ON LOCAL CHARACTER RELATIONS

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Introduction

Suppose that G is a reductive group over a p-adic field F of characteristic *0*. Langlands and Shelstad [22] conjectured the existence of a remarkable family of maps

$$f \longrightarrow f^{G'} \;=\; f' \; .$$

These maps transfer functions on $G(F)$ to functions on endoscopic groups $G'(F)$, certain quasi-split groups which are typically less complicated than G. The maps represent the analytic side of an algebraic phenomenon, namely, that stable conjugacy is weaker than conjugacy. In other words, nonconjugate elements in $G(F)$ could be conjugate over the algebraic closure $G(\overline{F})$. The transfer maps are expected to play an important role in the theory of automorphic forms, as well as the local harmonic analysis on $G(F)$.

Waldspurger [28] recently established the Langlands-Shelstad conjecture, under the hypothesis that the fundamental lemma holds for a Lie algebra. The fundamental lemma is a hard problem that remains unsolved. However, it has always been regarded as a separate question, and Waldspurger's result came as a considerable surprise. He used global methods on the Lie algebra, and was able to deduce the conjecture by imposing the fundamental lemma at the unramified places. In this paper we shall establish some new properties of the transfer mappings, under the assumption of the fundamental lemma on both a group and its Lie algebra.

Each map is defined by a transfer

$$f'(\sigma') \;=\; \sum_{\gamma} \Delta(\sigma',\gamma) f_G(\gamma) \; , \qquad f \in C_c^\infty\big(G(F)\big), \tag{1}$$

of orbital integrals. Here, $f_G(\gamma)$ is the orbital integral

$$|D(\gamma)|^{\frac{1}{2}} \int_{G_\gamma(F)\backslash G(F)} f(x^{-1}\gamma x)dx$$

of f over a (strongly regular) conjugacy class in $G(F)$, and $\Delta(\sigma',\gamma)$ is the Langlands-Shelstad transfer factor, an explicit function of a stable conjugacy class σ' in $G'(F)$ and the conjugacy class γ in $G(F)$. (We shall recall these notions in more detail in Sections 1 and 2. The most general case is actually a bit more complicated than we are indicating here in the introduction.) Now the space of orbital integrals

$$\mathcal{I}(G) \;=\; \{f_G(\gamma) :\; f \in C_c^\infty\big(G(F)\big)\}$$

comes with a natural filtration over the partially ordered set of Levi subgroups of G. By construction, the transfer map is compatible with this filtration. On the other hand, the representation theory of $G(F)$ actually determines a grading of $\mathcal{I}(G)$. The purpose of this paper is to show that the transfer map is also compatible with the grading. This amounts to establishing certain character identities between $G(F)$ and $G'(F)$.

The identities we seek are between characters on $G(F)$ and *stable* characters on $G'(F)$. The proper notion of a stable character on $G'(F)$ depends on the classification of representations of $G'(F)$ into L-packets $\Pi_{\phi'}$, parametrized by maps

$$\phi' :\; W_F \times SU(2,\mathbb{C}) \longrightarrow {}^L G' \;,$$

from the Langlands group $W_F \times SU(2,\mathbb{C})$ into the L-group of G'. The stable characters should be parametrized by the set $\Phi(G')$ of (equivalence classes of) such maps. The important subset of elliptic stable characters would correspond to the subset $\Phi_2(G')$ of cuspidal parameters, which factor through no proper parabolic subgroup of ${}^L G'$. There is a natural decomposition

$$\Phi(G') \;=\; \coprod_{\{M'\}} \big(\Phi_2(M')/W(M')\big)$$

of $\Phi(G')$ into cuspidal parameters for Levi subgroups M' of G'. None of this helps us, however, since the required classification is well beyond the scope of present understanding. We shall have to look for a substitute.

We take our lead from the orthogonality relations satisfied by elliptic tempered characters. In the paper [3], we introduced a family of virtual characters

$$\gamma \longrightarrow I(\gamma, \tau) \ , \qquad \tau \in T_{\mathrm{ell}}(G),$$

of $G(F)$ whose restrictions to the elliptic set form an orthogonal basis of the space $\mathcal{I}_{\mathrm{cusp}}(G)$ of orbital integrals of cuspidal functions. This allows us to identify a function $f_G \in \mathcal{I}_{\mathrm{cusp}}(G)$ with its set of Fourier coefficients

$$f_G(\tau) \ = \ \int f_G(\gamma) I(\gamma, \tau) d\gamma \ , \qquad \tau \in T_{\mathrm{ell}}(G),$$

relative to the orthogonal basis. Similarly, the set of cuspidal Langlands parameters should determine an orthogonal basis of the space $S\mathcal{I}_{\mathrm{cusp}}(G')$ of stable orbital integrals of cuspidal functions on $G'(F)$. It is this property that we shall use. We shall construct a suitable orthogonal basis of $S\mathcal{I}_{\mathrm{cusp}}(G')$, indexed by an abstract set which we shall take the liberty of denoting by $\Phi_2(G')$. Any function $f' \in S\mathcal{I}_{\mathrm{cusp}}(G')$ can then be identified with its set of Fourier coefficients

$$f'(\phi') \ , \qquad \phi' \in \Phi_2(G'),$$

relative to this orthogonal basis.

Suppose that f' is the image of a function $f_G \in \mathcal{I}_{\mathrm{cusp}}(G)$ under the transfer map. Then for each $\phi' \in \Phi_2(G')$, we can expand $f'(\phi')$ as a linear combination

$$f'(\phi') \ = \sum_{\tau \in T_{\mathrm{ell}}(G)} \Delta(\phi', \tau) f_G(\tau) \ , \tag{2}$$

of character values $f_G(\tau)$. The coefficients $\Delta(\phi', \tau)$ can be regarded as spectral analogues of the transfer factors $\Delta(\sigma', \gamma)$. They are uniquely determined by the linear forms $f'(\phi')$ and $f_G(\tau)$ on $\mathcal{I}_{\mathrm{cusp}}(G)$ in (2). The character identities arise when we try to extend (2) to arbitrary functions f_G in $\mathcal{I}(G)$. The right hand side of (2) is composed of virtual characters, that are of course defined on the full space $\mathcal{I}(G)$. The left hand side, however, is defined a priori only for functions in the subspace $\mathcal{I}_{\text{cusp}}(G)$. There are two questions. In the special case where $G' = G$ (and in particular, where G is quasi-split), we shall show that the right hand side of (2) is a stable distribution, in that it depends only on the stable orbital integrals of f_G (Theorem 6.1). This allows us to define $f'(\phi')$ for any ϕ' and any function $f_G \in \mathcal{I}(G)$. In the general case, we then have to show that the two sides of (2) are equal (Theorem 6.2).

The disjoint union

$$T(G) \ = \ \coprod_{\{M\}} \big(T_{\text{ell}}(M)/W(M)\big) \ ,$$

over conjugacy classes of Levi subgroups of G, parametrizes a general set of virtual characters of $G(F)$. These objects determine a natural grading of $\mathcal{I}(G)$ which is compatible with the filtration defined by orbital integrals. Similarly, if we construct orthogonal bases of $S\mathcal{I}_{\text{cusp}}(M')$ for Levi subgroups M' of G', we obtain an analogue

$$\Phi(G') \ = \ \coprod_{\{M'\}} \big(\Phi_2(M')/W(M')\big)$$

of the entire set of Langlands parameters for G'. It is a consequence of Theorem 6.1 that $\Phi(G')$ determines a grading on $S\mathcal{I}(G')$. Theorem 6.2 then implies that the transfer map $f_G \to f'$ preserves the two gradings.

A fundamental global problem is to stabilize the trace formula [21]. A second purpose of this paper is to lay some local foundations for the future study of this problem. The ultimate goal is to express the invariant distributions in the trace formula explicitly in terms of stable distributions on endoscopic groups. Locally this entails studying the transfer maps $f_G \to f'$ simultaneously. Let $\mathcal{I}^{\mathcal{E}}(G)$ be the image of $\mathcal{I}(G)$ in $\bigoplus_{G'} S\mathcal{I}(G')$ under the map

$$\mathcal{I}^{\mathcal{E}} : \ f_G \longrightarrow f_G^{\mathcal{E}} \ = \ \bigoplus_{G'} f' \ .$$

In Section 2 we shall construct a set $\Gamma^{\mathcal{E}}(G)$ from the stable conjugacy classes on the endoscopic groups $\{G'\}$. We shall characterize $\mathcal{I}^{\mathcal{E}}(G)$ in Section 3 as a space of functions on $\Gamma^{\mathcal{E}}(G)$ (actually, on a slightly different set $\widetilde{\Gamma}^{\mathcal{E}}(G)$ in the most general case). In Section 5 we shall construct a spectral analogue $T^{\mathcal{E}}(G)$ of $\Gamma^{\mathcal{E}}(G)$ from the sets $\{\Phi(G')\}$. Theorems 6.1 and 6.2 will allow us to characterize $\mathcal{I}^{\mathcal{E}}(G)$ as a space of functions on $T^{\mathcal{E}}(G)$.

The virtual characters

$$f \longrightarrow f_G(\tau) \ = \ I_G(\tau, f)$$

are part of a larger family of invariant distributions

$$f \longrightarrow I_M(\tau, f) \ , \qquad \tau \in T(M),$$

obtained from weighted characters on $G(F)$. Weighted characters (as well as weighted orbital integrals) are important components of the trace formula. The stabilization problem includes being able to describe how they behave under endoscopic transfer. In the last section of the paper, we shall state a conjectural transfer formula for the distributions $I_M(\tau, f)$. The formula relies intrinsically on the objects $T^{\mathcal{E}}(M)$ and on the spectral transfer factors we have defined.

The paper is organized as follows. In Section 2 we review some properties of the Langlands-Shelstad transfer factors. We shall introduce adjoint transfer factors, and we shall establish an inversion formula (Lemma 2.2) that was suggested by Kottwitz. In Section 3 we recall the fundamental lemmas we are taking as hypotheses. We also introduce the map $\mathcal{T}^{\mathcal{E}}$. The main step for determining the image of $\mathcal{T}^{\mathcal{E}}$ is Lemma 3.4, which is a consequence of Waldspurger's kernel formula [28] and the descent properties of transfer [23]. In Section 4 we review the virtual characters $f_G \rightarrow f_G(\tau)$ and we discuss some related properties of Langlands parameters as motivation for what follows. In Section 5 we construct our orthogonal bases of the spaces $S\mathcal{I}_{\text{cusp}}(G')$ (Proposition 5.1). We then introduce the general sets $\Phi(G')$ and $T^{\mathcal{E}}(G)$, and the spectral transfer factors $\Delta(\phi', \tau)$. The spectral objects of Section 5 are parallel to many of the geometric objects of Section 2. Once we have introduced them, we will be in a position to state our two main theorems at the beginning of Section 6. The rest of Section 6 will be devoted to some interpretations and consequences of the theorems.

The proofs of the two theorems will be taken up in Sections 7–9. The arguments are global. In Section 7 we recall the simple form of the global trace formula, as well as Langlands' stabilization of the regular elliptic terms [21]. We establish Theorem 6.1 in Section 8, and Theorem 6.2 by similar arguments in Section 9. The main technical ingredient is a description of an orbital integral $f_G \to f_G(\gamma)$ as a distribution on $T(G)$. For each γ, there is a smooth function $\tau \to I(\gamma, \tau)$ on $T(G)$ such that

$$f_G(\gamma) = \int_{T(G)} I(\gamma, \tau) f_G(\tau) d\tau , \qquad f_G \in \mathcal{I}(G),$$

for a natural measure $d\tau$ on $T(G)$ [4, Theorem 4.1]. We end up comparing such a function with a sum of Dirac measures on $T(G)_{\mathbb{C}}$ contributed by two spectral expansions (Lemmas 8.4 and 9.4). We will be able to conclude that the contributions of the spectral expansions cancel, and to deduce the theorems from the resulting identity of geometric expansions.

The global arguments of Sections 7–9 are certainly not new. They go back to Kazhdan's theorem on the density of characters [13, Appendix]. The arguments were successively refined in a series of three papers, beginning with the work of Clozel [8] on the fundamental lemma, and followed by the papers [9] of Hales and [28] of Waldspurger. The present article owes much to these papers, and especially to the work of Waldspurger.

We had originally thought of working with twisted groups, in order to take advantage of the constructions of Kottwitz and Shelstad [18], [19]. This would have relied on various results for twisted groups which, though undoubtedly known to experts, have not been published. In the end it seemed better just to work with connected groups. We have nevertheless tried to write the paper in a way that suggests at least some of the appropriate generalizations to twisted groups.

1. The spaces $\mathcal{I}(G)$ and $S\mathcal{I}(G)$

Let G be a reductive algebraic group over a field F of characteristic *0*. By a Levi subgroup of G, we mean an F-rational Levi component of a parabolic subgroup of G defined over F. Let M_0 be a minimal Levi subgroup of G, fixed for the duration of the paper, and let $\mathcal{L} = \mathcal{L}^G$ be the finite set of Levi subgroups of G which contain M_0. The Weyl group

$$W_0 \; = \; W^G(M_0) \; = \; \mathrm{Norm}_G(M_0)/M_0$$

of (G, M_0) acts by conjugation on $\mathcal{L}$. We write $\mathcal{L}/W_0$ for the set of orbits. For any $M \in \mathcal{L}$, we identify the quotient by $W^M(M_0)$ of the stabilizer of M in W_0 with the Weyl group

$$W(M) \; = \; W^G(M) \; = \; \mathrm{Norm}_G(M)/M$$

of (G, M). We shall be interested in the partial order on $\mathcal{L}$ defined by inclusion, as well as the induced partial order on the quotient $\mathcal{L}/W_0$.

We assume that F is a local non-Archimedean field. Recall that for each $M \in \mathcal{L}$, there is a homomorphism H_M from $M(F)$ to the real vector space

$$\mathfrak{a}_M \; = \; \mathrm{Hom}\big(X(M)_F, \mathbb{R}\big) \; ,$$

defined by

$$e^{\langle H_M(m), \chi \rangle} \; = \; |\chi(m)| \; , \qquad \chi \in X(M)_F, \; m \in M(F).$$

Since F is p-adic, the image of $M(F)$ is a lattice $\mathfrak{a}_{M,F}$ in $\mathfrak{a}_M$. If A_M denotes the split component of the center of M, the image of $A_M(F)$ under H_M is also a lattice $\widetilde{\mathfrak{a}}_{M,F}$ in $\mathfrak{a}_M$, that has finite index in $\mathfrak{a}_{M,F}$. We form the dual lattices

$$\mathfrak{a}^\vee_{M,F} \; = \; \mathrm{Hom}(\mathfrak{a}_{M,F}, 2\pi i\mathbb{Z})$$

and

$$\widetilde{\mathfrak{a}}^\vee_{M,F} \; = \; \mathrm{Hom}(\widetilde{\mathfrak{a}}_{M,F}, 2\pi i\mathbb{Z})$$

in the imaginary dual space $i\mathfrak{a}_M^*$. The quotient

$$i\mathfrak{a}_{M,F}^* \;=\; i\mathfrak{a}_M^*/\mathfrak{a}_{M,F}^\vee$$

is of course a compact torus, and is a finite covering space of the compact torus $i\mathfrak{a}_M^*/\widetilde{\mathfrak{a}}_{M,F}^\vee$.

We choose the Haar measures on the spaces $\mathfrak{a}_M$ and $i\mathfrak{a}_M^*$ for which the compact tori $\mathfrak{a}_M/\widetilde{\mathfrak{a}}_{M,F}$ and $i\mathfrak{a}_M^*/\widetilde{\mathfrak{a}}_{M,F}^\vee$ each have volume 1. The two measures are dual to each other. Combined with the corresponding measures on $\mathfrak{a}_G$ and $i\mathfrak{a}_G^*$, they determine Haar measures on the kernels $\mathfrak{a}_M^G$ and $i(\mathfrak{a}_M^G)^*$ of the canonical projections $\mathfrak{a}_M \to \mathfrak{a}_G$ and $i\mathfrak{a}_M^* \to i\mathfrak{a}_G^*$, which are dual to each other. While we are at it, we recall [4, p. 168] how we can normalize the Haar measures on maximal tori. Suppose that T is a maximal torus in G which is defined over F. Replacing T by a $G(F)$-conjugate if necessary, we can assume that T is an elliptic maximal torus in some group $M \in \mathcal{L}$. That is to say, T is contained in M, and $T(F)/A_M(F)$ is compact. The group $A_M(F)$ has a canonical Haar measure, since H_M maps $A_M(F)$ onto the discrete group $\widetilde{\mathfrak{a}}_{M,F}$ and has compact kernel. We choose the Haar measure on $T(F)$ determined by the measure on $A_M(F)$ and the normalized Haar measure on $T(F)/A_M(F)$.

We shall write $\Gamma(G) = \Gamma_{\mathrm{reg}}\big(G(F)\big)$ for the set of strongly regular, semisimple conjugacy classes in $G(F)$ and $\Gamma_{\mathrm{ell}}(G) = \Gamma_{\mathrm{reg,ell}}\big(G(F)\big)$ for the subset of elliptic conjugacy classes. Thus, $\Gamma_{\mathrm{ell}}(G)$ is the set of conjugacy classes γ in $G(F)$ such that G_γ is an elliptic maximal torus in G. (As usual, G_γ denotes the centralizer of γ in G, with γ being allowed to stand for an element in the conjugacy class as well as the class itself.) If G' is some other group, together with a G-conjugacy class of maps $T' \to G$ of one of its maximal tori into G, the notion of a strongly G-regular element in $G'(F)$ makes sense. We shall write $\Gamma_G(G') = \Gamma_{G\text{-reg}}\big(G'(F)\big)$ and $\Gamma_{G,\mathrm{ell}}(G') = \Gamma_{G\text{-reg,ell}}\big(G'(F)\big)$ for the corresponding sets of conjugacy classes in $G'(F)$. It is clear that $\Gamma(G)$ equals the set of W_0-orbits in the disjoint union $\coprod_{M\in\mathcal{L}}\big(\Gamma_{G,\mathrm{ell}}(M)\big)$ of elliptic, G-regular conjugacy classes in Levi subgroups. We can obviously replace this by a disjoint union over the orbits of $\mathcal{L}/W_0$. In general, we shall denote these orbits throughout the paper by $\{M\}$, often without comment. Thus

$$\Gamma(G) \;=\; \coprod_{\{M\}} \big(\Gamma_{G,\mathrm{ell}}(M)/W(M)\big) \; . \tag{1.1}$$

Suppose that f is a function in $\mathcal{H}\big(G(F)\big)$, the Hecke algebra of locally constant functions of compact support on $G(F)$. For any $\gamma \in \Gamma(G)$ we can form the orbital integral

$$f_G(\gamma) \;=\; |D(\gamma)|^{\frac{1}{2}} \int_{G_\gamma(F)\backslash G(F)} f(x^{-1}\gamma x)dx \; ,$$

normalized by the Weyl discriminant

$$D(\gamma) \;=\; D^G(\gamma) \;=\; \det\big(1 - \mathrm{Ad}(\gamma)\big)_{\mathfrak{g}/\mathfrak{g}_\gamma} \; .$$

We write $\mathcal{I}(G) = \mathcal{I}\big(G(F)\big)$ for the image of the map $f \to f_G$. Then $\mathcal{I}(G)$ is the space of functions on $\Gamma(G)$ of bounded support that satisfy a Shalika germ expansion around any semisimple conjugacy class in $G(F)$. Both $\mathcal{H}\big(G(F)\big)$ and $\mathcal{I}\big(G(F)\big)$ have natural topologies as direct limits of finite dimensional spaces, and $f \to f_G$ is an open, continuous map from $\mathcal{H}\big(G(F)\big)$ onto $\mathcal{I}\big(G(F)\big)$. It is known that for any invariant distribution I on $G(F)$, there is a unique continuous linear form $\widehat{I}$ on $\mathcal{I}\big(G(F)\big)$ such that

$$I(f) \;=\; \widehat{I}(f_G) \; , \qquad f \in \mathcal{H}\big(G(F)\big).$$

For any Levi subgroup $M \in \mathcal{L}$, there is a restriction map $a_G \to a_M$ from $\mathcal{I}\big(G(F)\big)$ to $\mathcal{I}\big(M(F)\big)$. This provides $\mathcal{I}\big(G(F)\big)$ with a natural filtration

$$\mathcal{F}^M\big(\mathcal{I}(G)\big) \;=\; \{a_G \in \mathcal{I}(G) : \; a_L = 0, \; L \subsetneq M\}$$

over the partially ordered set $\mathcal{L}$. The subspace $\mathcal{F}^M\big(\mathcal{I}(G)\big)$ depends only on the W_0-orbit of M, so the filtration is really over the quotient $\mathcal{L}/W_0$. The smallest subspace $\mathcal{F}^G\big(\mathcal{I}(G)\big)$ is denoted $\mathcal{I}_{\text{cusp}}(G) = \mathcal{I}_{\text{cusp}}\big(G(F)\big)$. We also write $\mathcal{H}_{\text{cusp}}\big(G(F)\big)$ for the preimage of $\mathcal{I}_{\text{cusp}}(G)$ in $\mathcal{H}\big(G(F)\big)$. Observe that the graded component

$$\mathcal{G}^M\big(\mathcal{I}(G)\big) \;=\; \mathcal{F}^M\big(\mathcal{I}(G)\big)/\sum_{L\supsetneq M}\mathcal{F}^L\big(\mathcal{I}(G)\big)$$

attached to M is canonically isomorphic to the space $\mathcal{I}_{\text{cusp}}(M)^{W(M)}$ of $W(M)$-invariant functions in $\mathcal{I}_{\text{cusp}}(M)$. By a grading on $\mathcal{I}(G)$, we mean an isomorphism of $\mathcal{I}(G)$ with the graded vector space

$$\mathcal{I}_{\text{gr}}(G) \;=\; \bigoplus_{\{M\}}\mathcal{I}_{\text{cusp}}(M)^{W(M)}\ ,$$

which is compatible with the filtrations, and induces the canonical isomorphism of $\mathcal{G}^M\big(\mathcal{I}(G)\big)$ with $\mathcal{I}_{\text{cusp}}(M)^{W(M)}$ for every $\{M\}$. Orbital integrals do not by themselves lead to such a structure, essentially because of their germ expansions. On the other hand, we shall observe in Section 4 that certain virtual characters on $G(F)$ do provide a natural grading on $\mathcal{I}(G)$.

There is a natural measure on $\Gamma_{\text{ell}}(G)$ given by

$$\int_{\Gamma_{\text{ell}}(G)}\alpha(\gamma)d\gamma \;=\; \sum_{\{T\}}\big|W\big(G(F),T(F)\big)\big|^{-1}\int_{T(F)}\alpha(t)dt\ ,$$

for any $\alpha \in C_c\big(\Gamma(G)\big)$. Here $\{T\}$ is a set of representatives of $G(F)$-conjugacy classes of elliptic maximal torus in G over F, $W\big(G(F),T(F)\big)$ is the Weyl group of $\big(G(F),T(F)\big)$, and dt is the Haar measure on $T(F)$ we have fixed. The corresponding measures on the sets $\Gamma_{\text{ell}}(M)$ then determine a measure

$$\int_{\Gamma(G)}\alpha(\gamma)d\gamma \;=\; \sum_{\{M\}}|W(M)|^{-1}\int_{\Gamma_{\text{ell}}(M)}\alpha(\gamma_M)d\gamma_M$$

on the larger set $\Gamma(G)$. (See [2, Section 2].) Since any function in $\mathcal{I}\big(G(F)\big)$ is bounded on $\Gamma(G)$ [10, Theorem 14], we can form the inner product

$$(a_G,b_G) \;=\; \int_{\Gamma(G)}a_G(\gamma)\overline{b_G(\gamma)}d\gamma$$

on $\mathcal{I}\big(G(F)\big)$. We will be concerned with the restriction of the inner product to $\mathcal{I}_{\mathrm{cusp}}\big(G(F)\big)$, in which case the formula reduces to an integral over $\Gamma_{\mathrm{ell}}(G)$.

In dealing with stable conjugacy classes we shall usually take G to be quasi-split, but this is not necessary. We shall write $\Sigma(G) = \Sigma_{\mathrm{reg}}\big(G(F)\big)$, $\Sigma_{\mathrm{ell}}(G) = \Sigma_{\mathrm{reg,ell}}\big(G(F)\big)$, $\Sigma_G(G') = \Sigma_{G\text{-reg}}\big(G'(F)\big)$, etc. for the set of stable conjugacy classes in $\Gamma(G)$, $\Gamma_{\mathrm{ell}}(G)$ and $\Gamma_G(G')$. Then

$$\Sigma(G) \;=\; \coprod_{\{M\}} \big(\Sigma_{G,\mathrm{ell}}(M)/W(M)\big)\ , \tag{1.2}$$

where $\{M\}$ ranges over the orbits in $\mathcal{L}/W_0$, as we have agreed. For any maximal torus T of G defined over F, we have [15, Section 7] the finite abelian group

$$\mathcal{K}(T) \;=\; \pi_0\big(\widehat{T}^{\Gamma}/Z(\widehat{G})^{\Gamma}\big)\ ,$$

where $\widehat{T}$ and $\widehat{G}$ are complex dual groups of T and G, $Z(\widehat{G})$ is the center of $\widehat{G}$, and

$$\Gamma \;=\; \Gamma_F \;=\; \mathrm{Gal}(\overline{F}/F)\ .$$

If γ is any element in $\Gamma(G)$, there is a bijection between the set of $G(F)$-conjugacy classes in the stable conjugacy class σ of γ, and the set of characters on the group

$$\mathcal{K}_\sigma \;=\; \mathcal{K}_\gamma \;=\; \mathcal{K}(G_\gamma)\ .$$

(See [20]. This assertion depends on F being a p-adic field.) In particular, $n(\sigma) = |\mathcal{K}_\sigma|$ is the number of such conjugacy classes.

If $f \in \mathcal{H}\big(G(F)\big)$ and $\sigma \in \Sigma(G)$, we can form the stable orbital integral

$$f^G(\sigma) \;=\; \sum_{\gamma\to\sigma} f_G(\gamma)\ ,$$

where the sum is over the conjugacy classes γ in σ. We write $S\mathcal{I}(G) = S\mathcal{I}\big(G(F)\big)$ for the image of the map $f \to f^G$. Then $S\mathcal{I}(G)$ is the space of functions on $\Sigma(G)$ of bounded support which satisfy a stable Shalika germ expansion around any stable semisimple conjugacy class. It has a natural topology, and $f \to f^G$ becomes an open continuous map from $\mathcal{H}\big(G(F)\big)$ onto $S\mathcal{I}\big(G(F)\big)$. A *stable distribution* on $G(F)$ can be defined as any distribution which is in the closed linear span of the stable orbital integrals $f \to f^G(\sigma)$. For any stable distribution S, there is a unique continuous linear form $\widehat{S}$ on $S\mathcal{I}\big(G(F)\big)$ such that

$$S(f) \;=\; \widehat{S}(f^G)\ , \qquad f \in \mathcal{H}\big(G(F)\big).$$

Other constructions for $\mathcal{I}\big(G(F)\big)$ are easily adapted to $S\mathcal{I}\big(G(F)\big)$. There are restriction maps $a^G \to a^M$ from $S\mathcal{I}\big(G(F)\big)$ to $S\mathcal{I}\big(M(F)\big)$, and these provide a natural filtration

$$\mathcal{F}^M\big(S\mathcal{I}(G)\big) \;=\; \{a^G \in S\mathcal{I}(G):\ a^L = 0,\ L \subsetneqq M\}$$

of $S\mathcal{I}\big(G(F)\big)$ over the partially ordered set $\mathcal{L}/W_0$. We again write $S\mathcal{I}_{\mathrm{cusp}}(G) = S\mathcal{I}_{\mathrm{cusp}}\big(G(F)\big)$ for the smallest space $\mathcal{F}^G\big(S\mathcal{I}(G)\big)$. Then the graded component

$$\mathcal{G}^M\big(S\mathcal{I}(G)\big) \;=\; \mathcal{F}^M\big(S\mathcal{I}(G)\big) \Big/ \sum_{L \supsetneqq M} \mathcal{F}^L\big(S\mathcal{I}(G)\big)$$

attached to $\{M\}$ is canonically isomorphic to the space $S\mathcal{I}_{\mathrm{cusp}}(M)^{W(M)}$. One of the purposes of this paper will be to define a natural grading of $S\mathcal{I}\big(G(F)\big)$, that is, an isomorphism of $S\mathcal{I}\big(G(F)\big)$ with the graded vector space

$$S\mathcal{I}_{\mathrm{gr}}(G) \;=\; \bigoplus_{\{M\}} S\mathcal{I}_{\mathrm{cusp}}(M)^{W(M)}$$

which for each $\{M\}$, induces the isomorphism above.

We define a measure on $\Sigma_{\mathrm{ell}}(G)$ by setting

$$\int_{\Sigma_{\mathrm{ell}}(G)} \beta(\sigma)d\sigma \;=\; \sum_{\{T\}_{\mathrm{stab}}} |W_F(G,T)|^{-1} \int_{T(F)} \beta(t)dt\ ,$$

for any $\beta \in C_c\big(\Sigma(G)\big)$. Here $\{T\}_{\mathrm{stab}}$ is a set of representatives of stable conjugacy classes of elliptic maximal tori in G over F, and $W_F(G,T)$ is the subgroup of elements in the absolute Weyl group of (G,T) defined over F. We then obtain a measure

$$\int_{\Sigma(G)} \beta(\sigma)d\sigma \ = \ \sum_{\{M\}} |W(M)|^{-1} \int_{\Sigma_{\mathrm{ell}}(M)} \beta(\sigma_M)d\sigma_M$$

on the larger set $\Sigma(G)$. It is easy to see that the measures on $\Gamma(G)$ and $\Sigma(G)$ are related by a formula

$$\int_{\Sigma(G)} \Big(\sum_{\gamma\to\sigma} \alpha(\gamma)\Big)d\sigma \ = \ \int_{\Gamma(G)} \alpha(\gamma)d\gamma \ , \tag{1.3}$$

valid for any $\alpha \in C_c\big(\Gamma(G)\big)$. Recalling that $n(\sigma) = |\mathcal{K}_\sigma|$, we can form the inner product

$$(a^G, b^G) \ = \ \int_{\Sigma(G)} n(\sigma)^{-1} a^G(\sigma)\overline{b^G(\sigma)}d\sigma$$

on $S\mathcal{I}\big(G(F)\big)$. Its restriction to the subspace $S\mathcal{I}_{\mathrm{cusp}}\big(G(F)\big)$ of cuspidal functions reduces to an integral over the elliptic elements $\Sigma_{\mathrm{ell}}(G)$.

We shall sometimes work with spaces of functions that are equivariant on some central subgroup rather than compactly supported. For simplicity, we shall confine our attention to central tori in G. Let Z be a connected subgroup of the center of G defined over F, and let ζ be a character on $Z(F)$. We write $\mathcal{H}\big(G(F),\zeta\big)$ for the space of locally constant functions f on $G(F)$ that are compactly supported modulo $Z(F)$, and such that

$$f(xz) \ = \ \zeta(z)^{-1} f(x) \ , \qquad x \in G(F),\ z \in Z(F).$$

Let $\mathcal{I}(G,\zeta) = \mathcal{I}\big(G(F),\zeta\big)$ be the image of $\mathcal{H}\big(G(F),\zeta\big)$ under the map which sends f to the function

$$f_G(\gamma) \ = \ |D(\gamma)|^{\frac{1}{2}} \int_{G_\gamma(F)\backslash G(F)} f(x^{-1}\gamma x)dx \ , \qquad \gamma \in \Gamma(G).$$

Similarly, let $S\mathcal{I}(G,\zeta) = S\mathcal{I}\big(G(F),\zeta\big)$ be the image of $\mathcal{H}\big(G(F),\zeta\big)$ under the map which sends f to the function

$$f^G(\sigma) \;=\; \sum_{\gamma\to\sigma} f_G(\gamma)\;, \qquad \sigma\in\Sigma(G).$$

Then $\mathcal{I}(G,\zeta)$ and $S\mathcal{I}(G,\zeta)$ are spaces of functions on $\Gamma(G)$ and $\Sigma(G)$, respectively, that transform by ζ^{-1} under translation by $Z(F)$. There is a projection $f\to f_\zeta$ from $\mathcal{H}\big(G(F)\big)$ into $\mathcal{H}\big(G(F),\zeta\big)$ defined by

$$f_\zeta(x) \;=\; \int_{Z(F)} f(xz)\zeta(z)dz\;,$$

which has obvious analogues for $\mathcal{I}\big(G(F)\big)$ and $S\mathcal{I}\big(G(F)\big)$. These projections allow us to carry the constructions above directly over to the spaces $\mathcal{H}\big(G(F),\zeta\big)$, $\mathcal{I}\big(G(F),\zeta\big)$ and $S\mathcal{I}\big(G(F),\zeta\big)$. We obtain, for example, inner products

$$(a_G,b_G) \;=\; \int_{\Gamma_{\mathrm{reg}}(G(F))/Z(F)} a_G(\gamma)\overline{b_G(\gamma)}d\gamma$$

and

$$(a^G,b^G) \;=\; \int_{\Sigma_{\mathrm{reg}}(G(F))/Z(F)} n(\sigma)^{-1}a^G(\sigma)\overline{b^G(\sigma)}d\sigma$$

on the spaces $\mathcal{I}_{\mathrm{cusp}}(G,\zeta) = \mathcal{I}_{\mathrm{cusp}}\big(G(F),\zeta\big)$ and $S\mathcal{I}_{\mathrm{cusp}}(G,\zeta) = S\mathcal{I}\big(G(F),\zeta\big)$ of cuspidal functions. Throughout the paper we shall pass back and forth between the two settings as necessary, with only minimal comment.

It will actually be sufficient to consider the case that Z is an *induced torus.* We mean by this that Z is a product of tori of the form $\mathrm{Res}_{E/F}(\mathbb{G}_m)$, for finite extensions E of F. For example Z could be the split component A_G of the center of G. If Z has this property, and E is any extension of F, the map $G(E)\to(G/Z)(E)$ is surjective [14, Lemma 1.1(3)]. This often allows us to work directly with the group G/Z. For example,

$$\Gamma_{\mathrm{reg}}\big(G(F)\big)/Z(F) \;=\; \Gamma_{\mathrm{reg}}\big((G/Z)(F)\big) \;=\; \Gamma(G/Z)\;,$$

and

$$\Sigma_{\mathrm{reg}}\big(G(F)\big)/Z(F) \;=\; \Sigma_{\mathrm{reg}}\big((G/Z)(F)\big) \;=\; \Sigma(G/Z)\ .$$

2. Geometric transfer functions

Let $\mathcal{E}_{\mathrm{ell}}(G)$ be the set of equivalence classes of elliptic endoscopic data $(G', \mathcal{G}', s', \xi')$ for G over F [22, (1.2)]. Then G' is a quasi-split reductive group over F, $\mathcal{G}'$ is a split extension of the Weyl group W_F by the dual group $\widehat{G}'$, s' is a semisimple element in $\widehat{G}$, and ξ': $\mathcal{G}' \to {}^LG$ is an L-homomorphism, all subject to conditions (a) and (b) on [22, p. 224]. Elliptic means that the image of $\mathcal{G}'$ in ${}^LG$ is contained in no proper parabolic subgroup of ${}^LG$, or equivalently, that

$$\big(Z(\widehat{G}')^\Gamma\big)^0 \;=\; \big(Z(\widehat{G})^\Gamma\big)^0\ .$$

Two data are equivalent if they are isomorphic in the sense defined on [22, p. 225]. As is customary, we generally denote an element in $\mathcal{E}_{\mathrm{ell}}(G)$ by G', even though G' is really only the first component of a representative $(G', \mathcal{G}', s', \xi')$ of an equivalence class. We shall also write $\mathrm{Out}_G(G')$ for the group of outer automorphisms of an element $G' \in \mathcal{E}_{\mathrm{ell}}(G)$. Then

$$\mathrm{Out}_G(G') \;\cong\; \mathrm{Aut}_G(G')/\xi'(\widehat{G}')\ ,$$

where $\mathrm{Aut}_G(G')$ is the group of elements $g \in \widehat{G}$ such that $gs'g^{-1}$ lies in $s'Z(\widehat{G})$, and such that $\mathrm{Int}(g)$ is an L-isomorphism of $\xi'(\mathcal{G}')$ onto itself. Any element in $\mathrm{Out}_G(G')$ can be identified with an outer automorphism of G' which is defined over F. (See [15, Section 7].)

Suppose that $(G', \mathcal{G}', s', \xi')$ is an elliptic endoscopic datum. The group $\mathcal{G}'$ need not be an L-group. That is, there might not be an L-isomorphism from $\mathcal{G}'$ to ${}^LG'$ which is the identity on $\widehat{G}'$. To deal with the problem, one uses the following construction. (See [22, (4.4)], [18, (2.2)].)

Lemma 2.1. *There is a central extension of groups*

$$1 \longrightarrow \widetilde{Z}' \longrightarrow \widetilde{G}' \longrightarrow G' \longrightarrow 1$$

over F with the following properties.

(i) The central subgroup $\widetilde{Z}'$ of $\widetilde{G}'$ is an induced torus.

(ii) The dual exact sequence

$$1 \longrightarrow \widehat{G}' \longrightarrow \widehat{\widetilde{G}'} \longrightarrow \widehat{\widetilde{Z}'} \longrightarrow 1$$

extends to a short exact sequence of L-homomorphisms

$$1 \longrightarrow \mathcal{G}' \xrightarrow{\tilde{\xi}'} {}^L\widetilde{G}' \longrightarrow {}^L\widetilde{Z}' \longrightarrow 1 \ .$$

(iii) Every element of $\mathrm{Out}_G(G')$ extends uniquely to an outer automorphism of $\widetilde{G}'$ over F which leaves $\widetilde{Z}'$ pointwise fixed.

Proof. This is essentially the construction [22, (4.4)] of Langlands and Shelstad. The required group $\widetilde{G}'$ can be obtained from a z-extension [14, Section 1]

$$1 \longrightarrow \widetilde{Z} \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1$$

of G. That is, $\widetilde{G}'$ is an endoscopic datum for $\widetilde{G}$, and the central subgroup $\widetilde{Z}'$ of $\widetilde{G}'$ equals $\widetilde{Z}$. Condition (i) is part of the definition of a z-extension. Condition (ii) follows from the main theorem of [20]. It remains to establish (iii).

The group $\widetilde{G}'$ can be written as a fibred sum

$$(\widetilde{Z}' \times G_1')/Z_1 \ ,$$

where Z_1 is a finite subgroup of $\widetilde{Z}'$ defined over F, and $G_1' \to G'$ is a finite central extension over F with kernel Z_1. An outer automorphism α of G' clearly has at most one extension to $\widetilde{G}'$ which leaves $\widetilde{Z}'$ pointwise fixed. Such an extension exists if and only if there is an extension of α to G_1' which leaves Z_1 pointwise fixed. This will be the case if and only if the dual outer automorphism $\widehat{\alpha}$ of $\widehat{G}'$ leaves the finite subgroup

$$\widehat{Z}_1 \;=\; \mathrm{Hom}(Z, \mathbb{G}_m)$$

of $Z(\widehat{G}')$ pointwise fixed. These remarks apply to any central extension $\widetilde{G}' \to G$. However, if $\widetilde{G}'$ is obtained from a z-extension of G as above, the finite group $\widehat{Z}_1$ will be contained in the subgroup $Z(\widehat{G})$ of $Z(\widehat{G}')$. Assume that α lies in $\mathrm{Out}_G(G')$. Then $\widehat{\alpha}$ is induced by conjugation in $\widehat{G}'$ by an element in $\widehat{G}$, and therefore leaves $Z(\widehat{G})$ pointwise fixed. In particular, $\widehat{\alpha}$ leaves $\widehat{Z}_1$ pointwise fixed. It follows that α extends to an outer automorphism of $\widetilde{G}'$ which leaves $\widetilde{Z}'$ pointwise fixed. $\square$

For each elliptic endoscopic datum, we fix a central extension $\widetilde{G}' \to G'$ and an L-embedding

$$\widetilde{\xi}' :\; \mathcal{G}' \longrightarrow {}^L\widetilde{G}' \ ,$$

which satisfy the conditions of the lemma. We can assume that these objects are compatible under isomorphisms of endoscopic data, and therefore depend only on the elements $G' \in \mathcal{E}_{\mathrm{ell}}(G)$. Fix G'. Condition (i) implies that the map $\widetilde{G}'(F) \to G'(F)$ is surjective. The composition

$$W_F \longrightarrow \mathcal{G}' \xrightarrow{\tilde{\xi}'} {}^L\widetilde{G}' \longrightarrow {}^L\widetilde{Z}' \ ,$$

determined by condition (ii) and any section $W_F \to \mathcal{G}'$, provides a Langlands parameter for the torus $\widetilde{Z}'$. This is dual to a character $\widetilde{\zeta}'$ on $\widetilde{Z}'(F)$. By condition (iii), $\mathrm{Out}_G(G')$ can be identified with a finite group of F-rational outer automorphisms of $\widetilde{G}'$ which leave $\widetilde{Z}'$ pointwise invariant. In particular, every element in $\mathrm{Out}_G(G')$ acts on $\widetilde{G}'(F)$ (up to inner automorphisms of $\widetilde{G}'$ which are defined over F), and fixes the central character $\widetilde{\zeta}'$ on $\widetilde{Z}'(F)$. Therefore $\mathrm{Out}_G(G')$ acts as a finite group of linear automorphisms of the vector space $S\mathcal{I}\big(\widetilde{G}'(F), \widetilde{\zeta}'\big)$.

Associated to the objects G', $\widetilde{G}'$, $\widetilde{\xi}'$ and $\widetilde{\zeta}'$, we have the transfer factor

$$\Delta(\sigma', \gamma) \;=\; \Delta_G(\sigma', \gamma)$$

of Langlands and Shelstad [22, especially (4.4)] (and Kottwitz and Shelstad [18] in the more general twisted case). It is a smooth function of (σ', γ) in $\Sigma_G(\widetilde{G}') \times \Gamma(G)$. As a function of σ', $\Delta(\sigma', \gamma)$ transforms under translation by $\widetilde{Z}'(F)$ according to the inverse of the character $\widetilde{\zeta}'$. More generally, suppose that Z is a central induced torus in G over F, as in Section 1. Then Z embeds canonically as a central torus in G'. Its preimage in $\widetilde{G}'$ is also a central induced torus over F, which we shall denote by $\widetilde{Z}'Z$. Then there is a character $\widetilde{\zeta}'_Z$ on $(\widetilde{Z}'Z)(F)$, whose restriction to $\widetilde{Z}'(F)$ equals $\widetilde{\zeta}'$, such that

$$\Delta(\sigma' z, \gamma z_G) \;=\; \widetilde{\zeta}'_Z(z)^{-1}\Delta(\sigma', \gamma)\ , \qquad z \in (\widetilde{Z}'Z)(F), \tag{2.1}$$

where z_G is the projection of z onto $Z(F) = (\widetilde{Z}'Z)(F)/\widetilde{Z}'(F)$. (See [22, (4.4)], [18, (5.1)].)

For any $G' \in \mathcal{E}_{\mathrm{ell}}(G)$ there is an injective linear map $\lambda \to \lambda'$ from $\mathfrak{a}^*_{G,\mathbb{C}}$ to $\mathfrak{a}^*_{\tilde{G}',\mathbb{C}}$. To see this, observe that $\mathfrak{a}^*_{G,\mathbb{C}}$ and $\mathfrak{a}^*_{\tilde{G}',\mathbb{C}}$ are the Lie algebras of the complex, connected abelian Lie groups $\big(Z(\widehat{G})^{\Gamma}\big)^0$ and $\big(Z(\widehat{\widetilde{G}'})^{\Gamma}\big)^0$. Since G' is elliptic, $\big(Z(\widehat{G})^{\Gamma}\big)^0$ equals $\big(Z(\widehat{G}')^{\Gamma}\big)^0$, and there is an injection from $\big(Z(\widehat{G}')^{\Gamma}\big)^0$ to $\big(Z(\widehat{\widetilde{G}'})^{\Gamma}\big)^0$ which is dual to the projection $\widetilde{G}' \to G'$. The map $\lambda \to \lambda'$ obtained in this way is in turn dual to a projection from $\mathfrak{a}_{\tilde{G}'}$ onto $\mathfrak{a}_G$. According to the construction in [22], $\Delta(\sigma', \gamma)$ vanishes unless the projection of σ' onto $G'(F)$ is an image of γ (in the language of [22]). In particular, the point $H_{\tilde{G}'}(\sigma')$ in $\mathfrak{a}_{\tilde{G}'}$ must project onto the point $H_G(\gamma)$ in $\mathfrak{a}_G$. It follows that

$$e^{\lambda'(H_{\tilde{G}'}(\sigma'))}\Delta(\sigma', \gamma) \;=\; \Delta(\sigma', \gamma)e^{\lambda(H_G(\gamma))}\ , \tag{2.2}$$

for any $\lambda \in \mathfrak{a}^*_{G,\mathbb{C}}$.

One of the purposes of this paper is to keep track of transfer mappings as G' varies. To this end, we introduce an "endoscopic" set $\widetilde{\Gamma}^{\mathcal{E}}_{\mathrm{ell}}(G)$ which is to be parallel to $\Gamma_{\mathrm{ell}}(G)$. We define $\widetilde{\Gamma}^{\mathcal{E}}_{\mathrm{ell}}(G)$ to be the set of isomorphism classes of pairs (G', σ'), where G' is an elliptic endoscopic datum for G and σ' in an element in $\Sigma_{G,\mathrm{ell}}(\widetilde{G}')$. By an isomorphism from (G',σ') to a second pair (G'_1,σ'_1), we mean an isomorphism from the datum G' to G'_1, which takes σ' to σ'_1. It is clear that $\widetilde{\Gamma}^{\mathcal{E}}_{\mathrm{ell}}(G)$ can be identified with the disjoint union over $G' \in \mathcal{E}_{\mathrm{ell}}(G)$ of the sets

$$\Sigma_{\mathrm{ell}}(\widetilde{G}',G) \ = \ \Sigma_{G,\mathrm{ell}}(\widetilde{G}')/\mathrm{Out}_G(G')$$

of $\mathrm{Out}_G(G')$-orbits in $\Sigma_{G,\mathrm{ell}}(\widetilde{G}')$. Observe that there is an internal action

$$(G',\sigma') \longrightarrow (G',z\sigma') \ , \qquad z \in \widetilde{Z}'(F),$$

of the groups $\widetilde{Z}'(F)$ on $\widetilde{\Gamma}^{\mathcal{E}}_{\mathrm{ell}}(G)$. We have reserved the symbol $\Gamma^{\mathcal{E}}_{\mathrm{ell}}(G)$ for the associated quotient. Thus $\Gamma^{\mathcal{E}}_{\mathrm{ell}}(G)$ is the disjoint union over $G' \in \mathcal{E}_{\mathrm{ell}}(G)$ of the sets

$$\Sigma_{\mathrm{ell}}(G',G) \ = \ \Sigma_{G,\mathrm{ell}}(G')/\mathrm{Out}_G(G') \ .$$

We shall usually denote a pair (G',σ') in either $\widetilde{\Gamma}^{\mathcal{E}}_{\mathrm{ell}}(G)$ or $\Gamma^{\mathcal{E}}_{\mathrm{ell}}(G)$ simply by σ', since the element $G' \in \mathcal{E}(G)$ is uniquely determined by σ'.

It is a direct consequence of the definitions in [22] that $\Delta(\sigma',\gamma)$ depends only on the isomorphism class of (G',σ'). The restriction of the transfer factors to the elliptic set can therefore be regarded as a single function on $\widetilde{\Gamma}^{\mathcal{E}}_{\mathrm{ell}}(G) \times \Gamma_{\mathrm{ell}}(G)$. We propose to study this function together with the adjoint function

$$\Delta(\gamma,\sigma') \ = \ |\mathcal{K}_\gamma|^{-1}\overline{\Delta(\sigma',\gamma)} \tag{2.3}$$

on $\Gamma_{\mathrm{ell}}(G) \times \widetilde{\Gamma}^{\mathcal{E}}_{\mathrm{ell}}(G)$. Observe that any product

$$\Delta(\gamma,\sigma')\Delta(\sigma',\gamma_1) \ , \qquad \gamma,\gamma_1 \in \Gamma_{\mathrm{ell}}(G),$$

is invariant under the action of the groups $\widetilde{Z}'(F)$ on σ', and can therefore be regarded as a function of σ' in the quotient $\Gamma^{\mathcal{E}}_{\mathrm{ell}}(G)$.

We shall write

$$\delta(\gamma,\gamma_1) \;=\; \begin{cases} 1, & \text{if } \gamma=\gamma_1, \\ 0, & \text{otherwise}, \end{cases}$$

for any elements $\gamma, \gamma_1 \in \Gamma_{\mathrm{ell}}(G)$, and

$$\widetilde{\delta}(\sigma',\sigma'_1) \;=\; \begin{cases} \widetilde{\zeta}'(z'), & \text{if } \sigma'_1 = z\sigma' \text{ for some } z' \in \widetilde{Z}'(F), \\ 0, & \text{otherwise}, \end{cases}$$

for any elements $\sigma', \sigma'_1 \in \widetilde{\Gamma}^{\mathcal{E}}_{\mathrm{ell}}(G)$.

Lemma 2.2. *The transfer factors satisfy*

$$\sum_{\sigma' \in \Gamma^{\mathcal{E}}_{\mathrm{ell}}(G)} \Delta(\gamma,\sigma')\Delta(\sigma',\gamma_1) \;=\; \delta(\gamma,\gamma_1), \qquad \gamma,\gamma_1 \in \Gamma_{\mathrm{ell}}(G), \tag{2.4}$$

and

$$\sum_{\gamma \in \Gamma_{\mathrm{ell}}(G)} \Delta(\sigma',\gamma)\Delta(\gamma,\sigma'_1) \;=\; \widetilde{\delta}(\sigma',\sigma'_1), \qquad \sigma',\sigma'_1 \in \widetilde{\Gamma}^{\mathcal{E}}_{\mathrm{ell}}(G). \tag{2.5}$$

Proof. We can assume that the extensions $\widetilde{G}' \to G'$ are all obtained from a fixed extension $\widetilde{G} \to G$, as in [22, (4.4)]. The definitions of [22, (4.4)] then allow us to reduce the problem to the case where this extension is trivial. We shall therefore assume that $\widetilde{G}' = G'$ for each $G' \in \mathcal{E}_{\mathrm{ell}}(G)$.

The required formulas reduce in the end to inversion on the finite abelian group $\mathcal{K}_\gamma$. To get to that point, one has to go through an argument like that of [22, Section 6.4]. Instead of embedding a group of rational points into a group of adèlic points, however, one considers the diagonal embedding $\gamma \to (\gamma,\gamma^{-1})$ of $G(F)$ into $G(F) \times G(F)$. That the transfer factors simplify in this situation was pointed out to me by Kottwitz.

We shall make free use of the language and notation of [22], often without comment. To define the general transfer factor $\Delta(\sigma',\gamma)$, it is necessary to fix elements $\overline{\sigma}'$ and $\overline{\gamma}$ such that $\overline{\sigma}'$ is an image of $\overline{\gamma}$ (in the language of [22, (1.3)]), and to specify $\Delta(\overline{\sigma}',\overline{\gamma})$ arbitrarily.

We will take it to be any complex number of absolute value 1. Then $\Delta(\sigma', \gamma)$ is defined to be the product of $\Delta(\overline{\sigma}', \overline{\gamma})$ with the factor

$$\Delta(\sigma', \gamma; \overline{\sigma}', \overline{\gamma}) \;=\; \frac{\Delta_I(\sigma', \gamma)}{\Delta_I(\overline{\sigma}', \overline{\gamma})} \frac{\Delta_{II}(\sigma', \gamma)}{\Delta_{II}(\overline{\sigma}', \overline{\gamma})} \frac{\Delta_2(\sigma', \gamma)}{\Delta_2(\overline{\sigma}', \overline{\gamma})} \Delta_1(\sigma', \gamma; \overline{\sigma}', \overline{\gamma}) \;.$$

There is an additional factor

$$\Delta_{IV}(\sigma', \gamma) \;=\; |D^G(\gamma)||D^{G'}(\gamma')|^{-1}$$

included in the definition of [22], but since we have already put these normalizing factors into our orbital integrals, we must leave them out here. The remaining factors are all constructed from the special values of unitary abelian characters, and therefore have absolute value 1. We can therefore write the summand

$$\Delta(\gamma, \sigma')\Delta(\sigma', \gamma_1) \;=\; |\mathcal{K}_\gamma|^{-1}\overline{\Delta(\sigma', \gamma)}\Delta(\sigma', \gamma_1)$$

in (2.4) as the product of $|\mathcal{K}_\gamma|^{-1}$ with

$$\Delta(\sigma', \gamma; \overline{\sigma}', \overline{\gamma})^{-1}\Delta(\sigma', \gamma_1; \overline{\sigma}', \overline{\gamma}) \;.$$

By [22, Lemma 4.1A], this last product can be written as

$$\Delta(\sigma', \gamma_1; \sigma', \gamma) \;=\; \frac{\Delta_I(\sigma', \gamma_1)}{\Delta_I(\sigma', \gamma)} \cdot \frac{\Delta_{II}(\sigma', \gamma_1)}{\Delta_{II}(\sigma', \gamma)} \cdot \frac{\Delta_2(\sigma', \gamma_1)}{\Delta_2(\sigma', \gamma)} \cdot \Delta_1(\sigma', \gamma_1; \sigma', \gamma) \;.$$

We shall examine the four terms in the product on the right.

Recall that one has to fix an inner twisting $\psi\colon G \to G^*$ of G with a quasi-split group G^* over F. The simply connected covering G^*_{sc} of the derived group of G^* then plays an important role in the definitions. We are implicitly assuming that σ' is an image of both γ and γ_1, since the corresponding summand in (2.4) would otherwise vanish. Let T' be the centralizer of σ' in G'. We choose an admissible embedding $T' \to T^*$ of T' into G^* [22, (1.3)], and we let γ^* denote the image of σ' in T^*. The factors $\Delta_I(\sigma', \gamma)$, $\Delta_{II}(\sigma', \gamma)$ and $\Delta_2(\sigma', \gamma)$ which occur in the first three terms in the product $\Delta(\sigma', \gamma_1; \sigma', \gamma)$ are defined in [22, (3.2), (3.3), (3.5)]. An examination of the definitions reveals that while they may depend on γ^* and σ', these factors are all independent of γ. The first three terms in the product $\Delta(\sigma', \gamma_1; \sigma', \gamma)$ are therefore all equal to 1. We conclude that

$$\Delta(\gamma, \sigma')\Delta(\sigma', \gamma_1) \; = \; |\mathcal{K}_\gamma|^{-1}\Delta_1(\sigma', \gamma_1; \sigma', \gamma) \; .$$

The definition of $\Delta_1(\sigma', \gamma_1; \sigma', \gamma)$ is given in [22, (3.4)]. Since the first and third arguments are the same, the definition reduces to a pairing

$$\Delta(\sigma', \gamma_1; \sigma', \gamma) \; = \; \langle \mu_{T^*}(\gamma, \gamma_1), s_{T^*}(\sigma') \rangle \; ,$$

of elements defined as follows. The first element $\mu_{T^*}(\gamma, \gamma_1)$ lies in $H^1(F, T^*_{\mathrm{sc}})$, where T^*_{sc} is the preimage of T^* in G^*_{sc}. It is the class of the cocycle

$$\tau \longrightarrow v_1(\tau)^{-1}v(\tau) \; , \qquad \tau \in \mathrm{Gal}(\overline{F}/F),$$

for elements $v(\tau) = hu(\tau)\tau(h)^{-1}$ and $v_1(\tau) = h_1u(\tau)\tau(h_1)^{-1}$ defined as on p. 245 of [22]. It depends only on γ, γ_1 and γ^*. For fixed γ_1 and γ^*, the map $\gamma \to \mu_{T^*}(\gamma, \gamma_1)$ is easily seen to be a bijection from the set of elements in $\Gamma(G)$ which lie in the stable conjugacy class of γ_1 onto $H^1(F, T^*_{\mathrm{sc}})$. This relies on the property that $H^1(F, G^*_{\mathrm{sc}}) = \{1\}$, and therefore holds only in the p-adic case at hand. The second element $s_{T^*} = s_{T^*}(\sigma')$ is defined on p. 241 of [22]. It is the image in

$$\mathcal{K}(T^*) \; = \; \pi_0\big((\widehat{T}^*)^\Gamma / Z(\widehat{G}^*)^\Gamma\big) \; ,$$

under the isomorphism $\widehat{T}' \to \widehat{T}^*$ which is dual to the admissible embedding $T' \to T^*$ of the preimage of s' in $\widehat{T}'$. For any γ^* and σ', there is a unique admissible embedding which maps σ' to γ^*. Since Δ_1 is independent of the admissible embedding [22, Lemma 3.4A], we can fix γ^* and allow the embedding to vary with σ'. We obtain a map $\sigma' \to s_{T^*}(\sigma')$ from the set of images of γ^* in $\Gamma^{\mathcal{E}}_{\mathrm{ell}}(G)$ to $\mathcal{K}(T^*)$, which is easily seen to be a bijection. (The argument here is identical to the proof of the corresponding global assertion, which was established in greater generality in [16, Lemma 9.7].) The finite abelian groups $H^1(F, T^*_{\mathrm{sc}})$ and $\mathcal{K}(T^*)$ are in duality with each other.

We can now establish (2.4) and (2.5). The left hand side of (2.4) equals

$$\begin{aligned}\sum_{\sigma' \in \Gamma^{\mathcal{E}}_{\mathrm{ell}}(G)} & \Delta(\gamma, \sigma')\Delta(\sigma', \gamma_1) \\ &= |\mathcal{K}_\gamma|^{-1} \sum_{\sigma'} \langle \mu_{T^*}(\gamma, \gamma_1), s_{T^*}(\sigma') \rangle \\ &= |\mathcal{K}(T^*)|^{-1} \sum_{\kappa \in \mathcal{K}(T^*)} \langle \mu_{T^*}(\gamma, \gamma_1), \kappa \rangle \ .\end{aligned}$$

By Fourier inversion on the finite abelian group $\mathcal{K}(T^*) \cong \mathcal{K}_\gamma$, this equals the right hand side $\delta(\gamma, \gamma_1)$ of (2.4), since $\mu_{T^*}(\gamma, \gamma_1) = 1$ if and only if $\gamma = \gamma_1$. To deal with (2.5), we observe that

$$\Delta(\sigma', \gamma_1)^{-1}\Delta(\sigma', \gamma) \ = \ \overline{\Delta(\sigma', \gamma_1)}\Delta(\sigma', \gamma) \ = \ \langle \mu_{T^*}(\gamma, \gamma_1), s_{T^*}(\sigma') \rangle^{-1} \ .$$

Consequently

$$\begin{aligned}&\Delta(\sigma', \gamma)\Delta(\gamma, \sigma'_1) \\ &\quad = |\mathcal{K}_\gamma|^{-1}\Delta(\sigma', \gamma)\Delta(\sigma'_1, \gamma)^{-1} \\ &\quad = |\mathcal{K}_\gamma|^{-1}\Delta(\sigma', \gamma_1)\Delta(\sigma'_1, \gamma_1)^{-1}\langle \mu_{T^*}(\gamma, \gamma_1), s_{T^*}(\sigma'_1)s_{T^*}(\sigma')^{-1} \rangle \ .\end{aligned}$$

Summing over γ in the stable conjugacy class of γ_1, we see that the left hand side of (2.5) equals

$$\begin{aligned}|\mathcal{K}(T^*)|^{-1}\Delta(\sigma', \gamma_1)\Delta(\sigma'_1, \gamma_1)^{-1} & \sum_{\mu \in H^1(F, T^*_{\mathrm{sc}})} \langle \mu, s_{T*}(\sigma'_1)s_{T^*}(\sigma')^{-1} \rangle \\ = \ & \Delta(\sigma', \gamma_1)\Delta(\sigma'_1, \gamma_1)^{-1}\delta(\sigma', \sigma'_1) \\ = \ & \delta(\sigma', \sigma'_1) \ ,\end{aligned}$$

again by Fourier inversion on $\mathcal{K}(T^*)$. $\square$

We pause for a moment to note that transfer factors can govern a change of variables of integration. Our set $\Gamma^{\mathcal{E}}_{\mathrm{ell}}(G)$ inherits a measure from the sets $\Sigma_{G,\mathrm{ell}}(G')$, or rather, from the quotient measures on the sets $\Sigma_{\mathrm{ell}}(G', G)$.

Lemma 2.3. *Suppose that* $\alpha \in C_c\big(\Gamma_{\mathrm{ell}}(G)\big)$, *and that* $\beta \in C\big(\widetilde{\Gamma}^{\mathcal{E}}_{\mathrm{ell}}(G)\big)$ *is such that the product* $\beta(\sigma')\Delta(\sigma',\gamma)$ *descends to a function of* $\sigma' \in \Gamma^{\mathcal{E}}_{\mathrm{ell}}(G)$. *Then*

$$\int_{\Gamma_{\mathrm{ell}}(G)} \sum_{\sigma' \in \Gamma^{\mathcal{E}}_{\mathrm{ell}}(G)} \beta(\sigma')\Delta(\sigma',\gamma)\alpha(\gamma)d\gamma \ = \ \int_{\Gamma^{\mathcal{E}}_{\mathrm{ell}}(G)} \sum_{\gamma \in \Gamma_{\mathrm{ell}}(G)} \beta(\sigma')\Delta(\sigma',\gamma)\alpha(\gamma)d\sigma' \ .$$

Proof. Let $\psi \colon G \to G^*$ be the underlying quasi-split inner twist of G. According to (1.3), the integral over $\Gamma_{\mathrm{ell}}(G)$ can be decomposed into an integral over $\sigma^* \in \Sigma_{\mathrm{ell}}(G^*)$ and a sum over the elements $\gamma \in \Gamma_{\mathrm{ell}}(G)$ which map to σ^*. Similarly, the integral over $\Gamma^{\mathcal{E}}_{\mathrm{ell}}(G)$ can be decomposed into an integral over $\sigma^* \in \Sigma_{\mathrm{ell}}(G^*)$ and a sum over the elements $\sigma' \in \Gamma^{\mathcal{E}}_{\mathrm{ell}}(G)$ which map to σ^*. This is a variant of (1.3), which can be established from the definitions of the measures on $\Sigma_{G,\mathrm{ell}}(G')$, and the fact that the map

$$\sigma' \longrightarrow s_{T^*}(\sigma') \ , \qquad T^* = G^*_{\sigma^*},$$

from the proof of the last lemma, is a bijection from the preimage of σ^* in $\Gamma^{\mathcal{E}}_{\mathrm{ell}}(G)$ onto $\mathcal{K}(T^*)$. We leave this point for the reader to check.

With the two decompositions, we can represent each side of the required identity as an integral over $\Sigma_{\mathrm{ell}}(G^*)$, and a double sum over σ' and γ. By its definition, the transfer factor $\Delta(\sigma',\gamma)$ vanishes unless σ' and γ have the same image in $\Sigma_{\mathrm{ell}}(G^*)$. The double sum in each case can therefore be taken over the preimages of σ^* in $\Gamma^{\mathcal{E}}_{\mathrm{ell}}(G) \times \Gamma_{\mathrm{ell}}(G)$. The identity follows. □

To complete the picture, we need to expand $\widetilde{\Gamma}^{\mathcal{E}}_{\mathrm{ell}}(G)$ into a larger set $\widetilde{\Gamma}^{\mathcal{E}}(G)$ which is parallel to $\Gamma(G)$. The simplest procedure is simply to copy the decomposition (1.1) of $\Gamma(G)$. This requires a brief word about non-elliptic endoscopic data.

We write $\mathcal{E}(G)$ for the set of equivalence classes of general endoscopic data for G. Elements in $\mathcal{E}(G)$ can be represented in two separate ways — as elliptic endoscopic data for Levi subgroups of G, or as Levi subgroups of elliptic endoscopic data for G. This provides two decompositions

$$\mathcal{E}(G) \;=\; \coprod_{\{M\}} \big(\mathcal{E}_{\mathrm{ell}}(M)/W(M)\big) \tag{2.6}$$

and

$$\mathcal{E}(G) \;=\; \Big(\coprod_{G' \in \mathcal{E}_{\mathrm{ell}}(G)} \mathcal{L}^{G'}\Big)/\sim\;, \tag{2.7}$$

where the last equivalence relation is defined by $\widehat{G}$-conjugacy. (Keep in mind that distinct elements $G' \in \mathcal{E}_{\mathrm{ell}}(G)$ can have a Levi subgroup in common, which of course contributes only one element to $\mathcal{E}(G)$.) For each element $(M', \mathcal{M}', s'_M, \xi'_M)$ in $\mathcal{E}_{\mathrm{ell}}(M)$, we choose an extension $\widetilde{M}' \to M'$ and an embedding $\widetilde{\xi}'_M$: $\mathcal{M}' \to {}^L\widetilde{M}'$ (and hence also a character $\widetilde{\zeta}'_M$ on the central subgroup $\widetilde{Z}'_M(F)$ of $\widetilde{M}'(F)$), as in Lemma 2.1. We can assume that if M' belongs to $\mathcal{L}^{G'}$, for $G' \in \mathcal{E}_{\mathrm{ell}}(G)$, then $\widetilde{M}'$ is a Levi subgroup of $\widetilde{G}'$ and ξ'_M is a restriction of the corresponding embedding for G'. For example, if the objects are all obtained from a fixed extension $\widetilde{G} \to G$, as in the proof of Lemma 2.1, all required compatibility conditions will hold. Moreover, each group

$$\mathrm{Out}_G(M') \;=\; \mathrm{Aut}_G(M')/\xi'_M(\widehat{M}')$$

acts by outer automorphisms of $\widehat{M}'$ over F which leave $\widetilde{Z}'_M$ pointwise fixed.

Motivated by (2.6), we define

$$\widetilde{\Gamma}^{\mathcal{E}}(G) \;=\; \coprod_{\{M\}} \big(\widetilde{\Gamma}^{\mathcal{E}}_{G,\mathrm{ell}}(M)/W(M)\big) \tag{2.8}$$

and

$$\Gamma^{\mathcal{E}}(G) \;=\; \coprod_{\{M\}} \big(\Gamma^{\mathcal{E}}_{G,\mathrm{ell}}(M)/W(M)\big)\;, \tag{2.9}$$

where the subscript G as usual stands for elements which are G-regular. We can easily extend the definition of the transfer factors to elements σ' and γ in the larger sets $\widetilde{\Gamma}^{\mathcal{E}}(G)$

and $\Gamma(G)$. We define $\Delta(\sigma', \gamma)$ and $\Delta(\gamma, \sigma')$ to be zero unless there is an M such that (σ', γ) belongs to the Cartesian product of $\widetilde{\Gamma}^{\mathcal{E}}_{G,\mathrm{ell}}(M)/W(M)$ with $\Gamma_{G,\mathrm{ell}}(M)/W(M)$. If there is such an M, (σ', γ) is the image of a pair (σ'_M, γ_M) in $\widetilde{\Gamma}^{\mathcal{E}}_{G,\mathrm{ell}}(M) \times \Gamma_{G,\mathrm{ell}}(M)$. In this case we set

$$\Delta(\sigma', \gamma) \;=\; \Delta_G(\sigma', \gamma) \;=\; \sum_{w \in W(M)} \Delta_M(\sigma'_M, w\gamma_M)$$

and

$$\Delta(\gamma, \sigma') \;=\; \Delta_G(\gamma, \sigma') \;=\; \sum_{w \in W(M)} \Delta_M(\gamma_M, w\sigma'_M)\ .$$

Each sum contains at most one nonzero term, and depends only on σ' and γ. If we apply (2.4) and (2.5) to each M, we obtain general inversion formulas

$$\sum_{\sigma' \in \Gamma^{\mathcal{E}}(G)} \Delta(\gamma, \sigma')\Delta(\sigma', \gamma_1) \;=\; \delta(\gamma, \gamma_1), \qquad \gamma, \gamma_1 \in \Gamma(G), \tag{2.10}$$

and

$$\sum_{\gamma \in \Gamma(G)} \Delta(\sigma', \gamma)\Delta(\gamma, \sigma'_1) \;=\; \widetilde{\delta}(\sigma', \sigma'_1), \qquad \sigma', \sigma'_1 \in \widetilde{\Gamma}^{\mathcal{E}}(G), \tag{2.11}$$

on $\Gamma(G)$ and $\widetilde{\Gamma}^{\mathcal{E}}(G)$.

The transfer factors we have just defined are no different from the original ones. Consider an arbitrary element $\sigma' \in \widetilde{\Gamma}^{\mathcal{E}}(G)$. This element actually stands for the $W(M)$-orbit of a pair (M', σ'_M) in $\Gamma^{\mathcal{E}}_{G,\mathrm{ell}}(M)$. In general, there can be several elliptic endoscopic groups $G' \in \mathcal{E}_{\mathrm{ell}}(G)$ that contain M' as a Levi subgroup. Pick one of them. The $W^{G'}(M')$-orbit of σ'_M, which we shall also denote by σ', is a class in $\Sigma_G(\widetilde{G}')$. The original transfer factor $\Delta(\sigma', \gamma)$ from G to G' is then defined. The point is that it matches the function above defined in terms of Levi subgroups. One sees this by examining the four terms [22, (3.2)–(3.5)] in the Langlands-Shelstad definition, and by observing that the χ-data in [22, (2.5)] can be chosen so that $\chi_\alpha = 1$ for each root α of (G, T) that is not a root of (M, T). In particular, the original transfer factor $\Delta(\sigma', \gamma)$ depends only on the image of σ' in $\widetilde{\Gamma}^{\mathcal{E}}(G)$. We shall denote the image of $\Sigma_G(\widetilde{G}')$ in $\widetilde{\Gamma}^{\mathcal{E}}(G)$ by $\Sigma(\widetilde{G}', G)$.

3. The transfer map and its adjoint

Suppose that a_G is a function on $\Gamma(G)$. If G' belongs to $\mathcal{E}_{\mathrm{ell}}(G)$, the transfer factors serve to define a function

$$a'(\sigma') \;=\; a^{G'}(\sigma') \;=\; \sum_{\gamma\in\Gamma(G)} \Delta(\sigma',\gamma)a_G(\gamma) \tag{3.1}$$

on $\Sigma_G(\widetilde{G}')$. In particular, we obtain a function $f' = f^{G'}$ on $\Sigma_G(\widetilde{G}')$ for any $f \in \mathcal{H}\big(G(F)\big)$ by taking $a_G = f_G$.

If Sections 7–9, we shall use global arguments to study the transfer mapping. These require the fundamental lemma (for units), which we shall have to take on as a hypothesis.

Hypothesis 3.1. *Suppose that F_1 is a p-adic field of characteristic 0, that G_1 is a connected reductive group over F_1 which is unramified, and that G_1' is an element in $\mathcal{E}_{\mathrm{ell}}(G_1)$, together with the auxiliary data $(\widetilde{G}_1', \widetilde{\zeta}_1')$, which is also unramified. Let $f \in \mathcal{H}\big(G_1(F_1)\big)$ and $g \in \mathcal{H}\big(\widetilde{G}_1'(F_1), \widetilde{\zeta}_1'\big)$ be the unit elements of unramified Hecke algebras relative to fixed hyperspecial maximal compact subgroups. Then the functions f' and g' on $\Sigma_{G_1}(\widetilde{G}_1')$ are equal.*

We will also rely heavily on Waldspurger's proof of the Langlands-Shelstad transfer conjecture. This depends on the fundamental lemma for Lie algebras, which we shall state as a second hypothesis. On the Lie algebra $\mathfrak{g}(F)$ of $G(F)$, one can define orbital integrals $\phi \to \phi_G$ and transfer factors $\Delta(S', X)$. These provide a transfer map $\phi \to \phi' = \phi^{G'}$ from $C_c^\infty\big(\mathfrak{g}(F)\big)$ to functions on $\Sigma_G(\mathfrak{g}') = \Sigma_G(\widetilde{\mathfrak{g}}'/\widetilde{z}')$, the Lie algebra analogue of $\Sigma_G(G')$. (See [27].)

Hypothesis 3.2. *Suppose that G_1/F_1 and G_1'/F_1 are as in Hypothesis 3.1, and that $\phi \in C_c^\infty\big(\mathfrak{g}_1(F_1)\big)$ and $\chi \in C_c^\infty\big(\mathfrak{g}_1'(F_1)\big)$ are the characteristic functions of fixed hyperspecial* *lattices in the corresponding Lie algebras $\mathfrak{g}_1(F_1)$ and $\mathfrak{g}_1'(F_1)$. Then the functions ϕ' and χ' on the G_1-regular, stable conjugacy classes in $\mathfrak{g}_1'(F)$ are equal.*

Under the assumption of Hypothesis 3.2, Waldspurger has established the fundamental transfer theorem.

Theorem 3.3. *(Waldspurger [28, 11.5]). For any $G' \in \mathcal{E}_{\mathrm{ell}}(G)$, the image of the transfer map*

$$f \longrightarrow f' \ , \qquad f \in \mathcal{H}\big(G(F)\big),$$

is contained in $S\mathcal{I}\big(\widetilde{G}'(F), \widetilde{\zeta}'\big)$.

From this point on, we shall generally use the more compact notation $\mathcal{I}(G)$, $\mathcal{I}_{\mathrm{cusp}}(G)$, $S\mathcal{I}(\widetilde{G}', \widetilde{\zeta}')$ etc., in order to focus on the relationships between various spaces of invariant functions. For each $G' \in \mathcal{E}_{\mathrm{ell}}(G)$, let us write $S\mathcal{I}(\widetilde{G}', G)$ for the subspace of functions in $S\mathcal{I}(\widetilde{G}', \widetilde{\zeta}')$ which depend only on the image in $\widetilde{\Gamma}^{\mathcal{E}}(G)$ of the variable $\sigma' \in \Sigma_G(\widetilde{G}')$. Then $S\mathcal{I}(\widetilde{G}', G)$ is a space of functions on the closed subset $\Sigma(\widetilde{G}', G)$ of $\widetilde{\Gamma}^{\mathcal{E}}(G)$. The intersection of $S\mathcal{I}(\widetilde{G}', G)$ with $S\mathcal{I}_{\mathrm{cusp}}(\widetilde{G}', \widetilde{\zeta}')$ equals the space

$$S\mathcal{I}_{\mathrm{cusp}}(\widetilde{G}', G) \ = \ S\mathcal{I}_{\mathrm{cusp}}(\widetilde{G}', \widetilde{\zeta}')^{\mathrm{Out}_G(G')}$$

of cuspidal functions which are symmetric under $\mathrm{Out}_G(G')$. It follows from Waldspurger's theorem and the definitions that $f \to f'$ maps $\mathcal{H}\big(G(F)\big)$ continuously into $S\mathcal{I}(\widetilde{G}', G)$ and maps $\mathcal{H}_{\mathrm{cusp}}\big(G(F)\big)$ continuously into $S\mathcal{I}_{\mathrm{cusp}}(\widetilde{G}', G)$. Equivalently, we can state things in terms of the map $a_G \to a'$. The space $\mathcal{I}(G)$ is sent continuously into $S\mathcal{I}(\widetilde{G}', G)$, while $\mathcal{I}_{\mathrm{cusp}}(G)$ is mapped continuously into $S\mathcal{I}_{\mathrm{cusp}}(\widetilde{G}', G)$.

We shall use the spaces $S\mathcal{I}_{\mathrm{cusp}}(\widetilde{G}', G)$ to construct an "endoscopic" space which is parallel to $\mathcal{I}_{\mathrm{cusp}}(G)$. Set

$$\mathcal{I}^{\mathcal{E}}_{\mathrm{cusp}}(G) \ = \bigoplus_{G' \in \mathcal{E}_{\mathrm{ell}}(G)} S\mathcal{I}_{\mathrm{cusp}}(\widetilde{G}', G) \ .$$

Then $\mathcal{I}^{\mathcal{E}}_{\text{cusp}}(G)$ is a topological vector space of smooth functions on $\Gamma^{\mathcal{E}}_{\text{ell}}(G)$. For any function $a_G \in \mathcal{I}_{\text{cusp}}(G)$, we define

$$a^{\mathcal{E}}_G \; = \bigoplus_{G' \in \mathcal{E}_{\text{ell}}(G)} a' \; ,$$

the direct sum of the images of a_G. Then

$$\mathcal{T}^{\mathcal{E}} : \; a_G \longrightarrow a^{\mathcal{E}}_G$$

is a continuous linear map from $\mathcal{I}_{\text{cusp}}(G)$ to $\mathcal{I}^{\mathcal{E}}_{\text{cusp}}(G)$. It is important for our purposes to know that the map is actually surjective. We shall use Waldspurger's results to establish this basic property.

Lemma 3.4. *$\mathcal{T}^{\mathcal{E}}$ maps $\mathcal{I}_{\text{cusp}}(G)$ onto $\mathcal{I}^{\mathcal{E}}_{\text{cusp}}(G)$.*

Proof. As in the proof of Lemma 2.2, we can appeal to the definitions of [22, (4.4)] to reduce the problem to the case of endoscopic data which are L-groups. We assume therefore that $\widetilde{G}' = G'$ for each $G' \in \mathcal{E}_{\text{ell}}(G)$. The lemma will be a corollary of Waldspurger's basic kernel formula [28, (1.2)]. We first recall this formula.

Fix a symmetric, nondegenerate G-invariant bilinear form B on $\mathfrak{g}$, and a nontrivial additive character ψ_0 on F. The Fourier transform

$$\widehat{\phi}(Y) \; = \; \int_{\mathfrak{g}(F)} \phi(X)\psi_0\big(B(X,Y)\big)dX \; , \qquad \phi \in C^\infty_c\big(\mathfrak{g}(F)\big),$$

is a linear isomorphism of $C^\infty_c\big(\mathfrak{g}(F)\big)$ onto itself. Moreover, there is a smooth function

$$i(X,Y) \; = \; i^G_G(X,Y)$$

of two variables in $\Gamma(\mathfrak{g})$, the space of regular $G(F)$-orbits in $\mathfrak{g}(F)$, such that

$$\phi_G(X) \; = \; \int_{\Gamma(\mathfrak{g})} i(X,Y)(\widehat{\phi})_G(Y)dY \; , \qquad X \in \Gamma(\mathfrak{g}),$$

for a certain measure dY on $\Gamma(\mathfrak{g})$ [11, Theorem 3]. If G is quasi-split, set

$$s(S,T) \;=\; s_G^G(S,T) \;=\; |\mathcal{K}_T|^{-1} \sum_{X \to S} \sum_{Y \to T} i(X,Y) \ ,$$

for regular stable G-orbits S and T in $\mathfrak{g}_{\mathrm{reg}}(F)$. The sums are over the $G(F)$-orbits in the given stable orbit, and $|\mathcal{K}_T|$ denotes the number of Y in the orbit of T. Waldspurger's kernel formula applies to any $G' \in \mathcal{E}_{\mathrm{ell}}(G)$. It is

$$\sum_{X \in \Gamma(\mathfrak{g})} \Delta(S',X) i(X,Y) \;=\; \gamma_0 \sum_{T' \in \Sigma_G(\mathfrak{g}')} s'(S',T')\Delta(T',Y) \ , \tag{3.2}$$

for elements $S' \in \Sigma_G(\mathfrak{g}')$ and $X \in \Gamma(\mathfrak{g})$, and a constant $\gamma_0 = \gamma_0(G,G')$. The set $\Sigma_G(\mathfrak{g}')$ here is the Lie algebra analogue of $\Sigma_G(G')$, and we have written $s'(S',T')$ for $s_{G'}^{G'}(S',T')$.

Fix $G' \in \mathcal{E}_{\mathrm{ell}}(G)$ and $S' \in \Sigma(\mathfrak{g}',G)$. If ϕ belongs to $C_c^\infty\big(\mathfrak{g}(F)\big)$, we can write

$$\begin{aligned} \phi'(S') \;&=\; \sum_{X \in \Gamma(\mathfrak{g})} \Delta(S',X)\phi_G(X) \\ &=\; \sum_X \Delta(S',X) \int_{\Gamma(\mathfrak{g})} i(X,Y)(\widehat{\phi})_G(Y)dY \\ &=\; \gamma_0 \int_{\Gamma(\mathfrak{g})} \Big(\sum_{T' \in \Sigma_G(\mathfrak{g}')} s'(S',T')\Delta(T',Y)\Big)(\widehat{\phi})_G(Y)dY \ , \end{aligned}$$

by (3.2) and the formula above for $\phi_G(X)$. A Lie algebra variant of Lemma 2.3 can be used to change the sum over T' and the integral over Y into a sum over Y and an integral over T'. We obtain a formula

$$\phi'(S') \;=\; \gamma_0 \int_{\Sigma_G(\mathfrak{g}')} s'(S',T')(\widehat{\phi})'(T')dT' \ . \tag{3.3}$$

We shall apply this formula with ϕ being a function supported on the regular elliptic set in $\mathfrak{g}(F)$.

Let $\mathcal{G}_{\mathrm{cusp}}(\mathfrak{g})$ be the space of germs

$$X \longrightarrow \phi_G(X) \ , \qquad \phi \in C_{c,\mathrm{cusp}}^\infty\big(\mathfrak{g}(F)\big),$$

around *0* of orbital integrals of cuspidal functions on $\mathfrak{g}(F)$. Then $\mathcal{G}_{\mathrm{cusp}}(\mathfrak{g})$ is a finite dimensional space of germs of functions of $X \in \Gamma(\mathfrak{g})$. For each $G' \in \mathcal{E}_{\mathrm{ell}}(G)$, we also define the space $S\mathcal{G}_{\mathrm{cusp}}(\mathfrak{g}')$ of germs of stable orbital integrals of cuspidal functions, and we set

$$S\mathcal{G}_{\mathrm{cusp}}(\mathfrak{g}', G) \;=\; S\mathcal{G}_{\mathrm{cusp}}(\mathfrak{g}')^{\mathrm{Out}_G(G')} \;.$$

Let $\tau^{\mathcal{E}}$ be the Lie algebra analogue of the map $\mathcal{T}^{\mathcal{E}}$. The main step is to show that $\tau^{\mathcal{E}}$ maps $\mathcal{G}_{\mathrm{cusp}}(\mathfrak{g})$ onto the finite dimensional vector space

$$\mathcal{G}^{\mathcal{E}}_{\mathrm{cusp}}(\mathfrak{g}) \;=\; \bigoplus_{G' \in \mathcal{E}_{\mathrm{ell}}(G)} S\mathcal{G}_{\mathrm{cusp}}(\mathfrak{g}', G) \;.$$

It is enough to consider elements in $\mathcal{G}^{\mathcal{E}}_{\mathrm{cusp}}(\mathfrak{g})$ which are supported on only one component. We therefore fix $G' \in \mathcal{E}_{\mathrm{ell}}(G)$, and take an arbitrary function

$$g'(S') \;, \qquad S' \in \Sigma_{\mathrm{ell}}(\mathfrak{g}', G) = \Sigma_{G,\mathrm{ell}}(\mathfrak{g}')/\mathrm{Out}_G(G'),$$

in $S\mathcal{G}_{\mathrm{cusp}}(\mathfrak{g}', G)$. Since $S\mathcal{G}_{\mathrm{cusp}}(\mathfrak{g}', G)$ is a finite dimensional vector space, it follows from (3.3) that we can represent $g'(S')$ as a finite linear combination

$$g'(S') \;=\; \sum_{i=1}^{n} \lambda_i s'(S', T'_i) \;,$$

for points $T'_i \in \Sigma_{G,\mathrm{ell}}(\mathfrak{g}')$. We can assume that the bilinear form $B'(\cdot,\cdot)$ on $\mathfrak{g}'$ is invariant under the group $\mathrm{Out}_G(G')$. It follows easily that the function $s'(S', T')$ is invariant under the diagonal action of $\mathrm{Out}_G(G')$ on the two variables. Since $g'(S')$ is invariant under $\mathrm{Out}_G(G')$, we can arrange matters so that $\mathrm{Out}_G(G')$ leaves invariant the finite set $\{T'_i\}$ and so that the coefficients $\{\lambda_i\}$ are constant on the $\mathrm{Out}_G(G')$-orbits. For each i, we can choose a compact neighbourhood V_i of T'_i in $\Sigma_{G,\mathrm{ell}}(\mathfrak{g}')$ such that

$$s'(S', T') \;=\; s'(S', T'_i)$$

for all $T' \in V_i$ and all S' sufficiently small. This is a straightforward consequence of Howe's finiteness theorem on the Lie algebra $\mathfrak{g}'(F)$ ([12], [11, Theorem 2]). We can therefore find a function $\alpha' \in C_c^\infty\big(\Sigma_{\mathrm{ell}}(\mathfrak{g}', G)\big)$ such that

$$g'(S') \;=\; \gamma_0 \int_{\Sigma_{G,\mathrm{ell}}(\mathfrak{g}')} s'(S', T')\alpha'(T')dT' \ ,$$

for all S' sufficiently small. Now the inversion formulas (2.4) and (2.5) from Lemma 2.2 have obvious Lie algebra analogues, which imply that the map

$$\tau^{\mathcal{E}} :\ C_c^\infty\big(\Gamma_{\mathrm{ell}}(\mathfrak{g})\big) \longrightarrow \bigoplus_{G_1' \in \mathcal{E}_{\mathrm{ell}}(G)} C_c^\infty\big(\Sigma_{\mathrm{ell}}(\mathfrak{g}_1', G)\big)$$

is actually a linear isomorphism. It follows from this that there is a function $\psi \in C_c^\infty\big(\mathfrak{g}_{\mathrm{reg,ell}}(F)\big)$ such that

$$\psi^{G_1'} \;=\; \begin{cases} \alpha', & \text{if } G_1' = G', \\ 0, & \text{otherwise,} \end{cases}$$

for any $G_1' \in \mathcal{E}_{\mathrm{ell}}(G)$. Let ϕ be the function in $C_c^\infty\big(\mathfrak{g}(F)\big)$ such that $\widehat{\phi} = \psi$. It follows from (3.3) that

$$\phi'(S') \;=\; \gamma_0 \int_{\Sigma_{G,\mathrm{ell}}(\mathfrak{g}')} s'(S', T')\alpha'(T')dT' \;=\; g'(S') \ ,$$

and that $\phi^{G_1'} = 0$ for any $G_1' \neq G'$. This establishes surjectivity for the map of germs on the Lie algebra.

The proof can now be completed using the results of Langlands-Shelstad [23] on descent. We shall just sketch the argument, referring the reader to Section 1 and Sections 2.1–2.3 of [23] for more details

Let $a_G^{\mathcal{E}} = \bigoplus_{G'} a'$ be an arbitrary function in $\mathcal{I}_{\mathrm{cusp}}^{\mathcal{E}}(G)$. According to the inversion formula (2.5), the function

$$a_G(\gamma) \;=\; \sum_{\sigma' \in \Gamma_{\mathrm{ell}}^{\mathcal{E}}(G)} \Delta(\gamma, \sigma')a_G^{\mathcal{E}}(\sigma') \ , \qquad \gamma \in \Gamma_{\mathrm{ell}}(G),$$

is such that

$$a_G^{\mathcal{E}}(\sigma') \;=\; \sum_{\gamma\in\Gamma_{\rm ell}(G)} \Delta(\sigma',\gamma)a_G(\gamma) \;=\; (\mathcal{T}^{\mathcal{E}}a_G)(\sigma')\ .$$

Together with the other formula (2.4), this tells us that $a_G^{\mathcal{E}}$ will be in the image of $\mathcal{I}_{\rm cusp}(G)$ if and only if the function a_G belongs to $\mathcal{I}_{\rm cusp}(G)$. We can assume that the components a' of $a_G^{\mathcal{E}}$ vanish except at one fixed element $G'\in\mathcal{E}_{\rm ell}(G)$. We have therefore to show that the function

$$a_G(\gamma) \;=\; \sum_{\sigma'\in\Sigma_{\rm ell}(G',G)} \Delta(\gamma,\sigma')a'(\sigma'), \qquad \gamma\in\Gamma_{\rm ell}(G), \tag{3.4}$$

belongs to $\mathcal{I}_{\rm cusp}(G)$.

The problem is a local one. Let ε be a fixed elliptic semisimple conjugacy class in $G(F)$, and let $\mathcal{G}_{\rm cusp}(G,\varepsilon)$ be the space of germs around ε of functions in $\mathcal{I}_{\rm cusp}(G)$. It is enough to show that for γ near ε, $a_G(\gamma)$ belongs to $\mathcal{G}_{\rm cusp}(G,\varepsilon)$. If $\varepsilon=1$, this follows from what we have already established on the Lie algebra, given the inversion formulas (2.4) and (2.5) and the fact that the exponential map is a linear bijection from $\mathcal{G}_{\rm cusp}(\mathfrak{g})$ to $\mathcal{G}_{\rm cusp}(G,1)$. The property also follows easily from the generalization [23, Lemma 3.5A] of (2.1) if ε is in the center of G. In general, let $\varepsilon_1',\dots,\varepsilon_n'$ be representatives in $G'(F)$ of the $\mathrm{Out}_G(G')$-orbits of stable conjugacy classes which are images of ε [23, Section 1.2], chosen so that each group

$$G_j' \;=\; G'_{\varepsilon_j'} \;=\; \mathrm{Cent}(G',\varepsilon_j')^0$$

is quasi-split [14, Lemma 3.3]. The image of a_G in $\mathcal{G}_{\rm cusp}(G,\varepsilon)$ depends only on the image of a' in the spaces $S\mathcal{I}_{\rm cusp}(G',\varepsilon_j')$ of germs of functions in $S\mathcal{I}_{\rm cusp}(G')$ around ε_j'. We can assume that these images all vanish except for one fixed j. In other words, we can restrict the sum in (3.4) to points $\sigma'\in\Sigma_{\rm ell}(G',G)$ which are close to ε_j'. Now the group G_j' determines an endoscopic datum in $\mathcal{E}_{\rm ell}(G_\varepsilon)$ [23, Section 1.4]. If γ and σ' are close to ε and ε_j' respectively, the terms $a_G(\gamma)$, $\Delta(\gamma,\sigma')$ and $a'(\sigma')$ can be identified with the corresponding objects for (G_ε,G_j') [23, Sections 1.6–1.7]. The fact that $a_G(\gamma)$ lies in

$\mathcal{G}_{\text{cusp}}(G, \varepsilon)$ then follows from the corresponding property we have established for the central element ε in G_ε. This completes the proof of the lemma. $\square$

We have introduced the inner product $(\cdot, \cdot)$ on the space $\mathcal{I}_{\text{cusp}}(G)$. We can also define an inner product on the second space $\mathcal{I}^{\mathcal{E}}_{\text{cusp}}(G)$ in terms of our inner products $(\cdot, \cdot)$ on the spaces $S\mathcal{I}_{\text{cusp}}(\widetilde{G}', \widetilde{\zeta}')$. If $a^{\mathcal{E}}_G = \bigoplus\limits_{G'} a'$ and $b^{\mathcal{E}}_G = \bigoplus\limits_{G'} b'$ are two functions in $\mathcal{I}^{\mathcal{E}}_{\text{cusp}}(G)$, set

$$(a^{\mathcal{E}}_G, b^{\mathcal{E}}_G) \;=\; \sum_{G' \in \mathcal{E}_{\text{ell}}(G)} \iota(G, G')(a', b') \;,$$

where

$$\iota(G, G') \;=\; |\text{Out}_G(G')|^{-1} |Z(\widehat{G}')^{\Gamma} / Z(\widehat{G})^{\Gamma}|^{-1} \;.$$

Proposition 3.5. *The map*

$$\mathcal{T}^{\mathcal{E}} : \; \mathcal{I}_{\text{cusp}}(G) \longrightarrow \mathcal{I}^{\mathcal{E}}_{\text{cusp}}(G)$$

is an isometric isomorphism.

Proof. It is obvious that $\mathcal{T}^{\mathcal{E}}$ is linear, and we have just shown that it is surjective. We can identify $\mathcal{I}_{\text{cusp}}(G)$ and $\mathcal{I}^{\mathcal{E}}_{\text{cusp}}(G)$ with spaces of functions on $\Gamma_{\text{ell}}(G)$ and $\widetilde{\Gamma}^{\mathcal{E}}_{\text{ell}}(G)$, respectively. It follows immediately from (2.4) and (2.5) that $\mathcal{T}^{\mathcal{E}}$ is an isomorphism, with inverse

$$b^{\mathcal{E}}_G \longrightarrow b_G \;, \qquad b^{\mathcal{E}}_G \in \mathcal{I}^{\mathcal{E}}_{\text{cusp}}(G),$$

given by

$$b_G(\gamma) \;=\; \sum_{\sigma' \in \Gamma^{\mathcal{E}}_{\text{ell}}(G)} \Delta(\gamma, \sigma') b^{\mathcal{E}}_G(\sigma'), \qquad \gamma \in \Gamma(G). \tag{3.5}$$

It remains to show that $\mathcal{T}^{\mathcal{E}}$ is an isometry.

Let a_G and $b^{\mathcal{E}}_G = \bigoplus\limits_{G'} b'$ be arbitrary functions in $\mathcal{I}_{\text{cusp}}(G)$ and $\mathcal{I}^{\mathcal{E}}_{\text{cusp}}(G)$. As above, we denote the functions $\mathcal{T}^{\mathcal{E}} a_G$ and $(\mathcal{T}^{\mathcal{E}})^{-1}(b^{\mathcal{E}}_G)$ by $a^{\mathcal{E}}_G = \bigoplus\limits_{G'} a'$ and b_G, respectively. We have to show that (a_G, b_G) equals $(a^{\mathcal{E}}_G, b^{\mathcal{E}}_G)$.

We can write the inner product $(a_G^{\mathcal{E}}, b_G^{\mathcal{E}})$ as

$$\sum_{G' \in \mathcal{E}_{\mathrm{ell}}(G)} \iota(G, G') \int_{\Sigma_{G,\mathrm{ell}}(G')} n(\sigma')^{-1} a'(\sigma') \overline{b'(\sigma')} d\sigma' \ ,$$

since the integrand depends only on the image of the element $\sigma' \in \Sigma_{G,\mathrm{ell}}(\widetilde{G}')$ in the set

$$\Sigma_{G,\mathrm{ell}}(\widetilde{G}')/\widetilde{Z}'(F) \ = \ \Sigma_{G,\mathrm{ell}}(\widetilde{G}'/\widetilde{Z}') \ = \ \Sigma_{G,\mathrm{ell}}(G') \ .$$

This equals

$$\sum_{G'} \iota(G, G') \int_{\Sigma_{G,\mathrm{ell}}(G')} \Big(\sum_{\gamma \in \Gamma_{\mathrm{ell}}(G)} |\mathcal{K}_{\sigma'}|^{-1} |\mathcal{K}_\gamma| a_G(\gamma) \overline{\Delta(\gamma, \sigma') b'(\sigma')} \Big) d\sigma' \ ,$$

by the definition of $\Delta(\gamma, \sigma')$. If $\Delta(\gamma, \sigma') \neq 0$, we have

$$\begin{aligned} & \iota(G, G') |\mathcal{K}_{\sigma'}|^{-1} |\mathcal{K}_\gamma| \\ & \quad = |\mathrm{Out}_G(G')|^{-1} \big| \pi_0 \big((\widehat{T}')^\Gamma / Z(\widehat{G})^\Gamma \big) \big|^{-1} \big| \pi_0 \big(\widehat{T}^\Gamma / Z(\widehat{G})^\Gamma \big) \big| \\ & \quad = |\mathrm{Out}_G(G')|^{-1} \ , \end{aligned}$$

since the tori $T' = G'_{\sigma'}$ and $T = G_\gamma$ are isomorphic. But the constant $|\mathrm{Out}_G(G')|^{-1}$ is exactly what is required to normalize the measure on the quotient $\Sigma_{\mathrm{ell}}(G', G)$ of $\Sigma_{G,\mathrm{ell}}(G')$. We obtain

$$\begin{aligned} (a_G^{\mathcal{E}}, b_G^{\mathcal{E}}) \ &= \ \sum_{G' \in \mathcal{E}_{\mathrm{ell}}(G)} \int_{\Sigma_{\mathrm{ell}}(G', G)} \sum_{\gamma \in \Gamma_{\mathrm{ell}}(G)} a_G(\gamma) \overline{\Delta(\gamma, \sigma') b'(\sigma')} d\sigma' \\ &= \int_{\Gamma_{\mathrm{ell}}^{\mathcal{E}}(G)} \sum_{\gamma \in \Gamma_{\mathrm{ell}}(G)} a_G(\gamma) \overline{\Delta(\gamma, \sigma') b_G^{\mathcal{E}}(\sigma')} d\sigma' \ . \end{aligned}$$

Applying Lemma 2.3, we see that this last expression can be written as

$$\int_{\Gamma_{\mathrm{ell}}(G)} a_G(\gamma) \sum_{\sigma' \in \Gamma_{\mathrm{ell}}^{\mathcal{E}}(G)} \overline{\Delta(\gamma, \sigma') b_G^{\mathcal{E}}(\sigma') d\gamma} \ ,$$

which is just

$$\int_{\Gamma_{\mathrm{ell}}(G)} a_G(\gamma) \overline{b_G(\gamma)} d\gamma \ = \ (a_G, b_G) \ ,$$

by (3.5). Therefore $\mathcal{T}^{\mathcal{E}}$ is an isometry. $\square$

The definitions at the end of Section 2 suggest a way to combine the noncuspidal spaces $S\mathcal{I}(\widetilde{G}', G)$ into a larger space $\mathcal{I}^{\mathcal{E}}(G)$, which contains $\mathcal{I}^{\mathcal{E}}_{\text{cusp}}(G)$, and is parallel to $\mathcal{I}(G)$. The set $\widetilde{\Gamma}^{\mathcal{E}}(G)$ is a union over $G' \in \mathcal{E}_{\text{ell}}(G)$ of subsets $\Sigma(\widetilde{G}', G)$ obtained from the stable conjugacy classes in $\widetilde{G}'(F)$. The subsets $\Sigma(\widetilde{G}', G)$ need not be disjoint, however, since they will intersect at Levi subgroups which are common to different G'. We define $\mathcal{I}^{\mathcal{E}}(G)$ to be the set of functions

$$a^{\mathcal{E}}_G = \bigoplus_{G' \in \mathcal{E}_{\text{ell}}(G)} a', \qquad a' \in S\mathcal{I}(\widetilde{G}', G),$$

with the property that a' and a'' have the same image in $S\mathcal{I}\big(\widetilde{M}'(F), \widetilde{\zeta}'\big)$, for any pair of elements G' and G'' in $\mathcal{E}_{\text{ell}}(G)$ which have the Levi subgroup M' in common. Then $\mathcal{I}^{\mathcal{E}}(G)$ can be identified with a space of smooth functions on $\widetilde{\Gamma}^{\mathcal{E}}(G)$. As with $\mathcal{I}(G)$, there is a natural filtration on $\mathcal{I}^{\mathcal{E}}(G)$ over the partially ordered set $\mathcal{L}/W_0$. Its associated graded vector space is

$$\mathcal{I}^{\mathcal{E}}_{\text{gr}}(G) = \bigoplus_{\{M\}} \mathcal{I}^{\mathcal{E}}_{\text{cusp}}(M)^{W(M)}.$$

For any function $a_G \in \mathcal{I}(G)$, we define a second function

$$a^{\mathcal{E}}_G = \bigoplus_{G' \in \mathcal{E}_{\text{ell}}(G)} a'$$

in $\mathcal{I}^{\mathcal{E}}(G)$ as a direct sum of endoscopic images of a_G. Then we have

$$a^{\mathcal{E}}_G(\sigma') = \sum_{\gamma \in \Gamma(G)} \Delta(\sigma', \gamma) a_G(\gamma), \qquad \sigma' \in \widetilde{\Gamma}^{\mathcal{E}}(G), \tag{3.6}$$

and

$$a_G(\gamma) = \sum_{\sigma' \in \Gamma^{\mathcal{E}}(G)} \Delta(\gamma, \sigma') a^{\mathcal{E}}_G(\sigma'), \qquad \gamma \in \Gamma(G), \tag{3.7}$$

for the transfer factors defined at the end of Section 2. It is clear that

$$\mathcal{T}^{\mathcal{E}} : a_G \longrightarrow a^{\mathcal{E}}_G$$

is a continuous linear map from $\mathcal{I}(G)$ into $\mathcal{I}^{\mathcal{E}}(G)$. By following the arguments from the proof of Lemma 3.4, we can establish that the map is surjective. (This assertion is also a direct consequence of the two theorems we will state in Section 6, and prove in Sections 8 and 9.) Therefore $\mathcal{T}^{\mathcal{E}}$ is a linear isomorphism from $\mathcal{I}(G)$ onto $\mathcal{I}^{\mathcal{E}}(G)$. The mapping is clearly compatible with the filtrations on the two spaces. One of our goals is to show that it is also compatible with gradings we shall define on the two spaces.

The discussion is easily adapted to the case of equivariant functions. As at the end of Section 1, suppose that Z is a central induced torus in G which is defined over F, and that ζ is a character on $Z(F)$. For any $G' \in \mathcal{E}_{\mathrm{ell}}(G)$, $\widetilde{Z}'Z$ is a central induced torus in $\widetilde{G}'$, and ζ pulls back to a character on the group $(\widetilde{Z}'Z)(F)$. Recall (2.1) that there is also a character $\widetilde{\zeta}'_Z$ on this group determined by the transfer factor. We shall write $\widetilde{\zeta}'\zeta$ for the product $\widetilde{\zeta}'_Z \otimes \zeta$ of the two characters on $(\widetilde{Z}'Z)(F)$. The transfer map sends $\mathcal{H}\big(G(F), \zeta\big)$ into $S\mathcal{I}(\widetilde{G}', \widetilde{\zeta}'\zeta)$. Its image is the subspace $S\mathcal{I}(\widetilde{G}', G, \zeta)$ of $S\mathcal{I}(\widetilde{G}', \widetilde{\zeta}'\zeta)$ composed of functions of $\sigma' \in \Sigma_G(\widetilde{G}')$ which depend only on the image of σ' in $\widetilde{\Gamma}^{\mathcal{E}}(G)$. Putting the spaces together as above, we obtain a general space $\mathcal{I}^{\mathcal{E}}(G, \zeta)$ and a transfer isomorphism $\mathcal{T}^{\mathcal{E}}$ from $\mathcal{I}(G, \zeta)$ onto $\mathcal{I}^{\mathcal{E}}(G, \zeta)$. The obvious variant of Lemma 3.5 asserts that $\mathcal{T}^{\mathcal{E}}$ maps $\mathcal{I}_{\mathrm{cusp}}(G, \zeta)$ isometrically onto the subspace $\mathcal{I}^{\mathcal{E}}_{\mathrm{cusp}}(G, \zeta)$ of cuspidal functions in $\mathcal{I}^{\mathcal{E}}(G, \zeta)$.

4. Virtual characters

We shall review the representation theoretic data which are dual to conjugacy classes. The irreducible tempered characters could well be regarded as the objects dual to semisimple conjugacy classes in $G(F)$. It is better, however, to take the family of virtual characters studied in [3] and [4].

This second family is parametrized by a set $T(G)$ introduced in [3, Section 3], and which we shall denote here by $\widetilde{T}(G)$. By definition then, $\widetilde{T}(G)$ is the set of W_0-orbits of (essential) triplets

$$\tau \;=\; (L,\pi,r)\;, \qquad L\in\mathcal{L},\ \pi\in\Pi_2(L),\ r\in\widetilde{R}_\pi,$$

where $\Pi_2(L)$ stands for the equivalence classes of irreducible unitary representations of $L(F)$ which are square integrable modulo the center, and $\widetilde{R}_\pi$ is a fixed central extension

$$1\longrightarrow Z_\pi\longrightarrow \widetilde{R}_\pi\longrightarrow R_\pi\longrightarrow 1$$

of the R-group of π. The purpose of the extension is to ensure that the normalized intertwining operators

$$r\longrightarrow \widetilde{R}_P(r,\pi)\;, \qquad r\in\widetilde{R}_\pi,\ P\in\mathcal{P}(L),$$

for the induced representation $\mathcal{I}_P(\pi)$ give a representation of $\widetilde{R}_\pi$ instead of just a projective representation of R_π. This representation of $\widetilde{R}_\pi$ has a central character on the central subgroup $Z_\tau = Z_\pi$, denoted by $\chi_\tau^{-1} = \chi_\pi^{-1}$. There is then a bijection $\rho\to\pi_\rho$ from $\Pi(\widetilde{R}_\pi,\chi_\pi)$, the set of irreducible representations of $\widetilde{R}_\pi$ whose central character on Z_π is equal to χ_π, onto the set of irreducible constituents of $\mathcal{I}_P(\pi)$, with the properties that

$$\mathrm{tr}\big(\widetilde{R}_P(r,\pi)\mathcal{I}_P(\pi,f)\big) \;=\; \sum_{\rho\in\Pi(\tilde{R}_\pi,\chi_\pi)} \mathrm{tr}\big(\rho^\vee(r)\big)\mathrm{tr}\big(\pi_\rho(f)\big)$$

and

$$\mathrm{tr}\big(\pi_\rho(f)\big) \;=\; |\widetilde{R}_\pi|^{-1}\sum_{r\in\tilde{R}_\pi}\mathrm{tr}\big(\rho(r)\big)\mathrm{tr}\big(\widetilde{R}_P(r,\pi)\mathcal{I}_P(\pi,f)\big)\;.$$

Of particular interest is the subset $\widetilde{T}_{\mathrm{ell}}(G)$ of (orbits of) elements $\tau=(L,\pi,r)$ in $\widetilde{T}(G)$ such that the kernel of $(1-r)$, acting on the space $\mathfrak{a}_L$, equals $\mathfrak{a}_G$. We can represent the original set $\widetilde{T}(G)$ as the set of W_0-orbits in the disjoint union $\coprod_{M\in\mathcal{L}}\big(\widetilde{T}_{\mathrm{ell}}(M)\big)$ of elliptic elements in Levi subgroups. We can therefore write

$$\widetilde{T}(G) \;=\; \coprod_{\{M\}}\big(\widetilde{T}_{\mathrm{ell}}(M)/W(M)\big)\;, \tag{4.1}$$

where $\{M\}$ as usual runs over the orbits in $\mathcal{L}/W_0$. There is an action

$$\tau \longrightarrow \tau_\lambda \;=\; (L, \pi_\lambda, r)\;, \qquad \tau \in \widetilde{T}_{\mathrm{ell}}(G),\; \lambda \in i\mathfrak{a}_G^*,$$

of $i\mathfrak{a}_G^*$ on $\widetilde{T}_{\mathrm{ell}}(G)$, where $\pi_\lambda(x)$ equals $\pi(x)e^{\lambda(H_L(x))}$ for any $x \in L(F)$. There is a similar action of $i\mathfrak{a}_M^*$ on $\widetilde{T}_{\mathrm{ell}}(M)$ for each M, and this gives $\widetilde{T}(G)$ the structure of a disjoint union of finite quotients of compact tori. We shall write $\widetilde{T}(G)_{\mathbb{C}}$ for the corresponding union of quotients of complex tori. Then $\widetilde{T}(G)_{\mathbb{C}}$ is the disjoint union over $\{M\}$ of the spaces of $W(M)$-orbits in

$$\widetilde{T}_{\mathrm{ell}}(M)_{\mathbb{C}} \;=\; \{\tau_\lambda :\; \tau \in \widetilde{T}_{\mathrm{ell}}(M),\; \lambda \in \mathfrak{a}_{M,\mathbb{C}}^*\}\;.$$

Observe that there is an internal action

$$\tau \longrightarrow z_\tau \tau \;=\; (L, \pi, z_\tau r)\;, \qquad z_\tau \in Z_\tau,$$

of the groups Z_τ on $\widetilde{T}(G)$. We have reserved the symbol $T(G)$ for the corresponding quotient of $\widetilde{T}(G)$. Then $T(G)$ is the set of W_0-orbits of triplets $\tau = (L, \pi, r)$, in which r simply lies in the R-group R_π. This can be written

$$T(G) \;=\; \coprod_{\{M\}} \big(T_{\mathrm{ell}}(M)/W(M)\big)\;. \tag{4.2}$$

We shall also write $T(G)_{\mathbb{C}}$ for the associated quotient of $\widetilde{T}(G)_{\mathbb{C}}$.

We are going to fix two objects which are a bit artificial in the present context. They will be needed in Section 5, but it is also useful to see here how they relate to the set $\widetilde{T}(G)$. The first object we fix is a finite group $\mathcal{O}$ of outer automorphisms of $G(F)$. We assume that there is an action $\tau \to \tau_\alpha$, $\alpha \in \mathcal{O}$, of $\mathcal{O}$ on $\widetilde{T}(G)$ that extends its canonical (right) action on $T(G)$, and commutes with the internal actions of the groups Z_τ. Since $\mathcal{O}$ also acts on $i\mathfrak{a}_{G,F}^*$, we can form the semi-direct product

$$\mathcal{O}^+ \;=\; i\mathfrak{a}_{G,F}^* \rtimes \mathcal{O}\;.$$

We then obtain a right action

$$\tau \longrightarrow \tau_\Lambda \ , \qquad \Lambda = (\lambda, \alpha) \in \mathcal{O}^+,$$

of $\mathcal{O}^+$ on $\widetilde{T}(G)$. The second object to fix is the central induced torus Z in G over F. We assume that Z contains A_G, and also that $\mathcal{O}$ normalizes $Z(F)$. The elements in $\widetilde{T}(G)$ have central characters on $Z(F)$. This provides a decomposition

$$\widetilde{T}(G) \ = \ \coprod_{\zeta} \widetilde{T}(G, \zeta)$$

parametrized by the characters ζ of $Z(F)$, in which $\widetilde{T}(G,\zeta)$ is the subset of elements in $\widetilde{T}(G)$ whose central character on $Z(F)$ equals ζ. We write $T(G,\zeta)$ and $T_{\mathrm{ell}}(G,\zeta)$ for the quotients of $\widetilde{T}(G,\zeta)$ and $\widetilde{T}_{\mathrm{ell}}(G,\zeta) = \widetilde{T}_{\mathrm{ell}}(G) \cap \widetilde{T}(G,\zeta)$ by the internal Z_τ-actions. We shall also sometimes write $\widetilde{T}_{\mathrm{par}}(G,\zeta)$ and $T_{\mathrm{par}}(G,\zeta)$ (and for that matter $\widetilde{T}_{\mathrm{par}}(G,\zeta)_{\mathbb{C}}$, $T_{\mathrm{par}}(G)$ etc.) for the "parabolic" elements in the given set, or in other words, the complement of the corresponding elliptic set.

The set $\widetilde{T}(G)$ parametrizes a family of locally integrable functions

$$\gamma \longrightarrow I(\tau, \gamma) \ , \qquad \tau \in \widetilde{T}(G),$$

of $\gamma \in \Gamma(G)$. The function attached to $\tau = (L, \pi, r)$ is just the normalized virtual character

$$I(\tau, \gamma) \ = \sum_{\rho \in \Pi(\tilde{R}_\pi, \chi_\pi)} \mathrm{tr}\big(\rho^\vee(r)\big) I(\pi_\rho, \gamma)$$

obtained from the normalized irreducible characters

$$I(\pi_\rho, \gamma) \ = \ |D(\gamma)|^{\frac{1}{2}} \Theta(\pi_\rho, \gamma) \ .$$

We summarize three properties of these functions.

(I_1) The action of the group $\mathcal{O}^+$ on $\widetilde{T}(G)$ satisfies

$$I(\tau_\Lambda, \gamma) \ = \ I\big(\tau, \alpha(\gamma)\big) e^{\lambda(H_G(\alpha(\gamma)))} \ , \qquad \Lambda = (\lambda, \alpha) \in \mathcal{O}^+.$$

(I_2) If τ belongs to a subset $\widetilde{T}(G,\zeta)$ of $\widetilde{T}(G)$, then

$$I(\tau,\gamma z) \;=\; I(\tau,\gamma)\zeta(z)\ , \qquad z\in Z(F).$$

(I_3) For any character ζ on $Z(F)$, the functions

$$\gamma \longrightarrow \overline{I(\tau,\gamma)}\ , \qquad \tau\in \widetilde{T}_{\mathrm{ell}}(G,\zeta),$$

taken up to equivalence in the quotient $T_{\mathrm{ell}}(G,\zeta)$, form an orthogonal basis of $\mathcal{I}_{\mathrm{cusp}}(G,\zeta)$. (See [3, Section 6].)

The internal actions of the groups Z_τ also contribute an identity

$$I(z_\tau\tau,\gamma) \;=\; \chi_\tau(z_\tau)^{-1}I(\tau,\gamma)\ , \qquad z_\tau\in Z_\tau.$$

Characters exist of course to be integrated against functions. To illustrate future constructions, let us tentatively write

$$a_{G,\mathrm{gr}}(\tau) \;=\; \int_{\Gamma(G)} I(\tau,\gamma)a_G(\gamma)d\gamma\ ,$$

for any $a_G\in\mathcal{I}(G)$ and $\tau\in\widetilde{T}(G)$. According to Kazhdan's theorem [13, Appendix, Theorem 1], $a_G\to a_{G,\mathrm{gr}}$ is a *injective* linear map from $\mathcal{I}(G)$ onto a space of functions on $\widetilde{T}(G)$. The trace Paley-Wiener theorem [6] in turn allows us to characterize the image. The two theorems together imply that the map is an isomorphism from $\mathcal{I}(G)$ onto the Paley-Wiener space on $\widetilde{T}(G)$, which can be defined as the space of functions α on $\widetilde{T}(G)$ such that

(i) α is supported on finitely many connected components of $\widetilde{T}(G)$.

(ii) On the connected component of any element $\tau\in\widetilde{T}_{\mathrm{ell}}(M)/W(M)$, $\alpha(\tau_\lambda)$ is a finite Fourier series in λ.

(iii) For any $\tau\in\widetilde{T}(G)$ and $z_\tau\in Z_\tau$, $\alpha(\tau z_\tau)$ equals $\chi_\tau(z_\tau)^{-1}\alpha(\tau)$. (See [3].)

The subspace $\mathcal{I}_{\mathrm{cusp}}(G)$ is mapped to the subspace of functions on $\widetilde{T}(G)$ which vanish on the complement $\widetilde{T}_{\mathrm{par}}(G)$ of $\widetilde{T}_{\mathrm{ell}}(G)$ in $\widetilde{T}(G)$. In other words, $\mathcal{I}_{\mathrm{cusp}}(G)$ can be identified with the Paley-Wiener space on $\widetilde{T}_{\mathrm{ell}}(G)$. Applied to each Levi subgroup M, this gives us a canonical isomorphism of the Paley-Wiener space on $\widetilde{T}(G)$ with the graded vector space

$$\mathcal{I}_{\mathrm{gr}}(G) \;=\; \bigoplus_{\{M\}} \mathcal{I}_{\mathrm{cusp}}(M)^{W(M)}$$

attached in Section 1 to the filtration on $\mathcal{I}(G)$. Therefore, $a_G \to a_{G,\mathrm{gr}}$ can be regarded as an isomorphism from $\mathcal{I}(G)$ onto $\mathcal{I}_{\mathrm{gr}}(G)$ which is compatible with the filtrations. With this understanding, we are free to drop the subscript gr, and to interpret $\mathcal{I}_{\mathrm{gr}}(G)$ simply as a grading in $\mathcal{I}(G)$ which is compatible with the filtration. In particular, an element $a_G \in \mathcal{I}(G)$ will be regarded as a function either on $\Gamma(G)$ or on $\widetilde{T}(G)$. Observe that if a_G is the image of a function $f \in \mathcal{H}\big(G(F)\big)$, we have the direct formula

$$f_G(\tau) \;=\; \mathrm{tr}\big(\widetilde{R}_P(r,\pi)\mathcal{I}_P(\pi,f)\big)\ , \qquad \tau = (L,\pi,r).$$

The spectral form of the inner product on $\mathcal{I}_{\mathrm{cusp}}(G)$ is given by the local trace formula, or rather the simple version of the local trace formula that applies to a pair of cuspidal functions [3, (4.15)]. We first define a measure on $T_{\mathrm{ell}}(G)$ by setting

$$\int_{T_{\mathrm{ell}}(G)} \alpha(\tau)d\tau \;=\; \sum_{\tau\in T_{\mathrm{ell}}(G)/i\mathfrak{a}^*_{G,\tau}} \int_{i\mathfrak{a}^*_{G,\tau}} \alpha(\tau_\lambda)d\lambda\ ,$$

for any function $\alpha \in C_c\big(T(G)\big)$. The integral on the right is over the compact torus

$$i\mathfrak{a}^*_{G,\tau} \;=\; i\mathfrak{a}^*_G/\mathfrak{a}^\vee_{G,\tau}\ ,$$

where $\mathfrak{a}^\vee_{G,\tau}$ is the stabilizer of τ in $i\mathfrak{a}^*_G$, a lattice that lies between $\mathfrak{a}^\vee_{G,F}$ and $\widetilde{\mathfrak{a}}^\vee_{G,F}$, and $d\lambda$ is the measure we have fixed on $i\mathfrak{a}^*_G$. For future reference, we also define a measure

$$\int_{T(G)} \alpha(\tau)d\tau \;=\; \sum_{\{M\}} |W(M)|^{-1}\int_{\Gamma_{\mathrm{ell}}(M)} \alpha(\tau_M)d\tau_M$$

on the larger space $T(G)$. A similar formula, based on the spaces $i(\mathfrak{a}_M^G)^*$ instead of $i\mathfrak{a}_M^*$, determines a measure on $T(G,\zeta)$. Now, for any element $\tau = (L,\pi,r)$ in $T_{\mathrm{ell}}(G)$, set

$$n(\tau) \;=\; |R_{\pi,r}||\det(1-r)_{\mathfrak{a}_L/\mathfrak{a}_G}|\ ,$$

where $R_{\pi,r}$ is the centralizer of r in the R-group R_π. The spectral form of our inner product on $\mathcal{I}_{\mathrm{cusp}}(G)$ is then given by

$$(a_G,b_G) \;=\; \int_{T_{\mathrm{ell}}(G)} n(\tau)^{-1}a_G(\tau)\overline{b_G(\tau)}d\tau\ , \qquad a_G,b_G\in\mathcal{I}_{\mathrm{cusp}}(G).$$

(See [3, Corollary 3.2]. We have made allowance for the fact that our measure on $T_{\mathrm{ell}}(G)$ differs from the one defined by [3, (3.5)] by a factor $|R_{\pi,r}|$.) This formula is equivalent to the orthogonality of the characters $\{I(\tau,\gamma)\}$, together with the formula

$$\int_{\Gamma_{\mathrm{ell}}(G/Z)} I(\tau,\gamma)\overline{I(\tau,\gamma)}d\gamma \;=\; n(\tau)$$

for their norms [3, Corollary 6.2].

We would also like to have a spectral interpretation of the space $\mathcal{I}^{\mathcal{E}}_{\mathrm{cusp}}(G)$. For this we require orthogonal bases of the spaces $S\mathcal{I}_{\mathrm{cusp}}(\widetilde{G}',G)$. We shall construct such objects abstractly in Section 5. In the meantime, we can motivate the construction by recalling some aspects of the conjectural theory of endoscopy. A reader could easily skip this discussion, and go directly to Section 5.

Assume that G is quasi-split for the rest of this section. Stable characters on $G(F)$ ought to be attached to Langlands parameters

$$\phi:\ L_F\longrightarrow {}^LG\ ,$$

which are maps from the Langlands group $L_F = W_F\times SU(2,\mathbb{C})$ into the L-group ${}^LG$. For this discussion, let $\Phi(G)$ denote the set of $\widehat{G}$-orbits of parameters which are tempered (the image of L_F in $\widehat{G}$ is bounded), and let $\Phi_2(G)$ be the subset of parameters in $\Phi(G)$ which are cuspidal (the image of L_F is contained in no proper parabolic subgroup). As in the earlier case of $T(G)$, there is a canonical decomposition

$$\Phi(G) \;=\; \coprod_{\{M\}} \big(\Phi_2(M)/W(M)\big)\ .$$

Recall that for any ϕ, S_ϕ denotes the centralizer in $\widehat{G}$ of the image of ϕ, and $\mathcal{S}_\phi$ stands for the group of connected components in $\overline{S}_\phi = S_\phi/Z(\widehat{G})^\Gamma$. Suppose that Z is a central induced torus in G over F. Any parameter $\phi \in \Phi(G)$ has a central character ζ on $Z(F)$, whose Langlands parameter is just the composition

$$W_F \xrightarrow{\ \phi\ } {}^LG \longrightarrow {}^LZ\ .$$

The entire set $\Phi(G)$ decomposes into a disjoint union over ζ of the subsets $\Phi(G,\zeta)$ of parameters with central character ζ. The set $\Phi(G)$ also comes with an action $\phi \to \phi_\lambda = \phi\cdot\rho_\lambda$ of $i\mathfrak{a}^*_{G,F}$, where ρ_λ is the unramified parameter which maps the Frobenius element to the image of λ in $\big(Z(\widehat{G})^\Gamma\big)^0$ under the exponential map. (Recall that $\mathfrak{a}^*_{G,\mathbb{C}}$ is the Lie algebra of $\big(Z(\widehat{G})^\Gamma\big)^0$.) If $\mathcal{O}$ is a finite group of outer automorphisms of G which are defined over F, we can extend this to an action

$$\phi \longrightarrow \phi_\Lambda\ , \qquad \Lambda = (\lambda,\alpha),$$

of the semi-direct product $\mathcal{O}^+ = i\mathfrak{a}^*_{G,F} \rtimes \mathcal{O}$ on $\Phi(G)$.

For each $\phi \in \Phi_2(G)$, we expect to have a canonical family of nonnegative integers $\{\Delta(\phi,\pi) :\ \pi \in \Pi_2(G)\}$ with the property that the function

$$S(\phi,\sigma) \;=\; \sum_{\pi\in\Pi_2(G)} \Delta(\phi,\pi) I(\pi,\gamma)\ , \qquad \gamma \in \Gamma(G), \tag{4.3}$$

depends only on the stable conjugacy class σ of γ. The set

$$\Pi_\phi \;=\; \{\pi \in \Pi_2(G) :\ \Delta(\phi,\pi) \;>\; 0\}$$

should have cardinality equal to that of the set $\widehat{\mathcal{S}}_\phi$ of irreducible characters of $\mathcal{S}_\phi$, with the number of particular values $\Delta(\phi,\pi)$ matching the number of characters having corresponding degrees. The packets Π_ϕ ought to be disjoint, and to have union equal to $\Pi_2(G)$. Finally, the "stable characters" $S(\phi,\sigma)$ should satisfy

$$S(\phi_\Lambda,\sigma) \;=\; S\big(\phi,\alpha(\sigma)\big)e^{\lambda(H_G(\alpha(\sigma)))}\ , \qquad \Lambda=(\lambda,\alpha)\in\mathcal{O}^+,$$

and

$$S(\phi,\sigma z) \;=\; S(\phi,\sigma)\zeta(z)\ , \qquad \phi\in\Phi_2(G,\zeta),\ z\in Z(F).$$

Suppose that all these properties hold. The function (4.3) can be regarded as a stable distribution, since it depends only on the stable conjugacy class σ of γ. This is not quite the same thing as a stable orbital integral, even on the elliptic set. Stable orbital integrals are defined by summing over γ in the stable class σ, which amounts to multiplying (4.3) by the integer $n(\sigma)=|\mathcal{K}_\sigma|$. By doing this, and taking the complex conjugate as well, we obtain a family of functions

$$\sigma \longrightarrow n(\sigma)\overline{S(\phi,\sigma)}\ , \qquad \phi\in\Phi_2(G,\zeta),\ \sigma\in\Sigma_{\mathrm{ell}}(G), \tag{4.4}$$

in $S\mathcal{I}_{\mathrm{cusp}}(G,\zeta)$. The inner product of two such functions equals

$$\begin{aligned}
&\int_{\Sigma_{\mathrm{ell}}(G/Z)} n(\sigma)^{-1}n(\sigma)S(\phi,\sigma)\overline{n(\sigma)S(\phi_1,\sigma)}d\sigma\\
&\qquad = \int_{\Gamma_{\mathrm{ell}}(G/Z)}\Big(\sum_{\pi\in\Pi_\phi}\Delta(\phi,\pi)I(\pi,\gamma)\Big)\overline{\Big(\sum_{\pi_1\in\Pi_{\phi_1}}\Delta(\phi_1,\pi_1)I(\pi_1,\gamma)\Big)}d\gamma\\
&\qquad = \sum_{\pi,\pi_1}\Delta(\phi,\pi)\Delta(\phi_1,\pi_1)\delta(\pi,\pi_1)\\
&\qquad = \delta(\phi,\phi_1)n(\phi)\ ,
\end{aligned}$$

where $n(\phi)=|\mathcal{S}_\phi|$, and $\delta(\cdot,\cdot)$ is the Kronecker delta. We have used the orthogonality relations for the characters $\{I(\pi,\gamma)\}$ and the disjointness of the packets $\{\Pi_\phi\}$. Thus, (4.4) is an orthogonal family of functions. It should in fact be an orthogonal basis of $S\mathcal{I}_{\text{cusp}}(G,\zeta)$. If this is so for each ζ, we can identify $S\mathcal{I}_{\text{cusp}}(G)$ with the space of functions

$$a^G(\phi) \;=\; \int_{\Sigma_{\text{ell}}(G)} S(\phi,\sigma)a^G(\sigma)d\sigma\ , \qquad a^G \in S\mathcal{I}_{\text{cusp}}(G), \tag{4.5}$$

of $\phi \in \Phi_2(G)$.

The theory of endoscopy would apply in this way to each of the quasi-split groups $G' \in \mathcal{E}_{\text{ell}}(G)$. Assume for simplicity that we can take $\widetilde{G}' = G'$ for every G'. We would then obtain parameters $\Phi_2(G')$, functions $S(\phi',\sigma')$ on $\Sigma_{\text{ell}}(G')$, linear forms $a'(\sigma')$ on $S\mathcal{I}_{\text{cusp}}(G')$, and the corresponding linear forms $f'(\phi')$ on $\mathcal{I}_{\text{cusp}}(G)$ obtained from the transfer map. The latter objects depend only on the image of ϕ' in $\Phi_2(G',G)$, the set of $\text{Aut}_G(G')$-orbits in $\Phi_2(G')$. They will have expansions

$$f'(\phi') \;=\; \sum_{\tau \in T_{\text{ell}}(G)} \Delta(\phi',\tau)f_G(\tau)\ ,$$

for coefficients $\Delta(\phi',\tau)$ on $\Phi_2(G',G)\times\widetilde{T}_{\text{ell}}(G)$ which are clearly to be regarded as representation theoretic analogues of the transfer factors $\Delta(\sigma',\gamma)$. As a function on the Cartesian product of

$$T^{\mathcal{E}}_{\text{ell}}(G) \;=\; \{(G',\phi') :\ G' \in \mathcal{E}_{\text{ell}}(G),\ \phi' \in \Phi_2(G',G)\}$$

with $\widetilde{T}_{\text{ell}}(G)$, $\Delta(\phi',\tau)$ is simply the matrix of the transfer map $\mathcal{T}^{\mathcal{E}}$ with respect to two bases. We shall give the details of these constructions in Section 5 for our formal replacements of the Langlands parameters.

Of course the conjectural theory of endoscopy would give much more than we will be able to deduce in this paper. What distinguishes the genuine Langlands parameters from the formal objects we shall define is the existence of a natural map from $\Phi_2(G',G)$ to $\Phi(G)$. The image of $\Phi_2(G',G)$ in $\Phi(G)$ does not have to be contained in $\Phi_2(G)$. We have reserved the symbol $\Phi_{\text{ell}}(G)$ for the union over G' of these images. Suppose that (G',ϕ') is a pair in $T^{\mathcal{E}}_{\text{ell}}(G)$, and ϕ is the image of ϕ' in $\Phi_{\text{ell}}(G)$. Then the semi-simple element s'

attached to G', taken modulo $Z(\widehat{G})^{\Gamma}$, lies in the subset $\overline{S}_{\phi,\mathrm{fin}}$ of semi-simple elements in $\overline{S}_{\phi}$ whose centralizer in $\overline{S}_{\phi}$ is finite. One can show that $(G',\phi')\to(\phi,s')$ is a bijection from $T^{\mathcal{E}}_{\mathrm{ell}}(G)$ onto the set

$$\{(\phi,s'):\ \phi\in\Phi_{\mathrm{ell}}(G),\ s'\in\overline{S}_{\phi,\mathrm{fin}}/\overline{S}_{\phi}\}\ ,$$

where $\overline{S}_{\phi,\mathrm{fin}}/\overline{S}_{\phi}$ is the set of $\overline{S}_{\phi}$-conjugacy classes in $\overline{S}_{\phi,\mathrm{fin}}$.

The set $\widetilde{T}_{\mathrm{ell}}(G)$ should have a parallel interpretation. Suppose that ϕ belongs to $\Phi_{\mathrm{ell}}(G)$. The identity component $\overline{S}^{0}_{\phi}$ of $\overline{S}_{\phi}$ must then be a torus in $\widehat{G}$, and the centralizer of $\overline{S}^{0}_{\phi}$ in $\widehat{G}$ will be the dual group of a Levi subgroup L of G. Let $\mathcal{S}^{1}_{\phi}$ be the group of components in $\mathcal{S}_{\phi}$ which centralize $\overline{S}^{0}_{\phi}$. Then $\mathcal{S}^{1}_{\phi}$ equals $\mathcal{S}_{\phi_L}$ for a parameter $\phi_L\in\Phi_2(L)$ whose composition with the embedding ${}^{L}L\subset{}^{L}G$ equals ϕ. There should be a bijection $\pi\to\langle\cdot,\pi\rangle$ from the L-packet Π_{ϕ_L} for L onto the set of characters $\widehat{\mathcal{S}}^{1}_{\phi}$ of $\mathcal{S}_{\phi}$. But the set $\widehat{\mathcal{S}}^{1}_{\phi}$ has a natural action of the quotient group $R_{\phi}=\mathcal{S}_{\phi}/\mathcal{S}^{1}_{\phi}$. The R-group R_{π} of any $\pi\in\Pi_{\phi_L}$ should simply be the stabilizer R_{ξ} in R_{ϕ} of the corresponding character $\xi=\langle\cdot,\pi\rangle$. There is no a priori reason why ξ should extend to an irreducible character on the preimage of R_{ξ} in $\mathcal{S}_{\phi}$. The obstruction will be a class in $H^2(R_{\xi},\mathbb{C}^*)$. The group $\widetilde{R}_{\pi}$ defined earlier ought to be any extension $\widetilde{R}_{\xi}$ of R_{ξ} which splits this cocycle. It follows that there should be a bijection $(L,\pi,r)\to(\phi,\xi,r)$ from $\widetilde{T}_{\mathrm{ell}}(G)$ onto the set

$$\{(\phi,\xi,r):\ \phi\in\Phi_{\mathrm{ell}}(G),\ \xi\in\widehat{S}^{1}_{\phi},\ r\in\widetilde{R}_{\xi,\mathrm{reg}}\}\ ,$$

where $\widetilde{R}_{\xi,\mathrm{reg}}$ is the set of elements in $\widetilde{R}_{\xi}$ which have a finite centralizer in $\overline{S}^{0}_{\phi}$. Notice that the question of whether any of the extensions $\widetilde{R}_{\pi}$ are nontrivial has been formulated purely in terms of the parameters. It appears to be quite accessible in this form, but I do not know the answer.

5. Spectral transfer factors

In the absence of the local theory of endoscopy, the parameters will have to be relegated to the role of abstract indices. We will simply choose a family of functions

$$\sigma \longrightarrow S(\phi, \sigma)\ , \qquad \sigma \in \Sigma_{\mathrm{ell}}(G),$$

which satisfy some general conditions. We will denote the set of indices ϕ by $\Phi_2(G)$, with the expectation that they may eventually be taken to be Langlands parameters.

For the time being, G will be quasi-split. We fix a finite group $\mathcal{O}$ of outer automorphisms of G defined over F. Then $\mathcal{O}$ acts by permutation on $\Sigma(G)$. We form the semi-direct product $\mathcal{O}^+ = i\mathfrak{a}^*_{G,F} \rtimes \mathcal{O}$ as before, and define the action

$$a^G \longrightarrow a^G_\Lambda\ , \qquad \Lambda = (\lambda, \alpha) \in \mathcal{O},$$

of $\mathcal{O}^+$ on the space of functions $\{a^G\}$ on $\Sigma(G)$ by setting

$$a^G_\Lambda(\sigma) \ = \ a^G\big(\alpha(\sigma)\big)e^{\lambda(H_G(\alpha(\sigma)))}\ , \qquad \sigma \in \Sigma(G).$$

We must incorporate this into our construction. We have also to keep track of central characters. We have fixed a central, induced torus Z of G over F, which contains A_G, and which we assume is normalized by $\mathcal{O}$. We begin with a fixed character ζ on $Z(F)$. The stabilizer of ζ in $\mathcal{O}^+$ is the finite group

$$\mathcal{O}^+_\zeta \ = \ (\widetilde{\mathfrak{a}}^\vee_{G,F}/\mathfrak{a}^\vee_{G,F}) \rtimes \mathcal{O}_\zeta\ ,$$

where $\mathcal{O}_\zeta$ is the stabilizer of ζ in $\mathcal{O}$. We require a family of functions in $S\mathcal{I}_{\mathrm{cusp}}(G, \zeta)$ on which $\mathcal{O}^+_\zeta$ acts by permutation.

Lemma 5.1. *We can choose a family of functions*

$$\sigma \longrightarrow S(\phi, \sigma)\ , \qquad \phi \in \Phi_2(G, \zeta),$$

of $\sigma \in \Sigma_{\mathrm{ell}}(G)$, indexed by a set $\Phi_2(G,\zeta)$, with the following two properties.

(i) The family remains invariant under the action of $\mathcal{O}_\zeta^+$.

(ii) The functions

$$\sigma \longrightarrow n(\sigma)\overline{S(\phi, \sigma)}\ , \qquad \phi \in \Phi_2(G, \zeta),$$

form an orthogonal basis of $S\mathcal{I}_{\mathrm{cusp}}(G,\zeta)$.

Proof. The group $\mathcal{O}_\zeta$ acts on functions through its permutation representation on $\Sigma_{\mathrm{ell}}(G)$. This action obviously leaves invariant the coefficients $n(\sigma) = |\mathcal{K}_\sigma|$. Our task, then, is to construct an orthogonal basis of functions

$$\sigma \longrightarrow S_\phi(\sigma) \ = \ n(\sigma)\overline{S(\phi,\sigma)}$$

in $S\mathcal{I}_{\mathrm{cusp}}(G,\zeta)$ on which $\mathcal{O}_\zeta^+$ acts by permutation. The idea is simple enough. Since $\mathcal{O}_\zeta^+$ preserves the inner product on $S\mathcal{I}_{\mathrm{cusp}}(G,\zeta)$, we can decompose the space into an orthogonal direct sum of finite dimensional subspaces on which $\mathcal{O}_\zeta^+$ acts irreducibly. What we want is a decomposition of $S\mathcal{I}_{\mathrm{cusp}}(G,\zeta)$ into an orthogonal direct sum of permutation representations of $\mathcal{O}_\zeta^+$. By exploiting the infinite dimensionality of $S\mathcal{I}_{\mathrm{cusp}}(G,\zeta)$, we will avoid having to deal with the finite dimensional representation of $\mathcal{O}_\zeta^+$ on the space $S\mathcal{G}_{\mathrm{cusp}}(G,1)$ of stable cuspidal germs.

The homomorphism

$$H_G:\ G(F) \longrightarrow \mathfrak{a}_{G,F}$$

maps $G(F)/Z(F)$ surjectively onto the finite abelian group $\mathcal{A} = \mathfrak{a}_{G,F}/\widetilde{\mathfrak{a}}_{G,F}$ which is dual to $\widetilde{\mathfrak{a}}_{G,F}^\vee/\mathfrak{a}_{G,F}^\vee$. For any $X \in \mathcal{A}$, let $S\mathcal{I}_{\mathrm{cusp}}(G,\zeta)^X$ be the space of functions in $S\mathcal{I}_{\mathrm{cusp}}(G,\zeta)$

that are supported on the inverse image of X. We shall say that X is *elliptic* if the space is nonempty, and we write $\mathcal{A}_E$ for the set of elliptic elements in $\mathcal{A}$. Then

$$S\mathcal{I}_{\mathrm{cusp}}(G,\zeta) \;=\; \bigoplus_{X\in\mathcal{A}_E} S\mathcal{I}_{\mathrm{cusp}}(G,\zeta)^X \;.$$

Observe that X is elliptic if and only if G/Z has an elliptic maximal torus T over F such that the set

$$T(F)^X \;=\; \{t\in T(F):\; H_G(t)=X\}$$

is nonempty. In particular, if X is elliptic, so is every element in the cyclic group generated by X, as well as every element in the $\mathcal{O}_\zeta$-orbit of X. We shall prove the lemma directly, without worrying about whether every element in $\mathcal{A}$ is elliptic.

Let $\mathcal{O}_\zeta^X$ be the stabilizer in $\mathcal{O}_\zeta$ of a given element $X\in\mathcal{A}_E$. We shall look at the linear representation of $\mathcal{O}_\zeta^X$ on $S\mathcal{I}_{\mathrm{cusp}}(G,\zeta)^X$. Let Σ_ζ^X be the subset of

$$\Sigma_{\mathrm{ell}}(G)^X \;=\; \{\sigma\in\Sigma_{\mathrm{ell}}(G):\; H_G(\sigma)=X\}$$

on which $\mathcal{O}_\zeta^X$ acts properly discontinuously. Keeping in mind how outer automorphisms act on maximal tori, and noting that $\Sigma_{\mathrm{ell}}(G)^X$ is an open subset of $\Sigma(G)$, we observe that Σ_ζ^X is an open dense subset of $\Sigma_{\mathrm{ell}}(G)^X$. In particular, the space $C_c^\infty(\Sigma_\zeta^X,\zeta)$ of smooth ζ^{-1}-equivariant functions on Σ_ζ^X with compact support modulo $Z(F)$, is an infinite dimensional subspace of $S\mathcal{I}_{\mathrm{cusp}}(G,\zeta)^X$. Since $\mathcal{O}_\zeta^X$ acts properly discontinuously on Σ_ζ^X, the corresponding linear action of $\mathcal{O}_\zeta^X$ on $C_c^\infty(\Sigma_\zeta^X,\zeta)$ is isomorphic to a countably infinite number of copies of the regular representation. We conclude that the multiplicity of any irreducible representation of $\mathcal{O}_\zeta^X$ in the $\mathcal{O}_\zeta^X$-module $S\mathcal{I}_{\mathrm{cusp}}(G,\zeta)^X$ is infinite. This is all we need.

We base the construction on the family $\mathcal{C}_E$ of subgroups C of $\mathcal{A}$ which are contained in $\mathcal{A}_E$. We choose an orthogonal basis $\mathcal{B}$ of $S\mathcal{I}_{\mathrm{cusp}}(G,\zeta)^0$ on which $\mathcal{O}_\zeta$ acts by permutation, together with a partition $\mathcal{B}=\coprod\limits_C \mathcal{B}_C$ indexed by the elements $C\in\mathcal{C}_E$, which is compatible with the action of $\mathcal{O}_\zeta$. In other words, $\alpha(\mathcal{B}_C)$ equals $\mathcal{B}_{\alpha C}$ for every $\alpha \in \mathcal{O}_\zeta$. We also assume that for each $X \in \mathcal{A}_E$, every irreducible representation of $\mathcal{O}_\zeta^X$ has infinite multiplicity in the linear representation attached to the permutation representation of $\mathcal{O}_\zeta^X$ on $\mathcal{B}(X) = \coprod_{\{C: X\in\mathcal{C}\}} \mathcal{B}_C$. This is clearly possible, given the condition above on the representation of $\mathcal{O}_\zeta$ on $S\mathcal{I}_{\text{cusp}}(G,\zeta)^0$. The same condition, with $X \neq 0$, allows us to choose orthogonal bases $\mathcal{B}^X$ of the spaces $S\mathcal{I}_{\text{cusp}}(G,\zeta)^X$, partitions

$$\mathcal{B}^X \ = \coprod_{\{C\in\mathcal{C}_E:\ X\in C\}} \mathcal{B}_C^X \ ,$$

and bijections

$$\beta_C^X :\ \mathcal{B}_C \longrightarrow \mathcal{B}_C^X \ , \qquad C \in \mathcal{C}_E,\ X \in C,$$

which are norm preserving and compatible with the action of $\mathcal{O}_\zeta$. The last condition, in more precise terms, is that any $\alpha \in \mathcal{O}_\zeta$ maps $\mathcal{B}_C^X$ bijectively to $\mathcal{B}_{\alpha C}^{\alpha X}$, and satisfies

$$\alpha\beta_C^X\alpha^{-1} \ = \ \beta_{\alpha C}^{\alpha X} \ .$$

We can now construct the required orthogonal basis of $S\mathcal{I}_{\text{cusp}}(G,\zeta)$. It will be parametrized by the set of triplets

$$\phi \ = \ (C, b, \mu) \ , \qquad C \in \mathcal{C}_E,\ b \in \mathcal{B}_C,\ \mu \in \widehat{C}.$$

The function in $S\mathcal{I}_{\text{cusp}}(G,\zeta)$ corresponding to $\phi = (C, b, \mu)$ is defined to be

$$S_\phi \ = \ \sum_{X\in C} \mu(X)\beta_C^X(b) \ .$$

If $\phi' = (C', b', \mu')$ is another triplet, the inner product $(S_\phi, S_{\phi'})$ vanishes by construction unless $C' = C$ and $b' = b$, in which case it equals

$$\begin{aligned}\sum_{X\in C} \big(\beta_C^X(b), \beta_C^X(b)\big)\mu(X)\overline{\mu'(X)} \ &= \ (b,b)\sum_X \mu(X)\overline{\mu'(X)} \\ &= \ (b,b)\delta(\mu,\mu') \ .\end{aligned}$$

It follows that the family of functions $\{S_\phi\}$ obtained in this way is an orthogonal set. Any element in $\mathcal{B}_C^X$ is of the form $\beta_C^X(b)$, and can be written as

$$|C|^{-1} \sum_{\mu \in \widehat{C}} \mu(X)^{-1} S_{(C,b,\mu)} \ .$$

Since the union over C and X of the sets $\mathcal{B}_C^X$ spans $S\mathcal{I}_{\mathrm{cusp}}(G,\zeta)$, so does our family $\{S_\phi\}$. Therefore, $\{S_\phi\}$ is an orthogonal basis.

Finally, consider elements $\lambda \in \widehat{\mathcal{A}}$ and $\alpha \in \mathcal{O}_\zeta$. If $\phi = (C, b, \mu)$, we have

$$(S_\phi)_\lambda \ = \ \sum_{X \in C} \lambda(X)\mu(X)\beta_C^X(b) \ = \ \sum_X (\lambda\mu)(X)\beta_C^X(b) \ = \ S_{\phi_\lambda} \ ,$$

where $\phi_\lambda = (C, b, \lambda\mu)$. Moreover,

$$\alpha(S_\phi) \ = \ \sum_{X \in C} \mu(X)\alpha\big(\beta_C^X(b)\big) \ = \ \sum_X (\alpha\mu)(\alpha X)\beta_{\alpha C}^{\alpha X}(\alpha b) \ = \ S_{\alpha\phi} \ ,$$

where $\alpha\phi = \phi_{\alpha^{-1}} = (\alpha C, \alpha b, \alpha\mu)$. It follows that the group $\mathcal{O}_\zeta^+ = \widehat{\mathcal{A}} \rtimes \mathcal{O}_\zeta$ acts by permutation on our basis $\{S_\phi\}$. The construction is complete. $\square$

If $\Lambda = (\lambda, \alpha)$ is an element in $\mathcal{O}^+$,

$$\zeta_\Lambda(z) \ = \ \zeta\big(\alpha(z)\big)e^{\lambda(H_G(\alpha(z)))} \ , \qquad z \in Z(F),$$

is another character on $Z(F)$. For any $\phi \in \Phi_2(G, \zeta)$, the function

$$\sigma \longrightarrow S\big(\phi, \alpha(\sigma)\big)e^{\lambda(H_G(\alpha(\sigma)))} \ , \qquad \sigma \in \Sigma_{\mathrm{ell}}(G),$$

is ζ_Λ-equivariant under translation by $Z(F)$. We shall denote the function by $S(\phi_\Lambda, \sigma)$. Thus, if we define a new indexing set formally by

$$\Phi_2(G, \zeta_\Lambda) \ = \ \{\phi_\Lambda : \ \phi \in \Phi_2(G, \zeta)\} \ ,$$

we obtain a corresponding family of functions

$$\sigma \longrightarrow S(\phi, \sigma) \ , \qquad \phi \in \Phi_2(G, \zeta_\Lambda).$$

In this way, we construct a family of functions for each character of $Z(F)$ in the $\mathcal{O}^+$-orbit of ζ. Carrying out the process for every $\mathcal{O}^+$-orbit, we obtain sets $\Phi_2(G,\zeta)$ in which ζ varies over all the characters of $Z(F)$. The disjoint union

$$\Phi_2(G) \ = \ \coprod_{\zeta} \Phi_2(G,\zeta)$$

parametrizes a larger family of functions

$$\sigma \longrightarrow S(\phi,\sigma) \ , \qquad \phi \in \Phi_2(G),$$

on $\Sigma_{\mathrm{ell}}(G)$, which has the following properties.

(S_1) The group $\mathcal{O}^+$ operates on $\Phi_2(G)$, and

$$S(\phi_\Lambda,\sigma) \ = \ S\big(\phi,\alpha(\sigma)\big)e^{\lambda(H_G(\alpha(\sigma)))} \ , \qquad \Lambda = (\lambda,\alpha) \in \mathcal{O}^+.$$

(S_2) If ϕ belongs to a subset $\Phi_2(G,\zeta)$ of $\Phi_2(G)$, then

$$S(\phi,\sigma z) \ = \ S(\phi,\sigma)\zeta(z) \ , \qquad z \in Z(F).$$

(S_3) For any character ζ of $Z(F)$, the functions

$$\sigma \longrightarrow n(\sigma)\overline{S(\phi,\sigma)} \ , \qquad \phi \in \Phi_2(G,\zeta),$$

form an orthogonal basis of $S\mathcal{I}_{\mathrm{cusp}}(G,\zeta)$.

Having fixed $\Phi_2(G)$ and its associated family of functions, we define

$$a^G(\phi) \ = \ \int_{\Sigma_{\mathrm{ell}}(G)} S(\phi,\sigma)a^G(\sigma)d\sigma \ , \qquad \phi \in \Phi_2(G),$$

for any function $a^G \in S\mathcal{I}_{\mathrm{cusp}}(G)$. This allows us to identify $S\mathcal{I}_{\mathrm{cusp}}(G)$ with a space of functions on $\Phi_2(G)$. As was the case with $T_{\mathrm{ell}}(G)$, the action $\phi \to \phi_\lambda$ of $i\mathfrak{a}_G^*$ makes $\Phi_2(G)$ into a disjoint union of compact tori of the form

$$i\mathfrak{a}^*_{G,\phi} \ = \ i\mathfrak{a}^*_G/\mathfrak{a}^\vee_{G,\phi} \ ,$$

where $\mathfrak{a}^\vee_{G,\phi}$ is the stabilizer of ϕ in $i\mathfrak{a}^*_G$. By expanding a^G in terms of the orthogonal bases $\{n(\sigma)S(\phi,\sigma)\}$, we see easily that $S\mathcal{I}_{\mathrm{cusp}}(G)$ becomes the Paley-Wiener space on $\Phi_2(G)$; that is, the space of functions on $\Phi_2(G)$ which are supported on finitely many connected components, and which on the component of any ϕ, pull back to a finite Fourier series on $i\mathfrak{a}^*_{G,\phi}$. In particular, functions in $S\mathcal{I}_{\mathrm{cusp}}(G)$ extend to the complexification

$$\Phi_2(G)_{\mathbb{C}} \;=\; \{\phi_\lambda :\; \phi \in \Phi_2(G),\; \lambda \in \mathfrak{a}^*_{G,\mathbb{C}}\}\ ,$$

which, as an abstract indexing set, parametrizes the functions

$$\{S(\phi_\lambda,\sigma) \;=\; S(\phi,\sigma)e^{\lambda(H_G(\sigma))} :\; \sigma \in \Sigma_{\mathrm{ell}}(G)\}\ .$$

It is easy to describe the spectral form of the inner product on $S\mathcal{I}_{\mathrm{cusp}}(G)$. We define a measure on $\Phi_2(G)$ by setting

$$\int_{\Phi_2(G)} \beta(\phi)d\phi \;=\; \sum_{\phi\in\Phi_2(G)/i\mathfrak{a}^*_G} \int_{i\mathfrak{a}^*_{G,\phi}} \beta(\phi_\lambda)d\lambda\ ,$$

for any function $\beta \in C_c\big(\Phi_2(G)\big)$. For any $\phi \in \Phi_2(G)$, set

$$n(\phi) \;=\; \int_{\Sigma_{\mathrm{ell}}(G/Z)} n(\sigma)S(\phi,\sigma)\overline{S(\phi,\sigma)}d\sigma\ ,$$

the inner product of the function in (S_3) with itself. Then the inner product on $S\mathcal{I}_{\mathrm{cusp}}(G)$ becomes

$$(a^G,b^G) \;=\; \int_{\Phi_2(G)} n(\phi)^{-1}a^G(\phi)\overline{b^G(\phi)}d\phi\ , \qquad a^G,b^G \in S\mathcal{I}_{\mathrm{cusp}}(G).$$

The parameters in $\Phi_2(G)$ are only the cuspidal elements in a larger set $\Phi(G)$. We can apply the construction of Lemma 5.1 to any Levi subgroup M, relative to the central induced torus $Z_M = A_M Z$, and any finite group of outer automorphisms $\mathcal{O}_M$ that contains the Weyl group $W(M)$, and normalizes Z_M. We then define

$$\Phi(G) \;=\; \coprod_{\{M\}} \big(\Phi_2(M)/W(M)\big)\ , \tag{5.1}$$

a disjoint union over the orbits of $\mathcal{L}/W_0$ as in (4.1), (1.2) and (1.1). We introduce the natural measure on $\Phi(G)$ by setting

$$\int_{\Phi(G)} \beta(\phi)d\phi \;=\; \sum_{\{M\}} |W(M)|^{-1} \int_{\Phi_2(G)} \beta(\phi_M)d\phi_M \ ,$$

for any $\beta \in C_c\big(\Phi(G)\big)$. Similarly, we can define measures on the spaces

$$\Phi(G,\zeta) \;=\; \coprod_{\{M\}} \big(\Phi_2(M,\zeta)/W(M)\big) \ ,$$

in terms of our measures on spaces $\{\mathfrak{a}_M^G\}$. We define the complex spaces $\Phi(G)_{\mathbb{C}}$ and $\Phi(G,\zeta)_{\mathbb{C}}$ as before, and we write $\Phi_{\mathrm{par}}(G)_{\mathbb{C}}$, $\Phi_{\mathrm{par}}(G,\zeta)$, $\Phi_{\mathrm{par}}(G)$ etc., for the complements of $\Phi_2(G)_{\mathbb{C}}$, $\Phi_2(G,\zeta)_{\mathbb{C}}$ and $\Phi_2(G)$ in corresponding ambient spaces. We shall get to the study of functions on $\Phi(G)$ in Section 6.

We now remove the hypothesis that G is quasi-split. We apply the construction of Lemma 5.1 to each of the quasi-split groups $G' \in \mathcal{E}_{\mathrm{ell}}(G)$, relative to the finite group $\mathrm{Out}_G(G')$ of outer automorphisms of G'. In this case we shall write

$$\Phi_2(\widetilde{G}',G) \;=\; \Phi_2(\widetilde{G}',\widetilde{\zeta}')/\mathrm{Out}_G(G')$$

for the set of $\mathrm{Out}_G(G')$-orbits in $\Phi_2(\widetilde{G}',\widetilde{\zeta}')$. More generally, suppose that $\mathcal{O}$ is a finite group of outer automorphisms of G over F, which has been embedded into the group of outer automorphisms of $G(F)$. Then $\mathcal{O}$ acts as a group of Γ-invariant outer automorphisms of $\widehat{G}$, and therefore acts on $\mathcal{E}_{\mathrm{ell}}(G)$. Let $\mathcal{O}'$ be the stabilizer of G' in $\mathcal{O}$. Then $\mathcal{O}'$ normalizes $\mathrm{Out}_G(G')$, and

$$\mathrm{Out}_G(G') \rtimes \mathcal{O}'$$

is a finite group of outer automorphisms of G'. We obtain a family

$$\Phi_2(\widetilde{G}',G)/\mathcal{O}' \;=\; \Phi_2(\widetilde{G}',\widetilde{\zeta}')/\mathrm{Out}_G(G') \rtimes \mathcal{O}'$$

of $\mathcal{O}'$-orbits in $\Phi_2(\widetilde{G}', G)$. We assume that for each $G' \in \mathcal{E}_{\mathrm{ell}}(G)$, the corresponding parameter set has been fixed. We would like to describe the linear isometry

$$\mathcal{T}^{\mathcal{E}} : \mathcal{I}_{\mathrm{cusp}}(G) \longrightarrow \mathcal{I}^{\mathcal{E}}_{\mathrm{cusp}}(G)$$

of Proposition 3.5 in spectral terms.

Let f be an arbitrary function in $\mathcal{H}_{\mathrm{cusp}}\big(G(F)\big)$. For any $G' \in \mathcal{E}_{\mathrm{ell}}(G)$, f' is a function in

$$S\mathcal{I}_{\mathrm{cusp}}(\widetilde{G}', G) \;=\; S\mathcal{I}_{\mathrm{cusp}}(\widetilde{G}', \widetilde{\zeta}')^{\mathrm{Out}_G(G')} ,$$

and $f'(\phi')$ is defined for every $\phi' \in \Phi_2(\widetilde{G}', G)$. As a linear form in f, $f'(\phi')$ can certainly be written in terms of the virtual characters $f_G(\tau)$. We obtain an expansion

$$f'(\phi') \;=\; \sum_{\tau \in T_{\mathrm{ell}}(G)} \Delta(\phi', \tau) f_G(\tau) , \tag{5.2}$$

for uniquely determined coefficients

$$\Delta(\phi', \tau) \;=\; \Delta_G(\phi', \tau) , \qquad \phi' \in \Phi_2(\widetilde{G}', G),\; \tau \in \widetilde{T}_{\mathrm{ell}}(G),$$

such that

$$\Delta(\phi', z_\tau \tau) \;=\; \chi_\tau(z_\tau) \Delta(\phi', \tau) , \qquad z_\tau \in Z_\tau.$$

We continue to work with an underlying central induced torus Z of G over F, which contains A_G and is normalized by $\mathcal{O}$. If τ belongs to $T_{\mathrm{ell}}(G, \zeta)$, for a character ζ of $Z(F)$, it follows easily from (2.1) that $\Delta(\phi', \tau)$ vanishes unless ϕ' belongs to $\Phi_2(\widetilde{G}', \widetilde{\zeta}'\zeta)$. From (2.2) we obtain the identity

$$\Delta(\phi', \tau) \;=\; \Delta(\phi'_{\lambda'}, \tau_\lambda)$$

for any $\lambda \in i\mathfrak{a}^*_G$. More generally, suppose that $\Lambda = (\lambda, \alpha)$ belongs to the semi-direct product of $i\mathfrak{a}^*_G$ with $\mathrm{Out}_G(G') \rtimes \mathcal{O}'$, and that $\Lambda' = (\lambda', \alpha)$. Then

$$\Delta(\phi, \tau) \;=\; \Delta(\phi'_{\Lambda'}, \tau_\Lambda) . \tag{5.3}$$

The set $\widetilde{T}_{\mathrm{ell}}(G)$ parametrizes an orthogonal basis of $\mathcal{I}_{\mathrm{cusp}}(G)$ (modulo the actions of Z_τ and $Z(F)$). Let us define

$$T^{\mathcal{E}}_{\mathrm{ell}}(G) \;=\; \{(G',\phi') :\; G' \in \mathcal{E}_{\mathrm{ell}}(G),\; \phi' \in \Phi_2(\widetilde{G}', G)\}\ .$$

Then $T^{\mathcal{E}}_{\mathrm{ell}}(G)$ parametrizes an orthogonal basis of $\mathcal{I}^{\mathcal{E}}_{\mathrm{cusp}}(G)$ (again up to the action of $Z(F)$). The coefficients $\{\Delta(\phi,\tau)\}$ give the matrix of the transfer map $\mathcal{T}^{\mathcal{E}}$ with respect to these bases. It is a straightforward matter to describe the matrix of the adjoint map $(\mathcal{T}^{\mathcal{E}})^*$. Arguments similar to those of the proof of Proposition 3.5 establish that if $a^{\mathcal{E}}_G = \bigoplus_{G'} a'$ is any function in $\mathcal{I}^{\mathcal{E}}_{\mathrm{cusp}}(G)$, the adjoint function

$$a_G \;=\; (\mathcal{T}^{\mathcal{E}})^{-1}(a^{\mathcal{E}}_G) \;=\; (\mathcal{T}^{\mathcal{E}})^{*}(a^{\mathcal{E}}_G)$$

is given by

$$a_G(\tau) \;=\; \sum_{\phi' \in T^{\mathcal{E}}_{\mathrm{ell}}(G)} \Delta(\tau,\phi')a^{\mathcal{E}}_G(\phi')\ , \tag{5.4}$$

where

$$\Delta(\tau,\phi') \;=\; \left|Z(\widehat{G}')^{\Gamma}/Z(\widehat{G})^{\Gamma}\right|^{-1} n(\tau)n(\phi')^{-1}\overline{\Delta(\phi',\tau)}\ . \tag{5.5}$$

Lemma 5.2. *Both $\Delta(\tau,\phi')$ and $\Delta(\phi,\tau)$ have finite support in ϕ' for fixed τ, and finite support in τ for fixed ϕ'. Moreover,*

$$\sum_{\phi' \in T^{\mathcal{E}}_{\mathrm{ell}}(G)} \Delta(\tau,\phi')\Delta(\phi',\tau_1) \;=\; \widetilde{\delta}(\tau,\tau_1), \qquad \tau,\tau_1 \in \widetilde{T}_{\mathrm{ell}}(G), \tag{5.6}$$

and

$$\sum_{\tau \in T_{\mathrm{ell}}(G)} \Delta(\phi',\tau)\Delta(\tau,\phi'_1) \;=\; \delta(\phi',\phi'_1), \qquad \phi',\phi'_1 \in T^{\mathcal{E}}_{\mathrm{ell}}(G), \tag{5.7}$$

for Kronecker delta functions $\widetilde{\delta}(\tau,\tau_1)$ and $\delta(\phi',\phi'_1)$ analogous to those of Section 2.

Proof. Fix $\tau \in \widetilde{T}_{\mathrm{ell}}(G)$. Then $\Delta(\phi',\tau)$ will vanish unless ϕ' and τ have the same central character on $Z(F)$. That is, the central character of ϕ' on $(\widetilde{Z}'Z)(F)$ must be equal to the product of $\widetilde{\zeta}'$ with the central character of τ on $Z(F)$. Let $a_G \in \mathcal{I}_{\mathrm{cusp}}(G)$ be a pseudo-coefficient of τ. By this we mean that a_G is supported on the $(Z_\tau \times i\mathfrak{a}_G^*)$-orbit of τ, and that

$$a_G(\tau_1) \;=\; \begin{cases} \chi_\tau(z_\tau)^{-1}, & \text{if } \tau_1 = z_\tau \tau,\ z_\tau \in Z_\tau, \\ 0, & \text{otherwise,} \end{cases}$$

if τ_1 is any element in the $(Z_\tau \times i\mathfrak{a}_G^*)$-orbit of τ with the same $Z(F)$-central character as τ. Then $a_G^{\mathcal{E}} = \mathcal{T}^{\mathcal{E}} a_G$ lies in $\mathcal{I}_{\mathrm{cusp}}^{\mathcal{E}}(G)$, and is supported on finitely many components in $T_{\mathrm{ell}}^{\mathcal{E}}(G)$. Since there are only finitely many ϕ' in any component with the same $Z(F)$-central character as τ, the definition (5.2) tells us that $\Delta(\phi', \tau)$ vanishes for all but finitely many ϕ'. On the other hand, if we take $a_G^{\mathcal{E}}$ to be a pseudo-coefficient of a fixed pair (G', ϕ') in $T_{\mathrm{ell}}^{\mathcal{E}}(G)$, we see from (5.4) that $\Delta(\tau, \phi')$ has finite support in τ. The first assertion of the lemma then follows in all cases from (5.5).

Since $\mathcal{T}^{\mathcal{E}}$ is a linear isometry, the matrices $\{\Delta(\phi', \tau)\}$ and $\{\Delta(\tau, \phi')\}$ are the inverses of each other. The second assertion of the lemma follows. $\square$

We shall need the spectral analogue of the change of variables in Lemma 2.3. Our set $T_{\mathrm{ell}}^{\mathcal{E}}(G)$ inherits a measure from the quotient measures on the sets $\Phi_2(\widetilde{G}', G)$.

Lemma 5.3. *Suppose that $\beta \in C_c\big(T_{\mathrm{ell}}^{\mathcal{E}}(G)\big)$, and that $\alpha \in C\big(\widetilde{T}_{\mathrm{ell}}(G)\big)$ is such that the product $\Delta(\phi', \tau)\alpha(\tau)$ descends to a function of $\tau \in T_{\mathrm{ell}}(G)$. Then*

$$\int_{T_{\mathrm{ell}}(G)} \sum_{\phi' \in T_{\mathrm{ell}}^{\mathcal{E}}(G)} \beta(\phi')\Delta(\phi', \tau)\alpha(\tau) d\tau \;=\; \int_{T_{\mathrm{ell}}^{\mathcal{E}}(G)} \sum_{\tau \in T_{\mathrm{ell}}(G)} \beta(\phi')\Delta(\phi', \tau)\alpha(\tau) d\phi' \,.$$

Proof. According to the definition of the measure $d\tau$, we can decompose the left hand side of the required identity into an expression

$$\sum_{\tau} \int_{i\mathfrak{a}_G^*/\tilde{\mathfrak{a}}_{G,F}^\vee} \sum_{\lambda_1} \sum_{\phi'} \sum_{\mu} \sum_{\mu_1} \beta(\phi_{\mu+\mu_1})\Delta(\phi'_{\mu+\mu_1}, \tau_{\lambda+\lambda_1})\alpha(\tau_{\lambda+\lambda_1}) d\lambda \,,$$

with sums over $\tau \in T_{\mathrm{ell}}(G)/i\mathfrak{a}_G^*$, $\lambda_1 \in \widetilde{\mathfrak{a}}_{G,F}^\vee/\mathfrak{a}_{G,\tau}^\vee$, $\phi' \in T_{\mathrm{ell}}^{\mathcal{E}}(G)/i\mathfrak{a}_G^*$, $\mu \in i\mathfrak{a}_G^*/\widetilde{\mathfrak{a}}_{G,F}^\vee$ and $\mu_1 \in \widetilde{\mathfrak{a}}_{G,F}^\vee/\mathfrak{a}_{G,\phi'}^\vee$. Recall that the transfer factor $\Delta(\phi'_{\mu+\mu_1}, \tau_{\lambda+\lambda_1})$ vanishes unless $\phi'_{\mu+\mu_1}$ and

$\tau_{\lambda+\lambda_1}$ have the same central character on $Z(F)$. The central characters remain invariant under translation of the elements μ_1 and λ_1. On the other hand, if we choose ϕ' and τ so that they have the same central character, we see that the sum over μ reduces to the one element $\mu = \lambda$. The expression becomes

$$\sum_{(\tau,\phi')} \int_{i\mathfrak{a}^*_G/\tilde{\mathfrak{a}}^\vee_{G,F}} \sum_{\lambda_1,\mu_1} \beta(\phi'_{\lambda+\mu_1})\Delta(\phi'_{\lambda+\mu_1}, \tau_{\lambda+\lambda_1})\alpha(\tau_{\lambda+\lambda_1})d\lambda \ ,$$

where (τ, ϕ') is summed over pairs in $\big(T_{\mathrm{ell}}(G) \times T^{\mathcal{E}}_{\mathrm{ell}}(G)\big)/i\mathfrak{a}^*_G$ with a common central character on $Z(F)$. From its obvious symmetry, we conclude that the expression must also be equal to the right hand side of the required identity. The identity is therefore valid. □

The basis of $\mathcal{I}_{\mathrm{cusp}}(G)$ given by $\widetilde{T}_{\mathrm{ell}}(G)$ is contained in the larger basis of $\mathcal{I}(G)$ parametrized by $\widetilde{T}(G)$. We would like to parametrize a parallel "endoscopic basis" by a set that contains $T^{\mathcal{E}}_{\mathrm{ell}}(G)$. The process is similar to the construction of the set $\widetilde{\Gamma}^{\mathcal{E}}(G)$ at the end of Section 2. Recalling that

$$\widetilde{T}(G) \ = \ \coprod_{\{M\}} \big(\widetilde{T}_{\mathrm{ell}}(M)/W(M)\big) \ ,$$

we simply define

$$T^{\mathcal{E}}(G) \ = \ \coprod_{\{M\}} \big(T^{\mathcal{E}}_{\mathrm{ell}}(M)/W(M)\big) \ . \tag{5.8}$$

In more direct terms, $T^{\mathcal{E}}(G)$ is the union over the W_0-orbits $\{M\}$ in $\mathcal{L}$, and the $W(M)$-orbits $\{M'\}$ in $\mathcal{E}_{\mathrm{ell}}(M)$, of parameter sets

$$\Phi_2(\widetilde{M}', M)/W(M)' \ = \ \Phi_2(\widetilde{M}', \widetilde{\zeta}')/\mathrm{Out}_M(M') \rtimes W(M)' \ ,$$

where $W(M)' = W(M)^{M'}$ denotes the stabilizer of M' in $W(M)$. It is a disjoint union of compact connected spaces, each of which is a quotient of a compact torus by a finite group. We are of course assuming that for each M', $\Phi_2(\widetilde{M}', \widetilde{\zeta}')$ has been chosen according to Lemma 5.1, as a set on which the group $\mathrm{Out}_M(M') \rtimes W(M)'$ acts by permutation.

We next extend the definition of the spectral transfer factors to elements ϕ' and τ in the larger sets $T^{\mathcal{E}}(G)$ and $\widetilde{T}(G)$. We define $\Delta(\phi',\tau)$ and $\Delta(\tau,\phi')$ to be zero unless there is an M such that (ϕ',τ) belongs to the Cartesian product of $T^{\mathcal{E}}_{\mathrm{ell}}(M)/W(M)$ with $\widetilde{T}_{\mathrm{ell}}(M)/W(M)$. If there is such an M, (ϕ',τ) is the image of a pair (ϕ'_M,τ_M) in $T^{\mathcal{E}}_{\mathrm{ell}}(M)\times\widetilde{T}_{\mathrm{ell}}(M)$. In this case, we set

$$\Delta(\phi',\tau) \;=\; \Delta_G(\phi',\tau) \;=\; \sum_{\widetilde{\tau}_M}\Delta_M(\phi'_M,\widetilde{\tau}_M)$$

and

$$\Delta(\tau,\phi') \;=\; \Delta_G(\tau,\phi') \;=\; \sum_{\widetilde{\phi}'_M}\Delta_M(\widetilde{\phi}'_M,\tau_M)\ ,$$

where $\widetilde{\tau}_M$ and $\widetilde{\phi}'_M$ are summed over the respective Weyl orbits $W(M)\tau_M$ and $W(M)\phi'_M$. It follows from (5.3) and (5.5) that the two sums are independent of the representatives ϕ'_M and τ_M. If we apply (5.6) and (5.7) to each M, we obtain general inversion formulas

$$\sum_{\phi'\in T^{\mathcal{E}}(G)}\Delta(\tau,\phi')\Delta(\phi',\tau_1) \;=\; \widetilde{\delta}(\tau,\tau_1), \qquad \tau,\tau_1\in\widetilde{T}(G), \tag{5.9}$$

and

$$\sum_{\tau\in T(G)}\Delta(\phi',\tau)\Delta(\tau,\phi'_1) \;=\; \delta(\phi,\phi'_1), \qquad \phi',\phi'_1\in T^{\mathcal{E}}(G), \tag{5.10}$$

on $\widetilde{T}(G)$ and $T^{\mathcal{E}}(G)$. Observe that $T^{\mathcal{E}}(G)$ is a disjoint union of subsets $T^{\mathcal{E}}(G,\zeta)$ parametrized by the characters ζ of $Z(F)$. The transfer factor $\Delta(\phi',\tau)$ will vanish unless ϕ' and τ belong to corresponding subsets $T^{\mathcal{E}}(G,\zeta)$ and $\widetilde{T}(G,\zeta)$.

Finally, we define the complex spaces

$$T^{\mathcal{E}}(G)_{\mathbb{C}} \;=\; \coprod_{\{M\}}\big(T^{\mathcal{E}}_{\mathrm{ell}}(M)_{\mathbb{C}}/W(M)\big)$$

and

$$T^{\mathcal{E}}(G,\zeta)_{\mathbb{C}} \;=\; \coprod_{\{M\}}\big(T^{\mathcal{E}}_{\mathrm{ell}}(M,\zeta)_{\mathbb{C}}/W(M)\big)\ .$$

There is an obvious way to extend the transfer factors to functions on $\widetilde{T}(G)_{\mathbb{C}} \times T^{\mathcal{E}}(G)_{\mathbb{C}}$. For example, given $\tau \in \widetilde{T}_{\text{ell}}(G)$, $\phi' \in T^{\mathcal{E}}_{\text{ell}}(G)$ and $\lambda, \mu \in \mathfrak{a}^*_G$, we define

$$\Delta(\phi'_\lambda, \tau_\mu) \;=\; \begin{cases} \Delta(\phi', \tau), & \text{if } \lambda = \mu, \\ 0, & \text{if } \lambda \neq \mu. \end{cases}$$

The inversion formulas (5.9) and (5.10) then hold for elements in the sets $\widetilde{T}(G)_{\mathbb{C}}$ and $T^{\mathcal{E}}(G)_{\mathbb{C}}$. As before, we shall write $T^{\mathcal{E}}_{\text{par}}(G)_{\mathbb{C}}$, $T^{\mathcal{E}}_{\text{par}}(G,\zeta)_{\mathbb{C}}$, $T^{\mathcal{E}}_{\text{par}}(G)$ etc., for the complements of $T^{\mathcal{E}}_{\text{ell}}(G)_{\mathbb{C}}$, $T^{\mathcal{E}}_{\text{ell}}(G,\zeta)_{\mathbb{C}}$ and $T^{\mathcal{E}}_{\text{ell}}(G)$ in the corresponding ambient spaces.

This is of course quite analogous to the discussion of the geometric transfer factors. As in the geometric case, we often want to focus on the subset of $T^{\mathcal{E}}(G)$ attached to a given $G' \in \mathcal{E}_{\text{ell}}(G)$. There is a natural map from the set $\Phi(\widetilde{G}', \widetilde{\zeta}')$ into $T^{\mathcal{E}}(G)$, whose image we shall denote by $\Phi(\widetilde{G}', G)$. Then $\Phi(\widetilde{G}', G)$ is a union of connected components of $T^{\mathcal{E}}(G)$. Observe that the intersection

$$\Phi(\widetilde{G}', G, \zeta) \;=\; \Phi(\widetilde{G}', G) \cap T^{\mathcal{E}}(G,\zeta)$$

equals the image of $\Phi(\widetilde{G}', \zeta'\zeta)$ in $T^{\mathcal{E}}(G,\zeta)$. Similarly, we define subsets $\Phi(\widetilde{G}', G)_{\mathbb{C}}$ and $\Phi(\widetilde{G}', G, \zeta)_{\mathbb{C}}$ of $T^{\mathcal{E}}(G)_{\mathbb{C}}$ and $T^{\mathcal{E}}(G,\zeta)_{\mathbb{C}}$ attached to G'. If we take the elements ϕ' above from any one of these subsets, we get spectral transfer factors for the pair (G, G').

6. Statement of two theorems

We described the map $f \to f'$ from $\mathcal{H}\big(G(F)\big)$ to $S\mathcal{I}(\widetilde{G}', \widetilde{\zeta}')$. It is defined by the transfer of orbital integrals. We can also define a map $f \to f'_{\text{gr}}$ from $\mathcal{H}\big(G(F)\big)$ to $S\mathcal{I}_{\text{gr}}(\widetilde{G}', \widetilde{\zeta}')$ by a transfer of characters. The problem is to relate the two maps.

For the appropriate level of generality, we need to fix an induced central torus Z of G over F, and a character ζ of $Z(F)$. Suppose for a moment that G is quasi-split. As we agreed in the last section, $S\mathcal{I}_{\text{cusp}}(G,\zeta)$ can be regarded as a space of functions on either

$\Sigma_{\text{ell}}(G)$ or $\Phi_2(G,\zeta)$. In particular, it can be identified with the natural Paley-Wiener space on $\Phi_2(G,\zeta)$. The larger graded vector space

$$S\mathcal{I}_{\text{gr}}(G,\zeta) \;=\; \bigoplus_{\{M\}} S\mathcal{I}_{\text{cusp}}(M,\zeta)^{W(M)}$$

can therefore be identified with the natural Paley-Wiener space on the set

$$\Phi(G,\zeta) \;=\; \coprod_{\{M\}} \bigl(\Phi_2(M,\zeta)/W(M)\bigr)\ .$$

For any $f \in \mathcal{H}\bigl(G(F),\zeta\bigr)$, we define a function

$$f^G_{\text{gr}}(\phi) \;=\; \sum_{\tau\in T(G,\zeta)} \Delta(\phi,\tau) f_G(\tau), \qquad \phi \in \Phi(G,\zeta), \tag{6.1}$$

on $\Phi(G,\zeta)$. Then $f \to f^G_{\text{gr}}$ is a continuous map from $\mathcal{H}\bigl(G(F),\zeta\bigr)$ to $S\mathcal{I}_{\text{gr}}(G,\zeta)$. In the case of general G, we have an induced central torus $\widetilde{Z}'Z$ in $\widetilde{G}'$ and a character $\widetilde{\zeta}'\zeta$ on $(\widetilde{Z}'Z)(F)$ for each $G' \in \mathcal{E}_{\text{ell}}(G)$, and we identify the graded vector space $S\mathcal{I}_{\text{gr}}(\widetilde{G}',\widetilde{\zeta}'\zeta)$ with the Paley-Wiener space on $\Phi(\widetilde{G}',\widetilde{\zeta}'\zeta)$. For any $f \in \mathcal{H}(G,\zeta)$, we again define a function

$$f'_{\text{gr}}(\phi') \;=\; \sum_{\tau\in T(G,\zeta)} \Delta(\phi',\tau) f_G(\tau), \qquad \phi' \in \Phi(\widetilde{G}',\widetilde{\zeta}'\zeta), \tag{6.2}$$

on $\Phi(\widetilde{G}',\widetilde{\zeta}'\zeta)$. Then $f \to f'_{\text{gr}}$ is a continuous linear map from $\mathcal{H}\bigl(G(F),\zeta)\bigr)$ to $S\mathcal{I}_{\text{gr}}(\widetilde{G}',\widetilde{\zeta}'\zeta)$.

Theorem 6.1. *Suppose that G is quasi-split and that $\phi \in \Phi(G,\zeta)$. Then the linear form*

$$f \longrightarrow f^G_{\text{gr}}(\phi)\ , \qquad f \in \mathcal{H}\bigl(G(F),\zeta\bigr),$$

is stable.

The theorem asserts that any element $\phi \in \Phi(G,\zeta)$ determines a stable, ζ-equivariant distribution on $G(F)$. The distribution has then to factor through the transfer map $f \to f^G$ of $\mathcal{H}\bigl(G(F),\zeta\bigr)$ to $\mathcal{I}(G,\zeta)$. We shall denote the corresponding linear form on $\mathcal{I}(G,\zeta)$ by $f^G(\phi)$. In other words, once we have proved the theorem we shall define

$$f^G(\phi) \;=\; f^G_{\text{gr}}(\phi), \qquad \phi \in \Phi(G,\zeta), \tag{6.3}$$

for any $f \in \mathcal{H}\bigl(G(F),\zeta)\bigr)$.

Theorem 6.2. *Suppose that G is arbitrary, that $G' \in \mathcal{E}_{\mathrm{ell}}(G)$, and that $\phi' \in \Phi(\widetilde{G}', \widetilde{\zeta}'\zeta)$. Then*

$$f'(\phi') \;=\; f'_{\mathrm{gr}}(\phi')\;,$$

for any $f \in \mathcal{H}\big(G(F), \zeta\big)$.

The special case that G is quasi-split and $G' = G$ is just the definition above. The general case is a separate question, however, since the maps f' and f'_{gr} are not obviously related. Before beginning the proofs, we shall discuss a few elementary consequences of the theorems.

If G is quasi-split, Theorem 6.1 asserts that the map $f \to f^G_{\mathrm{gr}}$ factors through the original transfer $f \to f^G$. We obtain a continuous linear map $a^G \to a^G_{\mathrm{gr}}$ from $S\mathcal{I}(G, \zeta)$ to $S\mathcal{I}_{\mathrm{gr}}(G, \zeta)$. The map is compatible with the filtrations on the two spaces, and for any M, induces the canonical isomorphism of $\mathcal{G}^M\big(S\mathcal{I}(G, \zeta)\big)$ with $\mathcal{I}_{\mathrm{cusp}}(M, \zeta)^{W(M)}$. The map is therefore a linear isomorphism of $S\mathcal{I}(G, \zeta)$ with $S\mathcal{I}_{\mathrm{gr}}(G, \zeta)$, and induces a grading on $S\mathcal{I}(G, \zeta)$ which is compatible with the filtration. Once the theorem has been established, we will be able to drop the subscript gr. In other words, we will identify $S\mathcal{I}(G, \zeta)$ with $S\mathcal{I}_{\mathrm{gr}}(G, \zeta)$ by means of the map $a_G \to a_{G,\mathrm{gr}}$. In particular, an element in $S\mathcal{I}(G, \zeta)$ will be regarded as a function on either $\Sigma(G)$ or $\Phi(G, \zeta)$.

Consider the general case, with a fixed element $G' \in \mathcal{E}_{\mathrm{ell}}(G)$. Then

$$\mathcal{F}^M\big(S\mathcal{I}(\widetilde{G}', \widetilde{\zeta}'\zeta)\big) \;=\; \sum_{M' \in (\mathcal{E}_{\mathrm{ell}}(M) \cap \mathcal{L}^{G'})/W(M)} \mathcal{F}^{M'}\big(S\mathcal{I}(\widetilde{G}', \widetilde{\zeta}'\zeta)\big)$$

is a filtration on $S\mathcal{I}(\widetilde{G}', \widetilde{\zeta}'\zeta)$ which is indexed by the elements $\{M\} \in \mathcal{L}/W_0$. The original transfer mapping $f \to f'$ determines a linear transformation $a_G \to a'$ from $\mathcal{I}(G, \zeta)$ to $S\mathcal{I}(\widetilde{G}', \widetilde{\zeta}'\zeta)$ which is compatible with the filtrations on the two spaces. On the other hand, the mapping $f \to f'_{\mathrm{gr}}$ determines a linear transformation $a_G \to a'_{\mathrm{gr}}$ from $\mathcal{I}(G, \zeta)$ to $S\mathcal{I}(\widetilde{G}', \widetilde{\zeta}'\zeta)$ which is compatible with the associated gradings. Theorem 6.2 asserts that the two maps are the same. In particular, the original transfer map is compatible with the gradings we have defined on the two spaces.

There is an equivalent way to say these things. At the end of Section 3 we defined a space of functions $\mathcal{I}^{\mathcal{E}}(G,\zeta)$ on $\widetilde{\Gamma}^{\mathcal{E}}(G)$. We also defined a linear isomorphism $a_G \to a_G^{\mathcal{E}}$ from $\mathcal{I}(G,\zeta)$ onto $\mathcal{I}^{\mathcal{E}}(G,\zeta)$ by the transfer of orbital integrals. Following our discussion of the space $\mathcal{I}^{\mathcal{E}}_{\mathrm{cusp}}(G)$ in Section 5, we may regard the graded vector space

$$\mathcal{I}^{\mathcal{E}}_{\mathrm{gr}}(G,\zeta) \;=\; \bigoplus_{\{M\}} \mathcal{I}^{\mathcal{E}}_{\mathrm{cusp}}(M,\zeta)^{W(M)}$$

associated with $\mathcal{I}^{\mathcal{E}}(G,\zeta)$ as the Paley-Wiener space on

$$T^{\mathcal{E}}(G,\zeta) \;=\; \coprod_{\{M\}} \big(T^{\mathcal{E}}_{\mathrm{ell}}(M,\zeta)/W(M)\big) \; .$$

We earlier agreed to write

$$\mathcal{I}(G,\zeta) \;=\; \bigoplus_{\{M\}} \mathcal{I}_{\mathrm{cusp}}(M,\zeta)^{W(M)} \; ,$$

thereby identifying $\mathcal{I}(G,\zeta)$ with the Paley-Wiener space on

$$T(G,\zeta) \;=\; \coprod_{\{M\}} \big(T_{\mathrm{ell}}(M,\zeta)/W(M)\big) \; .$$

We can therefore map $\mathcal{I}(G,\zeta)$ to $\mathcal{I}^{\mathcal{E}}_{\mathrm{gr}}(G,\zeta)$ by a transfer of characters. Using the transfer factors introduced in Section 5, we map any function $a_G \in \mathcal{I}(G,\zeta)$ to the function

$$a^{\mathcal{E}}_{G,\mathrm{gr}}(\phi') \;=\; \sum_{\tau \in T(G,\zeta)} \Delta(\phi',\tau) a_G(\tau) \tag{6.4}$$

on $T^{\mathcal{E}}(G,\zeta)$. Then

$$\mathcal{I}^{\mathcal{E}}_{\mathrm{gr}} : \; a_G \longrightarrow a^{\mathcal{E}}_{G,\mathrm{gr}}$$

is a continuous linear isomorphism from $\mathcal{I}(G,\zeta)$ onto $\mathcal{I}^{\mathcal{E}}_{\mathrm{gr}}(G,\zeta)$, whose inverse is given by

$$a_G(\tau) \;=\; \sum_{\phi' \in T^{\mathcal{E}}(G,\zeta)} \Delta(\tau,\phi') a^{\mathcal{E}}_{G,\mathrm{gr}}(\phi') \; . \tag{6.5}$$

Consider an element $G' \in \mathcal{E}_{\mathrm{ell}}(G)$. Then $\Sigma(\widetilde{G}', G)$ is a closed subset of $\widetilde{\Gamma}^{\mathcal{E}}(G)$, and the subspace $S\mathcal{I}(\widetilde{G}', G, \zeta)$ of $S\mathcal{I}(\widetilde{G}', \widetilde{\zeta}'\zeta)$ is simply the space of functions on $\Sigma(\widetilde{G}', G)$ obtained by restricting functions in $\mathcal{I}^{\mathcal{E}}(G, \zeta)$. In particular, for any $f \in \mathcal{H}\bigl(G(F), \zeta\bigr)$, f' is just the restriction of $f_G^{\mathcal{E}}$ from $\widetilde{\Gamma}^{\mathcal{E}}(G)$ to $\Sigma(\widetilde{G}', G)$. Moreover, $\Phi(\widetilde{G}', G, \zeta)$ is a closed subset of $T^{\mathcal{E}}(G, \zeta)$, and the subspace $S\mathcal{I}_{\mathrm{gr}}(\widetilde{G}', G, \zeta)$ of $S\mathcal{I}_{\mathrm{gr}}(\widetilde{G}', \widetilde{\zeta}'\zeta)$ is the space of functions on $\Phi(\widetilde{G}', G, \zeta)$ obtained by restricting functions in $\mathcal{I}_{\mathrm{gr}}^{\mathcal{E}}(G, \zeta)$. In particular, f'_{gr} is the restriction of $f_{G,\mathrm{gr}}^{\mathcal{E}}$ from $\widetilde{T}^{\mathcal{E}}(G, \zeta)$ to $\Phi(\widetilde{G}', G, \zeta)$. Theorem 6.1, applied to G', allows us to identify the space $S\mathcal{I}(\widetilde{G}', G, \zeta)$ with $S\mathcal{I}_{\mathrm{gr}}(\widetilde{G}', G, \zeta)$. Since $\widetilde{\Gamma}^{\mathcal{E}}(G)$ and $T^{\mathcal{E}}(G, \zeta)$ are each unions over G' of the associated closed subsets, we can therefore identify $\mathcal{I}^{\mathcal{E}}(G, \zeta)$ with the space $\mathcal{I}_{\mathrm{gr}}^{\mathcal{E}}(G, \zeta)$. Theorem 6.2 then tells us that the maps $\mathcal{T}^{\mathcal{E}}\colon a_G \to a_G^{\mathcal{E}}$ and $\mathcal{T}_{\mathrm{gr}}^{\mathcal{E}}\colon a_G \to a_{G,\mathrm{gr}}^{\mathcal{E}}$ can also be identified. Once again, we drop the subscripts gr. An element $a_G^{\mathcal{E}} \in \mathcal{I}^{\mathcal{E}}(G, \zeta)$ is to be regarded as a function on either $\widetilde{\Gamma}^{\mathcal{E}}(G)$ or $T^{\mathcal{E}}(G, \zeta)$. If $a_G^{\mathcal{E}}$ is the image of an element $a_G \in \mathcal{I}(G, \zeta)$ under the map defined in Section 3 by the transfer of orbital integrals, the values $\{a_G(\tau)\}$ and $\{a_G^{\mathcal{E}}(\phi')\}$ will be related by (6.4) and (6.5).

Another way to view Theorems 6.1 and 6.2 is through the group theoretic analogue of Waldspurger's kernel formula [28, 1.2]. There are actually two such analogues. One is a family of relations for the virtual characters $\{I(\tau, \gamma)\}$. By definition,

$$I(\tau, \gamma) \;=\; I_G^G(\tau, \gamma)$$

is the smooth function on $\widetilde{T}(G, \zeta) \times \Gamma(G)$ such that

$$f_G(\tau) \;=\; \int_{\Gamma(G/Z)} I(\tau, \gamma) f_G(\gamma) d\gamma, \qquad f \in \mathcal{H}\bigl(G(F), \zeta\bigr). \tag{6.6}$$

The other is a family of relations for the adjoint functions $\{I(\gamma, \tau)\}$ introduced in [4, Section 4]. By construction,

$$I(\gamma, \tau) \;=\; I_G^G(\gamma, \tau)$$

is a smooth function on $\Gamma(G) \times \widetilde{T}(G,\zeta)$ such that

$$f_G(\gamma) \;=\; \int_{T(G,\zeta)} I(\gamma,\tau) f_G(\tau) d\tau, \qquad f \in \mathcal{H}\big(G(F),\zeta\big). \tag{6.7}$$

(This is the special case of [4, Theorem 4.1 (4.1)] in which $M = G$. The kernels

$$\tau \longrightarrow I_M(\gamma,\tau)\ , \qquad \tau \in \widetilde{T}_{\mathrm{disc}}(L),\ L \in \mathcal{L},$$

of [4] are then supported on the subsets $\widetilde{T}_{\mathrm{ell}}(L)$ of $\widetilde{T}_{\mathrm{disc}}(L)$, and can be put together as a single smooth function on $\widetilde{T}(G)$. The kernel here actually differs from the one in [4] by a factor $|R_{\sigma,r}|$, since our measure on $T_{\mathrm{ell}}(G)$ differs from the one in [3, (3.5)] by the same factor.) Since we are restricting τ to the subset $\widetilde{T}(G,\zeta)$, the functions $I(\tau,\gamma)$ and $I(\gamma,\tau)$ are respectively ζ and ζ^{-1}-equivariant under translation of γ by $Z(F)$.

We shall state and prove the relations as a corollary of the two theorems. However, we must first construct the stable versions of the functions $I(\tau,\gamma)$ and $I(\gamma,\tau)$.

Lemma 6.3. *Suppose that G is quasi-split and that Theorem 6.1 holds for G. Then there are smooth functions $S(\phi,\sigma) = S^G_G(\phi,\sigma)$ and $S(\sigma,\phi) = S^G_G(\sigma,\phi)$ of $\phi \in \Phi(G,\zeta)$ and $\sigma \in \Sigma(G)$, which are respectively ζ and ζ^{-1}-equivariant under translation of σ by $Z(F)$, such that*

$$f^G(\phi) \;=\; \int_{\Sigma(G/Z)} S(\phi,\sigma) f^G(\sigma) d\sigma \tag{6.8}$$

and

$$f^G(\sigma) \;=\; \int_{\Phi(G,\zeta)} S(\sigma,\phi) f^G(\phi) d\phi\ , \tag{6.9}$$

for any $f \in \mathcal{H}\big(G(F),\zeta\big)$.

Proof. The arguments required to construct the two functions are dual to each other, so it will be enough to deal with the second one. Fix $\sigma \in \Sigma(G)$. Then for any $f \in \mathcal{H}\big(G(F),\zeta\big)$, we have

$$\begin{aligned} f^G(\sigma) \;&=\; \sum_{\gamma \in \Gamma(G)} \Delta(\sigma,\gamma) f_G(\gamma) \\ &=\; \int_{T(G,\zeta)} \Big(\sum_{\gamma \in \Gamma(G)} \Delta(\sigma,\gamma) I(\gamma,\tau) \Big) f_G(\tau) d\tau\ . \end{aligned}$$

Since we are assuming that Theorem 6.1 holds, we can identify $S\mathcal{I}(G,\zeta)$ with $S\mathcal{I}_{\mathrm{gr}}(G,\zeta)$. In particular, $f^G(\sigma)$ depends only on the function $f^G(\phi)$ on $\Phi(G,\zeta)$. But $f^G(\phi)$ is the restriction of $f^{\mathcal{E}}_{G,\mathrm{gr}}$, a function in the Paley-Wiener space on $T^{\mathcal{E}}(G,\zeta)$, to the closed subset $\Phi(G,\zeta)$ of $T^{\mathcal{E}}(G,\zeta)$. Since $\Phi(G,\zeta)$ is also open in $T^{\mathcal{E}}(G,\zeta)$, we can in fact assume that $f^{\mathcal{E}}_{G,\mathrm{gr}}$ is supported on $\Phi(G,\zeta)$. Therefore

$$f_G(\tau) \;=\; \sum_{\phi\in\Phi(G,\zeta)} \Delta(\tau,\phi) f^G(\phi) \;.$$

We substitute this into the formula above. By a variant of Lemma 5.3, we can convert the sum over ϕ and the integral over τ into a sum over τ and an integral over ϕ. Formula (6.9) follows, with

$$S(\sigma,\phi) \;=\; \sum_{\gamma\in\Gamma(G)} \sum_{\tau\in T(G,\zeta)} \Delta(\sigma,\gamma) I(\gamma,\tau)\Delta(\tau,\phi) \;.$$

The ζ^{-1}-equivariance of $\sigma \to S(\sigma,\phi)$ follows from (2.1) and the corresponding property of $I(\gamma,\tau)$. $\square$

We remark that if $\phi\in\Phi_2(G,\zeta)$, the restriction of the first function $S(\phi,\sigma)$ to elements σ in $\Sigma_{\mathrm{ell}}(G)$ is just the original function constructed in Proposition 5.1. Its extension to $\Sigma(G)$ is determined by the extensions of the elliptic virtual characters $\gamma \to I(\tau,\gamma)$ from $\Gamma_{\mathrm{ell}}(G)$ to $\Gamma(G)$.

Suppose now that G is arbitrary, and that Theorem 6.1 holds for each quasi-split group $\widetilde{G}'$, for $G'\in\mathcal{E}_{\mathrm{ell}}(G)$. We shall write

$$S'(\cdot,\cdot) \;=\; S^{\widetilde{G}'}_{\widetilde{G}'}(\cdot,\cdot)$$

for the functions of $\sigma'\in\Sigma_G(\widetilde{G}')$ and $\phi'\in\Phi(\widetilde{G}',\widetilde{\zeta}'\zeta)$ given by the last lemma. The two theorems together then have the following corollary.

Corollary 6.4. *Suppose that $G' \in \mathcal{E}_{\mathrm{ell}}(G)$. Then*

$$\sum_{\tau \in T(G,\zeta)} \Delta(\phi', \tau) I(\tau, \gamma) \; = \sum_{\sigma' \in \Sigma_G(G')} S'(\phi', \sigma')\Delta(\sigma', \gamma) \; , \tag{6.10}$$

for any $\phi' \in \Phi(\widetilde{G}', \widetilde{\zeta}'\zeta)$ and $\gamma \in \Gamma(G)$, while

$$\sum_{\gamma \in \Gamma(G)} \Delta(\sigma', \gamma) I(\gamma, \tau) \; = \sum_{\phi' \in \Phi(\widetilde{G}', \widetilde{\zeta}'\zeta)} S'(\sigma', \phi')\Delta(\phi', \tau) \; , \tag{6.11}$$

for any $\sigma \in \Sigma_G(\widetilde{G}')$ and $\tau \in \widetilde{T}(G, \zeta)$.

Proof. Observe that the summand on the right hand side of the first identity does indeed depend only on the image of σ' in the quotient $\Sigma_G(G')$ of $\Sigma_G(\widetilde{G}')$ by $\widetilde{Z}'(F)$. This follows from (2.1) and the $\widetilde{\zeta}'$-equivariance of $S'(\phi', \sigma')$. The two identities are dual to each other, and it will be enough to prove the second one.

At first we shall assume only that Theorem 6.1 holds for the quasi-split group $\widetilde{G}'$. Fix $\sigma' \in \Sigma_G(\widetilde{G}')$. Then for any $f \in \mathcal{H}\big(G(F), \zeta\big)$, we have

$$\begin{aligned} f'(\sigma') \; &= \sum_{\gamma \in \Gamma(G)} \Delta(\sigma', \gamma) f_G(\gamma) \\ &= \int_{T(G,\zeta)} \Big(\sum_{\gamma \in \Gamma(G)} \Delta(\sigma', \gamma) I(\gamma, \tau) \Big) f_G(\tau) d\tau \; . \end{aligned}$$

In addition to f', we have the function f'_{gr}, which belongs to a space $S\mathcal{I}_{\mathrm{gr}}(\widetilde{G}', \widetilde{\zeta}'\zeta)$ that we have agreed to identify with $S\mathcal{I}(\widetilde{G}', \widetilde{\zeta}'\zeta)$. The value $f'_{\mathrm{gr}}(\sigma')$ is therefore defined. It is given by

$$\begin{aligned} f'_{\mathrm{gr}}(\sigma') \; &= \int_{\Phi(\widetilde{G}', \widetilde{\zeta}'\zeta)} S'(\sigma', \phi') f'_{\mathrm{gr}}(\phi') d\phi' \\ &= \int_{\Phi(\widetilde{G}', \widetilde{\zeta}'\zeta)} \sum_{\tau \in T(G,\zeta)} S'(\sigma', \phi')\Delta(\phi', \tau) f_G(\tau) d\phi' \; . \end{aligned}$$

By a straightforward variant of Lemma 5.3, we can rewrite this last expression in the form

$$\int_{T(G,\zeta)} \Big(\sum_{\phi' \in \Phi(\widetilde{G}', \widetilde{\zeta}'\zeta)} S'(\sigma', \phi')\Delta(\phi', \tau) \Big) f_G(\tau) d\tau \; .$$

We conclude that

$$f'(\sigma') - f'_{\mathrm{gr}}(\sigma') \;=\; \int_{T(G,\zeta)} F(\sigma',\tau) f_G(\tau) d\tau \ ,$$

where

$$F(\sigma',\tau) \;=\; \sum_{\gamma\in\Gamma(G)} \Delta(\sigma',\gamma) I(\gamma,\tau) - \sum_{\phi'\in\Phi(\widetilde{G}',\widetilde{\zeta}'\zeta)} S'(\sigma',\phi')\Delta(\phi',\tau) \ . \tag{6.12}$$

Now assume that Theorem 6.2 also holds. The two functions f' and f'_{gr} in $S\mathcal{I}\big(\widetilde{G}'(F),\widetilde{\zeta}'\zeta\big)$ then take the same values at any ϕ', and are therefore equal. Thus

$$\int_{T(G,\zeta)} F(\sigma',\tau) f_G(\tau) d\tau \;=\; 0$$

for any $f \in \mathcal{H}\big(G(F),\zeta\big)$. It follows that $F(\sigma',\tau)$ vanishes for every σ' and τ. This establishes the identity (6.11). The derivation of the other identity (6.10) is similar. □

Having discussed some ramifications of the two theorems, we can now begin their proof. It will occupy most of the rest of the paper. We shall always assume that our induced torus contains A_G, since we can convert our results to the case of a smaller torus by taking a Fourier transform in ζ. We shall also assume inductively that both theorems hold if G is replaced by a group of smaller dimension. We begin by showing that with this assumption, the theorems hold for elements ϕ and ϕ' which are not cuspidal. We will then deal with the cuspidal case in Sections 8 and 9.

Suppose first that G is quasi-split. Assume that ϕ lies in the complement $\Phi_{\mathrm{par}}(G,\zeta)$ of $\Phi_2(G,\zeta)$ in $\Phi(G,\zeta)$. Then ϕ is the image of an element $\phi_M \in \Phi_2(M,\zeta)$, for a proper Levi subgroup M of G. Then

$$\begin{aligned} f^G_{\mathrm{gr}}(\phi) \;&=\; \sum_{\tau\in T(G,\zeta)} \Delta(\phi,\tau) f_G(\tau) \;=\; \sum_{\tau}\Big(\sum_{\tau_M\to\tau} \Delta_M(\phi_M,\tau_M)\Big) f_G(\tau) \\ &=\; \sum_{\tau_M\in T_{\mathrm{ell}}(M,\zeta)} \Delta_M(\phi_M,\tau_M) f_M(\tau_M) \;=\; f^M_{\mathrm{gr}}(\phi_M) \ , \end{aligned}$$

since $f_G(\tau) = f_M(\tau_M)$. Applying Theorem 6.1 inductively to the proper Levi subgroup M, we conclude that

$$f_M \longrightarrow f^M_{\mathrm{gr}}(\phi_M) \;=\; f^M(\phi_M) \ , \qquad f_M \in \mathcal{H}\big(M(F),\zeta\big),$$

is a stable distribution on $M(F)$. Since the pullback under the map $f_G \to f_M$ of a stable distribution on $M(F)$ is a stable distribution on $G(F)$, we see that

$$f_G \longrightarrow f^G_{\mathrm{gr}}(\phi) \;=\; f^G(\phi)\;, \qquad f \in \mathcal{H}\big(G(F),\zeta\big),$$

is a stable distribution on $G(F)$. This establishes Theorem 6.1 for the element $\phi \in \Phi_{\mathrm{par}}(G,\zeta)$. Observe that by construction, $f^G(\phi)$ equals $f^M(\phi_M)$.

Now suppose that G is arbitrary, that $G' \in \mathcal{E}_{\mathrm{ell}}(G)$ is given, and that ϕ' lies in the complement $\Phi_{\mathrm{par}}(\widetilde{G}',G,\zeta)$ of $\Phi_2(\widetilde{G}',G,\zeta)$ in $\Phi(\widetilde{G}',G,\zeta)$. Then ϕ' is the image of an element $\phi'_M \in \Phi_2(\widetilde{M}',M,\zeta)$, for proper Levi subgroups $M' \subsetneqq G'$ and $M \subsetneqq G$, with $M' \in \mathcal{E}_{\mathrm{ell}}(M)$. We have

$$\begin{aligned} f'_{\mathrm{gr}}(\phi') \;&=\; \sum_{\tau \in T(G,\zeta)} \Delta(\phi',\tau) f_G(\tau) \;=\; \sum_{\tau}\Big(\sum_{\tau_M \to \tau} \Delta_M(\phi'_M,\tau_M)\Big) f_G(\tau) \\ &=\; \sum_{\tau_M \in T_{\mathrm{ell}}(M,\zeta)} \Delta_M(\phi'_M,\tau_M) f_M(\tau_M) \;=\; f^{M'}_{\mathrm{gr}}(\phi'_M) \\ &=\; f^{M'}(\phi'_M) \;=\; f^{G'}(\phi')\;, \end{aligned}$$

by our induction assumption and the quasi-split case just treated. This establishes Theorem 6.2 for the element $\phi' \in \Phi_{\mathrm{par}}(\widetilde{G}',G,\zeta)$.

7. Simple trace formulas

It remains to prove the two theorems for cuspidal elements ϕ and ϕ'. For this, we shall need the global trace formula. We shall review the simple version of the ordinary trace formula [1, Section 7], and the corresponding form of the stable trace formula [21].

We reserve the symbols G and F for the local objects we have been considering. We shall denote corresponding global objects by the same symbols, augmented by a dot on top. Thus $\dot{G}$ stands for a reductive group over a number field $\dot{F}$. We take $(\dot{G},\dot{F})$ to be any such pair, with the property that $(\dot{G}_u,\dot{F}_u) = (G,F)$, for some fixed nonArchimedean valuation u of $\dot{F}$. We fix a central induced torus $\dot{Z}$ in $\dot{G}$, which is defined over $\dot{F}$ and contains $A_{\dot{G}}$. The adèlic quotient $\dot{Z}(\mathbb{A})\dot{G}(\dot{F})\backslash\dot{G}(\mathbb{A})$ then has finite volume. We also fix a character $\dot{\zeta}$ on $\dot{Z}(\dot{F})\backslash\dot{Z}(\mathbb{A})$. Our concern will be the representation of $\dot{G}(\mathbb{A})$ on the Hilbert space of $\dot{\zeta}$-equivariant functions on $\dot{G}(\dot{F})\backslash\dot{G}(\mathbb{A})$. This is the setting of [21], but is slightly different from that of [1]. We shall make what minor modification are necessary in the formulation of [1], without comment.

Having fixed $\dot{G}$, $\dot{Z}$ and $\dot{\zeta}$, we can form the adèlic Hecke algebra $\mathcal{H}\big(\dot{G}(\mathbb{A}),\dot{\zeta}\big)$ of smooth functions on $\dot{G}(\mathbb{A})$ which are compactly supported modulo $\dot{Z}(\mathbb{A})$, and which transform under $\dot{Z}(\mathbb{A})$ according to $\dot{\zeta}^{-1}$. We shall define a subspace of "simple functions". Let $\mathcal{H}_{\mathrm{simp}}\big(\dot{G}(\mathbb{A}),\dot{\zeta}\big)$ be the subspace of functions in $\mathcal{H}\big(\dot{G}(\mathbb{A}),\dot{\zeta}\big)$ spanned by functions $\dot{f}=\prod\limits_{v}\dot{f}_v$ such that $\dot{f}_v$ is cuspidal at two nonArchimedean places v, and $\dot{f}_{w,\dot{G}}$ is supported on the strongly regular elements at one nonArchimedean place w. The global space of invariant functions $\mathcal{I}\big(\dot{G}(\mathbb{A}),\dot{\zeta}\big)$ can be defined as a direct limit, over finite sets S of valuations of $\dot{F}$, of spaces $\mathcal{I}\big(\dot{G}(\dot{F}_S),\dot{\zeta}_S\big)$ [1, Section 1]. There is then a continuous map $\dot{f}\rightarrow\dot{f}_{\dot{G}}$ from $\mathcal{H}\big(\dot{G}(\mathbb{A}),\dot{\zeta}\big)$ onto $\mathcal{I}\big(\dot{G}(\mathbb{A}),\dot{\zeta}\big)$. We write $\mathcal{I}_{\mathrm{simp}}\big(\dot{G}(\mathbb{A}),\dot{\zeta}\big)$ for the image of $\mathcal{H}_{\mathrm{simp}}\big(\dot{G}(\mathbb{A}),\dot{\zeta}\big)$ in $\mathcal{I}\big(\dot{G}(\mathbb{A}),\dot{\zeta}\big)$.

The simple trace formula applies to any function $\dot{f}$ in $\mathcal{H}_{\mathrm{simp}}\big(\dot{G}(\mathbb{A}),\dot{\zeta}\big)$. It is an identity

$$\sum_{\gamma\in\Gamma_{\mathrm{ell}}(\dot{G}/\dot{Z})} a^{\dot{G}}(\dot{\gamma})\dot{f}_{\dot{G}}(\dot{\gamma}) \;=\; \sum_{\nu}\sum_{\dot{\pi}\in\Pi_{\mathrm{disc}}(\dot{G},\dot{\zeta},\nu)} a^{\dot{G}}_{\mathrm{disc}}(\dot{\pi})\dot{f}_{\dot{G}}(\dot{\pi}) \;, \tag{7.1}$$

whose terms we describe. On the left hand side,

$$\Gamma_{\mathrm{ell}}(\dot{G}/\dot{Z}) \;=\; \Gamma_{\mathrm{reg,ell}}\big(\dot{G}(\dot{F})\big)/\dot{Z}(\dot{F})$$

denotes the set of strongly regular elliptic conjugacy classes in the group $(\dot{G}/\dot{Z})(\dot{F}) = \dot{G}(\dot{F})/\dot{Z}(\dot{F})$, and

$$\dot{a}^{\dot{G}}(\dot{\gamma}) \;=\; m(\dot{\gamma})^{-1}\mathrm{vol}\big(\dot{G}_{\dot{\gamma}}(\dot{F})\dot{Z}(\mathbb{A})\backslash\dot{G}_{\dot{\gamma}}(\mathbb{A})\big) \;,$$

where $m(\dot{\gamma})$ is the number of points $\dot{z}\in\dot{Z}(\dot{F})$ such that $\dot{\gamma}\dot{z}$ is $\dot{G}(\dot{F})$-conjugate to $\dot{\gamma}$. The global orbital integral $\dot{f}_{\dot{G}}(\dot{\gamma})$ is left $\dot{Z}(\dot{F})$-invariant in $\dot{\gamma}$, and therefore depends only on

$\dot{\gamma}$ as an element in $\Gamma(\dot{G}/\dot{Z})$. On the other side, ν runs over Weyl orbits in the complex vector space $\mathfrak{h}_{\mathbb{C}}^*$ of Archimedean infinitesimal characters defined in [1, Section 4]. Thus, $\mathfrak{h}_{\mathbb{C}}$ is a Cartan subalgebra of the complexified Lie algebra of the real Lie group $\dot{G}_\infty = (\mathrm{Res}_{\dot{F}/\mathbb{Q}}\dot{G})(\mathbb{R})$. The coefficients $a_{\mathrm{disc}}^{\dot{G}}(\dot{\pi})$ are defined as on p. 517 of [1], while $\Pi_{\mathrm{disc}}(\dot{G}, \dot{\zeta}, \nu)$ is a set of irreducible representations of $\dot{G}(\mathbb{A})$ whose central character on $\dot{Z}(\mathbb{A})$ equals $\dot{\zeta}$, and whose Archimedean infinitesimal character equals ν. In particular, the term corresponding to ν in (7.1) vanishes unless the projection of ν into the Lie algebra of $\dot{Z}_\infty = (\mathrm{Res}_{\dot{F}/\mathbb{Q}}\dot{Z})(\mathbb{R})$ coincides with the differential of the Archimedean component $\dot{\zeta}_\infty$ of $\dot{\zeta}$. (In [1, p. 517], it is the norm $t = \|\mathrm{Im}(\nu)\|$ which is specified, rather than ν itself.) The convergence of the right hand side of (7.1) is as an iterated sum; both the inner and the outer sums converge absolutely. Actually, the results of Müller [24] can be used to show that the right hand side of (7.1) converges absolutely as a double sum, but the estimates we will use do not require this stronger assertion. We write $I(\dot{f})$ for the linear form on $\mathcal{H}_{\mathrm{simp}}\big(\dot{G}(\mathbb{A}), \dot{\zeta}\big)$ defined by either side of the identity (7.1).

We are trying to deduce local results at the place u. We follow the usual conventions of using a superscript u to denote a component (relative to $\mathbb{A}$) away from u, as well as the standard subscript u for a component at u. Thus $\dot{\zeta} = \dot{\zeta}^u \otimes \dot{\zeta}_u$ is the decomposition of $\dot{\zeta}$ into characters on $\dot{Z}(\mathbb{A}^u)$ and $\dot{Z}(\dot{F}_u)$. We shall write $\zeta = \dot{\zeta}_u$ and $Z = \dot{Z}_u$. We define the space $\mathcal{H}_{\mathrm{simp}}\big(\dot{G}(\mathbb{A}^u), \dot{\zeta}^u\big)$ of simple functions on $\dot{G}(\mathbb{A}^u)$ as above. Then if $\dot{f}^u$ lies in $\mathcal{H}_{\mathrm{simp}}(\dot{G}(\mathbb{A}^u), \dot{\zeta}^u\big)$, and $f = \dot{f}_u$ is an arbitrary function in $\mathcal{H}\big(G(F), \zeta\big)$, the product $\dot{f} = \dot{f}^u f$ belongs to $\mathcal{H}_{\mathrm{simp}}\big(\dot{G}(\mathbb{A}), \dot{\zeta}\big)$. In what follows, $\dot{f}^u$ will generally be fixed while f will be allowed to vary. To keep track of the dependence on f, we write the left hand side of (7.1) as

$$I(\dot{f}) \;=\; \sum_{\gamma \in \Gamma_{\mathrm{ell}}(G/Z)} I(\dot{f}^u, \gamma) f_G(\gamma)\ , \tag{7.2}$$

where

$$I(\dot{f}^u, \widetilde{\gamma}) \;=\; \sum_{\dot{\gamma}} a^{\dot{G}}(\dot{\gamma}) \dot{f}^u_{\dot{G}}(\dot{\widetilde{\gamma}}^u)\ ,$$

for any preimage $\widetilde{\gamma}$ of γ in $\Gamma_{\mathrm{ell}}(G)$. The last sum is over elements $\dot{\gamma} \in \Gamma_{\mathrm{ell}}(\dot{G}/\dot{Z})$ which have a preimage $\dot{\widetilde{\gamma}} \in \Gamma_{\mathrm{ell}}(\dot{G})$ such that $\dot{\widetilde{\gamma}}_u$ equals $\widetilde{\gamma}$, while the summand in (7.2) depends only on the element γ in $\Gamma_{\mathrm{ell}}(G/Z)$.

The right hand side of (7.1) can be written in a similar way. It equals

$$I(\dot{f}) \;=\; \sum_{\nu} I_\nu(\dot{f}) \;, \tag{7.3}$$

where

$$I_\nu(\dot{f}) \;=\; \sum_{\dot{\pi} \in \Pi_{\mathrm{disc}}(\dot{G}, \dot{\zeta}, \nu)} a^{\dot{G}}_{\mathrm{disc}}(\dot{\pi}) \dot{f}_{\dot{G}}(\dot{\pi}) \;.$$

We can certainly write

$$\dot{f}_{\dot{G}}(\dot{\pi}) \;=\; \dot{f}^u_{\dot{G}}(\dot{\pi}^u) f_G(\pi) \;,$$

for irreducible representations $\dot{\pi}^u \in \Pi\big(\dot{G}(\mathbb{A}^u), \dot{\zeta}^u, \nu\big)$ and $\pi \in \Pi(G, \zeta)_{\mathbb{C}}$ of $\dot{G}(\mathbb{A}^u)$ and $G(F)$ respectively. However, we would like to end up with a sum over $\tau \in T(G, \zeta)_{\mathbb{C}}$ rather than $\pi \in \Pi(G, \zeta)_{\mathbb{C}}$. To this end, we first define new coefficients $a^{\dot{G}}_{\mathrm{disc}}(\dot{\pi}^u, \tau)$ on

$$\Pi\big(\dot{G}(\mathbb{A}^u), \dot{\zeta}^u, \nu\big) \times \widetilde{T}(G, \zeta)_{\mathbb{C}}$$

by writing $I_\nu(\dot{f})$ in the form

$$\sum_{(\dot{\pi}^u, \tau)} a^{\dot{G}}_{\mathrm{disc}}(\dot{\pi}^u, \tau) \dot{f}^u_{\dot{G}}(\dot{\pi}^u) f_G(\tau) \;.$$

This is possible because $f_G(\pi)$ has a finite expansion in terms of values $f_G(\tau)$, and because the original sum over $\dot{\pi}$ can be taken over a finite set. We can therefore write

$$I_\nu(\dot{f}) \;=\; \sum_{\tau \in T(G, \zeta)_{\mathbb{C}}} I_\nu(\dot{f}^u, \tau) f_G(\tau) \;, \tag{7.4}$$

where

$$I_\nu(\dot{f}^u, \tau) \;=\; \sum_{\dot{\pi}^u \in \Pi(\dot{G}(\mathbb{A}^u), \dot{\zeta}, \nu)} a^{\dot{G}}_{\mathrm{disc}}(\dot{\pi}^u, \tau) \dot{f}^u_{\dot{G}}(\dot{\pi}^u) \;.$$

The summand in (7.4) (and in the expression preceding (7.4)) does indeed depend on τ only as an element in $T(G,\zeta)_{\mathbb{C}}$ (rather than $\widetilde{T}(G,\zeta)_{\mathbb{C}}$). We have already noted that the sum over ν in (7.3) is absolutely convergent. We recall the uniform description [1, Section 6] of this convergence.

The Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ is defined [1, Section 4] as the complexification of a real Lie algebra

$$\mathfrak{h} \ = \ i\mathfrak{h}_K \oplus \mathfrak{h}_0$$

which is invariant under the complex Weyl group W_∞ of $\dot{G}_\infty$. In this paper, a *multiplier* for $\dot{G}$ will be a function α in $C_c^\infty(\mathfrak{h})^{W_\infty}$. The Fourier transform $\widehat{\alpha}$ will then be a W_∞-invariant function in the Paley-Wiener space on $\mathfrak{h}_{\mathbb{C}}^*$. If $\dot{f}$ is any function in $\mathcal{H}\big(\dot{G}(\mathbb{A}),\dot{\zeta}\big)$, one can transform the Archimedean components of $\dot{f}$ by α. This provides a second function $\dot{f}_\alpha$ in $\mathcal{H}\big(\dot{G}(\mathbb{A}),\dot{\zeta}\big)$, which is characterized by the property that

$$\dot{f}_{\alpha,\dot{G}}(\dot{\pi}) \ = \ \widehat{\alpha}(\nu)\dot{f}_{\dot{G}}(\dot{\pi}) \ ,$$

for any representation $\dot{\pi} \in \Pi\big(\dot{G}(\mathbb{A}),\dot{\zeta}\big)$ with Archimedean infinitesimal character equal to ν. In particular,

$$I_\nu(\dot{f}^u_\alpha,\tau) \ = \ \widehat{\alpha}(\nu)I_\nu(\dot{f}^u,\tau) \tag{7.5}$$

for any $\dot{f}^u \in \mathcal{H}\big(\dot{G}(\mathbb{A}^u),\dot{\zeta}^u\big)$ and $\tau \in T(G,\zeta)_{\mathbb{C}}$. The convergence estimate is given by the values of $\widehat{\alpha}$ on a subset

$$\mathfrak{h}_u^*(\nu,T) \ = \ \{\nu \in \mathfrak{h}_u^* : \ \|\mathrm{Re}(\nu)\| \le r, \ \|\mathrm{Im}(\nu)\| \ge T\}$$

of $\mathfrak{h}_{\mathbb{C}}^*$. Here $\mathfrak{h}_u^*$ denotes the set of points $\nu \in \mathfrak{h}_{\mathbb{C}}^*$ such that $\overline{\nu} = \eta\nu$ for some element $\eta \in W_\infty\theta_\infty$ of order 2, where θ_∞ is the Cartan involution of $\dot{G}_\infty$ acting on $\mathfrak{h}_{\mathbb{C}}^*$. The infinitesimal character ν of any unitary representation of $\dot{G}(\mathbb{A})$ is known to lie in the subset $\mathfrak{h}_u^*$. (We neglected to include the Cartan involution in the original definition of [1,

p. 536].) The Hermitian norm $\|\cdot\|$ in the definition is dual to a fixed $W_\infty\theta_\infty$-invariant Euclidean norm on $\mathfrak{h}$.

The convergence estimate is given by the next lemma, which follows directly from [1, Lemma 6.3], in the same way as [1, Corollary 6.5].

Lemma 7.1. *For any function $\dot{f}$ in $\mathcal{H}_{\mathrm{simp}}\big(\dot{G}(\mathbb{A}),\dot{\zeta}\big)$, we can choose constants C, k and r with the following property. For any positive numbers T and N and any α in $C_N^\infty(\mathfrak{h})^{W_\infty}$, the space of multipliers with support of norm bounded by N,*

$$\sum_{\{\nu:\|\mathrm{Im}(\nu\|>T\}} |I_\nu(\dot{f}_\alpha)| \ \le\ Ce^{kN} \sup_{\nu\in\mathfrak{h}_u^*(r,T)} (|\widehat{\alpha}(\nu)|). \tag{7.6}$$

The role of the trace formula in this paper will be in its stabilization, a linear combination of stable distributions on global endoscopic groups. Suppose for a moment that $\dot{G}$ is quasi-split. We define $S\mathcal{I}\big(\dot{G}(\mathbb{A}),\dot{\zeta}\big)$ as a direct limit, over finite sets S of valuations of $\dot{F}$, of spaces $S\mathcal{I}\big(\dot{G}(\dot{F}_S),\dot{\zeta}_S\big)$. Then $\dot{f}\to\dot{f}^{\dot{G}}$ is a continuous map of $\mathcal{H}\big(\dot{G}(\mathbb{A}),\dot{\zeta}\big)$ onto $S\mathcal{I}\big(\dot{G}(\mathbb{A}),\dot{\zeta}\big)$, which sends $\mathcal{H}_{\mathrm{simp}}\big(\dot{G}(\mathbb{A}),\dot{\zeta}\big)$ to a closed subspace denoted by $S\mathcal{I}_{\mathrm{simp}}\big(\dot{G}(\mathbb{A}),\dot{\zeta}\big)$). If $\dot{\sigma}$ is a strongly regular, elliptic element in $\dot{G}(\dot{F})$, $\dot{f}^{\dot{G}}(\dot{\sigma})$ is defined, and depends only on the image of $\dot{\sigma}$ in the set

$$\Sigma_{\mathrm{ell}}(\dot{G}/\dot{Z}) \ =\ \Sigma_{\mathrm{reg,ell}}\big(\dot{G}(\dot{F})/\dot{Z}(\dot{F})\big) \ =\ \Sigma_{\mathrm{reg,ell}}\big(\dot{G}(\dot{F})\big)/\dot{Z}(\dot{F})$$

of strongly regular, stable elliptic conjugacy classes in $\dot{G}(\dot{F})/\dot{Z}(\dot{F})$. One of the purposes of [21] was to introduce a linear form $S=S^{\dot{G}}$ on $\mathcal{H}_{\mathrm{simp}}\big(\dot{G}(\mathbb{A}),\dot{\zeta}\big)$, defined by an expression

$$S(f) \ =\ \sum_{\dot{\sigma}\in\Sigma_{\mathrm{ell}}(\dot{G}/\dot{Z})} b^{\dot{G}}(\dot{\sigma})\dot{f}^{\dot{G}}(\dot{\sigma})\ ,$$

for certain coefficients $b^{\dot{G}}(\dot{\sigma})$. This linear form is a stable distribution (by definition), and can obviously be identified with the pullback

$$S(\dot{f}) \ =\ \widehat{S}(\dot{f}^{\dot{G}})$$

of a linear form $\widehat{S}$ on $S\mathcal{I}_{\mathrm{simp}}\big(\dot{G}(\mathbb{A}),\dot{\zeta}\big)$. Suppose that $\dot{f}=\dot{f}^u f$, for $\dot{f}^u\in\mathcal{H}_{\mathrm{simp}}\big(\dot{G}(\mathbb{A}^u),\dot{\zeta}^u\big)$ as above. Then we can write

$$S(\dot{f}) \;=\; \sum_{\sigma\in\Sigma_{\mathrm{ell}}(G/Z)} S(\dot{f}^u,\sigma)f^G(\sigma)\ , \tag{7.7}$$

where

$$S(\dot{f}^u,\widetilde{\sigma}) \;=\; \sum_{\dot{\sigma}} b^{\dot{G}}(\dot{\sigma})\dot{f}^{u,\dot{G}}(\dot{\widetilde{\sigma}}^u)\ ,$$

for any preimage $\widetilde{\sigma}$ of σ in $\Sigma_{\mathrm{ell}}(G)$. The last sum is over elements $\dot{\sigma}\in\Sigma_{\mathrm{ell}}(\dot{G}/\dot{Z})$ which have a preimage $\dot{\widetilde{\sigma}}\in\Sigma_{\mathrm{ell}}(\dot{G})$ such that $\dot{\widetilde{\sigma}}_u$ equals $\widetilde{\sigma}$, while the summand in (7.7) depends only on the element σ.

Returning to the case where $\dot{G}$ is arbitrary, we let $\mathcal{E}_{\mathrm{ell}}(\dot{G})$ denote the set of elliptic global endoscopic data for $\dot{G}$ over $\dot{F}$. Lemma 2.1 has an obvious analogue for the global field $\dot{F}$. For each $\dot{G}'\in\mathcal{E}_{\mathrm{ell}}(\dot{G})$, we fix a central extension

$$1\longrightarrow \dot{\widetilde{Z}}{}'\longrightarrow \dot{\widetilde{G}}{}'\longrightarrow \dot{G}'\longrightarrow 1$$

over $\dot{F}$ which satisfies the three conditions of the lemma. As with the local constructions of Section 2, we obtain a global Langlands parameter for $\dot{\widetilde{Z}}{}'$, and hence a character $\dot{\widetilde{\zeta}}{}'$ on $\dot{\widetilde{Z}}{}'(\dot{F})\backslash\dot{\widetilde{Z}}{}'(\mathbb{A})$. The datum also determines an extension $\dot{\widetilde{\zeta}}{}'_{\dot{Z}}$ of $\dot{\widetilde{\zeta}}{}'$ to an automorphic representation on the induced torus $\dot{\widetilde{Z}}{}'\dot{Z}$ obtained by taking the preimage of $\dot{Z}$ in $\dot{\widetilde{G}}$. Following the local notation further, we write $\dot{\widetilde{\zeta}}{}'\dot{\zeta}$ for the automorphic representation $\dot{\widetilde{\zeta}}{}'_{\dot{Z}}\otimes\dot{\zeta}$ on $\dot{\widetilde{Z}}{}'\dot{Z}$.

Suppose that v is a valuation for $\dot{F}$. Then $\dot{G}'$ determines a local endoscopic datum $\dot{G}'_v\in\mathcal{E}(\dot{G}_v)$. The choices above also determine auxiliary objects $\dot{\widetilde{G}}{}'_v$, $\dot{\widetilde{Z}}{}'_v$ and $\dot{\widetilde{\zeta}}{}'_v$ for $\dot{G}'_v$ which satisfy the conditions of Lemma 2.1. Appealing to the transfer theorem of Waldspurger [28] or Shelstad [25] (according to whether v is discrete or Archimedean), we obtain a map $\dot{f}_v\to\dot{f}'_v$ from $\mathcal{H}\big(\dot{G}(F_v),\dot{\zeta}_v\big)$ to $S\mathcal{I}\big(\dot{\widetilde{G}}{}'(\dot{F}_v),\dot{\widetilde{\zeta}}{}'_v\dot{\zeta}_v\big)$. If $\dot{G}_v$ and $\dot{\zeta}_v$ are unramified, and $\dot{f}_v$ is the element in a hyperspecial Hecke algebra determined by $\dot{\zeta}_v$, then

$\dot{f}'_v$ vanishes unless $\dot{G}'_v$ is also unramified [16, Theorem 7.5], in which case it is given by Hypothesis 3.1. Putting the local transfer maps together, we obtain a global transfer map $\dot{f} \to \dot{f}'$ from $\mathcal{H}\big(\dot{G}(\mathbb{A}), \dot{\zeta}\big)$ to $S\mathcal{I}\big(\widetilde{\dot{G}'}(\mathbb{A}), \dot{\tilde{\zeta}}'\dot{\zeta}\big)$. There are only finitely many $\dot{G}' \in \mathcal{E}_{\mathrm{ell}}(\dot{G})$ which are unramified outside any given finite set of places [21, Lemma 8.12]. It follows that for any $\dot{f}$, there are only finitely many $\dot{G}' \in \mathcal{E}(\dot{G})$ with $\dot{f}' \neq 0$.

For each $\dot{G}' \in \mathcal{E}_{\mathrm{ell}}(\dot{G})$, we have the stable linear form $S' = S^{\widetilde{\dot{G}}'}$ on $\mathcal{H}_{\mathrm{simp}}\big(\widetilde{\dot{G}'}(\mathbb{A}), \dot{\tilde{\zeta}}'\dot{\zeta}\big)$. If $\dot{f}$ belongs to $\mathcal{H}_{\mathrm{simp}}\big(\dot{G}(\mathbb{A}), \dot{\zeta}\big)$, $\dot{f}'$ lies in $S\mathcal{I}_{\mathrm{simp}}\big(\widetilde{\dot{G}'}(\mathbb{A}), \dot{\tilde{\zeta}}'\dot{\zeta}\big)$, and we can evaluate $\widehat{S}'$ at $\dot{f}'$. In [21], Langlands establishes an expansion

$$I(\dot{f}) \; = \sum_{\dot{G}' \in \mathcal{E}_{\mathrm{ell}}(\dot{G})} \iota(\dot{G}, \dot{G}')\widehat{S}'(\dot{f}') \; , \tag{7.8}$$

for certain coefficients $\iota(\dot{G}, \dot{G}')$. The formula is valid for any function $\dot{f}' \in \mathcal{H}_{\mathrm{simp}}\big(\dot{G}(\mathbb{A}), \dot{\zeta}\big)$, and for any such $\dot{f}$, the sum over $\dot{G}'$ can be taken over a finite set.

In order to exploit the stabilized trace formula (7.8), we have to apply a stable version of (7.6) to each of its terms. The dual groups $\widehat{\dot{G}}$ and $\widehat{\dot{G}'} \subset \widehat{\widetilde{\dot{G}'}}$ provide a linear embedding of $\mathfrak{h}^*$ into the corresponding space $(\widetilde{\mathfrak{h}}')^*$ for $\widetilde{\dot{G}'}$, which is defined up to the action of the Weyl group W_∞. There is also an affine linear embedding from $\mathfrak{h}^*_{\mathbb{C}}$ into $(\widetilde{\mathfrak{h}}')^*_{\mathbb{C}}$, of the form

$$\nu \longrightarrow \nu' \; = \; \nu + d\dot{\tilde{\zeta}}'_\infty \; ,$$

which is compatible with the Archimedean transfer map $\dot{f}_\infty \to \dot{f}'_\infty$. This can be established from the Archimedean analogue of (2.1) and the differential equations satisfied by orbital integrals. (See [26].) If $\alpha \in C^\infty_c(\mathfrak{h})^{W_\infty}$ is a multiplier for $\dot{G}$, let $\alpha' \in C^\infty_c(\widetilde{\mathfrak{h}}')^{W'_\infty}$ be any multiplier for $\widetilde{\dot{G}'}$ such that

$$\widehat{\alpha}'(\nu') \; = \; \widehat{\alpha}(\nu) \; , \qquad \nu \in \mathfrak{h}^*_{\mathbb{C}}.$$

Then

$$(\dot{f}_\alpha)' \; = \; \dot{f}'_{\alpha'} \; , \qquad \dot{f} \in \mathcal{H}\big(\dot{G}(\mathbb{A}), \dot{\zeta}\big). \tag{7.9}$$

Suppose again that $\dot{G}$ is quasisplit. Then we take $\dot{\widetilde{G}}' = \dot{G}'$, with the embedding $\dot{\widetilde{\xi}}'$ equal to the identity. In particular, $\nu' = \nu$ for any $\nu \in \mathfrak{h}_{\mathbb{C}}^*$. For each element ν which is compatible with $d\dot{\zeta}_\infty$, we define a distribution $S_\nu = S_\nu^{\dot{G}}$ on $\mathcal{H}_{\mathrm{simp}}\big(\dot{G}(\mathbb{A}), \dot{\zeta}\big)$ inductively by

$$S_\nu(\dot{f}) \;=\; I_\nu(\dot{f}) - \sum_{\dot{G}' \neq \dot{G}} \iota(\dot{G}, \dot{G}') \widehat{S}'_{\nu'}(\dot{f}') \;.$$

An expansion

$$S(\dot{f}) \;=\; \sum_\nu S_\nu(\dot{f}) \;, \qquad \dot{f} \in \mathcal{H}_{\mathrm{simp}}\big(\dot{G}(\mathbb{A}), \dot{\zeta}\big), \tag{7.10}$$

then follows inductively from (7.8) and the corresponding expansion for $I(\dot{f})$. We also have an identity

$$S_\nu(\dot{f}_\alpha) \;=\; \widehat{\alpha}(\nu) S_\nu(\dot{f}) \;, \tag{7.11}$$

for each ν and $\alpha \in C_c^\infty(\mathfrak{h})^{W_\infty}$. This follows inductively from the corresponding formula (7.5) for $I_\nu(\dot{f}_\alpha)$ and the relation (7.9).

Corollary 7.2. *For any function $\dot{f} \in \mathcal{H}_{\mathrm{simp}}\big(\dot{G}(\mathbb{A}), \dot{\zeta}\big)$, we can choose constants C, k and r such that for any positive numbers T and N, and any $\alpha \in C_N^\infty(\mathfrak{h})^{W_\infty}$,*

$$\sum_{\{\nu:\ \|\mathrm{Im}(\nu)\| > T\}} |S_\nu(\dot{f}_\alpha)| \;\leq\; Ce^{kN} \sup_{\nu \in \mathfrak{h}_u^*(r,T)} \big(|\widehat{\alpha}(\nu)|\big) \;. \tag{7.12}$$

Proof. It follows from (7.9) and the definition of $S_\nu(\dot{f})$ that the left hand side of (7.12) is bounded by

$$\sum_{\|\mathrm{Im}(\nu)\| > T} |I_\nu(\dot{f}_\alpha)| + \sum_{\dot{G}' \neq \dot{G}} \iota(\dot{G}, \dot{G}') \sum_{\|\mathrm{Im}(\nu)\| > T} |\widehat{S}'_{\nu'}(f'_{\alpha'})| \;.$$

We can assume inductively that the corollary holds for each of the quasi-split groups $\dot{\widetilde{G}}'$, with $\dot{G}' \neq \dot{G}$. The estimate follows without difficulty from Lemma 7.1. $\square$

8. Proof of stability

We shall now prove Theorem 6.1. For this section, the group G over F will be quasi-split. We are going to derive the local result from the global trace formula, so we fix global objects $\dot{G}$, $\dot{F}$, $\dot{Z}$ and $\dot{\zeta}$ as in Section 7. In addition to the requirement that $(\dot{G}_u, \dot{F}_u) = (G, F)$, we ask that $\dot{G}$ be quasi-split over $\dot{F}$, and that $A_{\dot{G},u}$ equal A_G. The torus $Z = \dot{Z}_u$ will then contain A_G. It is clearly possible to choose $\dot{G}$ so that these additional conditions are met.

We have to exploit the fact that the distribution $S(\dot{f})$ is stable. The first step is to verify that the same is true of $S_\nu(\dot{f})$.

Lemma 8.1. *For each ν, the linear form $S_\nu(\dot{f})$ on $\mathcal{H}_{\mathrm{simp}}\big(\dot{G}(\mathbb{A}), \dot{\zeta}\big)$ is stable.*

Proof. Choose any function $\dot{f} \in \mathcal{H}_{\mathrm{simp}}\big(\dot{G}(\mathbb{A}), \dot{\zeta}\big)$ with $\dot{f}^{\dot{G}} = 0$. Our task is to show that $S_\nu(\dot{f}) = 0$ for any ν. If $\alpha \in C_c^\infty(\mathfrak{h})^{W_\infty}$ is any multiplier, we have

$$\begin{aligned} \sum_\nu S_\nu(\dot{f}_\alpha) \ &= \ S(\dot{f}_\alpha) \\ &= \ \widehat{S}\big((\dot{f}_\alpha)^{\dot{G}}\big) \ = \ \widehat{S}\big((\dot{f}^{\dot{G}})_\alpha\big) \ = \ 0 \ , \end{aligned}$$

by (7.9) and (7.10). We shall combine this with Corollary 7.2. The argument is like that of [5, Section 2.15], but simpler.

Choose constants C, k and r so that the estimate (7.12) holds. Let ν_1 be a fixed W_∞-orbit in $\mathfrak{h}_u^*$. Enlarging the constant r in (7.12) if necessary, we may assume that ν_1 is contained in the cylinder $\mathfrak{h}_u^*(r) = \mathfrak{h}_u^*(r, 0)$. We can choose a function $\alpha_1 \in C_c^\infty(\mathfrak{h})^{W_\infty}$ such that $\widehat{\alpha}_1$ maps $\mathfrak{h}_u^*(r)$ to the unit interval, and such that the inverse image of 1 under $\widehat{\alpha}_1$ is the Weyl orbit ν_1. (See [5, Lemma II.15.2]. The proof of this lemma must be augmented with the fact that W_∞ normalizes the set $W_\infty\theta_\infty$.) Then α_1 belongs to $C_{N_1}^\infty(\mathfrak{h})^{W_\infty}$, for some positive integer N_1. Since α_1 is rapidly decreasing on $\mathfrak{h}_u^*(r)$, we can choose $T > 0$ such that

$$|\widehat{\alpha}_1(\nu)| \ \leq \ e^{-2kN_1} \ ,$$

for all W_∞-orbits ν in $\mathfrak{h}_u^*(r,T)$. Let α_m be the convolution of α_1 with itself m times. Then α_m lies in $C_{mN_1}^\infty(\mathfrak{h})^{W_\infty}$. Since $\widehat{\alpha}_m(\nu) = \big(\widehat{\alpha}_1(\nu)\big)^m$, the estimate (7.12) gives us

$$\sum_{\|\mathrm{Im}(\nu)\|>T} |S_\nu(\dot{f}_{\alpha_m})| \;\le\; C(e^{-kN_1})^m \; .$$

Therefore, the left hand side approaches 0 as m approaches infinity.

Now by (7.11),

$$S_\nu(\dot{f}_{\alpha_m}) \;=\; \widehat{\alpha}_1(\nu)^m S_\nu(\dot{f})$$

for any ν. In particular,

$$S_{\nu_1}(\dot{f}_{\alpha_m}) \;=\; \widehat{\alpha}_1(\nu_1)^m S_{\nu_1}(\dot{f}) \;=\; S_{\nu_1}(\dot{f}) \; .$$

Since $\sum\limits_{\nu} S_\nu(\dot{f}_{\alpha_m}) = 0$, we find that $|S_{\nu_1}(\dot{f})|$ is bounded by the sum of

$$\sum_{\{\nu:\ \|\mathrm{Im}(\nu)\|\le T,\ \nu\neq\nu_1\}} |S_\nu(\dot{f}_{\alpha_m})|$$

and

$$\sum_{\|\mathrm{Im}(\nu)\|>T} |S_\nu(\dot{f}_{\alpha_m})| \; .$$

We have just observed that the second sum goes to 0 as m approaches infinity. There are only finitely many nonzero terms in the first sum, and for each such term,

$$\lim_{m\to\infty} |S_\nu(\dot{f}_{\alpha_m})| \;=\; \lim_{m\to\infty} \big(|\alpha_1(\nu)|^m |S_\nu(\dot{f})|\big) \;=\; 0 \; ,$$

since $\alpha_1(\nu) < 1$ for any W_∞-orbit ν in $\mathfrak{h}_u^*$ not equal to ν_1. It follows that $S_{\nu_1}(\dot{f}) = 0$. Consequently, S_ν is a stable distribution for any ν. $\square$

As in Section 7, we take $\dot{f} = \dot{f}^u f$, for a fixed function $\dot{f}^u \in \mathcal{H}_{\mathrm{simp}}\big(\dot{G}(\mathbb{A}^u), \dot{\zeta}^u\big)$ and an arbitrary function $f \in \mathcal{H}\big(G(F), \zeta\big)$. We can assume that $\dot{f}^u = \prod\limits_{v\neq u} \dot{f}_v = \dot{f}^{u,w}\dot{f}_w$, where $\dot{f}_w$ is cuspidal for some nonArchimedean place $w \neq u$. Applying Proposition 3.5 to the nonArchimedean group $\dot{G}_w$, we can write $\dot{f}_w = \dot{f}_{w,1} + \dot{f}_{w,2}$ for functions $\dot{f}_{w,1}$ and $\dot{f}_{w,2}$ in

$\mathcal{H}_{\mathrm{cusp}}\big(\dot{G}(\dot{F}_w),\dot{\zeta}_w\big)$ such that $\dot{f}'_{w,1}=0$ for every $\dot{G}'\in\mathcal{E}(\dot{G})$ with $\dot{G}'\neq\dot{G}$, and such that $\dot{f}^{\dot{G}}_{w,2}=0$. (The image of $\dot{G}'$ in $\mathcal{E}(\dot{G}_w)$ need not be elliptic, but $\dot{f}'_{w,1}$ will of course still vanish, since $\dot{f}_{w,1}$ is cuspidal.) We write $\dot{f}$ as a sum $\dot{f}_1+\dot{f}_2$, where

$$\dot{f}_i \;=\; \dot{f}^{u,w}\dot{f}_{w,i}f \;=\; \dot{f}^u_i f\ , \qquad i=1,2.$$

Then $\dot{f}'_1=0$ for every $\dot{G}'\neq\dot{G}$, while

$$\dot{f}^{\dot{G}} \;=\; \dot{f}^{\dot{G}}_1+\dot{f}^{\dot{G}}_2 \;=\; \dot{f}^{\dot{G}}_1\ .$$

Since S_ν is stable, we have

$$\begin{aligned} S_\nu(\dot{f}) \;&=\; \widehat{S}_\nu(\dot{f}^{\dot{G}}) \;=\; \widehat{S}_\nu(\dot{f}^{\dot{G}}_1) \;=\; S_\nu(\dot{f}_1)\\ &=\; I_\nu(\dot{f}_1)-\sum_{\dot{G}'\neq\dot{G}}\iota(\dot{G},\dot{G}')\widehat{S}'_\nu(\dot{f}'_1)\\ &=\; I_\nu(\dot{f}_1)\ . \end{aligned}$$

It follows from (7.4) that

$$S_\nu(\dot{f}) \;=\; \sum_{\tau\in T(G,\zeta)_{\mathbb{C}}} S_\nu(\dot{f}^u,\tau)f_G(\tau)$$

where

$$S_\nu(\dot{f}^u,\tau) \;=\; I_\nu(\dot{f}^u_1,\tau)\ .$$

The support of the function $\tau\to S_\nu(\dot{f}^u,\tau)$ meets any connected component of $\widetilde{T}(G,\zeta)_{\mathbb{C}}$ in a finite set. Since $f_G(\tau)$ is supported on only finitely many components, the product of the two functions is supported on a finite subset of $T(G,\zeta)_{\mathbb{C}}$. It follows easily that the coefficients $S_\nu(\dot{f}^u,\tau)$ are independent of the decomposition $\dot{f}=\dot{f}_1+\dot{f}_2$, as the notation implies.

We would rather have a formula for $S_\nu(\dot{f})$ in terms of the basis $T^{\mathcal{E}}(G,\zeta)_{\mathbb{C}}$ instead of $T(G,\zeta)_{\mathbb{C}}$. To this end, we substitute the inversion formula (6.5) for $f_G(\tau)$ (or rather its analogue for τ in the complex domain $T(G,\zeta)_{\mathbb{C}}$) into the formula above. We obtain

$$S_\nu(\dot{f}) \;=\; \sum_{\phi'\in T^{\mathcal{E}}(G,\zeta)_{\mathbb{C}}} S_\nu(\dot{f}^u,\phi')f^{\mathcal{E}}_{G,\mathrm{gr}}(\phi')\ , \tag{8.1}$$

where

$$S_\nu(\dot{f}^u,\phi') \;=\; \sum_{\tau\in T(G,\zeta)_{\mathbb{C}}} S_\nu(\dot{f}^u,\tau)\Delta(\tau,\phi')\ .$$

The main point is to show that the coefficient $S_\nu(\dot{f}^u,\phi')$ vanishes if ϕ' lies in the complement of $\Phi(G,\zeta)_{\mathbb{C}}$ in $T^{\mathcal{E}}(G,\zeta)_{\mathbb{C}}$. We first deal with the case that ϕ' lies in the subset $T^{\mathcal{E}}_{\mathrm{ell}}(G,\zeta)$ of $T^{\mathcal{E}}(G,\zeta)_{\mathbb{C}}$.

Lemma 8.2. *If ϕ' lies in the complement of $\Phi_2(G,\zeta)$ in $T^{\mathcal{E}}_{\mathrm{ell}}(G,\zeta)$, then $S_\nu(\dot{f}^u,\phi')=0$ for any ν.*

Proof. Choose $f\in\mathcal{H}_{\mathrm{cusp}}\big(G(F),\zeta\big)$ so that the function $f^{\mathcal{E}}_{G,\mathrm{gr}}$ on $T^{\mathcal{E}}_{\mathrm{ell}}(G,\zeta)$ is the characteristic function of the point ϕ'. Since f is cuspidal, f^G_{gr} equals f^G, and since f^G_{gr} is the restriction of $f^{\mathcal{E}}_{G,\mathrm{gr}}$ to $\Phi_2(G,\zeta)$, f^G vanishes. Therefore the function $\dot{f}=\dot{f}^u f$ also has the property that $\dot{f}^{\dot{G}}$ vanishes. Having established that S_ν is stable, we can conclude that $S_\nu(\dot{f})=0$. But $f^{\mathcal{E}}_{G,\mathrm{gr}}$ vanishes on the complement of $T^{\mathcal{E}}_{\mathrm{ell}}(G,\zeta)$, so from (8.1) we see that $S_\nu(\dot{f})$ equals $S_\nu(\dot{f}^u,\phi')$. The lemma follows. □

The domain of the integral (8.1) can be represented as a disjoint union

$$T^{\mathcal{E}}(G,\zeta)_{\mathbb{C}} \;=\; \Phi(G,\zeta)_{\mathbb{C}}\cup\big(T^{\mathcal{E}}_{\mathrm{ell}}(G,\zeta)-\Phi_2(G,\zeta)\big)\cup\big(T^{\mathcal{E}}_{\mathrm{par}}(G,\zeta)_{\mathbb{C}}-\Phi_{\mathrm{par}}(G,\zeta)_{\mathbb{C}}\big)\ .$$

The lemma implies that the integral vanishes on $\Phi^{\mathcal{E}}_{\mathrm{ell}}(G,\zeta)-\Phi_2(G,\zeta)$, so (8.1) can be taken as the relation which identifies the two expressions

$$S_\nu(\dot{f})-\sum_{\phi\in\Phi(G,\zeta)_{\mathbb{C}}} S_\nu(\dot{f}^u,\phi)f^G_{\mathrm{gr}}(\phi) \tag{8.2}$$

and

$$\sum_{\phi'} S_\nu(\dot{f}^u,\phi')f^{\mathcal{E}}_{G,\mathrm{gr}}(\phi')\ , \tag{8.3}$$

where ϕ' is summed over $T^{\mathcal{E}}_{\mathrm{par}}(G,\zeta)_{\mathbb{C}}-\Phi_{\mathrm{par}}(G,\zeta)_{\mathbb{C}}$. We must prove that the coefficients $S_\nu(\dot{f}^u,\phi')$ in (8.3) also vanish.

Let $\Omega_{\mathbb{C}}$ be a connected component in the complement of $\Phi_{\mathrm{par}}(G,\zeta)_{\mathbb{C}}$ in $T^{\mathcal{E}}_{\mathrm{par}}(G,\zeta)_{\mathbb{C}}$. Set

$$\Omega \;=\; \Omega_{\mathbb{C}} \cap T^{\mathcal{E}}(G,\zeta)\ .$$

We can identify Ω with a set of orbits $\Omega_M/W(\Omega)$, where M is a proper Levi subgroup of G, Ω_M is a connected component in $T^{\mathcal{E}}_{\mathrm{ell}}(M,\zeta)$, and $W(\Omega)$ is the stabilizer of Ω_M in $W(M)$. If ϕ' is any point in Ω_M, the compact torus

$$i(\mathfrak{a}^G_{M,\phi'})^* \;=\; i(\mathfrak{a}^G_M)^*/i(\mathfrak{a}^G_M)^* \cap \mathfrak{a}^\vee_{M,\phi'}$$

acts simply transitively on Ω_M. Let $\mathcal{I}^{\mathcal{E}}(\Omega)$ be the space of functions on Ω which pull back to finite Fourier series on this torus. Then $\mathcal{I}^{\mathcal{E}}(\Omega)$ can be identified with the closed subspace of functions in $\mathcal{I}^{\mathcal{E}}_{\mathrm{gr}}(G,\zeta)$ which are supported on Ω. For any $\omega \in \mathcal{I}^{\mathcal{E}}(\Omega)$ there is a function $f_\omega \in \mathcal{H}\big(G(F),\zeta\big)$ such that

$$f^{\mathcal{E}}_{\omega,G,\mathrm{gr}}(\phi') \;=\; \begin{cases} \omega(\phi'), & \text{if } \phi' \in \Omega, \\ 0, & \text{otherwise,} \end{cases}$$

for any $\phi' \in T^{\mathcal{E}}(G,\zeta)$. If we take $\dot{f}$ to be the function $\dot{f}_\omega = \dot{f}^u f_\omega$, the expression (8.2) reduces to $S_\nu(\dot{f}_\omega)$ while (8.3) reduces to a sum over $\Omega_{\mathbb{C}}$. The identity of the two becomes

$$S_\nu(\dot{f}_\omega) \;=\; \sum_{\phi' \in \Omega_{\mathbb{C}}} S_\nu(\dot{f}^u,\phi')\omega(\phi')\ . \tag{8.4}$$

Lemma 8.3. *Suppose that S is a stable, ζ-equivariant distribution on $G(F)$. Then there is a smooth function F on Ω such that*

$$S(f_\omega) \;=\; \int_\Omega F(\phi')\omega(\phi')d\phi'\ ,$$

for any $\omega \in \mathcal{I}^{\mathcal{E}}(\Omega)$.

Proof. As a linear form on $\mathcal{H}\big(G(F),\zeta\big)$, S lies in the closed linear span of the stable orbital integrals

$$f \longrightarrow f^G(\sigma)\ , \qquad \sigma \in \Sigma(G).$$

Suppose that σ does not belong to $\Sigma_{\mathrm{ell}}(G)$. Then σ lies in $\Sigma_{G,\mathrm{ell}}(L)/W(L)$, for a proper Levi subgroup L of G. Applying our induction hypothesis that Theorem 6.1 holds for L, we obtain

$$f_\omega^G(\sigma) \ = \ f_\omega^L(\sigma) \ = \ f_{\omega,\mathrm{gr}}^L(\sigma) \ .$$

The right hand side depends only on the restriction of the function $f_{\omega,G,\mathrm{gr}}^{\mathcal{E}}$ to the subset $\Phi_2(L,\zeta)/W(L)$ of $\Phi(G,\zeta)$. Since the component Ω lies in the complement of $\Phi(G,\zeta)$ in $T^{\mathcal{E}}(G,\zeta)$, this restriction must vanish. Therefore $f_\omega^G(\sigma) = 0$.

We can therefore take σ to be in $\Sigma_{\mathrm{ell}}(G)$. Following the definition, we write

$$f_\omega^G(\sigma) \ = \ \sum_{\gamma \in \Gamma_{\mathrm{ell}}(G)} \Delta(\sigma,\gamma) f_{\omega,G}(\gamma) \ .$$

The dependence on ω is of course through the terms $f_{\omega,G}(\gamma)$, which are determined by the function

$$\begin{aligned} f_{\omega,G}(\tau) \ &= \ \sum_{\phi' \in T^{\mathcal{E}}(G,\zeta)} \Delta(\tau,\phi') f_{\omega,G,\mathrm{gr}}^{\mathcal{E}}(\phi') \\ &= \ \sum_{\phi' \in \Omega} \Delta(\tau,\phi')\omega(\phi') \end{aligned}$$

on $\widetilde{T}(G,\zeta)$. In fact, for any $f \in \mathcal{H}\big(G(F),\zeta\big)$ we can write

$$f_G(\gamma) \ = \ \int_{T(G,\zeta)} I(\gamma,\tau) f_G(\tau) d\tau \ ,$$

for the smooth function $I(\gamma,\tau)$ on $\Gamma(G) \times \widetilde{T}(G,\zeta)$ described in Section 6. If we combine these formulas, and use a variant of Lemma 5.3 to change variables in the integral-sum, we see that

$$f_\omega^G(\sigma) \ = \ \int_\Omega F(\sigma,\phi')\omega(\phi') d\phi' \ ,$$

where

$$F(\sigma,\phi') \ = \ \sum_{\gamma \in \Gamma_{\mathrm{ell}}(G)} \sum_{\tau \in T(G,\zeta)} \Delta(\sigma,\gamma) I(\gamma,\tau) \Delta(\tau,\phi) \ .$$

Since $\Sigma_{\text{ell}}(G)$ is compact modulo $Z(F)$, we can apply the Howe conjecture [7]. We deduce that the linear forms

$$\omega \longrightarrow f^G_\omega(\sigma) \ , \qquad \sigma \in \Sigma_{\text{ell}}(G),$$

span a finite dimensional space. The lemma follows, with $F(\phi')$ being a finite linear combination of the functions $\phi' \to F(\sigma, \phi')$. □

Lemma 8.4. *If ϕ' is any point in $\Omega_{\mathbb{C}}$, then $S_\nu(\dot{f}^u, \phi') = 0$ for any ν.*

Proof. According to Lemma 8.1, S_ν is a stable distribution on $\dot{G}(\mathbb{A})$. Therefore

$$f \longrightarrow S_\nu(\dot{f}^u f) \ , \qquad f \in \mathcal{H}\big(G(F), \zeta\big),$$

is a stable ζ-equivariant distribution on $G(F)$. We can therefore apply Lemma 8.3 to the left hand side of the identity (8.4). The identity becomes

$$\int_\Omega F_\nu(\dot{f}^u, \phi')\omega(\phi')d\phi' \ = \ \sum_{\phi' \in \Omega_{\mathbb{C}}} S_\nu(\dot{f}^u, \phi')\omega(\phi') \ , \tag{8.5}$$

for a smooth function $\phi' \to F_\nu(\dot{f}^u, \phi')$ on Ω.

The identity (8.5) is valid if ω is any function in the Paley-Wiener space on Ω_M which is symmetric under $W(\Omega)$. As a function of $\phi' \in \Omega_{M,\mathbb{C}}$, $S_\nu(\dot{f}^u, \phi')$ is certainly symmetric under $W(\Omega)$. The same is true of $F_\nu(\dot{f}^u, \phi')$, as a function of $\phi' \in \Omega_M$. It follows that (8.5) actually holds for any ω in the Paley-Wiener space on Ω_M. We shall show that both $S_\nu(\dot{f}^u, \phi')$ and $F_\nu(\dot{f}^u, \phi')$ vanish.

Fix $\phi' \in \Omega_M$. Then Ω_M can be identified with the compact torus $i(\mathfrak{a}^G_{M,\phi'})^*$ under the map $\lambda \to \phi'_\lambda$. We can take the function ω in (8.5) to be any finite Fourier series on $i(\mathfrak{a}^G_{M,\phi'})^*$. The Fourier transform of ω can then be identified with an arbitrary function of finite support on the dual lattice

$$\Lambda_{\phi'} \ = \ \mathfrak{a}_{M,\phi'}/(\mathfrak{a}_{M,\phi'} \cap \mathfrak{a}_G) \ .$$

Consider the Fourier transform of each side of (8.5) as a distribution on $\Lambda_{\phi'}$. The function

$$\lambda \longrightarrow S_\nu(\dot{f}^u, \phi'_\lambda)$$

is defined on the complexification of the torus $i(\mathfrak{a}^G_{M,\phi'})^*$, but is supported at finitely many points. Its Fourier transform is therefore a finite linear combination of complex exponential functions on the lattice $\Lambda_{\phi'}$. The Fourier transform of

$$\lambda \longrightarrow F(\dot{f}^u, \phi'_\lambda) ,$$

on the other hand, is a rapidly decreasing function on $\Lambda_{\phi'}$. The two can be equal only if they both vanish. Thus

$$F_\nu(\dot{f}^u, \phi') \;=\; S_\nu(\dot{f}^u, \phi') \;=\; 0 , \qquad \phi' \in \Omega_M,$$

and therefore $S_\nu(\dot{f}^u, \phi')$ vanishes for any $\phi' \in \Omega_{\mathbb{C}}$. □

Since $\Omega_{\mathbb{C}}$ is an arbitrary connected component in the domain of summation of (8.3), the summands in (8.3) are all equal to 0. Thus, the expression (8.3) vanishes, and so therefore does (8.2). We obtain

$$S_\nu(\dot{f}) \;=\; \sum_{\phi \in \Phi(G,\zeta)_{\mathbb{C}}} S_\nu(\dot{f}^u, \phi) f^G_{\mathrm{gr}}(\phi) , \tag{8.6}$$

for any ν and for $\dot{f} = \dot{f}^u f$ as above. We are going to use the global identity (8.6) to establish the local result that $f \to f^G_{\mathrm{gr}}(\phi)$ is stable.

Fix a function $f \in \mathcal{H}\big(G(F), \zeta\big)$ such that $f^G(\sigma) = 0$ for every $\sigma \in \Sigma(G)$. We want to show $f^G_{\mathrm{gr}}(\phi) = 0$ for any $\phi \in \Phi(G, \zeta)$. If ϕ belongs to the complement of $\Phi_2(G, \zeta)$ in $\Phi(G, \zeta)$, the result follows from our induction assumption, as we saw at the end of Section 6. In other words, the function $\phi \to f^G_{\mathrm{gr}}(\phi)$ on $\Phi(G, \zeta)$ is supported on $\Phi_2(G, \zeta)$. It has finite support on this set, so we can find a function $h \in \mathcal{H}_{\mathrm{cusp}}\big(G(F), \zeta\big)$ such that

$$h^G(\phi) \;=\; h^G_{\mathrm{gr}}(\phi) = f^G_{\mathrm{gr}}(\phi)$$

for every $\phi \in \Phi(G, \zeta)$.

We have a fixed function $\dot{f}^u \in \mathcal{H}_{\text{simp}}\bigl(\dot{G}(\mathbb{A}^u), \dot{\zeta}^u\bigr)$. The function f we have chosen determines the global function $\dot{f} = \dot{f}^u f$ in $\mathcal{H}_{\text{simp}}\bigl(\dot{G}(\mathbb{A}), \dot{\zeta}\bigr)$. The stable distribution S_ν must vanish on $\dot{f}$, and the identity (8.6) reduces to

$$\sum_{\phi \in \Phi_2(G,\zeta)_{\mathbb{C}}} S_\nu(\dot{f}^u, \phi) f_{\text{gr}}^G(\phi) \;=\; 0 \; .$$

From h we obtain a second global function

$$\dot{f}_h(\dot{x}) \;=\; \dot{f}^u(\dot{x}^u) h(x) \; , \qquad \dot{x} = \dot{x}^u x \in \dot{G}(\mathbb{A}),$$

in $\mathcal{H}_{\text{simp}}\bigl(\dot{G}(\mathbb{A}), \dot{\zeta}\bigr)$. Applying (8.6) to $\dot{f}_h$ and summing over ν, we see that

$$\begin{aligned} S(\dot{f}_h) \;&=\; \sum_\nu S_\nu(\dot{f}_h) \\ &=\; \sum_\nu \sum_{\phi \in \Phi(G,\zeta)_{\mathbb{C}}} S_\nu(\dot{f}^u, \phi) h_{\text{gr}}^G(\sigma) \\ &=\; \sum_\nu \sum_\phi S_\nu(\dot{f}^u, \phi) f_{\text{gr}}^G(\phi) \;=\; 0 \; . \end{aligned}$$

We then obtain

$$\sum_{\dot{\sigma} \in \Sigma_{\text{ell}}(\dot{G}/\dot{Z})} b^{\dot{G}}(\dot{\sigma}) (\dot{f}^u)^{\dot{G}}(\dot{\sigma}^u) h^G(\dot{\sigma}_u) \;=\; 0 \; , \tag{8.7}$$

from the original geometric expansion of $S(\dot{f}_h)$. This sum can be taken over a finite set that depends only on the support of $\dot{f}_h$. We shall use the identity (8.7) to show that h^G vanishes at any element $\sigma \in \Sigma_{\text{ell}}(G)$.

At this point we must impose another constraint on $(\dot{G}, \dot{F})$. We assume that $\dot{G}$ splits over some finite Galois extension $\dot{E}$ of $\dot{F}$ such that $\text{Gal}(\dot{E}/\dot{F})$ is equal to a decomposition group $\text{Gal}(\dot{E}_{w_1}/\dot{F}_{u_1})$, for a valuation $u_1 \neq u$ of $\dot{F}$ and a valuation w_1 of $\dot{E}$ over u_1. We can always modify our global data to make this additional condition hold. (See the discussion of this stage of the proof of Theorem 6.2 near the end of Section 9.) The condition implies that $\dot{E}_{u_1} = \dot{E} \otimes \dot{F}_{u_1}$ is a field. It follows from [17, Lemma 1(b)] that $\dot{G}(\dot{F})$ is dense in G(F). We can therefore approximate any $\sigma \in \Sigma_{\mathrm{ell}}(G)$ by $\dot{F}$-rational stable conjugacy classes $\dot{\sigma} \in \Sigma_{\mathrm{ell}}(\dot{G})$. Since h^G is continuous, we have only to show that $h^G(\dot{\sigma})$ vanishes for any $\dot{\sigma} \in \Sigma_{\mathrm{ell}}(\dot{G})$.

Fix $\dot{\sigma} \in \Sigma_{\mathrm{ell}}(\dot{G})$. We would like to choose the function $\dot{f}^u = \prod_{v \neq u} \dot{f}_v$ so that the summand corresponding to $\dot{\sigma}$ is the only nonvanishing term in (8.7). We are dealing with stable orbital integrals here, so it is enough to shrink the function at one place. We choose a finite place $v \neq u$, and vary f_v so that the function $\dot{f}_v^{\dot{G}}$ on $\Sigma_{\mathrm{ell}}(\dot{G})$ approaches the $\dot{\zeta}_v^{-1}$-equivariant Dirac measure on $\dot{\sigma}_v \dot{Z}(\dot{F}_v)$. There will then be at most one nonvanishing summand in (8.7). If we assume also that $(\dot{f}^u)(\dot{\gamma}) \neq 0$, as we may, the formula (8.7) leads to the conclusion that $h^G(\dot{\sigma}) = 0$.

We have established that the cuspidal function h^G vanishes on the dense subset $\Sigma_{\mathrm{ell}}(\dot{G})$ of its domain $\Sigma_{\mathrm{ell}}(G)$. Therefore h^G vanishes identically. It follows that

$$f_{\mathrm{gr}}^G(\phi) \; = \; h_{\mathrm{gr}}^G(\phi) \; = \; h^G(\phi) \; = \; 0 \, ,$$

for any cuspidal element $\phi \in \Phi_2(G, \zeta)$. This completes our proof of Theorem 6.1. □

We shall use the identity (8.6) again in our proof of Theorem 6.2. It was proved above under the local assumption that $A_{\dot{G},u} = A_G$, but this restriction, which was only for simplicity anyway, is no longer needed. Having established Theorem 6.1 for G, we can identify the spaces $\mathcal{I}(G, \zeta)$ and $\mathcal{I}_{\mathrm{gr}}(G, \zeta)$. The expansion (8.6) then follows easily from (8.1) and the fact that S_ν is stable. At the same time, we also see that the coefficients $S_\nu(\dot{f}^u, \phi)$ are stable linear forms in $\dot{f}^u \in \mathcal{H}_{\mathrm{simp}}\big(\dot{G}(\mathbb{A}^u), \dot{\zeta}^u\big)$.

9. Proof of the character relations

In this section we shall establish Theorem 6.2. We therefore take G to be a general group over F. We have established Theorem 6.1, and can apply it to the various quasi-split groups $\widetilde{G}'$, with $G' \in \mathcal{E}_{\mathrm{ell}}(G)$. As we agreed in Section 6, we shall identify the two spaces

$S\mathcal{I}(\widetilde{G}', \widetilde{\zeta}'\zeta)$ and $S\mathcal{I}_{\mathrm{gr}}(\widetilde{G}', \widetilde{\zeta}'\zeta)$. The problem is then to show that the corresponding maps f' and f'_{gr} are the same. Alternatively, we could treat all G' together, using Theorem 6.1 to identify the two spaces $\mathcal{I}^{\mathcal{E}}(G,\zeta)$ and $\mathcal{I}^{\mathcal{E}}_{\mathrm{gr}}(G,\zeta)$. In this setting, the problem is to show that the maps $f^{\mathcal{E}}_G$ and $f^{\mathcal{E}}_{G,\mathrm{gr}}$ are the same. The two problems are of course equivalent, since for any G', f' and f'_{gr} are the restrictions of the functions $f^{\mathcal{E}}_G$ and $f^{\mathcal{E}}_{G,\mathrm{gr}}$ to the closed subsets $\Sigma_G(\widetilde{G}')$ and $\Phi(\widetilde{G}', G, \zeta)$ of $\widetilde{\Gamma}^{\mathcal{E}}(G)$ and $T^{\mathcal{E}}(G,\zeta)$ respectively. We shall freely use notation from both viewpoints during the course of the proof.

It will be convenient to establish a dual assertion. Define

$$f^{\mathrm{gr}}_G(\tau) \;=\; \sum_{\phi' \in T^{\mathcal{E}}(G,\zeta)} \Delta(\tau,\phi') f^{\mathcal{E}}_G(\phi') \;,$$

for any $f \in \mathcal{H}\big(G(F),\zeta\big)$ and $\tau \in \widetilde{T}(G,\zeta)$. The inversion formula (5.10) then gives an identity

$$f^{\mathcal{E}}_G(\phi') \;=\; \sum_{\tau \in T(G,\zeta)} \Delta(\phi',\tau) f^{\mathrm{gr}}_G(\tau) \;,$$

for any $\phi' \in T^{\mathcal{E}}(G,\zeta)$, which can be compared with the definition

$$f^{\mathcal{E}}_{G,\mathrm{gr}}(\phi') \;=\; \sum_{\tau \in T(G,\zeta)} \Delta(\phi',\tau) f_G(\tau) \;.$$

To establish the required equality of $f^{\mathcal{E}}_G(\phi')$ and $f^{\mathcal{E}}_{G,\mathrm{gr}}(\phi')$, it is necessary and sufficient to prove that $f^{\mathrm{gr}}_G(\tau)$ equals $f_G(\tau)$. Again we shall derive this local result from the global trace formula. We shall describe the argument in steps which are parallel to those of Section 8.

We fix global objects $\dot{G}$, $\dot{F}$, $\dot{Z}$ and $\dot{\zeta}$ as in Section 7. The only local conditions we impose for the moment are that $(\dot{G}_u, \dot{F}_u) = (G,F)$ as always, and that the F-split tori $A_{\dot{G},u}$ and A_G are the same. The original distribution I on $\mathcal{H}_{\mathrm{simp}}\big(\dot{G}(\mathbb{A}),\dot{\zeta}\big)$ has an expansion

$$I(\dot{f}) \;=\; \sum_{\nu \in \mathfrak{h}^*_u} I_\nu(\dot{f})$$

indexed by Archimedean infinitesimal characters. It also has a parallel expansion

$$I(\dot{f}) = \sum_{\nu \in \mathfrak{h}_u^*} I_\nu^{\mathcal{E}}(\dot{f})$$

determined by (7.8) and (7.10), in which

$$I_\nu^{\mathcal{E}}(\dot{f}) = \sum_{\dot{G}' \in \mathcal{E}_{\mathrm{ell}}(\dot{G})} \iota(\dot{G}, \dot{G}') \widehat{S}'_{\nu'}(\dot{f}')$$

is defined in terms of endoscopic data.

Lemma 9.1. *For each ν, the linear forms $I_\nu(\dot{f})$ and $I_\nu^{\mathcal{E}}(\dot{f})$ on $\mathcal{H}_{\mathrm{simp}}\big(\dot{G}(\mathbb{A}), \dot{\zeta}\big)$ are equal.*

Proof. Suppose that $\alpha \in C_c^\infty(\mathfrak{h})^{W_\infty}$ is a multiplier. Then

$$I_\nu(\dot{f}_\alpha) = \widehat{\alpha}(\nu) I_\nu(\dot{f}) .$$

We also have

$$\begin{aligned} I_\nu^{\mathcal{E}}(\dot{f}_\alpha) &= \sum_{\dot{G}'} \iota(\dot{G}, \dot{G}') \widehat{S}'_{\nu'}(\dot{f}'_{\alpha'}) \\ &= \sum_{\dot{G}'} \iota(\dot{G}, \dot{G}') \widehat{\alpha}'(\nu') \widehat{S}'_{\nu'}(\dot{f}') \\ &= \widehat{\alpha}(\nu) I_\nu^{\mathcal{E}}(\dot{f}) , \end{aligned}$$

by (7.9). On the other hand,

$$\sum_\nu \big(I_\nu^{\mathcal{E}}(\dot{f}_\alpha) - I_\nu(\dot{f}_\alpha) \big) = I(\dot{f}_\alpha) - I(\dot{f}_\alpha) = 0 .$$

Moreover, the estimate of Lemma 7.1 holds if $I_\nu(\dot{f}_\alpha)$ is replaced by $I_\nu^{\mathcal{E}}(\dot{f}_\alpha)$, since by Corollary 7.2 the same estimate holds for the terms $\widehat{S}'_{\nu'}(f'_{\alpha'})$ indexed by $\dot{G}'$.

Fix $\nu_1 \in \mathfrak{h}_u^*/W_\infty$, $\alpha_1 \in C_{N_1}^\infty(\mathfrak{h})^{W_\infty}$ and T as in the proof of Lemma 8.1. Then

$$I_{\nu_1}^{\mathcal{E}}(\dot{f}) - I_{\nu_1}(\dot{f})$$

equals

$$I_{\nu_1}^{\mathcal{E}}(\dot{f}_{\alpha_m}) - I_{\nu_1}(\dot{f}_{\alpha_m})$$

for any m, and this is bounded in absolute value by the sum of

$$\sum_{\{\nu:\ \|\mathrm{Im}(\nu)\|\leq T,\ \nu\neq\nu_1\}} |I_\nu^{\mathcal{E}}(\dot{f}_{\alpha_m}) - I_\nu(\dot{f}_{\alpha_m})|$$

and

$$\sum_{\|\mathrm{Im}(\nu)\|>T} |I_\nu^{\mathcal{E}}(\dot{f}_{\alpha_m}) - I_\nu(\dot{f}_{\alpha_m})| \ .$$

As in the proof of Lemma 8.1, each of these two sums approaches 0 as m approaches infinity. The lemma follows. □

As before, we take $\dot{f} = \dot{f}^u f$, for a fixed function $\dot{f}^u \in \mathcal{H}_{\mathrm{simp}}\big(\dot{G}(\mathbb{A}^u), \dot{\zeta}^u\big)$ and a general function $f \in \mathcal{H}\big(G(F), \zeta\big)$. The distribution $I_\nu(\dot{f})$ can be expressed according to (7.4) as a sum over $T(G,\zeta)_{\mathbb{C}}$. To obtain a similar expression for $I_\nu^{\mathcal{E}}(\dot{f})$, we have first to substitute the formula

$$\widehat{S}'_{\nu'}(\dot{f}') \ = \sum_{\tilde{\phi}'\in\Phi(\tilde{G}',\tilde{\zeta}'\zeta)_{\mathbb{C}}} \widehat{S}'_{\nu'}\big((\dot{f}^u)', \widetilde{\phi}'\big) f'(\widetilde{\phi}')$$

obtained by applying (8.6) to $\dot{\widetilde{G}'}$ and $\dot{f}'$, into the definition above for $I_\nu^{\mathcal{E}}(\dot{f})$. The coefficients of $f'(\widetilde{\phi}')$ here come from stable linear forms on $\mathcal{H}_{\mathrm{simp}}\big(\dot{\widetilde{G}'}(\mathbb{A}^u), (\dot{\widetilde{\zeta}}'\dot{\zeta})^u\big)$, and are therefore well defined functions of $(\dot{f}^u)'$. (See the remarks at the end of Section 8.) We obtain a double sum over $\dot{G}' \in \mathcal{E}_{\mathrm{ell}}(\dot{G})$ and $\widetilde{\phi}' \in \Phi(\dot{\widetilde{G}}'_u, \widetilde{\zeta}'\zeta)_{\mathbb{C}}$. The linear form $f_G^{\mathcal{E}}(\phi') = f'(\widetilde{\phi}')$ depends only on the image ϕ' of $\widetilde{\phi}'$ in $T^{\mathcal{E}}(G,\zeta)_{\mathbb{C}}$, and is independent of $\dot{\widetilde{G}}'_u$. We can therefore write

$$I_\nu^{\mathcal{E}}(\dot{f}) \ = \sum_{\phi'\in T^{\mathcal{E}}(G,\zeta)_{\mathbb{C}}} I_\nu^{\mathcal{E}}(\dot{f}^u, \phi') f_G^{\mathcal{E}}(\phi') \ ,$$

where

$$I_\nu^{\mathcal{E}}(\dot{f}^u, \phi') \ = \sum_{\dot{G}'\in\mathcal{E}_{\mathrm{ell}}(\dot{G})} \sum_{\tilde{\phi}'} \iota(\dot{G}, \dot{G}')\widehat{S}'_{\nu'}\big((\dot{f}^u)', \widetilde{\phi}'\big) \ ,$$

with $\widetilde{\phi}'$ being summed over the preimage of ϕ' in $\Phi(\dot{\widetilde{G}}'_u, \widetilde{\zeta}'\zeta)_{\mathbb{C}}$. Applying the inversion formula for $f_G^{\mathrm{gr}}(\tau)$ above, we see that

$$I_\nu^{\mathcal{E}}(\dot{f}) \ = \sum_{\tau\in T(G,\zeta)_{\mathbb{C}}} I_\nu^{\mathcal{E}}(\dot{f}^u, \tau) f_G^{\mathrm{gr}}(\tau) \ ,$$

where

$$I_\nu^{\mathcal{E}}(\dot{f}^u,\tau) \;=\; \sum_{\phi'\in T^{\mathcal{E}}(G,\zeta)_{\mathbb{C}}} I_\nu^{\mathcal{E}}(\dot{f}^u,\phi')\Delta(\phi',\tau)\ .$$

But $I_\nu(\dot{f})$ equals $I_\nu^{\mathcal{E}}(\dot{f})$, by Lemma 9.1. Identifying the corresponding two expansions, we obtain an identity

$$\sum_{\tau\in T(G,\zeta)_{\mathbb{C}}} \big(I_\nu^{\mathcal{E}}(\dot{f}^u,\tau)f_G^{\mathrm{gr}}(\tau) - I_\nu(\dot{f}^u,\tau)f_G(\tau)\big) \;=\; 0\ . \tag{9.1}$$

Lemma 9.2. *If τ lies in $\widetilde{T}_{\mathrm{ell}}(G,\zeta)$, we have*

$$I_\nu^{\mathcal{E}}(\dot{f}^u,\tau) \;=\; I_\nu(\dot{f}^u,\tau)$$

for any ν.

Proof. Choose $f\in\mathcal{H}_{\mathrm{cusp}}\big(G(F),\zeta\big)$ so that the function f_G on $\widetilde{T}(G,\zeta)$ vanishes on the complement of τZ_τ, and is nonzero at τ. Since f is cuspidal, the functions $f_G^{\mathcal{E}}$ and $f_{G,\mathrm{gr}}^{\mathcal{E}}$ in $\mathcal{I}_{\mathrm{cusp}}^{\mathcal{E}}(G,\zeta)$ are equal. Therefore the functions f_G^{gr} and f_G in $\mathcal{I}_{\mathrm{cusp}}(G,\zeta)$ are also equal. The lemma follows without difficulty from (9.1). □

Lemma 9.2 simplifies the part of the sum in (9.1) that is over the subset $T_{\mathrm{ell}}(G,\zeta)$ of $T(G,\zeta)_{\mathbb{C}}$. Suppose that τ lies in the complement $\widetilde{T}_{\mathrm{par}}(G,\zeta)_{\mathbb{C}}$ of $\widetilde{T}_{\mathrm{ell}}(G,\zeta)$ in $\widetilde{T}(G,\zeta)_{\mathbb{C}}$. Then $f_G^{\mathrm{gr}}(\tau)$ equals $f_G(\tau)$, since our induction hypothesis implies that $f'(\phi')=f'_{\mathrm{gr}}(\phi')$ for every ϕ' in the complement of $T_{\mathrm{ell}}^{\mathcal{E}}(G,\zeta)$ in $T^{\mathcal{E}}(G,\zeta)_{\mathbb{C}}$. The formula (9.1) can therefore be taken as the relation which identifies the two expressions

$$\sum_{\tau\in T_{\mathrm{ell}}(G,\zeta)} I_\nu(\dot{f}^u,\tau)\big(f_G^{\mathrm{gr}}(\tau)-f_G(\tau)\big) \tag{9.2}$$

and

$$\sum_{\tau\in T_{\mathrm{par}}(G,\zeta)_{\mathbb{C}}} \big(I_\nu(\dot{f}^u,\tau)-I_\nu^{\mathcal{E}}(\dot{f}^u,\tau)\big)f_G(\tau)\ . \tag{9.3}$$

We must show that the identity of Lemma 9.2 also holds for the coefficients in (9.3).

Let $\Omega_{\mathbb{C}}$ be a connected component in $T_{\mathrm{par}}(G,\zeta)_{\mathbb{C}}$, and set $\Omega = \Omega_{\mathbb{C}} \cap T_{\mathrm{par}}(G,\zeta)$. Then $\Omega = \Omega_M/W(\Omega)$, where M is a proper Levi subgroup of G, Ω_M is a connected component of $T_{\mathrm{ell}}(M,\zeta)$, and $W(\Omega)$ is the stabilizer of Ω_M in $W(M)$. If τ is any point in Ω_M, the compact torus

$$i(\mathfrak{a}_{M,\tau}^G)^* \;=\; i(\mathfrak{a}_M^G)^*/i(\mathfrak{a}_M^G)^* \cap \mathfrak{a}_{M,\tau}^\vee$$

acts simply transitively on Ω_M. In the present context, we have to take the preimage $\widetilde{\Omega}$ of Ω in $\widetilde{T}(G,\zeta)$. Let $\mathcal{I}(\widetilde{\Omega})$ be the space of functions on $\widetilde{\Omega}$ which transform under the action of Z_τ by the character χ_τ^{-1}, and which pull back to finite Fourier series on $i(\mathfrak{a}_{M,\tau}^G)^*$. Then $\mathcal{I}(\widetilde{\Omega})$ can be identified with the closed subspace of functions in $\mathcal{I}(G,\zeta)$ which are supported on $\widetilde{\Omega}$. For any $\omega \in \mathcal{I}(\widetilde{\Omega})$, there is a function $f_\omega \in \mathcal{H}\big(G(F),\zeta\big)$ such that

$$f_{\omega,G}(\tau) \;=\; \begin{cases} \omega(\tau), & \text{if } \tau \in \widetilde{\Omega}, \\ 0, & \text{otherwise}, \end{cases}$$

for any $\tau \in \widetilde{T}(G,\zeta)$. If we take $\dot{f}$ to be the function $\dot{f}_\omega = \dot{f}^u f_\omega$, the expression (9.3) is just

$$\sum_{\tau \in \Omega_{\mathbb{C}}} \big(I_\nu(\dot{f}^u,\tau) - I_\nu^{\mathcal{E}}(\dot{f}^u,\tau)\big)\omega(\tau) \; . \tag{9.4}$$

To control (9.3) we need the following analogue of Lemma 8.3.

Lemma 9.3. *Suppose that I is an invariant, ζ-equivariant distribution on $G(F)$. Then there is a smooth function F on $\widetilde{\Omega}$, which transforms under Z_τ according to χ_τ, such that*

$$\widehat{I}(f_{\omega,G}^{\mathrm{gr}} - f_{\omega,G}) \;=\; \int_\Omega F(\tau)\omega(\tau)d\tau \; ,$$

for any $\omega \in \mathcal{I}(\widetilde{\Omega})$.

Proof. By definition, $f_G^{\mathrm{gr}} - f_G$ is the function

$$\tau \longrightarrow f_G^{\mathrm{gr}}(\tau) - f_G(\tau) \; , \qquad \tau \in \widetilde{T}(G,\zeta),$$

in $\mathcal{I}(G,\zeta)$. If τ belongs to a subset $\widetilde{T}_{\mathrm{ell}}(L,\zeta)/W(L)$ of $\widetilde{T}(G,\zeta)$, for a proper Levi subgroup L, our induction hypothesis implies that $f_G^{\mathrm{gr}}(\tau) - f_G(\tau) = 0$. It follows that $f_G^{\mathrm{gr}} - f_G$

belongs to $\mathcal{I}_{\mathrm{cusp}}(G,\zeta)$. Let J be the linear form on the associated space $\mathcal{I}^{\mathcal{E}}_{\mathrm{cusp}}(G,\zeta)$ given by

$$J\big(\mathcal{T}^{\mathcal{E}}(a_G)\big) \;=\; \widehat{I}(a_G)\ , \qquad a_G \in \mathcal{I}_{\mathrm{cusp}}(G,\zeta).$$

Recall that $\mathcal{T}^{\mathcal{E}}$ is an isomorphism from $\mathcal{I}_{\mathrm{cusp}}(G,\zeta)$ onto $\mathcal{I}^{\mathcal{E}}_{\mathrm{cusp}}(G,\zeta)$ which can be expressed in terms of either conjugacy classes (Section 3) or virtual characters (Section 5). The second description gives

$$\begin{aligned}\big(\mathcal{T}^{\mathcal{E}}(f^{\mathrm{gr}}_G - f_G)\big)(\phi') \;&=\; \sum_{\tau\in T_{\mathrm{ell}}(G,\zeta)} \Delta(\phi',\tau)\big(f^{\mathrm{gr}}_G(\tau) - f_G(\tau)\big)\\ &=\; f^{\mathcal{E}}_G(\phi') - f^{\mathcal{E}}_{G,\mathrm{gr}}(\phi')\ ,\end{aligned}$$

for any $\phi' \in T^{\mathcal{E}}_{\mathrm{ell}}(G,\zeta)$. Therefore $\mathcal{T}^{\mathcal{E}}(f^{\mathrm{gr}}_G - f_G)$ equals the difference of the two functions $f^{\mathcal{E}}_G$ and $f^{\mathcal{E}}_{G,\mathrm{gr}}$ in $\mathcal{I}^{\mathcal{E}}(G,\zeta)$. Now any continuous linear form on $\mathcal{I}^{\mathcal{E}}_{\mathrm{cusp}}(G,\zeta)$ is in the closed linear span of the evaluation maps

$$a^{\mathcal{E}}_G \longrightarrow a^{\mathcal{E}}_G(\sigma')\ , \qquad \sigma' \in \widetilde{\Gamma}^{\mathcal{E}}_{\mathrm{ell}}(G).$$

It follows easily from the Howe conjecture that the distribution

$$\omega \longrightarrow \widehat{I}(f^{\mathrm{gr}}_{\omega,G} - f_{\omega,G}) \;=\; J(f^{\mathcal{E}}_{\omega,G} - f^{\mathcal{E}}_{\omega,G,\mathrm{gr}})$$

is a finite linear combination of values

$$f^{\mathcal{E}}_{\omega,G}(\sigma') - f^{\mathcal{E}}_{\omega,G,\mathrm{gr}}(\sigma')\ , \qquad \sigma' \in \widetilde{\Gamma}^{\mathcal{E}}_{\mathrm{ell}}(G).$$

But we saw in the proof of Corollary 6.4 that

$$\begin{aligned}f^{\mathcal{E}}_{\omega,G}(\sigma') - f^{\mathcal{E}}_{\omega,G,\mathrm{gr}}(\sigma') \;&=\; \int_{T(G,\zeta)} F(\sigma',\tau) f_{\omega,G}(\tau)d\tau\\ &=\; \int_{\Omega} F(\sigma',\tau)\omega(\tau)d\tau\ ,\end{aligned}$$

for the smooth function $F(\sigma',\tau)$ given by (6.12). (Recall that this earlier argument depended only on the validity of Theorem 6.1.) The lemma follows, with $F(\tau)$ being a finite linear combination of the functions $\phi' \to F(\sigma,\phi')$. $\square$

Lemma 9.4. *If τ is any point in $\widetilde{\Omega}_{\mathbb{C}}$, the preimage of $\Omega_{\mathbb{C}}$ in $\widetilde{T}(G,\zeta)_{\mathbb{C}}$, we have*

$$I_{\nu}^{\mathcal{E}}(\dot{f}^{u},\tau) - I_{\nu}(\dot{f}^{u},\tau) \;=\; 0$$

for any ν.

Proof. By (7.4), the expression (9.2) is the value of the ζ-equivariant distribution

$$f \longrightarrow I_{\nu}(\dot{f}^{u} f)\ , \qquad f \in \mathcal{H}\big(G(F),\zeta\big),$$

(or rather its push-forward to $\mathcal{I}(G,\zeta)$) at the function $f_{G}^{\mathrm{gr}} - f_{G}$. We can apply the last lemma to this distribution. The value of (9.2) at $f = f_{\omega}$ takes the form

$$\int_{\Omega} F_{\nu}(\dot{f}^{u},\tau)\omega(\tau)d\tau\ , \tag{9.5}$$

for a smooth function $\tau \rightarrow F_{\nu}(\dot{f}^{u},\tau)$ on $\widetilde{\Omega}$ which transforms under Z_{τ} according to χ_{τ}. The expressions (9.4) and (9.5) are then equal for any function $\omega \in \mathcal{I}(\widetilde{\Omega})$.

The rest of the proof is similar to that of Lemma 8.4. The two functions $I_{\nu}^{\mathcal{E}}(\dot{f}^{u},\tau) - I_{\nu}(\dot{f}^{u},\tau)$ and $F_{\nu}(\dot{f}^{u},\tau)$ are both χ_{τ}-equivariant under translation by Z_{τ}. However, any choice of section $\Omega \rightarrow \widetilde{\Omega}$ allows us to regard the first one as a function of finite support on $\Omega_{M,\mathbb{C}}/W(\Omega)$, and the second one as a smooth function on $\Omega_{M}/W(\Omega)$. The identity of (9.4) with (9.5) then holds with ω being any function in the Paley-Wiener space on Ω_{M}. Fix $\tau \in \Omega$. This identifies Ω with the compact torus $i(\mathfrak{a}_{M,\tau}^{G})^{*}$, under the map $\lambda \rightarrow \tau_{\lambda}$. The Fourier transform of ω becomes an arbitrary function of finite support on the dual lattice

$$\Lambda_{\tau} \;=\; \mathfrak{a}_{M,\tau}/(\mathfrak{a}_{M,\tau} \cap \mathfrak{a}_{G})\ .$$

By taking Fourier transforms of (9.4) and (9.5), we obtain an identity of distributions on Λ_{τ}. As in the proof of Lemma 8.4, we deduce that

$$I_{\nu}^{\mathcal{E}}(\dot{f}^{u},\tau) - I_{\nu}(\dot{f}^{u},\tau) \;=\; 0\ ,$$

for any $\tau \in \widetilde{\Omega}_{\mathbb{C}}$. $\square$

Since $\Omega_{\mathbb{C}}$ is an arbitrary component in the domain of the sum (9.3), this sum has to vanish. Since (9.3) equals the expression (9.2), we obtain

$$\sum_{\tau \in T_{\mathrm{ell}}(G,\zeta)} I_\nu(\dot{f}^u, \tau)\big(f_G^{\mathrm{gr}}(\tau) - f_G(\tau)\big) \;=\; 0 \tag{9.6}$$

for any ν. We shall use this global identity to prove the local result that $f_G^{\mathrm{gr}}(\tau)$ equals $f_G(\tau)$.

Set

$$h_G(\tau) \;=\; f_G^{\mathrm{gr}}(\tau) - f_G(\tau)\ , \qquad \tau \in \widetilde{T}(G,\zeta).$$

According to our induction hypothesis, this function is supported on $\widetilde{T}_{\mathrm{ell}}(G,\zeta)$. Since f_G^{gr} and f_G both lie in $\mathcal{I}(G,\zeta)$, h_G belongs to $\mathcal{I}_{\mathrm{cusp}}(G,\zeta)$, and is the image of some function $h \in \mathcal{H}_{\mathrm{cusp}}\big(G(F),\zeta\big)$.

We are working with the function $\dot{f} = \dot{f}^u f$ in $\mathcal{H}_{\mathrm{simp}}\big(\dot{G}(\mathbb{A}),\dot{\zeta}\big)$ constructed from $\dot{f}^u \in \mathcal{H}_{\mathrm{simp}}\big(\dot{G}(\mathbb{A}^u),\dot{\zeta}^u\big)$. We can also form the second function $\dot{f}_h = \dot{f}^u h$ in $\mathcal{H}_{\mathrm{simp}}\big(\dot{G}(\mathbb{A}),\dot{\zeta}\big)$. Observe that

$$\begin{aligned} I(\dot{f}_h) \;&=\; \sum_\nu I_\nu(\dot{f}_h) \\ &=\; \sum_\nu \sum_{\tau \in T_{\mathrm{ell}}(G,\zeta)} I_\nu(\dot{f}^u,\tau) h_G(\tau) \;=\; 0\ , \end{aligned}$$

by (7.3), (7.4) and (9.6). It follows from (7.2) that

$$\sum_{\dot{\gamma} \in \Gamma_{\mathrm{ell}}(\dot{G}/\dot{Z})} a^{\dot{G}}(\dot{\gamma}) \dot{f}^u_{\dot{G}}(\dot{\gamma}^u) h_G(\dot{\gamma}_u) \;=\; 0\ . \tag{9.7}$$

The sum can be taken over a finite set which depends only on the support of $\dot{f}_h$. We can certainly assume that $\dot{f}^u$ splits into a product $\dot{f}_\infty \dot{f}^{\infty,u}$ of Archimedean and non-Archimedean components. The function $\dot{f}_\infty$ is supposed to lie in an Archimedean Hecke algebra $\mathcal{H}\big(\dot{G}(\dot{F}_\infty),\dot{\zeta}_\infty\big)$. However, $\mathcal{H}\big(\dot{G}(\dot{F}_\infty),\dot{\zeta}_\infty\big)$ is dense in $C_c^\infty\big(\dot{G}(\dot{F}_\infty),\dot{\zeta}_\infty\big)$, and as a linear form in $\dot{f}_\infty$, the left hand side of (9.7) extends continuously to $C_c^\infty\big(\dot{G}(\dot{F}_\infty),\dot{\zeta}_\infty\big)$.

The identity (9.7) therefore holds for any $\dot{f}_\infty \in C_c^\infty\bigl(\dot{G}(\dot{F}_\infty), \dot{\zeta}_\infty\bigr)$ and $\dot{f}^{\infty,u} \in \mathcal{H}_{\mathrm{simp}}\bigl(\dot{G}(\dot{F}^{\infty,u}), \dot{\zeta}^{\infty,u}\bigr)$. We shall use it to show that h_G vanishes at any element $\gamma \in \Gamma_{\mathrm{ell}}(G)$.

Fix $\gamma \in \Gamma_{\mathrm{ell}}(G)$. We shall also write γ for a representative of the conjugacy class in $G(F)$, so that $T = G_\gamma$ is a fixed, elliptic maximal torus in G over F. We must first impose some new conditions on the global objects $(\dot{G}, \dot{F})$.

We choose $(\dot{G}, \dot{F})$, together with an elliptic maximal torus $\dot{T}$ in $\dot{G}$ over $\dot{F}$, such that

$$(\dot{G}_{u_i}, \dot{T}_{u_i}, \dot{F}_{u_i}) \ \cong\ (G, T, F)\ , \qquad i = 0, 1, 2,$$

for three nonArchimedean places $u = u_0$, u_1 and u_2 of $\dot{F}$. To see that this is possible, let E be any finite Galois extension of F over which G and T split, and let $\dot{E}'$ be a global field such that $\dot{E}'_{w'} \cong E$ for some valuation w'. The local Galois group $\mathrm{Gal}(E/F)$ is a subgroup of the decomposition group at w' in $\mathrm{Gal}(\dot{E}'/\mathbb{Q})$. In other words, $\mathrm{Gal}(E/F)$ can be identified with a subgroup of $\mathrm{Gal}(\dot{E}'/\mathbb{Q})$ and therefore acts on $\dot{E}'$. Let $\dot{F}' \subset \dot{E}'$ be its fixed field, and let u' be any valuation on $\dot{F}'$ which w' divides. The completion $\dot{F}_{u'}$ is then isomorphic with F. The required global field $\dot{F}$ can be any finite Galois extension of $\dot{F}'$ of degree at least 3, in which u' splits completely, with $\{u_0, u_1, u_2\}$ being three valuations which divide u'. If $\dot{E}$ is the composite field $\dot{E}'\dot{F}$, then

$$\mathrm{Gal}(\dot{E}/\dot{F}) \ \cong\ \mathrm{Gal}(E/F) \ \cong\ \mathrm{Gal}(\dot{E}_{w_i}/\dot{F}_{u_i})\ , \qquad i = 0, 1, 2,$$

for valuations w_i of $\dot{E}$ which divide u_i. If follows easily that there is a pair $(\dot{G}, \dot{T})$ over $\dot{F}$, which splits over $\dot{E}$, and has the required properties. Our original local data also include a central torus $Z \supset A_G$ and a character ζ on $Z(F)$. The global construction is such that there is a central induced torus $\dot{Z} \supset A_{\dot{G}}$ over $\dot{F}$ and a character $\dot{\zeta}$ on $\dot{Z}(\dot{F})\backslash\dot{Z}(\mathbb{A})$, such that

$$(\dot{Z}_{u_i}, A_{\dot{G},u_i}, \dot{\zeta}_{u_i}) \ \cong\ (Z, A_G, \zeta)\ , \qquad i = 0, 1, 2.$$

Since $\dot{T}$ splits over the Galois extension $\dot{E}$ of $\dot{F}$, and since $\dot{E}_{u_1}$ is a field, for the valuation $u_1 \neq u$, $\dot{T}(\dot{F})$ is dense in $T(F)$ [17, Lemma 1(b)]. We can therefore approximate γ by $\dot{F}$-rational points $\dot{\gamma}$. Since h_G is continuous, we need only show that $h_G(\dot{\gamma}) = 0$ for any fixed class $\dot{\gamma} \in \Gamma_{\mathrm{ell}}(\dot{G})$ with a representative, which we shall also denote by $\dot{\gamma}$, in $\dot{T}(\dot{F})$.

Let V be a large finite set of valuations of $\dot{F}$ which contains the Archimedean places and the nonArchimedean valuations $\{u_i\}$. At the places $v \notin V$, we assume that $\dot{G}_v$, $\dot{T}_v$ and $\dot{\zeta}_v$ are unramified, and also that $|\alpha(\dot{\gamma}_v) - 1|_{\overline{v}} = 1$ for any root α of $(\dot{G}, \dot{T})$. ($\overline{v}$ here stands for the valuation over v in an algebraic closure of $\dot{F}_v$.) We choose the function

$$\dot{f}^u = \prod_{v \neq u} \dot{f}_v = \dot{f}_\infty \dot{f}^{\infty,u}$$

so that for each $v \in V$, $\dot{f}_{v,\dot{G}}$ approaches the $\dot{\zeta}_v^{-1}$-equivariant Dirac measure on $\dot{\gamma}_v \dot{Z}(\dot{F}_v)$. At the places $v \notin V$, we take $\dot{f}_v$ to be the unit in the appropriate unramified Hecke algebra. The functions $\dot{f}_{u_1}$ and $\dot{f}_{u_2}$ will be cuspidal, and $\dot{f}^{\infty,u}$ will belong to $\mathcal{H}_{\mathrm{simp}}\big(\dot{G}(\mathbb{A}^{\infty,u}), \dot{\zeta}^{\infty,u}\big)$. The identity (9.7) will therefore hold for $\dot{f}^u$. But the conditions on $\dot{f}^u$ imply that the left hand side of (9.7) is a nonzero multiple of $h_G(\dot{\gamma}_u)$. This is a consequence of [16, Proposition 7.1], and the fact that any class in $H^1(\dot{F}, \dot{T})$ which maps to 0 in $\bigoplus_{v \neq u} H^1(\dot{F}_v, \dot{T}_v)$ also maps to 0 in $H^1(\dot{F}_u, \dot{T}_u)$. (See [13, Appendix], [1, p. 528–529].) It follows that $h_G(\dot{\gamma}_u) = 0$.

We have established that h_G vanishes on a dense subset of $T(F)$. Therefore, $h_G(\gamma) = 0$ for the arbitrary element $\gamma \in \Gamma_{\mathrm{ell}}(G)$ which we fixed. Since h_G can be regarded as a function on either $\Gamma_{\mathrm{ell}}(G)$ or $\widetilde{T}_{\mathrm{ell}}(G, \zeta)$, we conclude that

$$f_G^{\mathrm{gr}}(\tau) - f_G(\tau) = h_G(\tau) = 0$$

for any element $\tau \in \widetilde{T}_{\mathrm{ell}}(G, \zeta)$. This is what we set out to prove. It implies that $f'_{\mathrm{gr}}(\phi')$ equals $f'(\phi')$ for any $G' \in \mathcal{E}_{\mathrm{ell}}(G)$ and any element $\phi' \in \Phi_2(\widetilde{G}', \widetilde{\zeta}'\zeta)$. This completes our proof of Theorem 6.2. $\square$

10. A problem on weighted characters

The virtual characters

$$f \longrightarrow f_G(\tau) \ = \ I_G(\tau, f) \ , \qquad \tau \in \widetilde{T}(G),$$

are included in a more general family of invariant distributions

$$f \longrightarrow I_M(\tau, f) \ , \qquad M \in \mathcal{L}, \ \tau \in \widetilde{T}(M),$$

which are attached to weighted characters on $G(F)$. Weighted characters (as well as weighted orbital integrals) arise in the general global trace formula. It is an important problem to understand how they behave under endoscopic transfer. We shall state a conjectural identity for these distributions in terms of the spectral transfer factors, which may be regarded as a generalization of Theorems 6.1 and 6.2.

Weighted characters are usually defined so that they depend on a choice of normalizing factors for intertwining operators. (See for example [4, Section 2].) To simplify the discussion, we shall not deal with the question of choosing compatible normalizing factors on different groups. Let us instead just consider weighted characters that are constructed from unnormalized intertwining operators. The resulting objects are then only defined on an open dense subset of the usual domain. We define

$$\mathcal{J}_M(\pi, P) \ = \ \lim_{\lambda \to 0} \sum_{Q \in \mathcal{P}(M)} J_{Q|P}(\pi)^{-1} J_{Q|P}(\pi_\lambda) \theta_Q(\lambda)^{-1} \ ,$$

in notation similar to that of [4, Section 2]. Here $M \in \mathcal{L}$ is a Levi subgroup, $P \in \mathcal{P}(M)$ is a fixed parabolic subgroup with Levi component M, $\pi \in \Pi_{\mathrm{temp}}\big(M(F)\big)$ is an irreducible tempered representation of $M(F)$, and

$$J_{Q|P}(\pi) : \ \mathcal{I}_P(\pi) \longrightarrow \mathcal{I}_Q(\pi) \ , \qquad Q \in \mathcal{P}(M),$$

are the unnormalized intertwining operators between corresponding induced representations. The operators are defined and analytic on an open dense subset of $\Pi_{\mathrm{temp}}\big(M(F)\big)$, and the limit exists at every π in this subset. The weighted character at π is given by the trace

$$J_M(\pi, f) \;=\; \mathrm{tr}\big(\mathcal{J}_M(\pi, P)\mathcal{I}_P(\pi, f)\big)\ , \qquad f \in \mathcal{H}\big(G(F)\big).$$

The weighted character at an element

$$\tau \;=\; (L, \pi, r)\ , \qquad L \subset M,\ \pi \in \Pi_2(L),\ r \in \widetilde{R}_\pi^M,$$

in $\widetilde{T}(M)$ is defined [4, (2.4)] to be

$$J_M(\tau, f) \;=\; \sum_{\rho \in \Pi(\widetilde{R}_\pi^M, \chi_\pi)} \mathrm{tr}\big(\rho^\vee(r)\big) J_M(\pi_\rho, f)\ , \qquad f \in \mathcal{H}\big(G(F)\big).$$

As with $J_M(\pi, f)$ above, and with other functions introduced below, $J_M(\tau, f)$ is to be regarded as an analytic function on an open dense subset of its domain.

As a distribution in f, $J_M(\tau, f)$ is not invariant, but it becomes invariant when it is modified by weighted orbital integrals. (See [4, Section 3].) The resulting linear form has to be defined, initially at least, on the subspace $C_c^\infty\big(G_{\mathrm{reg}}(F)\big)$ of $\mathcal{H}\big(G(F)\big)$. The image of this space under the map $f \to f_G$ is the subspace

$$I_c^\infty(G_{\mathrm{reg}}) \;=\; C_c^\infty\Big(\Gamma_{\mathrm{reg}}\big(G(F)\big)\Big)$$

of functions in $\mathcal{I}(G)$ of compact support on $\Gamma_{\mathrm{reg}}\big(G(F)\big)$. Similarly, the image of $C_c^\infty\big(G_{\mathrm{reg}}(F)\big)$ under the map $f \to f^G$ is the subspace

$$SI_c^\infty(G_{\mathrm{reg}}) \;=\; C_c^\infty\Big(\Sigma_{\mathrm{reg}}\big(G(F)\big)\Big)$$

of functions in $S\mathcal{I}(G)$ of compact support on $\Sigma_{\mathrm{reg}}\big(G(F)\big)$. The invariant distributions

$$I_M(\tau, f) \;=\; I_M^G(\tau, f)\ , \qquad \tau \in \widetilde{T}(M),\ f \in C_c^\infty\big(G_{\mathrm{reg}}(F)\big),$$

are defined [4, $(3.5^\vee)$] inductively by a formula

$$J_M(\tau, f) \;=\; \sum_{S \in \mathcal{L}(M)} \widehat{I}_M^S\big(\tau, \phi_S(f)\big)\ ,$$

where

$$\phi_S : C_c^\infty\big(G_{\mathrm{reg}}(F)\big) \longrightarrow I_c^\infty(G_{\mathrm{reg}})$$

is a map defined as a weighted orbital integral

$$\phi_S(f,\gamma) \;=\; J_S(\gamma,f) \;=\; |D(\gamma)|^{\frac{1}{2}} \int_{G_\gamma(F)\backslash G(F)} f(x^{-1}\gamma x)v_S(x)dx\ , \qquad \gamma\in\Gamma_G(S).$$

(See [4, Section 1].) We want an identity relating these invariant distributions with corresponding objects on endoscopic groups.

Fix the Levi subgroup $M \in \mathcal{L}$, and consider an element $M' \in \mathcal{E}_{\mathrm{ell}}(M)$. We choose a representative $(M', \mathcal{M}', s'_M, \xi'_M)$ within the given equivalence class so that $\mathcal{M}'$ is a subgroup of ${}^LM$, and the embedding ξ'_M is the identity. Then s'_M is a semisimple element in $\widehat{M}$ which centralizes $\mathcal{M}'$. Suppose that s' is an element in the set $s'_M Z(\widehat{M})^\Gamma$, and that $\widehat{G}'$ is the connected centralizer of s' in $\widehat{G}$. Then $\mathcal{G}' = \widehat{G}'\mathcal{M}'$ is a subgroup of ${}^LG$, and is a split extension of W_F by $\widehat{G}'$. Taking ξ' to be the identity embedding of $\mathcal{G}'$ into ${}^LG$, we obtain an endoscopic datum $(G', \mathcal{G}', s', \xi')$ for G. We shall write $\mathcal{E}_{M'}(G)$ for the set of such s', taken modulo the subgroup $Z(\widehat{G})^\Gamma$ of $Z(\widehat{M})^\Gamma$, for which the corresponding endoscopic datum for G is elliptic. Following the earlier convention, we shall represent a given element in $\mathcal{E}_{M'}(G)$ by its endoscopic group G'. We are not actually taking isomorphism classes of endoscopic data here, so different elements in $\mathcal{E}_{M'}(G)$ could give the same element in $\mathcal{E}_{\mathrm{ell}}(G)$. However, the ellipticity condition we have imposed at least means that there are only finitely many elements in $\mathcal{E}_{M'}(G)$. We can identify M' with a Levi subgroup of any given $G' \in \mathcal{E}_{M'}(G)$. We shall write

$$\iota_{M'}(G,G') \;=\; \big|\big(Z(\widehat{M})^\Gamma\big)^0 \cap Z(\widehat{G}')^\Gamma / \big(Z(\widehat{M})^\Gamma\big)^0 \cap Z(\widehat{G})^\Gamma\big|^{-1}\ .$$

To state the conjecture in the same form as the theorems of Section 6, we fix a central induced torus Z of G over F, and a character ζ of $Z(F)$. The weighted characters and the associated invariant distributions have obvious variants for functions in the subspace $C_c^\infty\big(G_{\mathrm{reg}}(F),\zeta\big)$ of $\mathcal{H}\big(G(F),\zeta\big)$.

Conjecture 10.1. *Suppose that $M \in \mathcal{L}$ and $\phi' \in T^{\mathcal{E}}(M,\zeta)$. Then the linear form*

$$I_M(\phi', f) \;=\; \sum_{\tau \in T(M,\zeta)} \Delta_M(\phi',\tau) I_M(\tau, f)\ , \qquad f \in C_c^\infty\big(G_{\mathrm{reg}}(F),\zeta\big),$$

equals an endoscopic *expression*

$$I_M^{\mathcal{E}}(\phi', f) \;=\; \sum_{G' \in \mathcal{E}_{M'}(G)} \iota_{M'}(G,G') \widehat{S}_{\tilde{M}'}^{\tilde{G}'}(\phi', f')\ , \tag{10.1}$$

for any fixed element $M' \in \mathcal{E}_{\mathrm{ell}}(M)$ such that $\phi' \in \Phi(\widetilde{M}', M, \zeta)$, and for stable linear forms $S_{\tilde{M}'}^{\tilde{G}'}(\phi'_\alpha)$ on $C_c^\infty\big(\widetilde{G}'_{\mathrm{reg}}(F), \widetilde{\zeta}'\zeta\big)$.

It is implicit in (10.1) that $I_M^{\mathcal{E}}(\phi', f)$ should be independent of the choice of M', as well as the representative of ϕ' is in $\Phi(\widetilde{M}', \widetilde{\zeta}'\zeta)$. Moreover, $S_{\tilde{M}'}^{\tilde{G}'}(\phi')$ is to depend only on $\widetilde{G}'$ and $\widetilde{M}'$ (and not on G or M). When G is quasisplit, the identity includes an inductive definition of these stable distributions. The assertion in this case is that

$$S_M^G(\phi, f) \;=\; I_M(\phi, f) - \sum_{\substack{G' \in \mathcal{E}_M(G) \\ G' \neq G}} \iota_M(G,G') \widehat{S}_M^{\tilde{G}'}(\phi, f') \tag{10.2}$$

is a stable linear form on $C_c^\infty\big(G_{\mathrm{reg}}(F),\zeta\big)$, for every $\phi \in \Phi(M,\zeta)$. If G is not quasisplit, or if ϕ' does not lie in $\Phi(M,\zeta)$, we can assume inductively that the terms on the right hand side of (10.1) have been defined. The assertion becomes a general identity between two different invariant distributions. Observe that when $M = G$, the conjectural assertion includes Theorem 6.1 if G is quasisplit, and is Theorem 6.2 in general.

As with the two earlier theorems, Conjecture 10.1 can be formulated in terms of integral kernels. It is known that there is a function

$$I_M(\tau,\gamma) \;=\; I_M^G(\tau,\gamma)$$

on $\widetilde{T}(M,\zeta) \times \Gamma(G)$ such that

$$I_M(\tau, f) \;=\; \int_{\Gamma(G)} I_M(\tau,\gamma) f_G(\gamma) d\gamma\ , \tag{10.3}$$

for any $\tau \in \widetilde{T}(M, \zeta)$ and $f \in C_c^\infty\big(G_{\mathrm{reg}}(F), \zeta\big)$ [4, Theorem 4.3]. As a function of $\gamma \in \Gamma(G)$, $I_M(\tau, \gamma)$ actually turns out to be locally integrable, so (10.3) provides a canonical extension of the distribution to $\mathcal{H}\big(G(F), \zeta\big)$. The conjectural assertion for (10.2) implies that the stable distributions $S_M^G(\phi)$ are also given by integral kernels. If G is quasisplit, it can be established inductively from (10.2) that

$$S_M^G(\phi, f) \; = \; \int_{\Sigma(G)} S_M^G(\phi, \sigma) f^G(\sigma) d\sigma \; , \tag{10.4}$$

for a function $S_M^G(\phi, \sigma)$ on $\Phi(M, \zeta) \times \Sigma(G)$. In fact, Conjecture 10.1 is equivalent to the following.

Conjecture 10.2. *Suppose that $M \in \mathcal{L}$, $\phi' \in T^{\mathcal{E}}(M, \zeta)$ and $\gamma \in \Gamma(G)$. Then the function*

$$I_M(\phi', \gamma) \; = \sum_{\tau \in T(M, \zeta)} \Delta_M(\phi', \tau) I_M(\tau, \gamma)$$

equals an endoscopic expression

$$I_M^{\mathcal{E}}(\phi', \gamma) \; = \sum_{G'} \sum_{\sigma' \in \Sigma_G(G')} \iota_{M'}(G, G') S_{\tilde{M}'}^{\tilde{G}'}(\phi', \sigma') \Delta_G(\sigma', \gamma) \; , \tag{10.5}$$

for M' and G' as in (10.1), and for functions $S_{\tilde{M}'}^{\tilde{G}'}(\cdot, \cdot)$ on $\Phi(\widetilde{M}', \widetilde{\zeta}'\zeta) \times \Sigma_G(G')$.

In an attempt to state things as clearly as possible, we have perhaps formulated the conjectures more precisely than is justified by the available evidence. A careful study of the descent and splitting properties of the distribution (10.1) should reveal, for example, whether our definition of the constants $\iota_{M'}(G, G')$ is correct. What does seem apparent is the essential role of the spectral transfer factors. This is one reason why we have spent the time we have taken to set them up.

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