories each give the classification of irreducible representations in terms of their highest weights.

The problem here, which has been posed by Dihua Jiang, is to understand a similar dichotomy for automorphic representations. I am posting it because it appears to be quite natural, and because I believe that it is very important. The question also bears upon an offhand comment of Wilfried from last November,

"... but what about the modules!"

or words to that effect. In [**A**], we describe a classification of automorphic representations of orthogonal and symplectic groups. It is based on a comparison of the trace formula with its stabilization (and is still conditional on the stabilization of the twisted trace formula for GL(N), part of work in progress by Moeglin and Waldspurger). The comparison of trace formulas is a theory that rests ultimately on the characters of representations. The theta correspondence is a complementary theory based on the actual modules of representations. It has the advantage of being very explicit. The disadvantage is that it does not directly classify representations into local and global packets from which one can deduce multiplicities. In fact, it does not give an exhaustion theorem for the representations it constructs. However, initial results suggest that one might be able to have the advantages of both theories by using them together.

The problem of comparing the theta correspondence with the endoscopic classification seems to be quite complex. It might require sustained efforts from a number of mathematicians. For a more detailed description of the problem, with the initial results mentioned above, we refer the reader to [**J**]. For local results, see [**M**].

## References

[A] J. Arthur, *The Endoscopic Classification of Representations*, to appear in *Automorphic Representations and L-functions*, Tata Institute of Fundamental Research. MR 3156849

[J] D. Jiang, *Automorphic integral transforms for classical groups I*: endoscopy correspondences, to appear in *Conference on Automorphic Forms and Related Geometry: Assessing the Legacy of I. I. Piatetskii-Shapiro.* MR 3220929

[M] C. Moeglin, *Conjecture d'Adams pour la correspondence de Howe et filtration de Kudla*, Adv. Lect. Math. **19**, 2011, 445–503. MR 2906916

[W] H. Weyl, *The Theory of Groups and Quantum Mechanics*, 1931, rept. Dover Publications, 1950. MR 3363447

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