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A TRACE FORMULA FOR REDUCTIVE GROUPS I TERMS ASSOCIATED TO CLASSES IN $G(\mathbf{Q})$

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Introduction

This paper, along with a subsequent one, is an attempt to generalize the Selberg trace formula to an arbitrary reductive group G defined over the rational numbers. Our main results have been announced in the lectures [1(*c*)].

In his original papers [8(*a*), (*b*)], Selberg gave a novel formula for the trace of a certain operator associated with a compact quotient of a semisimple Lie group and a discrete subgroup. When the discrete subgroup is arithmetic, the situation is essentially equivalent to the case that the group G is anisotropic. Then $G(\mathbf{A})$ is a locally compact topological group and $G(\mathbf{Q})$ is a discrete subgroup such that $G(\mathbf{Q})\backslash G(\mathbf{A})$, the space of cosets (with the quotient topology), is compact. The operator is convolution on $L^2(G(\mathbf{Q})\backslash G(\mathbf{A}))$ by a smooth, compactly supported function f on $G(\mathbf{A})$.

Let us recall how to derive the formula in this case. To understand the idea, it is not necessary to be an expert in algebraic groups, or even to be familiar with the notion of adèles. If $\phi \in L^2(G(\mathbf{Q})\backslash G(\mathbf{A})$, define

$$(R(y)\phi)(x) = \phi(xy), \qquad x, y \in G(\mathbf{A}).$$

Then R is a unitary representation of $G(\mathbf{A})$, and the convolution operator is defined by,

$$R(f) = \int_{G(\mathbf{A})} f(y)R(y)dy.$$

$(R(f)\phi)(x)$ can be written

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$$\begin{aligned}\int_{G(\mathbf{A})} f(y)\phi(xy)dy &= \int_{G(\mathbf{A})} f(x^{-1}y)\phi(y)dy \\ &= \int_{G(\mathbf{Q})\backslash G(\mathbf{A})} \sum_{\gamma \in G(\mathbf{Q})} f(x^{-1}\gamma y)\phi(\gamma y)dy \\ &= \int_{G(\mathbf{Q})\backslash G(\mathbf{A})} \left\{ \sum_{\gamma \in G(\mathbf{Q})} f(x^{-1}\gamma y) \right\} \phi(y)dy,\end{aligned}$$

by changing the variable of integration twice. Thus, $R(f)$ is an integral operator. Its kernel equals

$$\sum_{\mathfrak{o} \in \mathcal{O}} K_{\mathfrak{o}}(x, y),$$

where $\mathcal{O}$ is the set of conjugacy classes in the group $G(\mathbf{Q})$, and

$$K_{\mathfrak{o}}(x, y) = \sum_{\gamma \in \mathfrak{o}} f(x^{-1}\gamma y), \qquad \mathfrak{o} \in \mathcal{O}.$$

On the other hand, there is an even more formal way to write the kernel. It is not hard to show that R decomposes into a direct *sum* of irreducible representations of $G(\mathbf{A})$, each occurring with finite multiplicity. In other words

$$L^2(G(\mathbf{Q})\backslash G(\mathbf{A})) = \bigoplus_{\chi \in \mathfrak{X}} L^2(G(\mathbf{Q})\backslash G(\mathbf{A}))_\chi,$$

where $\mathfrak{X}$ is a set of unitary equivalence classes of irreducible representations of $G(\mathbf{A})$, and the restriction of the representation R to the subspace $L^2(G(\mathbf{Q})\backslash G(\mathbf{A})_\chi$ is equivalent to a finite number of copies of χ. For each $\chi \in \mathfrak{X}$, let $\mathcal{B}_\chi$ be a suitable orthonormal basis of $L^2(G(\mathbf{Q})\backslash G(\mathbf{A}))_\chi$. Then

$$K_\chi(x, y) = \sum_{\phi \in \mathcal{B}_\chi} (R(f))\phi(x) \cdot \overline{\phi(y)}$$

converges, and

$$\sum_{\chi \in \mathfrak{X}} K_\chi(x, y)$$

is a second formula for the kernel of $R(f)$. The Selberg trace formula comes from integrating both formulas for the kernel over the diagonal. (The necessary convergence arguments are easily established.) Thus

$$\sum_{\mathfrak{o} \in \mathcal{O}} J_{\mathfrak{o}}(f) = \sum_{\chi \in \mathfrak{X}} J_\chi(f),$$

where $J_{\mathfrak{o}}(f)$ is the integral over x in $G(\mathbf{Q})\backslash G(\mathbf{A})$ of

$$k_{\mathfrak{o}}(x, f) = K_{\mathfrak{o}}(x, x)$$

and $J_\chi(f)$ is the integral of

$$k_\chi(x, f) = K_\chi(x, x).$$

If $\{\gamma\}$ stands for a set of representatives of the conjugacy classes in $G(\mathbf{Q})$, and $G(\gamma)$ denotes the centralizer of γ in G, the formula can be written

$$\sum_{\{\gamma\}} \operatorname{vol}\,(G(\mathbf{Q}, \gamma)\backslash G(\mathbf{A}, \gamma)) \cdot \int_{G(\mathbf{A}, \gamma)\backslash G(\mathbf{A})} f(x^{-1}\gamma x)dx = \sum_{\chi \in \mathscr{X}} tr\, R(f)_\chi. \tag{1}$$

Thus, by simple formal manipulations, the trace of the operator $R(f)$ has been expressed as a sum of integrals of f over conjugacy classes.

If G is not anisotropic, $G(\mathbf{Q})\backslash G(\mathbf{A})$ is no longer compact and everything breaks down. Selberg suggested a program for obtaining a formula and studied some special cases. Further progress has since been made by others, but the program has been carried out completely for only a small number of groups, essentially just GL_2 and related groups.

What are the difficulties in the noncompact setting? In general $R(f)$ is still an integral operator. However, the natural definition of $\mathscr{O}$ seems to be the equivalence classes composed of those elements in $G(\mathbf{Q})$ whose *semisimple* components are $G(\mathbf{Q})$-conjugate. If G is anisotropic, all the elements in $G(\mathbf{Q})$ are semisimple so this relation is then just conjugacy. With $K_\mathfrak{o}(x, y)$ defined as above, the kernel of $R(f)$ can again be written

$$\sum_{\mathfrak{o} \in \mathscr{O}} K_\mathfrak{o}(x, y).$$

One of the main complications in the noncompact setting is that R no longer decomposes discretely. R contains a continuous family of representations for every parabolic subgroup of G defined over $\mathbf{Q}$. The intertwining operators are provided by Eisenstein series, whose main properties we recall in §3.

We now have to define the set $\mathscr{X}$. If $G = GL_n$ and investigations of Jacquet, Shalika and Piatetskii-Shapiro proceed as expected, $\mathscr{X}$ could be defined as the classes of irreducible unitary representations of $G(\mathbf{A})$ relative to a certain relation, weaker than unitary equivalence. Otherwise, $\mathscr{X}$ must be defined as in §3, in terms of cuspidal automorphic representations of Levi components of parabolic subgroups of G. At any rate we obtain an identity

$$\sum_{\mathfrak{o} \in \mathscr{O}} K_\mathfrak{o}(x, y) = \sum_{\chi \in \mathscr{X}} K_\chi(x, y) \tag{2}$$

in §4 by equating two different formulas for the kernel of $R(f)$. We can now set $x = y$ and ask whether the various functions on each side are integrable.

It turns out that $K_\mathfrak{o}(x, x)$ is integrable precisely when $\mathfrak{o}$ intersects no group $P(\mathbf{Q})$, with P a proper parabolic subgroup defined over $\mathbf{Q}$. The more parabolic subgroups that meet $\mathfrak{o}$, the worse will be the divergence of the integral. The integral of $K_\chi(x, x)$ behaves in the same sort of way. We are therefore forced in §5 to modify the identity of adding suitable correction terms to each side. The correction terms are indexed by the conjugacy classes of proper parabolic subgroups, P, of G, and they also depend on a point T in a positive Weyl chamber.

The construction of the correction terms proceeds as follows: one multiplies the identity

$$\sum_{\mathfrak{o} \in \mathcal{O}} K_{P,\mathfrak{o}}(x, x) = \sum_{\chi \in \mathscr{X}} K_{P,\chi}(x, x)$$

(the analogue for P of (2)), by the characteristic function of a subset of $P(\mathbf{Q})\backslash G(\mathbf{A})$ associated to T. The product is then summed over $P(\mathbf{Q})\backslash G(\mathbf{Q})$ to make the functions left $G(\mathbf{Q})$-invariant. We denote the corrected functions by $k_{\mathfrak{o}}^T(x, f)$ and $k_\chi^T(x, f)$ and obtain a new identity

$$\sum_{\mathfrak{o} \in \mathcal{O}} k_{\mathfrak{o}}^T(x, f) = \sum_{\chi \in \mathscr{X}} k_\chi^T(x, f), \qquad x \in G(\mathbf{Q})\backslash G(\mathbf{A}). \tag{3}$$

When T is large the correction terms all vanish on a large compact subset of $G(\mathbf{Q})\backslash G(\mathbf{A})$. On this set, $k_{\mathfrak{o}}^T(x, f)$ equals $K_{\mathfrak{o}}(x, x)$. Moreover, if $\mathfrak{o}$ does not intersect any proper parabolic subgroup, $k_{\mathfrak{o}}^T(x, f)$ equals $K_{\mathfrak{o}}(x, x)$ everywhere. Similar remarks apply to the functions on the right.

Our main goal will be to show that each side of the new identity (3) is integrable and that the integrals may be taken inside the sums. If $J_{\mathfrak{o}}^T(f)$ and $J_\chi^T(f)$ stand for the integrals of the summands, we will then have a formula

$$\sum_{\mathfrak{o} \in \mathcal{O}} J_{\mathfrak{o}}^T(f) = \sum_{\chi \in \mathscr{X}} J_\chi^T(f) \tag{4}$$

which generalizes the Selberg trace formula for compact quotient.

In this paper we will deal with the left hand side of (3). We prove the integrability in §7. The argument is partly geometric and partly combinatorial. It is a standard technique in classical reduction theory to partition the upper half plane modulo $SL_2(\mathbf{Z})$ into a compact region and a noncompact region whose topological properties are particularly simple. Imitating a construction in Langlands' manuscript on Eisenstein series, we introduce a similar partition of $G(\mathbf{Q})\backslash G(\mathbf{A})$. This partition and other geometric facts are discussed in §6. We begin §7 with some manipulations that yield different formulas for the functions $k_{\mathfrak{o}}(x, f)$. This amounts to studying the components of $k_{\mathfrak{o}}(x, f)$ on various subsets of the domain of integration. It turns out that all the components which are not integrable on a given subset cancel. What remains can be estimated by the Poisson summation formula as in GL_2 [5, §XVI], and is at length shown to be integrable.

The classes $\mathfrak{o}$ which do not meet any proper parabolic subgroup contain only semisimple elements, and are actual conjugacy classes. Thus $J_{\mathfrak{o}}^T(f)$ is independent of T and can be expressed as an orbital integral as on the left hand side of (1). In §8 we study the distributions $J_{\mathfrak{o}}^T$ for a wider collection of $\mathfrak{o}$, the classes we will term *unramified*. We will show that $J_{\mathfrak{o}}^T(f)$ can still be expressed as an orbital integral of f. However, it will be the orbital integral with respect to a measure which is not invariant, but weighted by a function obtained by taking the volume of a certain convex hull. It has been shown [1(b)] that this particular weighted orbital integral can be evaluated explicitly for certain special functions f.

The main use of the trace formula would be to obtain information about the right hand side from knowledge of the left hand side. One hopes that it will someday shed light on Langlands' functoriality conjecture, perhaps the fundamental problem in the whole area. Three special cases of this conjecture have already been solved [5], [6(c)], [7], by using the trace formula for GL_2. In each case, a trace formula for one group G was compared with the trace formula for $G' = GL_2$. An arbitrary funciton f on $G(\mathbf{A})$ was used to define a function f' on $G'(\mathbf{A})$ so as to make the left hand sides of the two formulas equal. The correspondence between automorphic representations was then extracted from the resulting equality of right hand sides. The formula (4), whose proof we will complete in the next paper, is not as explicit as the trace formula for GL_2 [5, pg. 516-517]. For example, if $\mathfrak{o}$ is the class consisting of unipotent elements, we cannot express $J^T_{\mathfrak{o}}(f)$ as a weighted orbital integral. Nevertheless, I believe that (4) can eventually yield the same kind of results for general G. One would characterize the distributions $J^T_{\mathfrak{o}}$ and J^T_{χ} by the properties required by the applications rather than by their explicit formulas.

§1. Preliminaries

Let G be a reductive algebraic group defined over $\mathbf{Q}$. We shall fix, for once and for all, a minimal parabolic subgroup P_0, and a Levi component, M_{P_0}, of P_0, both defined over $\mathbf{Q}$. In this paper we shall work only with standard parabolic subgroups of G; that is, parabolic subgroups P, defined over $\mathbf{Q}$, which contain P_0. Let us agree to refer to such groups simply as "parabolic subgroups." Fix P. Let N_P be the unipotent radical of P and let M_P be the unique Levi component of P which contains M_{P_0}. Denote the split component of the center of M_P by A_p. N_P, M_P and A_P are all defined over $\mathbf{Q}$. Let $X(M_P)_{\mathbf{Q}}$ be the group of characters of M_P defined over $\mathbf{Q}$. Then

$$\mathfrak{a}_P = \mathrm{Hom}\,(X(M_P)_{\mathbf{Q}}, \mathbf{R})$$

is a real vector space whose dimension equals that of A_p. Its dual space is

$$\mathfrak{a}_P^* = X(M_P)_{\mathbf{Q}} \otimes \mathbf{R}.$$

We shall denote the set of simple roots of (P, A) by Δ_P. They are elements in $X(A_P)_{\mathbf{Q}}$ and are canonically embedded in $\mathfrak{a}_P^*$. The set $\Delta_0 = \Delta_{P_0}$ is a base for a root system. In particular, we have the co-root $\alpha^\vee$ in $\mathfrak{a}_{P_0}$ for every root $\alpha \in \Delta_P$.

Suppose that P_1 and P_2 are parabolic subgroups with $P_1 \subset P_2$. $\mathfrak{a}_{P_2}^*$ comes with an embedding into $\mathfrak{a}_{P_1}^*$, while $\mathfrak{a}_{P_2}$ is a natural quotient vector space of $\mathfrak{a}_{P_1}$. The group $M_{P_2} \cap P_1$ is a parabolic subgroup of M_{P_2} with unipotent radical

$$N_{P_1}^{P_2} = N_{P_1} \cap M_{P_2}.$$

The set, $\Delta_{P_1}^{P_2}$, of simple roots of $(M_{P_2} \cap P_1, A_{P_1})$ is a subset of Δ_{P_1}. As is well known, $P_2 \to \Delta_{P_1}^{P_2}$ is a bijection between the set of parabolic subgroups P_2 which contain P_1, and the collection of subsets of Δ_{P_1}. Identify $\mathfrak{a}_{P_2}$ with the subspace

$$\{H \in \mathfrak{a}_{P_1} : \alpha(H) = 0,\ \alpha \in \Delta_{P_1}^{P_2}\}.$$

If $\mathfrak{a}_{P_1}^{P_2}$ is defined to be the subspace of $\mathfrak{a}_{P_1}$ annihilated by $\mathfrak{a}_{P_2}^*$, then

$$\mathfrak{a}_{P_1} = \mathfrak{a}_{P_1}^{P_2} \oplus \mathfrak{a}_{P_2}.$$

The subspace, $(\mathfrak{a}_{P_1}^{P_2})^*$, of $\mathfrak{a}_{P_1}^*$ spanned by $\Delta_{P_1}^{P_2}$ is in natural duality with $\mathfrak{a}_{P_1}^{P_2}$ and we have

$$\mathfrak{a}_{P_1}^* = (\mathfrak{a}_{P_1}^{P_2})^* \oplus \mathfrak{a}_{P_2}^*.$$

The space $\mathfrak{a}_{P_1}^{P_2}$ also embeds into $\mathfrak{a}_{P_0}^{P_2}$, and

$$\mathfrak{a}_{P_0}^{P_2} = \mathfrak{a}_{P_0}^{P_1} \oplus \mathfrak{a}_{P_1}^{P_2}.$$

Any root $\alpha \in \Delta_{P_1}^{P_2}$ is the restriction to $\mathfrak{a}_{P_1}^{P_2}$ of a unique root $\beta \in \Delta_{P_0}^{P_2}$. Define $\alpha^\vee$ to be the projection onto $\mathfrak{a}_{P_1}^{P_2}$ of the vector $\beta^\vee$ in $\mathfrak{a}_{P_0}^{P_2}$. Then

$$\{\alpha^\vee : \alpha \in \Delta_{P_1}^{P_2}\}$$

is a basis of $\mathfrak{a}_{P_1}^{P_2}$. Let

$$\hat{\Delta}_{P_1}^{P_2} = \{\varpi_\alpha : \alpha \in \Delta_{P_2}^{P_2}\}$$

be the corresponding dual basis for $(\mathfrak{a}_{P_1}^{P_2})^*$. If $\varpi \in \hat{\Delta}_{P_1}^{P_2}$, define $\varpi^\vee \in \mathfrak{a}_{P_1}^{P_2}$ by

$$\alpha(\varpi^\vee) = \varpi(\alpha^\vee), \qquad \alpha \in \Delta_{P_1}^{P_2}.$$

Then $\{\varpi_\alpha^\vee\}$ and $\{\alpha\}$ is another pair of dual bases for $\mathfrak{a}_{P_1}^{P_2}$ and $(\mathfrak{a}_{P_1}^{P_2})^*$.

If P_i is a parabolic subgroup, and P_i appears in our notation as a subscript or a superscript, we shall often use only i, instead of P_i, for the subscript or superscript. For example,

$$M_0 = M_{P_0}, \qquad \mathfrak{a}_1^2 = \mathfrak{a}_{P_1}^{P_2}, \quad \text{and} \quad N_1^2 = N_{P_1}^{P_2}.$$

If the letter P alone is used, we shall often omit it altogether as a subscript. Thus, $P = NM$, Δ is the set of simple roots of (P, A), and so on. G itself is a parabolic subgroup. We shall often write Z for A_G, and $\mathfrak{z}$ for $\mathfrak{a}_G$.

The following proposition is trivial to prove. However, it will be the ultimate justification for much of this paper so we had best draw attention to it.

PROPOSITION 1.1. *Suppose that P_1 and P_2 are parabolic subgroups, with $P_1 \subset P_2$. Then*

$$\sum_{\{P : P_1 \subset P \subset P_2\}} (-1)^{(\dim(A/A_2))} = \begin{cases} 1 & \text{if } P_1 = P_2 \\ 0 & \text{otherwise.} \end{cases}$$

(The sum is, of course, over parabolic subgroups P; $A = A_P$ is uniquely determined by P.)

Proof. The sum can be regarded as the sum over all subsets of Δ_1^2. The result follows from the binomial theorem. □

Let $\mathbf{A}$ (resp. $\mathbf{A}_f$) be the ring of adèles (resp. finite adèles) of $\mathbf{Q}$. Then

$$G(\mathbf{A}) = G(\mathbf{R}) \cdot G(\mathbf{A}_f)$$

is the restricted direct product over all valuations v, of the groups $G(\mathbf{Q}_v)$. Fix, for once and for all, a maximal compact subgroup

$$K = \prod_v K_v,$$

of $G(\mathbf{A})$ so that the following properties hold:

(i) For any embedding of G into GL_n, defined over $\mathbf{Q}$, $K_v = GL_n(\mathfrak{o}_v) \cap G(\mathbf{Q}_v)$ for almost all finite places v.

(ii) For every finite v, K_v is a special maximal compact subgroup.

(iii) The Lie algebras of $K_{\mathbf{R}}$ and $A_0(\mathbf{R})$ are orthogonal with respect to the Killing form.

Suppose that P is a parabolic subgroup. If $m = \prod_v m_v$ lies in $M(\mathbf{A})$, define a vector $H_M(m)$ in $\mathfrak{a}_P$ by

$$e^{\langle H_M(m), \chi\rangle} = |\chi(m)| = \prod_v |\chi(m_v)|_v, \ \chi \in X(m)_{\mathbf{Q}}.$$

H_M is a homomorphism of $M(\mathbf{A})$ to the additive group $\mathfrak{a}_P$. Let $M(\mathbf{A})^1$ be its kernal. Then $M(\mathbf{A})$ is the direct product of $M(\mathbf{A})^1$ and $A(\mathbf{R})^0$, the component of 1 in $A(\mathbf{R})$. By our conditions on K, $G(\mathbf{A}) = P(\mathbf{A})K$. Any $x \in G(\mathbf{A})$ can be written as

$$nmak, \ n \in N(\mathbf{A}), \ m \in M(\mathbf{A})^1, \ a \in A(\mathbf{R})^0, \ k \in K.$$

We define $H_P(x) = H(x)$ to be the vector $H_M(ma) = H_M(a)$ in $\mathfrak{a}_P$. Notice that if $P_1 \subset P_2$, $\mathfrak{a}_{P_1}^{P_2}$ is the image of $M_{P_2}(\mathbf{A})^1$ under H_{P_1}.

We shall denote the restricted Weyl group of (G, A_0) by Ω. Ω acts on $\mathfrak{a}_0$ and $\mathfrak{a}_0^*$ in the usual way. For every $s \in \Omega$ we shall fix a representative w_s in the intersection of $G(\mathbf{Q})$ with the normalizer of A_0. w_s is determined modulo $M_0(\mathbf{Q})$. If P_1 and P_2 are parabolic subgroups, let $\Omega(\mathfrak{a}_1, \mathfrak{a}_2)$ denote the set of distinct isomorphisms from $\mathfrak{a}_1$ onto $\mathfrak{a}_2$ obtained by restricting elements in Ω to $\mathfrak{a}_1$. P_1 and P_2 are said to be *associated* if $\Omega(\mathfrak{a}_1, \mathfrak{a}_2)$ is not empty. If s_1 belongs to $\Omega(\mathfrak{a}_1, \mathfrak{a}_2)$, there is a unique element s in Ω whose restriction to $\mathfrak{a}_1$ is s_1 and such that $s^{-1}\alpha$ is a positive root (that is to say, a root of (P_0, A_0)) for every $\alpha \in \Delta_0^{P_1}$. Thus, $\Omega(\mathfrak{a}_1, \mathfrak{a}_2)$ can be regarded as a subset of Ω; in particular, w_{s_1} is an element in $G(\mathbf{Q})$ for every $s_1 \in \Omega(\mathfrak{a}_1, \mathfrak{a}_2)$.

We shall need to adopt some conventions for choices of Haar measures. For any connected subgroup, V, of N_0, defined over $\mathbf{Q}$, we take the Haar measure on $V(\mathbf{A})$ which assigns $V(\mathbf{Q})\backslash V(\mathbf{A})$ the volume one. Similarly, we take the Haar measure of K to be one. Fix Haar measures on each of the vector spaces $\mathfrak{a}_P$. On the spaces $\mathfrak{a}_P^*$ we take the dual Haar measures. We then utilize the isomorphisms

$$H_P : A_P(\mathbf{R})^0 \rightarrow \mathfrak{a}_P$$

to define Haar measures on the groups $A_P(\mathbf{R})^0$. Finally, fix a Haar measure on $G(\mathbf{A})$. For any P, let

$$\mathfrak{a}_P^+ = \{H \in \mathfrak{a}_P : \alpha(H) > 0, \ \alpha \in \Delta_P\},$$

and

$$(\mathfrak{a}_P^*)^+ = \{\Lambda \in \mathfrak{a}_P^* : \Lambda(\alpha^\vee) > 0,\ \alpha \in \Delta_P\}.$$

There is a vector ρ_P in $\mathfrak{a}_P^*$ such the modular function

$$\delta_P(p) = |\det(Ad\, p)_{\mathfrak{n}_p(A)}|, \qquad p \in P(\mathbf{A}),$$

on $P(\mathbf{A})$ equals $e^{2\rho_P(H_p(p))}$. Here $\mathfrak{n}_P$ stands for the Lie algebra of N_P. There are unique Haar measures on $M(\mathbf{A})$ and $M(\mathbf{A})^1$ such that for any function $h \in C_c(G(\mathbf{A}))$,

$$\begin{aligned}\int_{G(\mathbf{A})} h(x)dx &= \int_{N(\mathbf{A})} \int_{M(\mathbf{A})} \int_K h(nmk)\, e^{-2\rho_P(H_P(m))}\, dndmdk \\ &= \int_{N(\mathbf{A})} \int_{M(\mathbf{A})^1} \int_{A(\mathbf{A})^0} \int_K h(nmak)\, e^{-2\rho_P(H_P(a))}\, dndadmdk.\end{aligned}$$

We should recall some properties of height functions associated to rational representations of G. Let V be a vector space defined over $\mathbf{Q}$. Suppose that $\{v_1, \cdots, v_n\}$ is a basis of $V(\mathbf{Q})$. If $\xi_v \in V(\mathbf{Q}_v)$, and

$$\xi_v = \sum_i \xi_v^i v_i,\ \xi_v^i \in \mathbf{Q}_v,$$

define

$$\|\xi_v\|_v = \max_i |\xi_v^i|_v$$

if v is finite, and

$$\|\xi_{\mathbf{R}}\|_{\mathbf{R}} = \left(\sum_i |\xi_{\mathbf{R}}^i|^2\right)^{\frac{1}{2}}$$

if $v = \mathbf{R}$. An element $\xi = \prod_v \xi_v$ in $V(\mathbf{A})$ is said to be primitive if $\|\xi_v\|_v = 1$ for almost all v, in which case we set

$$\|\xi\| = \prod_v \|\xi_v\|_v.$$

$\|\cdot\|$ is called the height function associated to the basis $\{v_1, \cdots, v_n\}$. Suppose that

$$\Lambda : G \to GL(V)$$

is a homomorphism defined over $\mathbf{Q}$. Let K_Λ be the group of elements $k \in K$ such that $\|\Lambda(k)v\| = \|v\|$ for any primitive $v \in V(\mathbf{A})$. It is possible to choose the basis $\{v_1, \cdots, v_n\}$ such that K_Λ is of finite index in K, and also so that for each $a \in A_0$, the operator $\Lambda(a)$ is diagonal. We shall always assume that for a given Λ, the basis has been chosen to satisfy these two conditions. From our basis on $V(\mathbf{Q})$ we obtain a basis for the vector space of endomorphisms of $V(\mathbf{Q})$. Every element in $G(\mathbf{A})$ is primitive with respect to the corresponding height function and for every primitive $v \in V(\mathbf{A})$, and every $x \in G(\mathbf{A})$,

$$\|\Lambda(x)v\| \leq \|\Lambda(x)\| \cdot \|v\|.$$

If $t > 0$, define

$$G_t = \{x \in G(\mathbf{A}) : \|\Lambda(x)\| \leq t\}.$$

Suppose that Λ has the further property that G_t is compact for every t. It is known that there are constants C and N such that for any t, the volume of G_t (with respect to our Haar measure) is bounded by Ct^N. For the rest of this paper we shall simply assume that some Λ, satisfying this additional property, has been fixed, and we shall write $\|x\|$ for $\|\Lambda(x)\|$. This "norm" function on $G(\mathbf{A})$ satisfies

$$\|x\| \geq 1,$$

$$\|k_1 x k_2\| = \|x\|,$$

$$\|xy\| \leq \|x\| \cdot \|y\|,$$

and

$$\|x^{-1}\| \leq C\|x\|^N,$$

for constants C and N, elements $x, y \in G(\mathbf{A})$, and k_1, k_2 belonging to a subgroup of finite index in K.

Once $\| \;\|$ has been fixed, we shall want to consider different rational representations Λ of G. In particular, suppose that the highest weight of Λ is λ, for some element λ in $\mathfrak{a}_0^*$. Then there are constants c_1 and c_2 such that

$$c_1 e^{-\lambda(H_0(x))} \leq \|\Lambda(x)^{-1}v\| \leq c_2\, e^{-\lambda(H_0(x))},$$

for all $x \in G(\mathbf{A})$. By varying the linear functional λ, we can then show that for any Euclidean norm $\|\cdot\|$ on ${}_0$ we can choose a constant c so that

$$\|H_0(x)\| \leq c(1 + \log \|x\|), \qquad x \in G(\mathbf{A}).$$

Suppose that P is a parabolic subgroup. Recall that there is a finite number of disjoint open subsets of $\mathfrak{a}$, called the *chambers* of $\mathfrak{a}$. Their union is the complement in $\mathfrak{a}$ of the set of hyperplanes which are orthogonal to the roots of (P, A). $\mathfrak{a}^+$ is one chamber. According to Lemma 2.13 of [6(*b*)] the set of chambers is precisely the collection, indexed by all P' and $s \in \Omega(\mathfrak{a}, \mathfrak{a}')$, of open subsets $s^{-1}(\mathfrak{a}')^+$. We shall write $n(A)$ for the number of chambers. More generally, if $P_1 \subset P$, let $n_P(A_1)$ be the number of connected components in the orthogonal complement in $\mathfrak{a}_1$ of the set of hyperplanes which are orthogonal to the roots of $(P_1 \cap M, A_1)$.

§2. The kernel $K_P(x, y)$

Let R be the regular representation of $G(\mathbf{A})^1$ on $L^2(G(\mathbf{Q})\backslash G(\mathbf{A})^1)$. The map which sends f to the operator

$$R(f) = \int_{G(\mathbf{A})^1} f(x)R(x)dx$$

gives us a representation of any reasonable convolution algebra of functions f on $G(\mathbf{A})^1$. For example, we could take $C_c(G(\mathbf{A})^1)$, which is defined as the topological direct limit over all compact subsets Γ of $G(\mathbf{A})^1$, of the spaces of continuous functions on $G(\mathbf{A})^1$ supported on Γ.

We shall be more interested in the smooth functions on $G(\mathbf{A})^1$. For any place v, let $G(\mathbf{Q}_v)^1$ denote the intersection of $G(\mathbf{Q}_v)$ with $G(\mathbf{A})^1$. Then $G(\mathbf{R})^1$ contains the connected component of 1 in $G(\mathbf{A})^1$. Notice that for any v, K_v is contained in $G(\mathbf{Q}_v)^1$. Suppose that K_0 is an open compact subgroup of $G(\mathbf{A}_f)^1$. Then the double coset space $K_0\backslash G(\mathbf{A})^1/K_0$ is a discrete union of countably many copies of $G(\mathbf{R})^1$. In particular it is a differentiable manifold. Suppose that Γ is a compact subset of $G(\mathbf{A})^1$ such that

$$\Gamma = K_0\Gamma K_0.$$

Let $C^\infty(G(\mathbf{A})^1, \Gamma, K_0)$ be the algebra of smooth functions on $K_0\backslash\Gamma/K_0$ which are supported on Γ. $\mathcal{U}(\mathfrak{g}(\mathbf{R})^1 \otimes \mathbf{C})$, the universal enveloping algebra of the complexification of the Lie algebra of $G(\mathbf{R})^1$, acts on this space on both the left and the right. We shall denote these actions by the convolution symbol. The seminorms

$$\|f\|_{X,Y} = \sup_{x \in G(\mathbf{A})^1} |(X * f * Y)(x)|, \qquad f \in C^\infty(G(\mathbf{A})^1, \Gamma, K_0),$$

indexed by elements $X, Y \in \mathcal{U}(\mathfrak{g}(\mathbf{R})^1 \otimes \mathbf{C})$, define a topology on $C^\infty(G(\mathbf{A})^1, \Gamma, K_0)$. Let $C_c^\infty(G(\mathbf{A})^1)$ be the topological direct limit over all pairs (Γ, K_0), of the spaces $C^\infty(G(\mathbf{A})^1, \Gamma, K_0)$. If r is any positive integer we can define $C_c^r(G(\mathbf{A})^1)$ the same way, except that we of course take only those seminorms for which the sum of the degrees of X and Y is no greater than r. Finally, any subgroup L of $G(\mathbf{A})^1$ acts on $C_c^r(G(\mathbf{A})^1)$ by

$$f^h(x) = f(hxh^{-1}), \qquad x \in G(\mathbf{A})^1, \qquad h \in L.$$

We shall write $C_c^r(G(\mathbf{A})^1)^L$ (or for that matter, we will write X^L, if X is any set on which L acts) for the set of L-invariant elements.

Suppose that $f \in C_c^r(G(\mathbf{A})^1)$. $R(f)$ is an integral operator on $G(\mathbf{Q})\backslash G(\mathbf{A})^1$ with kernel

$$K(x, y) = \sum_{\gamma \in G(\mathbf{Q})} f(x^{-1}\gamma y).$$

If we had the Selberg trace formula for compact quotient in mind, we would be inclined to decompose the formula for $K(x, x)$ into terms corresponding to conjugacy classes in $G(\mathbf{Q})$. It turns out, however, that an equivalence relation in $G(\mathbf{Q})$, weaker than conjugacy, is more appropriate to the non-compact setting. Any $\gamma \in G(\mathbf{Q})$ can be uniquely written as $\gamma_s\gamma_u$, where γ_s is semi-simple, γ_u is unipotent, and the two elements commute. We shall say that elements γ and γ'

in $G(\mathbf{Q})$ are *equivalent* if γ_s and γ'_s are $G(\mathbf{Q})$-conjugate. Let $\mathcal{O}$ be the set of equivalence classes. Every class in $\mathcal{O}$ contains one and only one conjugacy class of semi-simple elements in $G(\mathbf{Q})$.

If γ is in $G(\mathbf{Q})$, and H is a subgroup of G, defined over $\mathbf{Q}$, let $H(\gamma)$ be the centralizer of γ in H. Both $H(\gamma)$ and its identity component, $H(\gamma)^0$, are defined over $\mathbf{Q}$. If R is any ring containing $\mathbf{Q}$, let $H(R, \gamma)$ be the centralizer of γ in $H(R)$. Now suppose that $\mathfrak{o}$ is a class in $\mathcal{O}$. It is clearly possible to choose a parabolic subgroup P and a semisimple element γ in $\mathfrak{o}$ such that γ belongs to $M(\mathbf{Q})$, but such that no $M(\mathbf{Q})$-conjugate of γ lies in $P_1(\mathbf{Q})$, for P_1 a parabolic subgroup of G, $P_1 \subsetneq P$. In other words, γ lies is no proper parabolic subgroup of M, defined over $\mathbf{Q}$. Elements in $M(\mathbf{Q})$ with this property are called elliptic elements. $M(\gamma)^0$ is a reductive group, defined over $\mathbf{Q}$, which is anisotropic modulo A. The group P is not uniquely determined by $\mathfrak{o}$. However if (M', γ') is another pair, associated to $\mathfrak{o}$ as above, then $\gamma' = w\gamma w^{-1}$ for some element $w \in G(\mathbf{Q})$. Since A' is the split component of the center of $G(\gamma')^0$, it equals wAw^{-1}. Therefore $w = w_s\eta$, for $s \in \Omega(\mathfrak{a}, \mathfrak{a}')$, and $\eta \in M(\mathbf{Q})$. It follows that there is a bijection from $\mathcal{O}$ onto the set of equivalence classes of pairs (M, c), where $P = NM$ is a parabolic subgroup and c is a conjugacy class in $M(\mathbf{Q})$ of elliptic elements, two pairs (M, c) and (M', c') being defined to be equivalent if $c' = w_s c w_s^{-1}$ for some s in $\Omega(\mathfrak{a}, \mathfrak{a}')$.

Suppose that $\mathfrak{o}$ is a class in $\mathcal{O}$ and that P and γ are as above. Let Σ denote the set of roots of (P, A). Σ determines a decomposition

$$\mathfrak{n} = \bigoplus_{\alpha \in \Sigma} \mathfrak{n}_\alpha$$

of the Lie algebra of N. Let $\Sigma(\gamma)$ be the set of roots α in Σ such that the centralizer of γ in $\mathfrak{n}_\alpha$ is not zero. The elements in $\Sigma(\gamma)$ are, of course, characters on A. Let A' be the intersection of their kernels. We can choose parabolic subgroups $P_1 \subset P_2$, and an element $s \in \Omega(\mathfrak{a}, \mathfrak{a}_1)$, such that $A_2 = w_s A' w_s^{-1}$. Set $\gamma_1 = w_s \gamma w_s^{-1}$. It is an elliptic element in $M_1(\mathbf{Q})$. A_2 is the split component of the center of $G(\gamma_1)^0$, and $P_1(\gamma_1)^0$ is a minimal parabolic subgroup of $G(\gamma_1)^0$. Notice that P_1 and P_2 are equal if and only if every element in the class $\mathfrak{o}$ is semisimple. In general, any element in $\mathfrak{o}$ is conjugate to $\gamma_1\nu_1$, where ν_1 is a unipotent element in $P_1(\gamma_1)^0$. ν_1 must lie in the unipotent radical of $P_1(\gamma_1)^0$, so in particular it belongs to $N_1(\mathbf{Q})$.

LEMMA 2.1. *Suppose that $P = NM$ is a parabolic subgroup defined over $\mathbf{Q}$. Suppose that μ is in $M(\mathbf{Q})$. Then if $\phi \in C_c(N(\mathbf{A}))$,*

$$\sum_{\delta \in N(\mathbf{Q},\gamma_s)\backslash N(\mathbf{Q})} \ \sum_{\eta \in N(\mathbf{Q},\gamma_s)} \phi(\gamma^{-1}\delta^{-1}\gamma\eta\delta) = \sum_{\eta \in N(\mathbf{Q})} \phi(\eta).$$

Proof. Neither side of the putative formula changes if μ is replaced by an $M(\mathbf{Q})$-conjugate of itself. After noting that the previous discussion can be applied to classes in $M(\mathbf{Q})$ as well as $G(\mathbf{Q})$, we can assume that there is a parabolic subgroup P_1, with $P_1 \subset P$, such that $\gamma_s \in M_1(\mathbf{Q})$, and $\gamma_u \in M(\mathbf{Q}, \gamma_s) \cap N_1(\mathbf{Q})$. Now the Lie algebra of N can be decomposed into eigenspaces under the action of A_1. It follows that there exists a sequence

$$N = N_0 \supset N_1 \supset \cdots \supset N_r = \{1\}$$

of normal γ_s-stable subgroups of N which are defined over $\mathbf{Q}$ and satisfy the following properties:

(i) $N_{k+1} \backslash N_k$ is abelian for each k.

(ii) If $\delta \in N_k$, and η either belongs to N or equals γ_u, $\eta^{-1}\delta^{-1}\eta\delta$ belongs to N_{k+1}.

We shall show that for all k, $0 \leq k \leq r$,

$$\sum_{\delta \in N(\mathbf{Q},\gamma_s)N_k(\mathbf{Q})\backslash N(\mathbf{Q})} \quad \sum_{\eta \in N(\mathbf{Q},\gamma_s)N_k(\mathbf{Q})} \phi(\gamma^{-1}\delta^{-1}\gamma\eta\delta) \tag{2.1}$$

equals

$$\sum_{\delta \in N(\mathbf{Q},\gamma_s)\backslash N(\mathbf{Q})} \quad \sum_{\eta \in N(\mathbf{Q},\gamma_s)} \phi(\gamma^{-1}\delta^{-1}\gamma\eta\delta). \tag{2.2}$$

The assertion of the lemma is the case that $k = 0$. The equality is immediate if $k = r$. By decreasing induction on k, we assume that (2.2) equals

$$\sum_{\delta \in N(\mathbf{Q},\gamma_s)N_{k+1}(\mathbf{Q})\backslash N(\mathbf{Q})} \quad \sum_{\eta \in N(\mathbf{Q},\gamma_s)N_k(\mathbf{Q})} \phi(\gamma^{-1}\delta^{-1}\gamma\eta\delta).$$

This is the sum over $\delta_1 \in N(\mathbf{Q},\gamma_s)N_k(\mathbf{Q})\backslash N(\mathbf{Q})$ of

$$\sum_{\delta_2 \in N(\mathbf{Q},\gamma_s)N_{k+1}(\mathbf{Q})\backslash N(\mathbf{Q},\gamma_s)N_k(\mathbf{Q})} \quad \sum_{\eta \in N(\mathbf{Q},\gamma_s)N_{k+1}(\mathbf{Q})} \phi(\gamma^{-1}\delta_1^{-1}\delta_2^{-1}\gamma\eta\delta_2\delta_1)$$

$$= \sum_{\delta_2 \in N_k(\mathbf{Q},\gamma_s)N_{k+1}(\mathbf{Q})\backslash N_k(\mathbf{Q})} \quad \sum_{\eta} \phi(\gamma^{-1}\delta_1^{-1}\delta_2^{-1}\gamma\eta\delta_2\delta_1).$$

For fixed $\delta_2 \in N_k(\mathbf{Q})$, we change variables in the sum over η. We find that

$$\begin{aligned}\sum_{\eta \in N(\mathbf{Q},\gamma_s)N_{k+1}(\mathbf{Q})} \phi(\gamma^{-1}\delta_1^{-1}\delta_2^{-1}\gamma\eta\delta_2\delta_1) &= \sum_{\eta} \phi(\gamma^{-1}\delta_1^{-1}\gamma \cdot \gamma^{-1}\delta_2^{-1}\gamma\eta\delta_2 \cdot \delta_1) \\ &= \sum_{\eta} \phi(\gamma^{-1}\delta_1^{-1}\gamma \cdot \gamma_s^{-1}\delta_2^{-1}\gamma_s\eta\delta_2 \cdot \delta_1) \\ &= \sum_{\eta} \psi(\gamma_s^{-1}\delta_2^{-1}\gamma_s\delta_2),\end{aligned}$$

where

$$\psi(x) = \sum_{\eta \in N(\mathbf{Q},\gamma_s)N_{k+1}(\mathbf{Q})} \phi(\gamma^{-1}\delta_1^{-1}\gamma\eta \cdot x \cdot \delta_1)$$

is a compactly supported function on the discrete set

$$N_k(\mathbf{Q}, \gamma_s)\, N_{k+1}(\mathbf{Q}) \backslash N_k(\mathbf{Q}).$$

The map

$$y \to N_k(\gamma_s)N_{k+1} \cdot \gamma_s^{-1}y^{-1}\gamma_s y, \qquad y \in N_k(\gamma_s)N_{k+1}\backslash N_k,$$

is an isomorphism from $N_k(\gamma_s)N_{k+1}\backslash N_k$ onto itself which is defined over $\mathbf{Q}$. Therefore

$$\sum_{\delta_2 \in N_k(\mathbf{Q},\gamma_s)N_{k+1}(\mathbf{Q})\backslash N_k(\mathbf{Q})} \psi(\gamma_s^{-1}\delta_2^{-1}\gamma_s\delta_2)$$

equals

$$\sum_{\eta \in N(\mathbf{Q},\gamma_s)N_k(\mathbf{Q})} \phi(\gamma^{-1}\delta_1^{-1}\gamma\eta\delta_1).$$

It follows that (2.2) equals (2.1). □

It follows from the lemma that if $\mathfrak{o} \in \mathcal{O}$, and if $\gamma \in \mathfrak{o} \cap M(\mathbf{Q})$, then $\gamma\eta$ belongs to $\mathfrak{o}$ for each η in $N(\mathbf{Q})$. In other words

$$\mathfrak{o} \cap P(\mathbf{Q}) = (\mathfrak{o} \cap M(\mathbf{Q})) \cdot N(\mathbf{Q}).$$

A similar remark holds for the intersection of $\mathfrak{o}$ with any parabolic subgroup of M.

LEMMA 2.2 *Under the hypotheses of Lemma* 2.1,

$$\int_{N(\mathbf{A},\gamma_s)\backslash N(\mathbf{A})} \int_{N(\mathbf{A},\gamma_s)} \phi(\gamma^{-1}n_1^{-1}\gamma n_2 n_1)dn_2 dn_1 = \int_{N(\mathbf{A})} \phi(n)dn.$$

Proof. The proof can be transcribed from the proof of Lemma 2.1 by replacing each sum over a set of rational points by the integral over the corresponding set of $\mathbf{A}$-valued points. □

If $\mathfrak{o} \in \mathcal{O}$, define

$$K_{\mathfrak{o}}(x, y) = \sum_{\gamma \in \mathfrak{o}} f(x^{-1}\gamma y).$$

Then

$$K(x, y) = \sum_{\mathfrak{o}} K_{\mathfrak{o}}(x, y).$$

More generally, if P is a parabolic subgroup, define

$$K_{P,\mathfrak{o}}(x, y) = \sum_{\gamma \in M(\mathbf{Q}) \cap \mathfrak{o}} \int_{N(\mathbf{A})} f(x^{-1}\gamma n y)dn.$$

Then

$$K_P(x, y) = \sum_{\mathfrak{o} \in \mathcal{O}} K_{P,\mathfrak{o}}(x, y)$$

equals

$$\sum_{\gamma \in M(\mathbf{Q})} \int_{N(\mathbf{A})} f(x^{-1}\gamma n y) dn.$$

This is just the kernel of $R_P(f)$, where R_P is the regular representation of $G(\mathbf{A})^1$ on $L^2(N(\mathbf{A})M(\mathbf{Q})\backslash G(\mathbf{A})^1)$.

§3. A review of Eisenstein series

In this section we shall recall those results on Eisenstein series which are needed for the trace formula. They are due to Langlands; the main ideas are in the article [6(*a*)] while the details appear in [6(*b*)]. Suppose that P is a parabolic subgroup. Recall that the space of cusp forms, $L^2_{\text{cusp}}(M(\mathbf{Q})\backslash M(\mathbf{A})^1)$, on $M(\mathbf{A})^1$ is the space of functions ϕ in $L^2(M(\mathbf{Q})\backslash M(\mathbf{A})^1)$ such that for any parabolic subgroup P_1, with $P_1 \subsetneq P$,

$$\int_{N_1(\mathbf{Q}) \cap M(\mathbf{Q}) \backslash N_1(\mathbf{A}) \cap M(\mathbf{A})} \phi(nm) dn$$

equals 0 for almost all $m \in M(\mathbf{A})^1$. It is known that

$$L^2_{\text{cusp}}(M(\mathbf{Q})\backslash M(\mathbf{A})^1) = \bigoplus_{\rho} V_\rho,$$

where ρ ranges over all irreducible unitary representations of $M(\mathbf{A})^1$, and each V_ρ is an $M(\mathbf{A})^1$-invariant subspace of $L^2_{\text{cusp}}(M(\mathbf{Q})\backslash M(\mathbf{A})^1)$, isomorphic under the action of $M(\mathbf{A})^1$ to a finite number of copies of ρ. An irreducible unitary representation ρ of $M(\mathbf{A})^1$ is said to be *cuspidal* if $V_\rho \neq 0$. Suppose that P' is another parabolic subgroup, and that ρ' is an irreducible unitary cuspidal representation of $M'(\mathbf{A})^1$. We shall say that the pairs (M, ρ) and (M', ρ') are *equivalent* if there is an $s \in \Omega(\mathfrak{a}, \mathfrak{a}')$ such that the representation

$$(s\rho)(m') = \rho(w_s^{-1} m' w_s), \qquad m' \in M'(\mathbf{A})^1,$$

is unitarily equivalent to ρ'. Let $\mathscr{X}$ be the set of equivalence classes of pairs. Then corresponding to any $\chi \in \mathscr{X}$ we have a class, $\mathscr{P}_\chi$, of associated parabolic subgroups. If P is any parabolic subgroup and $\chi \in \mathscr{X}$, set

$$L^2_{\text{cusp}}(M(\mathbf{Q})\backslash M(\mathbf{A})^1)_\chi = \bigoplus_{\{\rho:(M,\rho) \in \chi\}} V_\rho.$$

This is a closed $M(\mathbf{A})^1$-invariant subspace of $L^2_{\text{cusp}}(M(\mathbf{Q})\backslash M(\mathbf{A}))^1)$, which is empty if P does not belong to $\mathscr{P}_\chi$. We have

$$L^2_{\text{cusp}}(M(\mathbf{Q})\backslash M(\mathbf{A})^1) = \bigoplus_{\chi \in \mathscr{X}} L^2_{\text{cusp}}(M(\mathbf{Q})\backslash M(\mathbf{A})^1)_\chi.$$

Again suppose that P is fixed and that $\chi \in \mathscr{X}$. Suppose first of all that there is a group P_1 in $\mathscr{P}_\chi$ which is contained in P. Let ψ be a smooth function on $N_1(\mathbf{A})M_1(\mathbf{Q})\backslash G(\mathbf{A})$ such that

$$\Psi_a(m, k) = \psi(amk),\ k \in K,\ m \in M_1(\mathbf{Q})\backslash M_1(\mathbf{A})^1,\ a \in A_1(\mathbf{Q})\backslash A_1(\mathbf{A}),$$

vanishes for a outside a compact subset of $A_1(\mathbf{Q})\backslash A_1(\mathbf{A})$, transforms under $K_{\mathbf{R}}$ according to an irreducible representation W, and as a function of m, belongs to $L^2_{\text{cusp}}(M_1(\mathbf{Q})\backslash M_1(\mathbf{A})^1)$. One of the first results in the theory of Eisenstein series is that the function

$$\hat{\psi}^M(m) = \sum_{\delta \in P_1(\mathbf{Q}) \cap M(\mathbf{Q})\backslash M(\mathbf{Q})} \psi(\delta m), \qquad m \in M(\mathbf{Q})\backslash M(\mathbf{A})^1,$$

is square integrable on $M(\mathbf{Q})\backslash M(\mathbf{A})^1$, ([6($a$), Lemma 1]). We define $L^2(M(\mathbf{Q})\backslash M(\mathbf{A})^1)_\chi$ to be the closed span of all functions of the form $\hat{\psi}^M$, where P_1 runs through those groups in $\mathscr{P}_\chi$ which are contained in P, and W is allowed to vary over all irreducible representations of $K_{\mathbf{R}}$. If there does not exist a group $P_1 \in \mathscr{P}_\chi$ which is contained in P, define $L^2(M(\mathbf{Q})\backslash M(\mathbf{A})^1)_\chi$ to be $\{0\}$. It follows from [6(a), Lemma 2], that $L^2(M(\mathbf{Q})\backslash M(\mathbf{A})^1)$ is the orthogonal direct sum over all $\chi \in \mathscr{X}$ of the spaces $L^2(M(\mathbf{Q})\backslash M(\mathbf{A})^1)_\chi$.

For any P, let $\Pi(M)$ denote the set of equivalence classes of irreducible unitary representations of $M(\mathbf{A})$. If $\zeta \in \mathfrak{a}^*_{\mathbf{C}}$ and $\pi \in \Pi(M)$, let π_ζ be the product of π with the quasicharacter

$$x \to e^{\zeta(H_P(x))}, \qquad x \in G(\mathbf{A}).$$

If ζ belongs to $i\mathfrak{a}^*$, π_ζ is unitary, so we obtain a free action of the group $i\mathfrak{a}^*$ on $\Pi(M)$. $\Pi(M)$ becomes a differentiable manifold whose connected components are the orbits of $i\mathfrak{a}^*$. We can also transfer our Haar measure on $i\mathfrak{a}^*$ to each of the orbits in $\Pi(M)$; this allows us to define a measure $d\pi$ on $\Pi(M)$. If $P_2 \supset P$, let $\Pi^{P_2}(M)$ be the space of orbits of $i\mathfrak{a}^*_2$ on $\Pi(M)$. $\Pi^{P_2}(M)$ inherits a measure from our measures on $\Pi(M)$ and $i\mathfrak{a}^*_2$.

For $\pi \in \Pi(M)$, let $\mathscr{H}^0_P$ be the space of smooth functions

$$\phi : N(\mathbf{A})M(\mathbf{Q})\backslash G(\mathbf{A}) \to \mathbf{C}$$

which satisfy the following conditions:

(i) ϕ is right K-finite.

(ii) For every $x \in G(\mathbf{A})$, the function

$$m \to \phi(mx), \qquad m \in M(\mathbf{A}),$$

is a matrix coefficient of π.

(iii) $$\|\phi\|^2 = \int_K \int_{M(\mathbf{Q})\backslash M(\mathbf{A})^1} |\phi(mk)|^2 dm dk < \infty.$$

Let $\mathscr{H}_P(\pi)$ be the completion of $\mathscr{H}^0_P(\pi)$. It is a Hilbert space. If $\phi \in \mathscr{H}_P(\pi)$, and $\zeta \in \mathfrak{a}^*_{\mathbf{C}}$, define

$$\phi_\zeta(x) = \phi(x)e^{\zeta(H_P(x))}, \qquad x \in G(\mathbf{A}),$$

and

$$(I_P(\pi_\zeta, y)\phi_\zeta)(x) = \phi_\zeta(xy) \cdot \delta_P(xy)^{\frac{1}{2}} \cdot \delta_P(x)^{-\frac{1}{2}}, \qquad x, y \in G(\mathbf{A}).$$

Then $I_P(\pi_\zeta)$ is a representation of $G(\mathbf{A})$ which is unitary if $\zeta \in i\mathfrak{a}^*$. Notice that $\mathscr{H}_P(\pi) = \{0\}$ unless there is a subrepresentation of the regular representation of $M(\mathbf{A})^1$ on $L^2(M(\mathbf{Q})\backslash M(\mathbf{A})^1)$ which is equivalent to the restriction of π to $M(\mathbf{A})^1$.

Given $\chi \in \mathscr{X}$, let $\mathscr{H}_P(\pi)_\chi$ be the closed subspace of $\mathscr{H}_P(\pi)$ consisting of those ϕ such that for all x the function

$$m \to \phi(mx), \qquad m \in M(\mathbf{Q})\backslash M(\mathbf{A})^1,$$

belongs to $L^2(M(\mathbf{Q})\backslash M(\mathbf{A})^1)_\chi$. Then

$$\mathscr{H}_P(\pi) = \bigoplus_{\chi \in \mathscr{X}} \mathscr{H}_P(\pi)_\chi.$$

(I do not know whether $\mathscr{H}_P(\pi)_\chi$ can be nonzero for more than one χ.) Suppose that K_0 is an open compact subgroup of $G(\mathbf{A}_f)$ and that W is an equivalence class of irreducible representations of $K_{\mathbf{R}}$. Let $\mathscr{H}_P(\pi)_{\chi,K_0}$ be the subspace of functions in $\mathscr{H}_P(\pi)_\chi$ which are invariant under $K_0 \cap K$, and let $\mathscr{H}_P(\pi)_{\chi,K_0,W}$ be the space of functions in $\mathscr{H}_P(\pi)_{\chi,K_0}$ which transform under $K_{\mathbf{R}}$ according to W. It is a consequence of the decomposition of the spaces $L^2(M(\mathbf{Q})\backslash M(\mathbf{A})^1)_\chi$, established by Langlands in §7 of [6(*b*)], that each of the spaces $\mathscr{H}_P(\pi)_{\chi,K_0,W}$ is finite dimensional. We shall need to have ortho-normal bases of the spaces $\mathscr{H}_P(\pi)_\chi$. Let us fix such a basis, $\mathscr{B}_P(\pi)_\chi$, for each π and χ such that

$$\mathscr{B}_P(\pi_\zeta)_\chi = \{\phi_\zeta : \phi \in \mathscr{B}_P(\pi)_\chi\}, \qquad \zeta \in i\mathfrak{a}^*,$$

and such that every $\phi \in \mathscr{B}_P(\pi)_\chi$ belongs to one of the spaces $\mathscr{H}_P(\pi)_{\chi,K_0,W}$. We shall need these bases in the next section, when we give a second formula for the kernal $K(x, y)$ in terms of Eisenstein series.

Suppose that $\pi \in \Pi(M)$, $\phi \in \mathscr{H}_P^0(\pi)$, $\zeta \in \mathfrak{a}^*_{P,\mathbf{C}}$ and $s \in \Omega(\mathfrak{a}, \mathfrak{a}')$. If $Re\,\zeta$ belongs to $\rho_P + (\mathfrak{a}_P^*)^+$, the Eisenstein series and global intertwining operators are defined by

$$E(x, \phi_\zeta) = \sum_{\delta \in P(\mathbf{Q})\backslash G(\mathbf{Q})} \phi_\zeta(\delta x) \cdot \delta_P(\delta x)^{\frac{1}{2}},$$

and

$$(M(s, \pi_\zeta)\phi_\zeta)(x) = \int_{N'(\mathbf{A}) \cap w_sN(\mathbf{A})w_s^{-1}\backslash N'(A)} \phi_\zeta(w_s^{-1}nx) \cdot \delta_P(w_s^{-1}nx)^{\frac{1}{2}} \cdot \delta_P(x)^{-\frac{1}{2}}\, dn.$$

The properties that we will need are all contained, at least implicitly, in [6(*b*)] (see especially Appendix II), and have been summarized in [1(*c*)]. For the convenience of the reader, we shall recall them again here.

$E(x, \phi_\zeta)$ and $M(s, \pi_\zeta)\phi_\zeta$ can be continued as meromorphic functions in ζ to $\mathfrak{a}^*_{\mathbf{C}}$. If $\zeta \in i\mathfrak{a}^*$, $E(x, \phi_\zeta)$ is a smooth function of x, and $M(s, \pi_\zeta)$ is a unitary operator from $\mathcal{H}_P(\pi_\zeta)$ to $\mathcal{H}_{P'}(s\pi_\zeta)$. There is an integer N such that for any ϕ,

$$\sup_{x \in G(\mathbf{A})} (\|x\|^{-N}|E(x, \phi_\zeta)|)$$

is a locally bounded function on the set of $\zeta \in \mathfrak{a}^*_{\mathbf{C}}$ at which $E(x, \phi_\zeta)$ is regular. If $h \in C_c(G(\mathbf{A}))^K$ and $t \in \Omega(\mathfrak{a}', \mathfrak{a}'')$, the following functional equations hold:

(i) $E(x, I_P(\pi, h)\phi) = \int_{G(\mathbf{A})} h(y)\, E(xy, \phi)dy.$

(ii) $E(x, M(s, \pi)\phi) = E(x, \phi).$

(iii) $M(ts, \pi) = M(t, s\pi)\, M(s, \pi).$

Let $\mathcal{P}$ be a class of associated parabolic subgroups. Let $\hat{L}_{\mathcal{P}}$ be the set of collections $F = \{F_P : P \in \mathcal{P}\}$ of functions

$$F_P : \pi \to \mathcal{H}_P(\pi), \qquad \pi \in \Pi(M),$$

such that

(i) $F_{P'}(s\pi) = M(s, \pi)F_P(\pi), \qquad s \in \Omega(\mathfrak{a}, \mathfrak{a}'),$

(ii) $\|F\|^2 = \sum_{P \in \mathcal{P}} n(A)^{-1} \int_{\Pi(M)} \|F_P(\pi)\|^2 d\pi < \infty.$

Then the map which sends F to the function

$$\sum_{P \in \mathcal{P}} n(A)^{-1} \int_{\Pi(M)} E(x, F_P(\pi))d\pi,$$

defined for F in a dense subspace of $\hat{L}_{\mathcal{P}}$, extends to a unitary map from $\hat{L}_{\mathrm{p}}$ onto a closed, $G(\mathbf{A})$-invariant subspace $L^2_{\mathcal{P}}(G(\mathbf{Q})\backslash G(\mathbf{A}))$ of $L^2(G(\mathbf{Q})\backslash G(\mathbf{A}))$. Moreover, there is an orthogonal decomposition

$$L^2(G(\mathbf{Q})\backslash G(\mathbf{A})) = \bigoplus_{\mathcal{P}} \hat{L}_{\mathcal{P}}(G(\mathbf{Q})\backslash G(\mathbf{A})).$$

We could equally well have defined subspaces $L^2_{\mathcal{P}}(G(\mathbf{Q})\backslash G(\mathbf{A})^1)$ of $L^2(G(\mathbf{Q})\backslash G(\mathbf{A})^1)$. The only change in the formulation would be to integrate over $\Pi^G(M)$ instead of $\Pi(M)$. This would allow us to decompose the representation R into a direct integral of the representations $I_P(\pi)$ (Let us agree to denote the restriction of the representation $I_P(\pi)$ to $G(\mathbf{A})^1$ by $I_P(\pi)$ as well.) If $\chi \in \mathcal{X}$, we could replace $\hat{L}_{\mathcal{P}}$ by a space of collections $F = \{F_P : P \in \mathcal{P}\}$ such that $F_P(\pi)$ belongs to $\mathcal{H}_P(\pi)_\chi$ for each π. We would obtain a decomposition of the space $L^2(G(\mathbf{Q})\backslash G(\mathbf{A})^1)_\chi$. More generally, if $P \subset P_2$, and if ζ is a suitable point in $\mathfrak{a}^*_{\mathbf{C}}$, define

$$E_{P_2}(x, \phi_\zeta) = \sum_{\delta \in P(\mathbf{Q})\backslash P_2(\mathbf{Q})} \phi_\zeta(\delta x) \cdot \delta_P(\delta x)^{\frac{1}{2}}.$$

The discussion above holds if the functions $E(x)$ are replaced by $E_{P_2}(x)$. It then amounts to a description of the decomposition of the representation R_{P_2} into a direct integral of $\{I_P(\pi) : P \subset P_2\}$.

We shall end this section with some simple remarks on representations of the universal enveloping algebra. $\mathcal{U}(\mathfrak{g}(\mathbf{R})^1 \otimes \mathbf{C})$ acts, through the representation $I_P(\pi)$, on the vector space $\mathcal{H}_P^0(\pi)$. There are two involutions of $\mathcal{U}(\mathfrak{g}(\mathbf{R})^1 \otimes \mathbf{C})$. The first,

$$X \rightarrow \bar{X},$$

is just conjugation with respect to the real form $\mathfrak{g}(\mathbf{R})^1$. The second, (actually an anti-involution),

$$X \rightarrow X^*,$$

is the adjoint map. For any $\pi \in \Pi(M)$, X, $Y \in \mathcal{U}(\mathfrak{g}(\mathbf{R})^1 \otimes \mathbf{C})$, and $h \in C_c^\infty(G(\mathbf{A})^1)$,

$$I_P(\pi, X)\, I_P(\pi, h)\, I_P(\pi, Y) = I_P(\pi, X * h * Y),$$

and

$$I_P(\pi, X)^* = I_P(\pi, X^*).$$

(We also have

$$I_P(\pi, h)^* = I_P(\pi, h^*),$$

where $h^*(x) = \overline{h(x^{-1})}$.) We shall denote the left invariant differential operator associated to X by $R(X)$. If we wish to emphasize the fact that we are differentiating with respect to a variable x, we shall write $R_x(X)$. Then

$$R_x(X)\, E(x, \phi) = E(x, I_P(\pi, X)\phi), \qquad \phi \in \mathcal{H}_P^0(\pi).$$

It follows that for every $\phi \in \mathcal{H}_P^0(\pi)$ and $X \in \mathcal{U}(\mathfrak{g}(\mathbf{R})^1 \otimes \mathbf{C})$ that there a locally bounded function $c(\zeta)$, defined on the set of points $\zeta \in \mathfrak{a}_{\mathbf{C}}^*$ at which $E(x, \phi_\zeta)$ is regular, such that

$$|R_x(X)\, E(x, \phi_\zeta) \leq c(\zeta)\, \|x\|^N, \qquad x \in G(\mathbf{A})^1. \tag{3.1}$$

§4. The second formula for the kernel

The theory of Eisenstein series yields another formula for $K(x, y)$. The equality of this formula with that of §2 is what will eventually lead to our trace formula. The following result is essentially due to Duflo and Labesse [2].

LEMMA 4.1. *For any $m \geq 1$ we can choose elements $g_{\mathbf{R}}^1 \in C_c^m(G(\mathbf{A})^1)^{K_{\mathbf{R}}}$, $g_{\mathbf{R}}^2 \in C_c^\infty(G(\mathbf{R})^{K_{\mathbf{R}}}$ and $Z \in \mathcal{U}\mathfrak{g}(\mathbf{R})^1 \otimes \mathbf{C}^{K_{\mathbf{R}}}$ such that $Z * g_{\mathbf{R}}^1 + g_{\mathbf{R}}^2$ is the Dirac distribution at the identity in $G(\mathbf{R})^1$.*

Proof. Let Δ be any elliptic element in $\mathcal{U}(\mathfrak{g}(\mathbf{R})^1 \otimes \mathbf{C})^{K_{\mathbf{R}}}$. For example, Δ could be obtained as a linear combination of the Casimir operators of $G(\mathbf{R})^1$ and

$K_{\mathbf{R}}$. We can assume that Δ is $K_{\mathbf{R}}$-invariant. Let Ω be a small open $K_{\mathbf{R}}$-invariant neighborhood of 1 in $G(\mathbf{R})^1$. By the existence of a fundamental solution [4, Pg. 174] we can find, for any $n > 0$, a function h on Ω such that $\Delta^n * h$ is the Dirac distribution. By the elliptic regularity theorem, we can choose n so large that h belongs to $C^m(\Omega)$. Since Δ is $K_{\mathbf{R}}$-invariant, we can assume that $h \in C^m(\Omega)^{K_{\mathbf{R}}}$. Moreover, h is infinitely differentiable away from the identity. Let ϕ be any function in $C_c^\infty(\Omega)^{K_{\mathbf{R}}}$ which equals 1 in a neighborhood of the identity. Then $\Delta * h\phi$ equals the Dirac distribution in a neighborhood of the identity, and is smooth away from the identity. Thus $g_{\mathbf{R}}^1 = h\phi$ is in $C_c^m(\Omega)^{K_{\mathbf{R}}}$, and $g_{\mathbf{R}}^2$, the difference between $\Delta^n * g_{\mathbf{R}}^1$ and the Dirac distribution, is actually in $C_c^\infty(\Omega)^{K_{\mathbf{R}}}$. The lemma is established with $Z = \Delta^n$. □

COROLLARY 4.2. *Suppose $r_0 > \deg Z$. Then any $h \in C_c^{r_0}(G(\mathbf{A})^1)$ equals*

$$\sum_{i=1}^{2} h_i * g_i,$$

where, in the notation of the lemma, $h_1 = h * Z$, $h_2 = h$, and g_i is the product of $g_{\mathbf{R}}^i$ with a multiple of the characteristic function of an open compact subgroup of $G(\mathbf{A}_f)$.

Proof. Let K_0 be any open compact subgroup of $G(\mathbf{A}_f)$ under which h is bi-invariant. Define

$$g_i(x_{\mathbf{R}} \cdot x_f), \quad x_{\mathbf{R}} \in G(\mathbf{R})^1, x_f \in G(\mathbf{A}_f)^1, i = 1, 2,$$

to be $g_{\mathbf{R}}^i(x_{\mathbf{R}})$ divided by the volume of K_0 if $x_f \in K_0$, and 0 otherwise. The corollary then follows from the lemma. □

LEMMA 4.3. *There is a constant N and a continuous semi-norm $\|\ \|_0$ on $C_c(G(\mathbf{A})^1)$ such that for any $f \in C_c(G(\mathbf{A})^1)$ and $x, y \in G(\mathbf{A})^1$,*

$$\left| \sum_{\gamma \in G(\mathbf{Q})} f(x^{-1}\gamma y) \right| \leq \|f\|_0 \|x\|^{2N}.$$

This lemma is well known, at least in the $\Gamma \backslash G(\mathbf{R})^1$ setting ([3, Lemma 9]). The extension to adèle groups is easy. □

Recall that a function f on $G(\mathbf{A})^1$ is said to be K-finite if the space spanned by the left and right K-translates of f is finite dimensional.

LEMMA 4.4. *There is an r_0 and a continuous seminorm $\|\cdot\|_{r_0}$ on $C_c^{r_0}(G(\mathbf{A})^1)$ such that if $X, Y \in \mathcal{U}(\mathfrak{g}(\mathbf{R})^1 \otimes \mathbf{C})$, $r \geq r_0 + \deg X + \deg Y$, and f is a K-finite function in $C_c^r(G(\mathbf{A})^1)$,*

$$(4.1) \quad \sum_{\chi} \sum_{P} n(A)^{-1} \int_{\Pi^G(M)} \left| \sum_{\phi \in \mathcal{B}_P(\pi)_\chi} R(X)E(x, I_P(\pi, f)\phi) \cdot \overline{R(Y^*)E(y, \phi)} \right| d\pi$$

is bounded by

$$\|X * f * Y\|_{r_0} \cdot \|x\|^N \cdot \|y\|^N.$$

Here N is as in Lemma 4.3.

Proof. Since f is K-finite, the sum over ϕ is finite. Let $E(x, \pi) = E(x, \pi)_\chi$ be the vector in the algebraic direct product

$$\prod_{\phi \in \mathscr{B}_P(\pi)_\chi} \mathbf{C}\phi$$

such that for any K-finite vector ψ in $\mathscr{H}_P(\pi)_\chi$,

$$(\psi, E(x, \pi)) = E(x, \psi).$$

Then

$$\sum_{\phi \in \mathscr{B}_P(\pi)_\chi} R(X)E(x, I_P(\pi, f)\phi) \cdot \overline{R(Y^*)E(y, \phi)} \tag{4.2}$$

equals

$$\begin{aligned}
&\sum_\phi E(x, I_P(\pi, X)I_P(\pi, f)\phi)\, \overline{E(y, I_P(\pi, Y^*)\phi)} \\
&= \sum_\phi (\phi, (I_P(\pi, X)I_P(\pi, f))^*E(x, \pi))\, (I_P(\pi, Y^*)^*E(y, \pi), \phi) \\
&= (I_P(\pi, X)\, I_P(\pi, f)\, I_P(\pi, Y)\, E(y, \pi), E(x, \pi)) \\
&= (I_P(\pi, X * f * Y)\, E(y, \pi), E(x, \pi)).
\end{aligned}$$

Apply Corollary 4.2 with $h = X * f * Y$. (The integer m, upon which Z depends, can be arbitrary.) Then $h = \sum_{i=1}^{2} h_i * g_i$. We can choose g_i to be in $C_c^m(G(\mathbf{A})^1)^K$. Since each function h_i is K-finite, we can assume that g_i is K-finite as well. The absolute value of (4.2) equals

$$\left| \sum_i (I_P(\pi, g_i)E(y, \pi), I_P(\pi, h_i)^*E(x, \pi)) \right|,$$

which is bounded by

$$\begin{aligned}
&\sum_i \|I_P(\pi, g_i)E(y, \pi)\| \cdot \|I_P(\pi, h_i)^*E(x, \pi)\| \\
&= \sum_i (I_P(\pi, g_i^* * g_i)E(y, \pi), E(y, \pi))^{\frac{1}{2}} \cdot (I_P(h_i * h_i^*)E(x, \pi), E(x, \pi))^{\frac{1}{2}}.
\end{aligned}$$

Consider the set

$$\mathscr{S} = \{(\chi, P, \pi) : x \in \mathscr{X}, \pi \in \Pi^G(M)\}.$$

Regarded as a disjoint union of copies of $\Pi^G(M)$, it comes equipped with a topology. The integral (4.1) defines a measure on $\mathscr{S}$. Suppose that S is a compact subset of $\mathcal{S}$ and that k is a K-finite function in $C_c(G(\mathbf{A})^1)$. It follows from the results on Eisenstein series summarized in §3 that

$$\int_{(\chi,P,\pi)\in S} (I_P(\pi, k * k^*) E(x, \pi)_\chi, E(y, \pi)_\chi)$$

is the kernel of the restriction of the operator $R(k * k^*)$ to an invariant subspace. Since the operator is positive semidefinite, the value at $x = y$ of this expression is bounded by

$$\sum_{\gamma \in G(\mathbf{Q})} (k * k^*)(x^{-1}\gamma x).$$

By Lemma 4.3 this is in turn bounded by

$$\|k * k^*\|_0 \cdot \|x\|^{2N}.$$

Since this last expression is independent of S it remains a bound if the original integral is taken over all of $\mathcal{S}$. It follows from Schwartz' inequality that (4.1) is bounded by

$$\|x\|^N \|y\|^N \cdot \sum_i \|g_i^* * g_i\|_0^{\frac{1}{2}} \cdot \|h_i * h_i^*\|_0^{\frac{1}{2}}.$$

Since $h_1 = h * Z$, $h_2 = h$ and $r_0 > \deg Z$, the map which sends h to

$$\|h\|_{r_0} = \sum_i \|g_i^* * g_i\|_0^{\frac{1}{2}} \cdot \|h_i * h_i^*\|_0^{\frac{1}{2}}$$

is a continuous seminorm on $C_c^{r_0}(G(\mathbf{A})^1)$. The lemma follows. □

The K-finite functions are not dense in $C_c^r(G(\mathbf{A})^1)$. However, there is a positive integer l_0, depending only on G, such that for any $r > l_0$, $C_c^r(G(\mathbf{A})^1)$ is contained in the closure in $C_c^{r-l_0}(G(\mathbf{A})^1)$ of the K-finite functions in $C_c^{r-l_0}(G(\mathbf{A})^1)$. If $\omega = (W_1, W_2)$ is a pair of equivalence classes of irreducible representations of $K_{\mathbf{R}}$, define

$$f_\omega(x) = \deg W_1 \cdot \deg W_2 \int_{K_{\mathbf{R}} \times K_{\mathbf{R}}} ch_{W_1}(k_1) f(k_1^{-1} x k_2^{-1}) ch_{W_2}(k_2) dk_1 dk_2,$$

if ch_{W_i} is the character of W_i. It follows easily from the representation theory of a compact Lie group that l_0 may be chosen so that if $r > l_0$ and $\|\cdot\|$ is any continuous seminorm on $C_c^{r-l_0}(G(\mathbf{A})^1)$,

$$f \to \sum_\omega \|f_\omega\|, \qquad f \in C_c^r(G(\mathbf{A})^1),$$

is a continuous seminorm on $C_c^r(G(\mathbf{A})^1)$, and $\sum_\omega f_\omega$ converges absolutely in $C_c^{r-l_0}(G(\mathbf{A})^1)$ to f.

Suppose that the measure space $\mathcal{S}$ is defined as in the proof of the last lemma. If S is a measurable subset of $\mathcal{S}$ and f is a K-finite function in $C_c^{r_0}(G(\mathbf{A})^1)$, define $I(S, f, x, y)$ to be

$$\int_{(\chi,P,\pi)\in S} \sum_{\phi\in\mathcal{B}_P(\pi)_\chi} E(x, I_P(\pi, f)\phi)\,\overline{E(y,\phi)}d\pi.$$

For fixed x and y this defines a continuous linear functional on a subspace of $C_c^{r_0}(G(\mathbf{A})^1)$, the closure of which contains $C_c^{r_0+l_0}(G(\mathbf{A})^1)$. We can therefore define $I(S, f, x, y)$ for any $f \in C_c^{r_0+l_0}(G(\mathbf{A})^1)$.

LEMMA 4.5. *If $f \in C_c^r(G(\mathbf{A})^1)$, $I(S, f, x, y)$ is continuously differentiable in either x or y of order $r - r_0 - l_0$. If $X, Y \in \mathcal{U}(\mathfrak{g}(\mathbf{R})^1 \otimes \mathbf{C})$ and $r \geq r_0 + l_0 + \deg X + \deg Y$,*

$$R_x(X)R_y(\bar{Y}^*)I(S, f, x, y)$$

equals $I(S, X * f * Y, x, y)$.

Proof. Suppose first of all that f is K-finite. If S is relatively compact the result follows from (3.1) and the proof of Lemma 4.4. In general S can be written as a disjoint union of relatively compact sets S_k. For any n,

$$\begin{aligned}&\Big|\sum_{k>n} R(\bar{Y}^*)\, I(S_k, X * f, x, y)\Big| \\ &= \Big|\sum_{k>n} I(S_k, X * f * Y, x, y)\Big|.\end{aligned}$$

In the notation of the proof of Lemma 4.4, this is bounded by

$$\begin{aligned}&\sum_i \Big(\sum_{k>n} I(S_k, g_i^* * g_i, y, y)\Big)^{\frac{1}{2}} \Big(\sum_{k>n} I(S_k, h_i * h_i^*, x, x)\Big)^{\frac{1}{2}} \\ &\leq \|y\|^N \cdot \sum_i \|g_i^* * g_i\|_0^{\frac{1}{2}} \Big(\sum_{k>n} I(S_k, h_i * h_i^*, x, x)\Big)^{\frac{1}{2}}.\end{aligned}$$

As n approaches ∞ this expression approaches 0 uniformly for y in compact subsets. It follows that the function

$$\sum_k I(S_k, X * f, x, y) = I(S, X * f, x, y)$$

is differentiable in y and its derivative with respect to $\bar{Y}^*$ equals

$$\sum_k I(S_k, X * f * Y, x, y) = I(S, X * f * Y, x, y).$$

The differentiability in x follows the same way.

Now suppose that f is an arbitrary function in $C_c^r(G(\mathbf{A})^1)$. Let f_n be a sequence of K-finite functions that converges to f in $C_c^{r-l_0}(G(\mathbf{A})^1$. If $n_1 > n_2$,

$$\begin{aligned}&|R_y(\bar{Y}^*)\, I(S, X * f_{n_1}, x, y) - R_y(\bar{Y}^*)I(S, X * f_{n_2}, x, y)| \\ &= |I(S, X * f_{n_1} * Y - X * f_{n_2} * Y, x, y)| \\ &\leq \|x\|^N\|y\|^N \cdot \|X * (f_{n_1} - f_{n_2}) * Y\|_{r_0},\end{aligned}$$

by Lemma 4.4. Therefore the sequence

$$R_y(\bar{Y}^*)\, I(S, X * f_n, x, y)$$

converges uniformly for x and y in compact sets. In particular,

$$I(S, X * f, x, y)$$

is differentiable y, and

$$\begin{aligned} R_y(\bar{Y}^*)I(S, X * f, x, y) &= \lim_{n \to \infty} R_y(\bar{Y}^*)I(S, X * f_n, x, y) \\ &= \lim_{n \to \infty} I(S, X * f_n * Y, x, y) \\ &= I(S, X * f * Y, x, y), \end{aligned}$$

since $X * f_n * Y$ approaches $X * f * Y$ in $C_c^{r_0}(G(\mathbf{A})^1)$.

The differentiability in x follows the same way. ∎

COROLLARY 4.6. *Suppose that $X, Y \in \mathcal{U}(\mathfrak{g}(\mathbf{R})^1 \otimes \mathbf{C}$ and $r = r_0 + l_0 + \deg X + \deg Y$. Then there is a continuous seminorm $\| \ \|$ on $C_c^r(G(\mathbf{A})^1)$ such that if $\{S_k\}$ is any sequence of disjoint subsets of S_1, and $f \in C_c^r(G(\mathbf{A})^1)$,*

$$\sum_k |R_x(X)R_y(\bar{Y}^*)I(S_k, f, x, y)| \leq \|f\| \cdot \|x\|^N \cdot \|y\|^N.$$

Proof. By the lemma, we may assume that $X = Y = 1$. The corollary then follows from Lemma 4.4 and the remarks immediately following its proof. □

COROLLARY 4.7. *Under the assumptions of Corollary 4.6,*

$$\sum_k R_x(X)R_y(\bar{Y}^*)I(S_k, f, x, y) = R_x(X)R_y(\bar{Y}^*)I(S, f, x, y),$$

where S is the union of the sets S_k.

Proof. Again by Lemma 4.5, we need only consider the case that $X = Y = 1$. According to Lemma 4.4 and the remark following its proof,

$$\begin{aligned} I(S, f, x, y) &= \sum_\omega I(S, f_\omega, x, y) \\ &= \sum_\omega \sum_k I(S_k, f_\omega, x, y). \end{aligned}$$

But

$$\sum_\omega \sum_k |I(S_k, f_\omega, x, y)|$$

is bounded by the product of $\|x\|^N \cdot \|y\|^N$ and

$$\sum_{\omega} \|f_\omega\|_{r_0}.$$

This last expression represents a continuous seminorm on $C_c^{r_0+t_0}(G(\mathbf{A})^1)$, and is, in particular, finite. Therefore the double series above converges absolutely. It equals

$$\sum_k \sum_\omega I(S_k, f_\omega, x, y) = \sum_k I(S_k, f, x, y),$$

as required. □

It follows from the discussion of §3 that to any closed subset S of $\mathcal{S}$ there corresponds a closed invariant subspace of $L^2(G(\mathbf{Q})\backslash G(\mathbf{A})^1)$. Let P_S be the orthogonal projection onto this subspace. Set

$$R_S(f) = P_S R(f) P_S = R(f) P_S.$$

LEMMA 4.8. *$I(S, f, x, y)$ is the integral kernel of $R_S(f)$.*

Proof. Suppose, first of all, that f is a K-finite function in $C_c^{r_0}(G(\mathbf{A})^1)$. If S is compact, the lemma follows from the results on Eisenstein series summarized in §3. In general, S can be expressed as a disjoint union of sets S_k such that for each n,

$$S^n = \bigcup_{k=1}^{n} S_k$$

is compact. If ϕ and ψ are functions in $C_c^\infty(G(\mathbf{Q})\backslash G(\mathbf{A})^1)$,

$$\begin{aligned}(R_S(f)\phi, \psi) &= \lim_{n\to\infty} (R_{S^n}(f)\phi, \psi)\\ &= \lim_{n\to\infty} \iint \bar{\psi}(x) I(S^n, f, x, y)\phi(y)dy\, dx\\ &= \sum_{k=1}^{\infty} \iint \bar{\psi}(x)\, I(S_k, f, x, y)\phi(y)dy\, dx.\end{aligned}$$

It follows from Lemma 4.4 and dominated convergence that this equals

$$\iint \bar{\psi}(x)\, I(S, f, x, y)\phi(y)dy\, dx$$

Now suppose that f is an arbitrary function in $C_c^{r_0+t_0}(G(\mathbf{A}))$. Let $\{f_n\}$ be a sequence of K-finite functions that converges in $C_c^{r_0}(G(\mathbf{A}))$ to f. By the dominated convergence theorem,

$$\iint \overline{\psi(x)}\, I(S, f, x, y)\phi(y)dy\, dx = \lim_{n\to\infty} \iint \overline{\psi(x)}\, I(S, f_n, x, y)\phi(y)dy\, dx$$

$$= \lim_{n \to \infty} (R_S(f_n)\phi, \psi)$$

$$= \lim_{n \to \infty} (R(f_n)P_S\phi, \psi).$$

Lemma 4.3 allows us to use, once again, the dominated convergence theorem. The above limit equals

$$(R_S(f)\phi, \psi).$$

We have proved that $I(S, f, x, y)$ is the kernel of $R_S(f)$. □

Suppose that χ is an element in $\mathscr{X}$ and that S is the set $\{(\chi, P, \pi)\}$ obtained by taking all P and π. Then $R_S(f)$ is the projection of $R(f)$ onto $L^2(G(\mathbf{Q})\backslash G\mathbf{A})^1)_\chi$. We shall write $R_\chi(f)$ and $K_\chi(x, y)$ for $R_S(f)$ and $I(S, f, x, y)$. It follows from Lemma 4.8 and Corollary 4.7 that

$$\sum_\chi K_\chi(x, y)$$

equals the kernel of $R(f)$. It therefore must equal $K(x, y)$ almost everywhere. However the difference between these two functions is continuous in x and y (separately). It follows easily that

$$K(x, y) = \sum_{\chi \in \mathscr{X}} K_\chi(x, y)$$

for all x and y.

Suppose that P is a parabolic subgroup. If $K(x, y)$ is replaced by $K_P(x, y)$ and $E(x, \phi)$ is replaced by $E_P(x, \phi)$, we can obtain obvious analogues of the definitions of this section, as well as of Lemmas 4.3 through 4.8. Then $K_{P,\chi}(x, y)$ is defined to be

$$\sum_{P_1 \subset P} n_P(A_1)^{-1} \int_{\Pi^0(M_1)} \sum_{\phi \in \mathscr{B}_{P_1}(\pi)_\chi} E_P(x, I_{P_1}(\pi, f)\phi)\, \overline{E_P(y, \phi)}\, d\pi$$

if f is K-finite. If f is an arbitrary function in $C_c^{r_0 + t_0}(G(\mathbf{A})^1)$,

$$K_P(x, y) = \sum_{\chi \in \mathscr{X}} K_{P,\chi}(x, y)$$

for all x and y.

§5. The modified kernel identity

After comparing the formulas of §2 and §4 for $K_P(x, y)$, we note that

$$\sum_{\mathfrak{o}} K_{P,\mathfrak{o}}(x, y) = \sum_\chi K_{P,\chi}(x, y) \tag{5.1}$$

for all x and y. In this section we shall modify each of the functions $K_{\mathfrak{o}}(x, x)$ and $K_\chi(x, x)$ so that the sum over $\mathfrak{o}$ remains equal to the sum over χ. We shall later find that all of the modified functions are integrable over $G(\mathbf{Q})\backslash G(\mathbf{A})^1$. First, we shall need a lemma.

If $P_1 \subset P_2$, let

$$\tau_{P_1}^{P_2} = \tau_1^2 \quad \text{and} \quad \hat{\tau}_{P_1}^{P_2} = \hat{\tau}_1^2$$

be the characteristic functions on $\mathfrak{a}_0$ of

$$\{H \in \mathfrak{a}_0 : \alpha(H) > 0,\ \alpha \in \Delta_1^2\}$$

and

$$\{H \in \mathfrak{a}_0 : \varpi(H) > 0,\ \varpi \in \hat{\Delta}_1^2\}.$$

We shall denote τ_P^G and $\hat{\tau}_P^G$ simply by τ_P and $\hat{\tau}_P$.

LEMMA 5.1. *Suppose that we are given a parabolic subgroup P, and a Euclidean norm $\| \ \|$ on $\mathfrak{a}_P$. Then there are constants c and N such that for all $x \in G(\mathbf{A})^1$ and $X \in \mathfrak{a}_P$,*

$$\sum_{\delta \in P(\mathbf{Q})\backslash G(\mathbf{Q})} \hat{\tau}_P(H(\delta x) - X) \leq c(\|x\| e^{\|X\|})^N.$$

In particular, the sum is finite.

Proof. Suppose that $\varpi \in \hat{\Delta}_0$. Let Λ be a rational representation of G on the vector space V, with highest weight $d\varpi$, $d > 0$. Choose a height function relative to a basis on $V(\mathbf{Q})$ as in §1. We can assume that the basis contains a highest weight vector v. According to the Bruhat decomposition, any element $\delta \in G(\mathbf{Q})$ can be written $\pi\, w_s \nu$, for $\pi \in P_0(\mathbf{Q})$, $\nu \in N_0(\mathbf{Q})$ and $s \in \Omega$. It follows that

$$\|\Lambda(\delta)^{-1} v\| \geq 1.$$

However, there are constants c_1, c_2 and N_1 such that for any $x \in G(\mathbf{A})^1$,

$$\begin{aligned} \|\Lambda(\delta)^{-1} v\| &= \|\Lambda(x)\ \Lambda(\delta x)^{-1} v\| \\ &\leq c_1 \|x\|^{N_1} \|\Lambda(\delta x)^{-1}\ v\| \\ &\leq c_2 \|x\|^{N_1} e^{-d\varpi(H_0(\delta x))}. \end{aligned}$$

It follows that there is a constant c such that

$$\varpi(H_0(\delta x)) \leq c(1 + \log \|x\|) \tag{5.2}$$

for all $\varpi \in \hat{\Delta}_0$, $x \in G(\mathbf{A})^1$ and $\delta \in G(\mathbf{Q})$.

For each x, let $\Gamma(x)$ be a fixed set of representatives of $P(\mathbf{Q})\backslash G(\mathbf{Q})$ in $G(\mathbf{Q})$ such that for any $\delta \in \Gamma(x)$, δx belongs to $\omega \mathfrak{S} A(\mathbf{R})^0 K$, where ω is a fixed compact subset of $N(\mathbf{A})$ and $\mathfrak{S}$ is a fixed Siegel set in $M(\mathbf{A})^1$. Then there is a compact subset ω_0 of $N_0(\mathbf{A})\ M_0(\mathbf{A})^1$ and a point T_0 in $\mathfrak{a}_0$ such that for any x, and any $\delta \in \Gamma(x)$, δx belongs to $\omega_0 A_0(\mathbf{R})^0 K$, and in addition,

$$\alpha(H_0(\delta x)) \geq \alpha(T_0), \tag{5.3}$$

for every $\alpha \in \Delta_0^P$. We are interested in those δ such that $\hat{\tau}_P(H_0(\delta x) - X) = 1$; that is, such that

$$\varpi(H_0(\delta x)) > \varpi(X) \tag{5.4}$$

for every $\varpi \in \hat{\Delta}_P$. The set of points $H_0(\delta x)$ in $\mathfrak{a}_0^G$ which satisfy (5.2), (5.3) and (5.4) is compact. In fact, it follows from our discussion that if $x \in G(\mathbf{A})^1$, $\delta \in \Gamma(x)$ and $\hat{\tau}_P(H(\delta x) - X) = 1$, then $\|\delta x\|$ is bounded by a constant multiple of a power of $\|x\| \cdot e^{\|X\|}$. Since

$$\|\delta\| \leq \|\delta x\| \cdot \|x^{-1}\| \leq c\|\delta x\| \cdot \|x\|^N,$$

for some c and N, $\|\delta\|$ too is bounded by a constant multiple of a power of $\|x\| \cdot e^{\|X\|}$. Because $G(\mathbf{Q})$ is a discrete subgroup of $G(\mathbf{A})^1$, the lemma follows from the fact that the volume in $G(\mathbf{A})^1$ of the set

$$\{y \in G(\mathbf{A})^1 : \|y\| \leq t\}$$

is bounded by a constant multiple of a power of t. □

COROLLARY 5.2. *Suppose that $T \in \mathfrak{a}_0$ and $N \geq 0$. Then we can find constants c' and N' such that for any function ϕ on $P(\mathbf{Q})\backslash G(\mathbf{A})^1$, and $x, y \in G(\mathbf{A})^1$,*

$$\sum_{\delta \in P(\mathbf{Q})\backslash G(\mathbf{Q})} |\phi(\delta x)|\, \hat{\tau}_P(H(\delta x) - H(y) - T) \tag{5.5}$$

is bounded by

$$c'\|x\|^{N'} \cdot \|y\|^{N'} \cdot \sup_{u \in G(\mathbf{A})^1} (|\phi(u)| \cdot \|u\|^{-N}).$$

Proof. The expression (5.5) is bounded by the product of

$$\sup_{u \in G(\mathbf{A})^1} (|\phi(u)| \cdot \|u\|^{-N})$$

and

$$\sum_{\delta \in \Gamma(x)} \|\delta x\|^N \cdot \hat{\tau}_P(H(\delta x) - H(y) - T).$$

We proved in the lemma that when $\hat{\tau}_P(H(\delta x) - H(y) - T)$ was equal to 1, $\|\delta x\|$ was bounded by a constant multiple of a power of $\|x\| e^{\|H_P(y) + T\|}$. The corollary therefore follows from the lemma itself. □

Suppose that T is a fixed point in $\mathfrak{a}_0^+$. We shall say that T is suitably regular if $\alpha(T)$ is sufficiently large for all $\alpha \in \Delta_0$. We shall make this assumption whenever it is convenient, often without further comment. For the rest of this paper, f will be a function in $C_c^r(G(\mathbf{A})^1)$, where $r \geq r_0 + l_0$. We shall also assume that r is as large as necessary at any given time, again, without further comment. Suppose that $x \in G(\mathbf{A})^1$. For $\mathfrak{o} \in \mathcal{O}$ and $\chi \in \mathcal{X}$, define

$$k_{\mathfrak{o}}^T(x, f) = \sum_P (-1)^{\dim(A_P/Z)} \sum_{\delta \in P(\mathbf{Q})\backslash G(\mathbf{Q})} K_{P,\mathfrak{o}}(\delta x, \delta x) \cdot \hat{\tau}_P(H(\delta x) - T),$$

and

$$k_\chi^T(x, f) = \sum_P (-1)^{\dim(A_P/Z)} \sum_{\delta \in P(\mathbf{Q})\backslash G(\mathbf{Q})} K_{P,\chi}(\delta x, \delta x) \cdot \hat{\tau}_P(H(\delta x) - T).$$

PROPOSITION 5.3. $\sum_{\mathfrak{o}} k_{\mathfrak{o}}^T(x, f) = \sum_\chi k_\chi^T(x, f)$.

Proof. The left hand side equals

$$\sum_P (-1)^{\dim(A_P/Z)} \sum_{\delta \in P(\mathbf{Q})\backslash G(\mathbf{Q})} \sum_{\mathfrak{o}} K_{P,\mathfrak{o}}(\delta x, \delta x)\hat{\tau}_P(H(\delta x) - T),$$

since the sums over P and δ are finite. By (5.1) this equals

$$\sum_P (-1)^{\dim(A_P/Z)} \sum_{\delta \in P(\mathbf{Q})\backslash G(\mathbf{Q})} \sum_\chi K_{P,\chi}(\delta x, \delta x)\hat{\tau}_P(H(\delta x) - T),$$

which is just $\sum_\chi k_\chi^T(x, f)$. □

§6. Some geometric lemmas

We want to show that each of the terms in the identity of Proposition 5.2 is integrable over $G(\mathbf{Q})\backslash G(\mathbf{A})^1$. In this section we shall collect some geometric lemmas which will be needed in estimating the integrals.

If $P_1 \subset P_2$, set

$$\sigma_1^2(H) = \sigma_{P_2}^{P_1}(H) = \sum_{\{P_3 : P_3 \supset P_2\}} (-1)^{\dim(A_3/A_2)} \tau_1^3(H) \cdot \hat{\tau}_3(H),$$

for $H \in \mathfrak{a}_0$. We hope that the reader will understand the motivation for this definition, as well as for the next lemma, after reading §7.

LEMMA 6.1. *If $P_2 \supset P_1$, σ_1^2 is the characteristic function of the set of $H \in \mathfrak{a}_1$ such that*

(i) $\alpha(H) > 0$ *for all* $\alpha \in \Delta_1^2$,

(ii) $\alpha(H) \leq 0$ *for all* $\sigma \in \Delta_1\backslash\Delta_1^2$, *and*

(iii) $\varpi(H) > 0$ *for all* $\varpi \in \hat{\Delta}_2$.

Proof. Fix $H \in \mathfrak{a}_1$. Consider the subset of those ϖ in $\hat{\Delta}_2$ for which $\varpi(H) > 0$. This subset is of the form $\hat{\Delta}_R$, for a unique parabolic subgroup $R \supset P_2$. Then

$$\sigma_1^2(H) = \sum_{\{P_3 : P_3 \supset R\}} (-1)^{\dim(A_2/A_3)} \tau_1^3(H).$$

Suppose that $\tau_1^3(H) = 1$ for a given $P_3 \supset R$. Then $\tau_1^3(H) = 1$ for all smaller P_3. It follows from Proposition 1.1 that the above sum vanishes unless the original P_3 equals R. Thus,

$$\sum_{\{P_3 : P_3 \supset R\}} (-1)^{\dim(A_2/A_3)} \tau_1^3(X)$$

is the characteristic function of

$$\{X \in \mathfrak{a}_1 : \alpha(X) > 0,\ \alpha \in \Phi_1^R;\ \alpha(X) \leq 0,\ \alpha \in \Delta_1 \setminus \Delta_1^R\} \tag{6.1}$$

If $R = P_2$, $\sigma_1^2(H) = 1$ if and only if H belongs to the set (6.1), as required. We must show that if R is strictly larger than P_2, $\sigma_1^2(H) = 0$. Suppose not. Then H belongs to the set (6.1). In particular, the projection of H onto $\mathfrak{a}_1^R$ lies in the positive chamber, which is contained in set of positive linear combinations of roots in Δ_1^R. Thus $\varpi(H) > 0$ for all $\varpi \in \hat{\Delta}_1^R$. By the definition of R, $\varpi(H) > 0$ for all $\varpi \in \hat{\Delta}_R$. From this, we shall show that if

$$H = \sum_{\alpha \in \Delta_1} c_\alpha \alpha^\vee,$$

each c_α is positive. Suppose that $\varpi \in \hat{\Delta}_R \subset \hat{\Delta}_1$, and that α_ϖ is the element in Δ_1 which is paired with ϖ. Then $c_{\alpha_\varpi} = \varpi(H)$ is positive. Therefore the projection of

$$H_R = \sum_{\varpi \in \hat{\Delta}_R} c_{\alpha_\varpi} \alpha_\varpi^\vee$$

onto $\mathfrak{a}_1^R$ is in the negative chamber, so that if $\nu \in \hat{\Delta}_1^R$, $\nu(H_R)$ is negative. If α_ν is the root in Δ_1^R corresponding to ν,

$$c_{\alpha_\nu} = \nu(H) - \nu(H_R)$$

is positive. Thus each c_α is positive. Therefore $\varpi(H)$ is positive for each $\varpi \in \hat{\Delta}_1$, and in particular, for each ϖ in $\hat{\Delta}_2$. Therefore $R = P_2$ so we have a contradiction. □

COROLLARY 6.2. *Fix $T \in \mathfrak{a}_0^+$. For any $H \in \mathfrak{a}_1^G$, let H_1^2 be the projection of H onto $\mathfrak{a}_1^2$. Then if $\sigma_1^2(H - T) \neq 0$, $\alpha(H_1^2)$ is positive for each $\alpha \in \Delta_1^2$, and*

$$\|H\| \leq c(1 + \|H_1^2\|),$$

for any Euclidean norm $\| \ \|$ on $\mathfrak{a}_0$, and some constant c.

Proof. The first condition follows directly from the lemma and the fact that T belongs to $\mathfrak{a}_0^+$. To prove the second one, write

$$H = H_1^2 + H_2.$$

The value at H_2 of any root in Δ_2 equals $\alpha(H_2)$ for some root α in $\Delta_1 \setminus \Delta_1^2$. But

$$\alpha(H_2) = \alpha(H - T) - \alpha(H_1^2) + \alpha(T) < -\alpha(H_1^2) + \alpha(T),$$

by the lemma. Since

$$\varpi(H_2) = \varpi(H) > 0,$$

for each $\varpi \in \hat{\Delta}_2$, H_2 belongs to a compact set. In fact the norm of H_2 is bounded by a constant multiple of $1 + \|H_1^2\|$ as required. □

This corollary will not be used until we begin to estimate the integrals of the functions defined in §5. A second consequence of Lemma 6.1 is the special case that $P_2 = P_1$, and $P_1 \neq G$. Since every functional in the set $\hat{\Delta}_1$ is negative on $-\mathfrak{a}_1^+$, $\sigma_{P_1}^{P_1}(H)$ is the characteristic function of the empty set. In other words,

$$\sum_{\{P_3 : P_3 \supset P_1\}} (-1)^{\dim(A_1/A_3)} \tau_1^3\,(H)\,\hat{\tau}_3(H) = 0$$

for all $H \in \mathfrak{a}_0$.

Suppose that Q and P are parabolic subgroups with $Q \subset P$. Fix $\Lambda \in \mathfrak{a}_0^*$. Let $\epsilon_Q^P(\Lambda)$ be (-1) raised to a power equal to the number of roots $\alpha \in \Delta_Q^P$ such that $\Lambda(\alpha^\vee) \leq 0$. Let

$$\phi_Q^P(\Lambda, H), \qquad H \in \mathfrak{a}_0,$$

be the characteristic function of the set of $H \in \mathfrak{a}_0$ such that for any $\alpha \in \Delta_Q^P$, $\varpi_\alpha(H) > 0$ if $\Lambda(\alpha^\vee) \leq 0$, and $\varpi_\alpha(H) \leq 0$ if $\Lambda(\alpha) > 0$. In the special case that none of the numbers $\Lambda(\alpha^\vee)$ or $\varpi_\alpha(H)$ is zero, these definitions give functions which occur in a combinatorial lemma of Langlands [1(b)]. It is desirable to have this lemma for general H and Λ, so we shall give a different proof, based only on Proposition 1.1.

LEMMA 6.3. $\displaystyle\sum_{\{R : Q \subset R \subset P\}} \epsilon_Q^R(\Lambda)\,\phi_Q^R(\Lambda, H)\,\tau_R^P(H)$

equals 0 *if* $\Lambda(\alpha^\vee) \leq 0$ *for some* $\alpha \in \Delta_Q^P$, *and equals* 1 *otherwise.*

Proof. If $R \neq P$, our remark following Corollary 6.2 implies that

$$\sum_{\{R_1 : R \subset P_1 \subset P\}} (-1)^{\dim(A_1/A_P)} \tau_R^1(H) \hat{\tau}_1^P(H)$$

vanishes for all H. Therefore

$$\sum_{\{R : Q \subset R \subset P\}} \epsilon_Q^R(\Lambda) \phi_Q^R(\Lambda, H) \tau_R^P(H)$$

is the difference between

$$\epsilon_Q^P(\Lambda)\phi_Q^P(\Lambda, H) \tag{6.2}$$

and

$$\sum_{\{R, P_1 : Q \subset R \subset P_1 \subsetneq P\}} \epsilon_Q^R(\Lambda)\phi_Q^R(\Lambda, H)\tau_R^1(H)\,(-1)^{\dim(A_1/A_P)}\hat{\tau}_1^P(H). \tag{6.3}$$

We shall prove the lemma by induction on $\dim(A_0/A_P)$. Define Δ_Q^Λ to be the set of roots $\alpha \in \Delta_Q^P$ such that $\alpha(\Lambda) > 0$. Associated to Δ_Q^Λ we have a parabolic subgroup P_Λ, with $Q \subset P_\Lambda \subset P$. By our induction assumption, the sum over R in (6.3) vanishes unless $P_1 \subset P_\Lambda$, in which case it equals $(-1)^{\dim(A_1/A_P)}\hat{\tau}_1^P(H)$. Thus, (6.3) equals

$$(6.4) \qquad \sum_{\{P_1: Q \subset P_1 \subset P_\Lambda\}} (-1)^{\dim(A_1/A_P)} \hat{\tau}_1^P(H)$$

if $P_\Lambda \neq P$, and equals (6.4) minus 1 if $P_\Lambda = P$. We need only show that (6.2) equals (6.4). This is a consequence of Proposition 1.1 and the definitions. $\square$

Our final aim for this section is to derive a partition of $G(\mathbf{Q})\backslash G(\mathbf{A})$ into disjoint subsets, one for each (standard) parabolic subgroup. The partition is similar to a construction from [6(b)], in which disjoint subsets of $G(\mathbf{Q})\backslash G(\mathbf{A})$ are associated to maximal parabolic subgroups. More generally, we shall partition $N(\mathbf{A})M(\mathbf{Q})\backslash G(\mathbf{A})$, where $P = NM$ is a parabolic subgroup. The result is little more than a restatement of a basic lemma from reduction theory, which we would do well to recall. Suppose that ω is a compact subset of $N_0(\mathbf{A})M_0(\mathbf{A})^1$ and that $T_0 \in -\mathfrak{a}_0^+$. For any parabolic subgroup P_1, let $\mathfrak{S}^{P_1}(T_0, \omega)$ be the set of

$$pak, \qquad p \in \omega,\ a \in A_0(\mathbf{R})^0, \qquad k \in K,$$

such that $\alpha(H_0(a) - T_0)$ is positive for each $\alpha \in \Delta_0^1$. By the properties of Siegel sets we can fix ω and T_0 so that for any P_1, $G(\mathbf{A}) = P_1(\mathbf{Q})\mathfrak{S}^{P_1}(T_0, \omega)$. The result from reduction theory is contained in the proof of [6(b), Lemma 2.12.]. Namely, any suitably regular point T in $\mathfrak{a}_0^+$ has the following property: suppose that $P_1 \subset P$ are parabolic subgroups, and that x and δx belong to $\mathfrak{S}^{P_1}(T_0, \omega)$ for points $x \in G(\mathbf{A})$ and $\delta \in P(\mathbf{Q})$. Then if $\alpha(H_0(x) - T) > 0$ for all α in $\Delta_0^P \backslash \Delta_0^{P_1}$, δ belongs to $P_1(\mathbf{Q})$.

Suppose that P_1 is given. Let $\mathfrak{S}^{P_1}(T_0, T, \omega)$ be the set of x in $\mathfrak{S}^{P_1}(T_0, \omega)$ such that $\varpi(H_0(x) - T) \leq 0$ for each $\varpi \in \hat{\Delta}_0^1$. Let

$$F^{P_1}(x, T) = F^1(x, T)$$

be the characteristic function of the set of $x \in G(\mathbf{A})$ such that δx belongs to $\mathfrak{S}^{P_1}(T_0, T, \omega)$ for some $\delta \in P_1(\mathbf{Q})$. $F^1(x, T)$ is left $A_1(\mathbf{R})^0 N_1(\mathbf{A})M_1(\mathbf{Q})$ invariant, and can be regarded as the characteristic function of the projection of $\mathfrak{S}^{P_1}(T_0, T, \omega)$ onto $A_1(\mathbf{R})^0 N_1(\mathbf{A})M_1(\mathbf{Q})\backslash G(\mathbf{A})$, a compact subset of $A_1(\mathbf{R})^0 N_1(\mathbf{A})M_1(\mathbf{Q})\backslash G(\mathbf{A})$.

LEMMA 6.4. *Fix P, and let T be any suitably regular point in $T_0 + \mathfrak{a}_0^+$. Then*

$$\sum_{\{P_1: P_0 \subset P_1 \subset P\}} \sum_{\delta \in P_1(\mathbf{Q})\backslash G(\mathbf{Q})} F^1(\delta x, T)\tau_1^P(H_0(\delta x) - T)$$

equals one for all x in $G(\mathbf{A})$.

Proof. Fix $x \in G(\mathbf{A})$. Choose $\delta \in P(\mathbf{Q})$ such that δx belongs to $\mathfrak{S}^P(T_0, \omega)$. Apply Lemma 6.3, with $Q = P_0$, $\Lambda \in (\mathfrak{a}_0^*)^+$, and $H = H_0(\delta x) - T$. Then there is a parabolic subgroup $P_1 \subset P$ such that $\varpi(H_0(\delta x) - T) \leq 0$ for all $\varpi \in \hat{\Delta}_0^1$ and $\alpha(H_0(\delta x) - T) > 0$ for $\alpha \in \Delta_1^P$. Therefore

$$F^1(\delta x, T)\tau_1^P(H_0(\delta x) - T) = 1,$$

so the given sum is at least one.

Suppose that there are elements $\delta_1, \delta_2 \in G(\mathbf{Q})$, and parabolic subgroups P_1 and P_2 contained in P such that

$$F^1(\delta_1 x, T)\tau_1^P(H_0(\delta_1 x) - T) = F^2(\delta_2 x, T)\tau_2^P(H_0(\delta_2 x) - T) = 1.$$

After left translating δ_i by an element in $P_i(\mathbf{Q})$ if necessary, we may assume that

$$\delta_i x \in \mathfrak{S}^{P_i}(T_0, T, \omega), \qquad i = 1, 2.$$

The projection of $H(\delta_i x) - T$ onto $\mathfrak{a}_0^P$ can be written

$$-\sum_{\alpha \in \Delta_0^i} c_\alpha \alpha^\vee + \sum_{\varpi \in \hat{\Delta}_i^P} c_\varpi \varpi^\vee,$$

where each c_α and c_ϖ is positive. It follows that $\alpha(H_0(\delta_i x) - T) > 0$ for every $\alpha \in \Delta_0^P \setminus \Delta_0^i$. In particular, since T lies in $T_0 + \mathfrak{a}_0^+$, $\delta_i x$ belongs to $\mathfrak{S}^P(T_0, \omega)$. The reduction theoretic result just quoted now implies that $\delta_2\delta_1^{-1}$ belongs to $P_1(\mathbf{Q})$ and $\delta_1\delta_2^{-1}$ belongs to $P_2(\mathbf{Q})$. In other words, $\delta_2 = \xi\delta_1$, for some element ξ in $P_1(\mathbf{Q}) \cap P_2(\mathbf{Q})$. Let $Q = P_1 \cap P_2$. Then $H_0(\delta_1 x) - T$ and $H_0(\delta_2 x) - T$ project onto the same point, say H_Q^P, on $\mathfrak{a}_Q^P$. If R equals either P_1 or P_2, we have $\varpi(H_Q^P) \leq 0$ for $\varpi \in \hat{\Delta}_Q^R$ and $\alpha(H_Q^P) > 0$ for $\alpha \in \Delta_R^P$. Applying Lemma 6.3, with $\Lambda \in (\mathfrak{a}_0^*)^+$, we see that there is exactly one R, with $Q \subset R \subset P$, for which these inequalities hold. Therefore $P_1 = P_2$, and δ_1 and δ_2 belong to the same $P_1(\mathbf{Q})$ coset in $G(\mathbf{Q})$. This proves that the given sum is at most one.

§7. Integrability of $k_{\mathfrak{o}}^T(x, f)$

The primary goal of this section is to establish the integrability of each function $k_{\mathfrak{o}}^T(x, f)$. In fact we will obtain

THEOREM 7.1. *For all sufficiently regular T,*

$$\sum_{\mathfrak{o} \in \mathcal{O}} \int_{G(\mathbf{Q})\setminus G(\mathbf{A})^1} |k_{\mathfrak{o}}^T(x, f)| dx$$

is finite.

Proof. For any x, $k_{\mathfrak{o}}^T(x, f)$ equals the sum over P and over $\delta \in P(\mathbf{Q})\setminus G(\mathbf{Q})$ of the product of

$$(-1)^{\dim(A/Z)} K_{P,\mathfrak{o}}(\delta x, \delta x) \cdot \hat{\tau}_P(H_0(\delta x) - T)$$

and

$$\sum_{\{P_1 : P_0 \subset P_1 \subset P\}} \sum_{\xi \in P_1(\mathbf{Q})\setminus P(\mathbf{Q})} F^1(\xi\delta x, T) \cdot \tau_1^P(H_0(\xi\delta x) - T).$$

This equals the sum over $\{P_1, P : P_0 \subset P_1 \subset P\}$ and $\{\delta \in P_1(\mathbf{Q})\setminus G(\mathbf{Q})\}$ of

$$(-1)^{\dim(A/Z)} F^1(\delta x, T)\tau_1^P(H_0(\delta x) - T)\hat{\tau}_P(H_0(\delta x) - T)\ K_{P,\mathfrak{o}}(\delta x, \delta x).$$

For any $H \in \mathfrak{a}_0$ we can certainly write

$$\tau_1^P(H)\hat{\tau}_P(H) = \sum_{\{P_2,P_3:P\subset P_2\subset P_3\}} (-1)^{\dim(A_2/A_3)}\tau_1^3(H)\hat{\tau}_3(H)$$

by Proposition 1.1. In the notation of §6, this equals

$$\sum_{\{P_2:P_2\supset P\}} \sigma_1^2(H).$$

We have shown that $k_{\mathfrak{o}}^T(x,f)$ is the sum over $\{P_1, P, P_2 : P_1 \subset P \subset P_2\}$ and over $\delta \in P_1(\mathbf{Q})\backslash G(\mathbf{Q})$ of

$$(-1)^{\dim(A/Z)}F^1(\delta x, T)\sigma_1^2(H_0(\delta x) - T)K_{P,\mathfrak{o}}(\delta x, \delta x).$$

Therefore

$$\sum_{\mathfrak{o}\in\mathcal{O}} \int_{G(\mathbf{Q})\backslash G(\mathbf{A})^1} |k_{\mathfrak{o}}^T(x,f)|dx \tag{7.2}$$

is bounded by the sum over $\{P_1, P_2 : P_1 \subset P_2\}$ and over $\mathfrak{o} \in \mathcal{O}$ of the integral over $P_1(\mathbf{Q})\backslash G(\mathbf{A})^1$ of the product of

$$F^1(x, T)\, \sigma_1^2(H_0(x) - T) \tag{7.3}$$

with the absolute value of

$$\sum_{\{P:P_1\subset P\subset P_2\}} (-1)^{\dim(A/Z)} \sum_{\gamma\in M(\mathbf{Q})\cap\mathfrak{o}} \int_{N(\mathbf{A})} f(x^{-1}\gamma nx)dn. \tag{7.4}$$

The critical part of this expression is the alternating sum over P. In order to exploit it we shall show that the sum over $\gamma \in M(\mathbf{Q}) \cap \mathfrak{o}$ in (7.4) can be taken over a smaller set. Fix P, and suppose that $x \in P_1(\mathbf{Q})\backslash G(\mathbf{A})^1$. We can assume that neither (7.3) nor

$$\sum_{\gamma\in M(\mathbf{Q})\cap\mathfrak{o}} \int_{N(\mathbf{A})} f(x^{-1}\gamma nx)dn$$

vanishes. We want to show that the sum over γ can be taken over the intersection of $\mathfrak{o}$ with the parabolic subgroup $P_1 \cap M$ of M.

We choose a representative of x in $G(\mathbf{A})^1$ of the form

$$n^*n_*mak,$$

where $k \in K$, n^*, n_* and m belong to fixed compact subsets of $N_2(\mathbf{A})$, $N_0^2(\mathbf{A})$ and $M_0(\mathbf{A})^1$ respectively, and $a \in A_0(\mathbf{R})^0 \cap G(\mathbf{A})^1$ has the property that

$$\alpha(H_0(a) - T_0) > 0, \qquad \alpha \in \Delta_0^1, \tag{7.5}$$

and

$$\varpi(H_0(a) - T) \le 0, \qquad \varpi \in \hat{\Delta}_0^1. \tag{7.6}$$

Here T_0 is as in §6. By Corollary 6.2, $\alpha(H_0(a) - T)$ is positive for any $\alpha \in \Delta_1^2$. It follows that the projection of $H_0(a) - T$ onto $\mathfrak{a}_0^2$ equals

$$\sum_{\varpi \in \hat{\Delta}_1^2} c_\varpi \varpi^\vee - \sum_{\alpha \in \Delta_0^1} c_\alpha \alpha^\vee,$$

where each c_α and c_ϖ is positive. Consequently,

$$\alpha(H_0(a) - T) > 0, \qquad \alpha \in \Delta_0^2 \setminus \Delta_0^1. \tag{7.7}$$

A well known lemma from reduction theory asserts that for any such a, $a^{-1}n_* m\, a$ belongs to a fixed compact subset of $N_0^2(\mathbf{A}) \times M_0(\mathbf{A})^1$ which is independent of T. Suppose that the assertion we are trying to prove is false. Then there is a γ in $M(\mathbf{Q}) \cap P_1(\mathbf{Q}) \setminus M(\mathbf{Q})$ such that

$$\int_{N(\mathbf{A})} f(k^{-1}a^{-1}m^{-1}n_*^{-1}n^{*\,-1} \cdot \gamma n \cdot n^* n_* m a k)dn$$

is not zero. This expression equals

$$\int_{N(\mathbf{A})} f(k^{-1}(a^{-1}mn_*a)^{-1} \cdot a^{-1}\gamma n a \cdot (a^{-1}mn_*a)k)dn.$$

Therefore there is a compact subset of $G(\mathbf{A})^1$ which meets $a^{-1}\gamma N(\mathbf{A})a$. Thus, $a^{-1}\gamma a$ belongs to a fixed compact subset of $M(\mathbf{A})^1$. According to the Bruhat decomposition for $M(\mathbf{Q})$, we can write $\gamma = \nu w_s \pi$, for $\nu \in N_0^P(\mathbf{Q})$, $\pi \in M(\mathbf{Q}) \cap P_0(\mathbf{Q})$, and $s \in \Omega^M$, the Weyl group of (M, A_0). s cannot belong to the Weyl group of (M_1, A_0), so we can find an element $\varpi \in \hat{\Delta}_1^P$ not fixed by s. Let Λ be a rational representation of M with highest weight $d\varpi$, $d > 0$. Let v be a highest weight vector in $V(\mathbf{Q})$, the space on which $G(\mathbf{Q})$ acts. Choose a height function $\|\ \|$ relative to a basis of $V(\mathbf{Q})$ as in §1. We can assume that the basis includes the vectors v and $\Lambda(w_s)v$. The component of $\Lambda(a^{-1}\nu w_s \pi a)v$ in the direction of $\Lambda(w_s)v$ is $e^{d(\varpi - s\varpi)(H_0(a))}\,\Lambda(w_s)v$. Therefore

$$\|\Lambda(a^{-1}\nu w_s \pi a)v\| \geq e^{d(\varpi - s\varpi)(H_0(a))}.$$

The left side of this inequality is bounded by a number which is independent of T. On the other hand, $\varpi - s\varpi$ is a nonnegative sum of roots in Δ_0^P, and at least one element in $\Delta_0^P \setminus \Delta_0^1$ has nonzero coefficient. It follows from (7.5) and (7.7) that the right hand side of the inequality can be made arbitrarily large by letting T be sufficiently regular. This is a contradiction.

We have shown that for T sufficiently regular, (7.4) equals

$$\left| \sum_{\{P: P_1 \subset P \subset P_2\}} (-1)^{\dim(A/Z)} \sum_{\gamma \in P_1(\mathbf{Q}) \cap M(\mathbf{Q}) \cap \mathfrak{o}} \int_{N(\mathbf{A})} f(x^{-1}\gamma n x)dn \right|.$$

According to the remark following Lemma 2.1,

$$P_1(\mathbf{Q}) \cap M(\mathbf{Q}) \cap \mathfrak{o} = (M_1(\mathbf{Q}) \cap \mathfrak{o})\, N_1^P(\mathbf{Q}),$$

so the absolute value of (7.4) is bounded by the sum over $\gamma \in M_1(\mathbf{Q}) \cap \mathfrak{o}$ of

$$\left| \sum_{\{P: P_1 \subset P \subset P_2\}} (-1)^{\dim(A/Z)} \sum_{\nu \in N_1^P(\mathbf{Q})} \int_{N(\mathbf{A})} f(x^{-1}\gamma \nu n x)dn \right|.$$

We would like to replace the sum over ν by a sum over the rational points of $\mathfrak{n}_1^P$, the Lie algebra of N_1^P. Let

$$e : \mathfrak{n}_0 \to N_0$$

be an isomorphism, defined over $\mathbf{Q}$, from the Lie algebra of N_0 onto N_0 which intertwines the action of A_0. The last expression equals

$$\left| \sum_{\{P: P_1 \subset P \subset P_2\}} (-1)^{\dim(A/Z)} \sum_{\zeta \in \mathfrak{n}_1^P(Q)} \int_{\mathfrak{n}(\mathbf{A})} f(x^{-1}\gamma e(\zeta + X)x)dX \right|.$$

Apply the Poisson summation formula to the sum over ζ. Let $\langle \cdot,\cdot \rangle$ be a positive definite bilinear form on $\mathfrak{n}_0(\mathbf{Q}) \times \mathfrak{n}_0(\mathbf{Q})$ for which the action of $A_0(\mathbf{Q})$ is self-adjoint, and let ψ be a nontrivial character on $\mathbf{A}/\mathbf{Q}$. We obtain

$$\left| \sum_{\{P: P_1 \subset P \subset P_2\}} (-1)^{\dim(A/Z)} \sum_{\gamma \in \mathfrak{n}_1^P(\mathbf{Q})} \int_{\mathfrak{n}_1(\mathbf{A})} f(x^{-1}\gamma e(X)x)\psi(\langle X, \zeta \rangle)dX \right|.$$

If $\mathfrak{n}_1^2(\mathbf{Q})'$ is the set of elements in $\mathfrak{n}_1^2(\mathbf{Q})$ which do not belong to any $\mathfrak{n}_1^P(Q)$, with $P_1 \subset P \subsetneq P_1$, this equals

$$\left| \sum_{\zeta \in \mathfrak{n}_1^2(\mathbf{Q})'} \int_{\mathfrak{n}_1(A)} f(x^{-1}\gamma e(X)x)\psi(\langle X, \zeta \rangle)dx \right|,$$

by Proposition 1.1. We have shown that (7.2) is bounded by the sum over $\{P_1, P_2 : P_1 \subset P_2\}$ of the integral over x in $P_1(\mathbf{Q})\backslash G(\mathbf{A})^1$ of

$$F^1(x, T)\sigma_1^2(H(x) - T) \sum_{\gamma \in M_1(\mathbf{Q})} \sum_{\zeta \in \mathfrak{n}_1^2(\mathbf{Q})'} \left| \int_{\mathfrak{n}_1(\mathbf{A})} f(x^{-1}\gamma e(X)x)\psi(\langle X, \zeta \rangle)dX \right|.$$

Set

$$x = n^* n_* mak,$$

where $k \in K$, $a \in A_1(\mathbf{R})^0 \cap G(\mathbf{A})^1$, and n^*, n_* and m lie in fixed fundamental domains in $N_2(\mathbf{A})$, $N_1^2(\mathbf{A})$ and $M_1(\mathbf{A})^1$ respectively. Of course, this change of variables will add a factor of $e^{-2\rho_1(H_0(a))}$ to the integrand. n^* is absorbed in the integral over X. We need only consider points for which the integrand does not vanish. Therefore m and $a^{-1}n_* a$ both remain in fixed compact sets. Thus (7.2) is bounded by a constant multiple of the quantity obtained by taking the sum over P_1, P_2 and γ, the supremum as y ranges over a fixed compact subset of $G(\mathbf{A})^1$, and the integral over a in $A_1(\mathbf{R})^0 \cap G(\mathbf{A})^1$ of the expression

$$e^{-2\rho_1(H_0(a))}\sigma_1^2(H_0(a) - T) \cdot \sum_{\zeta \in \mathfrak{n}_1^2(\mathbf{Q})'} \left| \int_{\mathfrak{n}_1(\mathbf{A})} f(y^{-1}a^{-1}\gamma e(X)ay)\psi(\langle X, \zeta \rangle)dX \right|.$$

This expression is the same as

$$\sigma_1^2(H_0(a) - T) \sum_{\zeta \in \mathfrak{n}_1^2(\mathbf{Q})'} \left| \int_{\mathfrak{n}_1(\mathbf{A})} f(y^{-1}\zeta e(X)y)\psi(\langle X, Ad(a)\zeta \rangle)dX \right|. \tag{7.8}$$

The sum over γ is finite. Our only remaining worry is the integral over a.

Let

$$\mathfrak{n}_1^2 = \bigoplus_\lambda \mathfrak{n}_\lambda$$

be the decomposition of $\mathfrak{n}_1^2$ into eigenspaces under the action of A_1^2. Each λ stands for a linear function on $\mathfrak{a}_1^2$. Choose a basis of $\mathfrak{n}_1^2(\mathbf{Q})$ such that each basis element lies in some $\mathfrak{n}_\lambda(\mathbf{Q})$. The basis gives us a Euclidean norm on $\mathfrak{n}_1^2(\mathbf{R})$. It also allows us to speak of $\mathfrak{n}_1^2(\mathbf{Z})$ and $\mathfrak{n}_\lambda(\mathbf{Z})$. Note that

$$Y \to \int_{\mathfrak{n}_1(\mathbf{A})} f(y^{-1}\gamma e(X)y)\psi(\langle X, Ad(a)\zeta\rangle)dX, \quad y \in \mathfrak{n}_Q^1(\mathbf{A}),$$

is the Fourier transform of a Schwartz-Bruhat function on $\mathfrak{n}_1^2(\mathbf{A})$ which varies smoothly with y. Therefore we can reduce the integral over X in (7.8) to a finite sum of integrals over a real vector space. It follows without difficulty that for every n we can choose N such that the supremum over y of (7.8) is bounded by a constant multiple of

$$\sigma_1^2(H_0(a) - T) \cdot \sum_{\zeta \in \mathfrak{n}_1^2\left(\frac{1}{N}\mathbf{Z}\right)'} \|Ad(a)\zeta\|^{-n}.$$

Every λ is a unique integral linear combination of elements in Δ_1^2. Suppose that S is a subset of roots λ with the property that for any $\alpha \in \Delta_1^2$, there is a λ in S whose α coordinate is positive. Let $\mathfrak{n}_S(\mathbf{Q})'$ be the set of elements in $\mathfrak{n}_1^2(\mathbf{Q})$ whose projections onto $\mathfrak{n}_\lambda$ are nonzero if λ belongs to S and are zero otherwise. Then the above sum over $\mathfrak{n}_1^2\left(\frac{1}{N}\mathbf{Z}\right)'$ can be replaced by the double sum over all such S and over ζ in $\mathfrak{n}_S\left(\frac{1}{N}\mathbf{Z}\right)'$. Clearly

$$\sigma_1^2(H_0(a) - T) \sum_{\zeta \in \mathfrak{n}_S\left(\frac{1}{N}\mathbf{Z}\right)'} \|Ad(a)\zeta\|^{-n}$$

is bounded by

$$\sigma_1^2(H_0(a) - T) \cdot \prod_{\lambda \in S} \sum_{\zeta \in \mathfrak{n}_\lambda\left(\frac{1}{N}\mathbf{Z}\right)'} \|Ad(a)\zeta\|^{-n_S},$$

where $\mathfrak{n}_\lambda\left(\frac{1}{N}\mathbf{Z}\right)'$ is the set of nonzero elements in $\mathfrak{n}_\lambda\left(\frac{1}{N}\mathbf{Z}\right)$ and n_S is the quotient of n by the number of roots in S. This last expression equals the product of

$$\prod_{\lambda \in S} \sum_{\zeta \in \mathfrak{n}_\lambda\left(\frac{1}{N}\mathbf{Z}\right)'} \|\zeta\|^{-n_S}$$

and

$$\sigma_1^2(H_0(a) - T) \cdot \prod_{\lambda \in S} e^{-n_s\lambda(H_0(a))}.$$

The first factor is finite for large enough n. The second factor equals

$$\sigma_1^2(H_0(a) - T) \prod_{\alpha \in \Delta_1^2} e^{-k_\alpha \alpha(H_0(a))}, \tag{7.9}$$

where each k_α is a positive real number.

The projection of $H_0(a) - T$ onto $\mathfrak{a}_1^G$ can be written

$$\sum_{\alpha \in \Delta_1^2} t_\alpha \varpi_\alpha^\vee + H^*,$$

where $H^* \in \mathfrak{a}_2^G$, and for each $\alpha \in \Delta_1^2$, t_α is a positive real number. If $\sigma_1^2(H_0(a) - T) \neq 0$, it follows from Corollary 6.2 that H^* belongs to a compact set whose volume is bounded by a polynomial in the numbers $\{t_\alpha\}$. Therefore, there is an N such that the integral of (7.9) is bounded by a multiple of

$$\prod_{\alpha \in \Delta_Q^1} \int_0^\infty (1 + |t_\alpha|)^N e^{-k_\alpha t_\alpha} dt_\alpha.$$

This last expression is finite. The proof of Theorem 7.1 is complete. □

§8. Weighted orbital integrals

For any $\mathfrak{o} \in \mathcal{O}$, set

$$J_\mathfrak{o}^T(f) = \int_{G(\mathbf{Q})\backslash G(\mathbf{A})^1} k_\mathfrak{o}^T(x, f)dx.$$

In this section we shall define another function $j_\mathfrak{o}^T(x, f)$. We shall show that its integral is equal to $J_\mathfrak{o}^T(f)$. Then we will, in some cases, reduce the integral of $j^T(x, f)$ to a weighted orbital integral of f.

Given $\mathfrak{o}$ and P, define the function

$$J_{P,\mathfrak{o}}(x, y) = \sum_{\gamma \in M(\mathbf{Q}) \cap \mathfrak{o}} \sum_{\eta \in N(\mathbf{Q},\gamma_s)\backslash N(\mathbf{Q})} \int_{N(\mathbf{A},\gamma_s)} f(x^{-1}\eta^{-1}\gamma n \eta x)dn.$$

It is obtained from $K_{P,\mathfrak{o}}(x, y)$ by replacing part of the integral over $N(\mathbf{A})$ by the corresponding sum over $\mathbf{Q}$-rational points. Define

$$j_\mathfrak{o}^T(x, f) = \sum_P (-1)^{\dim(A/Z)} \sum_{\delta \in P(\mathbf{Q})\backslash G(\mathbf{Q})} J_{P,\mathfrak{o}}(\delta x, \delta x) \cdot \hat{\tau}_P(H(\delta x) - T).$$

THEOREM 8.1. *For all sufficiently regular T,*

$$\sum_{\mathfrak{o} \in \mathcal{O}} \int_{G(\mathbf{Q})\backslash G(\mathbf{A})^1} |j_\mathfrak{o}^T(x, f)|dx$$

is finite. Moreover, for any $\mathfrak{o}$,

$$\int_{G(\mathbf{Q})\backslash G(\mathbf{A})^1} j^T_{\mathfrak{o}}(x, f)dx = J^T_{\mathfrak{o}}(f).$$

Proof. The argument used to prove the first statement is essentially the same as the proof of Theorem 7.1. The integral

$$\sum_{\mathfrak{o} \in \mathcal{O}} \int_{G(\mathbf{Q})\backslash G(\mathbf{A})^1} |j^T_{\mathfrak{o}}(x, f)|dx \tag{8.1}$$

is bounded by the sum over $\{P_1, P_2 : P_1 \subset P_2\}$ and $\mathfrak{o} \in \mathcal{O}$ of the integral over $P_1(\mathbf{Q})\backslash G(\mathbf{A})^1$ of the product of

$$F^1(x, T)\sigma_1^2(H_0(x) - T) \tag{8.2}$$

with the absolute value of

$$\sum_{\{P:P_1 \subset P \subset P_2\}} (-1)^{\dim(A/Z)} \left\{ \sum_{\gamma \in M(\mathbf{Q}) \cap \mathfrak{o}} \sum_{\eta \in N(\mathbf{Q},\gamma_s)\backslash N(\mathbf{Q})} \int_{N(\mathbf{A},\gamma_s)} f(x^{-1}\eta^{-1}\gamma n\eta x)dn \right\}. \tag{8.3}$$

As in the proof of Theorem 7.1 we observe that for T sufficiently regular the sum over γ may be taken over $P_1(\mathbf{Q}) \cap M(\mathbf{Q}) \cap \mathfrak{o}$. It follows from Lemma 2.1 and Poisson summation that the expression in the brackets in (8.3) equals the sum over γ in $M_1(\mathbf{Q}) \cap \mathfrak{o}$ of

$$\sum_{\nu \in N_1^P(\mathbf{Q})} \sum_{\eta \in N(\mathbf{Q},(\gamma\nu)_s)\backslash N(\mathbf{Q})} \int_{N(\mathbf{A},(\gamma\nu)_s)} f(x^{-1}\eta^{-1}\zeta\nu n\eta x)dn$$

$$= \sum_{\delta \in N_1^P(\mathbf{Q},\gamma_s)\backslash N_1^P(\mathbf{Q})} \sum_{\nu \in N_1^P(\mathbf{Q},\gamma_s)} \sum_{\eta \in N(\mathbf{Q},\delta^{-1}\gamma_s\delta)\backslash N(\mathbf{Q})} \int_{N(\mathbf{A},\delta^{-1}\gamma_s\delta)} f(x^{-1}\eta^{-1}\delta^{-1}\gamma\nu\delta n\eta x)dn$$

$$= \sum_{\eta \in N_1(\mathbf{Q},\gamma_s)\backslash N_1(\mathbf{Q})} \sum_{\nu \in N_1^P(\mathbf{Q},\zeta_s)} \int_{N(\mathbf{A},\zeta_s)} f(x^{-1}\eta^{-1}\gamma\nu n\eta x)dn$$

$$= \sum_{\eta \in N_1(\mathbf{Q},\gamma_s)\backslash N_1(\mathbf{Q})} \int_{\zeta \in \mathfrak{n}_1^P(\mathbf{Q},\gamma_s)} \int_{\mathfrak{n}_1(\mathbf{A},\gamma_s)} f(x^{-1}\eta^{-1} \cdot \gamma e(X) \cdot \eta x)\psi(\langle X, \zeta\rangle)dX.$$

Let $\mathfrak{n}_1^2(\mathbf{Q},\gamma_s)'$ be the set, possibly empty, of elements in $\mathfrak{n}_1^2(\mathbf{Q},\gamma_s)$ which do not belong to any $\mathfrak{n}_1^P(\mathbf{Q}, \gamma_s)$, with $P_1 \subset P \subsetneq P_2$. Then (8.1) is bounded by the sum over $\{P_1, P_2 : P_1 \subset P_2\}$ of the integral over x in $M_1(\mathbf{Q})N_1(\mathbf{A})\backslash G(\mathbf{A})^1$ of

$$F^1(x, T)\sigma_1^2(H_0(x) - T) \cdot \sum_{\gamma \in M_1(\mathbf{Q})} \int_{N_1(\mathbf{Q},\gamma_s)\backslash N_1(\mathbf{A})} dn \cdot \sum_{\zeta \in \mathfrak{n}_1^2(\mathbf{Q},\gamma_s)'}$$

$$\left| \int_{\mathfrak{n}_1(\mathbf{A},\gamma_s)} f(x^{-1}n^{-1}\gamma e(X)nx)\psi(\langle X, \zeta\rangle)dX \right|.$$

The integrand, as a function of n, is left $N_2(\mathbf{A}, \gamma_s)$ invariant. We can therefore write $n = n_2n_1$ where n_2 ranges over a relatively compact fundamental set for $N_1^2(\mathbf{Q}, \gamma_s)$ in $N_1^2(\mathbf{A}, \gamma_s)$ and n_1 belongs to $N_1(\mathbf{A}, \gamma_s)\backslash N_1(\mathbf{A})$. For any γ the integrand will vanish for n_1 outside a compact set. Next, set

$$x = mak,$$

where $k \in K$, $a \in A_1(\mathbf{R})^0 \cap G(\mathbf{A})^1$ and m lies in a fixed fundamental domain in $M_1(\mathbf{A})^1$. Note that a normalizes $N_1(\mathbf{A}, \gamma_s)\backslash N_1(\mathbf{A})$. If the integrand does not vanish m and $a^{-1}n_2a$ will both remain in fixed compact sets. Moreover, the sum over γ will be finite. It follows that (8.1) is bounded by a constant multiple of the quantity obtained by taking the sum over P_1, P_2 and γ, the supremum as y ranges over a fixed compact subset of $G(\mathbf{A})^1$ and the integral over $a \in A_1(\mathbf{R})^0 \cap G(\mathbf{A})^1$ of the expression

$$\sigma_1^2(H_0(a) - T) \cdot \sum_{\zeta \in \mathfrak{n}_1^2(\mathbf{Q},\gamma_s)'} \left| \int_{\mathfrak{n}_1(\mathbf{A},\gamma_s)} f(y^{-1}\gamma e(X)y)\psi(\langle X, Ad(a)\zeta\rangle)dX \right|$$

The finitude of (8.1) now follows from the arguments of §7.

Fix $\mathfrak{o}$. The integral of $j_{\mathfrak{o}}^T(x, f)$ is the sum over P_1 and P_2 of the integral over $P_1(\mathbf{Q})\backslash G(\mathbf{A})^1$ of the product of (8.2) and (8.3). Decompose the integral over $P_1(\mathbf{Q})\backslash G(\mathbf{A})^1$ into a double integral over $M_1(\mathbf{Q})N_1(\mathbf{A})\backslash G(\mathbf{A})^1$ and $N_1(\mathbf{Q})\backslash N_1(\mathbf{A})$. Then take the integral over $N_1(\mathbf{Q})\backslash N_1(\mathbf{A})$ inside the sum over P and γ in (8.3). The summand is then

$$\int_{N_1(\mathbf{Q})\backslash N_1(\mathbf{A})} dn_1 \cdot \sum_{\eta \in N(\mathbf{Q},\gamma_s)\backslash N(\mathbf{Q})} \int_{N(\mathbf{A},\gamma_s)} dn \cdot f(x^{-1}n_1^{-1}\eta^{-1} \cdot \gamma n \cdot \eta n_1 x)$$

$$= \int_{N_1(\mathbf{Q})\backslash N_1(\mathbf{A})} dn_1 \cdot \int_{N(\mathbf{Q})\backslash N(\mathbf{A})} dn_2$$

$$\cdot \sum_{\eta \in N(\mathbf{Q},\gamma_s)\backslash N(\mathbf{Q})N(\mathbf{A},\gamma_s)} dn \cdot f(x^{-1}n_1^{-1}n_2^{-1}\eta^{-1} \cdot \gamma n \cdot \eta n_2 n_1 x)$$

$$= \int_{N_1(\mathbf{Q})\backslash N_1(\mathbf{A})} dn_1 \cdot \int_{N(A,\gamma_s)\backslash N(\mathbf{A})} dn_2 \cdot \int_{N(\mathbf{A},\gamma_s)} dn \cdot f(x^{-1}n_1^{-1} \cdot n_2^{-1}\gamma n n_2 \cdot n_1 x)$$

$$= \int_{N_1(\mathbf{Q})\backslash N_1(\mathbf{A})} dn_1 \cdot \int_{N(\mathbf{A})} dn \cdot f(x^{-1}n_1^{-1} \cdot \gamma n \cdot n_1 x),$$

by Lemma 2.2. The integral over $N_1(\mathbf{Q})\backslash N_1(\mathbf{A})$ can now be taken back outside the sum over γ and P, and recombined with the integral over $M_1(\mathbf{Q})N_1(\mathbf{A})\backslash G(\mathbf{A})^1$. We must of course remember that (8.2) is a left $N_1(\mathbf{A})$-invariant function of x. We end up with the sum over P_1 and P_2 of the integral over $P_1(\mathbf{Q})\backslash G(\mathbf{A})^1$ of the product of (7.3) and (7.4). This is just the integral of $k_{\mathfrak{o}}^T(x,f)$. The theorem is proved. □

Let $\mathfrak{o}$ be a fixed equivalence class. Choose a semisimple element $\gamma_1 \in \mathfrak{o}$ and groups $P_1 \subset P_2$ as in §2. Assume that $P_1 = P_2$, so that $\mathfrak{o}$ consists entirely of semisimple elements. Any element in $G(\gamma_1)$ normalizes A_1, since it is the split component of the center of $G(\gamma_1)^0$. We obtain a map from $G(\gamma_1)$ to $\Omega(\mathfrak{a}_1, \mathfrak{a}_1)$, whose kernel is $G(\gamma_1) \cap M_1$. We shall say that the class $\mathfrak{o}$ is *unramified* if the map is trivial; in other words if $G(\gamma_1)$ is contained in M_1. It is clear that $\mathfrak{o}$ is unramified if and only if the only s in $\Omega(\mathfrak{a}_1, \mathfrak{a}_1)$ for which $w_s\gamma_1 w_s^{-1}$ is $M_1(\mathbf{Q})$-conjugate to γ_1 is the identity.

Let $\mathfrak{o}$ be an unramified class in $\mathcal{O}$, and let γ_1 and P_1 be as above. Since it consists entirely of semi-simple elements, $\mathfrak{o}$ is an actual conjugacy class in $G(\mathbf{Q})$. Suppose that P is a parabolic subgroup and that γ belongs to $M(\mathbf{Q}) \cap \mathfrak{o}$. Then there is a parabolic subgroup $P_2 \subset P$, and an element $\gamma_2 \in M_2(\mathbf{Q})$, which is $M(\mathbf{Q})$-conjugate to γ, such that the split component of the center of $G(\gamma_2)^0$ is A_2. Any element in $G(\mathbf{Q})$ which conjugates γ_1 to γ_2 will conjugate A_1 to A_2. It follows that for some $s \in \Omega(\mathfrak{a}_1, \mathfrak{a}_2)$, and $\eta \in M(\mathbf{Q})$,

$$\gamma = \eta w_s \gamma_1 w_s^{-1} \eta^{-1}. \tag{8.4}$$

Suppose that for a parabolic $P_3 \subset P$, and elements $s' \in \Omega(\mathfrak{a}_1, \mathfrak{a}_3)$ and $\eta' \in M(\mathbf{Q})$,

$$\gamma = \eta' w_{s'} \gamma_1 w_{s'}^{-1} (\eta')^{-1}.$$

Then there is an element $\zeta \in G(\mathbf{Q}, \gamma)$ such that

$$\eta' w_{s'} = \zeta \eta w_s.$$

Since $G(\gamma) \subset M$,

$$w_{s'} = \xi w_s,$$

for some element $\xi \in M(\mathbf{Q})$. Let $\Omega(\mathfrak{a}_1; P)$ be the set of elements s in the union over $\mathfrak{a}_2$ of the sets $\Omega(\mathfrak{a}_1, \mathfrak{a}_2)$ such that $s\mathfrak{a}_1 = \mathfrak{a}_2$ contains $\mathfrak{a}$ and $s^{-1}\alpha$ is positive for each $\alpha \in \Delta_2^P$. Then given γ and P, there is a unique s in $\Omega(\mathfrak{a}_1; P)$ such that (8.4) holds for some $\eta \in M(\mathbf{Q})$.

It follows from this discussion that $J_{P,\mathfrak{o}}(y, y)$ equals

$$\sum_{s \in \Omega(\mathfrak{a}_1;P)} \sum_{\eta \in M(\mathbf{Q}, w_s\gamma_1 w_s^{-1})\backslash M(\mathbf{Q})} \sum_{\zeta \in N(\mathbf{Q})} f(y^{-1}\zeta^{-1}\eta^{-1} w_s \gamma_1 w_s^{-1} \eta \zeta y)$$

$$= \sum_{s} \sum_{\zeta \in M(\mathbf{Q}, w_s\gamma_1 w_s^{-1})\backslash P(\mathbf{Q})} f(y^{-1}\zeta^{-1} w_s \gamma_1 w_s^{-1} \zeta y).$$

Therefore $j_{\mathfrak{o}}^T(x, f)$ equals

$$\sum_P (-1)^{\dim(A/Z)} \sum_{s \in \Omega(\mathfrak{a}_1;P)} \sum_{\delta \in M(\mathbf{Q}, w_s\gamma_1 w_s^{-1})\backslash G(\mathbf{Q})} f(x^{-1}\delta^{-1} w_s \gamma_1 w_s^{-1} \delta x) \hat{\tau}_P(H(\delta x) - T).$$

Since the centralizer of $w_s\gamma_1 w_s^{-1}$ in G is contained in M, this equals the sum over $\delta \in G(\mathbf{Q}, \gamma_1)\backslash G(\mathbf{Q})$ of the product of

$$f(x^{-1}\delta^{-1}\gamma_1 \delta x)$$

and

$$\sum_P (-1)^{\dim(A/Z)} \sum_{s \in \Omega(\mathfrak{a}_1;P)} \hat{\tau}_P(H_0(w_s \delta x) - T). \tag{8.5}$$

Suppose that λ is a point in $\mathfrak{a}_0^*$ such that $\lambda(\alpha^\vee) > 0$ for each $\alpha \in \Delta_0$. We shall show that (8.5) equals

$$\sum_{P_2} \sum_{s \in \Omega(\mathfrak{a}_1,\mathfrak{a}_2)} \epsilon_2(s\lambda)\phi_2(s\lambda,\ H_0(w_s\delta x) - T), \tag{8.6}$$

where we have written ϵ_2 and ϕ_2 for the functions denoted by $\epsilon^G_{P_2}$ and $\phi^G_{P_2}$ in §6. To see this, write (8.5) as the sum over P_2, over $s \in \Omega(\mathfrak{a}_1, \mathfrak{a}_2)$ and over those $P \supset P_2$ such that $s^{-1}\alpha > 0$ for all $\alpha \in \Delta^P_2$, of

$$(-1)^{\dim(A/Z)}\,\hat{\tau}_P\,(H_0(w_s\delta x) - T).$$

For a given s, define $P_s \supset P$ by

$$\Delta^{P_s}_{P_2} = \{\alpha \in \Delta_{P_2} : (s\lambda)(\alpha^\vee) > 0\}.$$

Then the sum over P is just the sum over $\{P : P_2 \subset P \subset P_s\}$. Since it is an alternating sum of characteristic functions, we can apply Proposition 1.1. The sum over P will vanish, for a given s, unless precisely one summand is nonzero. We have shown that for all $H \in \mathfrak{a}_0$,

$$\sum_{\{P \supset P_2 : s^{-1}\alpha > 0, \alpha \in \Delta^P_2\}} (-1)^{\dim(A/Z)}\,\hat{\tau}_P(H)$$

equals the product of $(-1)^{\dim(A_s/Z)}$ with the characteristic function of

$$\{H \in \mathfrak{a}_2 : \varpi(H) > 0,\ \varpi \in \hat{\Delta}_s;\ \varpi(H) \leq 0,\ \varpi \in \hat{\Delta}_2 \setminus \hat{\Delta}_s\}.$$

This is just the function

$$\epsilon_2(s\lambda)\phi_2(s\lambda,\ H).$$

We have shown that (8.5) equals (8.6).

After substituting (8.6) for (8.5) in the expression for $j^T_{\mathfrak{o}}(x, f)$, we must integrate over $G(\mathbf{Q})\backslash G(\mathbf{A})^1$. Since the integrand is left $Z(\mathbf{R})^0$-invariant, we can integrate over $Z(\mathbf{R})^0G(\mathbf{Q})\backslash G(\mathbf{A})$ instead. We could then combine the integral over x and the sum over δ if we were able to prove the resulting integral absolutely convergent. But the resulting integral can be written as

$$\operatorname{vol}(A_1(\mathbf{R})^0G(\mathbf{Q},\ \gamma_1)\backslash G(\mathbf{A},\ \gamma_1)) \int_{G(\mathbf{A},\ \gamma_1)\backslash G(\mathbf{A})} f(x^{-1}\gamma_1 x)v(x,\ T)dx, \tag{8.7}$$

where

$$v(x,\ T) = \int_{Z(\mathbf{R})^0\backslash A_1(\mathbf{R})^0} \Big(\sum_{P_2} \sum_{s \in \Omega(\mathfrak{a}_1,\ \mathfrak{a}_2)} \epsilon_2(s\lambda)\phi_2(s\lambda,\ H_0(w_s a x) - T)\Big)da.$$

The integral over x can be taken over a compact set. By [1(b), Corollary 3.3], the integral over a can also be taken over a compact set. It follows that $J^T_{\mathfrak{o}}(f)$ equals (8.7). We have expressed $J^T_{\mathfrak{o}}(f)$ as a weighted orbital integral of f whenever $\mathfrak{o}$ is unramified. We note that $v(x, T)$, the weight factor, equals the volume of the convex hull of the projection of

$$\{s^{-1}T - s^{-1}H_0(w_s x) : s \in \bigcup_{P_2} \Omega(\mathfrak{a}_1,\ \mathfrak{a}_2)\}$$

onto $\mathfrak{a}_1/\mathfrak{z}$, [1(b), Corollary 3.5].

REFERENCES

1. J. ARTHUR
   (a) *The Selberg trace formula for groups of F-rank one*, Ann. of Math. **100**(1974), 326-385.
   (b) *The characters of discrete series as orbital integrals*, Inv. Math. **32**(1976), 205-261.
   (c) "Eisenstein series and the trace formula," to appear in *Automorphic Forms, Representations and L-functions*, Amer. Math. Soc.
2. M. DUFLO AND J. P. LABESSE, *Sur la formule des traces de Selberg*, Ann. Sci. Ecole Norm. Sup. **4**(1971), 193-284.
3. HARISH-CHANDRA, *Automorphic forms on semisimple Lie groups*, Springer-Verlag, 1968.
4. L. HÖRMANDER, *Linear Partial Differential Operators*, Springer-Verlag, 1963.
5. H. JACQUET AND R. LANGLANDS, *Automorphic Forms on GL*(2), Springer-Verlag, 1970.
6. R. LANGLANDS
   (a) "Eisenstein series" in *Algebraic Groups and Discontinuous Subgroups*, Amer. Math. Soc., 1966.
   (b) *On the Functional Equations Satisfied by Eisenstein series*, Springer-Verlag, 1976.
   (c) *Base charge for* GL_2, mimeographed notes.
7. J. P. LABESSE AND R. LANGLANDS, *L-indistinguishability for* SL_2, preprint.
8. A. SELBERG
   (a) *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, J. Indian Math. Soc. **20**(1956), 47-87.
   (b) *Discontinuous groups and harmonic analysis*, Proc. Int. Cong. Math. 1962, 177-189.

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