Problems Beyond Endoscopy

James Arthur

To Roger Howe on the occasion of his seventieth birthday

Abstract. We give a short introduction to Beyond Endoscopy, a proposal by Langlands for attacking the general principle of functoriality. We shall try to motivate the proposal by emphasizing its structural similarities with the actual theory of endoscopy. We shall then discuss a few of the many problems that will need to be solved, some of which are suggested by the recent work of A. Altuğ .

Mathematics Subject Classification (2010). Primary 22E55, 11F66; Secondary 22E50.

Keywords. functoriality, endoscopy, Beyond Endoscopy, trace formula, stabilization, L-functions, primitization.

Contents

Introduction

This expository paper represents a short (and partial) introduction to Beyond Endoscopy, a proposal by Langlands for attacking the general principle of functoriality. The strategy is quite specific. It entails a direct application of the stable trace formula to automorphic L-functions.

The difficulties are enormous. They will require completely new methods from both analytic and algebraic number theory, as well as applications of the trace formula that go well beyond anything so far attempted. However, there has been considerable elaboration [L6], [FLN], [L7], [L8], [Al1], and [Al2] - [Al4] of the proposal in the years since Langlands first put it forward.

There have also been various other ideas proposed ([BK], [V], [Laf], [Sak], [N2], [BNS], [H], [GH], and [G]) which relate to Beyond Endoscopy (and to each other). We are now faced with a number of new directions to explore, including some that seem to be reasonably accessible.

In this paper, we shall describe Beyond Endoscopy from the point of view of the theory of endoscopy itself. More precisely, we shall describe Langlands' proposal as a direct analogue of the stabilization of the invariant trace formula, and its twisted analogue. This is a slightly different way of organizing some of the ideas of Langlands. It is implicit in his paper, but it is also related to the appendix of [A6].

We shall also describe a few concrete problems, some of which are suggested by recent work [Al2], [Al3], [Al4] of A. Altuğ. Altuğ treats a very special case of Beyond Endoscopy. His main results, which first appeared in his thesis [Al1], were established by other means some time ago. However, the thesis contains new analytic methods, which support the premises of the program. The thesis also presents some new phenomena for the special case it treats. The problems we describe include generalizations of these phenomena.

1. Endoscopy

Endoscopy amounts to a series of precise conjectures, also by Langlands, on the general structure of automorphic representations. The local part of the conjectures classifies irreducible representations of a local group into finite local L-packets. The global part consists of a formula for the multiplicity in the automorphic discrete spectrum of any representation in an associated global packet. I later supplemented these conjectures by introducing larger local packets, with a corresponding global multiplicity formula that accounts for the full automorphic discrete spectrum.

There has been recent progress in the conjectural theory of endoscopy. It has led to the conjectured local and global classification of representations of quasisplit classical groups, which is to say, quasisplit orthogonal, symplectic and unitary groups [A9], [M]. The methods are largely global. They rest on a comparison of global trace formulas. More precisely, the classifications ultimately follow from the stabilization of the invariant trace formula for classical groups, and the stabilization of the twisted trace formula for general linear groups[1]. We shall recall these endoscopic stabilizations, as motivation for the speculative relationships among stable trace formulas that would be at the centre of the theory Beyond Endoscopy.

Suppose that G is a connected, reductive algebraic group over a number field F. The trace formula for G depends on a test function f in the global Hecke algebra $\mathcal{H}(G) = \mathcal{H}(G(\mathbb{A}))$ on the adelic group $G(\mathbb{A})$. This algebra is a

[1] One also has to make limited use of the twisted trace formula for the quasisplit, special orthogonal groups $SO(2n)$.

topological direct limit

$$\mathcal{H}(G) = \varinjlim_{V} \mathcal{H}(G_V)$$

over finite sets

$$\mathrm{val}_{\mathrm{ram}}(F) \subset V \subset \mathrm{val}(F)$$

of valuations of F that contain the places at which G is ramified (a set that by agreement includes the set S_∞ of archimedean valuations). By definition, the algebra

$$\mathcal{H}(G_V) = \prod_{v \in V} \mathcal{H}(G_v) = \prod_{v} \mathcal{H}(G(F_v))$$

is a finite product of local Hecke algebras. It embeds injectively into $\mathcal{H}(G)$ under the mapping

$$f \longrightarrow f \cdot u^V, \qquad f \in \mathcal{H}(G_V),$$

where u^V is the characteristic function of a suitably fixed maximal compact subgroup K^V of $G^V = G(\mathbb{A}^V)$, the adelic group complement of G_V.

The invariant trace formula for G is the identity given by two different expansions

$$\sum_M |W(M)|^{-1} \sum_{\gamma \in \Gamma(M,V)} a^M(\gamma) I_M(\gamma, f) \tag{1.1}$$

and

$$\sum_M |W(M)|^{-1} \int_{\Pi(M,V)} a^M(\pi) I_M(\pi, f) d\pi \tag{1.2}$$

of a certain invariant linear form[2]

$$I(f) = I^G(f), \qquad f \in \mathcal{H}(G_V),$$

on $\mathcal{H}(G_V)$ [A1], [A6]. We will not review the classifications to which this identity and its refinements ultimately lead. Our purpose is rather to provide a context for Beyond Endoscopy. However, we shall say a few words about the individual terms in the two expansions, if only because they are the explicit objects on which everything else rests. A reader unfamiliar with the details of the trace formula can skip this discussion, and proceed directly to the statement of Theorem 1.1 below. Alternatively, one can consult the references [A1], [A2] or [A6] for further details.

In both expansions, M is summed over the finite set of conjugacy classes of Levi subgroups of G. For any such M, $W(M)$ represents the associated Weyl group in G. The corresponding coefficients $a^M(\gamma)$ and $a^M(\pi)$ in each expression depend only on M (rather than G). In the *geometric expansion* (1.1), $\Gamma(M, V)$ stands for a certain set of conjugacy classes in the group M_V. For any $\gamma \in \Gamma(M, V)$, $I_M(\gamma, f)$ is the invariant distribution attached to the weighted orbital integral over the G_V-conjugacy class of γ. In the *spectral*

[2] It is simplest to think of $I(f)$ as an object with no independent characterization. In other words, it is defined explicitly by either of the two expansions.

expansion (1.2), $\Pi(M, V)$ stands for a certain set of irreducible unitary representations of the group M_V. For any $\pi \in \Pi(M, V)$, $I_M(\pi, f)$ is an invariant distribution on $\mathcal{H}(G)$ attached to the weighted characters of the representation of G_V obtained by parabolic induction from π. To be a little more precise, it is what is left over after the noninvariant part of the weighted character has been removed, and transfered to the geometric side in order to construct the invariant distribution $I_M(\gamma, f)$. Finally, $d\pi$ is a natural measure on the set $\Pi(M, V)$. The main point is that the coefficients $a^M(\gamma)$ and $a^M(\pi)$ are fundamentally global objects, while the linear forms $I_M(\gamma, f)$ and $I_M(\pi, f)$ are defined by local harmonic analysis. Thus, despite the complexity of some of the terms, the trace formula comes with a clean delineation of its local and global constituents.

The primary terms are familiar and easy to describe. On the geometric side assume that $M = G$, and that γ is the G_V-component of a strongly regular, F-elliptic conjugacy class $\dot{\gamma}$ in $G(F)$, which is integral outside V. Then the global coefficient is the volume

$$a^G(\gamma) = a^G_{\mathrm{ell}}(\gamma) = \mathrm{vol}(G_{\dot{\gamma}}(F)\backslash G_{\dot{\gamma}}(\mathbb{A})^1), \tag{1.3}$$

where $G_{\dot{\gamma}}$ is the centralizer of $\dot{\gamma}$ in G. The group $G_{\dot{\gamma}}(\mathbb{A})^1$ is the analogue for $G_{\dot{\gamma}}$ of $G(\mathbb{A})^1$, the canonical group theoretic complement in $G(\mathbb{A})$ of the connected, central group

$$A^+_{G,\infty} = (\mathrm{Res}_{F/\mathbb{Q}} A_G)(\mathbb{R})^0 \subset G(\mathbb{A}).$$

(As usual, A_G denotes the split component over F of the centre of G.) The local invariant distribution is the invariant orbital integral

$$I_G(\gamma, f) = f_G(\gamma) = |D^G(\gamma)|^{1/2} \int_{G_\gamma(F_V)\backslash G(F_V)} f(x^{-1}\gamma x)dx, \tag{1.4}$$

where D^G is the Weyl discriminant of G.

On the spectral side, assume again that $M = G$, and that π is the G_V-component of an irreducible subrepresentation $\dot{\pi}$ of $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A})^1)$ that is unramified outside of V. As usual, $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A})^1)$ is the subspace of $L^2(G(F)\backslash G(\mathbb{A})^1)$ that decomposes discretely under right translation by $G(\mathbb{A})^1$. The global coefficient is then the multiplicity

$$a^G(\pi) = a^G_{\mathrm{disc}}(\pi) = \mathrm{mult}(\dot{\pi}, L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A})^1))$$

with which π occurs in this subspace. The local invariant linear form is the character

$$I_G(\pi, f) = f_G(\pi) = \mathrm{tr}(\pi(f)).$$

An important point is that the discrete part of the general measure $d\pi$ on the spectral side (1.2) is supported on more than just the discrete spectrum $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A})^1)$ of $L^2(G(F)\backslash G(\mathbb{A})^1)$. The *discrete part of the trace formula*, namely the contribution to (1.2) of the entire discrete part of the measure $d\pi$, is an invariant linear form

$$I_{\mathrm{disc}}(f) = \sum_{M_1} |W(M_1)|^{-1} \sum_{w \in W(M_1)_{\mathrm{reg}}} |\det(w-1)|^{-1} \mathrm{tr}(M_{P_1}(w)\mathcal{I}_{P_1}(f)), \tag{1.5}$$

in which M_1 and $W(M_1)$ are as in (1.2) (but with M_1 in place of M). For any $w \in W(M_1)$, $\det(w-1)$ is the determinant of $(w-1)$ as a linear operator on the Lie algebra of the real group $A^+_{M_1,\infty}/A^+_{G,\infty}$, and $W(M_1)_{\mathrm{reg}}$ is the subset of elements $w \in W(M_1)$ for which this determinant is nonzero. In the right hand term, P_1 is a parabolic subgroup of G with Levi component M_1, and $\mathcal{I}_{P_1}$ is the representation of $G(\mathbb{A})^1 \cong G(\mathbb{A})/A^+_{G,\infty}$ obtained by parabolic induction from the representation of $M_1(\mathbb{A})$ on $L^2_{\mathrm{disc}}((M_1(F)A^+_{M_1,\infty})\backslash M_1(\mathbb{A}))$. Finally,

$$M_{P_1}(w) : \mathcal{I}_{P_1} \longrightarrow \mathcal{I}_{P_1}$$

is the global intertwining operator attached to w that is at the heart of Langlands' theory of Eisenstein series. The actual discrete spectrum of G is given simply by the term with $M_1 = G$ in (1.5). We note that the general coefficients $a^M(\pi)$ in (1.2) (and the measure $d\pi$) are constructed from coefficients obtained from (1.5) (with G replaced by a variable Levi subgroup of M) in a simple manner. (See [A1, (4.5)], [A9, §3.1]).

The ultimate goal is to understand the coefficients $a^G_{\mathrm{disc}}(\pi)$ obtained by expanding the term with $M_1 = G$ in (1.5) as a linear combination of characters. However, the terms with $M_1 \neq G$ are often hard to separate from this in the study of endoscopy, so in practice it is the entire linear form $I_{\mathrm{disc}}(f)$ defined by (1.5) that must be treated. One wants to understand the global information in $I_{\mathrm{disc}}(f)$ from the local (and global) information in the geometric expansion (1.1). (The terms in the spectral expansion (1.2) that are complementary to $I_{\mathrm{disc}}(f)$ can often be treated by induction, and will be ignored in our discussion here.)

There was actually no compelling reason for us to review the terms in (1.1) and (1.2). We have done so in order to place Theorems 1.1 and 1.2 below in context. The first theorem is known as the *stabilization of the trace formula.*

Theorem 1.1. ([A2] - [A4]) *Suppose that the group G is quasisplit over F. Then there exist unique stable linear forms $S = S^G$ and $S_{\mathrm{disc}} = S^G_{\mathrm{disc}}$ on $\mathcal{H}(G_V)$ such that*

$$I^G(f) = \sum_{G'} \iota(G,G')\hat{S}^{\tilde{G}'}(f') \tag{1.6}$$

and

$$I^G_{\mathrm{disc}}(f) = \sum_{G'} \iota(G,G')\hat{S}^{\tilde{G}'}_{\mathrm{disc}}(f'). \tag{1.7}$$

The objects G' are *endoscopic groups*, which are supplementary quasisplit groups over F attached to G. As indices of summation in (1.6) and (1.7), they represent by convention a slightly broader set of data, namely isomorphism classes of elliptic endoscopic data $(G', \mathcal{G}', s', \xi')$ for G. The corresponding coefficients $\iota(G,G')$ in the two expansions are defined by simple explicit formulas. For each G', $\tilde{G}'$ is further reductive group over F, which represents an auxiliary datum $(\tilde{G}', \tilde{\xi}')$ for G'. It comes with a central torus

$\tilde{C}'$ over F and a character $\tilde{\eta}'$ on $\tilde{C}'(F)\backslash\tilde{C}'(A)$, and provides a minor technical modification in case $\mathcal{G}'$ (as an extension of W_F by $\hat{G}'$) is not actually equal to the L-group of G'. (See [A9, pages 130 – 131]). Finally, $f' = f^{\tilde{G}'}$ represents the Langlands-Shelstad transfer of the given function $f \in \mathcal{H}(G_V)$ from G to $\tilde{G}'$. These various objects were introduced and developed by Langlands, Shelstad and Kottwitz ([L5], [LS], [KS]). A reader unfamiliar with the details of endoscopy need not be concerned with them at this point. In the next section, we shall mention endoscopic data again in proposing "beyond endoscopic" analogues.

We will however recall here the notion of stability. For any given $v \in V$, a linear form S_v in $f_v \in \mathcal{H}(G_v)$ is *stable* if $S_v(f_v)$ depends only on the set of stable orbital integrals

$$f_v^G(\delta_v) = \sum_{\gamma_v \to \delta_v} f_{v,G}(\gamma_v), \qquad \delta_v \in \Delta_{\mathrm{reg}}(G_v),$$

where $\Delta_{\mathrm{reg}}(G_v)$ denotes the set of strongly regular, stable conjugacy classes in $G(F_v)$. An element $\delta_v \in \Delta_{\mathrm{reg}}(G_v)$ is thus the intersection of $G(F_v)$ with a strongly regular conjugacy class in $G(\bar{F}_v)$, and as such, is a union of finitely many strongly regular conjugacy classes $\gamma_v \in \Gamma_{\mathrm{reg}}(G_v)$ in $G(F_v)$. For the stable linear form S_v, there is then a unique linear form $\hat{S}_v$ on the space

$$\mathcal{S}(G_v) = \mathrm{span}\{\delta_v \longrightarrow f_v^{G_v}(\delta_v) : \ \delta_v \in \Delta_{\mathrm{reg}}(G_v)\}$$

such that

$$S_v(f_v) = \hat{S}_v(f_v^G), \qquad f_v \in \mathcal{H}(G_v).$$

Similar notation would obviously also apply to the equivariant spaces $\mathcal{H}(G_v, \eta_v)$ and $\mathcal{S}(G_v, \eta_v)$, relative to a character η_v on a central torus C_v in G_v. It is used in the right hand terms of (1.6) and (1.7) (with $(\tilde{G}'_v, \tilde{\eta}'_v)$ in place of (G_v, η_v)).

We recall also that the Langlands-Shelstad transfer $f'_v = f_v^{\tilde{G}'}$ of $f_v \in \mathcal{H}(G_v)$ is an $\tilde{\eta}'_v$-equivariant function

$$f'_v(\delta'_v) = \sum_{\gamma_v \in \Gamma_{\mathrm{reg}}(G_v)} \Delta(\delta'_v, \gamma_v) f_{v,G}(\gamma_v)$$

of a strongly G-regular element $\delta'_v \in \Delta_{\mathrm{reg}}(\tilde{G}'_v)$. The coefficient $\Delta(\delta'_v, \gamma_v)$ is the Langlands-Shelstad transfer factor, an explicit function of two variables, whose definition represents a remarkable combination of class field theory and the theory of algebraic groups [LS]. The key property that drives everything is that f'_v lies in the space $\mathcal{S}(\tilde{G}'_v, \tilde{\eta}'_v)$, rather than just being a general function of δ'_v. This difficult but fundamental theorem was proved by Shelstad for archimedean v, and then conjectured by Langlands and Shelstad for general v. It was finally completed for p-adic v by Waldspurger [W1], [W2] and Ngô [N1] with the proof of the Fundamental Lemma (which amounts to a more precise version of a special case).

Given now the various objects that go into the statement of Theorem 1.1, we shall say a few words about its proof. The groups G' consist of the maximal element $G' = G$, together with groups G' of dimension smaller than G. One assumes inductively that the linear forms $S^{\tilde{G}'}$ and $S^{\tilde{G}'}_{\rm disc}$ are defined and stable if $G' \neq G$. One then defines

$$S^G(f) = I^G(f) - \sum_{G' \neq G} \iota(G, G')\hat{S}^{\tilde{G}'}(f') \tag{1.8}$$

and

$$S^G_{\rm disc}(f) = I^G_{\rm disc}(f) - \sum_{G' \neq G} \iota(G, G')\hat{S}^{\tilde{G}'}_{\rm disc}(f'). \tag{1.9}$$

The problem is to show that the linear forms S^G and $S^G_{\rm disc}$ are stable.

The proof is indirect. One must first define stable analogues $\Delta(M, V)$, $b^M(\delta)$, $S_M(\delta, f)$, $\Phi(M, V)$, $b^M(\phi)$, $S_M(\phi, f)$ and $d\phi$ of $\Gamma(M, V)$, $a^M(\gamma)$, $I_M(\gamma, f)$, $\Pi(M, V)$, $a^M(\pi)$, $I_M(\pi, f)$ and $d\pi$ respectively, such that the sum

$$\sum_M |W(M)|^{-1} \sum_{\delta \in \Delta(M,V)} b^M(\delta)S_M(\delta, f) \tag{1.10}$$

equals

$$\sum_M |W(M)|^{-1} \int_{\Phi(M,V)} b^M(\phi)S_M(\phi, f)d\phi. \tag{1.11}$$

These objects are defined inductively in terms of their counterparts in (1.1) and (1.2), by various concrete analogues of (1.8). After much effort, one finally shows by induction that the linear forms $S_M(\delta, f)$ and $S_M(\phi, f)$ are all stable. One then takes $S^G(f)$ to be either of the two equal expansions (1.10) and (1.11), and $S^G_{\rm disc}(f)$ to be the component of the spectral expansion (1.11) corresponding to the discrete part of the measure $d\phi$. From this it follows at length from the various constructions that they are stable, and that they satisfy (1.8) and (1.9), and hence also (1.6) and (1.7).

Theorem 1.1 was established in [A4]. The results of this paper actually apply more generally to the case that G is an inner twist of a quasisplit group G^* over F. They assert that Theorem 1.1 remains valid as stated for the more general group G. The endoscopic groups G' for G are still quasisplit, so the linear forms on the right hand sides of (1.6) and (1.7) are defined and stable, by application of Theorem 1.1 to G^*. In this case, the problem is to establish the two identities, rather than show that a distribution is stable.

The second theorem is a major generalization of the first. It is known as the *stabilization of the twisted trace formula.*

Theorem 1.2. ([W3]-[W10], [MW2]-[MW4]) *Suppose that G is a G^0-space over the number field F, which is to say a bitorsor under a connected, reductive group G^0 over F, equipped with a character w on $G^0(F)\backslash G^0(A)$. Then the identities* (1.6) *and* (1.7) *of Theorem* (1.1) *remain valid as stated.*

In this context, the indices of summation G' in (1.1) and (1.2) are endoscopic data for the bitorsor G, or in earlier terminology, twisted endoscopic data for the group G^0. In particular, they remain quasisplit, connected groups over G. The remarks for inner twists above apply here as well. In particular, the linear forms on the right hand sides of (1.6) and (1.7), as they apply to Theorem 1.2, are defined and stable by application of Theorem 1.1. In this case, the problem is again to establish the two identities rather than to prove that a distribution is stable.

Theorems 1.1 and 1.2 were applied to the classification of representations in [A9] and [M]. Suppose that G is a connected quasisplit orthogonal, symplectic, or unitary group over the global field F. The global results of [A9] and [M] may be regarded informally as a classification of automorphic representation of G in terms of global packets Π_ψ. These should be parameterized in turn by $\hat{G}$-conjugacy classes of (elliptic) L-homomorphisms

$$\psi : L_F \times SU(2) \longrightarrow {}^L G \tag{1.12}$$

of bounded image. In general, L_F stands for the global Langlands group, a hypothetical extension of the global Weil group W_F by a compact, connected group (see [L3, §2], [K, §9] and [A5]). In the special case of a classical group here, L_F must be replaced by the ad hoc group defined in §1.4 in [A9] in order that the results be unconditional. We refer the reader to §1.5 in [A9] for the construction of a global packet Π_ψ in terms of finite local packets Π_{ψ_v}, and for the multiplicity formula with which an arbitrary representation $\pi \in \Pi_\psi$ occurs in the automorphic discrete spectrum of G.

2. Beyond Endoscopy

We shall now begin our discussion of Langlands' Beyond Endoscopy program. As we noted in the introduction, it is a proposal for attacking the general principle of functoriality. There are enormous problems to be solved, so functoriality has still to be regarded as a distant goal. However, the strategy is quite specific. It is based on the stable trace formula, and more precisely, a direction application of the stable trace formula to automorphic L-functions.

As before, G is a connected reductive algebraic group over a number field F, which we again take to be quasisplit. We also fix our function $f \in \mathcal{H}(G_V)$, for a large, finite set of valuations $V \subset \mathrm{val}(F)$ of F. The discrete part $S_{\mathrm{disc}}(f)$ of the stable trace formula is then a linear combination of irreducible unitary characters on $\mathcal{H}(G_V)$. We recall that there are irreducible constituents of the discrete spectrum $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A})^1)$ that are nontempered. In other words, π can have local constituents π_v whose characters, as invariant distributions on $G(F_v)$, are not tempered in the sense of Harish-Chandra. These are the automorphic representations that violate the analogue of Ramanujan's conjecture for G. In terms of the (hypothetical) global parameters (1.12), they should be the automorphic constituents of those packets Π_ψ for which the global parameter

$$\psi : L_F \times SU(2) \longrightarrow {}^L G$$

is nontrivial on the second factor $SU(2)$

Langlands' ideas are designed for the automorphic representations π of G that are tempered. In order to gain access to them, we must first remove the nontempered constituents of $S_{\text{disc}}(f)$. We shall write

$$S^1_{\text{cusp}}(f) = S^{1,G}_{\text{cusp}}(f)$$

for the *cuspidal part*[3] of $S_{\text{disc}}(f) = S^G_{\text{disc}}(f)$. It is defined as what remains in the linear combination of irreducible characters that constitute $S_{\text{disc}}(f)$ after we subtract those characters that are *either* nontempered *or* do not lie in the discrete spectrum of G. Now, we have a trace formula for $S_{\text{disc}}(f)$. For we can write $S_{\text{disc}}(f)$ as the difference between the geometric expansion (1.10) and the contribution to the spectral expansion (1.11) of the continuous part of the measure $d\phi$. We therefore also have a trace formula for $S^1_{\text{cusp}}(f)$. It equals the difference between the expression just described for $S_{\text{disc}}(f)$ and the contribution to $S_{\text{disc}}(f)$ of those characters that are either nontempered or do not lie in the discrete spectrum of G.

We should be a little more clear on this point. When we speak of removing the nontempered summands from $S_{\text{disc}}(f)$, we mean that we subtract the contribution from the nontempered parameters (1.12). That is, we remove the expected contribution of the global packets Π_ψ indexed by parameters ψ that are nontrivial on the second factor $SU(2)$. The tempered component of any such ψ, namely the restriction of ψ to the first factor L_F, would factor through a proper Levi subgroup ${}^LM$ of ${}^LG$, and ought therefore to be subject to a suitable induction hypothesis. In particular, we would be free to assume that its contribution to the discrete spectrum of M satisfies the analogue of Ramanujan's conjecture, and therefore actually is tempered. One of the aims of Beyond Endoscopy is to complete this argument by establishing Ramanujan's conjecture for G, insofar as it applies to the contribution of those parameters (1.12) for G that are trivial on $SU(2)$.

We should also say that the trace formula for $S^1_{\text{disc}}(f)$ we have described is highly artificial. It is given by a difference between geometric and spectral quantities that bear little resemblance to each other. Is it possible to effect any cancellation between these quantities? This question is an important concern in the theory, one that is emphasized in particular in the paper [FLN].

A central goal of Beyond Endoscopy might be formulated as the following rather vague question. Can one find a spectral decomposition of $S^1_{\text{cusp}}(f)$, as a linear combination of stable (sums of) characters $f_G(\pi)$, that accounts for the functorial origins of π. In other words, can we label each π according to its functorial images under L-homomorphisms

$$\rho : {}^LG' \longrightarrow {}^LG?$$

The principle of functoriality was introduced by Langlands as the centrepiece of his famous 1967 letter to Weil [L1]. As we recall, it postulates a correspondence $\pi' \longrightarrow \pi$ of automorphic representations between groups G' and G for

[3] This language is slightly misleading. It refers to the parameters ψ rather than the representations in the expected packets Π_ψ, which will often remain cuspidal for parameters that are nontrivial on the factor $SU(2)$ in (1.12).

any L-homomorphism ρ of their L-groups as above. Langlands' proposal for Beyond Endoscopy was to make use of the automorphic L-functions $L(s,\pi,r)$ of π attached to finite dimensional representations

$$r : {}^L G \longrightarrow GL(N,\mathbb{C}) \tag{2.1}$$

of the L-group of G. Phrased in terms of the question above, the goal would be to construct a stable generalization $S^r_{\text{cusp}}(f) = S^{r,G}_{\text{cusp}}(f)$ of $S^1_{\text{cusp}}(f) = S^{1,G}_{\text{cusp}}(f)$ for any r, in which the stable multiplicities of representations π that occur in $S^1_{\text{cusp}}(f)$ are weighted by the orders of poles

$$m_\pi(r) = -\text{ord}_{s=1}L(s,\pi,r) = \text{res}_{s=1}\left(-\frac{d}{ds}\log L(s,\pi,r)\right) \tag{2.2}$$

at $s=1$ of the L-function attached to r. Notice that $S^1_{\text{cusp}}(f)$ would then be the special case that r equals 1, the trivial 1-dimensional representation. For in this case, $L(s,\pi,r)$ equals the Dedekind zeta function $\zeta_F(s)$ (completed, let us say, at the archimedean places), which has a simple pole at $s=1$, and weight factors $m_1(\pi)$ therefore equal to 1.

The proposed stable distribution $S^r_{\text{cusp}}(f)$ should come with a decomposition

$$S^r_{\text{cusp}}(f) = \sum_{G'} \iota(r,G')\hat{P}^{\tilde{G}'}_{\text{cusp}}(f') \tag{2.3}$$

into what we can call *primitive*, stable distributions $P^{\tilde{G}'}_{\text{cusp}}$ on groups $\tilde{G}'(\mathbb{A})$. The indices of summation G' in (2.3) would represent isomorphism classes of elliptic "beyond endoscopic data" $(G',\mathcal{G}',\xi')$ in which G' (the first component) is again a quasisplit group over G, $\mathcal{G}'$ is a split extension

$$1 \longrightarrow \hat{G}' \longrightarrow \mathcal{G}' \longrightarrow W_F \longrightarrow 1$$

of the global Weil group W_F by the dual group $\hat{G}'$, and $\xi' : \mathcal{G}' \longrightarrow {}^L G$ is an L-embedding such that

$$\left|\text{Cent}(\mathcal{G}',\hat{G})/Z(\hat{G})^\Gamma\right| < \infty,$$

in the standard notation of [A9] in §3.2, for example. The superscript $\tilde{G}'$ in (2.3) would represent an "auxiliary datum" $(\tilde{G}',\tilde{\xi}')$ for G', where $\tilde{G}'$ (the first component) is a split central extension

$$1 \longrightarrow \tilde{C}' \longrightarrow \tilde{G}' \longrightarrow G' \longrightarrow 1$$

of G' by an "induced torus" $\tilde{C}'$ over F, and $\tilde{\xi}' : \mathcal{G}' \longrightarrow {}^L\tilde{G}'$ is an L-embedding, which as in the case of endoscopy from §1, comes with a character $\tilde{\eta}'$ on $\tilde{C}'(F)\backslash\tilde{C}'(\mathbb{A})$. The coefficient in (2.3) should be a product

$$\iota(r,G') = m'(r)\iota(G,G') \tag{2.4}$$

of a coefficient $\iota(G,G')$ that is independent of r and the *dimension datum* $m'(r) = m_{G'}(r)$ of G' at r, which is to say, the multiplicity of the trivial representation of $\mathcal{G}'$ in the composition $r \circ \xi'$.

In the linear form $\hat{P}^{\tilde{G}'}_{\mathrm{cusp}}(f')$, the function f' is the *stable transfer* of the given function f from G to $\tilde{G}'$. More precisely, it is the image of f under the mapping

$$f \longrightarrow f' = f^{\tilde{G}'}$$

from $\mathcal{H}(G_V)$ to $\mathcal{S}(\tilde{G}'_V, \tilde{\eta}'_V)$ defined by

$$f'(\phi') = f^G(\tilde{\xi}' \circ \phi') \tag{2.5}$$

where ϕ' ranges over the bounded, $\tilde{\eta}'_V$-equivariant Langlands parameters for $\tilde{G}_V$, a set

$$\Phi_{\mathrm{bdd}}(\tilde{G}'_V, \tilde{\eta}'_V) = \left\{ \phi' = \prod_{v \in V} \phi'_v : \ \phi'_v \in \Phi_{\mathrm{bdd}}(\tilde{G}'_v, \tilde{\eta}'_v) \right\}$$

that is bijective with the set of $\tilde{G}'_V$-orbits in the family of L-homomorphisms

$$\mathrm{Hom}_{\mathrm{bdd}}(W_{F_V}, \mathcal{G}'_V) = \prod_{v \in V} \mathrm{Hom}_{\mathrm{bdd}}(W_{F_v}, \mathcal{G}'_v)$$

with bounded image in $\tilde{G}'_V$. This definition presupposes the local Langlands correspondence for the groups G'_v (and $\tilde{G}'_v$). One would hope to establish it along the way.

Finally, to describe the primitive linear form $P^{\tilde{G}'}_{\mathrm{cusp}}$, we can obviously suppose that $\tilde{G}' = G' = G$. In this case, it would be the "primitive part"

$$S^G_{\mathrm{prim}}(f) = P^G_{\mathrm{cusp}}(f)$$

of $S^G(f)$ (or $S^{1,G}_{\mathrm{cusp}}(f)$). By this, we mean that $P^G_{\mathrm{cusp}}(f)$ is the spectral contribution to the stable trace formula of those tempered, cuspidal automorphic representations that are primitive, in the sense that they are not functorial images from some smaller group. Of course, we do not have functoriality at this point, so this description of $P^G_{\mathrm{cusp}}(f)$ cannot serve as a definition. Instead, we define $P^G_{\mathrm{cusp}}(f)$ inductively by setting

$$P^G_{\mathrm{cusp}}(f) = S^1_{\mathrm{cusp}}(f) - \sum_{G' \neq G} \iota(1, G') \hat{P}^{\tilde{G}'}_{\mathrm{cusp}}(f'). \tag{2.6}$$

For any representation r, we would then have a definition for all of the terms on the right hand side of (2.3). In the case $r \neq 1$, the decomposition (2.3) then represents an identity that would have to be proved.

Notice the analogies of our formulation of Beyond Endoscopy with the endoscopic construction of the last section. Indeed, the equation (2.3) is completely parallel to (1.7), with the interpretation of (1.7) from Theorem 1.2 in which G stands for a G^0-space. (We could have made the notation completely parallel by letting G represent the pair (G, r) in (2.3)). In particular, the objects S^r_{cusp}, G', $\iota(r, G')$, $\tilde{G}'$ and $P^{\tilde{G}'}_{\mathrm{cusp}}$ in (2.3) assume the formal roles of I^G_{disc}, G', $\iota(G, G')$, $\tilde{G}'$ and $S^{\tilde{G}'}_{\mathrm{disc}}$ in (1.7). The case $r = 1$ of (2.3) is of course parallel to the case $G = G^0$ of (1.7). The definition (2.6) is therefore parallel to the endoscopic definition (1.9).

Observe that we have not yet discussed analogues of the original invariant trace formula (the identity of (1.1) and (1.2)) and its stable refinement (the identity of (1.10) and (1.11)). They would consist respectively of a geometric expansion for $S^r_{\text{cusp}}(f)$, and another geometric expansion for $P^G_{\text{cusp}}(f)$. The resulting identities would amount to new trace formulas, a third and fourth kind, which we could call the *r-trace formula* and the *primitive trace formula.* Their role would be to establish and interpret the general expansion (2.3). If the analogy with endoscopy is taken further, we might expect the primitive trace formula to follow from the r-trace formula by a process of "primitization", parallel to the stabilization of the invariant trace formula. From this perspective, the r-trace formula would therefore be the central result from which the rest should follow.

Langlands' idea is to construct the distribution $S^r_{\text{cusp}}(f)$ from the special case $S^1_{\text{cusp}}(f)$ in which $r = 1$. The finite set $V \subset \text{val}(F)$, the function f in $\mathcal{H}(G_V) \subset \mathcal{H}(G)$, and the representation r of ${}^LG$ remain fixed. Given any valuation $w \notin V$, we define a new function f^r_w in $\mathcal{H}(G)$ by setting

$$f^r_w(xx_w) = f(x)h^r_w(x_w), \qquad\qquad x \in G_V, x_w \in G(F_w),$$

where h^r_w is the unramified spherical function on $G_w = G(F_w)$ such that

$$\hat{h}^r_w(c_w) = \text{tr}(r(c_w)),$$

for any Frobenius-Hecke conjugacy class[4] (FH-class) c_w in ${}^LG_w$. We are writing $\hat{h}^r_w$ for the Satake transform of h^r_w and as usual, q_w will be the order of the residue field of F_w. Consider an irreducible unitary representation π of $G(\mathbb{A})$ that is unramified outside of V. The associated (incomplete) L-function is given by an Euler product

$$L^V(s, \pi, r) = \prod_{w \notin V} \det\left(1 - r(c(\pi_w))q_w^{-s}\right)^{-1},$$

where $c(\pi_w)$ denotes the FH-class of the w-component π_w of π. It converges for s in some right half plane. Suppose that $L^V(s, \pi, r)$ has meromorphic continuation to the complex plane, is analytic for $\Re(s) > 1$, and has no poles away from $s = 1$ on the line $\Re(s) = 1$. Using a Tauberian theorem, one can then show that the order of the pole[5] of $L^V(s, \pi, r)$ at $s = 1$ equals

$$m_\pi(r) = -\lim_{N \to \infty} |V_N|^{-1} \sum_{w \in V_N} \log(q_w)\text{tr}(r(c(\pi_w))),$$

if

$$V_N = \{w \notin V : \ q_w \leq N\}.$$

(See the appendix to §2.1 of [Se]). If we apply this identity to the spectral definition of $S^r_{\text{cusp}}(f)$, taking also the definition of h^r_w into account, we obtain

---

[4] Also called a Satake class.

[5] We are assuming that $L(s, \pi, r)$ and $L^V(s, \pi, r)$ have the same order at $s = 1$, as is expected.

a formula

$$S^r_{\text{cusp}}(f) = \lim_{N\to\infty} |V_N|^{-1} \sum_{w\in V_N} \log(q_w) S^1_{\text{cusp}}(f^r_w). \tag{2.7}$$

The limit formula (2.7) would thus be a consequence of the meromorphic continuation of the L-functions attached to r. As Langlands pointed out in his original article [L2], the meromorphic continuation and functional equation of general automorphic L-functions would follow from the principle of functoriality. But functoriality is the ultimate goal of Beyond Endoscopy, so we are certainly not in a position to assume meromorphic continuation of L-functions. All we can say is that we expect the limit formula (2.7) to be valid.

Langlands' proposal is to try to establish the limit in (2.7) from the geometric side of the trace formula for $S^1_{\text{cusp}}(f^r_w)$. Recall that the trace formula for $S^1_{\text{cusp}}(f)$ is given by the geometric expansion for $S(f)$ minus some spectral terms. We have observed, following Langlands, that these spectral terms might be treated by an induction hypothesis. However, we would want more. We would like to see some explicit cancellation of the spectral terms from the geometric expansion, at least insofar as the limit (2.7) is concerned. This would leave a purely geometric expansion to which one could try to apply the limit. The main concern here is the contribution of the nontempered stable characters in the spectral formula for $S_{\text{disc}}(f)$. As we have noted, these should be attached to the global packets Π_ψ for which the parameter ψ is nontrivial on the factor $SU(2)$ in (1.12).

An important new technique was introduced in the paper [FLN] by Frenkel, Langlands and Ngô. Motivated no doubt by Ngô's use of the Hitchin fibration in his proof of the Fundamental Lemma for a function field, they proposed a parameterization of the geometric terms in the stable trace formula over our number field F by what they called the base of the Steinberg-Hitchin fibration (SH-base). Its introduction is likely among other things to make the geometric contribution of nontempered characters more transparent.

Suppose for simplicity that G is simply connected, and split over F. In this case, the SH-base (over F) is an F-vector space $\mathcal{A}(F)$ of dimension equal to $\text{rk}(G)$, the rank of G. Its coordinates are in fact parameterized by the irreducible, finite dimensional representations r_i of G attached to fundamental dominant weights (relative to some splitting of G). The mapping

$$\delta \longrightarrow \bigoplus_i \text{tr}(r_i(\delta)) \tag{2.8}$$

is then a bijection, from the set of stable semisimple conjugacy classes δ in $G(F)$ onto the SH-base $\mathcal{A}(F)$. Now the set of geometric terms in the stable trace formula of G fibres over the set $\{\delta\}$. (Any such term is a stable distribution that is actually supported on a more general set of conjugacy classes, but each of these projects to the same δ.) For any point $a \in \mathcal{A}(F)$ with preimage δ under (2.8), let $\theta_f(a)$ be the sum of the geometric terms for $S^G(f)$ that belong to the fibre of δ. We can then write the geometric expansion for $S^G(f)$ as

$$\sum_{a\in\mathcal{A}(F)} \theta_f(a) = S^G(f), \tag{2.9}$$

following the definition from [FLN].

As a vector space over F, the SH-base has points with values in any extension of F, and in particular, in the ring of adeles $\mathbb{A}$ of F. The set $\mathcal{A}(\mathbb{A})$ is of course a locally compact abelian group, in which $\mathcal{A}(F)$ is a discrete, cocompact subgroup. The real import of the formulation (2.9) from [FLN] is the suggestion by the authors of a possible application of the Poisson summation formula, which we write informally[6] as

$$\sum_{a\in\mathcal{A}(F)} \theta_f(a) \sim \sum_{b\in\mathcal{A}(F)} \hat{\theta}_f(b). \tag{2.10}$$

This would give the geometric expansion of $S^G(f)$ something of a spectral interpretation, which one might then try to compare with the terms in the spectral expansion of $S^G(f)$. The application of Poisson summation has long been an intriguing idea in the study of trace formulas. In the past, it has generally been considered multiplicatively, on a maximal torus of G over F. For example, it was applied to the maximal split torus in $GL(2)$ in [L4] to deal with weighted orbital integrals. However, further progress proved to be elusive. The additive application of Poisson summation in (2.10) now raises interesting new possibilities.

There are a number of serious analytic difficulties to be solved before a general Poisson formula (2.10) can be applied. Frenkel, Langlands and Ngô established a modified version of it in [FLN]. They then suggested a striking relation of (2.10) to the assimilation problem for nontempered spectral terms. It pertains to the most severely nontempered term, the trace of the trivial, 1-dimensional representation π_1 of $G(\mathbb{A})$. This is the contribution to the discrete spectrum of the parameter $\psi = \psi_1$ in (1.12) that is trivial on the first factor L_F, and that as a homomorphism from the second factor $SU(2)$ to $\hat{G}$, corresponds to the principal unipotent conjugacy class in $\hat{G}$. The authors of [FLN] conjectured that it contributes only to $\hat{\theta}_f(0)$, the summand with $b = 0$ on the right hand side of (2.10). They then provide some evidence for the conjecture, which Langlands strengthens in the subsequent paper [L7].

3. Questions and Problems

There appear to be a number of accessible questions related to Beyond Endoscopy, as well as an inevitable supply of interesting but considerably more challenging problems. We shall discuss seven general questions. Rather than attempting to order the questions/problems by difficulty, we shall pose them

[6]The formula cannot be literally true, since the summands $\theta_f(a)$ are not the restriction to $a \in \mathcal{A}(F)$ of a natural Schwartz function on $\mathcal{A}(\mathbb{A})$. Since we are only trying to describe the basic ideas, we shall ignore this question.

in the natural order they might arise from the discussion of the last section. We shall be very brief.

Some of the questions are suggested by the thesis[7] of A. Altuğ [Al1]. He works in the case that $F = \mathbb{Q}$, G equals $GL(2)$, r is the standard 2-dimensional representation of $\hat{G} = GL(2, \mathbb{C})$, and V equals the single archimedean valuation $V_{\mathbb{R}} = \{\mathbb{R}\}$. The function f is then of the form

$$f_{\mathbb{R}} \cdot u^{\mathbb{R}}, \qquad f_{\mathbb{R}} \in \mathcal{H}(G_{\mathbb{R}}). \tag{3.1}$$

Despite the fact that the limited applications of this case to functoriality have long been known, the methods of Altuğ are quite powerful. They lead to new phenomena, which will be the basis of our problems **III - VI**.

In this section, G remains a quasisplit group over F. We will sometimes specialize it to a simply-connected split group, as at the end of the last section, and sometimes to the group $GL(2)$ over $\mathbb{Q}$ of [Al1]. This is often just for simplicity, and is generally to be inferred implicitly from the context.

I. It should be possible to determine the precise nature of the coefficients $\iota(r, G')$ in the decomposition (2.3). The problem is to obtain a formula for the first factor $\iota(G, G')$ in (2.4). We have essentially copied the standard notation used in the coefficients (1.6) and (1.7) from endoscopy, since there will certainly be similarities. But there is something new to be aware of here. It is the failure [AYY] of an index of summation G' in (2.3) (as an isomorphism class of "beyond endoscopic data") to be determined by its dimension datum (the function $m'(r)$ in (2.4)). My understanding is that this lack of uniqueness is characterized by the results of the recent paper [Y] of Jun Yu. However, the phenomenon does not appear to bear on the narrow problem here of describing the factor $\iota(G, G')$. Rather, it relates to the question of how we would interpret the entire primitive decomposition (2.3), once it has been established. We shall say a few words on this broader question in a final problem **VII**.

II. A basic goal of Beyond Endoscopy is to establish a primitive decomposition (2.3) for the linear form $S^r_{\text{cusp}}(f)$, just as a goal of endoscopy was to establish the stable decomposition (1.7) for the linear form $I^G_{\text{disc}}(f)$, with the broader interpretation of Theorem 1.2. Guided by endoscopy, we would first try to establish a corresponding geometric decomposition, which in this case we would hope to construct eventually as a limit (2.7). Now (2.7) was based on the orders $m_\pi(r)$ of poles of L-functions, which come from the residues of logarithmic derivatives in (2.2). It was pointed out by Sarnak [Sa] that this would actually lead to intractible estimates on the geometric side. The problem is the irregular nature of the volumes (1.3). He suggested replacing the residues $m_\pi(r)$ of logarithmic derivatives simply by the residues

$$\text{res}_{s=1} L(s, \pi, r)$$

[7] We shall generally quote from this thesis, although the results from Chapter 5 of the thesis are easier to extract from the subsequent paper [Al2].

of the L-functions themselves. This suggestion was adopted in the later papers [FLN], [L7] and [L8], and was also the perspective of Altuğ's thesis.

Can we reconcile the two possible decompositions? Will the geometric decomposition of $S^r_{\mathrm{cusp}}(f)$ based on residues of L-functions, whatever form it might take, imply the spectral decomposition (2.3) based on residues of corresponding logarithmic derivatives? These questions are for the moment hypothetical. Something close to spectral decomposition (2.3) must surely be true, informed as we are by the principle of functoriality. A geometric decomposition is something we would hope to prove directly. The problem is to determine how its spectral implications might include (2.3).

III. This question is less specific. It falls under the general heading of real harmonic analysis. Altuğ's estimates are based on the harmonic analysis of $GL(2,\mathbb{R})$. This would seem to be a relatively simple and well travelled field. But combined in [Al1] with analytic number theory arising from the approximate functional equation, and interpreted in terms of the SH-base for the group $SL(2)$, it becomes a formidable technique.

Can we generalize the estimates to other groups G and more general functions f? Of course, if we take the finite set V to include nonarchimedean places, we will need to develop new p-adic methods. Even for the unramified case $V = V_{\mathbb{R}}$, there will be p-adic questions that are presumably Beyond Endoscopic analogues of the Fundamental Lemma. They are undoubtedly related to recent work [N2] of Ngô (see also [BNS]). However, one could also try to isolate the archimedean problems without attempting to deal directly with the p-adic questions. This at least is how the proof of the general endoscopic identities (1.6) and (1.7) evolved.

The terms to be considered are the local distributions $S_M(\delta, f)$ on the geometric side (1.10) of the general stable trace formula. In the case $G = GL(2)$ of Altuğ, they are the terms (i), (ii), (iv), (v) and (viii) on pages 516 - 517 of [JL]. Altuğ's study of these terms (some of which do not appear explicitly), and his use of the approximate functional equation [IK] to control the volumes in the term (ii) of [JL], eventually lead him in Chapter 5 of [Al1] to a Poisson formula[8] (2.10) for this setting. The problem is to try to extend some portion of these results to more general settings. A reason to be optimistic is that we have Harish-Chandra's great legacy of harmonic analysis on real groups, with Shelstad's extensions to stable distributions, and perhaps also the techniques of [A7], [A8] and [A10] for dealing with stable, weighted orbital integrals.

IV. This is the problem of absorbing the nontempered characters in the automorphic discrete spectrum of G into the geometric expansion (1.10) of $S^G(f)$. If we are to be guided by the experience of Langlands and Altuğ, we would first want to establish the Poisson formula (2.10) for G. This already

[8] Altuğ actually truncates the left hand side of (2.10) in order not to deal with weighted orbital integrals, so it is actually a slight simplification of (2.10) that is established.

raises formidable analytic difficulties, as we mentioned in the last section, some of which would fall under the general questions in **III** on archimedean harmonic analysis. Once he established a form of (2.10) in his setting for $GL(2)$, Altuğ then showed in Chapter 5 [Al1] that the trivial 1-dimensional representation does indeed contribute only to the term $\hat{\theta}_f(0)$, as was conjectured for general G in [FLN]. The question is whether there are natural results of this sort for other nontempered characters in the automorphic discrete spectrum of G.

As we noted at the end of the last section, the trivial 1-dimensional representation of $G(\mathbb{A})$ corresponds to the parameter ψ_1 attached to the principal unipotent u_1 in $\hat{G}$. For the group $G = SL(2)$, it is believed to be the only nontempered constituent of the discrete spectrum of G. For general G, we expect packets Π_ψ of nontempered automorphic representations for the parameters $\psi = \psi_u$ attached to other unipotent classes u in $\hat{G}$, or more generally, to any parameter ψ that is nontrivial on the $SU(2)$ factor in (1.12). If these packets are also to give some simple contribution to (2.10), we might try to guess what it is. We could then try to find evidence for the guess from the archimedean estimates we could hope to obtain from the discussion in **III** above. What seems clear, as has been emphasized in [L6], is that we cannot expect to establish any form of the limit (2.7) without first accounting for the nontempered automorphic representations of G.

V. There is another set of terms in the stable trace formula that will have an important bearing on (2.7). For $G = SL(2)$, it consists of the single stable distribution

$$\mathrm{tr}(M(w)\mathcal{I}_P(0,f)), \qquad w \neq 1, \tag{3.2}$$

or what amounts to the same thing, the term (vi) for $GL(2)$ in [JL]. For $GL(2)$, this distribution is tempered, but because it is associated with the Riemann zeta function, it will give a nonzero contribution to the limit (2.7). Altuğ shows in Chapter 5 [Al1] that like the trivial representation of $GL(2)$ in **IV**, it also contributes only to the term $\hat{\theta}_f(0)$ in (2.10). What can possibly be going on?

The question is particularly pressing when we note that the generalizations of the distributions (3.2) to arbitrary (quasisplit) groups are often nontempered. In fact, if we think of the stabilization (1.7) of the right hand side of (1.5) (and include the terms with $w = 1$ in (1.5)), we are forced to agree that these general terms will also include the nontempered constituents of the general discrete spectrum discussed in **IV**. A general answer to the question here will thus also provide answers for **IV**. These questions seem to be related to the general injective mapping

$$u \longrightarrow w$$

from unipotent classes in $\hat{G}$ to conjugacy classes in the Weyl group of $\hat{G}$ or G, defined by Kazhdan and Lusztig in §9 in the paper [KL]. They are probably also related to the stable multiplicity formula Theorem 4.1.2 in [A8], which was proved for quasisplit orthogonal and symplectic groups G in [A8]. We hope to return to the questions in **IV** and **V** in another paper.

VI. This question concerns the interpretation of the limit (2.7). Consider the case studied by Altuğ, in which r is the standard 2-dimensional representation of the group $\hat{G} = GL(2, \mathbb{C})$. For any cuspidal automorphic representation π of the group G, the L-function

$$L(s, \pi) = L(s, \pi, r)$$

is then entire. In particular, it has no pole at $s = 1$. Therefore, by its spectral definition (either taken with the order $m_\pi(r)$ from §2 or modified with the actual residues as in II above), the linear form $S^r_{\mathrm{cusp}}(f)$ should vanish. Therefore the limit on the right hand side of (2.7) (modified or not) must also vanish. On the other hand, Altuğ has established a geometric formula for $S^1_{\mathrm{cusp}}(f)$, which according to (2.7), should lead to a geometric formula for $S^r_{\mathrm{cusp}}(f)$. If the premises of Beyond Endoscopy are to hold, one would certainly have to be able to establish the vanishing of the limit (2.7) directly from the geometric formula for $S^1_{\mathrm{cusp}}(f)$. Altuğ does this in Chapter 9 [Al1], under a certain constraint on the function f.

Altuğ's constraint on f is that its archimedean component $f_\mathbb{R}$ is a cuspidal function that is supported on the space of classical modular forms of weight $k > 2$. With this restriction, the trivial representation of $G(\mathbb{A})$ and the linear form (3.2) both vanish on f, so the estimates described in **IV** and **V** are not required. The distribution $S^1_{\mathrm{cusp}}(f^r_w)$ in (2.7) reduces simply to the classical formula of Selberg for the traces of Hecke operators. Recall that this formula has a parabolic contribution from boundary points. These come from the weighted orbital integrals of $f_\mathbb{R}$, the first time in [Al1] that these more exotic components of the trace formula are present. The problem is to show that the limit of a sum of terms (given by a minor modification of (2.7) according to the remarks in **II**) actually vanishes. But here there is a surprise. Altuğ shows that the limit of the elliptic terms and the limit of the parabolic terms both exist in their own right, but that *they are each nonzero*[9]. He shows also that the two limits sum to 0, and therefore that the (modified) limit (2.7) does vanish, as required.

How general is this phenomena? It seems remarkable that a new identity should arise from the very familiar, sixty year old formula of Selberg. The first thing that one might attempt would be to remove the cuspidal constraint on $f_\mathbb{R}$. The elliptic orbital integrals and the weighted orbital integrals then become considerably more complex, even for the group $GL(2)$. Do the two

[9]This phenomenon applies only to the modification of the limit (2.7) discussed in **II**, wherein the logarithmic derivative of (2.2) is replaced by the L-function itself. I thank Altuğ for pointing out that if one sticks with the original logarithmic derivative, the elliptic and parabolic limits are in fact both equal to zero. (See [L6, §2.3].) The dichotomy here, which entails an examination of weighted orbital integrals for $GL(2)$, might be a good place to begin a study of the questions raised in **II**.

individual limits still exist? What is the nature of the corresponding linear form in f? What happens if, for example, $GL(2)$ is replaced by $GL(N)$, but r remains the standard (N-dimensional) representation? We refer the reader to later sections of [L6] for a discussion of archimedean weighted orbital integrals in a somewhat more general setting.

VII. Among the expected properties we have described, there has been an obvious omission. It is the principle of functoriality itself. Given an r-trace formula for $S^r_{\mathrm{cusp}}(f)$ with primitization (2.3), for each representation r of ${}^LG$, how might one try to deduce functoriality?

The problem would be to establish functoriality between G' and G, for any of the indices G' on the right hand side of the primitive decomposition (2.3) attached to G and r. In the paper [L6], Langlands suggested trying to isolate the contribution of G' (which we have formulated as the primitive distribution $\hat{P}'_{\mathrm{cusp}}(f')$ in (2.3)) by letting the representation r of ${}^LG$ vary. In particular, he raised the question of how close the dimension data $m'(r)$ in (2.4) come to determining the contribution of G'. As we mentioned above in **I**, this question has since been settled [AYY], [Y], with a conclusion that the dimension data are generally not sufficient to isolate G'. What should be the next step? This is a question that calls for considerably more thought. I raise it only because a problem that seems similar had to be solved in the classification [A9] for representations of orthogonal and symplectic groups.

Suppose that G is the $G^0 = GL(2n)$-space defined by the standard outer automorphism of $GL(2n)$. Let G' be one of the two split groups G'_1 and G'_2 whose dual groups are $\hat{G}'_1 = Sp(2n, \mathbb{C})$ and $\hat{G}'_2 = SO(2n, \mathbb{C})$ respectively. We are now thinking of the twisted stabilization (1.6) for endoscopy, rather than the beyond endoscopic primitization (2.3). In particular, the two groups G' represent two elliptic endoscopic data for G, which therefore correspond to two terms on the right hand side of (1.6). The problem from [A9] was to separate the contributions of these terms to the left hand side of (1.6). The issue in this case was not in the dimension data for the two groups G', which actually do separate the two contributions. (See [A9, Theorem 1.5.3 (a)].) The problem was to establish this from the limited information at hand, and in particular, to establish functoriality between G' and G^0 in this special case. It required a further technique.

The solution in [A9] was to extend the application of (1.6). For any global parameter ψ for G, one can attach larger groups G'_+ to the two groups $G' = G'_1$ and $G' = G'_2$, and a larger G^0_+-space G_+ to the G^0-space G. (See [A9, §5.3].) The stabilization (1.6) for G_+ then leads to further information about the correspondence between G' and G. For the most critical parameters (which we called simple in [A9], but which really act as the primitive objects in that setting), further enlargements G'_{++} and G_{++} are needed in order to complete the underlying proof by induction. (See [A9, §8.2].) The argument requires considerable care, since the larger groups G'_+ and G'_{++} are outside the domain of the basic induction hypothesis. We shall say nothing further about the technique, except to ask whether an enlargement in this spirit of the fixed groups G and G' in (2.3) might provide useful supplementary information.

References

[Al1] S.A. Altuğ, *Beyond Endoscopy via the Trace Formula*, PhD Thesis, Princeton University, (2013).

[Al2] ______, *Beyond endoscopy via the trace formula-I: Poisson summation and isolation of special representations*, Compositio Mathematica, (2015), 1–30.

[Al3] ______, *Beyond endoscopy via the trace formula-II: Asymptotic expansions of Fourier transforms of orbital integrals and bounds towards the Ramanujan's conjecture*, submitted.

[Al4] ______, *Beyond endoscopy via the trace formula-III*, in preparation.

[AYY] J. An, J.-K. Yu, and J. Yu, *On the dimension datum of a subgroup and its application to isopectral manifolds*, Journal of Differential Geometry, **94** (2013), 59–85.

[A1] J. Arthur, *The invariant trace formula II. Global theory*, Journal of American Mathematical Society, **1** (1988), 223–293.

[A2] ______, *A stable trace formula I. General expansions*, Journal of the Institute of Mathematics of Jussieu, **1** (2002), 175–277.

[A3] ______, *A stable trace formula II. Global descent*, Inventiones Mathematicae, **143** (2001), 157–220.

[A4] ______, *A stable trace formula III. Proof of the main theorems*, Annals of Mathematics, **158** (2003), 769–873.

[A5] ______, *A note on the automorphic Langlands group*, Canadian Mathematical Bulletin, **45** (2002), 466–482.

[A6] ______, *An introduction to the trace formula*, in *Harmonic Analysis, the Trace Formula, and Shimura Varieties*, Clay Mathematics Proceedings, vol. 4, (2005), 1–263.

[A7] ______, *An asymptotic formula for real groups*, Journal für die Reine und Angewandte Mathematik, **601** (2006), 163–230.

[A8] ______, *Parabolic transfer for real groups*, Journal of American Mathematical Society, **21** (2008), 171–234.

[A9] ______, *The Endoscopic Classification of Representations: Orthogonal and Symplectic Groups*, Colloquium Publications, **61** (2013), American Mathematical Society.

[A10] ______, *Germ expansions for real groups*, preprint (under revision).

[BNS] A. Bouthier, B.C. Ngô, and Y. Sakellaridis, *On the formal arc space of a reductive monoid*, preprint.

[BK] A. Braverman, and D. Kazhdan, *γ-functions of representations and lifting*, in *Visions in Mathematics*, 2000, 237–278.

[FLN] E. Frenkel, R.P. Langlands, and B.C. Ngô, *La formule des traces et la functorialité, Le début d'un programme*, Annales des Sciences Mathématiques du Québec, **34** (2010), 199–243.

[G] J. Getz, *Nonabelian Fourier transforms over the complex numbers*, preprint.

[GH] J. Getz, and P.E. Herman, *A nonabelian trace formula*, preprint.

[H] P.E. Herman, *Quadratic base change and the analytic continuation of the Asai L-function: A new trace formula approach*, preprint.

[IK] H. Iwaniec, and E. Kowalski, *Analytic Number Theory*, Colloquium Publications, **53** (2004), American Mathematical Society.

[JL] H. Jacquet, and R.P. Langlands, *Automorphic forms on $GL(2)$*, Lecture Notes in Mathematics, **114** (1970), Springer Verlag.

[KL] D. Kazhdan, and G. Lusztig, *Fixed point varieties on affine flag manifolds*, Israel Journal of Mathematics, **62** (1988), 129–168.

[K] R. Kottwitz, *Stable trace formula: cuspidal tempered terms*, Duke Mathematical Journal, **51** (1984), 611–650.

[KS] R. Kottwitz, and D. Shelstad, *Foundations of Twisted Endoscopy*, Astérisque, **255** (1999), Societe Mathématique de France.

[Laf] L. Lafforgue, *Noyaux du transfert automorphe de Langlands et formules de Poisson non linéaires*, Japanese Journal of Mathematics, **9** (2014), 1–68.

[L1] R. Langlands, *Letter to André Weil*, 1967, publications.ias.edu.

[L2] ———, *Problems in the theory of automorphic forms*, in *Lectures in Modern Analysis and Applications*, Lecture Notes in Mathematics, vol. 170, Springer, New York, 1970, 18–61.

[L3] ———, *Automorphic representations, Shimura varieties, and motives. Ein Märchen*, in *Automorphic Forms, Representations and L-functions*, Proceedings of Symposium in Pure Mathematics, vol. 33, Part 2, American Mathematical Society, 1979, 205–246.

[L4] ———, *Base Change for $GL(2)$*, Annals of Mathematics Studies, **96** (1980), Princeton University Press.

[L5] ———, *Les débuts d'une formule des traces stable*, Publications mathématiques de l'Université Paris VII, **13** (1983).

[L6] ———, *Beyond endoscopy*, in *Contributions to Automorphic Forms, Geometry, and Number Theory*, John Hopkins University Press, 2004, 611–698.

[L7] ———, *Singularités et transfert*, Annales mathématiques du Québec, **37** (2013), 173–253.

[L8] ———, *A prologue to functoriality and reciprocity: Part 1*, Pacific Journal of Mathematics, **260** (2012), 583–663.

[LS] R. Langlands, and D. Shelstad, *On the definition of transfer factors*, Mathematische Annalen, **278** (1987), 219–271.

[MW1] C. Mœglin, and J.-L. Waldspurger, *Le spectre résiduel de $GL(n)$*, Annales scientifiques de l'École Normale Supérieure, **22** (1989), 605–674.

[MW2] ———, *Stabilisation de la formule des traces tordue VI: la partie géométrique de cette formule*, preprint.

[MW3] ———, *Stabilisation de la formule des traces tordue X: stabilisation spectrale*, preprint.

[MW4] ______, *Représentations elliptiques: caractérisation et formule de transfert de caractéres*, preprint.

[M] C.P. Mok, *Endoscopic classification of representations of quasi-split unitary groups*, Memoirs of the American Mathematical Society, **235** (2015).

[N1] B.C. Ngô, *Le lemme fondamental pour les algébres de lie*, Publications mathématiques de l'IHÉS, **111** (2010), 1–169.

[N2] ______, *On a certain sum of automorphic L-functions*, in *Automorphic Forms and Related Geometry*, Contemporary Mathematics, **614** (2014), 337–343.

[Sak] Y. Sakellaridis, *Beyond endoscopy for the relative trace formula I: local theory*, in *Automorphic Representations and L-functions*, Tata Institute of Fundamental Research, 2013, 521–590.

[Sa] P. Sarnak, *Comments on Langlands' Lecture: "Endoscopy and Beyond"*, 2001, publications.ias.edu.

[Se] J.-P. Serre, *Abelian l-adic representations and elliptic curves*, Benjamin, New York, 1968.

[V] A. Venkatesh, *Beyond endoscopy and special forms on* $GL(2)$, Journal für die Reine und Angewandte Mathematik, **577** (2004), 23–80.

[W1] J.-L. Waldspurger, *Le lemme fondamental implique le transfert*, Compositio Mathematica, **105** (1997), 153–236.

[W2] ______, *Endoscopie et changement de caractéristique*, Journal of the Institute of Mathematics of Jussieu, **5** (2006), 423–525.

[W3] ______, *Stabilisation de la formule des traces tordue I: endoscopie tordue sur un corps local*, preprint.

[W4] ______, *Stabilisation de la formule des traces tordue II: intégrales orbitales et endoscopic sur un corps local non-archimédien; définitions et énonceés des résultats*, preprint.

[W5] ______, *Stabilisation de la formule des traces tordue III: intégrales orbitales et endoscopie sur un corps local non-archimédien; réductions et preuves*, preprint.

[W6] ______, *Stabilisation de la formule des traces tordue IV: transfert spectral archimédien*, preprint.

[W7] ______, *Stabilisation de la formule des traces tordue V: intégrales orbitales et endoscopie sur le corps réel*, preprint.

[W8] ______, *Stabilisation de la formule des traces tordue VII: descente globale*, preprint.

[W9] ______, *Stabilisation de la formule des traces tordue VIII: l'application* $\epsilon_{\tilde{M}}$ *sur un corps de base local non-archimédien*, preprint.

[W10] ______, *Stabilisation de la formule des traces tordue IX: propriétés des intégrales orbitales pondérées omega-équivariantes sur le corps réel*, preprint.

[Y] J. Yu, *On the dimension datum of a subgroup*, preprint.

James Arthur
Department of Mathematics
University of Toronto
40 St. George Street, Toronto, Ontario, Canada
M5S 2E4
e-mail: arthur@math.utoronto.ca