# Towards a Stable Trace Formula

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Abstract. The paper is a report on the problem of stabilizing the trace formula. The goal is the construction and analysis of a stable trace formula that can be used to compare automorphic representations on different groups.

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1. It is an important problem to place the automorphic representation theory of classical groups on an equal footing with that of $GL(n)$. Thirty years after the study of $GL(2)$ by Jacquet-Langlands [12], the theory for $GL(n)$ is now in pretty good shape. It includes an understanding of the relevant $L$-functions [13], a classification of the discrete spectrum [21] and cyclic base change [10]. One would like to establish similar things for orthogonal, symplectic and unitary groups. A satisfactory solution would have many applications to number theory, the extent of which is hard to even guess at present.

A general strategy has been known for some time. One would like to compare trace formulas for classical groups with a twisted trace formula for $GL(n)$. There is now a trace formula that applies to any group [4]. However, it contains terms that are complicated, and are hard to compare with similar terms for other groups. The general comparison problem has first to be formulated more precisely, as that of stabilizing the trace formula [18]. In this form, the problem is to construct a stable trace formula, a refined trace formula whose individual terms are stable distributions. It includes also the further analysis required to establish identities between terms in the original trace formula and their stable counterparts on other groups. This would allow a cancellation of all the geometric and residual terms from the relevant trace formulas, leaving only terms that describe automorphic spectra. The resulting identity given by these remaining terms would lead to reciprocity laws for automorphic spectra on different groups. In the case of classical groups, such identities would provide the means for attacking the original classification problem.

The purpose of this report is to discuss the construction and deeper analysis of a stable trace formula. I can say nothing about the fundamental lemma (or

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its analogue for weighted orbital integrals), which is one of the key problems to be solved. The reader can consult [11] and [25] for some special cases that have been resolved. Furthermore, I shall stick to the ordinary trace formula, since the twisted trace formula presents extra difficulties [16]. With these caveats, I believe that the general problem has been essentially solved. Since there are still a number of things to be written out, I shall organize the report conservatively as a series of stabilization problems for the various constituents of the trace formula. The solutions, all being well, will appear in the papers [8] and [9].

2. Let $G$ be a connected, reductive algebraic group over a number field $F$. To simplify the discussion, we shall actually assume that $G$ is semisimple and simply connected. If $V$ is a finite set of valuations of $F$, $\mathcal{H}\big(G(F_V)\big)$ will denote the Hecke algebra of functions on $G(F_V)$, the product over $v \in V$ of the groups $G(F_v)$. We shall usually take $V$ to be a large finite set outside of which $G$ is unramified. A function in $\mathcal{H}\big(G(F_V)\big)$ can then be identified with the function on the adèle group $G(\mathbb{A})$ obtained by taking its product with the characteristic function of a maximal compact subgroup $K^V$ of $G(\mathbb{A}^V)$. The trace formula is to be regarded as two different expansions of a certain linear form $I$ on $\mathcal{H}\big(G(F_V)\big)$.

The first expansion

$$I(f) = \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma(M,V)} a^M(\gamma) I_M(\gamma, f) \tag{1}$$

is in terms of geometric data. As usual, $\mathcal{L} = \mathcal{L}^G$ denotes the set of Levi subgroups (over $F$) that contain a fixed minimal one, and $W_0^G$ is the restricted Weyl group of $G$. For any $M \in \mathcal{L}$, $\Gamma(M, V)$ is a set of conjugacy classes in $M(F_V)$. The coefficient $a^M(\gamma)$ depends only on $M$, and is really a global object. It is constructed from rational conjugacy classes in $M(F)$ that project onto $\gamma$, and are integral outside of $V$. The linear form $I_M(\gamma, f)$ on the other hand is a local object. It is an invariant distribution constructed from the weighted orbital integral of $f$ over the induced conjugacy class of $\gamma$ in $G(F_V)$.

The second expansion

$$I(f) = \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi(M,V)} a^M(\pi) I_M(\pi, f) d\pi \tag{2}$$

is in terms of spectral data, and is entirely parallel to the first one. For any $M \in \mathcal{L}$, $\Pi(M, V)$ is a certain set of equivalence classes of irreducible unitary representations of $M(F_V)$, equipped with a natural measure $d\pi$. The coefficient $a^M(\pi)$ is again a global object that depends only on $M$. It is constructed from automorphic representations of $M(\mathbb{A})$ that project onto $\pi$, and are integral outside of $V$. Similarly, the linear form $I_M(\pi, f)$ is a local object. It is an invariant distribution obtained from residues of weighted characters of $f$ at unramified twists $\pi_\lambda$ of $\pi$. The integral over $\Pi(M, V)$ is actually only known to be conditionally convergent. However, this is sufficient for present purposes, and in any case, could probably be strengthened with the results of Müller [22].

The trace formula is thus the identity obtained by equating the right hand sides of (1) and (2). It is perhaps difficult for a general reader to get a feeling for the situation, since we have not defined the various terms precisely. We would simply like to stress the general structure of the two expansions, and to note that it is the term with $M = G$ in the second expansion that contains the basic information on the automorphic discrete spectrum. For example, if $G$ is anisotropic, this term is just the trace of the right convolution of $f$ on $L^2\big(G(F)\backslash G(\mathbb{A})/K^V\big)$. The term is more complicated for general $G$, but it includes a discrete part

$$I_{\mathrm{disc}}(f) \;=\; \sum_{\pi\in\Pi_{\mathrm{disc}}(G,V)} a^G_{\mathrm{disc}}(\pi) f_G(\pi)\,, \tag{3}$$

that comes from the discrete spectrum of $L^2\big(G(F)\backslash G(\mathbb{A})/K^V\big)$ as well as induced discrete spectra of proper Levi subgroups [4, (4.3), (4.4)]. The ultimate goal for the trace formula is to deduce information about the multiplicities $a^G_{\mathrm{disc}}(\pi)$. In particular, the other terms — those with $M \neq G$ in the spectral expansion and those with any $M$ in the geometric expansion — are to be regarded as objects one would analyze in some fashion to gain information about the discrete part $I_{\mathrm{disc}}(f)$ of the first term.

We have actually reformulated somewhat the trace formula from [4]. The invariant distributions $I_M(\gamma, f)$ and $I_M(\pi, f)$ here are defined in terms of the weighted characters of [6], and are independent of the choice of normalizing factors for intertwining operators implicit in [4]. On the geometric side, this modification has the effect of including values of weighted orbital integrals of the characteristic function of $K^V \cap M(\mathbb{A}^V)$ in the global coefficients of [2, (8.1)]. On the spectral side, the effect is to replace the complete automorphic $L$-functions in the global coefficients of [4, §4] with partial, unramified $L$-functions.

3. It is hard to extract arithmetic information from the trace formula for $G$ by studying it in isolation. One should try instead to compare it with trace formulas for certain other groups. The groups in question are the *endoscopic* groups for $G$, a family of quasisplit groups over $F$ attached to $G$ that includes the quasisplit inner form of $G$. One actually has to work with endoscopic data, which are endoscopic groups with extra structure [18], [19]. We write $\mathcal{E}_{\mathrm{ell}}(G, V)$ for the set of isomorphism classes of elliptic endoscopic data for $G$ over $F$ that are unramified outside of $V$.

Suppose that $G' \in \mathcal{E}_{\mathrm{ell}}(G, V)$ and that $v$ belongs to $V$. In [19], Langlands and Shelstad define a map from functions $f_v \in \mathcal{H}\big(G(F_v)\big)$ to functions $f'_v = f_v^{G'}$ on the strongly $G$-regular stable conjugacy classes $\delta'_v$ of $G'(F_v)$. We recall that stable conjugacy is the equivalence relation on the strongly regular elements in $G'(F_v)$ defined by conjugacy over an algebraic closure of $F_v$. The map is defined by

$$f'_v(\delta'_v) = \sum_{\gamma v} \Delta_G(\delta'_v, \gamma_v) f_{v,G}(\gamma_v),$$

where $\gamma_v$ ranges over the ordinary conjugacy classes in $G(F_v)$, $\Delta_G(\delta'_v, \gamma_v)$ is the transfer factor defined in [19], and $f_{v,G}(\gamma_v) = J_G(\gamma_v, f_v)$ is the invariant orbital integral of $f_v$ over the conjugacy class $\gamma_v$.

The Langlands-Shelstad transfer conjecture asserts that for any $f_v$, there is a function $h_v \in \mathcal{H}\big(G'(F_v)\big)$, not necessarily unique, whose stable orbital integral at any $\delta'_v$ equals $f'_v(\delta'_v)$. The fundamental lemma is a supplementary conjecture. It asserts that if $G$ and $G'$ are unramified at $v$, and $f_v$ is the characteristic function of a hyperspecial maximal compact subgroup of $G(F_v)$, then $h_v$ can be chosen to be the characteristic function of a hyperspecial maximal compact subgroup of $G'(F_v)$. Waldspurger [26] has shown, roughly speaking, that the fundamental lemma implies the transfer conjecture. We shall assume from now on that they both hold. A linear form $S' = S^{G'}$ on $\mathcal{H}\big(G'(F_V)\big)$ is said to be *stable* if its value at any $h \in \mathcal{H}\big(G'(F_V)\big)$ depends only on the stable orbital integrals of $h$. If this is so, there is a linear form $\widehat{S}'$ on the space of stable orbital integrals of functions in $\mathcal{H}\big(G'(F_V)\big)$ such that $S'(h)$ equals $\widehat{S}'(h')$. In particular, we obtain a linear form $f \to \widehat{S}'(f')$ in $f \in \mathcal{H}\big(G(F_V)\big)$.

We can now begin to describe the basic problem. The ultimate goal would be to stabilize the distribution $I_{\mathrm{disc}}$ in (3).

Problem 1: *Construct a stable linear form $S^G_{\mathrm{disc}}$ on $\mathcal{H}\big(G(F_V)\big)$, for $G$ quasisplit over $F$, such that for any $G$ at all, $I_{\mathrm{disc}}(f)$ equals the endoscopic expression*

$$I^{\mathcal{E}}_{\mathrm{disc}}(f) = \sum_{G' \in \mathcal{E}_{\mathrm{ell}}(G,V)} \iota(G,G')\widehat{S}'_{\mathrm{disc}}(f'), \qquad f \in \mathcal{H}\big(G(F_V)\big).$$

*Here $\iota(G,G')$ is a coefficient, introduced by Langlands [18], that can be defined by the formula of [14, Theorem 8.3.1].*

The problem has a general structure that is common to many stabilization questions. If we take $G$ to be quasisplit, the required formula amounts to an inductive definition of $S^G_{\mathrm{disc}}$. Since $G$ belongs to $\mathcal{E}_{\mathrm{ell}}(G,V)$ in this case, and is the endoscopic group of greatest dimension, we can assume inductively that the linear form $S' = S^{G'}$ is defined and stable for any $G' \in \mathcal{E}_{\mathrm{ell}}(G,V)$ not equal to $G$. We can therefore set

$$S^G_{\mathrm{disc}}(f) = I_{\mathrm{disc}}(f) - \sum_{G' \neq G} \iota(G,G')\widehat{S}'_{\mathrm{disc}}(f').$$

The problem then has two parts. If $G$ is quasisplit, one has to show that $S^G_{\mathrm{disc}}$ is stable. This is needed to complete the inductive definition. If $G$ is not quasisplit, the summands in the expression $I^{\mathcal{E}}_{\mathrm{disc}}(f)$ are all defined inductively in terms of groups $G'$ distinct from $G$. In this case, it is the identity itself that has to be proved.

The problem has been solved completely only for $SL(2)$ and $U(3)$ (and related groups) [17], [23]. A general solution of Problem 1 would be a milestone. It would relate fundamental global data on different groups by means of a transfer map $f \to f'$ defined in purely local terms. The resulting information would be particularly powerful if it could be combined with a property of strong multiplicity one, either for individual representations, or for packets of representations. For example, a twisted form of the identity in Problem 1 would relate many classical groups $G$ to $GL(n)$. Together with the identity for $G$ itself, this would provide a powerful tool for dealing with the classification problem discussed earlier.

However, Problem 1 is unlikely to be solved directly. The strategy should be to consider similar problems for the various other terms in the trace formula. Towards this end, we first pose a parallel problem for the entire geometric expansion.

Problem 2: *Construct a stable linear form $S^G$ on $\mathcal{H}\big(G(F_V)\big)$, for $G$ quasisplit over $F$, such that for any $G$, $I(f)$ equals the endoscopic expression*

$$I^{\mathcal{E}}(f) \;=\; \sum_{G' \in \mathcal{E}_{\mathrm{ell}}(G,V)} \iota(G,G')\widehat{S}'(f'), \qquad\qquad f \in \mathcal{H}\big(G(F_V)\big).$$

4. To deal with Problem 2, we would have to set up a series of stabilization problems for the various terms in the geometric expansion (1). Such problems have been solved for some terms in [18] and [15].

If $v$ is a place of $F$, we write $\Gamma(G_v)$ for the set of conjugacy classes in $G(F_v)$. Assuming that each such class has been equipped with an invariant measure, we identify $\Gamma(G_v)$ with a set of invariant distributions on $G(F_v)$. In the case of archimedean $v$, examples of Assem [1, §1.10] suggest that elements in $\Gamma(G_v)$ do not always behave well under endoscopic transfer. We are forced to consider a larger family of distributions. Let us define $\mathcal{D}(G_v)$ to be the space spanned by invariant distributions on $G(F_v)$ of the form

$$\int_{G_c(F)\backslash G(F)} I_c(f_v^x)dx, \qquad\qquad f_v \in C_c^\infty\big(G(F_v)\big),$$

where $c$ is a semisimple element in $G(F_v)$, $G_c$ is the centralizer of $c$ in $G$, $I_c$ is an invariant distribution on $G_c(F_v)$ that is supported on the unipotent set, and $f_v^x(y) = f_v(x^{-1}cyx)$, for $y \in G_c(F_v)$. We then let $\Gamma_+(G_v)$ be a fixed basis of $\mathcal{D}(G_v)$ that contains $\Gamma(G_v)$. If $v$ is $p$-adic, $\Gamma(G_v)$ actually equals $\Gamma_+(G_v)$, but $\Gamma(G_v)$ is a proper subset of $\Gamma_+(G_v)$ if $G_v$ is archimedean. We also fix a basis $\Sigma_+(G'_v)$ of the stable distributions in $\mathcal{D}(G'_v)$, for each endoscopic datum $G'_v$ of $G$ over $F_v$. Among various compatability conditions, we assume that $\Sigma_+(G'_v)$ contains the set of stable strongly $G$-regular orbital integrals on $G'(F_v)$. Extending the earlier notation, we write $f'_v(\delta'_v)$ for the pairing obtained from elements $f_v \in \mathcal{H}(G_v)$ and $\delta'_v \in \Sigma_+(G'_v)$. Then we can write $f'_v(\delta'_v)$ as a finite linear combination of distributions $f_{v,G}(\gamma_v)$ in $\Gamma_+(G_v)$, with coefficients $\Delta(\delta'_v, \gamma_v)$ that reduce to the Langlands-Shelstad transfer factors in the special case that $\delta'_v$ is strongly $G$-regular.

If $V$ is a finite set of valuations as before, we define $\Gamma(G_V)$, $\Gamma_+(G_V)$ etc., by the appropriate products. Thus, if $M'_V = \prod M'_v$ is a product of local endoscopic data for a Levi subgroup $M$ of $G$, and $\delta' = \prod \delta'_v$ belongs to $\Sigma_+(M'_V)$, $\Delta_M(\delta', \gamma)$ equals the product over $v \in V$ of the factors $\Delta_M(\delta'_v, \gamma_v)$, for each $\gamma = \prod \gamma_v$ in $\Gamma_+(M_V)$. We shall take $M'_V$ to be the image of a global endoscopic datum $M' \in \mathcal{E}_{\mathrm{ell}}(M,V)$ in what follows.

Consider first the local terms $I_M(\gamma, f)$ in the geometric expansion. They are defined at this point only for $\gamma \in \Gamma(M_V)$. However, we shall assume that we can construct $I_M(\gamma, f)$ for any $\gamma$ in the larger set $\Gamma_+(M_V)$, by some variant of the techniques in [3, §3-5].

Problem 3: *Construct stable linear forms $S^G_M(\delta)$ on $\mathcal{H}\big(G(F_V)\big)$, for $G$ quasisplit over $F$ and $\delta \in \Sigma_+(M_V)$, such that for any $G$, $M$, $M'$ and $\delta'$, the linear form*

$$I_M(\delta', f) = \sum_{\gamma \in \Gamma_+(M_V)} \Delta_M(\delta', \gamma) I_M(\gamma, f)$$

*equals the endoscopic expression*

$$I^{\mathcal{E}}_M(\delta', f) = \sum_{G' \in \mathcal{E}_{M'}(G)} \iota_{M'}(G, G') \widehat{S}^{G'}_{M'}(\delta', f').$$

*Here, $\mathcal{E}_{M'}(G)$ is a set of global endoscopic data for $G$ and $\iota_{M'}(G, G')$ is a simple coefficient, both defined as in [7, §3].*

Consider now the global coefficients $a^M(\gamma)$. We define $a^M$ on the larger set $\Gamma_+(M_V)$ by setting $a^M(\gamma) = 0$ for any $\gamma$ in the complement of $\Gamma(M, V)$ in $\Gamma_+(M_V)$.

Problem 4: *Construct coefficients $b^M(\delta)$, for $M$ quasisplit over $F$ and $\delta \in \Sigma_+(M_V)$, such that for any $M$ and $\gamma$, $a^M(\gamma)$ equals the endoscopic coefficient*

$$a^{M,\mathcal{E}}(\gamma) = \sum_{M' \in \mathcal{E}_{\mathrm{ell}}(M,V)} \sum_{\delta' \in \Sigma_+(M'_V)} \iota(M, M') b^{M'}(\delta') \Delta_M(\delta', \gamma).$$

We now sketch how to solve Problem 2 in terms of Problems 3 and 4. If $G$ is quasisplit, let us define

$$S^G(f) = \sum_{M \in \mathcal{L}} |W^M_0||W^G_0|^{-1} \sum_{\delta \in \Sigma_+(M_V)} b^M(\delta) S^G_M(\delta, f).$$

According to Problem 3, this is a stable linear form on $\mathcal{H}\big(G(F_V)\big)$, and so satisfies the requirement of Problem 2. It remains to show that with this definition, the endoscopic identity of Problem 2 holds.

Suppose that $G$ is arbitrary. The endoscopic expression of Problem 2 equals

$$I^{\mathcal{E}}(f) = \sum_{G' \in \mathcal{E}_{\mathrm{ell}}(G,V)} \iota(G, G') \sum_{R' \in \mathcal{L}^{G'}} |W^{R'}_0||W^{G'}_0|^{-1} S_{R'}(G'),$$

where $S_{R'}(G')$ is the sum over $\sigma' \in \Sigma_+(R'_V)$ of $b^{R'}(\sigma')\widehat{S}^{G'}_{R'}(\sigma', f')$. By a variant of [7, Lemma 9.2], this in turn equals

$$\begin{aligned} &\sum_{R \in \mathcal{L}^{G^*}} |W^R_0||W^{G^*}_0|^{-1} \sum_{R' \in \mathcal{E}_{\mathrm{ell}}(R,V)} \iota(R, R') \sum_{G' \in \mathcal{E}_{R'}(G^*)} \iota_{R'}(G^*, G') S_{R'}(G^*) \\ &= \sum_R |W^R_0||W^{G^*}_0|^{-1} \sum_{R'} \iota(R, R') \sum_{\sigma' \in \Sigma(R'_V)} b^{R'}(\sigma') I^{\mathcal{E}}_R(\sigma', f), \end{aligned}$$

where $G^*$ is a quasisplit inner form of $G$, and $I_R^{\mathcal{E}}(\sigma', f)$ is defined as in Problem 3, but with $(G, M, M', \delta')$ replaced by $(G^*, R, R', \sigma')$. A global analogue of the vanishing property [7, Theorem 8.3] asserts that $I_R^{\mathcal{E}}(\sigma', f)$ vanishes unless $R$ comes from $G$. If we identify $\mathcal{L}$ with a subset of $\mathcal{L}^{G^*}$, this means that $I_R^{\mathcal{E}}(\sigma', f)$ vanishes unless $R$ is $W_0^{G^*}$-conjugate to a group $M \in \mathcal{L}$. In case $R$ is conjugate to $M$, there are elements $M' \in \mathcal{E}_{\mathrm{ell}}(M, V)$ and $\delta' \in \Sigma_+(M'_V)$ such that $I_R^{\mathcal{E}}(\sigma', f)$ equals the endoscopic expression $I_M^{\mathcal{E}}(\delta', f)$ of Problem 3. Since we also have $b^{R'}(\sigma') = b^{M'}(\delta')$ and $\iota(R, R') = \iota(M, M')$ in this case, the expression for $I^{\mathcal{E}}(f)$ can be written

$$\sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{M'\in\mathcal{E}_{\mathrm{ell}}(M,V)} \iota(M, M') \sum_{\delta'\in\Sigma_+(M'_V)} b^{M'}(\delta') I_M^{\mathcal{E}}(\delta', f).$$

But the identities of Problems 3 and 4 imply that

$$\sum_{M'\in\mathcal{E}_{\mathrm{ell}}(M,V)} \sum_{\delta'\in\Sigma_+(M'_V)} \iota(M, M') b^{M'}(\delta') I_M^{\mathcal{E}}(\delta', f) = \sum_{\gamma\in\Gamma(M,V)} a^M(\gamma) I_M(\gamma, f).$$

We can therefore conclude that $I^{\mathcal{E}}(f)$ equals

$$\sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma\in\Gamma(M,V)} a^M(\gamma) I_M(\gamma, f),$$

which is just $I(f)$. This is the required identity of Problem 2.

5. Problems 3 and 4 thus imply Problem 2. To relate Problem 2 to the basic Problem 1, we would need to solve spectral analogues of Problems 3 and 4.

Suppose that $v$ is a place of $F$. We write $\Pi(G_v)$ for the set of equivalence classes of irreducible representations of $G(F_v)$. If $G'_v$ is a local endoscopic datum for $G$, we shall write $\Phi(G'_v)$ for a fixed basis of the space of all stable distributions on $G'_v(F_v)$ spanned by irreducible characters. If $v$ is archimedean, we take the elements in $\Phi(G'_v)$ to be analytic continuations (in the appropriate unramified spectral variable) of the stable tempered characters in [24]. In this case, elements in $\Phi(G'_v)$ correspond to Langlands parameters $\phi$: $W_{F_v} \to {}^L G'_v$. If $v$ is $p$-adic, we have to take $\Phi(G'_v)$ to be an abstract basis, obtained by analytic continuation from elements in the basis of tempered stable distributions chosen in [5, Proposition 5.1 and (5.1)]. Extending earlier notation, we write $f'_v(\phi'_v)$ for the pairing obtained from elements $f_v \in \mathcal{H}(G_v)$ and $\phi'_v \in \Phi(G'_v)$. By results in [24] and [5], we can write $f'_v(\phi'_v)$ as a finite linear combination of characters $f_{v,G}(\pi_v)$, with coefficients $\Delta_G(\phi'_v, \pi_v)$ that are spectral analogues of the original transfer factors.

We also extend notation we used earlier for the finite set $V$ of valuations. Thus, if $M'_V = \prod M'_v$ is a product of local endoscopic data for a Levi subgroup $M$ of $G$, and $\phi' = \prod \phi'_v$ belongs to $\Phi(M'_V) = \prod \Phi(M'_v)$, $\Delta_M(\delta', \gamma)$ equals the product over $v \in V$ of the factors $\Delta_M(\phi'_v, \gamma_v)$, for each $\pi = \prod \pi_v$ in $\Pi(M_V) = \prod \Pi(M_v)$. As before, we shall take $M'_V$ to be the image of a global endoscopic datum $M' \in \mathcal{E}_{\mathrm{ell}}(M, V)$.

Problem 5: *Construct stable linear forms $S_M^G(\phi)$ on $\mathcal{H}\big(G(F_V)\big)$, for $G$ quasisplit over $F$ and $\phi \in \Phi(M_V)$, such that for any $G$, $M$, $M'$ and $\phi'$, the linear form*

$$I_M(\phi', f) = \sum_{\pi \in \Pi(M_V)} \Delta_M(\phi', \pi) I_M(\pi, f)$$

*equals the endoscopic expression*

$$I_M^{\mathcal{E}}(\phi', f) = \sum_{G' \in \mathcal{E}_{M'}(G)} \iota_{M'}(G, G') \widehat{S}_{M'}^{G'}(\delta', f').$$

Problem 6: *Construct coefficients $b^M(\phi)$, for $M$ quasisplit over $F$ and $\phi \in \Phi(M_V)$, such that for any $M$ and $\pi$, $a^M(\pi)$ equals the endoscopic coefficient*

$$a^{M,\mathcal{E}}(\pi) = \sum_{M' \in \mathcal{E}_{\mathrm{ell}}(M,V)} \sum_{\phi' \in \Phi(M'_V)} \iota(M, M') b^{M'}(\phi') \Delta_M(\phi', \pi).$$

Set

$$I_{\mathrm{aut}}(f) = \int_{\Pi(G,V)} a^G(\pi) f_G(\pi) d\pi, \qquad f \in \mathcal{H}\big(G(F_V)\big).$$

Since $I_G(\pi, f) = \mathrm{tr}\big(\pi(f)\big) = f_G(\pi)$, for any $\pi \in \Pi(G_V)$, this is just the term with $M = G$ in the spectral expansion of $I(f)$. It can be regarded as the purely automorphic part of the trace formula. The discrete part of $I_{\mathrm{aut}}(f)$, regarded as a distribution on $\Pi(G, V)$, is just the distribution $I_{\mathrm{disc}}(f)$ of Problem 1.

If $M = G$, Problem 5 simply reduces to the expansion of $f'(\phi')$ above. However, Problem 6 is serious in this case, being closely related to Problem 1. It is really the other cases of Problems 5 and 6, those with $M \neq G$, that would be our immediate concern. Assume that these cases have been solved. It is then not hard to show from Problem 6 that if $M$ is quasisplit, $b^M$ is supported on a subset $\Phi(M, V)$ of $\Phi(M_V)$ that has a natural measure $d\phi$. Assuming that Problem 2 has also been solved, we set

$$S_{\mathrm{aut}}^G(f) = S^G(f) - \sum_{M \neq G} |W_0^G||W_0^G|^{-1} \int_{\Phi(M,V)} b^M(\phi) S_M^G(\phi, f) d\phi,$$

for any quasisplit group $G$ and any $f \in \mathcal{H}\big(G(F_V)\big)$. According to Problems 2 and 5, this is a stable linear form on $\mathcal{H}\big(G(F_V)\big)$. If $G$ is arbitrary, we consider the endoscopic expression

$$I_{\mathrm{aut}}^{\mathcal{E}}(f) = \sum_{G' \in \mathcal{E}_{\mathrm{ell}}(G,V)} \iota(G, G') \widehat{S}'_{\mathrm{aut}}(f'), \qquad f \in \mathcal{H}\big(G(F_V)\big).$$

Substituting for $\widehat{S}'_{\mathrm{aut}}(f')$ in this expression, we obtain a term to which Problem 2 applies, and a spectral expansion that can be treated by the argument we applied in §4 to the geometric expansion of $I^{\mathcal{E}}(f)$. We arrive in the end at a formula that identifies $I^{\mathcal{E}}_{\mathrm{aut}}(f)$ with $I_{\mathrm{aut}}(f)$.

We have just sketched a solution of what would be Problem 1 if $I_{\mathrm{disc}}$, $S^G_{\mathrm{disc}}$ and $I^{\mathcal{E}}_{\mathrm{disc}}$ were replaced by $I_{\mathrm{aut}}$, $S^G_{\mathrm{aut}}$, and $I^{\mathcal{E}}_{\mathrm{aut}}$. But $I_{\mathrm{disc}}$ is just the discrete part of $I_{\mathrm{aut}}$. Using a well known argument that separates a suitable distribution into its continuous and discrete parts, one could obtain a solution of Problem 1 from what we have established.

6. We have not really proved anything. We have tried only to argue that Problems 3–6 are at the heart of stabilizing the trace formula. We shall conclude with a few words on the strategy for attacking these problems.

One begins by fixing $G$, and assuming inductively that all the problems can be solved if $G$ is replaced by a proper subgroup. Since the coefficients $a^M(\gamma)$ and $a^M(\pi)$ depend only on $M$, this takes care of the global Problems 4 and 6, except for the case $M = G$. As for Problem 5, the residual distributions $I_M(\pi, f)$ are not independent of the distributions $I_M(\gamma, f)$ of Problem 3. The proof of [10, Theorem II.10.2] can likely be generalized to show that Problem 3 implies Problem 5. Now, the representations $\pi \in \Pi(M, V)$ that occur in the the spectral expansion (2) are unitary. In this case, there are descent and splitting formulas that express $I_M(\pi, f)$ in terms of related distributions on proper Levi subgroups $M$. Therefore, a solution of Problem 5 for the local terms in the spectral expansion would also follow from our induction assumption. (See [10, p. 145].)

It is Problem 3, then, that becomes the main concern. One has first to state the problem in a more elaborate form, one that generalizes the conjectures in [6, §4] and [7, §3], and clearly separates the inductive definitions from what is to be proved. This entails introducing adjoint transfer factors $\Delta_M(\gamma, \delta')$, that depend only on the image of $\delta'$ in a certain set $\Sigma^{\mathcal{E}}_{+}(M_V)$ attached to $M$. We cannot go into any detail, but the construction is a generalization of the discussion of [5, §2] and [7, §2] for strongly regular conjugacy classes. To have adjoint relations, and for that matter, the global vanishing theorem mentioned in §4, one has actually to take $G$ to be a certain disjoint union of connected groups — a global $K$-group, in language suggested in [7]. At any rate, once we have the factor $\Delta_M(\gamma, \delta')$, we can set

$$I^{\mathcal{E}}_M(\gamma, f) = \sum_{\delta' \in R^{\mathcal{E}}(M_V)} \Delta_M(\gamma, \delta') I^{\mathcal{E}}_M(\delta', f), \qquad \gamma \in \Gamma_+(M_V),$$

as in [7, (5.5)]. The required identity of Problem 3 becomes the assertion that $I^{\mathcal{E}}_M(\gamma, f)$ equals $I_M(\gamma, f)$.

The terms in the endoscopic expression $I^{\mathcal{E}}(f)$ of Problem 2 can be defined inductively. An elaboration of the argument sketched in §4, and which is the global analogue [7, Theorem 9.1(a)], then leads to a geometric expansion for $I^{\mathcal{E}}(f)$ that is parallel to the expansion (1) for $I(f)$. The general strategy is to compare these two expansions. In particular, one obtains an explicit geometric expansion for the difference $I^{\mathcal{E}}(f) - I(f)$. On the other hand, similar considerations lead to a spectral expansion of $I^{\mathcal{E}}(f) - I^{\mathcal{E}}_{\mathrm{aut}}(f)$. The cases of Problems 5 and 6 implied by the induction hypothesis actually identify the terms in this latter expansion with corresponding terms in the original spectral expansion for $I(f) - I_{\mathrm{aut}}(f)$. The result is a formula

$$I^{\mathcal{E}}(f) - I(f) = I^{\mathcal{E}}_{\mathrm{aut}}(f) - I_{\mathrm{aut}}(f). \tag{4}$$

To be able to exploit the last formula, one has to extend most of the techniques of Chapter II of [10], (as well as add a few new ones, based on the local trace formula). We mention just one, the problem of descent for the global coefficients. There is a simple descent formula for the coefficients $a^G(\gamma)$ at arbitrary $\gamma$ in terms of coefficients evaluated at unipotent elements [2, (8.1)]. Using the main theorem of [20], one can establish a parallel descent formula for $a^{G,\mathcal{E}}(\gamma)$. Together with the fundamental lemma, which takes care of the spherical weighted orbital integrals we have built into the definition of these coefficients, this reduces the identity of Problem 4 (with $M = G$) to the case of unipotent $\gamma$. It allows one to collapse the terms with $M = G$ in the geometric expansion of the left hand side of (4) to a sum over unipotent elements. Similarly, there is a descent formula for coefficients $a^G(\pi)$ at arbitrary $\pi$ in terms of discrete parts $a^M_{\mathrm{disc}}(\pi)$ and unramified partial $L$-functions. Using simple combinatorial arguments, one can establish a parallel descent formula for $a^{G,\mathcal{E}}(\pi)$. This reduces the identity of Problem 6 (with $M = G$) to the case of the coefficients $a^G_{\mathrm{disc}}(\pi)$, and allows one to replace the right hand side of (4) with the distribution $I^{\mathcal{E}}_{\mathrm{disc}}(f) - I_{\mathrm{disc}}(f)$. It is this revised form of (4) that should eventually yield the required identities of the various problems.

If $G$ is quasisplit, a global analogue of [7, Theorem 9.1(b)] gives a geometric expansion of the distribution $S^G(f)$ of Problem 2. One has to carry out an analysis of this expansion that is largely parallel to the discussion above. Similar techniques should eventually yield the required stability assertions of the various problems.

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