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On the Fourier transforms of weighted orbital integrals

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Introduction

Suppose for a moment that G is a finite group. There are two canonical bases for the vector space of class functions on G. One is parametrized by the set $\Gamma(G)$ of conjugacy classes in G, the other by the set $\Pi(G)$ of (equivalence classes of) irreducible representations. Consider the elements of these bases as G-invariant linear functionals on $C(G)$. In other words, set

$$f_G(\gamma) = |G|^{-1} \sum_{x \in G} f(x^{-1}\gamma x), \quad \gamma \in \Gamma(G),$$

and

$$g_G(\pi) = |G|^{-1} \operatorname{tr}\Big(\sum_{x \in G} g(x)\pi(x)\Big), \quad \pi \in \Pi(G),$$

for functions $f, g \in C(G)$. Then the two families of linear functionals satisfy inversion formulas

$$f_G(\gamma) = \sum_{\pi \in \Pi(G)} I_G(\gamma, \pi) f_G(\pi) \tag{1}$$

and

$$g_G(\pi) = \sum_{\gamma \in \Gamma(G)} I_G(\pi, \gamma) g_G(\gamma) |G_\gamma|^{-1}, \tag{1$^\vee$}$$

where

$$I_G(\pi, \gamma) = \operatorname{tr}(\pi(\gamma))$$

is the character of π, and

$$I_G(\gamma, \pi) = I_G(\pi^\vee, \gamma) = I_G(\pi, \gamma^{-1}). \tag{2}$$

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These formulas are of course immediate consequences of the definitions and the orthogonality relations for irreducible characters.

Suppose now that G is a connected reductive algebraic group over a local field F of characteristic 0. The purpose of this paper is to find the natural analogues for $G(F)$ of the relations (1) and $(1)^\vee$. There is already a partial solution to the problem which comes from the orthogonality relations satisfied by elliptic tempered characters. However, the solution is valid only for cuspidal functions. We are looking for a general solution, which applies to hyperbolic as well as elliptic elements.

The two families of linear forms carry over to $G(F)$, and are defined on the Schwartz space $\mathscr{C}(G(F))$. The first family consists of the invariant orbital integrals

$$f_G(\gamma) = I_G(\gamma, f) = |D(\gamma)|^{\frac{1}{2}} \int_{G_\gamma(F)\backslash G(F)} f(x^{-1}\gamma x)\,dx,$$

parametrized by regular conjugacy classes $\gamma \in \Gamma(G(F)) \cap G_{\mathrm{reg}}(F)$. For the second family, one might consider taking the tempered characters

$$g_G(\pi) = I_G(\pi, g) = \mathrm{tr}(\pi(g))$$

associated to irreducible tempered representations $\pi \in \Pi_{\mathrm{temp}}(G(F))$. However, it is more appropriate to choose the slightly different family of virtual characters

$$g_G(\tau) = I_G(\tau, g) = \sum_{\varrho} \mathrm{tr}(\varrho^\vee(r))\, g_G(\pi_\varrho)$$

parametrized by the triplets $\tau = (M_1, \sigma, r) \in T(G)$ of the paper [8]. For example, the elliptic elements $T_{\mathrm{ell}}(G)$ in $T(G)$ are more natural than the elliptic representations in $\Pi_{\mathrm{temp}}(G(F))$; it is known ([20], [22]) that $T_{\mathrm{ell}}(G)$ provides a basis of the virtual characters which are *supertempered* in the sense of Harish-Chandra. A cuspidal function $f \in \mathscr{C}(G(F))$ is one for which $f_G(\tau)$ vanishes on the complement of $T_{\mathrm{ell}}(G)$ in $T(G)$. The partial solution we have mentioned applies to cuspidal functions f and g. It takes the form

$$I_G(\gamma, f) = \int_{T(G)} I_G(\gamma, \tau) f_G(\tau)\,d\tau, \tag{3}$$

and

$$I_G(\tau, g) = \int_{\Gamma(G(F)) \cap G_{\mathrm{reg}}(F)} I_G(\tau, \gamma) g_G(\gamma)\,d\gamma, \tag{3$^\vee$}$$

where

$$I_G(\tau, \gamma) = |D(\gamma)|^{\frac{1}{2}} \Theta(\tau, \gamma)$$

is the virtual character associated to τ (normalized by the Weyl discriminant), and

$$I_G(\gamma, \tau) = i^G(\tau) I_G(\tau^\vee, \gamma). \tag{4}$$

Because f and g are cuspidal, each integrand will actually be supported on the appropriate set of elliptic elements; it is for these elements that the formula (4) is given. (We leave the precise description of the constant $i^G(\tau)$ and the measures $d\tau$ and $d\gamma$, and for that matter the triplets τ, until the text.)

The identities (3) and $(3)^\vee$ will actually hold without the restrictions on f, g and τ. The second one is essentially Harish-Chandra's theorem that an irreducible character is given by a locally integrable function. The first identity is the assertion that the Fourier transform of an invariant orbital integral, regarded as a tempered distribution on $T(G)$, is also given by a function. This will be a special case of Theorem 4.1. The adjoint relation (4) between the two functions, however, does not hold in general. It is peculiar to the elliptic case. What could be its general analogue? To answer this question, we must enlarge the families of invariant distributions under consideration.

We shall have to take into account the weighted orbital integrals

$$J_M(\gamma,f) = |D(\gamma)|^{\frac{1}{2}} \int_{G_\gamma(F)\backslash G(F)} f(x^{-1}\gamma x)\, v_M(x)\, dx\,,$$

which we shall study as linear forms on $\mathscr{C}(G(F))$. These distributions are parametrized by Levi subgroups $M \in \mathscr{L}$ of G, and by conjugacy classes $\gamma \in \Gamma(M(F)) \cap G_{\mathrm{reg}}(F)$. If M is a proper Levi subgroup, the weight factor $v_M(x)$ is not a constant, and $J_M(\gamma,f)$ is a noninvariant distribution. In § 3 we shall review the formal procedure for constructing an invariant distribution $I_M(\gamma,f)$ from $J_M(\gamma,f)$. Weighted orbital integrals, and the associated invariant distributions, are important objects for the study of automorphic forms. They are among the principal terms in the trace formula. We are going to investigate their Fourier transforms. Our main result will be Theorem 4.1, which provides a qualitative description of the Fourier transform of $I_M(\gamma,\cdot)$ as a tempered distribution on $T(G)$. It expresses the Fourier transform as a finite linear combination of smooth functions

$$I_M(\gamma,\tau)\,, \quad \tau \in T_{\mathrm{disc}}(L),\ L \in \mathscr{L}\,,$$

on certain submanifolds $T_{\mathrm{disc}}(L)$ of $T(G)$. This result can be regarded as the natural generalization of the expansions (1) and (3).

As dual analogues of weighted orbital integrals, one might choose the weighted characters

$$J_L(\pi,g) = \mathrm{tr}\big(\mathscr{R}_L(\pi,P)\,\mathscr{I}_P(\pi,g)\big)\,,$$

also familiar from the trace formula, which are parametrized by Levi subgroups $L \in \mathscr{L}$ and representations $\pi \in \Pi_{\mathrm{temp}}(L(F))$. Following our earlier lead, we shall instead take the weighted virtual characters

$$J_L(\tau,g) = \sum_{\varrho} \mathrm{tr}\big(\varrho^\vee(r)\big)\, I_L(\pi_\varrho,g)$$

attached to triplets $\tau = (L_1,\sigma,r)$ in $T(L)$. This gives us a second family of noninvariant distributions. The formal procedure for constructing invariant distributions, or rather its dual analogue, runs into difficulty in this case. The problem comes from the behaviour of weighted orbital integrals near singular points. However, if we are prepared to replace $\mathscr{C}(G(F))$ by the smaller space $C_c^\infty(G_{\mathrm{reg}}(F))$, we can still construct an invariant distribution $I_L(\tau, g)$ from $J_L(\tau, g)$. Theorem 4.3 provides a qualitative description of $I_L(\tau, g)$ as an invariant distribution on $G_{\mathrm{reg}}(F)$. It asserts simply that the distribution can be identified with a smooth function

$$I_L(\tau, \gamma), \quad \gamma \in \Gamma_{\mathrm{ell}}(M(F)) \cap G_{\mathrm{reg}}(F),\ M \in \mathscr{L},$$

on $G_{\mathrm{reg}}(F)$. This result can be regarded as the natural generalization of the expansions $(1)^\vee$ and $(3)^\vee$.

Thus, Theorems 4.1 and 4.3 provide us with two functions $I_M(\gamma, \tau)$ and $I_L(\tau, \gamma)$ with which we can expand the two kinds of invariant distributions $I_M(\gamma, f)$ and $I_L(\tau, g)$. In Theorem 4.5, we shall establish a simple adjoint relation

$$I_M(\gamma, \tau) = (-1)^{\dim(A_M \times A_L)} i^L(\tau)\, I_L(\tau^\vee, \gamma) \tag{5}$$

between the two functions. It is this result which is the natural generalization of the formulas (2) and (4).

Most of the paper will be taken up with the proof of the three theorems we have described. Our main tool will be the local trace formula of [7] and [8]. In fact, given the qualitative assertions that comprise Theorems 4.1 and 4.3, the relation (5) is precisely equivalent to the local trace formula. We shall discuss the local trace formula in § 5. We shall actually derive a different version of the formula (Theorem 5.1), that makes use of both families of invariant distributions. The earlier invariant version of the local trace formula [8] employed only the distributions $\{I_M(\gamma, f)\}$, and is less suited to the present purpose. We shall then use the formula, in § 6, to derive Theorems 4.3 and 4.5 from Theorem 4.1. This leaves us with our primary task, to establish Theorem 4.1.

Suppose that T is an elliptic maximal torus in M, and that θ is a function in $C_c^\infty(T_{\mathrm{reg}}(F))$. The local trace formula can be translated (Lemma 6.1) into a spectral expansion

$$I_M(\theta, f) = \sum_{L \in \mathscr{L}} |W_0^L| |W_0^G|^{-1} \int_{T_{\mathrm{disc}}(L)} I_M(\theta, \tau) f_L(\tau)\, d\tau \tag{6}$$

for the inner product

$$I_M(\theta, f) = \int_{T(F)} \theta(\gamma) I_M(\gamma, f)\, d\gamma\,.$$

If we let θ approach the Dirac measure at a point γ in $T_{\mathrm{reg}}(F)$, the identity (6) ought to approach the expansion for $I_M(\gamma, f)$ required by Theorem 4.1. However, to make this work, we have to be able to control the behaviour of the linear form $I_M(\theta, \tau)$. In the case $F = \mathbb{R}$, we can take care of the difficulties by using the differential equations satisfied by $I_M(\gamma, f)$. We shall establish Theorem 4.1 for $F = \mathbb{R}$ in § 7. When F is a p-adic field, we have to use something else. In § 8, we shall prove an analogue of the Howe conjecture (Theorem 8.1) for the distributions $\{I_M(\gamma, f)\}$. Our method will be to combine the identity (6) with the kind of residue argument that is familiar from Paley-Wiener theorems. This theorem, which was originally proposed as a problem in [6], will allow us to establish Theorem 4.1 for p-adic F in § 9. Theorem 8.1 actually gives us control over singular set in the p-adic case. For example, we shall show (Corollary 9.3) that the smooth function $\gamma \to I_L(\tau, \gamma)$ on $G_{\mathrm{reg}}(F)$ is locally integrable on $G(F)$. In other words, the distribution $g \to I_L(\tau, g)$ is given by a locally integrable function on $G(F)$. This is a generalization of the character theorem of Harish-Chandra and Howe.

Our original interest in the paper is reflected in the title. We wanted to find the obstruction to the Fourier transform of $I_M(\gamma, f)$ being a smooth function on $T(G)$. The idea actually goes back to an observation of Kottwitz on the results in [24]. As we have already noted, there is no obstruction when $M = G$. For arbitrary M, the situation is described by Theorem 4.1. In general, the complement of $T_{\mathrm{ell}}(L)$ in the stratum $T_{\mathrm{disc}}(L)$ is properly embedded in $T(G)$. It is on this complement that the Fourier transform fails to be a smooth function, since it is a multiple of the Dirac measure in the normal directions. We shall give an explicit expression (4.7) for the singular part of $I_M(\gamma, f)$ in terms of the regular parts of the corresponding distributions on Levi subgroups of G. In particular, we will obtain a simple formula for the "discrete part" of $I_M(\gamma, f)$, that is, the contribution to the Fourier transform from the subset $T_{\mathrm{disc}}(G)$ of $T(G)$. This formula may be useful for comparison problems that arise from endoscopy.

§ 1. Weighted orbital integrals

Let G be a reductive algebraic group over a local field F of characteristic 0. Weighted orbital integrals on $G(F)$ are among the principal terms in the global trace formula. Our purpose is to study these objects as tempered distributions. In other words, we shall consider them as linear forms on the Schwartz space $\mathscr{C}(G(F))$ of $G(F)$. In this section we shall review a few of their elementary properties.

We first recall the basic objects on $G(F)$ which go into the construction of weighted orbital integrals. Let K be a fixed maximal compact subgroup of $G(F)$, and let M_0 be a fixed F-rational Levi component of some minimal parabolic subgroup of G defined over F. We assume that K and $M_0(F)$ are in good relative position [7], § 1. It is also understood that K is special in the case that F is a p-adic field. Any parabolic subgroup P of G which is defined over F, and contains M_0, has a unique Levi component M_P which contains M_0. Both M_P and the unipotent radical N_P of P are defined over F. We write $\mathscr{L} = \mathscr{L}^G$ for the finite set of subgroups of G of the form M_P, and we refer to the elements in $\mathscr{L}$ simply as Levi subgroups of G. As usual, $\mathscr{L}(M) = \mathscr{L}^G(M)$ denotes the set of Levi subgroups which contain a given $M \in \mathscr{L}$. Similarly, $\mathscr{F}(M) = \mathscr{F}^G(M)$ stands for the set of parabolic subgroups P of G over F such that M_P contains M, and $\mathscr{P}(M) = \mathscr{P}^G(M)$ denotes the subset of groups $P \in \mathscr{F}(M)$ with $M_P = M$. If M is any group in $\mathscr{L}$, $K_M = K \cap M(F)$ is a maximal compact subgroup of $M(F)$. The triplet (M, K_M, M_0) then satisfies the same hypotheses as (G, K, M_0).

Suppose that $M \in \mathscr{L}$ is a Levi subgroup. We have the canonical homomorphism H_M from $M(F)$ to the real vector space

$$\mathfrak{a}_M = \mathrm{Hom}(X(M)_F, \mathbb{R})$$

which is defined by

$$e^{\langle H_M(m), \chi \rangle} = |\chi(m)|, \quad m \in M(F), \ \chi \in X(M)_F.$$

Let A_M be the split component of the center of M. Then $\mathfrak{a}_{M,F} = H_M(M(F))$ and $\tilde{\mathfrak{a}}_{M,F} = H_M(A_M(F))$ are closed subgroups of $\mathfrak{a}_M$, while $\mathfrak{a}_{M,F}^\vee = \mathrm{Hom}(\mathfrak{a}_{M,F}, 2\pi i\mathbb{Z})$ and $\tilde{\mathfrak{a}}_{M,F}^\vee = \mathrm{Hom}(\tilde{\mathfrak{a}}_{M,F}, 2\pi i\mathbb{Z})$ are closed subgroups of $i\mathfrak{a}_M^*$. The quotient

$$i\mathfrak{a}_{M,F}^* = i\mathfrak{a}_M^* / \mathfrak{a}_{M,F}^\vee$$

is a compact torus if F is p-adic, and simply equals $i\mathfrak{a}_M^*$ if F is Archimedean.

For each $M \in \mathscr{L}$, we fix a Haar measure on the vector space $\mathfrak{a}_M$, and we choose the dual Haar measure on the real vector space $i\mathfrak{a}_M^*$. In the case that F is a p-adic field, we require that the measures be normalized so that the quotients $\mathfrak{a}_M/\tilde{\mathfrak{a}}_{M,F}$ and $i\mathfrak{a}_M^*/\tilde{\mathfrak{a}}_{M,F}^\vee$ each have volume 1. The kernel of H_M in $A_M(F)$ is compact, and therefore has a canonical normalized Haar measure. Since the group $\tilde{\mathfrak{a}}_{M,F} = H_M(A_M(F))$ is either discrete or equal to $\mathfrak{a}_M$, it too has an assigned Haar measure. These two Haar measures in turn determine a unique Haar measure on $A_M(F)$. We have so far just lifted the conventions from §1 of the paper [8]. In this paper we shall also normalize the Haar measures on maximal tori. If T is an elliptic maximal torus in M over F, the quotient $T(F)/A_M(F)$ is compact. The normalized Haar measure on this compact group, together with the Haar measure on $A_M(F)$, then determines a Haar measure on $T(F)$. This in turn provides a Haar measure on the group of rational points of any maximal torus in G over F, since any such torus is $G(F)$-conjugate to an elliptic maximal torus in some M. In particular, if γ lies in the set $G_{\mathrm{reg}}(F)$ of (strongly) regular points in $G(F)$, and G_γ denotes the centralizer of γ in G, then $G_\gamma(F)$ will have a fixed Haar measure.

Fix the Levi subgroup $M \in \mathscr{L}$. If P belongs to $\mathscr{P}(M)$, we have the map

$$H_P : G(F) \to \mathfrak{a}_M,$$

which is defined for any element

$$x = mnk, \quad m \in M(F), \ n \in N_P(F), \ k \in K,$$

in $G(F)$ by

$$H_P(x) = H_M(x).$$

The functions

$$v_P(\Lambda, x) = e^{-\Lambda(H_P(x))}, \quad \Lambda \in i\mathfrak{a}_M^*, \ P \in \mathscr{P}(M),$$

form a (G, M)-family [2], §6–7. Writing

$$\theta_P(\Lambda) = \mathrm{vol}\left(\mathfrak{a}_M^G / \mathbb{Z}(\Delta_P^\vee)\right)^{-1} \prod_{\alpha \in \Delta_P} \lambda(\alpha^\vee),$$

in the usual notation [7], (6.7), we take the limit

$$v_M(x) = \lim_{\Lambda \to 0} \sum_{P \in \mathscr{P}(M)} v_P(\Lambda, x)\,\theta_P(\Lambda)^{-1}.$$

We recall that the limit exists ([2], Lemma 6.2), and in fact equals the volume in $\mathfrak{a}_M/\mathfrak{a}_G$ of the convex hull of the points

$$\{-H_P(x) : P \in \mathscr{P}(M)\}.$$

As a function of $x \in G(F)$, $v_M(x)$ is left $M(F)$-invariant. It is used to define noninvariant measures on conjugacy classes.

Suppose that γ lies in $M(F) \cap G_{\mathrm{reg}}(F)$. The weighted orbital integral at γ is the linear functional

$$J_M(\gamma) = J_M^G(\gamma) : f \to J_M(\gamma, f), \quad f \in \mathscr{C}(G(F)),$$

on $\mathscr{C}(G(F))$ defined by

$$J_M(\gamma, f) = |D(\gamma)|^{\frac{1}{2}} \int_{G_\gamma(F)\backslash G(F)} f(x^{-1}\gamma x)\, v_M(x)\, dx. \tag{1.1}$$

As usual,

$$D(\gamma) = \det\left(1 - \mathrm{Ad}(\gamma)\right)_{\mathfrak{g}/\mathfrak{g}_\gamma}$$

denotes the Weyl discriminant. The convergence of the integral follows from [2], Lemma 8.1, as does the fact that the linear functional $f \to J_M(\gamma, f)$ on $\mathscr{C}(G(F))$ is continuous. Thus, $J_M(\gamma)$ is a tempered distribution on $G(F)$. It depends on a choice of Haar measure on $G(F)$, as well as the measure on $G_\gamma(F)$ which we fixed above.

Following [8], §1 we write $\Gamma(M(F))$ for the set of conjugacy classes in $M(F)$. Since the integral in (1.1) depends only on the image of γ in $\Gamma(M(F))$, we can regard $J_M(\cdot, f)$ as a function on $\Gamma(M(F)) \cap G_{\mathrm{reg}}(F)$ instead of $M(F) \cap G_{\mathrm{reg}}(F)$. We shall in fact use the two interpretations interchangeably. Observe that when $M = G$, $J_M(\gamma, f)$ is just the invariant orbital integral over the $G(F)$-conjugacy class of γ. In this case we shall write

$$f_G(\gamma) = J_G(\gamma, f), \quad \gamma \in G_{\mathrm{reg}}(F), \tag{1.2}$$

when we want to emphasize the dependence on γ. If Q belongs to $\mathscr{P}(M)$, and f_Q is the familiar Schwartz function

$$f_Q(m) = \delta_Q(m)^{\frac{1}{2}} \int_K \int_{N_Q(F)} f(k^{-1} m n k)\, dn\, dk, \quad m \in M(F),$$

on $M(F)$, the function $(f_Q)_M$ is independent of Q. We denote it simply by f_M. One sees easily that

$$f_M(\gamma) = f_G(\gamma), \quad \gamma \in M(F) \cap G_{\mathrm{reg}}(F). \tag{1.3}$$

Weighted orbital integrals have natural symmetry and descent properties that we should recall. There is an obvious action

$$(M, \gamma) \to (\omega M, \omega\gamma), \quad \omega \in W_0^G,$$

of W_0^G on the set of pairs

$$(M, \gamma), \quad M \in \mathscr{L}, \gamma \in \Gamma(M(F)) \cap G_{\mathrm{reg}}(F).$$

The symmetry condition

$$J_{wM}(w\gamma, f) = J_M(\gamma, f), \quad w \in W_0^G, \tag{1.4}$$

is an immediate consequence of the fact that $v_{wM}(wx)$ equals $v_M(x)$. For the descent condition, we assume that γ lies in $M_1(F) \cap G_{\mathrm{reg}}(F)$, where $M_1 \in \mathscr{L}$ is a subgroup of M. It then follows without difficulty from [4], Corollary 7.2 that

$$J_M(\gamma, f) = \sum_{S \in \mathfrak{L}(M_1)} d_{M_1}^G(M, S) J_{M_1}^S(\gamma, f_{Q_S}), \tag{1.5}$$

where $d_{M_1}^G(M, S)$ is the constant described in [4], p. 356, and

$$S \to Q_S \in \mathscr{P}(S),$$

is the retraction defined on p. 357 of [4]. Let $\Gamma_{\mathrm{ell}}(M(F))$ denote the set of $M(F)$-conjugacy classes in the F-elliptic set $M(F)_{\mathrm{ell}}$ of $M(F)$. Any class in the complement of $\Gamma_{\mathrm{ell}}(M(F))$ in $\Gamma(M(F)) \cap G_{\mathrm{reg}}(F)$ will intersect a proper Levi subgroup $M_1(F)$ of $M(F)$. In this case, the right hand side of (1.5) is a sum over proper subgroups of G. The descent formula therefore reduces the study of $J_M(\gamma, f)$ to the case that γ is elliptic in $M(F)$.

§ 2. Weighted characters

We are going to study weighted orbital integrals in conjunction with a parallel family of tempered distributions, the weighted characters on $G(F)$. We really ought to say weighted "virtual" characters, for the elements in this second family will be parametrized by the virtual tempered characters on $M(F)$ discussed in [8]. We shall review some of the constructions of [8], with minor adjustments to allow us to work with the Schwartz space instead of the Hecke algebra.

We shall generally follow the notation of [8]. If $M \in \mathscr{L}$ is a fixed Levi subgroup, $\Pi_{\mathrm{temp}}(M(F))$ denotes the set of (equivalence classes of) irreducible tempered representations of $M(F)$, and $\Pi_2(M(F))$ stands for the subset of representations in $\Pi_{\mathrm{temp}}(M(F))$

which are square integrable modulo $A_M(F)$. On each of these sets we have the contragredient involution $\pi \to \pi^\vee$. We also have the natural action

$$\pi_\lambda(m) = \pi(m) e^{\lambda(H_M(m))}, \quad \pi \in \Pi_{\mathrm{temp}}(M(F)),\ \lambda \in i\mathfrak{a}_M^*,\ m \in M(F),$$

of $i\mathfrak{a}_M^*$. Recall that if F is a p-adic field, the stabilizer $\mathfrak{a}_{M,\pi}^\vee$ of π in $i\mathfrak{a}_M^*$ is a lattice. If $F = \mathbb{R}$ on the other hand, $\mathfrak{a}_{M,\pi}^\vee$ is trivial. In fact any representation in $\Pi_{\mathrm{temp}}(M(\mathbb{R}))$ can be written uniquely in the form π_λ, where λ lies in $i\mathfrak{a}_M^*$ and π belongs to the subset $\Pi_{\mathrm{temp}}(M(\mathbb{R})/A_M(\mathbb{R})^0)$ of representations in $\Pi_{\mathrm{temp}}(M(\mathbb{R}))$ whose central character is trivial on the connected Lie group $A_M(\mathbb{R})^0$. In this case, we shall write μ_{π_λ} for the linear form that determines the infinitesimal character of π_λ. Thus, μ_{π_λ} is a Weyl orbit of elements in the dual of a complex Cartan subalgebra, which we assume is equipped with a suitable Hermitian norm $\|\cdot\|$. Then

$$\|\mu_{\pi_\lambda}\| = \|\mu_\pi + \lambda\| = \|\mu_\pi\| + \|\lambda\|.$$

For any parabolic subgroup $P \in \mathscr{P}(M)$, we can form the induced representation

$$\mathscr{I}_P(\pi_\lambda) : x \to \mathscr{I}_P(\pi_\lambda, x), \quad x \in G(F),$$

of $G(F)$. It acts on a Hilbert space $\mathscr{H}_P(\pi)$ of vector valued functions on K which is independent of λ. Weighted characters depend on normalized intertwining operators between these induced representations. However, the factors that one can use to normalize the intertwining operators are not unique. In fact it will be important in this paper to be able to vary the normalizing factors. We shall summarize the properties of the normalizing factors that we require.

Fix $M \in \mathscr{L}$ and $\pi \in \Pi_{\mathrm{temp}}(M(F))$. The unnormalized intertwining operators

$$J_{P'|P}(\pi_\lambda) : \mathscr{H}_P(\pi) \to \mathscr{H}_{P'}(\pi), \quad P, P' \in \mathscr{P}(M),$$

are defined by analytic continuation as meromorphic, operator valued functions of $\lambda \in \mathfrak{a}_{M,C}^*$. We want to choose scalar valued meromorphic functions $r_{P'|P}(\pi_\lambda)$ so that the normalized operators

$$R_{P'|P}(\pi_\lambda) = r_{P'|P}(\pi_\lambda)^{-1} J_{P'|P}(\pi_\lambda)$$

have certain properties. We ask that the normalizing factors be subject to the following list of conditions.

(r.1) $$r_{P'|P}(\pi_\lambda) = \prod_{\beta \in \Sigma_{P'}^r \cap \Sigma_{\bar{P}}^r} r_\beta(\pi_\lambda) = r_{\bar{P}|\bar{P}'}(\pi_\lambda),$$

where each $r_\beta(\pi_\lambda)$ is a meromorphic function that depends only on the projection $\lambda(\beta^\vee)$.

(r.2) If π is an irreducible constituent of an induced representation

$$\mathscr{I}_R(\sigma), \quad \sigma \in \Pi_2(M_1(F)),\ R \in \mathscr{P}^M(M_1),\ M_1 \subset M,$$

then

$$r_{P'|P}(\pi_\lambda) = r_{P'(R)|P(R)}(\sigma_\lambda)\,.$$

(r.3) $$r_{P'|P}(\pi_\lambda)\, r_{P|P'}(\pi_\lambda) = J_{P'|P}(\pi_\lambda)\, J_{P|P'}(\pi_\lambda)\,.$$

(r.4) $$\overline{r_{P'|P}(\pi_\lambda)} = r_{P|P'}(\pi_{-\bar{\lambda}})\,.$$

(r.5) $$r_{P'|P}(\pi_\lambda) = r_{P|P'}(\pi^\vee_{-\lambda})\,.$$

(r.6) $$r_{wP'|wP}(w \cdot \pi_\lambda) = r_{P'|P}(\pi_\lambda)\,, \quad w \in W_0^G\,.$$

(r.7) Suppose that F is a p-adic field with residue field of order q. Then $r_{P'|P}(\pi_\lambda)$ is a rational function in the variables

$$\{q^{\lambda(\beta^\vee)} : \beta \in \Sigma^r_{P'} \cap \Sigma^r_{\bar{P}}\}\,.$$

(r.8) Suppose that $F = \mathbb{R}$, and that

$$q_{P'|P}(\pi_\lambda) = \prod_{\beta \in \Sigma^r_{P'} \cap \Sigma^r_{\bar{P}}} \lambda(\beta^\vee)^{n_\beta(\pi)}, \quad \pi \in \Pi_{\mathrm{temp}}(M(\mathbb{R})/A_m(\mathbb{R})^0),\ \lambda \in i\mathfrak{a}^*_M\,,$$

where $n_\beta(\pi)$ is the order of the pole of $r_\beta(\pi_\lambda)$ of $\lambda = 0$. (It is known that $n_\beta(\pi)$ equals 0 or 1.) Then if D_λ is an invariant differential operator on $i\mathfrak{a}^*_M$, there are constants C and N such that

$$|D_\lambda(q_{P'|P}(\pi_\lambda)\, r_{P'|P}(\pi_\lambda))^{-1}| \leqq C(1 + \|\mu_\pi + \lambda\|)^N\,,$$

for every π and λ.

The conditions we have just described are quite familiar, and follow standard notation. Thus, $\Sigma^r_{P'}$ denotes the set of reduced roots of (P', A_M), $\bar{P}$ is the parabolic subgroup opposite to P, and in (r.2), $P(R) \in \mathscr{P}(M_1)$ is the parabolic subgroup with $P(R) \cap M = R$ and $P(R) \subset P$. When combined with the corresponding properties ([5], § 1) of the operators $J_{P'|P}(\pi_\lambda)$, these conditions lead to the properties we expect of the normalized operators $R_{P'|P}(\pi_\lambda)$. For example, as analogues of (r.2)–(r.7) we have

(R.2) $$R_{P'|P}(\pi_\lambda) = R_{P'(R)|P(R)}(\sigma_\lambda), \quad \pi = \mathscr{I}_R(\sigma)\,.$$

(R.3) $$R_{P''|P}(\pi_\lambda) = R_{P''|P'}(\pi_\lambda)\, R_{P'|P}(\pi_\lambda), \quad P, P', P'' \in \mathscr{P}(M)\,.$$

(R.4) $$R_{P'|P}(\pi_\lambda)^* = R_{P|P'}(\pi_{-\bar{\lambda}})\,.$$

(R.5) $$R_{P'|P}(\pi_\lambda)^\vee = R_{P|P'}(\pi^\vee_{-\lambda})\,,$$

where $R_{P'|P}(\pi_\lambda)^\vee$ denotes the transpose of $R_{P'|P}(\pi_\lambda)$.

(R.6) $$R_{wP'|wP}(\tilde{w} \cdot \pi_\lambda) = A(\tilde{w})\, R_{P'|P}(\pi_\lambda)\, A(\tilde{w})^{-1}\,, \quad w \in W_0^G\,,$$

where $A(\tilde{w})$ is the map from $\mathscr{H}_P(\pi)$ to $\mathscr{H}_{wP}(\tilde{w}\pi)$ determined by a representative $\tilde{w}$ of w in K.

(R.7) If F is a p-adic field with residue field of order q, the matrix coefficients of $R_{P'|P}(\pi_\lambda)$ are rational functions in the variables $\{q^{\lambda(\beta^\vee)}\}$ of (r.7).

The analogue of (r.8) is a growth condition ([2], (7.6)) that was quoted from an unpublished manuscript. For convenience we shall reproduce the proof of the estimate here.

Lemma 2.1. *Assume that $F = \mathbb{R}$, and that the normalizing factors satisfy the conditions* (r.1)– (r.8). *For each irreducible K-type $\delta \in \Pi(K)$, let $\mathscr{H}_P(\pi)_\delta$ be the δ-isotypical subspace of $\mathscr{H}_P(\pi)$, and let $R_{P'|P}(\pi_\lambda)_\delta$ be the restriction of $R_{P'|P}(\pi_\lambda)$ to $\mathscr{H}_P(\pi)_\delta$. Then if D_λ is an invariant differential operator on $i\mathfrak{a}_M^*$, there are constants C and N such that*

$$\text{(R.8)} \qquad \|D_\lambda R_{P'|P}(\pi_\lambda)_\delta\| \leqq C(1 + \|\mu_\pi + \lambda\|)^N (1 + \|\mu_\delta\|)^N ,$$

for all $\pi \in \Pi_{\mathrm{temp}}(M(\mathbb{R})/A_M(\mathbb{R})^0)$, $\lambda \in i\mathfrak{a}_M^*$, and $\delta \in \Pi(K)$.

Proof. Since any $\pi \in \Pi_{\mathrm{temp}}(M(\mathbb{R}))$ is a constituent of a representation induced from discrete series, the formula (R.2) leads to an immediate reduction of the problem. We need only establish the estimate for representations π which belong to $\Pi_2(M(\mathbb{R})/A_M(\mathbb{R})^0)$. Moreover, from the analogues of (r.1) and (r.3) for $R_{P'|P}(\pi_\lambda)$, we can reduce the problem further to the case that $\dim(A_M/A_G) = 1$ and $P' = \bar{P}$. We shall in fact estimate the Hilbert-Schmidt norms

$$\|D_\lambda R_{\bar{P}|P}(\pi_\lambda) T\|_2, \quad \pi \in \Pi_2(M(\mathbb{R})/A_M(\mathbb{R})^0),\ \lambda \in i\mathfrak{a}_M^* ,$$

in which T ranges over linear operators on $\mathscr{H}_P(\pi)_\delta$.

Harish-Chandra has defined a linear isometry $T \to \psi_T$ from $\mathrm{End}(\mathscr{H}_P(\pi)_\delta)$ onto the space $\mathscr{C}_\pi(M(\mathbb{R})/A_M(\mathbb{R})^0, \delta \times \delta)$ of δ-spherical Schwartz functions attached to π. Thus,

$$\|D_\lambda R_{\bar{P}|P}(\pi_\lambda) T\|_2 = \|D_\lambda \psi_{T_1}\| ,$$

where

$$T_1 = R_{\bar{P}|P}(\pi_\lambda) T = r_{\bar{P}|P}(\pi_\lambda)^{-1} J_{\bar{P}|P}(\pi_\lambda) T .$$

But ψ_{T_1} has an expression

$$\psi_{T_1} = c_0\, r_{\bar{P}|P}(\pi_\lambda)^{-1} c_{\bar{P}|P}(1, \lambda) \psi_T$$

in terms of Harish-Chandra's c-functions, with c_0 being a constant that depends only on M ([19], Corollary to Lemma 18.1). Moreover, one can use Harish-Chandra's techniques of harmonic analysis to estimate the c-functions. By [9], Lemma 4.5, there are constants C' and N' such that

$$\|D_\lambda q_{\bar{P}|P}(\pi_\lambda)\, c_{\bar{P}|P}(1, \lambda) \psi\| \leqq C'(1 + \|\mu_\pi + \lambda\|)^{N'} (1 + \|\mu_\delta\|)^{N'} \|\psi\| ,$$

for all $\pi \in \Pi_2(M(\mathbb{R})/A_M(\mathbb{R}^0))$, $\lambda \in i\mathfrak{a}_M^*$, $\delta \in \Pi(K)$, and $\psi \in \mathscr{C}_\pi(M(\mathbb{R})/A_M(\mathbb{R})^0, \delta \times \delta)$. It follows from (r.8) that there are constants C'' and N'' such that

$$\|D_\lambda R_{\bar{P}|P}(\pi_\lambda)T\|_2 \leqq C''(1 + \|\mu_\pi + \lambda\|)^{N''}(1 + \|\mu_\delta\|)^{N''}\|T\|_2,$$

for all π, λ and δ, and all $T \in \mathrm{End}(\mathscr{H}_P(\pi)_\delta)$. This clearly implies the reduced form of the required estimate, and therefore the required estimate itself. □

Lemma 2.2. *The normalizing factors $\{r_{P'|P}(\pi_\lambda)\}$ can be chosen so that the conditions* (r.1)–(r.8) *all hold. Moreover, if $\{r_{P'|P}(\pi_\lambda)\}$ satisfy the eight conditions, so do the complementary functions*

$$r^\vee_{P'|P}(\pi_\lambda) = r_{P|P'}(\pi_\lambda).$$

Proof. In [5], Theorem 2.1, the intertwining operators were normalized subject to a slightly different set of conditions. However, the only conditions here which are not implied by the earlier ones are (r.5) and (r.8). It is easy to see that these additional constraints are also satisfied by the normalizing factors chosen in the proof of [5], Theorem 2.1. For example, if $F = \mathbb{R}$, the condition (r.5) follows without difficulty from the construction [5], (3.2) of $r_{P'|P}(\pi_\lambda)$ in terms of L-functions. Moreover, (r.8) follows from standard properties of the gamma function. In the case of a p-adic field, the general existence argument of Langlands implies (r.5). Therefore, the normalizing factors of [5], Theorem 2.1 satisfy the conditions (r.1)–(r.8).

For the second assertion of the lemma, recall that Harish-Chandra's μ-functions

$$\mu_{P'|P}(\pi_\lambda) = (J_{P'|P}(\pi_\lambda)J_{P|P'}(\pi_\lambda))^{-1}$$

are symmetric in P and P'. This implies that (r.3) holds for the functions $r^\vee_{P'|P}(\pi_\lambda)$. All the other conditions for $r^\vee_{P'|P}(\pi_\lambda)$ are obvious. □

Remark. There is a condition (R_7) in [5], Theorem 2.1 which asserts that $r_{P'|P}(\pi_\lambda)$ has no zeros or poles with the real part of λ in the positive chamber attached to P. This property, which is useful for studying Langlands quotients, is stronger than the conditions here. It cannot be satisfied by the functions $r_{P'|P}(\pi_\lambda)$ and $r^\vee_{P'|P}(\pi_\lambda)$ simultaneously.

Fix a family

$$r_{P'|P}(\pi_\lambda), \quad P, P' \in \mathscr{P}(M),\ \pi \in \Pi_{\mathrm{temp}}(M(F)),\ \lambda \in i\mathfrak{a}_M^*,$$

of functions which satisfy the conditions (r.1)–(r.8). If $\{R_{P'|P}(\pi_\lambda)\}$ is the corresponding set of normalized intertwining operators, we can form the (G, M)-family

$$\mathscr{R}_Q(\Lambda, \pi, P) = R_{Q|P}(\pi)^{-1}R_{Q|P}(\pi_\Lambda), \quad \Lambda \in i\mathfrak{a}_M^*,\ Q \in \mathscr{P}(M),$$

of operator valued functions. We can then take the associated operator

$$\mathscr{R}_M(\pi, P) = \lim_{\Lambda \to 0} \sum_{Q \in \mathscr{P}(M)} \mathscr{R}_Q(\Lambda, \pi, P)\theta_Q(\Lambda)^{-1}.$$

(See [2], §7.) The weighted character at π is the linear functional in $\mathscr{C}(G(F))$ defined by

$$J_M(\pi, f) = \operatorname{tr}(\mathscr{R}_M(\pi, P)\mathscr{I}_P(\pi, f)), \quad f \in \mathscr{C}(G(F)). \tag{2.1}$$

It is easy to see that the operator on the right is of trace class. If F is Archimedean, this follows from (R.8), while if F is p-adic, the operator is actually of finite rank. In either case, $f \to J_M(\pi, f)$ is a continuous linear functional on $\mathscr{C}(G(F))$, and is therefore a tempered distribution on $G(F)$.

We want to focus on the basis of virtual tempered characters introduced in [8] rather than on the set of irreducible tempered characters. The elements in this second basis are determined by the set $\tilde{T}(G)$ of essential triplets

$$\tau = (M_1, \sigma, r), \quad M_1 \in \mathscr{L}, \ \sigma \in \Pi_2(M_1(F)), \ r \in \tilde{R}_\sigma,$$

defined in [8], §3. Recall that $\tilde{R}_\sigma = \tilde{R}_\sigma^G$ is a fixed central extension

$$1 \to Z_\sigma \to \tilde{R}_\sigma \to R_\sigma \to 1$$

of the R-group of σ which splits a certain 2-cocycle. The process singles out a character χ_σ of Z_σ, and we write $\Pi(\tilde{R}_\sigma, \chi_\sigma)$ as in [8] for the set of irreducible representations of $\tilde{R}_\sigma$ whose central character on Z_σ equals χ_σ. We shall also sometimes write $Z_\tau = Z_\sigma$ and $\chi_\tau = \chi_\sigma$, as well as

$$\tau_\lambda = (M, \sigma_\lambda, r), \quad \lambda \in \mathfrak{a}^*_{G,C},$$

$$z\tau = (M, \sigma, zr), \quad z \in Z_\tau,$$

and

$$\tau^\vee = (M, \sigma^\vee, r).$$

In §2 of [8] we used the normalized intertwining operators to construct a representation

$$r \to \tilde{R}(r, \sigma) = \xi_\sigma(r)^{-1} A(\sigma_r) R_{r^{-1}P_1|P_1}(\sigma), \quad P_1 \in \mathscr{P}(M_1),$$

of $\tilde{R}_\sigma$ on $\mathscr{H}_{P_1}(\sigma)$. This representation then determined a bijection $\varrho \to \pi_\varrho$, from $\Pi(\tilde{R}_\sigma, \chi_\sigma)$ onto the set of irreducible constituents of $\mathscr{I}_{P_1}(\sigma)$, with the properties that

$$\operatorname{tr}(\tilde{R}(r, \sigma)\mathscr{I}_{P_1}(\sigma, f)) = \sum_{\varrho \in \Pi(\tilde{R}_\sigma, \chi_\sigma)} \operatorname{tr}(\varrho^\vee(r)) \operatorname{tr}(\pi_\varrho(f)) \tag{2.2}$$

and

$$\operatorname{tr}(\pi_\varrho(f)) = |\tilde{R}_\sigma|^{-1} \sum_{r \in \tilde{R}_\sigma} \operatorname{tr}(\varrho(r)) \operatorname{tr}(\tilde{R}(r, \sigma)\mathscr{I}_{P_1}(\sigma, f)). \tag{2.3}$$

We dealt only with the Hecke algebra in [8]. However (2.2) and (2.3) are clearly identities between tempered distributions, and are valid for any $f \in \mathscr{C}(G(F))$. The distributions on the left hand side of (2.2) provide a basis for the virtual tempered characters on $G(F)$. The identities (2.2) and (2.3) are simply the transition matrices between this basis and the original basis of irreducible tempered characters.

Suppose that

$$\tau = (M_1, \sigma, r), \quad M_1 \subset M, \, \sigma \in \Pi_2(M_1(F)), \, r \in \tilde{R}_\sigma^M,$$

is a triplet in $\tilde{T}(M)$. The weighted character at τ is the linear functional

$$J_M(\tau) = J_M^G(\tau) : f \to J_M(\tau, f), \quad f \in \mathscr{C}(G(F)),$$

on $\mathscr{C}(G(F))$ defined by

$$(2.4) \qquad J_M(\tau, f) = \sum_{\varrho \in \Pi(\tilde{R}_\sigma^M, \chi_\sigma)} \mathrm{tr}\left(\varrho^\vee(r)\right) J_M(\pi_\varrho, f).$$

It is clear that $J_M(\tau, f)$ is a tempered distribution on $G(F)$. It depends on a choice of Haar measure on $G(F)$.

As in [8], §3, we write $T(M)$ for the set of W_0^M-orbits in $\tilde{T}(M)$. Since the sum in (2.4) depends only on the image of τ in $T(M)$, we can regard $J_M(\cdot, f)$ as a function on $T(M)$ instead of $\tilde{T}(M)$. We shall again use the two interpretations interchangeably. Notice that when $M = G$, $J_M(\tau, f)$ is just the virtual character defined by (2.2). As in §1, we write

$$(2.5) \qquad f_G(\tau) = J_G(\tau, f), \quad \tau \in T(G).$$

Thus, f_G stands for a function either on regular conjugacy classes in $G(F)$ or on W_0^G-orbits in $\tilde{T}(G)$. Recalling that $f_M = (f_Q)_M$ for any group $Q \in \mathscr{F}(M)$, we see easily that

$$(2.6) \qquad f_M(\tau) = f_G(\tau), \quad \tau \in \tilde{T}(M).$$

Weighted characters also have descent and symmetry properties. There is an action of W_0^G on the set of pairs

$$(M, \tau), \quad M \in \mathscr{L}, \, \tau \in T(M).$$

There is also an equivalence relation on $T(M)$ determined by orbits of the groups Z_τ. One sees directly from (R.6) and the definition (2.4) that

$$(2.7) \qquad J_{wM}(wz\tau, f) = \chi_\tau(z)^{-1} J_M(\tau, f), \quad w \in W_0^G, \, z \in Z_\tau.$$

The descent condition is parallel to (1.5). Suppose that $\tau \in T(M)$ lies in the image of $T(M_1)$, for a Levi subgroup M_1 of M. (In other words, τ is the W_0^M-orbit of a triplet in the subset $\tilde{T}(M_1)$ of $\tilde{T}(M)$.) Then

$$(2.8) \qquad J_M(\tau, f) = \sum_{S \in \mathscr{L}(M_1)} d_{M_1}^G(M, S) J_{M_1}^S(\tau, f_{Q_S}).$$

This formula follows without difficulty from [4], Corollary 7.2. Recall [8], § 3 that $T_{\mathrm{ell}}(M)$ stands for the set of W_0^M-orbits $\tau = (M_1, \sigma, r)$ in $T(M)$ for which r lies in the subset $\tilde{R}^M_{\sigma,\mathrm{reg}}$ of regular elements in $\tilde{R}^M_\sigma$. (That is, the space $\mathfrak{a}^r_{M_1}$ of vectors in $\mathfrak{a}_{M_1}$ left fixed by r equals $\mathfrak{a}_M$.) Any τ in the complement of $T_{\mathrm{ell}}(M)$ in $T(M)$ lies in (the image of) $T(M_1)$, for a proper Levi subgroup M_1 of M. In this case, the right hand side of (2.8) is a sum over proper subgroups of G. The descent formula therefore reduces the study of $J_M(\tau, f)$ to the case that τ is elliptic.

§ 3. Invariant distributions

The distributions we have described are not generally invariant. Their values change when they are evaluated at conjugates

$$f^y(x) = f(yxy^{-1}), \quad x, y \in G(F),$$

of a given function f in $\mathscr{C}(G(F))$. Take any group $M \in \mathscr{L}$, and let $J_M = J_M^G$ stand for either a weighted orbital integral $J_M(\gamma)$ or a weighted character $J_M(\tau)$. Then

$$J_M(f^y) = \sum_{Q \in \mathscr{F}(M)} J_M^{M_Q}(f_{Q,y}), \tag{3.1}$$

where $f_{Q,y}$ is the function

$$m \to \delta_Q(m)^{\frac{1}{2}} \int_K \int_{N_Q(F)} f(k^{-1}mnk)\, u'_Q(k,y)\, dn\, dk,$$

in $\mathscr{C}(M_Q(F))$ defined in [2], (3.3). (See [2], Lemmas 8.2 and 8.3.) We first recall the formalism by which one can attach an invariant distribution to J_M.

To account for general spaces of test functions, let us suppose that for each $S \in \mathscr{L}$, we have been given a topological vector space $C(S)$ which injects continuously into $\mathscr{C}(S(F))$. We assume that the subspaces $C(S) \subset \mathscr{C}(S(F))$ are invariant under conjugation, and also under the transformations

$$f \to f_{Q,y}, \quad f \in C(S),\ Q \in \mathscr{F}^S(M_0),\ y \in S(F).$$

The notion of an invariant distribution on $C(S)$ then makes sense. Moreover, each J_M can be regarded as a distribution on $C(G)$ for which the property (3.1) holds. We suppose in addition that we have been given a second family $\{I(S)\}$ of topological vector spaces, as well as open, continuous maps $\mathscr{T}_S: C(S) \to I(S)$ which are surjective, and which are invariant under conjugation in $C(S)$. We shall say that a continuous, conjugation-invariant map θ from $C(S)$ to some other topological vector space V is *supported on* $I(S)$ if it vanishes on the kernel of $\mathscr{T}_S$. If θ has this property, there is a unique continuous map $\hat{\theta}: I(S) \to V$ such that $\theta = \hat{\theta} \circ \mathscr{T}_S$. Of course, the most important case is when $V = \mathbb{C}$. Then θ is supported on $I(S)$ if and only if it lies in the image of the injective transpose map $\mathscr{T}'_S: I'(S) \hookrightarrow C'(S)$ between dual topological vector spaces. The distribution $\hat{\theta}$ on $I(S)$ is then equal to the inverse image of θ under $\mathscr{T}'_S$.

Having been given the spaces $\{C(S)\}$ and $\{I(S)\}$, we then suppose that we have been able to construct a family of continuous maps

$$\phi_S^{S'} : C(S') \to I(S), \quad S \subset S',$$

for which the obvious analogues of (3.1) are valid. In particular, $\phi_S = \phi_S^G$ satisfies

$$\phi_S(f^y) = \sum_{Q \in \mathscr{F}(S)} \phi_S^{M_Q}(f_{Q,y}), \quad f \in C(G),\ y \in G(F). \tag{3.2}$$

The invariant distribution $I_M = I_M^G$ attached to J_M can then be defined inductively by a formula

$$J_M(f) = \sum_{S \in \mathscr{L}(M)} \hat{I}_M^S(\phi_S(f)). \tag{3.3}$$

Part of the inductive definition includes the hypothesis that if $M \subset S \subsetneq G$, the distribution I_M^S on $C(S)$ is supported on $I(S)$. To complete the definition, one is then faced with having to show that the distribution

$$I_M(f) = J_M(f) - \sum_{\substack{S \in \mathscr{L}(M) \\ S \neq G}} \hat{I}_M^S(\phi_S(f))$$

on $C(G)$ is supported on $I(G)$.

A process similar to what we have described was carried out for the Hecke algebra in [8] (as well as in various papers on the global trace formula). We shall consider two other families of spaces here.

For the first example, we take $C(G)$ to be the Schwartz space $\mathscr{C}(G(F))$ itself. We take $I(G)$ to be the space $\mathscr{I}(G(F))$ of functions ϕ on $T(G)$ which satisfy the innocuous symmetry condition

$$\phi(z\tau) = \chi_\tau(z)^{-1}\phi(\tau), \quad z \in Z_\tau,\ \tau \in T(G),$$

and which, as W_0^G-invariant functions on $\tilde{T}(G)$, lie in $\mathscr{S}(\tilde{T}(G))$ in case F is Archimedean, and in $C_c^\infty(\tilde{T}(G))$ if F is p-adic. In the latter case, $\tilde{T}(G)$ is a disjoint union of compact tori, and $C_c^\infty(\tilde{T}(G))$ has a standard meaning. In the former case, $\tilde{T}(G)$ is a disjoint union of Euclidean spaces, on which one can define a Schwartz space. To give a more precise definition, we write $\mu_\tau = \mu_\sigma$ if $\tau = (M_1, \sigma, r)$ is any element in $\tilde{T}(G)$. Then $\mathscr{S}(\tilde{T}(G))$ is the space of smooth functions ϕ on $\tilde{T}(G)$ such that for each $L \in \mathscr{L}$, each integer n, and each invariant differential operator D_λ on $i\mathfrak{a}_L^*$, transferred in the obvious way

$$D_\tau \phi(\tau) = \lim_{\lambda \to 0} D_\lambda \phi(\tau_\lambda), \quad \tau \in \tilde{T}_{\mathrm{ell}}(L),$$

to $\tilde{T}_{\mathrm{ell}}(L)$, the semi-norm

$$\|\phi\|_{L,D_\tau,n} = \sup_{\tau \in \tilde{T}_{\mathrm{ell}}(L)} \left(|D_\tau \phi(\tau)|(1 + \|\mu_\tau\|)^n\right)$$

is finite. In each case, it is clear how to assign a topology to the space. We can actually identify $\mathscr{I}(G(F))$ with the topological vector space of functions on $\Pi_{\text{temp}}(G(F))$ defined in [2], §5, and also denoted by $\mathscr{I}(G(F))$. The passage back and forth is through the formulas

$$\phi(\tau_r) = \sum_{\varrho \in \Pi(\tilde{R}_\sigma, \chi_\sigma)} \operatorname{tr}(\varrho^\vee(r))\,\phi(\pi_\varrho),$$

and

$$\phi(\pi_\varrho) = |\tilde{R}_\sigma|^{-1} \sum_{r \in \tilde{R}_\sigma} \operatorname{tr}(\varrho(r))\,\phi(\tau_r), \quad \tau_r = (M_1, \sigma, r),$$

obtained from (2.2) and (2.3).

Given the spaces $\mathscr{C}(G(F))$ and $\mathscr{I}(G(F))$, we set

$$(\mathscr{T}_G f)(\tau) = f_G(\tau) = J_G(\tau, f), \quad \tau \in T(G),$$

for any $f \in \mathscr{C}(G(F))$. It is clear that $\mathscr{T}_G f$ is invariant under conjugation of f. It is also known [10] that $\mathscr{T}_G$ is an open, continuous and surjective map from $C(G(F))$ onto $\mathscr{I}(G(F))$. Replacing G by an arbitrary Levi subgroup, we obtain maps

$$\mathscr{T}_S : \mathscr{C}(S(F)) \to \mathscr{I}(S(F)), \quad S \in \mathscr{L},$$

with the required properties. We then define the map ϕ_S, as in earlier papers, by simply taking $\phi_S(f)$ to be the function $J_S(\tau, f)$ of $\tau \in T(S)$. According to [2], Corollary 9.2, ϕ_S maps $\mathscr{C}(G(F))$ continuously to $\mathscr{I}(L(F))$. (The proof relies on the estimate [2], (7.6), which is the inequality (R.8) we established in Lemma 2.1.) Moreover, the required formula (3.2) follows from the corresponding property (3.1) for $J_S(\tau, f)$. By the general construction above, then, we obtain a family of invariant distributions $\{I_M(\gamma)\}$ on $\mathscr{C}(G(F))$ from the weighted orbital integrals $\{J_M(\gamma)\}$. These distributions are quite complicated and contain interesting information. The invariant distributions $\{I_M(\tau)\}$ corresponding to the weighted characters $\{J_M(\tau)\}$, on the other hand, are trivial. It follows inductively from (3.3) that $I_M(\tau)$ vanishes if $M \neq G$, while $I_G(\tau, f)$ simply equals $f_G(\tau)$.

For purposes of comparison, we shall restate the construction for this example as a formal definition.

Definition 3.1. If

$$\phi_S : \mathscr{C}(G(F)) \to \mathscr{I}(S(F)), \quad S \in \mathscr{L},$$

is the map whose value at any $f \in \mathscr{C}(G(F))$ is the function

$$\phi_S(f) : \tau \to J_S(\tau, f), \quad \tau \in T(S), \tag{3.4}$$

we define invariant distributions

$$I_M(\gamma) = I_M^G(\gamma), \quad \gamma \in M(F) \cap G_{\text{reg}}(F),$$

on $\mathscr{C}(G(F))$ inductively by

$$I_M(\gamma, f) = J_M(\gamma, f) - \sum_{\substack{S \in \mathscr{L}(M) \\ S \neq G}} \hat{I}_M^S(\gamma, \phi_S(f)) . \quad \square \tag{3.5}$$

In § 5 we shall complete the inductive definition by showing that $I_M(\gamma)$ is supported on $\mathscr{I}(G(F))$.

The second example will be dual to the first one, in that we shall interchange the roles of conjugacy classes and characters. However, $J_S(\gamma, f)$ is a badly behaved function of γ near the singular set. It is not generally the invariant orbital integral of a Schwartz function on $S(F)$. To get around this difficulty, we take $C(G)$ to be $C_c^\infty(G_{\mathrm{reg}}(F))$, the space of smooth compactly supported functions on the open set of regular elements in $G(F)$, equipped with the usual topology. We take $I(G)$ to be the space $I_c^\infty(G_{\mathrm{reg}}(F))$ of class functions on $\Gamma(G(F))$ whose restrictions to any maximal torus T of G over F are smooth functions of compact support on

$$T_{\mathrm{reg}}(F) = T(F) \cap G_{\mathrm{reg}}(F) .$$

The topology on $I_c^\infty(G_{\mathrm{reg}}(F))$ is defined by the topologies on the various spaces $C_c^\infty(T_{\mathrm{reg}} F))$.

Given the spaces $C_c^\infty(G_{\mathrm{reg}}(F))$ and $I_c^\infty(G_{\mathrm{reg}}(F))$, we set

$$(\mathscr{T}_G g)(\gamma) = g_G(\gamma) = J_G(\gamma, g), \quad \gamma \in \Gamma(G(F)) ,$$

for any $g \in C_c^\infty(G_{\mathrm{reg}}(F))$. It follows easily from the fact that $G_{\mathrm{reg}}(F)$ is a finite union of fibre bundles over spaces $T_{\mathrm{reg}}(F)$ that $\mathscr{T}_G$ is an open, continuous and surjective map from $C_c^\infty(G_{\mathrm{reg}}(F))$ onto $I_c^\infty(G_{\mathrm{reg}}(F))$. We therefore obtain maps

$$\mathscr{T}_S : C_c^\infty(S_{\mathrm{reg}}(F)) \to I_c^\infty(S_{\mathrm{reg}}(F)), \quad S \in \mathscr{L} ,$$

with the required properties. We then define the map ϕ_S by taking $\phi_S(g)$ to be the function $J_S(\gamma, g)$ of $\gamma \in \Gamma(S(F)) \cap G_{\mathrm{reg}}(F)$. The smoothness properties of weighted orbital integrals imply that ϕ_S maps $C_c^\infty(G_{\mathrm{reg}}(F))$ continuously to $I_c^\infty(S_{\mathrm{reg}}(F))$. Moreover, the required formula (3.2) again follows from the corresponding property (3.1) for $J_S(\gamma, g)$. It will help to maintain the distinction between the two examples if we write L here instead of M. Then our general construction yields a family of invariant distributions $\{I_L(\tau)\}$ on $C_c^\infty(G_{\mathrm{reg}}(F))$ from the weighted characters $\{J_L(\tau)\}$. These are the distributions which in this second example are complicated. The invariant distributions $\{I_L(\gamma)\}$ corresponding to weighted orbital integrals $\{J_L(\gamma)\}$ are the ones which are trivial. It follows inductively from (3.3) that $I_L(\gamma)$ vanishes if $L \neq G$, while $I_G(\gamma, g)$ simply equals $g_G(\gamma)$.

In summary, we have

Definition 3.2. If

$$\phi_S : C_c^\infty(G_{\mathrm{reg}}(F)) \to I_c^\infty(S_{\mathrm{reg}}(F)), \quad S \in \mathscr{L} ,$$

is the map whose value at any $g \in C_c^\infty(G_{\mathrm{reg}}(F))$ is the function

$$(3.4)^\vee \qquad \phi_S(g): \gamma \to J_S(\gamma, g), \quad \gamma \in \Gamma(S(F)) \cap G_{\mathrm{reg}}(F),$$

we define invariant distributions

$$I_L(\tau) = I_L^G(\tau), \quad \tau \in T(L),$$

on $C_c^\infty(G_{\mathrm{reg}}(F))$ inductively by

$$(3.5)^\vee \qquad I_L(\tau, g) = J_L(\tau, g) - \sum_{\substack{S \in \mathscr{L}(L) \\ S \neq G}} \hat{I}_L^S(\tau, \phi_S(g)). \quad \square$$

We shall complete this inductive definition in § 5 as well, by showing that $I_L(\tau)$ is supported on $I_c^\infty(G_{\mathrm{reg}}(F))$.

Remarks. 1. It would actually be easy to show directly that $I_L(\tau)$ is supported on $I_c^\infty(G_{\mathrm{reg}}(F))$. If $F = \mathbb{R}$, for example, this would follow easily from a much more general result of Bouaziz [12], Corollaire 3.3.2(a). If F is p-adic, the result could be deduced from [21], Proposition 4. The induction hypothesis of Definition 3.2 is therefore simpler than that of Definition 3.1. However, for the sake of symmetry, we may as well complete the two inductive definitions together.

2. The distributions $I_M(\tau)$ of Definition 3.2 have been studied in slightly different guise by Labesse [23]. For if ϕ is any function in $I_c^\infty(G_{\mathrm{reg}}(F))$, there is an $f \in C_c^\infty(G(F))$ such that

$$J_S(\gamma, f) = \begin{cases} \phi(\gamma), & \text{if } S = G, \\ 0, & \text{if } S \subsetneqq G. \end{cases}$$

(See the proof of [23], Lemma 2.1.) One obtains

$$\hat{I}_L(\tau, \phi) = J_L(\tau, f)$$

for any such f. The results of Labesse show that these distributions are natural objects to use in comparison of global trace formulas.

There is some overlap in the notation between the two examples. Thus, ϕ_S stands for the two different maps (3.4) and $(3.4)^\vee$. In future contexts, we shall generally specify the meaning of any ambiguous notation. However, unless stated otherwise, $\{I_M(\gamma)\}$ will stand for the distributions of Definition 3.1, while $\{I_L(\tau)\}$ will stand for the distributions of Definition 3.2.

The invariant distributions $I_M(\gamma)$ and $I_L(\tau)$ inherit symmetry properties from $J_M(\gamma)$ and $J_L(\tau)$. It follows easily from (1.4) and (2.7) that

$$(3.6) \qquad I_{wM}(w\gamma, f) = I_M(\gamma, f), \quad w \in W_0^G,$$

and

$$(3.6)^\vee \qquad I_{wL}(wz\tau, g) = \chi_\tau(z)^{-1} I_L(\tau, g), \quad w \in W_0^G, \ z \in Z_\tau.$$

Moreover, the descent properties (1.5) and (2.8) are also reflected in the invariant distributions. If γ lies in a Levi subgroup M_1 of M, the proof of [4], Corollary 8.3 yields a formula

$$(3.7)\qquad I_M(\gamma,f) = \sum_{S\in\mathscr{L}(M_1)} d_{M_1}^G(M,S)\hat{I}_{M_1}^S(\gamma,f_S),\quad f\in\mathscr{C}(G(F)).$$

As in the notation (2.6) of §2, f_S is the image in $\mathscr{I}(S(F))$ of any of the functions f_Q, $Q\in\mathscr{P}(S)$. If τ lies in the image of $T(L_1)$ in $T(L)$, for a Levi subgroup L_1 of L, we can apply a similar argument to $I_L(\tau,g)$. The formula is

$$(3.7)^\vee\qquad I_L(\tau,g) = \sum_{S\in\mathscr{L}(L_1)} d_{L_1}^G(L,S)\hat{I}_{L_1}^S(\tau,g_S),\quad g\in C_c^\infty(G_{\mathrm{reg}}(F)),$$

where, as in the notation (1.3) of §1, g_S is the image in $I_c^\infty(S_{\mathrm{reg}}(F))$ of any of the functions f_Q.

§ 4. Statement of three theorems

We can now give a precise description of the main results. We shall state three theorems on the distributions $\{I_M(\gamma)\}$ and $\{I_L(\tau)\}$, together with two corollaries on the noninvariant distributions $\{J_M(\gamma)\}$ and $\{J_L(\tau)\}$. We shall then make some general remarks, interpreting some aspects of the results, before we begin to discuss the proofs.

Notice that both families of invariant distributions depend on choices of normalizations for the intertwining operators. For the first family $\{I_M(\gamma)\}$, the dependence is through the maps $\phi_S(f)$ defined by (3.4). For the second family $\{I_L(\tau)\}$, it is through the noninvariant distribution $J_L(\tau)$ which occurs on the right hand side of the definition $(3.5)^\vee$. We are going to impose a restriction on these two possible choices. We require that the two families of normalizing factors be complementary in the sense of Lemma 2.2. In other words, if $\{r_{P'|P}(\pi_\lambda)\}$ is one family, the second family must be $\{r_{P'|P}^\vee(\pi_\lambda) = r_{P|P'}(\pi_\lambda)\}$.

As in [8], §3, we shall write $T_{\mathrm{disc}}(G)$ for the set of W_0^G-orbits

$$(L_1,\sigma,r),\quad L_1\in\mathscr{L},\ \sigma\in\Pi_2(L_1(F)),\ r\in\tilde{R}_\sigma,$$

in $T(G)$ for which the set

$$W_\sigma(r)_{\mathrm{reg}} = \{w\in W_\sigma(r) : \mathfrak{a}_{L_1}^w = \mathfrak{a}_G\}$$

of regular elements in $W_\sigma(r)$ is nonempty. (Recall [8], §3 that $W_\sigma(r) = W_\sigma^0\cdot r$ is the subset of elements in $W(\mathfrak{a}_{L_1})$ which stabilize σ and which have the same projection onto the R-group as r. For any w in this set, we write $\varepsilon_\sigma(w)$ for the sign of the element wr^{-1} in the Weyl group W_σ^0.) The function

$$i(\tau) = i^G(\tau) = |W_\sigma^0|^{-1}\sum_{w\in W_\sigma(r)_{\mathrm{reg}}}\varepsilon_\sigma(w)|\det(1-w)_{\mathfrak{a}_{L_1}/\mathfrak{a}_G}|^{-1},$$

defined on elements $\tau = (L_1, \sigma, r)$ in $T_{\mathrm{disc}}(G)$, will play an interesting role in our results. Observe that $T_{\mathrm{disc}}(G)$ contains the set $T_{\mathrm{ell}}(G)$ of elliptic triplets. As we noted in [8], § 3, the group W_σ^0 is trivial if τ lies in $T_{\mathrm{ell}}(G)$. In this case $i(\tau)$ equals the positive number $|d(\tau)|^{-1}$, where

$$d(\tau) = d^G(\tau) = \det(1-r)_{\mathfrak{a}_{L_1}/\mathfrak{a}_G}.$$

In general, $T_{\mathrm{disc}}(G)$ is a countable disjoint union of orbits under the group $i\mathfrak{a}^*_{G,F}$. We shall use the measure $d\tau$ on $T_{\mathrm{disc}}(G)$ defined in [8], (3.5) by

$$\int_{T_{\mathrm{disc}}(G)} \theta(\tau)\,d\tau = \sum_{\tau \in T_{\mathrm{disc}}(G)/i\mathfrak{a}^*_G} |\tilde{R}_{\sigma,r}|^{-1} |\mathfrak{a}^\vee_{G,\sigma}/\mathfrak{a}^\vee_{G,F}|^{-1} \int_{i\mathfrak{a}^*_{G,F}} \theta(\tau_\lambda)\,d\lambda,$$

where θ is any function in $C_c(T_{\mathrm{disc}}(G))$, and $\tilde{R}_{\sigma,r}$ is the centralizer of r in $\tilde{R}_\sigma$.

Theorem 4.1. *The invariant distribution $I_M(\gamma)$ on $\mathscr{C}(G(F))$ has an expansion*

$$I_M(\gamma, f) = \sum_{L \in \mathscr{L}} |W_0^L| |W_0^G|^{-1} \int_{T_{\mathrm{disc}}(L)} I_M(\gamma, \tau) f_L(\tau)\,d\tau, \tag{4.1}$$

for a smooth function

$$I_M(\gamma, \tau) = I_M^G(\gamma, \tau), \quad \gamma \in \Gamma(M(F)) \cap G_{\mathrm{reg}}(F),\ \tau \in T_{\mathrm{disc}}(L),$$

which satisfies the symmetry condition

$$I_{wM}(w\gamma, z w^\vee \tau) = \chi_\tau(z) I_M(\gamma, \tau), \quad z \in Z_\tau,\ w, w^\vee \in W_0^G. \tag{4.2}$$

If γ lies in a Levi subgroup M_1 of M, the function also satisfies a descent condition

$$I_M(\gamma, \tau) = \sum_{S \in \mathscr{L}(M_1)} d_{M_1}^G(M, S) \Big(\sum_{\substack{w \in W_0^S \backslash W_0^G \\ wL \subset S}} I_{M_1}^S(\gamma, w\tau) \Big). \tag{4.3}$$

Finally, if F is Archimedean, the function satisfies a growth condition

$$|D_\lambda D_\gamma I_M(\gamma, \tau)| \leqq c(\gamma)(1 + \|\mu_\tau\|)^n, \tag{4.4}$$

where n is a positive integer and $c(\gamma)$ is a locally bounded function on $\Gamma(M(F)) \cap G_{\mathrm{reg}}(F)$, both depending on a given pair of invariant differential operators D_γ and D_τ transferred from $A_M(F)$ and $i\mathfrak{a}^*_L$ respectively.

Corollary 4.2. *The weighted orbital integral $J_M(\gamma)$ on $\mathscr{C}(G(F))$ has an expansion*

$$J_M(\gamma, f) = \sum_{L \in \mathscr{L}} \sum_{S \in \mathscr{L}(L) \cap \mathscr{L}(M)} \int_{T_{\mathrm{disc}}(L)} |W_0^L| |W_0^S|^{-1} I_M^S(\gamma, \tau) J_S(\tau, f)\,d\tau. \tag{4.5}$$

Recall that $\Gamma_{\mathrm{ell}}(G(F))$ denotes the set of elliptic conjugacy classes in $G(F)$. We shall use the measure $d\gamma$ on $\Gamma_{\mathrm{ell}}(G(F))$ defined in [8], §1 by

$$\int_{\Gamma_{\mathrm{ell}}(G(F))} \phi(\gamma)\,d\gamma = \sum_{\{T\}} |W(G(F), T(F))|^{-1} \int_{T(F)} \phi(t)\,dt, \quad \phi \in C_c(\Gamma_{\mathrm{ell}}(G(F))),$$

where $\{T\}$ is a set of representatives of $G(F)$-conjugacy classes of elliptic maximal tori in G over F, and $W(G(F), T(F))$ is the Weyl group of $(G(F), T(F))$.

Theorem 4.3. *The invariant distribution $I_L(\tau)$ on $C_c^\infty(G_{\mathrm{reg}}(F))$ has an expansion*

$$(4.1)^\vee \qquad I_L(\tau, g) = \sum_{M \in \mathscr{L}} |W_0^M| |W_0^G|^{-1} \int_{\Gamma_{\mathrm{ell}}(M(F))} I_L(\tau, \gamma) g_M(\gamma) \, d\gamma ,$$

for a smooth function

$$I_L(\tau, \gamma) = I_L^G(\tau, \gamma), \quad \tau \in T(L),\ \gamma \in \Gamma_{\mathrm{ell}}(M(F)) \cap G_{\mathrm{reg}}(F),$$

which satisfies the symmetry condition

$$(4.2)^\vee \qquad I_{w^\vee L}(z w^\vee \tau, w\gamma) = \chi_\tau(z)^{-1} I_L(\tau, \gamma), \quad z \in Z_\tau,\ w^\vee,\ w \in W_0^G .$$

The function also satisfies a descent condition

$$(4.3)^\vee \qquad I_L(\tau, \gamma) = \sum_{S \in \mathscr{L}(L_1)} d_{L_1}^G(L, S) \Big(\sum_{\substack{w \in W_0^S \backslash W_0^G \\ wL \subset S}} I_{L_1}^S(\tau, w\gamma) \Big),$$

if τ lies in $T(L_1)$ for a Levi subgroup L_1 of L, and a growth condition

$$(4.4)^\vee \qquad |D_\tau D_\gamma I_L(\tau, \gamma)| \leqq c(\gamma)(1 + \|\mu_\tau\|)^n ,$$

if F is Archimedean and D_τ, D_γ, $c(\gamma)$ and n are as in (4.4).

Corollary 4.4. *The weighted character $J_L(\tau)$ on $C_c^\infty(G_{\mathrm{reg}}(F))$ has an expansion*

$$(4.5)^\vee \quad J_L(\tau, g) = \sum_{M \in \mathscr{L}} \int_{\Gamma_{\mathrm{ell}}(M(F))} \sum_{S \in \mathscr{L}(M) \cap \mathscr{L}(L)} |W_0^M| |W_0^S|^{-1} I_L^S(\tau, \gamma) J_S(\gamma, g) \, d\gamma .$$

The structure of the two theorems we have stated is obviously parallel. Our third theorem asserts that the connection is more than formal.

Theorem 4.5. *The functions defined by Theorems* 4.1 *and* 4.3 *satisfy a reciprocity relation*

$$(4.6) \qquad I_M(\gamma, \tau) = (-1)^{\dim(A_M \times A_L)} i^L(\tau) I_L(\tau^\vee, \gamma),$$

for any pair of points $\gamma \in \Gamma_{\mathrm{ell}}(M(F)) \cap G_{\mathrm{reg}}(F)$ and $\tau \in T_{\mathrm{disc}}(L)$.

Remarks. 1. The function $I_M(\gamma, \tau)$ of Theorem 4.1 will be uniquely determined by the expansion (4.1) and the part of the symmetry condition (4.2) that applies to τ. This is because the functions $f_L(\tau)$ in the integrand of (4.1) are subject to a parallel symmetry condition, but are otherwise free to range over the natural Schwartz space of test functions on $T_{\mathrm{disc}}(L)$. Similarly, the function $I_L(\tau, \gamma)$ of Theorem 4.3 will be uniquely determined by $(4.1)^\vee$ and the part of $(4.2)^\vee$ that applies to γ.

2. If F is a p-adic field, the integrands in (4.1) and (4.5) are smooth, compactly supported functions of τ, so the integrals converge. If F is Archimedean, it is the growth condition (4.4) which insures the convergence of the integrals. As for the expressions $(4.1)^\vee$ and $(4.5)^\vee$, the integrands are smooth, compactly supported functions of γ for any F.

3. The notation in Theorem 4.1 is slightly ambiguous. We might have denoted the postulated function by $I_{M,L}(\gamma, \tau)$ instead of $I_M(\gamma, \tau)$, since the sets $\{T_{\mathrm{disc}}(L)\}$ need not be disjoint. For the purpose of (4.1), however, we have only to consider points $\tau \in T_{\mathrm{disc}}(L)$ which are G-regular in the obvious sense. Any such τ then determines L uniquely, and the notation $I_M(\gamma, \tau)$ makes more sense.

4. (a) Theorem 4.1 describes the (invariant) Fourier transform $\hat{I}_M(\gamma)$ of $I_M(\gamma)$, as a tempered distribution on the (multidimensional) manifold $T(G)$. The assertion is that this tempered distribution is a sum of C^∞-functions on the strata $\{T_{\mathrm{disc}}(L)\}$ of $T(G)$. In particular, the singular support of $\hat{I}_M(\gamma)$ is contained in the union

$$\bigcup_{L \in \mathscr{L}} (T_{\mathrm{disc}}(L) - T_{\mathrm{ell}}(L)),$$

or rather the image of this set in $T(G)$.

(b) To describe the singular values of $\hat{I}_M(\gamma)$, consider a point τ in $T_{\mathrm{disc}}(L) - T_{\mathrm{ell}}(L)$. Then there is a proper Levi subgroup L_1 of L such that τ lies in $T_{\mathrm{ell}}(L_1)$. We have noted earlier that the number $i^{L_1}(\tau)$ equals $|d^{L_1}(\tau)|^{-1}$. By combining $(4.3)^\vee$ with (4.6), we can deduce a formula

$$I_M(\gamma, \tau) = (-1)^{\dim(A_{L_1}/A_L)} i^L(\tau) |d^{L_1}(\tau)| \sum_{S \in \mathscr{L}(L_1)} d^G_{L_1}(L, S) \Big(\sum_{\substack{w \in W_0^S \backslash W_0^G \\ wM \subset S}} I^S_{wM}(w\gamma, \tau) \Big) \tag{4.7}$$

of descent relative to the second variable. In particular, the values of $I_M(\gamma, \tau)$ for general τ are determined by functions $I^S_{M'}(\gamma', \tau)$ in which τ is elliptic.

(c) Consider the special case of (4.7) in which $M = G$. Then S must equal G, and we have

$$d^G_{L_1}(L, S) = d^G_{L_1}(L, G) = \begin{cases} 0, & \text{if } L_1 \neq L, \\ 1, & \text{if } L_1 = L. \end{cases}$$

Consequently, $I_G(\gamma, \tau)$ vanishes if τ lies in the complement of $T_{\mathrm{ell}}(L)$ in $T_{\mathrm{disc}}(L)$. Therefore, the Fourier transform of the invariant orbital integral $I_G(\gamma)$ has no singular support. It can be regarded as a C^∞-function on $T(G)$.

5. Theorem 4.3 describes $I_M(\tau)$ as an invariant tempered distribution on $G_{\mathrm{reg}}(F)$. The assertion is that the distribution is actually a C^∞-function on $G_{\mathrm{reg}}(F)$. Since there is no singular support, we do not have to look for an analogue of the descent formula (4.7).

6. Corollary 4.2 can be regarded as a description of the (noninvariant) Fourier transform of a weighted orbital integral. It follows directly from (4.5) that the Fourier transform of $J_M(\gamma)$ is a sum of smooth functions on strata defined by $\{T_{\mathrm{disc}}(L)\}$. Finally, Corollary 4.4

tells us about the restriction of a weighted character to $G_{\mathrm{reg}}(F)$. It follows easily from $(4.5)^{\vee}$ that as a distribution on $G_{\mathrm{reg}}(F)$, $J_L(\tau)$ is a C^{∞}-function.

The proofs of the theorems will take up the rest of the paper. The two corollaries, on the other hand, are easy consequences of the corresponding theorems. To see this, take any group $S \in \mathscr{F}(M)$. Applying (4.1) to $I_M^S(\gamma)$, and recalling the definition (3.4) of $\phi_S(f)$, we see that

$$\hat{I}_M^S(\gamma, \phi_S(f)) = \sum_{L \in \mathscr{L}^S} |W_0^L| |W_0^S|^{-1} \int_{T_{\mathrm{disc}}(L)} I_M^S(\gamma, \tau) \phi_S(f, \tau) d\tau$$
$$= \sum_{L \in \mathscr{L}^S} |W_0^L| |W_0^S|^{-1} \int_{T_{\mathrm{disc}}(L)} I_M^S(\gamma, \tau) J_S(\tau, f) d\tau .$$

Similarly, if $S \in \mathscr{F}(L)$, we obtain a formula

$$\hat{I}_L^S(\tau, \phi_S(g)) = \sum_{M \in \mathscr{L}^S} |W_0^M| |W_0^S|^{-1} \int_{\Gamma_{\mathrm{ell}}(M(F))} I_L^S(\tau, \gamma) J_S(\gamma, g) d\gamma$$

from $(4.1)^{\vee}$ and $(3.4)^{\vee}$. The required expansions (4.5) and $(4.5)^{\vee}$ of Corollaries 4.2 and 4.4 then follow from (3.5) and $(3.5)^{\vee}$ respectively.

We shall conclude this section by observing that the symmetry and descent assertions of Theorems 4.1 and 4.3 are easy consequences of the assertions. The half of the symmetry condition (4.2) that pertains to τ is actually part of the definition of $I_M(\gamma, \tau)$. It serves to determine the function uniquely, once we have established an expansion (4.1). The other half of (4.2), that which applies to γ, follows from (3.6), (4.1) and the uniqueness of $I_M(\gamma, \tau)$. Similar remarks apply to the symmetry condition $(4.2)^{\vee}$. The descent conditions (4.3) and $(4.3)^{\vee}$ are equally straightforward. For example, we see easily from (3.7) that if $I_M(\gamma, \tau)$ is replaced by the right hand side of (4.3), the expansion (4.1) remains unchanged. But the right hand side of (4.3) is symmetric under translation of τ by Z_τ and W_0^G. It must therefore equal the uniquely determined function $I_M(\gamma, \tau)$. The formula $(4.3)^{\vee}$ follows in the same way from $(3.7)^{\vee}$.

Keep in mind that an element $\gamma \in \Gamma(M(F)) \cap G_{\mathrm{reg}}(F)$ will lie in $\Gamma_{\mathrm{ell}}(M_1(F))$, for some Levi subgroup M_1 of M. If M_1 is a proper Levi subgroup of M, we can use (4.3) to define $I_M(\gamma, \tau)$ inductively in terms of the functions $I_{M_1}^S(\gamma, w\tau)$. The growth condition (4.4), as well as the smoothness of $I_M(\gamma, \tau)$, then follows from the corresponding property for $I_{M_1}^S(\gamma, w\tau)$. Moreover, the descent formula (3.7) yields the expansion (4.1) for $I_M(\gamma, f)$ in terms of its analogues for $\hat{I}_{M_1}^S(\gamma, f_S)$. Therefore, it will be sufficient to establish what remains to be proved of Theorem 4.1 in the special case that γ lies in $\Gamma_{\mathrm{ell}}(M(F))$. Similarly, it is enough to establish the rest of Theorem 4.3 when τ lies in the subset $T_{\mathrm{ell}}(L)$ of $T(L)$. To deal with the basic cases, we shall use the local trace formula on $G(F)$.

§ 5. The local trace formula

The theorems stated in § 4 will be proved by means of the local trace formula. In the paper [8] we constructed an invariant local trace formula from the original noninvariant version [7] and the distributions of Definition 3.1. In this section we shall modify the construction by including the distributions of Definition 3.2. The result will be a different version of the invariant local trace formula, one which is particularly simple, and which is more easily applied to the theorems of § 4.

The conventions of § 4 are to remain in force for the rest of the paper. Thus, the intertwining operators implicit in the construction of the invariant distributions $I_M(\gamma)$ and $I_L(\tau)$ of Definitions 3.1 and 3.2 will be assigned complementary normalizations.

Theorem 5.1. *For any pair of functions $g \in C_c^\infty(G_{\mathrm{reg}}(F))$ and $f \in \mathscr{C}(G(F))$, the expression*

$$\sum_{M \in \mathscr{L}} |W_0^M| \, |W_0^G|^{-1} (-1)^{\dim(A_M/A_G)} \int_{\Gamma_{\mathrm{ell}}(M(F))} I_M(\gamma, f) \, g_M(\gamma) \, d\gamma \tag{5.1}$$

equals

$$\sum_{L \in \mathscr{L}} |W_0^L| \, |W_0^G|^{-1} (-1)^{\dim(A_L/A_G)} \int_{T_{\mathrm{disc}}(L)} i^L(\tau) \, I_L(\tau, g) \, f_L(\tau^\vee) \, d\tau \, . \tag{5.2}$$

Proof. We shall derive the identity from the noninvariant trace formula of [7], Theorem 12.2. The results of [7] were established only for functions in the Hecke algebra $\mathscr{H}(G(F))$ of K-finite functions in $C_c^\infty(G(F))$, so we shall have to assume for the moment that g and f lie in this space. As interpreted in [8], Proposition 4.1, the noninvariant trace formula asserts that the geometric expansion

$$\sum_{M \in \mathscr{L}} |W_0^M| \, |W_0^G|^{-1} (-1)^{\dim(A_M/A_G)} \int_{\Gamma_{\mathrm{ell}}(M(F))} J_M(\gamma, g \times f) \, d\gamma \tag{5.3}$$

equals the spectral expansion

$$\sum_{L \in \mathscr{L}} |W_0^L| \, |W_0^G|^{-1} (-1)^{\dim(A_L/A_G)} \int_{T_{\mathrm{disc}}(L)} i^L(\tau) \, J_L(\tau, g \times f) \, d\tau \, . \tag{5.4}$$

The notation here follows [7], § 12 and [8], § 4. Thus, $J_M(\gamma, g \times f)$ is a weighted orbital integral ([7], (12.2)) on the product $G(F) \times G(F)$. According to [4], Corollary 7.4, it can be decomposed in terms of the distributions (1.1) of § 1 by the splitting formula

$$J_M(\gamma, g \times f) = \sum_{M_1, M_2 \in \mathscr{L}(M)} d_M^G(M_1, M_2) \, J_M^{M_1}(\gamma, g_{Q_1}) \, J_M^{M_2}(\gamma, f_{Q_2}) \, , \tag{5.5}$$

where

$$(M_1, M_2) \to (Q_1, Q_2) \in \mathscr{P}(M_1) \times \mathscr{P}(M_2)$$

is the retraction defined in [4], pp. 357–358. Similarly, if $\tau = (L_1, \sigma, r)$, $J_L(\tau, g \times f)$ is the weighted character

$$\sum_{\varrho, \varrho' \in \Pi(\tilde{R}_\sigma^L, \chi_\sigma)} \operatorname{tr}(\varrho'(r)) \operatorname{tr}(\varrho^\vee(r)) \, J_L(\pi_{\varrho'}^\vee \otimes \pi_\varrho, g \times f)$$

on $G(F) \times G(F)$ defined in [8], § 4. This weighted character was defined in terms of *unnormalized* intertwining operators. We claim that it nonetheless has a decomposition

$$J_L(\tau, g \times f) = \sum_{L_1, L_2 \in \mathscr{L}(L)} d_L^G(L_1, L_2) \, J_L^{L_1}(\tau^\vee, g_{Q_1}) \, J_L^{L_2}(\tau, f_{Q_2}) \tag{5.6}$$

in terms of the distributions (2.1), (2.4) constructed in § 2 by means of normalized operators. The normalizations implicit in the two terms $J_L^{L_1}(\tau^\vee, f_{Q_1})$ and $J_L^{L_2}(\tau, g_{Q_2})$ are understood to be complementary.

To prove the claim, we recall from [8], § 4 that

$$J_L(\pi_1^\vee \otimes \pi_2, g \times f) = \operatorname{tr}\big(\mathscr{J}_L(\pi_1^\vee \otimes \pi_2, P)\mathscr{I}_P(\pi_1^\vee \otimes \pi_2, g \times f)\big),$$

if π_1 and π_2 are irreducible constituents of the induced representations

$$\sigma^L = \mathscr{I}_R(\sigma), \quad R \in \mathscr{P}^L(L_1),$$

of $L(F)$. Here, $\mathscr{J}_L(\pi_1^\vee \otimes \pi_2, P)$ is the operator

$$\lim_{\Lambda \to 0} \sum_{Q \in \mathscr{P}(L)} \mathscr{J}_Q(\Lambda, \pi_1^\vee \otimes \pi_2, P)\theta_Q(\Lambda)^{-1}$$

obtained from the (G, L)-family

$$\mathscr{J}_Q(\Lambda, \pi_1^\vee \otimes \pi_2, P) = \big(J_{\bar{Q}|P}(\pi_1^\vee) \otimes J_{Q|P}(\pi_2)\big)^{-1}\big(J_{\bar{Q}|P}(\pi_{1,-\Lambda}^\vee) \otimes J_{Q|P}(\pi_{2,\Lambda})\big),$$

$\Lambda \in i\mathfrak{a}_L^*$, $Q \in \mathscr{P}(L)$. Let $\{\mathscr{R}_Q(\Lambda, \pi_1^\vee \otimes \pi_2, P)\}$ be a second (G, L)-family obtained by assigning complementary normalizations to the intertwining operators. That is,

$$\mathscr{J}_Q(\Lambda, \pi_1^\vee \otimes \pi_2, P) = r_Q(\Lambda, \pi_1^\vee \otimes \pi_2, P)\mathscr{R}_Q(\Lambda, \pi_1^\vee \otimes \pi_2, P),$$

where

$$r_Q(\Lambda, \pi_1^\vee \otimes \pi_2, P) = \big(r_{\bar{Q}|P}^\vee(\pi_1^\vee) r_{Q|P}(\pi_2)\big)^{-1}\big(r_{\bar{Q}|P}^\vee(\pi_{1,-\Lambda}^\vee) r_{Q|P}(\pi_{2,\Lambda})\big).$$

Now

$$r_{\bar{Q}|P}^\vee(\pi_1^\vee) = r_{P|\bar{Q}}(\pi_1^\vee) = r_{\bar{Q}|P}(\pi_1) = r_{\bar{P}|Q}(\pi_1),$$

by the properties (r. 5) and (r. 1) of the normalizing factors. Since π_1 and π_2 are irreducible constituents of σ^L, the corresponding normalizing factors are the same as the ones for σ^L. Therefore

$$r_{\bar{Q}|P}^\vee(\pi_1^\vee) r_{Q|P}(\pi_2) = r_{\bar{P}|Q}(\sigma^L) r_{Q|P}(\sigma^L) = r_{\bar{P}|P}(\sigma^L).$$

It follows that

$$r_Q(\Lambda, \pi_1^\vee \otimes \pi_2, P) = r_{\bar{P}|P}(\sigma^L)^{-1} r_{\bar{P}|P}(\sigma_\Lambda^L),$$

a function which is independent of Q, and which equals 1 at $\Lambda = 0$. We conclude that $\mathscr{J}_L(\pi_1^\vee \otimes \pi_2, P)$ equals

$$\mathscr{R}_L(\pi_1^\vee \otimes \pi_2, P) = \lim_{\Lambda \to 0} \sum_{Q \in \mathscr{P}(L)} \mathscr{R}_Q(\Lambda, \pi_1^\vee \otimes \pi_2, P)\theta_Q(\Lambda)^{-1}.$$

In other words, the weighted character $J_L(\pi_1^\vee \otimes \pi_2, g \times f)$ above may also be constructed from normalized intertwining operators, provided that we use complementary normalizations for π_1 and π_2. Since $\mathcal{R}_Q(\Lambda, \pi_1^\vee \otimes \pi_2, P)$ is a product of (G, L)-families, the splitting formula [4], Corollary 7.4 gives us a decomposition

$$J_L(\pi_1^\vee \otimes \pi_2, g \times f) = \sum_{L_1, L_2 \in \mathcal{L}(L)} d_L^G(L_1, L_2) J_L^{L_1}(\pi_1^\vee, g_{Q_1}) J_L^{L_2}(\pi_2, f_{Q_2})$$

into distributions constructed in § 2. The required formula (5.6) then follows from the definition (2.4).

Before we go on, we shall show that the identity of (5.3) with (5.4) remains valid if g and f are taken from the given spaces $C_c^\infty(G_{\mathrm{reg}}(F))$ and $\mathscr{C}(G(F))$ rather than from $\mathscr{H}(G(F))$. In fact, the identity holds for any pair of Schwartz functions. We have only to show that (5.3) and (5.4) both extend to continuous bilinear forms on $\mathscr{C}(G(F))$. To deal with (5.4) first, consider the splitting formula (5.6). The integrand in (5.4) is the product of $i^L(\tau)$ with a finite linear combination of products

$$J_L^{L_1}(\tau^\vee, g_{Q_1}) J_L^{L_2}(\tau, f_{Q_2}),$$

each of which we can write as

$$\phi_L^{L_1}(g_{Q_1}, \tau^\vee) \phi_L^{L_2}(f_{Q_2}, \tau)$$

in the notation (3.4). As we have already noted in § 3, ϕ_L maps $\mathscr{C}(G(F))$ continuously to $\mathscr{I}(L(F))$. It follows easily from the definition of $\mathscr{I}(L(F))$ that

$$(g, f) \to \int_{T_{\mathrm{disc}}(L)} i^L(\tau) \phi_L^{L_1}(g_{Q_1}, \tau^\vee) \phi_L^{L_2}(f_{Q_2}, \tau) \, d\tau$$

is a continuous bilinear form on $\mathscr{C}(G(F))$. The same is therefore true of (5.4). As for (5.3), we use the other splitting formula (5.5) to write the integrand as a finite linear combination of products

$$J_M^{M_1}(\gamma, g_{Q_1}) J_M^{M_2}(\gamma, f_{Q_2}).$$

If F is Archimedean, [1], Corollary 7.4 gives an estimate

$$(5.7) \qquad |J_M(\gamma, f)| \leqq \nu(f)(1 + |\log|D(\gamma)||)^p (1 + \|H_M(\gamma)\|)^{-n}, \quad f \in \mathscr{C}(G(F)),$$

where ν is a continuous semi-norm on $\mathscr{C}(G(F))$ which depends on an arbitrarily chosen positive integer n. A similar estimate holds if F is p-adic. One needs the Howe conjecture [14], § 2, Corollary 2 to handle the case that $M = G$, but the proof is otherwise the same as that of [1], Corollary 7.4. It is easy to see that for any $p \in \mathbb{R}$, the function $(1 + |\log|D(\gamma)||)^p$ is locally integrable and tempered on any maximal torus in $M(F)$. It follows without difficulty that

$$(g, f) \to \int_{\Gamma_{\mathrm{ell}}(M(F))} J_M^{M_1}(\gamma, g_{Q_1}) J_M^{M_2}(\gamma, f_{Q_2}) \, d\gamma$$

is a continuous bilinear form on $\mathscr{C}(G(F))$. The same is therefore true of (5.3). Since $\mathscr{H}(G(F))$ is dense in $\mathscr{C}(G(F))$, the identity of (5.3) with (5.4) holds if g and f are Schwartz functions.

Our main task is to derive an invariant formula by converting the terms in (5.3) and (5.4) into invariant distributions. We shall follow the general procedure of § 3, but with maps ϕ_S acting on functions on the product $G(F) \times G(F)$. Instead of defining these maps purely by weighted characters, as was done in [8], § 4, we shall put the dual maps (3.4) and $(3.4)^\vee$ together.

If $S \in \mathscr{L}$, we must construct a map ϕ_S from the (algebraic) tensor product

$$C_c^\infty(G_{\mathrm{reg}}(F)) \otimes \mathscr{C}(G(F)) \quad \text{into} \quad I_c^\infty(S_{\mathrm{reg}}(F)) \otimes \mathscr{I}(S(F)).$$

Turning once again to the splitting formula [4], Corollary 7.4, this time for motivation, we define

$$\phi_S(g \times f), \quad g \in C_c^\infty(G_{\mathrm{reg}}), f \in \mathscr{C}(G(F)),$$

to be the function

$$\phi_S(g \times f, \gamma \times \tau) = \sum_{S_1, S_2 \in \mathscr{L}(S)} d_S^G(S_1, S_2) J_S^{S_1}(\gamma, g_{Q_1}) J_S^{S_2}(\tau, f_{Q_2})$$

of $(\gamma \times \tau)$ in $\big(\Gamma(S(F)) \cap S_{\mathrm{reg}}(F)\big) \times T(S)$. The splitting formula, in fact, implies that we could have equally well expressed $\phi_S(g \times f, \gamma \times \tau)$ directly in terms of the product of the two (G, S)-families that occur in the definitions of $J_S(\gamma)$ and $J_S(\tau)$. This second formulation leads directly to the expansion of

$$\phi_S((g \times f)^y), \quad y \in G(F) \times G(F),$$

which is analogous to (3.2). Since similar expansions hold for conjugates $J_M(\gamma, (g \times f)^y)$ and $J_L(\tau, (g \times f)^y)$ of the distributions in (5.3) and (5.4), we can apply the general construction in § 3. We define invariant distributions $I_L(\tau) = I_L^G(\tau)$ and $I_M(\gamma) = I_M^G(\gamma)$ on

$$C_c^\infty(G_{\mathrm{reg}}(F)) \otimes \mathscr{C}(G(F))$$

inductively by the prescriptions

$$I_M(\gamma, g \times f) = J_M(\gamma, g \times f) - \sum_{\substack{S \in \mathscr{L}(M) \\ S \neq G}} \hat{I}_M^S(\gamma, \phi_S(g \times f)), \tag{5.8}$$

and

$$I_L(\tau, g \times f) = J_L(\tau, g \times f) - \sum_{\substack{S \in \mathscr{L}(L) \\ S \neq G}} \hat{I}_L^S(\tau, \phi_S(g \times f)). \tag{5.9}$$

Of course we have to assume inductively that if $S \subsetneq G$, then $I_M^S(\gamma)$ and $I_L^S(\tau)$ are supported on $I_c^\infty(S_{\mathrm{reg}}(F)) \otimes \mathscr{I}(S(F))$. We shall presently establish that the same property holds if $S = G$.

We claim that the invariant versions

$$\sum_{M \in \mathscr{L}} |W_0^M| |W_0^G|^{-1} (-1)^{\dim(A_M/A_G)} \int_{\Gamma_{\mathrm{ell}}(M(F))} I_M(\gamma, g \times f) \, d\gamma \tag{5.10}$$

and

$$(5.11) \qquad \sum_{L \in \mathscr{L}} |W_0^L| |W_0^G|^{-1} (-1)^{\dim(A_L/A_G)} \int_{T_{\mathrm{disc}}(L)} i^L(\tau) I_L(\tau, g \times f)\, d\gamma$$

of (5.3) and (5.4) respectively, are equal to each other. Assume inductively that this is so if G is replaced by any group of smaller dimension. Writing $I^G_{\mathrm{geom}}(g \times f)$ and $J^G_{\mathrm{geom}}(g \times f)$ for the respective expressions (5.10) and (5.3), and substituting the expansion (5.8) into (5.10), we see that

$$I^G_{\mathrm{geom}}(g \times f) = J^G_{\mathrm{geom}}(g \times f) - \sum_{S \subsetneq G} |W_0^S| |W_0^G|^{-1} (-1)^{\dim(A_S/A_G)} \hat{I}^S_{\mathrm{geom}}(\phi_S(g \times f)) .$$

Similarly, we have a parallel relation

$$I^G_{\mathrm{spec}}(g \times f) = J^G_{\mathrm{spec}}(g \times f) - \sum_{S \subsetneq G} |W_0^S| |W_0^G|^{-1} (-1)^{\dim(A_S/A_G)} \hat{I}^S_{\mathrm{spec}}(\phi_S(g \times f))$$

between the spectral expressions (5.11) and (5.4). Since $J^S_{\mathrm{geom}}(g \times f)$ equals $J^G_{\mathrm{spec}}(g \times f)$, the claim follows from the induction assumption.

Invariant distributions on $G(F) \times G(F)$ such as those in (5.10) and (5.11) are also subject to a splitting formula. The usual version [4], Proposition 9.1 is for the case that the map ϕ_S is defined entirely by weighted characters as in (3.4). However, the same proof applies if ϕ_S is defined entirely by weighted orbital integrals as in $(3.4)^\vee$, or by a combination of the two, as is the case here. The only requirement is that ϕ_S itself have the splitting property

$$\phi_S(g \times f) = \sum_{S_1, S_2 \in \mathscr{L}(S)} d_S^G(S_1, S_2)\, \phi_S^{S_1}(g_{Q_1})\, \phi_S^{S_2}(f_{Q_2}) ,$$

which in the present situation was actually part of the definition. The splitting property (5.5) for the noninvariant distribution $J_M(\gamma)$ then leads to the formula

$$(5.12) \qquad I_M(\gamma, g \times f) = \sum_{M_1, M_2 \in \mathscr{L}(M)} d_M^G(M_1, M_2)\, \hat{I}_M^{M_1}(\gamma, g_{M_1})\, \hat{I}_M^{M_2}(\gamma, f_{M_2})$$

for the integrand in (5.10). Similarly, the splitting property (5.6) for the noninvariant distribution $J_L(\tau)$ leads to the formula

$$(5.13) \qquad I_L(\tau, g \times f) = \sum_{L_1, L_2 \in \mathscr{L}(L)} d_L^G(L_1, L_2)\, \hat{I}_L^{L_1}(\tau^\vee, g_{L_1})\, \hat{I}_L^{L_2}(\tau, f_{L_2})$$

for the integrand in (5.11). Now the invariant distributions $\hat{I}_M^{M_1}(\gamma, g_{M_1})$ in (5.12) are defined by means of the maps $(3.4)^\vee$. They are essentially trivial. According to the remarks preceding Definition 3.2, $\hat{I}_M^{M_1}(\gamma, g_{M_1})$ vanishes unless $M_1 = M$, in which case it equals $g_M(\gamma)$. Moreover, if $M_1 = M$, the constant $d_M^G(M_1, M_2)$ vanishes unless $M_2 = G$, in which case it equals 1. The formula (5.12) reduces to

$$(5.14) \qquad I_M(\gamma, g \times f) = I_M(\gamma, f)\, g_M(\gamma) .$$

Substituting this into the geometric expansion (5.10), we obtain the original expression (5.1). In (5.13) it is the distributions $\hat{I}_L^{L_2}(\tau, f_{L_2})$ which are essentially trivial. According to the remarks preceding Definition 3.1, $\hat{I}_L^{L_2}(\tau, f_{L_2})$ vanishes unless $L_2 = L$, in which case it equals $f_L(\tau)$. The formula (5.13) reduces to

$$I_L(\tau, g \times f) = I_L(\tau^\vee, g) f_L(\tau) . \tag{5.15}$$

Substituting this into the spectral expansion (5.11), and changing the variable of summation from τ to $\tau^\vee$, we obtain the original expression (5.2). The given expressions (5.1) and (5.2) are therefore equal.

We have proved the theorem, except to complete the induction argument. What remains is to show that the invariant distributions (5.8) and (5.9) are supported on $I_c^\infty(G_{\mathrm{reg}}(F)) \otimes \mathscr{I}(G(F))$. From the splitting formulas (5.14) and (5.15), we see that it is enough to show that the distributions $I_M(\gamma, f)$ and $I_L(\tau, g)$ are supported on $\mathscr{I}(G(F))$ and $I_c^\infty(G_{\mathrm{reg}}(F))$ respectively. This is the induction assumption we have been carrying since § 3. We shall establish it now from the identity of (5.1) and (5.2) we have just proved.

Suppose first that $f \in \mathscr{C}(G(F))$ is such that the function f_G in $\mathscr{I}(G(F))$ vanishes. We shall show that $I_M(\gamma, f) = 0$ for any M and any point $\gamma \in \Gamma(M(F)) \cap G_{\mathrm{reg}}(F)$. The descent formula (3.7) together with our induction hypothesis reduces the problem immediately to the case that γ lies in $\Gamma_{\mathrm{ell}}(M(F)) \cap G_{\mathrm{reg}}(F)$. Now the condition f means that $f_L(\tau^\vee) = 0$ for every $L \in \mathscr{L}$ and $\tau \in T_{\mathrm{disc}}(L)$. The expression (5.2) then vanishes, and so therefore does (5.1). Choose $g \in C_c^\infty(G_{\mathrm{reg}}(F))$ such that the function $g_G \in I_c^\infty(G_{\mathrm{reg}}(F))$ approaches the delta function at (the image in $\Gamma(G(F))$ of) a fixed G-regular class γ in $\Gamma_{\mathrm{ell}}(M(F))$. The expression (5.1) then converges to a nonzero multiple of $I_M(\gamma, f)$ as we can plainly see from (1.3) and (1.4). It follows that $I_M(\gamma, f)$ vanishes.

Similarly, suppose that $g \in C_c^\infty(G_{\mathrm{reg}}(F))$ is such that the function g_G in $I_c^\infty(G_{\mathrm{reg}}(F))$ vanishes. We shall show that $I_L(\tau, g) = 0$ for any L and any $\tau \in T(L)$. Again, by the descent formula $(3.7)^\vee$ and the induction hypothesis, we can assume that $I_L(\tau, g)$ vanishes if τ lies in the complement of $T_{\mathrm{ell}}(L)$. In particular, the integral in (5.2) can be taken over the subset $T_{\mathrm{ell}}(L)$ of $T_{\mathrm{disc}}(L)$. The condition on g implies that $g_M(\gamma) = 0$ for every M and γ. The expression (5.1) then vanishes, and so therefore does (5.2). Choose $f \in \mathscr{C}(G(F))$ such that the function $f_G \in \mathscr{I}(G(F))$ approaches the (Z_τ-equivariant) delta function at (the image in $T(G)$ of) a fixed point $\tau^\vee \in T_{\mathrm{ell}}(L)$. Since τ is elliptic, the constant $i^L(\tau)$ is positive. The expression (5.2) then converges to a nonzero multiple of $I_L(\tau, g)$, as we can see from (2.6) and (2.7). It follows that $I_L(\tau, g)$ vanishes.

We have shown that the invariant distributions (5.8) and (5.9) are supported on $I_c^\infty(G_{\mathrm{reg}}(F)) \otimes \mathscr{I}(G(F))$. This establishes the induction hypothesis, and completes the proof of the theorem. □

We have also established the following corollary, which serves to complete the inductive Definitions 3.1 and 3.2.

Corollary 5.2. *The invariant distributions $I_M(\gamma, f)$ and $I_L(\tau, g)$ are supported on $\mathscr{I}(G(F))$ and $I_c^\infty(G_{\mathrm{reg}}(F))$ respectively.* □

Let us state as a second corollary a result obtained earlier in the proof of the theorem, even though we shall have no further need for it in the paper.

Corollary 5.3. *The noninvariant trace formula holds for any pair of functions in the Schwartz space. That is, the identity of* (5.3) *with* (5.4) *is valid if f and g lie in $\mathscr{C}(G(F))$.* □

Remark. Corollary 5.3 depends on the inequality (5.7), which requires the Howe conjecture if F is p-adic. The theorem, however, does not depend on the Howe conjecture. This is because $C_c^\infty(G(F))$ is already contained in the Hecke algebra when F is p-adic. In this case, one requires only a weaker version of (5.7), in which γ ranges over a compact subset of $M(F) \cap G_{\mathrm{reg}}(F)$. In § 8 we shall use Theorem 5.1 to prove a generalization of the Howe conjecture.

§ 6. Proof of the theorems: first steps

We can now begin to prove the three theorems stated in § 4. With the version of the local trace formula we have just established, it will be an easy matter to derive Theorems 4.3 and 4.5 from Theorem 4.1. We shall do this first. We shall then discuss the initial stages of the proof of Theorem 4.1.

Suppose that Theorem 4.1 is valid. We are then free to combine the expansion (4.1) with the local trace formula. Choose functions $f \in \mathscr{C}(G(F))$ and $g \in C_c^\infty(G_{\mathrm{reg}}(F))$ as in the statement of Theorem 5.1, and substitute the formula (4.1) into the geometric expansion (5.1). The growth condition (4.4), combined with the fact that the functions f_L and g_M lie in $\mathscr{I}(L(F))$ and $I_c^\infty(M_{\mathrm{reg}}(F))$ respectively, insures that the resulting double integrals over $\Gamma_{\mathrm{ell}}(M(F)) \times T_{\mathrm{disc}}(L)$ converge absolutely. The geometric expansion (5.1) then becomes

$$\sum_{L \in \mathscr{L}} |W_0^L| \, |W_0^G|^{-1} (-1)^{\dim(A_L/A_G)} \int_{T_{\mathrm{disc}}(L)} I'_L(\tau, g) f_L(\tau^\vee) \, d\tau, \tag{6.1}$$

where

$$I'_L(\tau, g) = \sum_{M \in \mathscr{L}} |W_0^M| \, |W_0^G|^{-1} (-1)^{\dim(A_M \times A_L)} \int_{\Gamma_{\mathrm{ell}}(M(F))} I_M(\gamma, \tau^\vee) g_M(\gamma) \, d\gamma.$$

The spectral expansion (5.2) must therefore equal (6.1). Consider these two expressions as distributions in f_G. Their difference is a finite sum of smooth symmetric functions on the strata $\{T_{\mathrm{disc}}(L)\}$ of $T(G)$. Since f_G ranges over the entire space $\mathscr{I}(G(F))$, we can separate the contributions of the various strata. We deduce without difficulty that

$$i^L(\tau) I_L(\tau, g) = I'_L(\tau, g), \quad L \in \mathscr{L}, \, \tau \in T_{\mathrm{disc}}(L). \tag{6.2}$$

Suppose that τ lies in $T_{\mathrm{ell}}(L)$. Then $i^L(\tau)$ equals $|d^L(\tau)|^{-1}$, and in particular, does not vanish. The required expansion $(4.1)^\vee$ of Theorem 4.3 then follows from (6.2) and the definition of $I'_L(\tau, g)$. As we remarked at the end of § 4, this in turn implies that $(4.1)^\vee$ holds for any $\tau \in T(L)$.

Consider the identity (6.2) again, for an element τ in $T_{\mathrm{disc}}(L)$. We can expand the left hand side according to $(4.1)^{\vee}$, and the right hand side according to the definition of $I'_L(\gamma, g)$. Comparing coefficients of

$$g_M(\gamma) = g_G(\gamma), \quad \gamma \in \Gamma_{\mathrm{ell}}(M(F)) \cap G_{\mathrm{reg}}(F),$$

while taking account of the relevant symmetry conditions, we see immediately that

$$(-1)^{\dim(A_M \times A_L)} I_M(\gamma, \tau^{\vee}) = i^L(\tau) I_L(\tau, \gamma).$$

Since $i^L(\tau^{\vee}) = i^L(\tau)$, this yields the required identity (4.6) for Theorem 4.5. Observe that if $\tau \in T_{\mathrm{ell}}(L)$, the growth condition $(4.4)^{\vee}$, and also the smoothness of $I_L(\tau, \gamma)$, follow from the corresponding properties for $I_M(\gamma, \tau)$. In view of the remarks at the end of §4, this establishes Theorem 4.3 in complete generality. We have shown that Theorems 4.3 and 4.5 follow from Theorem 4.1.

The proof of Theorem 4.1 is more difficult. The local trace formula will again be our main tool, but it will have to be combined with something else. For the moment we shall be content to restate Theorem 5.1 in the form we shall eventually use.

The principal assertion of Theorem 4.1 is the existence of the expansion (4.1) for $I_M(\gamma, f)$. As we noted at the end of §4, we can assume that γ is an elliptic element in $\Gamma(M(F)) \cap G_{\mathrm{reg}}(F)$. We shall therefore confine our attention to points γ in $T_{\mathrm{reg}}(F)$, where T is a fixed elliptic, maximal torus in M over F. (It is unfortunate that we now have a second meaning for the symbol T. There is probably not much consolation in knowing that the original T is really an upper case τ.) Suppose that θ is a smooth function of compact support on $T_{\mathrm{reg}}(F)$. Theorem 5.1 will provide us with an expansion akin to (4.1), but for the inner product

$$I_M(\theta, f) = \int_{T(F)} \theta(\gamma) I_M(\gamma, f)\, d\gamma \tag{6.3}$$

of θ with $I_M(\cdot, f)$.

To see this, define a function

$$\phi_\theta(\gamma) = \sum_{w \in W(G(F), T(F))} \theta(w\gamma), \quad \gamma \in T(F), \tag{6.4}$$

which is symmetric under the Weyl group of $T(F)$. Then ϕ_θ can be identified with a function in $I_c^\infty(G_{\mathrm{reg}}(F))$ which is supported on the conjugacy classes which meet $T_{\mathrm{reg}}(F)$. We can consequently choose g_θ in $C_c^\infty(G_{\mathrm{reg}}(F))$ such that

$$\mathscr{T}_G g_\theta = \phi_\theta.$$

This means that the function

$$g_{\theta, M_1}(\gamma_1), \quad M_1 \in \mathscr{L}, \ \gamma_1 \in \Gamma_{\mathrm{ell}}(M_1(F)) \cap G_{\mathrm{reg}}(F),$$

equals $\phi_\theta(\gamma)$ if (M_1, γ_1) is $G(F)$-conjugate to a pair (M, γ) with γ in $T(F)$, and that the function vanishes otherwise. Counting the number of conjugates of M in $\mathscr{L}$, and keeping in mind that $I_M(\gamma, f)$ itself is symmetric under translation of γ by $W(G(F), T(F))$, we see that

$$\begin{aligned} I_M(\theta, f) &= |W(G(F), T(F))|^{-1} \int_{T(F)} \phi_\theta(\gamma) I_M(\gamma, f)\, d\gamma \\ &= |W(\mathfrak{a}_M)|^{-1} \int_{\Gamma_{\mathrm{ell}}(M(F)\cap T(F))} \phi_\theta(\gamma) I_M(\gamma, f)\, d\gamma \\ &= \sum_{M_1 \in \mathscr{L}} |W_0^{M_1}||W_0^G|^{-1} \int_{\Gamma_{\mathrm{ell}}(M_1(F))} I_{M_1}(\gamma_1, f)\, g_{\theta, M_1}(\gamma_1)\, d\gamma_1 . \end{aligned}$$

This last expression equals the product of $(-1)^{\dim(A_M/A_G)}$ with the geometric expansion (5.1). It follows from Theorem 5.1 that $I_M(\theta, f)$ equals

$$\sum_{L \in \mathscr{L}} |W_0^L||W_0^G|^{-1}(-1)^{\dim(A_M \times A_L)} \int_{T_{\mathrm{disc}}(L)} i^L(\tau) I_L(\tau, g_\theta) f_L(\tau^\vee)\, d\tau .$$

It is convenient to change the variable of integration from τ to $\tau^\vee$, and to set

$$I_M(\theta, \tau) = (-1)^{\dim(A_M \times A_L)} i^L(\tau) I_L(\tau^\vee, g_\theta) . \tag{6.5}$$

Observe that the notation makes sense; since the distribution $I_L(\tau^\vee)$ is supported on $I_c^\infty(G_{\mathrm{reg}}(F))$, the right hand side of (6.5) depends only on ϕ_θ, which is determined in turn by θ. We can then state what we have obtained from Theorem 5.1 as follows.

Lemma 6.1. *The identity*

$$I_M(\theta, f) = \sum_{L \in \mathscr{L}} |W_0^L||W_0^G|^{-1} \int_{T_{\mathrm{disc}}(L)} I_M(\theta, \tau) f_L(\tau)\, d\tau$$

holds for any pair of functions $f \in \mathscr{C}(G(F))$ and $\theta \in C_c^\infty(T_{\mathrm{reg}}(F))$. □

Recall that $T_{\mathrm{disc}}(L)$ is a disjoint union of $i\mathfrak{a}_L^*$-homogeneous spaces. Let us write $\mathscr{C}(T_{\mathrm{disc}}(L))$ for the topological vector space of functions ϕ on $T_{\mathrm{disc}}(L)$ which satisfy the usual symmetry condition

$$\phi(z\tau) = \chi_\tau(z)^{-1}\phi(\tau), \quad \tau \in T_{\mathrm{disc}}(L),\ z \in Z_\tau ,$$

and which lie in $\mathscr{S}(T_{\mathrm{disc}}(L))$ if F is Archimedean, and in $C_c^\infty(T_{\mathrm{disc}}(L))$ if F is p-adic. These conditions are familiar from the definition of $\mathscr{I}(L(F))$ in § 3. In fact, $\mathscr{C}(T_{\mathrm{disc}}(L))$ is just the space obtained by restricting functions in $\mathscr{I}(L(F))$ to the submanifold $T_{\mathrm{disc}}(L)$ of $T(L)$.

Lemma 6.2. *The map which sends $\theta \in C_c^\infty(T_{\mathrm{reg}}(F))$ to the function*

$$I_M(\theta, \tau), \quad \tau \in T_{\mathrm{disc}}(L),$$

is a continuous linear transformation from $C_c^\infty(T_{\mathrm{reg}}(F))$ to $\mathscr{C}(T_{\mathrm{disc}}(L))$.

Proof. We shall have to examine the right hand side of (6.5) as a function of θ. We claim that the map which sends $\phi \in I_c^\infty(G_{\mathrm{reg}}(F))$ to the function

$$\psi_\phi(\tau) = \hat{I}_L(\tau, \phi), \quad \tau \in T(L),$$

is a continuous linear transformation from $I_c^\infty(G_{\mathrm{reg}}(F))$ to $\mathscr{I}(L(F))$. Assume inductively that this is so if G is replaced by a proper Levi subgroup S in $\mathscr{L}(L)$. Since the map ϕ_S defined by $(3.4)^\vee$ sends $C_c^\infty(G_{\mathrm{reg}}(F))$ continuously to $I_c^\infty(L_{\mathrm{reg}}(F))$, we see that

$$g \to \hat{I}_L^S(\tau, \phi_S(g)), \quad g \in C_c^\infty(G_{\mathrm{reg}}(F)),\ \tau \in T(L),$$

is a continuous linear transformation from $C_c^\infty(G_{\mathrm{reg}}(F))$ to $\mathscr{I}(L(F))$. Moreover, the map

$$g \to J_L(\tau, g), \quad g \in C_c^\infty(G_{\mathrm{reg}}(F)),\ \tau \in T(L),$$

taken from the definition $(3.5)^\vee$, is the composition of the continuous embedding of $C_c^\infty(G_{\mathrm{reg}}(F))$ into $\mathscr{C}(G(F))$ with the continuous map of $\mathscr{C}(G(F))$ to $\mathscr{I}(L(F))$ defined by (3.4). It follows from the definition $(3.5)^\vee$ that

$$g \to I_L(\tau, g), \quad g \in C_c^\infty(G_{\mathrm{reg}}(F)),\ \tau \in T(L),$$

is a continuous linear transformation from $C_c^\infty(G_{\mathrm{reg}}(F))$ to $\mathscr{I}(L(F))$. But the distribution $I_L(\tau)$ is supported on $I_c^\infty(G_{\mathrm{reg}}(F))$. It follows from the general remarks of § 3 that

$$\phi \to \psi_\phi, \quad \phi \in I_c^\infty(G_{\mathrm{reg}}(F)), \tag{6.6}$$

is a continuous linear transformation from $I_c^\infty(G_{\mathrm{reg}}(F))$ to $\mathscr{I}(L(F))$, as claimed.

Now, it follows directly from the definitions that

$$\theta \to \phi_\theta, \quad \theta \in C_c^\infty(T_{\mathrm{reg}}(F)), \tag{6.7}$$

is a continuous linear transformation from $C_c^\infty(T_{\mathrm{reg}}(F))$ to $I_c^\infty(G_{\mathrm{reg}}(F))$. Similarly, if

$$\omega_\psi(\tau) = (-1)^{\dim(A_M \times A_L)} i^L(\tau)\psi(\tau^\vee), \qquad \tau \in T_{\mathrm{disc}}(L),$$

the map

$$\psi \to \omega_\psi, \quad \psi \in \mathscr{I}(L(F)), \tag{6.8}$$

is a continuous linear transformation from $\mathscr{I}(L(F))$ to $\mathscr{C}(T_{\mathrm{disc}}(L))$. Substituting

$$I_L(\tau^\vee, g_\theta) = \hat{I}_L(\tau^\vee, \phi_\theta)$$

into the definition (6.5), we see that the map given in the statement of the lemma is the composition of (6.7), (6.6) and (6.8). The lemma follows. □

Notice the similarity between the required expansion (4.1) of Theorem 4.1 and the formula in Lemma 6.1. The obvious strategy would be to let θ approach the Dirac measure at a regular point γ in $T_{\mathrm{reg}}(F)$. However, Lemma 6.2 is not strong enough to give us control over the functions $I_M(\theta, \tau)$. We are going to have to treat the real and p-adic cases separately. For real groups we shall use the differential equations satisfied by the distributions $I_M(\gamma, f)$. For p-adic groups we shall have to establish an analogue of the Howe conjecture for these distributions.

§ 7. Completion of the proof (Archimedean F)

In this section we shall finish the proof of the theorems of § 4 in the case F is Archimedean. We may as well take $F = \mathbb{R}$. What remains is to establish Theorem 4.1, for elliptic G-regular elements γ in $M(\mathbb{R})$. Observe that if M is replaced by a strictly larger Levi subgroup S, no such γ will be elliptic in $S(\mathbb{R})$. The distribution $I_S(\gamma, f)$ can then be expanded according to (3.7), as a sum over proper subgroups of G for which we can assume inductively that Theorem 4.1 holds. We may therefore assume that the assertions of Theorem 4.1 are valid for $I_S(\gamma, f)$, if γ is an elliptic G-regular element in $M(\mathbb{R})$.

We fixed an elliptic maximal torus T of M in the last section. Our task is to construct the function $I_M(\gamma, \tau)$ for $\gamma \in T_{\mathrm{reg}}(\mathbb{R})$, and to establish the expansion (4.1) and the growth condition (4.4). We shall combine Lemma 6.1 with the differential equations satisfied by $I_M(\gamma, f)$.

The expansion

$$I_M(\theta, f) = \sum_{L \in \mathscr{L}} |W_0^L| |W_0^G|^{-1} \int_{T_{\mathrm{disc}}(L)} I_M(\theta, \tau) f_L(\tau)\, d\tau \tag{7.1}$$

of Lemma 6.1 holds for any functions $f \in \mathscr{C}(G(\mathbb{R}))$ and $\theta \in C_c^\infty(T_{\mathrm{reg}}(\mathbb{R}))$. Before discussing the differential equations, let us look at a concrete estimate for $I_M(\theta, \tau)$ implied by the assertion of Lemma 6.2. Suppose that D_τ is a differential operator on $T_{\mathrm{disc}}(L)$ as in (4.4).

Then

$$\sup_{\tau \in T_{\mathrm{disc}}(L)} (|D_\tau \omega(\tau)|), \quad \omega \in \mathscr{C}(T_{\mathrm{disc}}(L)),$$

is certainly a continuous semi-norm on $\mathscr{C}(T_{\mathrm{disc}}(L))$. If $\omega(\tau) = I_M(\theta, \tau)$, Lemma 6.2 asserts that the resulting function of θ is a continuous semi-norm on $C_c^\infty(T_{\mathrm{reg}}(\mathbb{R}))$. It follows that there is a locally bounded function $b(\gamma)$ on $T_{\mathrm{reg}}(\mathbb{R})$, and a finite set of invariant differential operators D_i on $T(\mathbb{R})$, such that

$$|D_\tau I_M(\theta, \tau)| \leqq \sup_{\gamma \in T_{\mathrm{reg}}(\mathbb{R})} \Big(b(\gamma) \sum_i |D_i \theta(\gamma)|\Big), \tag{7.2}$$

for every $\theta \in C_c^\infty(T_{\mathrm{reg}}(\mathbb{R}))$ and $\tau \in T_{\mathrm{disc}}(L)$. If the total degree of D_τ is bounded by a non-negative integer N, we can clearly choose the differential operators $\{D_i\}$ to have total degree bounded by an integer d_N which depends only on N. It follows from this estimate that the right hand side of (7.1) extends to a continuous linear functional of θ in the topological vector space $C_c^{d_0}(T_{\mathrm{reg}}(\mathbb{R}))$. The same is clearly true of the inner product (6.3) which appears on the left hand side of (7.1). The expansion (7.1) therefore holds if θ is any function in $C_c^{d_0}(T_{\mathrm{reg}}(\mathbb{R}))$.

The differential equations

$$I_M(\gamma, zf) = \sum_{S \in \mathscr{L}(M)} \partial_M^S(\gamma, z_S) I_S(\gamma, f), \quad \gamma \in T_{\mathrm{reg}}(\mathbb{R}),$$

were described in [3], §11–12 and [4], (2.6). (The weighted orbital integrals $J_M(\gamma)$ were studied in [3] only on $C_c^\infty(G(\mathbb{R}))$, and the invariant distributions $I_M(\gamma)$ were defined in [4] only on the Hecke algebra of $G(\mathbb{R})$. However, the results in [3], §11–12 extend to $\mathscr{C}(G(\mathbb{R}))$ by [1], Lemma 8.5, while the one line justification of [4], (2.6) applies equally to our Definition 3.1.) The elements z lie in the center $\mathscr{Z} = \mathscr{Z}_G$ of the universal enveloping algebra of the Lie algebra of $G(\mathbb{C})$. For each such z, z_S is the image of z in $\mathscr{Z}_S$, and $\partial_M^S(\gamma, z_S)$ is a differential operator on $T_{\mathrm{reg}}(\mathbb{R})$ whose coefficients are analytic functions of γ. In the case that $S = M$, we have

$$\partial_M^M(\gamma, z_M) = \partial(h_T(z)),$$

where

$$h_T : \mathscr{Z} \to S(\mathfrak{t}(\mathbb{C}))$$

is the Harish-Chandra homomorphism, and $\partial(\omega)$ stands for the invariant differential operator on $T(\mathbb{R})$ attached to any ω in the symmetric algebra $S(\mathfrak{t}(\mathbb{C}))$ on the Lie algebra of $T(\mathbb{C})$. The differential equations become

$$\partial(h_T(z))I_M(\gamma, f) = I_M(\gamma, zf) - \sum_{\substack{S \in \mathscr{L}(M)\\ S \neq M}} \partial_M^S(\gamma, z_S) I_S(\gamma, f). \tag{7.3}$$

To establish (4.1) we shall use the technique [17], Lemma 48 of Harish-Chandra in the form that applies to the present situation. (See [1], §8, [15], lecture 3, p.13, [3], Lemma 13.2.)

Let ω be the element in $S(\mathfrak{t}(\mathbb{C}))$ such that $\partial(\omega)$ is the Laplacian on $T(\mathbb{R})$ relative to a suitable invariant metric. Suppose that m is an arbitrary positive integer, which for the moment will be fixed. Since $S(\mathfrak{t}(\mathbb{C}))$ is a finite module over $h_T(\mathscr{Z})$, we can find a positive integer r and elements $\{z_j : 1 \leqq j \leqq r\}$ in $\mathscr{Z}$ such that

$$\omega^{mr} - \sum_{j=1}^{r} h_T(z_j)\,\omega^{m(r-j)} = 0.$$

The adjoint involution on $S(\mathfrak{t}(\mathbb{C}))$ induced by the automorphism $X \to -X$ of $\mathfrak{t}(\mathbb{C})$ fixes ω, so we can assume that it also fixes the elements $h_T(z_j)$. Let δ_1 be the Dirac delta distribution at 1 in $T(\mathbb{R})$. From the theory of elliptic operators, we know that we can find a function θ_1 in $C^{2mr-m_0}(T(\mathbb{R}))$, for some positive integer m_0 which depends only on the dimension of T, such that

$$\delta_1 = \partial(\omega)^{mr}\theta_1.$$

This function is in fact infinitely differentiable on the complement of 1 in $T(\mathbb{R})$. Let α be a smooth, compactly supported function on the Lie algebra of $T(\mathbb{R})$ which equals 1 in a neighbourhood of 0, and set

$$\theta_\varepsilon(\gamma) = \theta_1(\gamma)\,\alpha(\varepsilon^{-1}\log\gamma), \quad \gamma \in T(\mathbb{R}),$$

for any small positive number ε. Then θ_ε also belongs to $C^{2mr-m_0}(T(\mathbb{R}))$, and we have

$$\delta_1 = \partial(\omega)^{mr}\theta_\varepsilon + \eta_\varepsilon,$$

where η_ε belongs to $C_c^\infty(T(\mathbb{R}))$. Both θ_ε and η_ε are supported on a ball about 1 in $T(\mathbb{R})$ whose diameter depends linearly on ε. We can also assume that these functions are invariant under the automorphism $\gamma \to \gamma^{-1}$. Substituting the equation above for ω^{mr}, we obtain

$$\delta_1 = \sum_{j=1}^{r} h_T(z_j)\,\partial(\omega)^{m(r-j)}\theta_\varepsilon + \eta_\varepsilon\,. \tag{7.4}$$

We are trying to establish the expansion (4.1) for $I_M(\gamma, f)$, with f in $\mathscr{C}(G(\mathbb{R}))$. As we have already observed, Lemma 8.5 of [1] and the inductive nature of Definition 3.1 insure that $I_M(\gamma, f)$ is a smooth function of $\gamma \in T_{\mathrm{reg}}(\mathbb{R})$. This function of course does not change if we take its convolution with δ_1. To describe the resulting contributions from the compactly supported distributions on the right hand side of (7.4), we write $\theta_{\varepsilon,\gamma}$ and $\eta_{\varepsilon,\gamma}$ for the translates of θ_ε and η_ε by a point γ^{-1} in $T_{\mathrm{reg}}(\mathbb{R})$. We fix ε so that as γ varies over some preassigned region which is relatively compact in $T_{\mathrm{reg}}(\mathbb{R})$, the support of each of the functions $\theta_{\varepsilon,\gamma}$ and $\eta_{\varepsilon,\gamma}$ is contained in $T_{\mathrm{reg}}(\mathbb{R})$. Then the value at γ of the convolution of $I_M(\cdot, f)$ with η_ε is just the inner product of $\eta_{\varepsilon,\gamma}$ with $I_M(\cdot, f)$. Moreover, the convolution of $I_M(\cdot, f)$ with the jth summand in (7.4) has value at γ equal to

$$\int_{T_{\mathrm{reg}}(\mathbb{R})} \big(\partial(\omega)^{m(r-j)}\theta_{\varepsilon,\gamma}\big)(\gamma')\big(\partial(h_T(z_j))\,I_M(\gamma', f)\big)\,d\gamma'\,.$$

According to the differential equations (7.3), this in turn equals

$$\int_{T_{\mathrm{reg}}(\mathbb{R})} \big(\partial(\omega)^{m(r-j)}\theta_{\varepsilon,\gamma}\big)(\gamma')\Big(I_M(\gamma', z_j f) - \sum_{S \supsetneqq M} \partial_j^S I_S(\gamma', f)\Big)\,d\gamma'\,,$$

where

$$\partial_j^S = \partial_M^S(\gamma', z_{j,s})\,.$$

We can therefore write $I_M(\gamma, f)$ as the sum of

$$\int_{T_{\mathrm{reg}}(\mathbb{R})} \eta_{\varepsilon,\gamma}(\gamma')\,I_M(\gamma', f)\,d\gamma'\,,$$

$$\sum_{j=1}^{r} \int_{T_{\mathrm{reg}}(\mathbb{R})} \big(\partial(\omega)^{m(r-j)}\theta_{\varepsilon,\gamma}\big)(\gamma')\,I_M(\gamma', z_j f)\,d\gamma'\,,$$

and

$$-\sum_{j=1}^{r} \sum_{S \supsetneqq M} \int_{T_{\mathrm{reg}}(\mathbb{R})} \big(\partial(\omega)^{m(r-j)}\theta_{\varepsilon,\gamma}\big)(\gamma')\big(\partial_j^S I_S(\gamma', f)\big)\,d\gamma'\,.$$

We shall deal with these expressions in turn.

The first expression is just the inner product $I_M(\eta_{\varepsilon,\gamma}, f)$ defined by (6.3). By (7.1) it has an expansion

$$\sum_L |W_0^L|\,|W_0^G|^{-1} \int_{T_{\mathrm{disc}}(L)} I_M^1(\gamma, \tau)\,f_L(\tau)\,d\tau\,,$$

in which we have written

$$I_M^1(\gamma, \tau) = I_M(\eta_{\varepsilon,\gamma}, \tau)\,.$$

To deal with the second expression, we have to put a condition on the original integer m. We assume that $2m - m_0 \geqq d_0$. The functions

$$\big(\partial(\omega)^{m(r-j)}\theta_{\varepsilon,\gamma}\big)(\gamma'), \quad 1 \leqq j \leqq r,$$

then all lie in $C_c^{d_0}(T_{\mathrm{reg}}(\mathbb{R}))$, and may consequently be substituted for θ in the identity (7.1). Since

$$(z_j f)_L(\tau) = \langle h_T(z_j), \mu_\tau \rangle f_L(\tau),$$

the second expression equals

$$\sum_L |W_0^L| |W_0^G|^{-1} \int_{T_{\mathrm{disc}}(L)} I_M^2(\gamma,\tau) f_L(\tau)\, d\tau,$$

where

$$I_M^2(\gamma,\tau) = \sum_{j=1}^{r} \langle h_T(z_j), \mu_\tau \rangle I_M\big(\partial(\omega)^{m(r-j)}\theta_{\varepsilon,\gamma}, \tau\big).$$

The third expression involves distributions $I_S(\gamma', f)$ for which we are assuming Theorem 4.1 is valid. In particular, we are assuming that the functions $I_S(\gamma', \tau)$ have been defined and satisfy the required properties. Applying the expansion (4.1) in this case, we write the third expression as

$$\sum_L |W_0^L| |W_0^G|^{-1} \int_{T_{\mathrm{disc}}(L)} I_M^3(\gamma,\tau) f_L(\tau)\, d\tau,$$

where

$$I_M^3(\gamma,\tau) = -\sum_{j=1}^{r} \sum_{S \supsetneqq M} \int_{T_{\mathrm{reg}}(\mathbb{R})} \big(\partial(\omega)^{m(r-j)}\theta_{\varepsilon,\gamma}\big)(\gamma')\big(\partial_j^S I_S(\gamma',\tau)\big)\, d\gamma'.$$

The sum of these three expressions gives the required expansion (4.1) for $I_M(\gamma, f)$. The expansion holds if we set

$$I_M(\gamma,\tau) = I_M^1(\gamma,\tau) + I_M^2(\gamma,\tau) + I_M^3(\gamma,\tau). \tag{7.5}$$

The function $I_M(\gamma,\tau)$ appears to depend on the choices of m and ε. However, it is clear from (6.5) and $(3.6)^\vee$ that $I_M^1(\gamma,\tau)$ and $I_M^2(\gamma,\tau)$ are symmetric in τ, in the sense of (4.2). (We are using the property that $\chi_{\tau^\vee} = \chi_\tau^{-1}$, as in the beginning of § 3 of [8].) Our induction hypothesis implies that the same thing is true of $I_M^3(\gamma,\tau)$. It follows that

$$I_M(\gamma, zw^\vee\tau) = \chi_\tau(z)^{-1} I_M(\gamma,\tau), \quad z \in Z_r,\ w^\vee \in W_0^G.$$

As we noted in § 4, the expansion (4.1) and this symmetry property together characterize $I_M(\gamma,\tau)$ uniquely. The function therefore does not depend on the various objects that went into its definition. In particular, we are free to let the integer m vary when we want to establish other properties of $I_M(\gamma,\tau)$.

We must show that $I_M(\gamma,\tau)$ is smooth and that the growth condition (4.4) holds. The proofs are both straightforward consequences of the estimate (7.2). Each property is local in γ, so we can restrict γ to points in some relatively compact, open subset of $T_{\mathrm{reg}}(\mathbb{R})$. To establish the smoothness, we shall show that all partial derivatives of $I_M(\gamma, \tau)$ in γ and τ, each taken with total degree up to an arbitrary given integer N, exist. Suppose that D_τ is a differential operator on $T_{\text{disc}}(L)$ as in (4.4), with total degree bounded by N. It follows from (7.2) that $D_\tau I_M(\theta, \tau)$ extends continuously to a linear functional of θ in $C_c^{d_N}(T_{\text{reg}}(\mathbb{R}))$. If θ lies in $C_c^{N+d_N}(T_{\text{reg}}(\mathbb{R}))$, and if the translates θ_γ of θ by the points in some given open set are supported on a compact subset of $T_{\text{reg}}(\mathbb{R})$, the function $D_\tau I_M(\theta_\gamma, \tau)$ has continuous partial derivatives in γ of degree up to N. This also follows easily from (7.2). Now, the functions

$$\partial(\omega)^{m(r-j)}\theta_\varepsilon, \quad 1 \leqq j \leqq r,$$

all lie in $C_c^{2m-m_0}(T_{\text{reg}}(\mathbb{R}))$. If m is large enough, they therefore belong to $C_c^{N+d_N}(T_{\text{reg}}(\mathbb{R}))$. Since

$$\partial(\omega)^{m(r-j)}\theta_{\varepsilon,\gamma} = \left(\partial(\omega)^{m(r-j)}\theta_\varepsilon\right)_\gamma,$$

we see from the definition that the function $I_M^2(\gamma, \tau)$ is continuously differentiable in (γ, τ) of order up to N. Similarly, since η_ε lies in $C_c^\infty(T_{\text{reg}}(\mathbb{R}))$, we see that $I_M^1(\gamma, \tau)$ is in fact infinitely differentiable. As for the third function $I_M^3(\gamma, \tau)$, we use our induction assumption to assert that the functions

$$\partial_j^S I_S(\gamma, \tau), \quad S \supsetneqq M, \ 1 \leqq j \leqq r,$$

are smooth in (γ, τ). Since $I_M^3(\gamma, \tau)$ is a sum of terms obtained by convolving each of these functions of γ with functions of compact support, it too is a smooth function of (γ, τ). It follows that $I_M(\gamma, \tau)$ is differentiable of order up to N. Since N is arbitrary, we obtain the required smoothness of $I_M(\gamma, \tau)$.

The growth estimate is proved in a similar way. Choose differential operators D_γ and D_τ on $T_{\text{reg}}(\mathbb{R})$ and $T_{\text{disc}}(L)$ as in (4.4). If the total degree of each of them is bounded by N, we choose m so that

$$2m - m_0 \geqq N + d_N.$$

Arguing as above, we can estimate the function

$$D_\gamma D_\tau I_M^1(\gamma, \tau)$$

by means of (7.2). To deal with the second function, we choose positive integers c and n (depending on m, and hence on D_γ and D_τ) such that

$$\langle h_T(z_j), \mu_\tau\rangle \leqq c(1 + \|\mu_\tau\|)^n, \quad 1 \leqq j \leqq r, \ \tau \in T_{\text{disc}}(L), \ L \in \mathscr{L}.$$

We can then estimate

$$D_\gamma D_\tau I_M^2(\gamma, \tau)$$

in the same manner. Finally, we can estimate the third function

$$D_\gamma D_\tau I_M^3(\gamma, \tau)$$

from our induction hypothesis. The required growth condition (4.4) follows. Our proof of Theorem 4.1 in the case $F = \mathbb{R}$ is complete. $\square$

§ 8. A variant of the Howe conjecture

We turn now to the p-adic case. The differential equations are no longer available to control the behaviour of $I_M(\gamma, \tau)$. For ordinary orbital integrals (the case that $M = G$), the Howe conjecture is known to be a useful replacement for the differential equations. (See [14].) A similar principle applies to weighted orbital integrals. We shall establish a generalization of the Howe conjecture, which applies to the distributions $I_M(\gamma, f)$, and which will allow us to prove Theorem 4.1 in § 9.

For the next two sections, F will be a p-adic field whose residue field has order q. As before, $M \in \mathscr{L}$ will be a fixed Levi subgroup, and T will be a fixed elliptic maximal torus in M over F. If Δ is any subset of $T(F)$, we set

$$\Delta_{\text{reg}} = \Delta \cap G_{\text{reg}}(F) = \Delta \cap T_{\text{reg}}(F) .$$

Theorem 8.1. *Suppose that K_0 is an open compact subgroup of $G(F)$ and that Δ is an open compact subset of $T(F)$. Then the restriction of the set of invariant distributions*

$$f \to I_M(\gamma, f), \quad \gamma \in \Delta_{\text{reg}} ,$$

to the space $\mathscr{C}(G(F)//K_0)$ of K_0-bi-invariant functions in $\mathscr{C}(G(F))$ spans a finite dimensional space.

Proof. Although the distribution $I_M(\gamma, f)$ is invariant, it does not generally have compact invariant support. Therefore the Howe conjecture, which is actually equivalent to the case that $M = G$, does not imply Theorem 8.1. It does not seem possible to extend Clozel's proof [13] to the case at hand, although we shall use a key step [13], Lemma 5 from his argument. We shall obtain the theorem instead as another consequence of the local trace formula, or rather its reformulation as Lemma 6.1.

If θ is a function in $C_c^\infty(T_{\text{reg}}(F))$ which is supported on Δ, we can construct the linear form

$$I_M(\theta, f) = \int_{T(F)} \theta(\gamma) I_M(\gamma, f)\, d\gamma, \quad f \in \mathscr{C}(G(F)//K_0) ,$$

on $\mathscr{C}(G(F)//K_0)$. It will be sufficient for us to show that the space spanned by all such linear forms is finite dimensional. Lemma 6.1 asserts that

$$I_M(\theta, f) = \sum_{L \in \mathscr{L}} |W_0^L| |W_0^G|^{-1} \int_{T_{\text{disc}}(L)} I_M(\theta, \tau) f_L(\tau)\, d\tau ,$$

where

$$I_M(\theta, \tau) = (-1)^{\dim(A_M \times A_L)} i^L(\tau) I_L(\tau^\vee, g_\theta) .$$

Substituting the definition of the measure $d\tau$, we obtain an identity

$$(8.1) \qquad I_M(\theta, f) = \sum_L \sum_{\tau \in T_{\mathrm{disc}}(L)/i\mathfrak{a}_L^*} c_L(\tau) \int_{i\mathfrak{a}_{L,F}^*} I_L(\tau_\lambda, g_\theta) f_L(\tau_{-\lambda}^\vee)\, d\lambda,$$

where

$$c_L(\tau) = |W_0^L| |W_0^G|^{-1} (-1)^{\dim(A_M \times A_L)} |\tilde{R}_{\sigma,r}^L|^{-1} |\mathfrak{a}_{L,\sigma}^\vee / \mathfrak{a}_{L,F}^\vee|^{-1} i^L(\tau),$$

if $\tau = (L_1, \sigma, r)$. The function $g_\theta \in C_c^\infty(G_{\mathrm{reg}}(F))$ was described in § 6. We can in fact always arrange that g_θ lies in the subspace $C_\Delta^\infty(G_{\mathrm{reg}}(F))$ of functions in $C_c^\infty(G_{\mathrm{reg}}(F))$ which are supported on $\mathrm{Ad}(G(F))\Delta$. Moreover, as long as f remains in $\mathscr{C}(G(F)//K_0)$, the sum over τ in (8.1) can be taken over a fixed finite set. It is enough, then, to study the bilinear form

$$(8.2) \qquad \int_{i\mathfrak{a}_{L,F}^*} I_L(\tau_\lambda, g) f_L(\tau_{-\lambda}^\vee)\, d\lambda, \quad g \in C_\Delta^\infty(G_{\mathrm{reg}}(F)),\ f \in \mathscr{C}(G(F)//K_0),$$

for any fixed $\tau \in T_{\mathrm{disc}}(L)$.

Appealing to the Plancherel formula on $i\mathfrak{a}_{L,F}^*$, we write (8.2) as

$$\sum_{X \in \mathfrak{a}_{L,F}} I_L(\tau, X, g) f_L(\tau^\vee, X),$$

where

$$I_L(\tau, X, g) = \int_{i\mathfrak{a}_{L,F}^*} I_L(\tau_\lambda, g) e^{-\lambda(X)}\, d\lambda$$

and

$$f_L(\tau^\vee, X) = \int_{i\mathfrak{a}_{L,F}^*} f_L(\tau_\lambda^\vee) e^{-\lambda(X)}\, d\lambda.$$

The next lemma is clearly the essential point.

Lemma 8.2. *The set of distributions*

$$g \to I_L(\tau, X, g), \quad g \in C_\Delta^\infty(G_{\mathrm{reg}}(F)),\ X \in \mathfrak{a}_{L,F},$$

on $C_\Delta^\infty(G_{\mathrm{reg}}(F))$ spans a finite dimensional space.

Proof. The assertion is evidently equivalent to the finite dimensionality of the space spanned by all linear functionals

$$\phi \to \hat{I}_L(\tau, X, \phi), \quad \phi \in I_\Delta^\infty(G_{\mathrm{reg}}(F)),\ X \in \mathfrak{a}_{L,F},$$

on the image $I_\Delta^\infty(G_{\mathrm{reg}}(F))$ of $C_\Delta^\infty(G_{\mathrm{reg}}(F))$ under $\mathscr{T}_G$. Set

$$J_L(\tau, X, g) = \int_{i\mathfrak{a}_{L,F}^*} J_L(\tau_\lambda, g) e^{-\lambda(X)}\, d\lambda.$$

Then the original definition $(3.5)^\vee$ can be written

$$I_L(\tau, X, g) = J_L(\tau, X, g) - \sum_{\substack{S \in \mathscr{L}(L) \\ S \neq G}} \hat{I}_L^S(\tau, X, \phi_S(g)),$$

where ϕ_S is the map given by $(3.4)^\vee$. We assume inductively that the lemma holds if G is replaced by any $S \subsetneqq G$. Since ϕ_S maps $C_\Delta^\infty(G_{\mathrm{reg}}(F))$ into $I_\Delta^\infty(S_{\mathrm{reg}}(F))$, we can then conclude that the distributions

$$g \to \hat{I}_L^S(\tau, X, \phi_S(g)), \quad g \in C_\Delta^\infty(G_{\mathrm{reg}}(F)),\ S \subsetneqq G,\ X \in \mathfrak{a}_{L,F},$$

span a finite dimensional space. It is therefore sufficient to prove the lemma with $I_L(\tau, X, g)$ replaced by $J_L(\tau, X, g)$.

To deal with $J_L(\tau, X, g)$, we shall use a Paley-Wiener argument. It is convenient first of all to write

$$J_L(\tau, X, g) = \int_{\varepsilon + i\mathfrak{a}_{L,F}^*} J_L(\tau_\lambda, g) e^{-\lambda(X)} d\lambda, \tag{8.3}$$

where ε is a small point in $\mathfrak{a}_L^*$ in general position. This is of course possible, since $J_L(\tau_\lambda, g)$ is a meromorphic function of $\lambda \in \mathfrak{a}_{L,\mathbb{C}}^*$, which is analytic in a neighbourhood of $i\mathfrak{a}_L^*$. To prepare for more serious changes of contour, we recall the scheme ([5], § 10, [8], § 7) that is used to keep track of residues. For each group $Q \in \mathscr{F}(L)$, let $\mu_Q \in \mathfrak{a}_Q^*$ be a fixed point in the chamber $(\mathfrak{a}_Q^*)^+$ associated to Q which is very far from any of the walls. Then if X is any point in $\mathfrak{a}_{L,F}$, we set

$$\mu(X) = \mu_Q,$$

where $Q \in \mathscr{F}(L)$ is the unique group such that X lies in $\mathfrak{a}_Q^+$. Finally, given ε and X, and also a group $S \in \mathscr{L}(L)$, we define

$$\nu_S = \mu(X_S) + \varepsilon_S,$$

where X_S and ε_S are the projections of X and ε onto $\mathfrak{a}_S$. The integral over $\varepsilon + i\mathfrak{a}_{L,F}^*$ above can then be written as a sum of integrals over the contours $\nu_S + i\mathfrak{a}_{S,F}^*$. More precisely, for each S there is a set $R_S(\varepsilon, \mathscr{N}_S)$ of residue data for (S, L), which depends only on the family

$$\mathscr{N}_S = \{\nu_{S'} : L \subset S' \subset S\},$$

with the property that the integral on the right hand side of (8.3) equals the sum over $S \in \mathscr{L}(L)$ of

$$\int_{\nu_S + i\mathfrak{a}_{S,F}^*} \Bigg(\sum_{\Omega \in R_S(\varepsilon, \mathscr{N}_S)} \operatorname*{Res}_{\Omega, \lambda \to \Lambda_\Omega + \nu} \big(e^{-\lambda(X)} J_L(\lambda_\lambda, g)\big) \Bigg) d\nu. \tag{8.4}$$

(See [5], Proposition 10.1. We are following notation and terminology introduced in [5], § 8.) We can therefore analyze $J_M(\tau, X, g)$ by studying the integrals (8.4). We must show that the family of linear functionals in $g \in C_\Delta^\infty(G_{\mathrm{reg}}(F))$ given by (8.4), with X ranging over $\mathfrak{a}_{L,F}$, spans a finite dimensional space.

The point ν_S depends on X. Since it is a highly regular point in the chamber $(\mathfrak{a}_Q^*)^+$ for which X_S lies in $\mathfrak{a}_Q^+$, we can use the classical Paley-Wiener argument to bound the norm of X_S. The discussion at this stage is similar to the proofs of [5], Theorem 12.1 and [4], Lemma 4.2, so we shall not go into detail. The point is that

$$\lambda \to J_L(\tau_\lambda, g), \quad \lambda \in \mathfrak{a}_{L,\mathbb{C}}^*,$$

is a function of exponential type, apart from a finite set of poles, and the exponent of growth depends only on the support of g. We conclude that (8.4) vanishes unless X_S lies in a compact subset of $\mathfrak{a}_S$ which depends only on Δ. Since X_S lies in a lattice in $\mathfrak{a}_S$, it will in fact take on only a finite set of values.

What remains is to show that as X ranges over the subset of $\mathfrak{a}_{L,F}(Y)$ of elements in $\mathfrak{a}_{L,F}$ which project onto a given point $Y \in \mathfrak{a}_S$, the space spanned by the corresponding linear functionals (8.4) is finite dimensional. The problem is that the set $R_S(\varepsilon, \mathcal{N}_S)$ could a priori be infinite, even though for a given g, all but finitely many summands in (8.4) vanish. We must show that $R_S(\varepsilon, \mathcal{N}_S)$ is actually a finite set. The set is of course determined by the multiple residues encountered in changing the contours of integration in (8.3). We must therefore show that the poles (with multiplicities) of the function

$$\lambda \to J_L(\tau_\lambda, g), \quad \lambda \in \mathfrak{a}_{L,\mathbb{C}}^*,$$

can be taken from a finite set which is independent of g. As Clozel remarks in [13], this phenomenon is peculiar to p-adic groups.

Suppose that π is a representation in $\Pi_{\text{temp}}(L(F))$. Recall (R.7) that the matrix coefficients of the normalized intertwining operators $R_{P'|P}(\pi_\lambda)$ are rational functions of the variables

$$\{q^{\lambda(\beta^\vee)} : \beta \in \Sigma_{P'}^r \cap \Sigma_{\bar{P}}^r\}. \tag{8.5}$$

We shall show that these rational functions all have a common denominator.

Lemma 8.3. *Fix groups $P', P \in \mathcal{P}(L)$. Then there is a product*

$$\varrho_{P'|P}(\lambda) = \prod_{\beta \in \Sigma_{P'}^r \cap \Sigma_{\bar{P}}^r} \varrho_\beta(q^{\lambda(\beta^\vee)}), \quad \lambda \in \mathfrak{a}_{L,\mathbb{C}}^*, \tag{8.6}$$

of polynomials of one variable with the property that all of the matrix coefficients of the operators

$$\varrho_{P'|P}(\lambda)\, R_{P'|P}(\pi_\lambda) \tag{8.7}$$

are polynomials in the variables (8.5).

Proof. It is known that there is an open compact subgroup K_1 of K with the property that any composition factor of any of the induced representations $\mathcal{I}_P(\pi_\lambda)$, $\lambda \in \mathfrak{a}_{L,\mathbb{C}}^*$, has a K_1-fixed vector. (See [13], Lemma 5 and [11], 3.5.2.) Let $\mathcal{H}_P(\pi)_1$ be the finite dimensional subspace of vectors in $\mathscr{H}_P(\pi)$ which are K_1-fixed, and let Γ_1 be the canonical projection of $\mathscr{H}_P(\pi)$ onto $\mathscr{H}_P(\pi)_1$ which commutes with the action of K_1. Since $R_{P'|P}(\pi_\lambda)$ also commutes with the action of K_1, and since the matrix coefficients of this operator are rational functions, there is a polynomial $\varrho_{P'|P}(\lambda)$ in the variables (8.5) with the property that the matrix coefficients of the operators

$$\varrho_{P'|P}(\lambda)\,\Gamma_1\,R_{P'|P}(\pi_\lambda)\,\Gamma_1 = \varrho_{P'|P}(\lambda)\,R_{P'|P}(\pi_\lambda)\,\Gamma_1$$

are all polynomials in the variables (8.5). Since the singularities of $R_{P'|P}(\pi_\lambda)$ lie along hyperplanes of the form

$$q^{\lambda(\beta^\vee)} - c = 0, \quad \beta \in \Sigma^r_{P'} \cap \Sigma^r_{\bar{P}},\ c \in \mathbb{C}^*, \tag{8.8}$$

we can take $\varrho_{P'|P}(\lambda)$ to be of the form (8.6).

We claim that $\varrho_{P'|P}(\lambda)$ satisfies the required property of the lemma. It is enough to show that any matrix coefficient of the operator (8.7) is an entire function. Suppose that this is not so. Then there is a matrix coefficient which has a pole of order $k \geqq 1$ along some hyperplane of the form (8.8). If μ is a fixed generic point on the hyperplane, the operator

$$R_\mu = \lim_{\lambda \to \mu} (q^{\lambda(\beta^\vee)} - c)^k \varrho_{P'|P}(\lambda)\,R_{P'|P}(\pi_\lambda)$$

is well defined and is not equal to 0. Observe that

$$R_\mu \mathscr{I}_P(\pi_\mu, h) = \mathscr{I}_{P'}(\pi_\mu, h)\,R_\mu, \quad h \in C_c^\infty(G(F)).$$

It follows that the kernel $N(R_\mu)$ of R_μ is an invariant subspace of $\mathscr{H}_P(\pi)$ under the representation $\mathscr{I}_P(\pi_\mu)$. But $N(R_\mu)$ contains $\mathscr{H}_P(\pi)_1$, by construction. Therefore, the quotient of the representation $\mathscr{I}_P(\pi_\mu)$ on $\mathscr{H}_P(\pi)/N(R_\mu)$ has no K_1-fixed vector. This contradicts the definition of K_1. □

Having established Lemma 8.3, we can now complete the proof of Lemma 8.2. We are trying to control the singularities of the functions $J_L(\tau_\lambda, g)$ in (8.4). Recall that $J_L(\tau_\lambda, g)$ is a finite linear combination (2.4) of distributions

$$J_L(\pi_\lambda, g) = \operatorname{tr}\big(\mathscr{R}_L(\pi_\lambda, P)\,\mathscr{I}_P(\pi_\lambda, g)\big), \quad \pi \in \Pi_{\mathrm{temp}}(L(F)).$$

The operator $\mathscr{R}_L(\pi_\lambda, P)$ is obtained from the (G, L)-family

$$R_{P'|P}(\pi_\lambda)^{-1} R_{P'|P}(\pi_{\lambda+\Lambda}) = R_{P|P'}(\pi_\lambda)\,R_{P'|P}(\pi_{\lambda+\Lambda}), \quad P \in \mathscr{P}(L), \Lambda \in i\mathfrak{a}_L^*.$$

According to the general formula [2], (6.5), we can write

$$\mathscr{R}_L(\pi_\lambda, P) = \frac{1}{p!} \sum_{P' \in \mathscr{P}(L)} R_{P|P'}(\pi_\lambda)\left(\left(\frac{d}{dt}\right)^p R_{P'|P}(\pi_{\lambda+t\Lambda})\right)\theta_{P'}(\Lambda)^{-1},$$

where $p = \dim(A_L/A_G)$ and Λ is a fixed point in $i\mathfrak{a}_L^*$. We therefore obtain an expression

$$J_L(\pi_\lambda, g) = \sum_{P' \in \mathscr{P}(L)} \varrho_{P|P'}(\lambda)^{-1} \left(\frac{d}{dt}\right)^p \left(\varrho_{P'|P}(\lambda + t\Lambda)^{-1} \tilde{J}_{P'}(\pi_\lambda, t\Lambda, g)\right),$$

in which $\tilde{J}_{P'}(\pi_\lambda, t\Lambda, g)$ is the entire function

$$\frac{1}{p!} \operatorname{tr}\left(\varrho_{P|P'}(\lambda) R_{P|P'}(\pi_\lambda) \cdot \varrho_{P'|P}(\lambda + t\Lambda) R_{P'|P}(\pi_{\lambda + t\Lambda}) \cdot \mathscr{I}_P(\pi_\lambda, g)\right) \theta_{P'}(\Lambda)^{-1}$$

of $\lambda \in \mathfrak{a}_{L,\mathbb{C}}^*$. Thus, $J_L(\pi_\lambda, g)$ is a finite sum of terms, each of which is the product of an entire function of λ which depends on g, and a rational function in the variables $\{q^{\lambda(\beta^\vee)}\}$ which is independent of g. It follows that the poles (with multiplicities) of $J_L(\pi_\lambda, g)$ can be taken from a finite set which is independent of g. The same assertion is therefore true for the function $J_L(\tau_\lambda, g)$ in (8.4). This is what we had to prove. We conclude that as X varies over $\mathfrak{a}_{L,F}$, the integrals (8.4), and hence also the functionals $J_L(\tau, X, g)$, span a finite dimensional space of distributions on $C_\Delta^\infty(G_{\text{reg}}(F))$. The proof of Lemma 8.2 is complete. □

Now that we have verified our pair of embedded lemmas, we shall finish up the proof of the theorem. Lemma 8.2 implies that as f varies over $\mathscr{C}(G(F)//K_0)$, the functionals

$$\sum_{X \in \mathfrak{a}_{L,F}} I_L(\tau, X, g) f_L(\tau^\vee, X) \tag{8.9}$$

span a finite dimensional space of linear forms in $g \in C_\Delta^\infty(G_{\text{reg}}(F))$. This is the same as saying that as g varies, the functionals (8.9) span a finite dimensional space of linear forms in f. But (8.9) equals our earlier bilinear form (8.2). When $g = g_\theta$, (8.2) in turn appears as the integral in the expression (8.1) for $I_M(\theta, f)$. Recall that θ was allowed to be any function in $C_c^\infty(T_{\text{reg}}(F))$ with support in Δ. We have established that as θ varies, the linear forms

$$f \to I_M(\theta, f), \quad f \in \mathscr{C}(G(F)//K_0),$$

span a finite dimensional space. This yields the assertion of Theorem 8.1. □

Corollary 8.4. *Suppose that K_0 and Δ are as in Theorem* 8.1. *Then the weighted orbital integrals*

$$f \to J_M(\gamma, f), \quad \gamma \in \Delta_{\text{reg}},$$

span a finite dimensional space of linear forms on $\mathscr{C}(G(F)//K_0)$.

Proof. According to the definition (3.5), $J_M(\gamma, f)$ equals

$$\sum_{S \in \mathscr{S}(M)} \hat{I}_M^S(\gamma, \phi_S(f)),$$

where ϕ_S is the map (3.4). The image of $\mathscr{C}(G(F)//K_0)$ under ϕ_S is contained in a subspace of functions in $\mathscr{I}(S(F))$ which are supported on a fixed compact subset of $T(S)$. Any such subspace is in turn contained in $\mathcal{T}_S(\mathscr{C}(S(F)//K_0^S))$, for some open compact subgroup K_0^S of $S(F)$. This follows from the fact [10] that $\mathcal{T}_S : \mathscr{C}(S(F)) \to \mathscr{I}(S(F))$ is an open surjective map. The corollary then follows from the theorem. □

§ 9. Completion of the proof (p-adic F)

For this section, F will continue to be a p-adic field. Our last task is to prove Theorem 4.1 in this case. We shall actually be able to do more. The theorem of the last section is strong enough to give us some control of the function $\gamma \to I_M(\gamma, \tau)$ near the singular set. This will lead to an expansion (4.1) if γ is a singular element in $M(F)$. It will also allow us to define the distributions $I_L(\tau, g)$ for *any* function in $C_c^\infty(G(F))$.

Recall that $T(G)$ is a disjoint union of finite quotients of compact tori. Let Ω be a fixed connected component in $T(G)$. We shall write $\mathscr{I}(G(F), \Omega)$ for the closed subspace of functions ϕ in $\mathscr{I}(G(F))$ which are supported on Ω. By the properties [10] of the map $\mathcal{T}_G$, we can find an open compact subgroup $K(\Omega)$ of $G(F)$ such that the closed subspace

$$(9.1) \qquad \mathscr{C}(G(F), \Omega) = \mathcal{T}_G^{-1}(\mathscr{I}(G(F), \Omega)) \cap \mathscr{C}(G(F)//K(\Omega))$$

of $\mathscr{C}(G(F))$ is mapped surjectively onto $\mathscr{I}(G(F), \Omega)$ by $\mathcal{T}_G$.

In § 6 we fixed an elliptic maximal torus T in M over F. To prove Theorem 4.1 it remains for us only to establish the expansion (4.1) for elements γ in $T_{\mathrm{reg}}(F)$. Let Δ be an open compact subset of $T(F)$. We shall consider the space of linear functionals on $\mathscr{C}(G(F), \Omega)$ spanned by the invariant distributions

$$f \to I_M(\gamma, f), \quad \gamma \in \Delta_{\mathrm{reg}} = \Delta \cap G_{\mathrm{reg}}(F).$$

By Theorem 8.1, this vector space is finite dimensional. Taking a fixed basis

$$\{I_\alpha : \alpha \in A(\Delta, \Omega)\}$$

of the space, we obtain functions

$$\{t_M^\alpha : \alpha \in A(\Delta, \Omega)\}$$

on Δ_{reg} such that

$$I_M(\gamma, f) = \sum_{\alpha \in A(\Delta, \Omega)} t_M^\alpha(\gamma) I_\alpha(f)$$

for any $\gamma \in \Delta_{\mathrm{reg}}$ and $f \in \mathscr{C}(G(F), \Omega)$. The functions are smooth and linearly independent. We can therefore choose functions $\{\theta_\beta : \beta \in A(\Delta, \Omega)\}$ in $C_c^\infty(\Delta_{\mathrm{reg}})$ such that for any α and β,

$$\int_{T(F)} t_M^\alpha(\gamma) \theta_\beta(\gamma) \, d\gamma = \begin{cases} 1, & \text{if } \alpha = \beta, \\ 0, & \text{otherwise}. \end{cases}$$

Then the form

$$I_M(\theta_\alpha, f) = \int_{T(F)} \theta_\alpha(\gamma) I_M(\gamma, f) \, d\gamma, \quad f \in \mathscr{C}(G(F), \Omega),$$

defined in (6.3), equals $I_\alpha(f)$. Applying Lemma 6.1, we obtain an expansion

$$I_\alpha(f) = \sum_{L \in \mathscr{L}} |W_0^L| |W_0^G|^{-1} \int_{T_{\mathrm{disc}}(L)} I_M(\theta_\alpha, \tau) f_L(\tau)\, d\tau$$

for $I_\alpha(f)$ in terms of the functions $I_M(\theta_\alpha, \tau)$ defined by (6.5). Let us write Ω_L for the pre-image of Ω under the canonical map $T(L) \to T(G)$. It is a finite (possibly empty) union of connected components in $T(L)$. If f lies in $\mathscr{C}(G(F), \Omega)$, f_L will be supported on Ω_L, and the integral above can be taken over $T_{\mathrm{disc}}(L) \cap \Omega_L$. For any such f, and for $\gamma \in \Delta_{\mathrm{reg}}$, we have

$$\begin{aligned} I_M(\gamma, f) &= \sum_{\alpha \in A(\Delta, \Omega)} t_M^\alpha(\gamma) I_\alpha(f) \\ &= \sum_L |W_0^L| |W_0^G|^{-1} \int_{T_{\mathrm{disc}}(L) \cap \Omega_L} I_M(\gamma, \tau) f_L(\tau)\, d\tau, \end{aligned}$$

where

$$I_M(\gamma, \tau) = \sum_\alpha t_M^\alpha(\gamma) I_M(\theta_\alpha, \tau), \quad \tau \in T_{\mathrm{disc}}(L) \cap \Omega_L. \tag{9.2}$$

The function (9.2) is defined and smooth on the open compact subset

$$\Delta_{\mathrm{reg}} \times (T_{\mathrm{disc}}(L) \cap \Omega_L) \quad \text{of} \quad T_{\mathrm{reg}}(F) \times T_{\mathrm{disc}}(L).$$

As the notation suggests, it will be the restriction to this subset of the required function of Theorem 4.1. Since $T(L)$ is the disjoint union of sets Ω_L, we can certainly define a smooth function $I_M(\gamma, \tau)$ on $\Delta_{\mathrm{reg}} \times T_{\mathrm{disc}}(L)$ so that its restriction to any $\Delta_{\mathrm{reg}} \times (T_{\mathrm{disc}}(L) \cap \Omega_L)$ is given by (9.2). We would like to show that the expansion (4.1) holds for any $\gamma \in \Delta_{\mathrm{reg}}$ if f is any function in $\mathscr{C}(G(F))$. It certainly holds if f lies in $\mathscr{C}(G(F), \Omega)$ as above, since f_L is then supported on Ω_L. An arbitrary function f will still be bi-invariant under some open compact subgroup of $G(F)$, so the function $f_G = \mathscr{T}_G f$ on $T(G)$ will be supported on finitely many components $\{\Omega_i : 1 \leqq i \leqq r\}$. For each i, choose a function $f_i \in \mathscr{C}(G(F), \Omega_i)$ such that $\mathscr{T}_G f_i$ and $\mathscr{T}_G f$ have the same restrictions to Ω_i. Since $\mathscr{T}_G f_i$ vanishes on the complement of Ω_i, the function $f - \sum f_i$ lies in the kernel of $\mathscr{T}_G$. But the distribution $I_M(\gamma)$ is supported on $\mathscr{I}(G(F))$, which implies that

$$I_M(\gamma, f) = \sum_{i=1}^r I_M(\gamma, f_i).$$

We also have $f_L = \sum f_{i,L}$, for any $L \in \mathscr{L}$. The expansion (4.1) then holds for f, since it holds for each f_i.

Our definition of

$$I_M(\gamma, \tau), \quad \gamma \in \Delta_{\mathrm{reg}}, \ \tau \in T_{\mathrm{disc}}(L),$$

depends on the set Δ. However, we may certainly assume that the function satisfies the required symmetry condition (4.2). As we remarked in §4, the conditions (4.1) and (4.2) determine the function uniquely. In particular, if Δ' is an open compact subset of $T(F)$ which contains Δ, the restriction to Δ_{reg} of the function defined for Δ' equals the function defined for Δ. Letting Δ run over an increasing sequence of sets whose union is $T(F)$, we obtain smooth functions $I_M(\gamma, \tau)$ on $T_{\mathrm{reg}}(F) \times T_{\mathrm{disc}}(L)$ for which the required conditions (4.1) and (4.2) hold. According to the remarks at the end of §4, this finishes the proof of Theorem 4.1. □

We have established the theorems of §4 in complete generality. In the p-adic case we are now considering, our version of the Howe conjecture provides further qualitative information. Let us summarize what we can say about the kernels $I_M(\gamma, \tau)$.

Changing notation slightly, we take Δ to be an open compact subset of $M(F)$, and we set $\Delta_{\mathrm{reg}} = \Delta \cap G_{\mathrm{reg}}(F)$. If Ω is a connected component of $T(G)$, and $\mathscr{C}(G(F), \Omega)$ is the subspace (9.1) of $\mathscr{C}(G(F)//K(\Omega))$, the larger family of distributions

$$\{I_S(\gamma) : S \in \mathscr{L}(M),\ \gamma \in \Delta_{\mathrm{reg}}\} \tag{9.3}$$

still spans a finite dimensional space of linear functionals on $\mathscr{C}(G(F), \Omega)$. Let

$$\{I_\alpha : \alpha \in A'(\Delta, \Omega)\}$$

be a fixed basis of this space. We can then write

$$I_S(\gamma, f) = \sum_{\alpha \in A'(\Delta, \Omega)} t_S^\alpha(\gamma) I_\alpha(f), \quad f \in \mathscr{C}(G(F), \Omega), \tag{9.4}$$

for uniquely determined smooth functions

$$\{t_S^\alpha(\gamma) : \alpha \in A'(\Delta, \Omega)\}$$

on Δ_{reg}. Each I_α is the restriction to $\mathscr{C}(G(F), \Omega)$ of a distribution in the space (9.3). Theorem 4.1 then gives us an expansion

$$I_\alpha(f) = \sum_L |W_0^L| |W_0^G|^{-1} \int_{T_{\mathrm{disc}}(L)} I_\alpha(\tau) f_L(\tau)\, d\tau \tag{9.5}$$

for any $\alpha \in A'(\Delta, \Omega)$ and $f \in \mathscr{C}(G(F), \Omega)$, in which

$$I_\alpha(\tau), \quad \alpha \in A'(\Delta, \Omega),\ \tau \in T_{disc}(L),$$

are uniquely determined smooth functions on $T_{\mathrm{disc}}(L)$ which vanish on the complement of Ω_L, and which satisfy the symmetry condition

$$I_\alpha(z w^\vee \tau) = \chi_\tau(z) I_\alpha(\tau), \quad z \in Z_\tau,\ w^\vee \in W_0^G.$$

Combining the formulas above for $I_S(\gamma, f)$ and $I_\alpha(f)$ with the fact that $\mathscr{T}_G$ maps $\mathscr{C}(G(F), \Omega)$ surjectively onto $\mathscr{I}(G(F), \Omega)$, we obtain a locally finite presentation

$$I_S(\gamma, \tau) = \sum_n \sum_{\alpha \in A'(\Delta, \Omega)} t_S^\alpha(\gamma) I_\alpha(\tau), \quad \gamma \in \Delta_{\mathrm{reg}},\ \tau \in T_{\mathrm{disc}}(L), \tag{9.6}$$

for our kernel.

The linear forms I_α were defined on $\mathscr{C}(G(F), \Omega)$. It will be convenient to extend them to invariant distributions on the entire Schwartz space. We do so by simply requiring that the expansion (9.5) for $I_\alpha(f)$ hold for any f in $\mathscr{C}(G(F))$. We then have a formula

$$I_S(\gamma, f) = \sum_{\Omega} \sum_{\alpha \in A'(\Delta, \Omega)} t_S^\alpha(\gamma) I_\alpha(f),$$

which is valid for any $f \in \mathscr{C}(G(F))$.

We shall now deduce some corollaries of the theorems of § 4 and § 8, and of the formula (9.6) in particular, which we are able to prove only in the p-adic case. The first corollary concerns the distributions $I_M(\gamma, f)$ for singular γ. In earlier papers, we considered these distributions only as linear functionals on the Hecke algebra $C_c^\infty(G(F))$. However, we did define them for arbitrary elements γ in $M(F)$ ([3], § 2, § 5, [4], § 2). For singular elements, the distributions were defined by a limiting process from their values at G-regular points in $M(F)$.

Corollary 9.1. *Suppose that γ is an arbitrary element in $M(F)$. Then the invariant distribution $I_M(\gamma)$ on $C_c^\infty(G(F))$ is tempered. For each function $f \in \mathscr{C}(G(F))$, $I_M(\gamma, f)$ has an expansion* (4.1), *for uniquely determined smooth functions*

$$I_M(\gamma, \tau), \quad \tau \in T_{\mathrm{disc}}(L),\ L \in \mathscr{L},$$

which satisfy the symmetry condition (4.2).

Proof. As above, take Δ to be an arbitrary open compact subset of $M(F)$. It is of course enough to prove the result if γ equals a given point γ_1 in Δ. We shall first show that the distribution $I_M(\gamma_1)$ is tempered. In fact, by the construction in [3], § 2, § 5 and [4], § 2, $I_M(\gamma_1)$ lies in the (weak) closure of the space of continuous linear forms on $C_c^\infty(G(F))$ spanned by the distributions (9.3). However, if K_0 is any open, compact subgroup of $G(F)$, Theorem 8.1 tells us that the space of linear forms on $C_c^\infty(G(F)//K_0)$ spanned by (9.3) is finite dimensional. The restriction of $I_M(\gamma_1)$ to $C_c^\infty(G(F)//K_0)$ is therefore in this space. Since the distributions in (9.3) are tempered, the restriction of $I_M(\gamma_1)$ to $C_c^\infty(G(F)//K_0)$ coincides with a tempered distribution. It therefore extends to a continuous linear form on $\mathscr{C}(G(F)//K_0)$. The group K_0 is arbitrary, so by the definition of the Schwartz space, $I_M(\gamma_1)$ extends to a continuous linear form on $\mathscr{C}(G(F))$ which remains in the (weak) closure of the space spanned by (9.3). In particular, $I_M(\gamma_1)$ is tempered.

To establish (4.1), we have only to observe that a decomposition of the form (9.4) holds if γ equals our general point γ_1 in Δ. Indeed, since $I_M(\gamma_1)$ lies in the weak closure of the space spanned by (9.3), the restriction of $I_M(\gamma_1)$ to $\mathscr{C}(G(F), \Omega)$ is spanned by the finite set $\{I_\alpha : \alpha \in A'(\Delta, \Omega)\}$. This means that

$$I_M(\gamma_1, f) = \sum_{\alpha \in A'(\Delta, \Omega)} t_M^\alpha(\gamma_1) I_\alpha(f), \quad f \in \mathscr{C}(G(F), \Omega),$$

for uniquely determined functions t_M^α on the entire set Δ. It follows that

$$I_M(\gamma_1, f) = \sum_{\Omega} \sum_{\alpha \in A'(\Delta, \Omega)} t_M^\alpha(\gamma_1) I_\alpha(f)$$

if f is an arbitrary function in $\mathscr{C}(G(F))$. Substituting the expansion (9.5) for $I_\alpha(f)$, we obtain the required expansion (4.1) for $I_M(\gamma_1, f)$ if we define

$$I_M(\gamma_1, \tau) = \sum_{\Omega} \sum_{\alpha \in A'(\Delta, \Omega)} t_M^\alpha(\gamma_1) I_\alpha(\tau), \quad \tau \in T_{\mathrm{disc}}(L).$$

The symmetry condition (4.2) for $I_M(\gamma_1, \tau)$ follows easily from the corresponding symmetry conditions for $I_M(\gamma_1, f)$ and $I_\alpha(\tau)$. □

Remark. It is a consequence of the discussion that the limiting process by which one defines $I_M(\gamma, f)$ for singular γ, holds without change for the functions $I_M(\gamma, \tau)$. The process has two stages. If the centralizers M_γ and G_γ of γ in M and G are the same, one sees that the function

$$\delta \to I_M(\delta, f), \quad \delta \in M(F) \cap G_{\mathrm{reg}}(F),$$

is equal to the invariant orbital integral of some function $h_\gamma \in C_c^\infty(M(F))$, as long as δ is close to γ. The function $I_M(\gamma, \tau)$ can then be defined from $I_M(\delta, \tau)$ by inverting the Shalika germ expansion. If γ is an arbitrary element in $M(F)$, one must take a limit over points $a\gamma$, with $a \in A_M(F)$. (For any a which is close to 1, but in general position, $M_{a\gamma}$ equals $G_{a\gamma}$.) In this case, one can show that

$$I_M(\gamma, \tau) = \lim_{a \to 1} \sum_{S \in \mathscr{L}(M)} r_M^S(\gamma, a) I_S(a\gamma, \tau),$$

where $\{r_M^S(\gamma, a)\}$ are the functions defined in [3], § 5.

Corollary 9.2. *There is a constant p with the following property. For any open compact subset Δ of $M(F)$ and any connected component Ω of $T(G)$, there is a constant $c(\Delta, \Omega)$ such that*

$$|I_M(\gamma, \tau)| \leqq c(\Delta, \Omega)(1 + |\log|D(\gamma)||)^p,$$

for all $\gamma \in \Delta_{\mathrm{reg}}$, and all $\tau \in T_{\mathrm{disc}}(L) \cap \Omega_L$ with $L \in \mathscr{L}$.

Proof. We shall apply the formula

$$I_M(\gamma, \tau) = \sum_{\alpha \in A'(\Delta, \Omega)} t_M^\alpha(\gamma) I_\alpha(\tau), \quad \gamma \in \Delta_{\mathrm{reg}}, \tau \in T_{\mathrm{disc}}(L) \cap \Omega_L,$$

obtained from (9.6) by restricting the elements τ to be in $T_{\mathrm{disc}}(L) \cap \Omega_L$. The distributions $\{I_\alpha : \alpha \in A'(\Delta, \Omega)\}$ provide a finite set of linearly independent forms on the space $\mathscr{C}(G(F), \Omega)$. We can therefore find functions $\{f_\beta : \beta \in A'(\Delta, \Omega)\}$ in $\mathscr{C}(G(F), \Omega)$ with the property that

$$I_\alpha(f_\beta) = \begin{cases} 1, & \text{if } \alpha = \beta, \\ 0, & \text{otherwise}. \end{cases}$$

It follows from the formula (9.4) (with $S = M$) that

$$t_M^\alpha(\gamma) = I_M(\gamma, f_\alpha), \quad \gamma \in \Delta_{\mathrm{reg}}.$$

Now, recall the estimate (5.7) stated during the proof of Theorem 5.1. Combined with the inductive definition (3.5), it yields an estimate of the form

$$|I_M(\gamma, f)| \leqq \nu(f)(1+|\log|D(\gamma)||)^p(1+\|H_M(\gamma)\|)^{-n},$$

for any f and γ. If we specialize to $f = f_\alpha$ and $\gamma \in \Delta_{\mathrm{reg}}$, we obtain an inequality

$$|t_M^\alpha(\gamma)| \leqq c_\Delta^\alpha(1+|\log|D(\gamma)||)^p, \quad \gamma \in \Delta_{\mathrm{reg}},$$

for some constant c_Δ^α. The required estimate follows with

$$c(\Delta, \Omega) = \sum_{\alpha \in A'(\Delta, \Omega)} c_\Delta^\alpha \left(\sup_{L,\tau} |I_\alpha(\tau)|\right). \quad \square$$

Corollary 9.3. *The distribution*

$$g \to I_L(\tau, g), \quad g \in C_c^\infty(G_{\mathrm{reg}}(F)), \ \tau \in T(L),$$

is given by a function which is locally integrable on $G(F)$. In particular, the distribution has a canonical extension to functions g in $C_c^\infty(G(F))$.

Proof. Theorem 4.3 tells us that as a distribution on $G_{\mathrm{reg}}(F)$, $I_L(\tau)$ is given by the smooth class function

$$F(\gamma) = |D(\gamma)|^{-\frac{1}{2}} I_L(\tau, \gamma), \quad \gamma \in \Gamma_{\mathrm{ell}}(M(F)) \cap G_{\mathrm{reg}}(F), \ M \in \mathscr{L}.$$

We must show that the integral

$$\int_{G(F)} |F(x)|\, d(x)\, dx$$

is finite, for any nonnegative function g in $C_c^\infty(G(F))$.

By the descent formula $(3.7)^\vee$, it will be sufficient to prove the corollary when τ lies in $T_{\mathrm{ell}}(L)$. In this case, $i^L(\tau)$ equals the positive real number $|d^L(\tau)|^{-1}$, and the formula of Theorem 4.5 becomes

$$I_L(\tau, \gamma) = (-1)^{\dim(A_M \times A_L)} |d^L(\tau)|\, I_M(\gamma, \tau^\vee).$$

It follows from the Weyl character formula that

$$\int_{G(F)} |F(x)|\, g(x)\, dx$$

$$= \sum_{M \in \mathscr{L}} |W_0^M| \, |W_0^G|^{-1} \int_{\Gamma_{\mathrm{ell}}(M(F)) \cap G_{\mathrm{reg}}(F)} |D(\gamma)|^{\frac{1}{2}} |F(\gamma)|\, g_M(\gamma)\, d\gamma$$

$$= \sum_{M \in \mathscr{L}} |W_0^M| \, |W_0^G|^{-1} \sum_{\{T\}} |W(M(F), T(F))|^{-1} \int_{T_{\mathrm{reg}}(F)} |d^L(\tau)| \, |I_M(\gamma, \tau^\vee)|\, g_M(\gamma)\, d\gamma,$$

where $\{T\}$ is a set of representatives of $M(F)$-conjugacy classes of elliptic maximal tori in M over F. According to a basic theorem of Harish-Chandra ([16], Theorem 2, [18], Theorem 14), the orbital integral

$$g_M(\gamma) = J_G(\gamma, g) = |D(\gamma)|^{\frac{1}{2}} \int_{T(F)\backslash G(F)} f(x^{-1}\gamma x)\, dx, \quad \gamma \in T_{\mathrm{reg}}(F),$$

is bounded. But Corollary 9.2 gives us a bound for $|I_M(\gamma, \tau^\vee)|$ in terms of

$$(1 + |\log|D(\gamma)||)^p,$$

a function which is locally integrable on $T(F)$. It follows that the original integral is finite, and therefore that $F(\gamma)$ is a locally integrable function on $G(F)$.

The second assertion is clear. One simply defines

$$I_L(\tau, g) = \int_{G(F)} F(x) g(x)\, dx,$$

for any function $g \in C_c^\infty(G(F))$. □

Remark. In case $L = G$, we remark that

$$I_G(\tau, g) = J_G(\tau, g) = g_G(\tau), \quad g \in C_c^\infty(G(F)).$$

With the transition formulas (2.2) and (2.3) in mind, we can think of this distribution as being essentially an irreducible tempered character on $G(F)$. Corollary 9.3 can therefore be regarded as a generalization of the theorem of Harish-Chandra and Howe that an irreducible (tempered) character is a locally integrable function. A parallel generalization would be the assertion that the weighted character

$$g \to J_L(\tau, g), \quad g \in C_c^\infty(G(F)),\ \tau \in T(L),$$

is given by a locally integrable function on $G(F)$. This assertion can be established from Corollary 9.3, the formula $(4.5)^\vee$, and the estimate (5.7) for $J_S(\gamma, g)$.

We are not going to attempt to establish Corollaries 9.1–9.3 for Archimedean F. The proof is undoubtedly more difficult. One would probably need Archimedean germ expansions of the functions $I_M(\gamma, \tau)$ about a singular point γ_1.

§ 10. The distributions $I_{M,\mathrm{disc}}(\gamma)$ and $I_{L,\mathrm{ell}}(\tau)$

We shall conclude with a few brief comments on the extremal terms in the expansions of Theorems 4.1 and 4.3. Let us write $I_{M,\mathrm{disc}}(\gamma, f)$ for the summand corresponding to $L = G$ in the expansion (4.1) for $I_M(\gamma, f)$. Similarly, we write $I_{L,\mathrm{ell}}(\tau, g)$ for the summand corresponding to $M = G$ in the expansion $(4.1)^\vee$ for $I_L(\tau, g)$. Thus

$$I_{M,\mathrm{disc}}(\gamma, f) = \int_{T_{\mathrm{disc}}(G)} I_M(\gamma, \tau) f_G(\tau)\, d\tau, \quad f \in \mathscr{C}(G(F)),$$

and

$$I_{L,\mathrm{ell}}(\tau, g) = \int_{\Gamma_{\mathrm{ell}}(G(F))} I_L(\tau, \gamma) g_G(\gamma)\, d\gamma, \quad g \in C_c^\infty(G_{\mathrm{reg}}(F)).$$

These distributions can be described in more elementary terms.

In the first case, suppose that $\gamma \in \Gamma_{\mathrm{ell}}(M(F)) \cap G_{\mathrm{reg}}(F)$ and that $\tau \in T_{\mathrm{disc}}(G)$. It then follows from (4.6) that

$$I_M(\gamma, \tau) = (-1)^{\dim(A_M/A_G)} i^G(\tau) I_G(\tau^\vee, \gamma).$$

We obtain

$$(10.1) \qquad I_{M,\mathrm{disc}}(\gamma, f) = (-1)^{\dim(A_M/A_G)} \int_{T_{\mathrm{disc}}(G)} i^G(\tau) I_G(\tau^\vee, \gamma) f_G(\tau)\, d\tau,$$

for any function $f \in \mathscr{C}(G(F))$. As we observe directly from $(4.1)^\vee$, $I_G(\tau^\vee, \gamma)$ is just the value at γ of the virtual character on $G(F)$ attached to $\tau^\vee$ (normalized by the Weyl discriminant). More precisely, if

$$\tau = (L_1, \sigma, r), \quad L_1 \in \mathscr{L}, \ \sigma \in \Pi_2(L_1(F)), \ r \in \tilde{R}_\sigma,$$

we have

$$I_G(\tau^\vee, \gamma) = \sum_{\varrho \in \Pi(\tilde{R}_\sigma, \chi_\sigma)} \operatorname{tr}(\varrho(r)) |D(\gamma)|^{\frac{1}{2}} \Theta(\pi_\varrho^\vee, \gamma),$$

where $\Theta(\pi_\varrho^\vee, \cdot)$ is the character of the irreducible representation $\pi_\varrho^\vee$ of $G(F)$. (See the discussion at the beginning of [8], § 3.) The integral in (10.1) is a discrete sum (except for an insignificant integral over $i\mathfrak{a}_G^*$.) In other words, $I_{M,\mathrm{disc}}(\gamma)$ is essentially a linear combination of irreducible characters.

In the second case, suppose that $\tau \in T_{\mathrm{ell}}(L)$ and that $\gamma \in \Gamma_{\mathrm{ell}}(G(F)) \cap G_{\mathrm{reg}}(F)$. It then follows from (4.6) that

$$I_L(\tau, \gamma) = (-1)^{\dim(A_L/A_G)} |d^L(\tau)| I_G(\gamma, \tau^\vee),$$

since $i^L(\tau) = |d^L(\tau)|^{-1}$. We obtain

$$(10.1)^\vee \quad I_{L,\mathrm{ell}}(\tau, g) = (-1)^{\dim(A_L/A_G)} |d^L(\tau)| \int_{\Gamma_{\mathrm{ell}}(G(F))} I_G(\gamma, \tau^\vee) g_G(\gamma)\, d\gamma,$$

for any function $g \in C_c^\infty(G_{\mathrm{reg}}(F))$. As we see from (4.1), $I_G(\gamma, \tau^\vee)$ is the value at $\tau^\vee$ of the tempered function on $T(G)$ obtained by inverting the elliptic orbital integral $I_G(\gamma)$. Therefore, $I_{L,\mathrm{ell}}(\tau)$ is a continuous linear combination of functions obtained from elliptic orbital integrals.

A function $f \in \mathscr{C}(G(F))$ is said to be *cuspidal* if $f_L = 0$ for every $L \subsetneq G$. Suppose that this is so. Then the summands in (4.1) with $L \neq G$ all vanish, and $I_{M,\mathrm{disc}}(\gamma, f)$ equals

$I_M(\gamma, f)$. Moreover $f_G(\tau)$ equals 0 unless τ lies in the subset $T_{\mathrm{ell}}(G)$ of $T_{\mathrm{disc}}(G)$. Since $i^G(\tau) = |d^G(\tau)|^{-1}$ for any $\tau \in T_{\mathrm{ell}}(G)$, the formula (10.1) becomes

$$(10.2) \qquad I_M(\gamma, f) = (-1)^{\dim(A_M/A_G)} \int_{T_{\mathrm{ell}}(G)} |d^G(\tau)|^{-1} I_G(\tau^\vee, \gamma) f_G(\tau)\, d\tau\,.$$

This relationship between weighted orbital integrals and elliptic tempered characters is just [8], Theorem 5.1. Similarly, suppose that $g \in C_c^\infty(G_{\mathrm{reg}}(F))$ is cuspidal. Then the summands in $(4.1)^\vee$ with $L \neq G$ all vanish, and $I_{L,\mathrm{ell}}(\tau, g)$ equals $I_L(\tau, g)$. The formula $(10.1)^\vee$ becomes a relationship

$$(10.2)^\vee \qquad I_L(\tau, g) = (-1)^{\dim(A_L/A_G)} |d^L(\tau)| \int_{\Gamma_{\mathrm{ell}}(G(F))} I_G(\gamma, \tau^\vee) g_G(\gamma)\, d\gamma$$

between weighted characters and elliptic orbital integrals. If $G = GL(n)$ and τ is unramified, $(10.2)^\vee$ is essentially Waldspurger's formula [24], § II, Théorème.

The distribution $I_{M,\mathrm{disc}}(\gamma)$ is of particular interest. It is to be regarded as the "discrete part" of $I_M(\gamma)$ (or more precisely, of the distribution $\hat{I}_M(\gamma)$ on $T(G)$). The fact that $I_M(\gamma)$ has a discrete part is relevant to the general problem of comparing global trace formulas. In order to establish reciprocity relations between automorphic representations on different groups, one would try to prove identities between the terms of the corresponding global trace formulas. The distributions $I_M(\gamma)$ occur on the geometric side, but the discrete part $I_{M,\mathrm{disc}}(\gamma)$ has the potential to interfere with information obtained from the spectral side. To preclude this possibility, one would like to establish identities between certain linear combinations of distributions $\{I^H_{M_H,\mathrm{disc}}(\gamma_H)\}$ on endoscopic groups. It is likely that at least some of these identities could be established from the explicit formula (10.1) for $I_{M,\mathrm{disc}}(\gamma)$. The coefficients $i^G(\tau)$ that occur in (10.1) are of course an essential part of the problem.

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