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On elliptic tempered characters

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Contents

Introduction

Suppose that $G(F)$ is a real or p-adic group. That is, G is a connected reductive algebraic group over a local field F, which we take to be of characteristic 0. Harmonic analysis on $G(F)$ is built upon the set $\Pi_{\mathrm{temp}}(G(F))$ of irreducible tempered representations of $G(F)$. These representations include the discrete series for $G(F)$, and consist in general of irreducible constituents of representations induced from discrete series. We shall be interested in the subset of elliptic representations in $\Pi_{\mathrm{temp}}(G(F))$. The elliptic tempered representations also include the discrete series, and can be regarded as basic building blocks in $\Pi_{\mathrm{temp}}(G(F))$. The purpose of this paper is to study some properties of their characters.

We should recall that in general a representation $\pi \in \Pi_{\mathrm{temp}}(G(F))$ is infinite dimensional, and does not have a character in the classical sense. One of the cornerstones of

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the work of Harish-Chandra was his theory of characters of infinite dimensional representations. In general, the character of π is first defined as a distribution

$$\Theta(\pi, f) = \mathrm{tr}\left(\int_{G(F)} f(x)\pi(x)\,dx\right), \quad f \in C_c^\infty(G(F)),$$

which can then be identified with a function on $G(F)$. In other words,

$$\Theta(\pi, f) = \int_{G(F)} f(x)\Theta(\pi, x)\,dx, \quad f \in C_c^\infty(G(F)),$$

where $\Theta(\pi, x)$ is a locally integrable function on $G(F)$ that is smooth on the open dense subset $G_{\mathrm{reg}}(F)$ of regular elements. The elliptic representations $\Pi_{\mathrm{temp,ell}}(G(F))$ are the ones for which $\Theta(\pi, x)$ does not vanish on the elliptic set in $G_{\mathrm{reg}}(F)$. We would like to study the functions

$$\Phi(\pi, \gamma) = |D(\gamma)|^{1/2}\Theta(\pi, \gamma), \quad \pi \in \Pi_{\mathrm{temp,\,ell}}(G(F)),\ \gamma \in G_{\mathrm{reg}}(F),$$

where

$$D(\gamma) = \det(1 - \mathrm{Ad}(\gamma))_{\mathfrak{g}/\mathfrak{g}_\gamma}$$

is the Weyl discriminant.

In §2 we shall discuss the classification of elliptic tempered representations. This is well known for real groups, and is based in general on standard results. The problem is essentially that of decomposing an induced representation $\mathcal{I}_P(\sigma)$, where $P(F) = M(F)N(F)$ is a parabolic subgroup and σ is an irreducible tempered representation of the Levi component $M(F)$. In fact it is enough to treat the case that σ is square integrable, modulo the split component $A_M(F)$ of the center of $M(F)$. Let W_σ be the set of elements in the Weyl group of (G, A_M) which stabilize σ. For every element $w \in W_\sigma$, one can define a (normalized) self-intertwining operator $R(w, \sigma)$ of $\mathcal{I}_P(\sigma)$. The group W_σ itself has a decomposition

$$W_\sigma = W_\sigma' \rtimes R_\sigma,$$

where W_σ' is generated by reflections, and consists of elements $w \in W_\sigma$ such that the operator $R(w, \sigma)$ is a multiple of the identity. It is the complementary subgroup R_σ which determines the decomposition of $\mathcal{I}_P(\sigma)$. This is because the operators $R(w, \sigma)$, $w \in R_\sigma$, are known to be a $\mathbf{C}$-basis of the space of self-intertwining operators of $\mathcal{I}_P(\sigma)$.

Suppose for a moment that the map

$$w \to R(w, \sigma), \quad w \in R_\sigma,$$

is a homomorphism. The basis theorem then yields a bijection $\varrho \to \pi_\varrho$, from the set $\{\varrho\}$ of irreducible representations of R_σ onto the set of irreducible constituents of $\mathcal{I}_P(\sigma)$. To describe which of the representations π_ϱ are elliptic, we introduce a subset $R_{\sigma,\mathrm{reg}}$ of R_σ. The group R_σ acts on a real vector space

$$\mathfrak{a}_M = \mathrm{Hom}(X(M)_F, \mathbf{R}).$$

We define $R_{\sigma,\mathrm{reg}}$ to be the subset of elements in R_σ which leave pointwise invariant only the subspace $\mathfrak{a}_G$ of $\mathfrak{a}_M$. In Proposition 2.1 we shall see that the elliptic constituents of $\mathcal{I}_P(\sigma)$ correspond to the irreducible characters

$$\theta(\varrho, r) = \mathrm{tr}(\varrho(r)), \quad r \in R_\sigma,$$

of R_σ which do not vanish on $R_{\sigma,\mathrm{reg}}$. Proposition 2.1 actually applies to the general case, in which the map $r \to R(r,\sigma)$ is only a projective representation of R_σ. To deal with this complication, we take a suitable central extension

$$1 \to Z_\sigma \to \widetilde{R}_\sigma \to R_\sigma \to 1$$

of the R-group. The correspondence $\varrho \to \pi_\varrho$ then carries over verbatim, except that ϱ now ranges over irreducible representations of $\widetilde{R}_\sigma$ with a fixed central character on Z_σ. The classification of elliptic tempered representations follows. There is a bijection between $\Pi_{\mathrm{temp,ell}}(G(F))$ and the set of $G(F)$-orbits of triplets

$$\pi = (M, \sigma, \varrho),$$

where ϱ is an irreducible representation of $\widetilde{R}_\sigma$, with a certain central character χ_σ^{-1} on Z_σ, such that $\theta(\varrho)$ does not vanish on $\widetilde{R}_{\sigma,\mathrm{reg}}$.

With the preliminary discussion of §2 out of the way, we will turn to our study of characters. One of the simplest questions to describe concerns orthogonality relations. Suppose that π_ϱ and $\pi_{\varrho'}$ are two irreducible constituents of the induced representation $\mathcal{I}_P(\sigma)$. Given the characters $\Theta(\pi_\varrho)$ and $\Theta(\pi'_\varrho)$, we can form the elliptic inner product

$$\sum_{\{T\}} |W(G(F),T(F))|^{-1} \int_{T(F)/A_G(F)} |D(\gamma)| \Theta(\pi_\varrho, \gamma) \overline{\Theta(\pi_{\varrho'}, \gamma)}\, d\gamma, \tag{1}$$

where T runs over $G(F)$-conjugacy classes of elliptic maximal tori, and $d\gamma$ is the normalized Haar measure on $T(F)/A_G(F)$. The problem is to express this inner product in terms of the characters $\theta(\varrho)$ and $\theta(\varrho')$. In Corollary 6.3 we shall show that it equals a parallel inner product

$$|R_\sigma|^{-1} \sum_{r \in R_{\sigma,\mathrm{reg}}} |d(r)| \theta(\varrho, r) \overline{\theta(\varrho', r)} \tag{1*}$$

on the R-group, in which

$$d(r)=\det(1-r)_{\mathfrak{a}_M/\mathfrak{a}_G}.$$

The analogy of $d(r)$ with the Weyl discriminant is rather remarkable. It is a further example of a general correspondence between objects on $G(F)$ and objects on the R-group.

The identity of (1) with (1*) suggests a different point of view. Instead of irreducible representations of $\widetilde{R}_\sigma$ we ought to be working with conjugacy classes. Let $T(G)$ be the set of $G(F)$-orbits of triplets

$$\tau=(M,\sigma,r),$$

with M and σ as above, and r an element in $\widetilde{R}_\sigma$. Set $T_{\mathrm{ell}}(G)$ equal to the subset of orbits for which r lies in $\widetilde{R}_{\sigma,\,\mathrm{reg}}$. For any such τ, define

$$\Theta(\tau,f)=\mathrm{tr}(\widetilde{R}(r,\sigma)\mathcal{I}_P(\sigma,f)),\quad f\in C_c^\infty(G(F)),$$

where $\widetilde{R}(\,\cdot\,,\sigma)$ is the representation of $\widetilde{R}_\sigma$ obtained from the projective representation $R(\,\cdot\,,\sigma)$. Then $\Theta(\tau)$ is a virtual tempered character, in that it is a finite linear combination of irreducible tempered characters. The subset $\{\Theta(\tau):\tau\in T_{\mathrm{ell}}(G)\}$ presumably spans the space of virtual characters which are *supertempered* in the sense Harish-Chandra [22]. In any case, each distribution $\Theta(\tau)$ is represented by a locally integrable function $\Theta(\tau,x)$ on $G(F)$. The identity of (1) and (1*) will be a consequence of Corollary 6.2, which establishes orthogonality relations for the functions

$$\Phi(\tau,\gamma)=|D(\gamma)|^{1/2}\Theta(\tau,\gamma),\quad \tau\in T_{\mathrm{ell}}(G),\ \gamma\in G_{\mathrm{reg}}(F).$$

(In the paper we shall actually define $T(G)$ to be a slightly smaller set, by discarding certain triplets τ for which $\Theta(\tau)$ vanishes identically.)

The orthogonality relations (as well as most of the other results of the paper) will ultimately be consequences of the local trace formula introduced in [11]. The local trace formula can be regarded as an expansion of a certain distribution $I_{\mathrm{disc}}(f',f)$ in terms of weighted orbited integrals and weighted characters. Notice that there are two test functions f' and f, which we take to be in the Hecke algebra $\mathcal{H}(G(F))$ on $G(F)$. The distribution I_{disc} is defined by an elementary, but not uninteresting, linear combination of irreducible characters. The weighted orbital integrals and weighted characters are transcendental objects, related to noninvariant harmonic analysis on $G(F)$. We shall discuss the local trace formula in §3 and §4. The virtual characters $\Theta(\tau)$ are made to order for this purpose. In Proposition 3.1 we shall convert the definition [11, §12] of $I_{\mathrm{disc}}(f',f)$ into a simple expansion

$$I_{\mathrm{disc}}(f',f)=\int i(\tau)\Theta(\tau^\vee,f')\Theta(\tau,f)\,d\tau$$

in terms of these virtual characters, where $\tau^\vee = (M, \sigma^\vee, r)$ denotes the contragredient of $\tau = (M, \sigma, r)$. In §4 we shall convert the (noninvariant) trace formula of [11] into a local trace formula whose terms are all invariant distributions on $\mathcal{H}(G(F)) \times \mathcal{H}(G(F))$. As a bi-product of this discussion, we will obtain a local proof in Corollary 5.3 of the theorem of Kazhdan, which asserts that the invariant orbital integrals

$$I_G(\gamma, f) = |D(\gamma)|^{1/2} \int_{G_\gamma(F)\backslash G(F)} f(x^{-1}\gamma x)\, dx, \quad \gamma \in G_{\mathrm{reg}}(F),\ f \in \mathcal{H}(G(F)),$$

on $G(F)$ are supported on characters.

The invariant orbital integrals $I_G(\gamma, f)$ belong to a more general family of invariant distributions

$$I_M(\gamma, f), \quad M \subset G,\ \gamma \in G_{\mathrm{reg}}(F) \cap M(F),$$

attached to the (noninvariant) weighted orbital integrals

$$J_M(\gamma, f) = |D(\gamma)|^{1/2} \int_{G_\gamma(F)\backslash G(F)} f(x^{-1}\gamma x) v_M(x)\, dx.$$

These invariant distributions are the primary terms of the invariant local trace formula. (They play a similar role in the global trace formula [5].) One of the principal results of this paper is a formula (Theorem 5.1) for $I_M(\gamma, f)$ when f is a cuspidal function. Suppose that γ is a G-regular element in the elliptic set of the Levi subgroup $M(F)$. The formula is then an expansion

$$I_M(\gamma, f) = (-1)^{\dim(A_M/A_G)} \int_{T_{\mathrm{ell}}(G)} |d(\tau)|^{-1} \Phi(\tau^\vee, \gamma) \Theta(\tau, f)\, d\tau \tag{2}$$

of $I_M(\gamma, f)$ in terms of the elliptic tempered (virtual) characters $\Phi(\tau^\vee)$, evaluated at the (hyperbolic) element γ. We shall establish Theorem 5.1 from the invariant local trace formula, or rather the simple version (Corollary 4.3) that applies to the case that f is cuspidal. If we take f to be a "pseudo-coefficient" of a fixed element $\tau \in T_{\mathrm{ell}}(G)$, we will recognize Theorem 5.1 as an extension to elliptic representations of formulas [2], [3] for characters of discrete series as weighted orbital integrals. The applications to the proof of Kazhdan's theorem (Corollary 5.3) and the orthogonality relations (§6) will both be obtained from special cases of the formula (2).

The weighted orbital integral $J_M(\gamma, f)$ is a compactly supported function of the conjugacy classes γ in $M(F)$. However, the corresponding invariant distribution $I_M(\gamma, f)$ does not have compact support. To take care of this problem, and the resulting difficulties for the application to base change, we introduced a parallel family $\{{}^cI_M(\gamma, f)\}$ of distributions in [4, §4]. The distribution ${}^cI_M(\gamma, f)$ is invariant in f and compactly supported in γ. It is obtained from the definition of $I_M(\gamma, f)$ by changing a certain contour of integration, in a fashion suggested by the Paley–Wiener theorem. Changes of contour lead naturally to residues. In this case, one defines two further families

$$\{D_M(\sigma, X, f), {}^cD_M(\sigma, X, f) : \sigma \in \Pi_{\text{temp}}(M(F)), X \in \mathfrak{a}_{M,F}\} \tag{3}$$

of invariant distributions in terms of the residues in $\lambda \in \mathfrak{a}^*_{M,\mathbf{C}}$ of weighted characters

$$e^{-\lambda(X)} J_M(\sigma_\lambda, f) = e^{-\lambda(X)} \operatorname{tr}(\mathcal{R}_M(\sigma_\lambda, P)\mathcal{I}_P(\sigma_\lambda, f)).$$

We shall review all of these distributions, and various related objects, in §7. Since most of the ideas have been treated elsewhere, the discussion will be quite brief.

A second principal result of this paper is a parallel formula (Theorem 8.1) for ${}^cI_M(\gamma, f)$ when f is cuspidal. As in (2), suppose that γ is a G-regular element in the elliptic set of $M(F)$. The formula is then an expansion

$${}^cI_M(\gamma, f) = (-1)^{\dim(A_M/A_G)} \int_{T_{\text{ell}}(G)} |d(\tau)|^{-1}\, {}^c\Phi(\tau^\vee, \gamma)\Theta(\tau, f)\, d\tau \tag{4}$$

of ${}^cI_M(\gamma, f)$ in terms of truncated (virtual) characters ${}^c\Phi(\tau^\vee)$. The truncated character ${}^c\Phi(\tau^\vee, \gamma)$ is defined to be $\Phi(\tau^\vee, \gamma)$ or 0, according to whether or not the image $H_M(\gamma)$ of γ in $\mathfrak{a}_M$ lies in the subspace $\mathfrak{a}_G$. We shall establish Theorem 8.1 from Theorem 5.1 and properties of characters that apply separately to real and p-adic groups. For real groups we use the usual differential equations as in [7, §6], while for p-adic groups we use Casselman's formula for characters in terms of Jacquet modules.

We shall give two applications of Theorem 8.1. The first (Corollary 8.2) is a precise description of the support of a certain contour integral

$$\int_{\mu(X)+i\mathfrak{a}^*_{M,F}} e^{-\lambda(X)} \operatorname{tr}(\mathcal{R}_M(\sigma_\lambda, P)\mathcal{I}_P(\sigma_\lambda, f))\, d\lambda,$$

regarded as a function of $X \in \mathfrak{a}_{M,F}$. Here $\mu(X)$ is any point in the chamber $(\mathfrak{a}^*_Q)^+$ which is far from the walls, and $Q \supset M$ is the unique parabolic subgroup such that X lies in $\mathfrak{a}^+_Q$. This contour integral occurs in the definition of ${}^cI_M(\gamma, f)$. The required support property follows inductively from the definition of the functions ${}^cI_M(\tau^\vee, \gamma)$ on the right hand side of (4). Corollary 8.2 is valid only under quite restrictive conditions on f, which appear effectively to limit its use to the p-adic case. However, it may be relevant in this setting for the study of p-adic orbital integrals of spherical functions.

The second application of Theorem 8.1 is in §9. We shall establish two formulas relating characters to residues. The distributions (3) can easily be defined with σ replaced by an element $\tau_M \in T(M)$. Restricted to cuspidal functions f, the resulting distributions can be written as linear combinations of our basic virtual elliptic characters at f, with coefficients

$$\{D_M(\tau_M, X, \tau), {}^cD_M(\tau_M, X, \tau) : \tau \in T_{\mathrm{ell}}(G)\}.$$

It is these coefficients which contain the information about residues. Theorem 9.1 consists of one formula for the character $\Phi_M(\tau^\vee, \gamma)$ in terms of the coefficients

$${}^cD_L(\tau_L, H_L(\gamma), \tau), \quad L \supset M,\ \tau_L \in T_{\mathrm{ell}}(L),$$

and an inverse formula for the truncated character ${}^c\Phi_M(\tau^\vee, \gamma)$ in terms of the coefficients

$${}^cD_L(\tau_L, H_L(\gamma), \tau), \quad L \supset M,\ \tau_L \in T_{\mathrm{ell}}(L).$$

It would be interesting to investigate the relationship of these formulas with Osborne's conjecture (proved by Hecht and Schmid) on characters of real groups, and Casselman's formula for characters of p-adic groups.

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1. Elliptic tempered representations

Let G be a connected, reductive algebraic group over a local field F of characteristic 0. As in the paper [11], our concern will be the harmonic analysis of the locally compact group $G(F)$. We fix a maximal compact subgroup K of $G(F)$, which is hyperspecial if F is a p-adic field. For convenience, we recall some notation from [11, §1] that we will use in this paper.

As usual, one forms the real vector space

$$\mathfrak{a}_G = \mathrm{Hom}(X(G)_F, \mathbf{R})$$

from the module $X(G)_F$ of F-rational characters on G. There is a canonical homomorphism

$$H_G : G(F) \to \mathfrak{a}_G$$

defined by

$$e^{\langle H_G(x), \chi\rangle} = |\chi(x)|, \quad x \in G(F),\ \chi \in X(G)_F,$$

where $|\cdot|$ is the normalized valuation on F. Let A_G be the split component of the center of G. Then

$$\mathfrak{a}_{G,F} = H_G(G(F))$$

and

$$\tilde{\mathfrak{a}}_{G,F} = H_G(A_G(F))$$

are closed subgroups of $\mathfrak{a}_G$, while

$$\mathfrak{a}^\vee_{G,F} = \mathrm{Hom}(\mathfrak{a}_{G,F}, 2\pi i\mathbf{Z})$$

and

$$\tilde{\mathfrak{a}}^\vee_{G,F} = \mathrm{Hom}(\tilde{\mathfrak{a}}_{G,F}, 2\pi i\mathbf{Z})$$

are closed subgroups of $i\mathfrak{a}^*_G$. If F is a p-adic field, these four groups are all lattices. However, if F is Archimedean, $\tilde{\mathfrak{a}}_{G,F} = \mathfrak{a}_{G,F} = \mathfrak{a}_G$ and $\tilde{\mathfrak{a}}^\vee_{G,F} = \mathfrak{a}^\vee_{G,F} = \{0\}$.

It is convenient to fix a Haar measure on $\mathfrak{a}_G$. This determines a dual Haar measure on the real vector space $i\mathfrak{a}^*_G$. If F is a p-adic field, we normalize the measures so that the quotients $\mathfrak{a}_G/\tilde{\mathfrak{a}}_{G,F}$ and $i\mathfrak{a}^*_G/\tilde{\mathfrak{a}}^\vee_{G,F}$ each have volume 1. In this case, the volume of the quotient

$$i\mathfrak{a}^*_{G,F} = i\mathfrak{a}^*_G/\mathfrak{a}^\vee_{G,F}$$

equals the index $|\mathfrak{a}_{G,F}/\tilde{\mathfrak{a}}_{G,F}|$ of $\tilde{\mathfrak{a}}_{G,F}$ in $\mathfrak{a}_{G,F}$. In general, the kernel of H_G in $A_G(F)$ is compact, and therefore has a canonical normalized Haar measure. Since the group $\tilde{\mathfrak{a}}_{G,F} = H_G(A_G(F))$ is either discrete or equal to $\mathfrak{a}_G$, it also has a fixed Haar measure. These two Haar measures then determine a unique Haar measure on $A_G(F)$.

Let M_0 be a fixed F-rational Levi component of some minimal parabolic subgroup of G defined over F. We assume that K and $M_0(F)$ are in good relative position [11, §1]. Any parabolic subgroup P of G which is defined over F, and contains M_0, has a unique Levi component M_P which contains M_0. Both M_P and the unipotent radical N_P of P are defined over F. We write $\mathcal{L}$ for the finite set of subgroups of G of the form M_P, and we refer to the elements in $\mathcal{L}$ simply as *Levi subgroups* of G. Given any $M \in \mathcal{L}$, we write $\mathcal{F}(M) = \mathcal{F}^G(M)$ for the set of parabolic subgroups P of G over F such that M_P contains M, and $\mathcal{P}(M) = \mathcal{P}^G(M)$ for the subset of groups $P \in \mathcal{F}(M)$ with $M_P = M$. We also write $\mathcal{L}(M) = \mathcal{L}^G(M)$ for the set of Levi subgroups which contain M.

Suppose that $M \in \mathcal{L}$ is a Levi subgroup. Then $K_M = M(F) \cap K$ is a maximal compact subgroup of $M(F)$, and the triplet (M, K_M, M_0) satisfies the same hypotheses as (G, K, M_0). Any construction we make for G of course has an analogue for M. In particular, we can form the objects $\mathfrak{a}_M, H_M, A_M$, and so on. If P belongs to $\mathcal{P}(M)$ it is sometimes convenient to write $\mathfrak{a}_P = \mathfrak{a}_M$ and $A_P = A_M$. The symbol H_P, however, is reserved for the usual map from

$$G(F) = P(F)K = M_P(F)N_P(F)K$$

to $\mathfrak{a}_M$, defined for any element

$$x = mnk, \quad m \in M_P(F),\ n \in N_P(F),\ k \in K,$$

in $G(F)$ by

$$H_P(x) = H_M(m).$$

As we observed in [11, §1], the embeddings

$$A_G(F) \subset A_M(F) \subset M(F) \subset G(F)$$

give rise to mappings

$$\tilde{\mathfrak{a}}_{G,F} \hookrightarrow \tilde{\mathfrak{a}}_{M,F} \subset \mathfrak{a}_{M,F} \twoheadrightarrow \mathfrak{a}_{G,F}.$$

These in turn provide an embedding $\mathfrak{a}_G \hookrightarrow \mathfrak{a}_M$ and a surjection $\mathfrak{a}_M \twoheadrightarrow \mathfrak{a}_G$, from which we obtain a canonical decomposition $\mathfrak{a}_M = \mathfrak{a}_M^G \oplus \mathfrak{a}_G$. We fix Haar measures on the groups $\mathfrak{a}_M$, $i\mathfrak{a}_M^*$, $i\mathfrak{a}_{M,F}^*$ and $A_M(F)$ by following the prescriptions above for G. The Haar measures on $\mathfrak{a}_M$ and $\mathfrak{a}_G$ then induce a Haar measure on $\mathfrak{a}_M^G$. We may assume that as M varies over the finite set $\mathcal{L}$, the measures are all compatible with the transformations induced by elements in the Weyl group W_0^G of (G, A_{M_0}).

Let $M(F)_{\mathrm{ell}}$ denote the set of elements γ in $M(F)$ whose centralizer $M_\gamma(F)$ in $M(F)$ is compact modulo $A_M(F)$. We write $\Gamma_{\mathrm{ell}}(M(F))$ for the set of $M(F)$-conjugacy classes in $M(F)_{\mathrm{ell}}$. As in [11], we shall only be concerned with the intersection of $\Gamma_{\mathrm{ell}}(M(F))$ with $M_{\mathrm{reg}}(F)$, the set of M-regular elements in $M(F)$. The Haar measure on $A_M(F)$ determines a canonical measure on $\Gamma_{\mathrm{ell}}(M(F))$, which is supported on the intersection of $\Gamma_{\mathrm{ell}}(M(F))$ with $M_{\mathrm{reg}}(F)$, such that

$$\int_{\Gamma_{\mathrm{ell}}(M(F))} \phi(\gamma)\, d\gamma = \sum_{\{T\}} |W(M(F), T(F))|^{-1} \int_{T(F)} \phi(t)\, dt,$$

for any continuous function ϕ of compact support on $\Gamma_{\mathrm{ell}}(M(F)) \cap M_{\mathrm{reg}}(F)$. Here $\{T\}$ is a set of representatives of $M(F)$-conjugacy classes of maximal tori in M over F with $T(F)/A_M(F)$ compact, $W(M(F), T(F))$ is the Weyl group of $(M(F), T(F))$, and dt is the Haar measure on $T(F)$ determined by the Haar measure on $A_M(F)$ and the normalized Haar measure on the compact group $T(F)/A_M(F)$. We can then use the measures on $\Gamma_{\mathrm{ell}}(M(F))$, $M \in \mathcal{L}$, to write the Weyl integration formula as in [11, (2.2)]. The result is

$$\int_{G(F)} f(x)\, dx = \sum_{M \in \mathcal{L}} |W_0^M|\, |W_0^G|^{-1} \int_{\Gamma_{\mathrm{ell}}(M(F))} |D(\gamma)| \left(\int_{A_M(F) \backslash G(F)} f(x^{-1}\gamma x)\, dx \right) d\gamma,$$

where f is any function in $C_c^\infty(G(F))$, and

$$D(\gamma) = \det(1 - \mathrm{Ad}(\gamma))_{\mathfrak{g}/\mathfrak{g}_\gamma}$$

is the Weyl discriminant.

There is a simple notational convention, suggested by the work of Harish-Chandra, that we shall use regularly throughout the paper. Suppose that Θ is an invariant distribution on $G(F)$ which coincides with a function. In other words,

$$\Theta(f) = \int_{G(F)} \Theta(x) f(x)\, dx, \quad f \in C_c^\infty(G(F)),$$

where $\Theta(x)$ is a locally integrable function on $G(F)$. Suppose also that the restriction on $\Theta(x)$ to the regular set $G_{\mathrm{reg}}(F)$ is smooth. Then we shall write

$$\Phi(\gamma) = |D(\gamma)|^{1/2}\Theta(\gamma), \quad \gamma \in G_{\mathrm{reg}}(F), \tag{1.1}$$

for the "normalized" function on the regular set. In addition, if M belongs to $\mathcal{L}$, we let Φ_M denote the function on $M(F) \cap G_{\mathrm{reg}}(F)$ defined by

$$\Phi_M(\gamma) = \begin{cases} \Phi(\gamma), & \text{if } \gamma \in M(F)_{\mathrm{ell}} \cap G_{\mathrm{reg}}(F), \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

The functions Φ and Φ_M are of course invariant under conjugation by $G(F)$ and $M(F)$ respectively. In this situation the Weyl integration formula takes the form

$$\Theta(f) = \sum_{M \in \mathcal{L}} |W_0^M|\,|W_0^G|^{-1} \int_{\Gamma_{\mathrm{ell}}(M(F)) \cap G_{\mathrm{reg}}(F)} \Phi_M(\gamma) I(\gamma, f)\, d\gamma, \tag{1.3}$$

in which the distribution

$$I(\gamma, f) = I_G(\gamma, f), \quad \gamma \in G_{\mathrm{reg}}(F),\ f \in C_c^\infty(G(F)),$$

is the normalized orbital integral

$$|D(\gamma)|^{1/2} \int_{G_\gamma(F)\backslash G(F)} f(x^{-1}\gamma x)\, dx.$$

The main example is a *virtual character*, by which we mean a finite linear combination of characters of irreducible representations of $G(F)$. Many of our results will be stated in terms of normalized (virtual) characters (1.1) and (1.2).

In connection with distributions, it will actually be convenient for us to work with the Hecke algebra $\mathcal{H}(G(F))$ rather than the full space $C_c^\infty(G(F))$. Recall that $\mathcal{H}(G(F))$ is the space of functions in $C_c^\infty(G(F))$ which are left and right K-finite. We shall refer to a continuous linear functional on $\mathcal{H}(G(F))$, somewhat incorrectly, as a *distribution on* $\mathcal{H}(G(F))$. This of course is a more general object than a distribution on $G(F)$.

Finally, we recall a few standard notions from representation theory that we will need. Let $\Pi_{\text{temp}}(G(F))$ be the set of (equivalence classes of) irreducible tempered representations of $G(F)$. For any $\pi \in \Pi_{\text{temp}}(G(F))$, we write $\mathfrak{a}_{G,\pi}^{\vee}$ for the stabilizer of π in $i\mathfrak{a}_G^*$, relative to the locally free action

$$\pi_\lambda(x) = \pi(x)e^{\lambda(H_G(x))}, \quad \lambda \in i\mathfrak{a}_G^*, \, x \in G(F),$$

of $i\mathfrak{a}_G^*$ on $\Pi_{\text{temp}}(G(F))$. Then

$$\mathfrak{a}_{G,F}^{\vee} \subset \mathfrak{a}_{G,\pi}^{\vee} \subset \tilde{\mathfrak{a}}_{G,F}^{\vee},$$

and we have

$$\mathfrak{a}_{G,\pi}^{\vee} = \mathrm{Hom}(\mathfrak{a}_{G,\pi}, 2\pi i\mathbf{Z}),$$

for a subgroup $\mathfrak{a}_{G,\pi}$ in $\mathfrak{a}_G$ such that

$$\tilde{\mathfrak{a}}_{G,F} \subset \mathfrak{a}_{G,\pi} \subset \mathfrak{a}_{G,F}.$$

Suppose that $M \in \mathcal{L}$ and that π belongs to $\Pi_{\text{temp}}(M(F))$. Given $P \in \mathcal{P}(M)$ and $\lambda \in i\mathfrak{a}_M^*$, we can form the parabolically induced representation

$$\mathcal{I}_P(\pi_\lambda, x), \quad x \in G(F),$$

of $G(F)$. It acts on a Hilbert space $\mathcal{H}_P(\pi)$, of vector valued functions on K, which is independent of λ. The most important case is when $\pi = \sigma$ belongs to the subset $\Pi_2(M(F))$ of representations in $\Pi_{\text{temp}}(M(F))$ which are square integrable modulo $A_M(F)$. We shall write $\Pi_\sigma(G(F))$ for the set of irreducible constituents of the induced representation $\mathcal{I}_P(\gamma)$. This is a finite subset of $\Pi_{\text{temp}}(G(F))$ which is independent of P.

It is a fundamental consequence of the work of Harish-Chandra that as M and σ range over $\mathcal{L}$ and $\Pi_2(M(F))$ respectively, the sets $\Pi_\sigma(G(F))$ exhaust $\Pi_{\text{temp}}(G(F))$. The problem of classifying $\Pi_{\text{temp}}(G(F))$ is then reduced to classifying the representations in the finite sets $\Pi_\sigma(G(F))$, and to determining the intersection of any two such sets. This second question is answered by the proposition below, another consequence of work of Harish-Chandra.

We shall write $\widetilde{w}$ for any representative in K of an element w in the Weyl group W_0^G. If M and σ are as above, $wM = \widetilde{w}M\widetilde{w}^{-1}$ is another Levi subgroup, and

$$(w\sigma)(m') = \sigma(\widetilde{w}^{-1}m'\widetilde{w}), \quad m' \in (wM)(F),$$

is a representation in $\Pi_2((wM)(F))$. We obtain an action

$$(M, \sigma) \to (wM, w\sigma), \quad w \in W_0^G,$$

of W_0^G on the set of pairs

$$(M, \sigma), \quad M \in \mathcal{L}, \, \sigma \in \Pi_2(M(F)).$$

PROPOSITION 1.1. *Let (M,σ) and (M',σ') be any two pairs. If (M',σ') equals $(wM,w\sigma)$ for an element $w\in W_0^G$, the subsets $\Pi_\sigma(G(F))$ and $\Pi_{\sigma'}(G(F))$ of $\Pi_{\mathrm{temp}}(G(F))$ are identical. Conversely, if the sets $\Pi_\sigma(G(F))$ and $\Pi_{\sigma'}(G(F))$ have a representation in common, there is an element $w\in W_0^G$ such that $(M',\sigma')=(wM,w\sigma)$.*

The first assertion follows easily from the formula for the characters of the induced representations $\mathcal{I}_P(\sigma)$ and $\mathcal{I}_{P'}(\sigma')$. The second assertion is a deeper result, which is a consequence of Harish-Chandra's asymptotic estimates for matrix coefficients [19], [21], [36]. This was first observed by Langlands [32, §3], who used the property in his classification of admissible representations in terms of tempered representations. □

The proposition tells us that $\Pi_{\mathrm{temp}}(G(F))$ is the disjoint union, over all W_0^G-orbits of pairs (M,σ), of the sets $\Pi_\sigma(G(F))$. The remaining classification problem, apart from that of the square integrable representations σ in $\Pi_2(M(F))$, is then to determine the structure of the finite sets $\Pi_\sigma(G(F))$. Its solution is provided by the theory of the R-group.

A representation of $G(F)$ is said to be *elliptic* if its character does not vanish on the regular elliptic set. We shall denote the character of a general representation $\pi\in\Pi_{\mathrm{temp}}(G(F))$ by $\Theta(\pi)$. That is,

$$\mathrm{tr}(\pi(f))=\Theta(\pi,f)=\int_{G(F)}\Theta(\pi,x)f(x)\,dx,\qquad f\in\mathcal{H}(G(F)).$$

Then π is elliptic if and only if the normalized character $\Phi_G(\pi)$ does not vanish. We write $\Pi_{\mathrm{temp,\,ell}}(G(F))$ for the set of elliptic representations in $\Pi_{\mathrm{temp}}(G(F))$. It is the disjoint union, over all W_0^G-orbits of pairs (M,σ), of the sets

$$\Pi_{\sigma,\,\mathrm{ell}}(G(F))=\Pi_\sigma(G(F))\cap\Pi_{\mathrm{temp,\,ell}}(G(F)).$$

In the next section we shall describe the sets $\Pi_\sigma(G(F))$ and the subsets $\Pi_{\sigma,\,\mathrm{ell}}(G(F))$ in terms of the corresponding R-groups. The rest of the paper will be devoted to a study of the functions

$$\Phi_M(\pi),\quad M\in\mathcal{L},\ \pi\in\Pi_{\mathrm{temp,\,ell}}(G(F)),$$

and their relationships with other objects that arise naturally in the harmonic analysis on $G(F)$.

2. The *R*-group

We shall review the theory of the R-group, which provides a classification of the representations in the sets $\Pi_\sigma(G(F))$ and $\Pi_{\sigma,\,\mathrm{ell}}(G(F))$. These results are well known, at least in the Archimedean case [29]. There are some complications for p-adic groups, but the general ideas are similar. In particular, the classification follows from Harish-Chandra's commuting algebra theorem and Silberger's dimension theorem.

The decomposition of induced representations is of course determined by intertwining operators. For each $M \in \mathcal{L}$ and $\pi \in \Pi_{\mathrm{temp}}(M(F))$, one can construct normalized intertwining operators

$$R_{Q|P}(\pi) = r_{Q|P}(\pi)^{-1} J_{Q|P}(\pi) : \mathcal{H}_P(\pi) \to \mathcal{H}_Q(\pi), \quad P, Q \in \mathcal{P}(M),$$

between the induced representations $\mathcal{I}_P(\pi)$ and $\mathcal{I}_Q(\pi)$. The scalar normalizing factors $r_{Q|P}(\pi)$ are not unique, although there has been progress [35] towards constructing the canonical normalizations conjectured by Langlands [31, Appendix II]. The normalizing factors are, however, canonically determined from the case that $\pi = \sigma$ belongs to $\Pi_2(M(F))$. We fix them so that the conditions of [6, Theorem 2.1] all hold.

Let σ be a representation in $\Pi_2(M(F))$. We shall consider the stabilizer

$$W_\sigma = \{ w \in W(\mathfrak{a}_M) : w\sigma \cong \sigma \}$$

of σ in the Weyl group of $\mathfrak{a}_M$. For every w in W_σ, we must define a normalized intertwining operator

$$R(w, \sigma) = R_{P|P}(w, \sigma), \quad P \in \mathcal{P}(M),$$

from $\mathcal{I}_P(\sigma)$ to itself. Observe that σ can be extended to a representation of the group $M_w^+(F)$ generated by $M(F)$ and $\widetilde{w}$. If σ_w is such an extension, we define an intertwining operator

$$A(\sigma_w) : \mathcal{H}_{\widetilde{w}^{-1} P \widetilde{w}}(\sigma) \to \mathcal{H}_P(\sigma)$$

between $\mathcal{I}_{\widetilde{w}^{-1} P \widetilde{w}}(\sigma)$ and $\mathcal{I}_P(\sigma)$ by setting

$$(A(\sigma_w)\phi')(x) = \sigma_w(\widetilde{w}) \phi'(\widetilde{w}^{-1} x), \quad \phi' \in \mathcal{H}_{\widetilde{w}^{-1} P \widetilde{w}}(\sigma)$$

The composition

$$R(w, \sigma) = A(\sigma_w) R_{\widetilde{w}^{-1} P \widetilde{w} | P}(\sigma)$$

is then the desired intertwining operator for $\mathcal{I}_P(\sigma)$.

In general, we shall denote the contragredient of any representation π by $\pi^\vee$. In the case that $\pi = \sigma$ belongs to $\Pi_2(M(F))$ as above, the scalar normalizing factors can be chosen so that $r_{Q|P}(\sigma^\vee)$ equals $r_{P|Q}(\sigma)$. We shall also occasionally write $A^\vee$ for the transpose of an operator A if there is no danger of confusion. One finds that

$$\begin{aligned} R(w, \sigma)^\vee &= r_{\widetilde{w}^{-1} P \widetilde{w} | P}(\sigma)^{-1} J_{\widetilde{w}^{-1} P \widetilde{w} | P}(\sigma)^\vee A(\sigma_w)^\vee \\ &= r_{P | \widetilde{w}^{-1} P \widetilde{w}}(\sigma^\vee)^{-1} J_{P | \widetilde{w}^{-1} P \widetilde{w}}(\sigma^\vee) A(\sigma_w^\vee)^{-1} \\ &= R_{P | \widetilde{w}^{-1} P \widetilde{w}}(\sigma^\vee) A(\sigma_w^\vee)^{-1} \\ &= R(w, \sigma^\vee)^{-1}, \end{aligned}$$

for any $w \in W_\sigma$. We file the resulting formula

$$R(w, \sigma^\vee) = (R(w, \sigma)^{-1})^\vee \tag{2.1}$$

for future reference.

Fix a representation σ in $\Pi_2(M(F))$. We write W_σ^0 for the subgroup of elements w in W_σ such that the operator $R(w, \sigma)$ is a scalar. Then W_σ^0 is a normal subgroup of W_σ. The quotient

$$R_\sigma = W_\sigma / W_\sigma^0$$

is the R-group of σ. It is known that W_σ^0 is the Weyl group of a root system, composed of scalar multiples of those reduced roots α of (G, A_M) for which the reflection w_α belongs to W_σ^0. (See [28, §13], [37]. The proof makes essential use of the separate characterization of $\{\alpha\}$ as the roots whose corresponding Plancherel density vanishes.) These roots divide the vector space $\mathfrak{a}_M$ into chambers. Fixing such a chamber $\mathfrak{a}_\sigma^+$, we identify R_σ with the subgroup of elements in W_σ which preserve $\mathfrak{a}_\sigma^+$. We can then write W_σ as a semi-direct product

$$W_\sigma = W_\sigma^0 \rtimes R_\sigma.$$

According to Harish-Chandra's commuting algebra theorem ([20, Theorem 38.1], [36, Theorem 5.5.3.2]), the operators

$$\{R(r, \sigma) : r \in R_\sigma\}$$

span the algebra of intertwining operators of $\mathcal{I}_P(\sigma)$. The dimension theorem ([27, Theorem 2], [28, Theorem 13.4], [37]) asserts further that these operators are linearly independent. Before we can exploit these facts, however, we must deal with the possibility that the map $r \to R(r, \sigma)$ is not a homomorphism. In general, we have only a formula

$$R(r_1 r_2, \sigma) = \eta_\sigma(r_1, r_2) R(r_1, \sigma) R(r_2, \sigma), \quad r_1, r_2 \in R_\sigma,$$

where

$$\eta_\sigma(r_1, r_2) = A(\sigma_{r_1 r_2}) A(\sigma_{r_2})^{-1} A(\sigma_{r_1})^{-1} = \sigma_{r_1 r_2}(\tilde{r}_1 \tilde{r}_2) \sigma_{r_2}(\tilde{r}_2)^{-1} \sigma_{r_1}(\tilde{r}_1)^{-1}$$

is a 2-cocycle for R_σ with values in $\mathbf{C}^*$. The image $\bar{\eta}_\sigma$ of η_σ in $H^2(R_\sigma, \mathbf{C}^*)$ is the obstruction to being able to extend the representation σ to the group generated by $M(F)$ and $\{\tilde{r} : r \in R_\sigma\}$. For real groups the cocycle always splits [26, Theorem 7.1]. However, for p-adic groups the question is presently unresolved [25]. The expected parameterization of representations in L-packets in fact suggests that the cocycle might sometimes be nontrivial.

We shall deal with the problem by fixing a finite central extension

$$1 \rightarrow Z_\sigma \rightarrow \widetilde{R}_\sigma \rightarrow R_\sigma \rightarrow 1$$

over which η_σ splits. (See [16, Theorem 53.7]. For example, one could take Z_σ to be the cyclic group generated by $\bar{\eta}_\sigma$ in $H^2(R_\sigma, \mathbf{C}^*)$.) We then choose a function $\xi_\sigma : \widetilde{R}_\sigma \rightarrow \mathbf{C}^*$ which splits η_σ. This means that

$$\eta_\sigma(r_1, r_2) = \xi_\sigma(r_1 r_2)\xi_\sigma(r_2)^{-1}\xi_\sigma(r_1)^{-1}, \quad r_1, r_2 \in \widetilde{R}_\sigma,$$

where η_σ is obviously identified with its pullback to $\widetilde{R}_\sigma \times \widetilde{R}_\sigma$. It follows easily that

$$\xi_\sigma(zr) = \chi_\sigma(z)\xi_\sigma(r), \quad z \in Z_\sigma,\ r \in \widetilde{R}_\sigma,$$

for a linear character χ_σ on the central subgroup Z_σ. We can use ξ_σ to twist the intertwining operators. The result is a homomorphism

$$\widetilde{R}(r, \sigma) = \xi_\sigma(r)^{-1} R(r, \sigma), \quad r \in \widetilde{R}_\sigma, \tag{2.2}$$

of $\widetilde{R}_\sigma$ into the group of unitary intertwining operators for $\mathcal{I}_P(\sigma)$, with the property that

$$\widetilde{R}(zr, \sigma) = \chi_\sigma(z)^{-1}\widetilde{R}(r, \sigma), \quad z \in Z_\sigma,\ r \in \widetilde{R}_\sigma.$$

In fact, the map $R \rightarrow \widetilde{R}$ can be defined for any projective representation R of R_σ with multiplier η_σ. It determines a bijection from the set of such objects onto the set of ordinary representations of $\widetilde{R}_\sigma$ whose central character on Z_σ equals χ_σ^{-1}.

Observe that

$$\mathcal{R}(r, x) = \widetilde{R}(r, \sigma)\mathcal{I}_P(\sigma, x), \quad r \in \widetilde{R}_\sigma,\ x \in G(F),$$

is a representation of $\widetilde{R}_\sigma \times G(F)$ on $\mathcal{H}_P(\sigma)$. It has a decomposition into irreducible representations, which we write in the form

$$\mathcal{R} = \bigoplus_{\varrho, \pi} m_{\varrho,\pi}(\varrho^\vee \otimes \pi).$$

Here ϱ ranges over the set $\Pi(\widetilde{R}_\sigma, \chi_\sigma)$ of irreducible representations of $\widetilde{R}_\sigma$ with Z_σ-central character χ_σ, π ranges over $\Pi_\sigma(G(F))$, and each $m_{\varrho,\pi}$ is a nonnegative integer. The commuting algebra theorem implies that each integer $m_{\varrho,\pi}$ equals 0 or 1, and in addition, that for any ϱ there is at most one π with $m_{\varrho,\pi} = 1$, and that for any π there is at most one ϱ with $m_{\varrho,\pi} = 1$. The dimension theorem provides a bound in the other direction. It tells us that the map $r \to \widetilde{R}(r,\sigma)$ of $\widetilde{R}_\sigma$ into $\mathrm{End}_{G(F)}(\mathcal{H}_P(\sigma))$ must be equivalent to the representation obtained by inducing χ_σ^{-1} from Z_σ to $\widetilde{R}_\sigma$. We conclude that for every $\varrho \in \Pi(\widetilde{R}_\sigma, \chi_\sigma)$, there is a unique $\pi_\varrho \in \Pi_\sigma(G(F))$ such that $m_{\varrho,\pi_\varrho} = 1$. In other words, there is a bijection $\varrho \to \pi_\varrho$ of $\Pi(\widetilde{R}_\sigma, \chi_\sigma)$ onto $\Pi_\sigma(G(F))$ such that

$$\mathcal{R} = \bigoplus_{\varrho \in \Pi(\widetilde{R}_\sigma, \chi_\sigma)} (\varrho^\vee \otimes \pi_\varrho).$$

Expressed in terms of characters, the bijection is a formula

$$\mathrm{tr}(\widetilde{R}(r,\sigma)\mathcal{I}_P(\sigma,f)) = \sum_{\varrho \in \Pi(\widetilde{R}_\sigma, \chi_\sigma)} \mathrm{tr}(\varrho^\vee(r))\,\mathrm{tr}(\pi_\varrho(f)), \tag{2.3}$$

for any $r \in \widetilde{R}_\sigma$ and $f \in \mathcal{H}(G(F))$. We have thus obtained a classification of the representations in $\Pi_\sigma(G(F))$. We have still to determine which of these are elliptic.

If Π is a set of equivalence classes of irreducible representations of some group, let us write $\mathbf{C}(\Pi)$ for the complex vector space of virtual characters generated by Π. In particular, we can form the finite dimensional vector spaces $\mathbf{C}(\Pi(\widetilde{R}_\sigma, \chi_\sigma))$ and $\mathbf{C}(\Pi_\sigma(G(F)))$. From the bijection $\varrho \to \pi_\varrho$ we obtain an isomorphism $\theta \to \Theta$ from $\mathbf{C}(\Pi(\widetilde{R}_\sigma, \chi_\sigma))$ onto $\mathbf{C}(\Pi_\sigma(G(F)))$. To describe Θ in terms of θ, we have only to invert the formula (2.3). The result is

$$\Theta(f) = |\widetilde{R}_\sigma|^{-1} \sum_{r \in \widetilde{R}_\sigma} \theta(r)\,\mathrm{tr}(\widetilde{R}(r,\sigma)\mathcal{I}_P(\sigma,f)), \quad f \in \mathcal{H}(G(F)). \tag{2.4}$$

The correspondence $\theta \to \Theta$ behaves well under induction. To describe this, we must first introduce a family of subgroups of $\widetilde{R}_\sigma$.

Consider a Levi subgroup $L \in \mathcal{L}(M)$ with the property that the closure $\overline{\mathfrak{a}_\sigma^+}$ of the chamber $\mathfrak{a}_\sigma^+$ contains an open subset of $\mathfrak{a}_L$. Set

$$R_\sigma^L = W^L(\mathfrak{a}_M) \cap R_\sigma,$$

where $W^L(\mathfrak{a}_M)$ denotes the Weyl group of $\mathfrak{a}_M$ relative to L instead of G. We claim that R_σ^L can be identified with the R-group of σ relative to L. To see this, take any element w in the stabilizer

$$W_\sigma^L = W_\sigma \cap W^L(\mathfrak{a}_M)$$

of σ in $W^L(\mathfrak{a}_M)$, and consider the decomposition

$$w = w_0 r, \quad w_0 \in W_\sigma^0,\ r \in R_\sigma.$$

The conditions on r and w imply that the element $w_0^{-1}=rw^{-1}$ maps $\mathfrak{a}_L\cap\overline{\mathfrak{a}_\sigma^+}$ into $\overline{\mathfrak{a}_\sigma^+}$. Since $\overline{\mathfrak{a}_\sigma^+}$ is a fundamental domain for W_σ^0, we see that w_0^{-1} leaves the open subset $\mathfrak{a}_L\cap\overline{\mathfrak{a}_\sigma^+}$ of $\mathfrak{a}_L$ pointwise fixed. Therefore the whole space $\mathfrak{a}_L$ is left pointwise fixed by w_0^{-1}, and hence also by w_0 and r. In other words, w_0 belongs to the group $(W_\sigma^L)^0=W_\sigma^0\cap W^L(\mathfrak{a}_M)$, and r belongs to R_σ^L. It follows that $W_\sigma^L=(W_\sigma^L)^0\rtimes R_\sigma^L$, so R_σ^L can indeed be identified with the R-group relative to L. It is the complement of $(W_\sigma^L)^0$ in W_σ^L determined by the chamber $(\mathfrak{a}_\sigma^+ + \sigma_L)$ for the action of $(W_\sigma^L)^0$ on $\mathfrak{a}_M$. Having defined R_σ^L, we take $\widetilde{R}_\sigma^L$ to be the inverse image of R_σ^L in $\widetilde{R}_\sigma$. Then

$$1\to Z_\sigma\to \widetilde{R}_\sigma^L\to R_\sigma^L\to 1$$

is a central extension of R_σ^L which splits the restriction of the cocycle η_σ to R_σ^L. We shall consider representations of $\widetilde{R}_\sigma$ induced from subgroups $\widetilde{R}_\sigma^L$.

Given L as above, let ϱ_L be a representation in $\Pi(\widetilde{R}_\sigma^L,\chi_\sigma)$. We can induce this representation from $\widetilde{R}_\sigma^L$ to $\widetilde{R}_\sigma$, thereby obtaining a character θ in $\mathbf{C}(\Pi(\widetilde{R}_\sigma,\chi_\sigma))$. On the other hand, ϱ_L determines a representation $\pi_L=\pi_{\varrho_L}$ in $\Pi_\sigma(L(F))$, which for any $Q\in\mathcal{P}(L)$ we can induce from $Q(F)$ to $G(F)$. This gives a character Θ in $\mathbf{C}(\Pi_\sigma(G(F)))$ which is independent of Q. We claim that θ and Θ correspond under the bijection described above, and in particular, are related by (2.4). To see this, we first apply (2.4) to the characters of π_L and ϱ_L. Taking $P\in\mathcal{P}(M)$ to be any group contained in Q, we obtain

$$\begin{aligned}\Theta(f) &= \operatorname{tr}(\mathcal{I}_Q(\pi_L,f))\\ &= |\widetilde{R}_\sigma^L|^{-1}\sum_{r\in\widetilde{R}_\sigma^L}\operatorname{tr}(\varrho_L(r))\operatorname{tr}(\widetilde{R}(r,\sigma)\mathcal{I}_P(\sigma,f)),\quad f\in\mathcal{H}(G(F)),\end{aligned}$$

from the transitivity properties of induction. Since

$$r\to \operatorname{tr}(\widetilde{R}(r,\sigma)\mathcal{I}_P(\sigma,f)),\quad r\in\widetilde{R}_\sigma,$$

is a class function on $\widetilde{R}_\sigma$, the last expression becomes the right hand side of (2.4) when we apply the standard formula for the induced character θ.

Let us write $\mathbf{C}_{\mathrm{ind}}(\Pi(\widetilde{R}_\sigma,\chi_\sigma))$ for the submodule of $\mathbf{C}(\Pi(\widetilde{R}_\sigma,\chi_\sigma))$ generated by all characters θ of $\widetilde{R}_\sigma$ induced from representations $\varrho_L\in\Pi(\widetilde{R}_\sigma^L,\chi_\sigma)$, where $L\in\mathcal{L}(M)$ ranges over proper Levi subgroups of G with our condition that $\overline{\mathfrak{a}_\sigma^+}$ contains an open subset of $\mathfrak{a}_L$. Let us also write $\mathbf{C}_{\mathrm{ind}}(\Pi_\sigma(G(F)))$ for the submodule of $\mathbf{C}(\Pi_\sigma(G(F)))$ generated by all characters

$$\Theta(f)=\operatorname{tr}(\mathcal{I}_Q(\pi_L,f)),\quad Q\in\mathcal{P}(L),\ \pi_L\in\Pi_\sigma(L(F)),$$

where $L\in\mathcal{L}(M)$ ranges over all proper Levi subgroups of G. For any such Θ, we can replace π_L by a representation $w\pi_L$ in $\Pi_\sigma((wL)(F))$ for an element $w\in W_\sigma^0$. This means that we can replace the space $\mathfrak{a}_L$ by $w\mathfrak{a}_L$. We can therefore assume that L satisfies our condition that $\overline{\mathfrak{a}_\sigma^+}$ contains an open subset of $\mathfrak{a}_L$. It follows that the bijection $\theta \to \Theta$ maps a set of generators of $\mathbf{C}_{\mathrm{ind}}(\Pi(\widetilde{R}_\sigma, \chi_\sigma))$ to a set of generators of $\mathbf{C}_{\mathrm{ind}}(\Pi_\sigma(G(F)))$. Therefore, the image of $\mathbf{C}_{\mathrm{ind}}(\Pi(\widetilde{R}_\sigma, \chi_\sigma))$ is $\mathbf{C}_{\mathrm{ind}}(\Pi_\sigma(G(F)))$.

An element in $\mathbf{C}_{\mathrm{ind}}(\Pi_\sigma(G(F)))$, regarded as a locally integrable class function on $G_{\mathrm{reg}}(F)$, vanishes on the elliptic set $\Gamma_{\mathrm{ell}}(G(F))$. Therefore there is a map from the quotient

$$\mathbf{C}(\Pi_\sigma(G(F)))/\mathbf{C}_{\mathrm{ind}}(\Pi_\sigma(G(F)))$$

into a space of functions on $\Gamma_{\mathrm{ell}}(G(F)) \cap G_{\mathrm{reg}}(F)$. The map is actually injective, a fact that is implied by the stronger result [24, Theorem A]. It follows that the elliptic representations $\Pi_{\sigma,\mathrm{ell}}(G(F))$ are precisely the representations in $\Pi_\sigma(G(F))$ whose characters do not lie in $\mathbf{C}_{\mathrm{ind}}(\Pi_\sigma(G(F)))$.

It is easy to describe the irreducible representations in $\Pi(\widetilde{R}_\sigma, \chi_\sigma)$ which correspond to elliptic representations in $\Pi_\sigma(G(F))$. Let $\widetilde{R}_{\sigma,\mathrm{reg}}$ be the inverse image in $\widetilde{R}_\sigma$ of the set

$$R_{\sigma,\mathrm{reg}} = \{r \in R_\sigma : \mathfrak{a}_M^r = \mathfrak{a}_G\}, \tag{2.5}$$

where

$$\mathfrak{a}_M^w = \{H \in \mathfrak{a}_M : wH = H\}$$

denotes the space of fixed vectors of an element $w \in W(\mathfrak{a}_M)$. If r is an arbitrary element in R_σ, the space $\mathfrak{a}_M^r$ is of the form $\mathfrak{a}_L$ for some Levi subgroup $L \in \mathcal{L}(M)$. Moreover, it is a straightforward consequence of the invariance of $\mathfrak{a}_\sigma^+$ under r that $\overline{\mathfrak{a}_\sigma^+}$ contains an open subset of $\mathfrak{a}_L$, as above. It follows that $\widetilde{R}_\sigma$ is the disjoint union of $\widetilde{R}_{\sigma,\mathrm{reg}}$ with the set

$$\widetilde{R}'_\sigma = \bigcup_{L \neq G} \widetilde{R}_\sigma^L.$$

Now $\mathbf{C}_{\mathrm{ind}}(\Pi(\widetilde{R}_\sigma, \chi_\sigma))$ is the space of χ_σ-equivariant class functions on $\widetilde{R}_\sigma$ which are supported on $\widetilde{R}'_\sigma$. This follows from the usual formula for an induced character, and a simple induction argument based on the stratification of $\widetilde{R}'_\sigma$ by the subgroups $\{\widetilde{R}_\sigma^L\}$. There is consequently an isomorphism from the quotient

$$\mathbf{C}(\Pi(\widetilde{R}_\sigma, \chi_\sigma))/\mathbf{C}_{\mathrm{ind}}(\Pi(\widetilde{R}_\sigma, \chi_\sigma))$$

onto the space of χ_σ-equivariant class functions on $\widetilde{R}_{\sigma,\mathrm{reg}}$. Since $\mathbf{C}_{\mathrm{ind}}(\Pi(\widetilde{R}_\sigma, \chi_\sigma))$ corresponds with $\mathbf{C}_{\mathrm{ind}}(\Pi_\sigma(G(F)))$, we conclude that the elliptic representations in $\Pi_\sigma(G(F))$ are given by the irreducible characters in $\Pi(\widetilde{R}_\sigma, \chi_\sigma)$ which do not vanish on $\widetilde{R}_{\sigma,\mathrm{reg}}$. □

In summary we have

PROPOSITION 2.1. (a) *There is a unique bijection* $\varrho \to \pi_\varrho$ *from* $\Pi(\widetilde{R}_\sigma, \chi_\sigma)$ *onto* $\Pi_\sigma(G(F))$ *which satisfies the character identity* (2.3).

(b) *A sum of characters in* $\Pi(\widetilde{R}_\sigma, \chi_\sigma)$ *is induced from a proper subgroup* $\widetilde{R}^L_\sigma$ *of* $\widetilde{R}_\sigma$ *if and only if the corresponding sum of characters in* $\Pi_\sigma(G(F))$ *is induced from a parabolic subgroup with Levi component* $L(F)$.

(c) *A representation* π_ϱ *in* $\Pi_\sigma(G(F))$ *is elliptic if and only if the character of* ϱ *does not vanish on* $\widetilde{R}_{\sigma,\mathrm{reg}}$. □

Remarks. (1) For elements w_0 in W^0_σ, one can use the scalar valued operators $R(w_0, \sigma)$ to normalize the extensions σ_{w_0}. Each extension can then be chosen so that the corresponding operator $R(w_0, \sigma)$ is the identity. One obtains a representation

$$\widetilde{R}(w_0 r, \sigma) = \widetilde{R}(r, \sigma), \quad w_0 \in W^0_\sigma,\ r \in \widetilde{R}_\sigma,$$

of the group

$$\widetilde{W}_\sigma = W^0_\sigma \rtimes \widetilde{R}_\sigma$$

on $\mathcal{H}_P(\sigma)$. In particular, the operators $\widetilde{R}(r, \sigma)$ do not have to depend on the embedding of R_σ into W_σ.

(2) In the paper [13, §5], Clozel conjectured various relationships between representations in $\Pi_\sigma(G(F))$ and induced representations from the sets $\Pi_\sigma(L(F))$. More recently, D. Goldberg and R. Herb have discovered some unexpected phenomena for p-adic groups. (See R. Herb, "Elliptic representations for Sp($2n$) and SO(n)", preprint.) In general, Proposition 2.1 will be a good vehicle for studying the questions raised by Clozel. One has only to look for parallel relationships between representations in $\Pi(\widetilde{R}_\sigma, \chi_\sigma)$ and induced representations from the sets $\Pi(\widetilde{R}^L_\sigma, \chi_\sigma)$.

3. The distribution I_{disc}

We have described the classification of irreducible tempered characters in terms of irreducible representations of $\widetilde{R}$-groups. For some purposes it is better to work instead with the objects determined by conjugacy classes in $\widetilde{R}$-groups. These objects are virtual tempered characters, and provide a second basis for the complex vector space spanned by the irreducible tempered characters. They are particularly suited to the study of the discrete part of the local trace formula. The local trace formula, we will recall, is an expansion of a certain distribution $I_{\mathrm{disc}}(f', f)$ on $\mathcal{H}(G(F)) \times \mathcal{H}(G(F))$ in terms of weighted orbital integrals and weighted characters [11, §12]. We shall first discuss the basic virtual tempered characters. We will then be able to give a simple description of $I_{\mathrm{disc}}(f', f)$.

In §2 we attached certain objects to representations σ in $\Pi_2(M(F))$ which were not uniquely determined. These were the scalar normalizing factors $\{r_{Q|P}(\sigma)\}$, the extensions $\{\sigma_w\}$ of σ to the groups $\{M_w^+(F)\}$, the chamber $\mathfrak{a}_\sigma^+$, the extension $\widetilde{R}_\sigma \to R_\sigma$, and the function $\xi_\sigma : \widetilde{R}_\sigma \to \mathbf{C}^*$ which splits the cocycle η_σ. The degree to which other objects, such as the bijection $\varrho \to \pi_\varrho$, depend on these choices is minimal. In fact many of the objects we will look at are completely independent of the choices. In any case, we assume from now on that these choices have all been made, for every M and σ, subject only to any obvious compatibility conditions. For example, we will want a symmetry condition with respect to the action of W_0^G. We require that conjugation of R_σ by an element w in W_0^G extends to an isomorphism $r \to wr$ from $\widetilde{R}_\sigma$ onto $\widetilde{R}_{w\sigma}$. Another condition involves the contragredient. We can clearly take $\widetilde{R}_{\sigma^\vee} = \widetilde{R}_\sigma$ and $\chi_{\sigma^\vee} = \chi_\sigma^{-1}$, and by (2.1), we can also assume that the representation of $\widetilde{R}_\sigma \times G(F)$ attached to $\sigma^\vee$ is the contragredient of the representation attached to σ. This means that the correspondence (2.3) for $\sigma^\vee$ takes the form

$$\operatorname{tr}(\widetilde{R}(r, \sigma^\vee)\mathcal{I}_P(\sigma^\vee, f)) = \sum_{\varrho \in \Pi(\widetilde{R}_\sigma, \chi_\sigma)} \operatorname{tr}(\varrho(r)) \operatorname{tr}(\pi_\varrho^\vee(f)).$$

In what follows, we shall generally not distinguish in our notation between objects defined on R_σ and the corresponding Z_σ-invariant objects on $\widetilde{R}_\sigma$. For example, $X \to rX$ could stand for the action of $\widetilde{R}_\sigma$ on $\mathfrak{a}_M$ with isotropy subgroup Z_σ, as well as the underlying R_σ-action from which it is obtained.

We shall consider triplets

$$\tau = (M, \sigma, r), \quad M \in \mathcal{L},\ \sigma \in \Pi_2(M(F)),\ r \in \widetilde{R}_\sigma.$$

For any such τ, we define a distribution

$$\Theta(\tau, f) = \operatorname{tr}(\widetilde{R}(r, \sigma)\mathcal{I}_P(\sigma, f)), \quad f \in \mathcal{H}(G(F)). \tag{3.1}$$

This is the virtual character whose decomposition is given by (2.3). Set $Z_\tau = Z_\sigma$ and $\chi_\tau = \chi_\sigma$. Then if z belongs to Z_τ,

$$z\tau = (M, \sigma, zr)$$

is another triplet, which satisfies

$$\Theta(z\tau, f) = \chi_\tau(z)^{-1}\Theta(\tau, f).$$

There is also an action

$$\tau \to w\tau = (wM, w\sigma, wr), \quad w \in W_0^G,$$

of W_0^G on the set of all triplets, with the property that

$$\Theta(w\tau, f) = \Theta(\tau, f).$$

These two conditions force some of the distributions $\Theta(\tau)$ to vanish. We shall say that $\tau = (M, \sigma, r)$ is *essential* if the subgroup of elements in Z_σ which stabilize the $\widetilde{R}_\sigma$-conjugacy class of r is contained in the kernel of χ_σ. (This is always the case if the cocycle η_σ splits.) The inessential distributions $\Theta(\tau)$ are then zero and can be discarded. We shall write $\widetilde{T}(G)$ for the remaining set of essential triplets, and we define $T(G)$ to be the set of W_0^G-orbits in $\widetilde{T}(G)$. Our basic objects are then the distributions

$$\{\Theta(\tau) : \tau \in T(G)\}.$$

Taken up to the equivalence relation defined by the action of the groups Z_τ, these distributions form a basis of the vector space of all virtual tempered characters. This follows from Proposition 1.1 and the formula (2.3) for the irreducible constituents of $\mathcal{I}_P(\sigma)$. We are particularly concerned with the subset

$$T_{\mathrm{ell}}(G) = \{\tau = (M, \sigma, r) \in T(G) : r \in \widetilde{R}_{\sigma, \mathrm{reg}}\}$$

of orbits in $T(G)$ which are elliptic. These triplets correspond to distributions which do not vanish on $G(F)_{\mathrm{ell}}$. They are the elliptic tempered (virtual) characters of the title.

Observe that $\widetilde{T}(G)$ has a natural structure of an analytic manifold. For if $\tau = (M, \sigma, r)$ is any triplet, the isotropy subspace $\mathfrak{a}_M^r$ of $\mathfrak{a}_M$ equals $\mathfrak{a}_L$ for some $L \in \mathcal{L}(M)$. There is a locally free action

$$\tau \to \tau_\lambda = (M, \sigma_\lambda, r), \quad \lambda \in i\mathfrak{a}_L^*,$$

of $i\mathfrak{a}_L^*$ on the subset of elements $\tau \in \widetilde{T}(G)$ of this form. In this way, $\widetilde{T}(G)$ becomes an analytic manifold which is homeomorphic to either a disjoint union of Euclidean spaces (Archimedean case), or a disjoint union of compact tori (p-adic case). The set $T(G)$ then acquires the quotient topology from the action of W_0^G.

One place where the set $T(G)$ is simpler to use than $\Pi_{\mathrm{temp}}(G(F))$ is in the formulation of the trace Paley–Wiener theorem. Let $\mathcal{I}(G(F))$ be the space of functions

$$\phi : \widetilde{T}(G) \to \mathbf{C}$$

which satisfy the following four conditions.

(i) ϕ is supported on finitely many components of $\widetilde{T}(G)$.

(ii) $\phi(z\tau) = \chi_\tau(z)^{-1}\phi(\tau)$, $\tau \in \widetilde{T}(G)$, $z \in Z_\tau$.

(iii) ϕ is symmetric under W_0^G.

(iv) The function obtained by restricting ϕ to any connected component of $\widetilde{T}(G)$ belongs to the Paley–Wiener space; that is, the function is a finite Fourier series in the p-adic case, or the Fourier transform of a smooth function of compact support if F is Archimedean.

There is a natural topology which makes $\mathcal{I}(G(F))$ into a complete topological vector space. By means of the inversion formula (2.4), we can in fact identify $\mathcal{I}(G(F))$ with the topological vector space of functions on $\Pi_{\text{temp}}(G(F))$ introduced in [6, §11], and also denoted by $\mathcal{I}(G(F))$. The trace Paley–Wiener theorem [12] [14] is equivalent to the assertion that the map which sends $f \in \mathcal{H}(G(F))$ to the function

$$f_G(\tau) = \Theta(\tau, f), \quad \tau \in T(G),\ f \in \mathcal{H}(G(F)),$$

is a continuous surjective map from $\mathcal{H}(G(F))$ onto $\mathcal{I}(G(F))$. Observe that if τ is an element in $T_{\text{ell}}(G)$, there is a function f in $\mathcal{H}(G(F))$ with $f_G(\tau)=1$, and such that f_G vanishes away from the $(Z_\tau \times i\mathfrak{a}_G^*)$-orbit of τ in $T(G)$. We shall call such an f a *pseudocoefficient* for τ.

For the local trace formula it is useful to take a set which lies between $T_{\text{ell}}(G)$ and $T(G)$. If r belongs to an R-group R_σ, we write $W_\sigma(r)_{\text{reg}}$ for the intersection of the W_σ^0-coset

$$W_\sigma(r) = W_\sigma^0 r$$

in W_σ with the set

$$W_{\sigma,\,\text{reg}} = \{w \in W_\sigma : \mathfrak{a}_M^w = \mathfrak{a}_G\}$$

of regular elements. This also serves to define $W_\sigma(r)$ and $W_\sigma(r)_{\text{reg}}$ for elements $r \in \widetilde{R}_\sigma$, as we have agreed earlier. We define $T_{\text{disc}}(G)$ to be the set of orbits (M, σ, r) in $T(G)$ such that $W_\sigma(r)_{\text{reg}}$ is not empty. It is clear that

$$T_{\text{ell}}(G) \subset T_{\text{disc}}(G) \subset T(G).$$

To each element $\tau = (M, \sigma, r)$ in $T_{\text{disc}}(G)$ we attach a number

$$i(\tau) = i^G(\tau) = |W_\sigma^0|^{-1} \sum_{w \in W_\sigma(r)_{\text{reg}}} \varepsilon_\sigma(w) |\det(1-w)_{\mathfrak{a}_M^G}|^{-1}. \tag{3.2}$$

As in [11, p. 139], $\varepsilon_\sigma(w)$ stands for the sign of the projection of w onto the Weyl group W_σ^0, taken relative to the decomposition $W_\sigma = W_\sigma^0 \rtimes R_\sigma$. The numbers $i(\tau)$ encode combinatorial data from Weyl groups that is relevant to the comparison of global trace formulas [9]. We shall see in a moment that the numbers also arise in the local trace formula.

We can now describe the distribution I_{disc}. Fix functions f', $f \in \mathcal{H}(G(F))$. According to the definition [11, (12.4)], $I_{\mathrm{disc}}(f', f)$ equals the expression

$$\sum_{(M,\sigma,w)} |W_0^M|\,|W_0^G|^{-1}|\det(1-w)_{\mathfrak{a}_M^G}|^{-1}\varepsilon_\sigma(w)|\mathfrak{a}_{G,\sigma}^\vee/\mathfrak{a}_{G,F}^\vee|^{-1}\int_{i\mathfrak{a}_{G,F}^*}\tilde{J}_G(\sigma_\lambda, w, f'\times f)\,d\lambda,$$

where

$$\mathfrak{a}_{G,\sigma}^\vee = \mathrm{Hom}(\mathfrak{a}_{G,\sigma}, 2\pi i\mathbf{Z}) = \mathfrak{a}_{M,\sigma}^\vee \cap i\mathfrak{a}_G^*,$$

and

$$\tilde{J}_G(\sigma_\lambda, w, f'\times f) = \mathrm{tr}(R(w,\sigma_{-\lambda}^\vee)\mathcal{I}_P(\sigma_{-\lambda}^\vee, f'))\,\mathrm{tr}(R(w,\sigma_\lambda)\mathcal{I}_P(\sigma_\lambda, f)).$$

The sums are over $M\in\mathcal{L}$, $\sigma\in\Pi_2(M(F))/i\mathfrak{a}_G^*$ (the set of $i\mathfrak{a}_G^*$-orbits in $\Pi_2(M(F))$), and $w\in W_{\sigma,\mathrm{reg}}$. We would like to write this expression in a simpler form.

The essential step is to change the sum over $w\in W_{\sigma,\mathrm{reg}}$ into a double sum over $r\in R_\sigma$ and $w\in W_\sigma(r)_{\mathrm{reg}}$. We shall in fact sum over $r\in\tilde{R}_r$, at the same time dividing the summand by the quotient $|Z_\sigma| = |\tilde{R}_\sigma|\,|R_\sigma|^{-1}$. By (2.1) we can write

$$R(w,\sigma_{-\lambda}^\vee) = (R(w,\sigma_\lambda)^{-1})^\vee.$$

(This formula was used in [11, p. 136] without comment to write I_{disc} in the form above.) Combining this with the definitions of W_σ^0 and $\tilde{R}(r,\sigma)$, we see that if w lies in $W_\sigma(r)_{\mathrm{reg}}$, the operators $R(w,\sigma_{-\lambda}^\vee)$ and $R(w,\sigma_\lambda)$ differ from $\tilde{R}(r,\sigma_{-\lambda}^\vee)$ and $\tilde{R}(r,\sigma_\lambda)$ respectively by scalar multiples which are inverses of each other. It follows that

$$\begin{aligned}\tilde{J}_G(\sigma_\lambda, w, f'\times f) &= \mathrm{tr}(\tilde{R}(r,\sigma_{-\lambda}^\vee)\mathcal{I}_P(\sigma_{-\lambda}^\vee, f'))\,\mathrm{tr}(\tilde{R}(r,\sigma_\lambda)\mathcal{I}_P(\sigma_\lambda, f))\\ &= \Theta(\tau_{-\lambda}^\vee, f')\Theta(\tau_\lambda, f),\end{aligned}$$

where τ is the triplet (M,σ,r) and

$$\tau^\vee = (M,\sigma^\vee, r). \tag{3.3}$$

In particular, this term depends only on r. The sum over $W_\sigma(r)_{\mathrm{reg}}$ then leads directly to the number $i(\tau)$ defined in (3.2). We find that $I_{\mathrm{disc}}(f', f)$ equals the sum over all triplets

$$\tau = (M,\sigma,r),\quad M\in\mathcal{L},\ \sigma\in\Pi_2(M(F))/i\mathfrak{a}_G^*,\ r\in\tilde{R}_\sigma,$$

of the expression

$$i(\tau)|W_\sigma^0|\,|W_0^M|\,|W_0^G|^{-1}|Z_\sigma|^{-1}|\mathfrak{a}_{G,\sigma}^\vee/\mathfrak{a}_{G,F}^\vee|^{-1}\int_{i\mathfrak{a}_{G,F}^*}\Theta(\tau_{-\lambda}^\vee, f')\Theta(\tau_\lambda, f)\,d\lambda. \tag{3.4}$$

The summand (3.4) depends only on the W_0^G-orbit of τ. Moreover, the summand vanishes unless the triplet τ is essential, and the set $W_\sigma(r)_{\mathrm{reg}}$ is nonempty. The collection of all such W_0^G-orbits is just $\Pi_{\mathrm{disc}}(G)$, or rather, the set $\Pi_{\mathrm{disc}}(G)/i\mathfrak{a}_G^*$ of orbits of $i\mathfrak{a}_G^*$ in $\Pi_{\mathrm{disc}}(G)$. We can therefore take the sum over $\Pi_{\mathrm{disc}}(G)/i\mathfrak{a}_G^*$, provided that we replace $|W_0^G|^{-1}$ in the summand by the inverse of the order of the W_0^G-stabilizer of τ. The stabilizer of M in W_0^G is the subgroup $W_0^M \cdot W(\mathfrak{a}_M)$. The stabilizer of σ in this subgroup is $W_0^M \cdot W_\sigma$. Finally, the stabilizer of r in this second subgroup is $W_0^M \cdot W_\sigma^0 \cdot R_{\sigma,r}$, where $R_{\sigma,r}$ is the stabilizer of r in R_σ. Therefore, the stabilizer of $\tau=(M,\sigma,r)$ in W_0^G is $W_0^M \cdot W_\sigma^0 \cdot R_{\sigma,r}$. The order of this last group equals

$$|W_0^M|\,|W_\sigma^0|\,|\widetilde{R}_{\sigma,r}|\,|Z_\sigma|^{-1},$$

where $\widetilde{R}_{\sigma,r}$ is the centralizer of r in $\widetilde{R}_\sigma$. It follows that $I_{\mathrm{disc}}(f',f)$ equals

$$\sum i(\tau)|\widetilde{R}_{\sigma,r}|^{-1}|\mathfrak{a}_{G,\sigma}^\vee/\mathfrak{a}_{G,F}^\vee|^{-1}\int_{i\mathfrak{a}_{G,F}^*}\Theta(\tau_{-\lambda}^\vee,f')\Theta(\tau_\lambda,f)\,d\lambda,$$

where the sum is over elements $\tau=(M,\sigma,r)$ in $\Pi_{\mathrm{disc}}(G)/i\mathfrak{a}_G^*$.

Let us define a measure $d\tau$ on $T_{\mathrm{disc}}(G)$ by setting

$$\int_{T_{\mathrm{disc}}(G)}\theta(\tau)\,d\tau=\sum_{\tau\in\Pi_{\mathrm{disc}}(G)/i\mathfrak{a}_G^*}|\widetilde{R}_{\sigma,r}|^{-1}|\mathfrak{a}_{G,\sigma}^\vee/\mathfrak{a}_{G,F}^\vee|^{-1}\int_{i\mathfrak{a}_{G,F}^*}\theta(\tau_\lambda)\,d\lambda \tag{3.5}$$

for any function $\theta\in C_c(\Pi_{\mathrm{disc}}(G))$. We can then express the formula for I_{disc} as follows.

PROPOSITION 3.1. *Suppose that f' and f are functions in $\mathcal{H}(G(F))$. Then*

$$I_{\mathrm{disc}}(f',f)=\int_{T_{\mathrm{disc}}(G)}i(\tau)\Theta(\tau^\vee,f')\Theta(\tau,f)\,d\tau. \tag{3.6}$$

□

Remark. We mentioned that the numbers $i(\tau)$ have occurred elsewhere [9]. They satisfy a combinatorial identity [9, Theorem 8.1] which is related to endoscopy. In that context they can be expected to play a significant role in the derivation of multiplicity formulas for automorphic representations. The occurence of the numbers in formula (3.6) here seems to be a separate issue. It is not clear what implication it might have for local harmonic analysis.

In this paper we shall use the special case of Proposition 3.1 in which one of the functions is cuspidal. A function f in $\mathcal{H}(G(F))$ is said to be *cuspidal* if for every proper Levi subgroup L of G, the function

$$f_L(\pi_L)=\mathrm{tr}(\pi_L(f_Q))=\mathrm{tr}(\mathcal{I}_Q(\pi_L,f)),\quad \pi_L\in\Pi_{\mathrm{temp}}(L(F)),$$

vanishes identically. (Here, f_Q is the usual function

$$m \to \delta_Q(m)^{1/2} \int_K \int_{N_Q(F)} f(k^{-1}mnk)\, dn\, dk, \quad m \in L(F),$$

on $L(F)$, defined for any group $Q \in P(L)$.) Notice that the map $f \to f_L$ factors through the space $\mathcal{I}(G(F))$, so the same condition defines cuspidal functions in $\mathcal{I}(G(F))$. Suppose that $f \in \mathcal{H}(G(F))$ is cuspidal. Take a triplet $\tau = (M, \sigma, r)$ in $\Pi_{\mathrm{disc}}(G)$, and let $L \in \mathcal{L}(M)$ be the Levi subgroup such that $\mathfrak{a}_M^r$ equals $\mathfrak{a}_L$. Applying the formula (2.3) to L, and keeping in mind the transitivity properties of induction, we see that the distribution

$$\Theta(\tau, f) = \operatorname{tr}(\widetilde{R}(r, \sigma)\mathcal{I}_P(\sigma, f))$$

is a linear combination of values $f_L(\pi_L)$, with π_L ranging over the representations in $\Pi_\sigma(L(F))$. Since f is cuspidal, the distribution vanishes unless $L = G$. In other words, the distribution vanishes unless r belongs to $\widetilde{R}_{\sigma,\mathrm{reg}}$, which is to say that τ belongs to $T_{\mathrm{ell}}(G)$. Thus, the integrand in the formula (3.6) for $I_{\mathrm{disc}}(f', f)$ is supported on the subset $T_{\mathrm{ell}}(G)$ of $T_{\mathrm{disc}}(G)$.

For a given $\sigma \in \Pi_2(M(F))$, the set $\widetilde{R}_{\sigma,\mathrm{reg}}$ could of course be empty. We claim that if it is not empty, then the subgroup W_σ^0 of W_σ is trivial. To see this, recall that W_σ^0 is the Weyl group of a system of roots on the real vector space $\mathfrak{a}_M$. The decomposition $W_\sigma = W_\sigma^0 \rtimes R_\sigma$ is determined by the chamber $\mathfrak{a}_\sigma^+$ for W_σ^0 in $\mathfrak{a}_M$, and R_σ acts on $\mathfrak{a}_M$ as a group of automorphisms of W_σ^0 which preserve the chamber. In particular, the elements in R_σ preserve the positive roots in the root system. They leave invariant the vector in $\mathfrak{a}_\sigma^+$ obtained in the usual way as half the sum of the positive co-roots. If $W_\sigma^0 \neq \{1\}$, this vector is not zero. In particular, any element in R_σ has an invariant vector in the complement of $\mathfrak{a}_G$ in $\mathfrak{a}_M$, and $R_{\sigma,\mathrm{reg}}$ is therefore empty. This establishes the claim.

We have just seen that if $\tau = (M, \sigma, r)$ belongs to $T_{\mathrm{ell}}(G)$, then $W_\sigma^0 = \{1\}$. The coset $W_\sigma(r)$ then equals r itself. Since $\varepsilon_\sigma(r) = 1$, the formula (3.2) for the number $i(\tau)$ reduces simply to the inverse of the absolute value of the number

$$d(\tau) = d(r) = \det(1 - r)_{\mathfrak{a}_M/\mathfrak{a}_G}. \tag{3.7}$$

We have established

COROLLARY 3.2. *Suppose that f' and f are functions in $\mathcal{H}(G(F))$ and that f is cuspidal. Then*

$$I_{\mathrm{disc}}(f', f) = \int_{T_{\mathrm{ell}}(G)} |d(\tau)|^{-1} \Theta(\tau^\vee, f')\Theta(\tau, f)\, d\tau. \qquad \square$$

4. The invariant local trace formula

The formula (3.6) can serve as a definition of $I_{\text{disc}}(f',f)$. The local trace formula provides a second formula for this distribution in terms of weighted orbital integrals and weighted characters. The terms in this second expression are distributions which are not invariant under conjugation. We are going to need a formula whose constituents are all invariant. We shall therefore describe how to convert the original (noninvariant) local trace formula into an invariant formula. The process is similar to that of the global trace formula [5, §§3–4]. In particular, we will obtain in the end a local proof of the theorem of Kazhdan [24] that (invariant) orbital intergrals are supported on characters. The results of this section have been sketched elsewhere [8, §8], [10, §3], so we can afford to be rather brief.

We should first review the noninvariant trace formula in the context of the expression (3.6) for $I_{\text{disc}}(f',f)$. As in [11], it is best to present the formula as an identity of two distributions, evaluated at a function

$$f'\times f, \quad f',f\in\mathcal{H}(G(F)),$$

on $G(F)\times G(F)$. One distribution is given as an expansion in terms of weighted orbital integrals (the geometric side), while the other distribution is a parallel expansion in terms of weighted characters (the spectral side). In this setting, $I_{\text{disc}}(f',f)$ is just the leading term in the spectral expansion.

The geometric side is the expansion

$$\sum_{M\in\mathcal{L}}|W_0^M|\,|W_0^G|^{-1}(-1)^{\dim(A_M/A_G)}\int_{\Gamma_{\text{ell}}(M)}J_M(\gamma,f'\times f)\,d\gamma, \tag{4.1}$$

whose constituents are defined as in [11, §12]. In particular, $\Gamma_{\text{ell}}(M)$ stands for the set $\Gamma_{\text{ell}}(M(F))$ of F-elliptic conjugacy classes in $M(F)$, but embedded diagonally in $M(F)\times M(F)$. The measure $d\gamma$ is the image of the measure on $\Gamma_{\text{ell}}(M(F))$ defined in §1. The integrand $J_M(\gamma,f'\times f)$ is the weighted orbital integral

$$|D(\gamma)|\int_{A_M(F)\backslash G(F)}\int_{A_M(F)\backslash G(F)}f'(x_1^{-1}\gamma x_1)f(x_2^{-1}\gamma x_2)v_M(x_1,x_2)\,dx_1\,dx_2,$$

where $v_M(x_1,x_2)$ is the number

$$\lim_{\Lambda\to 0}\sum_{Q\in\mathcal{P}(M)}v_Q(\Lambda,x_1,x_2)\theta_Q(\Lambda)^{-1},$$

obtained from the (G,M)-family

$$v_Q(\Lambda,x_1,x_2)=e^{-\Lambda(H_Q(x_2)-H_{\bar{Q}}(x_1))},\quad \Lambda\in i\mathfrak{a}_M^*,\ Q\in\mathcal{P}(M),$$

is the usual way [2, Lemma 6.2].

The spectral side will be an expansion

$$\sum_{M\in\mathcal{L}} |W_0^M|\,|W_0^G|^{-1}(-1)^{\dim(A_M/A_G)}\int_{T_{\mathrm{disc}}(M)} i^M(\tau)J_M(\tau,f'\times f)\,d\tau, \tag{4.2}$$

where $J_M(\tau,f'\times f)$ is a weighted virtual character attached to τ. It is a linear combination of the weighted characters

$$J_M(\pi_1^\vee\otimes\pi_2,f'\times f),\quad \pi_1,\pi_2\in\Pi_{\mathrm{temp}}(M(F)),$$

defined by [11, (12.8)], with coefficients determined by the analogue for M of (2.3). More precisely, if

$$\tau=(M_1,\sigma,r),\quad M_1\subset M,\ \sigma\in\Pi_2(M_1(F)),\ r\in\widetilde{R}_\sigma^M,$$

we take the formula

$$J_M(\tau,f'\times f)=\sum_{\varrho',\,\varrho\in\Pi(\widetilde{R}_\sigma^M,\chi_\sigma)}\mathrm{tr}(\varrho'(r))\,\mathrm{tr}(\varrho^\vee(r))J_M(\pi_{\varrho'}^\vee\otimes\pi_\varrho,f'\times f) \tag{4.3}$$

as a definition of $J_M(\tau,f'\times f)$. We recall for convenience that

$$J_M(\pi_1^\vee\otimes\pi_2,f'\times f)=\mathrm{tr}(\mathcal{J}_M(\pi_1^\vee\otimes\pi_2,P)\mathcal{I}_P(\pi_1^\vee\otimes\pi_2,f'\times f)),$$

where $\mathcal{J}_M(\pi_1^\vee\otimes\pi_2,P)$ is the operator

$$\lim_{\Lambda\to 0}\sum_{Q\in\mathcal{P}(M)}\mathcal{J}_Q(\Lambda,\pi_1^\vee\otimes\pi_2,P)\theta_Q(\Lambda)^{-1}$$

obtained from the (G,M)-family

$$\mathcal{J}_Q(\Lambda,\pi_1^\vee\otimes\pi_2,P)=(J_{\overline{Q}|P}(\pi_1^\vee)\otimes J_{Q|P}(\pi_2))^{-1}(J_{\overline{Q}|P}(\pi_{1,-\Lambda}^\vee)\otimes J_{Q|P}(\pi_{2,\Lambda})),$$

$\Lambda\in i\mathfrak{a}_M^*$, $Q\in\mathcal{P}(M)$. The operators

$$J_{Q|P}(\pi)\colon\mathcal{H}_P(\pi)\to\mathcal{H}_Q(\pi),\quad \pi\in\Pi_{\mathrm{temp}}(M(F)),$$

are the unnormalized intertwining operators from $\mathcal{I}_P(\pi)$ to $\mathcal{I}_Q(\pi)$. They can have poles in $\pi\in\Pi_{\mathrm{temp}}(M(F))$, so the functions $\mathcal{J}_Q(\Lambda,\pi_1^\vee\otimes\pi_2,P)$ are defined only for $\pi_1^\vee\otimes\pi_2$ in general position. However, the operator $\mathcal{J}_M(\pi_1^\vee\otimes\pi_2,P)$ is regular at $\pi_1^\vee\otimes\pi_2$ if π_1 and π_2 belong to $\Pi_\sigma(M(F))$ as above [11, Lemma 12.1]. The formula (4.3) used to define $J_M(\tau,f'\times f)$ therefore makes sense.

Stating our version of the noninvariant local trace formula formally, we have

PROPOSITION 4.1. *For any functions f' and f in $\mathcal{H}(G(F))$, the geometric expansion* (4.1) *equals the spectral expansion* (4.2).

Proof. The proposition is a restatement of the main result (Theorem 12.2) of [11]. The geometric sides (4.1) and [11, (12.9)] are the same. We need only reconcile our expansion (4.2) with

$$\sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)}\int_{\Pi_{\mathrm{disc}}(M)} a^M_{\mathrm{disc}}(\pi_1^\vee\otimes\pi_2)J_M(\pi_1^\vee\otimes\pi_2,f'\times f)d(\pi_1^\vee\otimes\pi_2), \tag{4.4}$$

the original spectral side [11, (12.10)]. The numbers $a^G_{\mathrm{disc}}(\pi_1^\vee\otimes\pi_2)$ are defined as coefficients of an expansion [11, (12.6)] of $I_{\mathrm{disc}}(f',f)$. Comparing this with the expansion (3.6), and taking into account the measures [11, (12.5)] and (3.5) on $\Pi_{\mathrm{disc}}(G)$ and $T_{\mathrm{disc}}(G)$, one finds that $a^G_{\mathrm{disc}}(\pi_1^\vee\otimes\pi_2)$ vanishes unless

$$\pi_1^\vee\otimes\pi_2=\pi_{\varrho'}^\vee\otimes\pi_\varrho,\quad \varrho',\varrho\in\Pi(\widetilde{R}_\sigma,\chi_\sigma),$$

for some $\sigma\in\Pi_2(M_1(F))$, with $M_1\in\mathcal{L}$, and that

$$a^G_{\mathrm{disc}}(\pi_{\varrho'}^\vee\otimes\pi_\varrho)=\sum_r|\widetilde{R}_{\sigma,r}|^{-1}i(M_1,\sigma,r)\,\mathrm{tr}(\varrho'(r))\,\mathrm{tr}(\varrho^\vee(r)),$$

the sum being over conjugacy classes in $\widetilde{R}_\sigma$. The coefficients $a^M_{\mathrm{disc}}(\pi_1^\vee\otimes\pi_2)$ are of course given by the specialization of this formula from G to M. Combining this with the definition (4.3), and taking into account the measures on $\Pi_{\mathrm{disc}}(M)$ and $T_{\mathrm{disc}}(M)$, we see that the expressions (4.2) and (4.4) are equal. The proposition follows from [11, Theorem 12.2]. □

The primary ingredients of the invariant local trace formula are to be invariant distributions

$$I_M(\gamma,f'\times f)=I^G_M(\gamma,f'\times f)$$

of two variable functions attached to the weighted orbital integrals $J_M(\gamma,f'\times f)$. They are defined inductively by a formula

$$I_M(\gamma,f'\times f)=J_M(\gamma,f'\times f)-\sum_{\substack{L\in\mathcal{L}(M)\\ L\neq G}}\hat{I}^L_M(\gamma,\phi_L(f'\times f)) \tag{4.5}$$

[4, (2.1)], [8, (8.1)], and they are related by a splitting formula

$$I_M(\gamma,f'\times f)=\sum_{L',L\in\mathcal{L}(M)} d^G_M(L',L)I^{L'}_M(\gamma,f'_{Q'})I^L_M(\gamma,f_Q), \tag{4.6}$$

$Q' \in \mathcal{P}(L')$, $Q \in \mathcal{P}(L)$, to the analogous distributions in one variable [4, Proposition 9.1]. These formulas contain some undefined terms from earlier papers. In particular, $\phi_L(f' \times f)$ could be defined as the function on $\Pi_{\mathrm{temp}}(L(F)) \times \Pi_{\mathrm{temp}}(L(F))$ whose value at $\pi_1^\vee \otimes \pi_2$ equals

$$\mathrm{tr}(\mathcal{R}_L(\pi_1^\vee \otimes \pi_2, P)\mathcal{I}_P(\pi_1^\vee \otimes \pi_2, f' \times f)), \quad P \in \mathcal{P}(L),$$

where $\mathcal{R}_L(\pi_1^\vee \otimes \pi_2, P)$ is the operator

$$\lim_{\Lambda \to 0} \sum_{Q \in \mathcal{P}(L)} \mathcal{R}_Q(\Lambda, \pi_1^\vee \otimes \pi_2, P)\theta_Q(\Lambda)^{-1}$$

obtained from the (G, L)-family

$$\mathcal{R}_Q(\Lambda, \pi_1^\vee \otimes \pi_2, P) = (R_{\overline{Q}|P}(\pi_1^\vee) \otimes R_{Q|P}(\pi_2))^{-1}(R_{\overline{Q}|P}(\pi_{1,-\Lambda}^\vee) \otimes R_{Q|P}(\pi_{2,\Lambda})),$$

$\Lambda \in i\mathfrak{a}_L^*$, $Q \in \mathcal{P}(L)$. There is a technical problem that $\phi_L(f' \times f)$ does not quite belong to the space $\mathcal{I}(L(F) \times L(F))$ discussed in §3. This can be resolved by working with the spaces $\mathcal{H}_{\mathrm{ac}}(L(F) \times L(F))$ and $\mathcal{I}_{\mathrm{ac}}(L(F) \times L(F))$ introduced in [6, §11]. In the present situation it is simpler to observe that $J_M(\gamma, f' \times f)$ depends only on the restriction of $f' \times f$ to the subgroup

$$(G(F) \times G(F))^1 = \{(x', x) \in G(F) \times G(F) : H_G(x') = H_G(x)\}.$$

Consequently, $J_M(\gamma, \cdot)$ can be regarded as a distribution on the associated Hecke algebra $\mathcal{H}((G(F) \times G(F))^1)$. The corresponding space $\mathcal{I}((G(F) \times G(F))^1)$ consists of Paley–Wiener functions on the set of $i\mathfrak{a}_G^*$-orbits in either $T(G) \times T(G)$ or $\Pi_{\mathrm{temp}}(G(F)) \times \Pi_{\mathrm{temp}}(G(F))$. From [6, Theorem 12.1], one can interpret ϕ_L as a continuous map from $\mathcal{H}((G(F) \times G(F))^1)$ to $\mathcal{I}((L(F) \times L(F))^1)$. The inductive definition (4.5) then gives $I_M(\gamma, \cdot)$ as (the pullback to $G(F) \times G(F)$ of) a distribution on $\mathcal{H}((G(F) \times G(F))^1)$.

In general, an invariant distribution I on $\mathcal{H}(G(F))$ is said to be *supported on characters* if $I(f) = 0$ for every function $f \in \mathcal{H}(G(F))$ such that $f_G = 0$. If this is so, there is a unique distribution $\hat{I}$ on the topological vector space $\mathcal{I}(G(F))$ such that

$$\hat{I}(f_G) = I(f), \quad f \in \mathcal{H}(G(F)).$$

Similar definitions apply to invariant distributions on the spaces $\mathcal{H}((G(F) \times G(F))^1)$, $\mathcal{H}_{\mathrm{ac}}(G(F) \times G(F))$ or $\mathcal{H}_{\mathrm{ac}}(G(F))$. In particular, the Fourier transform $\hat{I}_M^L(\gamma)$ in (4.5) is defined provided that $I_M^L(\gamma)$ is supported on characters. This has been proved by global means [5, Theorem 5.1], but we prefer to establish the property here by local means. We shall therefore make only the induction assumption that for any $L\in\mathcal{L}(M)$ with $L\supsetneq G$, and any point $\gamma\in M_{\mathrm{reg}}(F)$, the distribution $I_M^L(\gamma)$ on $\mathcal{H}(L(F))$ is supported on characters. It follows from the splitting formula (4.6) that the corresponding distributions on $\mathcal{H}(L(F)\times L(F))$ are also supported on characters. Therefore, the formula (4.5) makes sense.

The secondary ingredients of the invariant local trace formula are simpler. They will be the invariant distributions attached to the weighted characters $J_M(\tau, f'\times f)$. Suppose that $\pi_1^\vee\otimes\pi_2$ is a representation in $\Pi_{\mathrm{temp}}(M(F))\times\Pi_{\mathrm{temp}}(M(F))$, and that $P\in\mathcal{P}(M)$ is fixed. The (G,M)-families $\{\mathcal{J}_Q(\Lambda,\pi_1^\vee\otimes\pi_2,P)\}$ and $\{\mathcal{R}_Q(\Lambda,\pi_1^\vee\otimes\pi_2,P)\}$ described above are related by the formula

$$\mathcal{J}_Q(\Lambda,\pi_1^\vee\otimes\pi_2,P)=r_Q(\Lambda,\pi_1^\vee\otimes\pi_2,P)\mathcal{R}_Q(\Lambda,\pi_1^\vee\otimes\pi_2,P), \tag{4.7}$$

$Q\in\mathcal{P}(M)$, $\Lambda\in i\mathfrak{a}_M^*$, where

$$r_Q(\Lambda,\pi_1^\vee\otimes\pi_2,P)=\left(r_{\bar{Q}|P}(\pi_1^\vee)r_{Q|P}(\pi_2)\right)^{-1}\left(r_{\bar{Q}|P}(\pi_{1,-\Lambda}^\vee)r_{Q|P}(\pi_{2,\Lambda})\right)$$

is a (G,M)-family constructed from the scalar normalizing factors. As with the (G,M)-family $\mathcal{J}_Q(\Lambda,\pi_1^\vee\otimes\pi_2,P)$, the function $r_Q(\Lambda,\pi_1^\vee\otimes\pi_2,P)$ is well defined only for $\pi_1^\vee\otimes\pi_2$ in general position. However, if $\tau=(M_1,\sigma,r)$ belongs to $T_{\mathrm{disc}}(M)$, the function

$$r_M(\pi_1^\vee\otimes\pi_2,P)=\lim_{\Lambda\to 0}\sum_{Q\in\mathcal{P}(M)} r_Q(\Lambda,\pi_1^\vee\otimes\pi_2,P)\theta_Q(\Lambda)^{-1}$$

is regular at $\pi_1^\vee\otimes\pi_2$ whenever π_1 and π_2 belong to $\Pi_\sigma(M(F))$. This follows from the analogous property for $\mathcal{J}_M(\pi_1^\vee\otimes\pi_2,P)$. It can also be deduced directly from the fact that the normalizing factors $r_{Q|P}(\pi)$, $\pi\in\Pi_\sigma(M(F))$, depend only on σ. We shall write

$$r_M(\tau,P)=r_M(\pi_1^\vee\otimes\pi_2,P),\quad \pi_1,\pi_2\in\Pi_\sigma(M(F)), \tag{4.8}$$

since the special value on the right is independent of $\pi_1^\vee\otimes\pi_2$. The invariant distributions

$$r_M(\tau,f'\times f)=r_M(\tau,P)\Theta(\tau^\vee,f'_P)\Theta(\tau,f_P),\quad f',f\in\mathcal{H}(G(F)), \tag{4.9}$$

on $\mathcal{H}(G(F)\times G(F))$ will occur on the spectral side of the invariant trace formula.

THEOREM 4.2. *For any functions f' and f in $\mathcal{H}(G(F))$, the expression*

$$\sum_{M\in\mathcal{L}}|W_0^M|\,|W_0^G|^{-1}(-1)^{\dim(A_M/A_G)}\int_{\Gamma_{\mathrm{ell}}(M)} I_M(\gamma,f'\times f)\,d\gamma \tag{4.10}$$

equals

$$\sum_{M\in\mathcal{L}} |W_0^M|\,|W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \int_{T_{\mathrm{disc}}(M)} i^M(\tau) r_M(\tau, f'\times f)\, d\tau. \tag{4.11}$$

Proof. Observe that (4.11) is equal to the expression

$$\sum_{M\in\mathcal{L}} |W_0^M|\,|W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \sum_{\tau\in T_{\mathrm{disc}}(M)/i\mathfrak{a}_M^*} i^M(\tau)\bar{r}_M(\tau, f'\times f), \tag{4.11*}$$

where

$$\bar{r}_M(\tau, f'\times f) = |\widetilde{R}_{\sigma,r}|^{-1} |\mathfrak{a}_{M,\sigma}^\vee/\mathfrak{a}_{M,F}^\vee|^{-1} \int_{i\mathfrak{a}_{M,F}^*} r_M(\tau_\lambda, f'\times f)\, d\lambda, \tag{4.9*}$$

for $\tau=(M_1,\sigma,r)$. Similarly, the noninvariant spectral expansion (4.2) equals

$$\sum_{M\in\mathcal{L}} |W_0^M|\,|W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \sum_{\tau\in T_{\mathrm{disc}}(M)/i\mathfrak{a}_M^*} i^M(\tau)\bar{J}_M(\tau, f'\times f), \tag{4.2*}$$

where

$$\bar{J}_M(\tau, f'\times f) = |\widetilde{R}_{\sigma,r}|^{-1} |\mathfrak{a}_{M,\sigma}^\vee/\mathfrak{a}_{M,F}^\vee|^{-1} \int_{i\mathfrak{a}_{M,F}^*} J_M(\tau_\lambda, f'\times f)\, d\lambda. \tag{4.3*}$$

The first step is to check that the invariant distributions

$$\bar{r}_M(\tau, f'\times f) = \bar{r}_M^G(\tau, f'\times f)$$

are related to weighted characters by a formula

$$\bar{J}_M(\tau, f'\times f) = \sum_{L\in\mathcal{L}(M)} \hat{\bar{r}}_M^L(\tau, \phi_L(f'\times f)) \tag{4.12}$$

which is parallel to (4.5). The distributions $\bar{r}_M^L(\tau)$ can be regarded as functionals on $(L(F)\times L(F))^1$ which are supported on characters, so the summands on the right hand side of (4.12) are well defined.

To establish (4.12) we apply the decomposition property [2, Lemma 6.5] to the product (4.7) of (G,M)-families. The result is

$$\mathcal{J}_M(\pi_1^\vee\otimes\pi_2, P) = \sum_{L\in\mathcal{L}(M)} r_M^L(\pi_1^\vee\otimes\pi_2, P)\mathcal{R}_L(\pi_1^\vee\otimes\pi_2, P),$$

in the notation of [2, §6]. This in turn leads to a formula

$$J_M(\pi_1^\vee\otimes\pi_2, f'\times f) = \sum_{L\in\mathcal{L}(M)} r_M^L(\tau, P)\,\mathrm{tr}(\mathcal{R}_L(\pi_1^\vee\otimes\pi_2, P)\mathcal{I}_P(\pi_1^\vee\otimes\pi_2, f'\times f)),$$

in which $\tau=(M_1,\sigma,r)$ belongs to $T_{\mathrm{disc}}(M)$ and π_1,π_2 lie in $\Pi_\sigma(M(F))$. The definitions (4.3) and (4.3*) express $\bar{J}_M(\tau,f'\times f)$ as an integral (over λ) of a linear combination of functions on the left hand side of the last formula. The same operations, applied to a summand on the right hand side, given the corresponding summand $\hat{\bar{r}}{}^L_M(\tau,\phi_L(f'\times f))$ for the right hand side of (4.12). This follows from the definitions (4.9) and (4.9*) of $\bar{r}^L_M(\tau)$, together with the definitions and an obvious descent property [2, (7.8)] of the map ϕ_L. In this way we derive the required formula (4.12).

Theorem 4.2 is a rather formal consequence of Proposition 4.1 and the identities (4.5) and (4.12). For convenience, we repeat the argument from [8, Proposition 8.1]. Write $J^G(f'\times f)$ and $I^G(f'\times f)$ for the respective geometric expansions (4.1) and (4.10). Substituting the identity (4.5) into (4.10), we find that $I^G(f'\times f)$ equals

$$J^G(f'\times f)-\sum_{L\neq G}|W_0^L|\,|W_0^G|^{-1}(-1)^{\dim(A_L/A_G)}\hat{I}^L(\phi_L(f'\times f)).$$

Our induction assumption insures that for $L\neq G$, the distribution I^L is supported on characters. Proposition 4.1 tells us that $J^G(f'\times f)$ is also equal to the original spectral expansion (4.2). Write $r^G(f'\times f)$ for either invariant spectral expansion (4.11) or (4.11*). Substituting the identity

$$\bar{r}_M(\tau,f'\times f)=\bar{J}_M(\tau,f'\times f)-\sum_{\substack{L\in\mathcal{L}(M)\\ L\neq G}}\hat{\bar{r}}{}^L_M(\tau,\phi_L(f'\times f))$$

obtained from (4.12) into (4.11*), we find that $r^G(f'\times f)$ equals

$$J^G(f'\times f)-\sum_{L\neq G}|W_0^L|\,|W_0^G|^{-1}(-1)^{\dim(A_L/A_G)}\hat{r}^L(\phi_L(f'\times f)).$$

We are trying to show that the distributions I^G and r^G are equal. We are certainly free to assume inductively that this is so if G is replaced by any proper Levi subgroup. In particular $\hat{I}^L(\phi_L(f'\times f))$ equals $\hat{r}^L(\phi_L(f'\times f))$ for any $L\in\mathcal{L}$ with $L\neq G$. The two expressions we have obtained for $I^G(f'\times f)$ and $r^G(f'\times f)$ are therefore equal. In other words, (4.10) equals (4.11), as required. □

Our use of the local trace formula in this paper will be confined to a simpler version, in which one or both of the functions is cuspidal.

COROLLARY 4.3. *Suppose that f' and f are functions in $\mathcal{H}(G(F))$, and that f is cuspidal. Then the invariant local trace formula reduces to the identity of an expression*

$$\sum_{M\in\mathcal{L}}|W_0^M|\,|W_0^G|^{-1}(-1)^{\dim(A_M/A_G)}\int_{\Gamma_{\mathrm{ell}}(M(F))}I_G(\gamma,f')I_M(\gamma,f)\,d\gamma \tag{4.13}$$

with

$$\int_{T_{\mathrm{ell}}(G)} |d(\tau)|^{-1}\Theta(\tau^\vee, f')\Theta(\tau, f)\, d\tau. \tag{4.14}$$

If both f' and f are cuspidal, the formula simplifies further to

$$\int_{\Gamma_{\mathrm{ell}}(G(F))} I_G(\gamma, f')I_G(\gamma, f)\, d\gamma = \int_{T_{\mathrm{ell}}(G)} |d(\tau)|^{-1}\Theta(\tau^\vee, f')\Theta(\tau, f)\, d\tau. \tag{4.15}$$

Proof. We start with the formula of Theorem 4.2, taking f to be the given cuspidal function. Consider first the geometric expansion (4.10). We shall apply the splitting formula (4.6) to the integrand $I_M(\gamma, f'\times f)$ in (4.10). If L is any group in $\mathcal{L}(M)$ with $L\neq G$, the distribution $I^L_M(\gamma)$ on $\mathcal{H}(L(F))$ is assumed to be supported on characters. We obtain

$$I^L_M(\gamma, f_Q) = \hat{I}^L_M(\gamma, (f_Q)_L) = \hat{I}^L_M(\gamma, f_L) = 0, \quad Q\in\mathcal{P}(L),$$

from the cuspidality of f. In other words, all the terms in (4.6) with $L\neq G$ vanish. Moreover, the coefficient $d^G_M(L', G)$ in (4.6) vanishes unless $L'=M$, in which case it equals 1. (See [4, §7].) The splitting formula reduces simply to

$$I_M(\gamma, f'\times f) = I^M_M(\gamma, f'_{P'})I_M(\gamma, f), \quad P'\in\mathcal{P}(M).$$

Since

$$I^M_M(\gamma, f'_{P'}) = I_G(\gamma, f'), \quad \gamma\in M(F)\cap G_{\mathrm{reg}}(F), \tag{4.16}$$

by a standard change of variables formula, the geometric expansion (4.10) is equal to (4.13).

The terms in the spectral expansion (4.11) are easily dealt with. The distributions $\Theta(\tau)$ are supported on characters, as we see directly from (2.3). If τ belongs to $T_{\mathrm{disc}}(M)$, for some $M\neq G$, we obtain

$$\begin{aligned} r_M(\tau, f'\times f) &= r_M(\tau, P)\Theta(\tau^\vee, f'_P)\Theta(\tau, f_P)\\ &= r_M(\tau, P)\widehat{\Theta}(\tau^\vee, f'_M)\widehat{\Theta}(\tau, f_M) = 0, \end{aligned}$$

from the definition (4.9) and the cuspidality of f. Consequently, the summands with $M\neq G$ in (4.11) all vanish. This leaves only the leading term, which by construction is just $I_{\mathrm{disc}}(f', f)$. (See Proposition 3.1.) It follows from Corollary 3.2 that the spectral expansion (4.11) equals (4.14). This establishes the first assertion of the corollary.

Suppose that f' is also cuspidal. Then if $M\neq G$, $I_G(\gamma, f')$ vanishes for any element $r\in M(F)\cap G_{\mathrm{reg}}(F)$, as we observe from (4.16). The summands with $M\neq G$ in (4.13) therefore vanish, and (4.13) reduces to the left hand side of (4.15). This establishes the formula (4.15), the second assertion of the corollary. □

To complete the original induction argument, we have to show that the distributions $I_M(\gamma)$ on $\mathcal{H}(G(F))$ are supported on characters. This will be an immediate consequence (Corollary 5.3) of the next theorem.

5. Characters and weighted orbital integrals

The elliptic tempered (virtual) characters $\Theta(\tau)$ are locally integrable functions on $G(F)$. We shall establish a formula which relates their values with the invariant distributions $I_M(\gamma)$ on $\mathcal{H}(G(F))$. The formula is a general analogue of earlier results [1], [3] relating the characters of discrete series to weighted orbital integrals of their matrix coefficients. It will be a consequence of the invariant local trace formula, or rather the simple version of Corollary 4.3.

The contragredient

$$\tau = (M_1, \sigma, r) \rightarrow \tau^\vee = (M_1, \sigma^\vee, r)$$

defines an involution on the set $T_{\mathrm{ell}}(G)$ of basic elliptic virtual characters. If γ lies in $M(F) \cap G_{\mathrm{reg}}(F)$, for some Levi subgroup $M \in \mathcal{L}$, we have

$$\Phi_M(\tau^\vee, \gamma) = \begin{cases} |D(\gamma)|^{1/2}\Theta(\tau^\vee, \gamma), & \text{if } \gamma \in M(F)_{\mathrm{ell}}, \\ 0, & \text{otherwise}, \end{cases} \tag{5.1}$$

in the notation of §1. Thus $\Phi_M(\tau^\vee)$ expresses the values of the normalized character of $\tau^\vee$ on (noncompact) tori in G which are elliptic in M.

THEOREM 5.1. *Suppose that f is a cuspidal function in $\mathcal{H}(G(F))$. Then*

$$I_M(\gamma, f) = (-1)^{\dim(A_M/A_G)} \int_{T_{\mathrm{ell}}(G)} |d(\tau)|^{-1} \Phi_M(\tau^\vee, \gamma)\Theta(\tau, f)\, d\tau, \tag{5.2}$$

for any group $M \in \mathcal{L}$ and any G-regular point γ in $M(F)$.

Proof. Suppose that γ does not lie in $M(F)_{\mathrm{ell}}$. We are still carrying the earlier induction hypothesis that the distributions $I_M^L(\gamma)$, $L \subsetneq G$, are supported on characters. It then follows from a descent formula [4, Corollary 8.3] and the cuspidality of f that $I_M(\gamma, f)$ vanishes. The right hand side of (5.2) vanishes by definition, so the formula holds in this case. It is therefore enough to establish (5.2) when γ lies in $M(F)_{\mathrm{ell}}$.

To deal with elliptic points in $M(F)$ we apply the simple version of the local trace formula. Consider the two expressions (4.13) and (4.14) in Corollary 4.3, with f the given cuspidal function, and f' a variable function in $\mathcal{H}(G(F))$. The expressions depend on f' through different distributions $I_G(\gamma, f')$ and $\Theta(\tau^\vee, f')$. However, $\Theta(\tau^\vee, f')$ is given by a locally integrable function, and has an expansion

$$\Theta(\tau^\vee, f') = \sum_{M \in \mathcal{L}} |W_0^M|\, |W_0^G|^{-1} \int_{\Gamma_{\mathrm{ell}}(M(F))} \Phi_M(\tau^\vee, \gamma) I_G(\gamma, f')\, d\gamma$$

as in (1.3). Substituting this into (4.14), we collect the coefficients of $I_G(\gamma, f')$ in the resulting identity of (4.13) with (4.14). We see that if

$$\delta_M(\gamma, f) = I_M(\gamma, f) - (-1)^{\dim(A_M/A_G)} \int_{T_{\mathrm{ell}}(G)} |d(\tau)|^{-1} I_M(\tau^\vee, \gamma)\Theta(\tau, f)\, d\tau,$$

then the expression

$$\sum_{M \in \mathcal{L}} |W_0^M|\, |W_0^G|^{-1} (-1)^{\dim(A_M/A_G)} \int_{\Gamma_{\mathrm{ell}}(M(F))} \delta_M(\gamma, f) I_G(\gamma, f')\, d\gamma \tag{5.3}$$

vanishes.

We must show that $\delta_M(\gamma, f)$ equals 0 for all M and γ. We have so far only established that the expression (5.3) vanishes if f' is any function in $\mathcal{H}(G(F))$. To handle the approximation argument we first enlarge the family of test functions.

LEMMA 5.2. *The expression* (5.3) *vanishes if f' is any function in $C_c^\infty(G(F))$.*

Proof. If F is p-adic, the spaces $C_c^\infty(G(F))$ and $\mathcal{H}(G(F))$ are the same, and there is nothing to prove. For Archimedean F, however, $\mathcal{H}(G(F))$ is only dense in $C_c^\infty(G(F))$. We must show that as a function of f', the absolute value of (5.3) extends to a continuous semi-norm on $C_c^\infty(G(F))$. This will be a consequence of the estimates in [11, §4] (as was the convergence of the various geometric expansions discussed in §4).

Observe that the difference

$$\delta'_M(\gamma, f) = \delta_M(\gamma, f) - I_M(\gamma, f), \quad \gamma \in M(F)_{\mathrm{ell}},$$

is essentially a finite linear combination of values of (normalized) irreducible characters. It follows from the local integrability of characters that the contribution of $\delta'_M(\gamma, f)$ to the absolute value of (5.3) extends to a continuous semi-norm on $C_c^\infty(G(F))$.

To deal with the contribution of $I_M(\gamma, f)$, we first consider the weighted orbital integral

$$J_M(\gamma, h) = |D(\gamma)|^{1/2} \int_{A_M(F)\backslash G(F)} h(x^{-1}\gamma x) v_M(x)\, dx,$$

for $h \in \mathcal{H}(G(F))$ and $\gamma \in M(F)_{\mathrm{ell}} \cap G_{\mathrm{reg}}(F)$. It is not hard to show that there are positive constants c_h and d such that

$$|J_M(\gamma, h)| \leqslant c_h (1 + |\log |D(\gamma)||)^d$$

for all $\gamma \in M(F)_{\mathrm{ell}}$. This estimate is a consequence of the proof of [11, Lemma 4.3], which we leave to the reader. (The result from [11] actually dealt with more complicated weighted orbital integrals $J^T(\gamma, h' \times h)$ on $\mathcal{H}(G(F)) \times \mathcal{H}(G(F))$. However, the technique applies equally well here.) Recall [4, §2] that the invariant distributions on $\mathcal{H}(G(F))$ are defined by a formula

$$I_M(\gamma, h) = J_M(\gamma, h) - \sum_{\substack{L \in \mathcal{L}(M) \\ L \neq G}} \hat{I}_M^L(\gamma, \phi_L(f))$$

which is similar to (4.5), and in which ϕ_L is the continuous map between the spaces $\mathcal{H}_{\mathrm{ac}}(G(F))$ and $\mathcal{I}_{\mathrm{ac}}(L(F))$ introduced in [6, §§11–12]. We obtain an estimate

$$|I_M(\gamma, h)| \leqslant c_h(H_M(\gamma))(1 + |\log |D(\gamma)||)^d, \quad \gamma \in M(F)_{\mathrm{ell}},$$

where $c_h(\cdot)$ now is a locally bounded function on $\mathfrak{a}_M$. This follows by induction from the definition of $\mathcal{I}_{\mathrm{ac}}(L(F))$, and the estimate above for $|J_M(\gamma, h)|$. We shall apply this inequality with $h = f$. We shall also apply the fundamental bound ([17, Theorem 2], [18, Theorem 14]) of Harish-Chandra on orbital integrals that was used in [11, §4]. This result implies an estimate

$$|I_G(\gamma, f')| \leqslant \sigma(f'), \quad \gamma \in G_{\mathrm{reg}}(F),\ f' \in \mathcal{H}(G(F)),$$

where $\sigma(\cdot)$ is a continuous semi-norm on $C_c^\infty(G(F))$. Since

$$\gamma \to (1 + |\log |D(\gamma)||)^d$$

is a locally integrable function on any maximal torus in $G(F)$, and $|I_G(\gamma, f')|$ is compactly supported, the contribution of $I_M(\gamma, f)$ to the absolute value of (5.3) also extends to a continuous semi-norm on $C_c^\infty(G(F))$.

We have shown that the absolute value of (5.3) extends to a continuous semi-norm on $C_c^\infty(G(F))$. Since (5.3) vanishes for any function f' in the dense subspace $\mathcal{H}(G(F))$ of $C_c^\infty(G(F))$, it vanishes for any f' in $C_c^\infty(G(F))$. □

We can now finish the proof of the theorem. Fix a group $M' \in \mathcal{L}$ and a G-regular element $\gamma' \in M'(F)_{\mathrm{ell}}$. The centralizer $T' = G_{\gamma'}$ of γ' in G is of course a maximal torus. We take $f' \in C_c^\infty(G(F))$ to be supported on the open set of elements in $G(F)$ which are conjugate to points in $T'(F) \cap G_{\mathrm{reg}}(F)$. The function

$$\gamma \to I_G(\gamma, f'), \quad \gamma' \in T'(F) \cap G_{\mathrm{reg}}(F),$$

is of course symmetric under the Weyl group $W(G(F), T'(F))$ of $T'(F)$. However, we are free to vary f' so that this function approaches the sum of Dirac measures on $T'(F)$

at the $W(G(F),T'(F))$-translates of γ'. On the other hand, $\delta_M(\gamma,f)$ is smooth in γ. Moreover, as a function on the set

$$\{(M,\gamma): M\in\mathcal{L}, \gamma\in\Gamma_{\mathrm{ell}}(M(F))\cap G_{\mathrm{reg}}(F)\},$$

$\delta_M(\gamma,f)$ is symmetric under the natural action of W_0^G. The set of W_0^G-orbits of such pairs is bijective with the set of G-regular conjugacy classes in $G(F)$. Therefore, the symmetry condition on $I_G(\gamma,f')$ in (5.3) is matched by a symmetry condition on $\delta_M(\gamma,f)$. As $I_G(\gamma,f')$ approaches the sum of Dirac measures, the expression (5.3) approaches

$$(-1)^{\dim(A_{M'}/A_G)}|W(G(F),T'(F))|\delta_{M'}(\gamma',f).$$

Since (5.3) vanishes for all such f', we can assert that $\delta_{M'}(\gamma',f)$ also vanishes. In other words,

$$I_{M'}(\gamma',f)=(-1)^{\dim(A_{M'}/A_G)}\int_{T_{\mathrm{ell}}(G)}|d(\tau)|^{-1}\Phi_{M'}(\tau^\vee,\gamma')\Theta(\tau,f)\,d\tau.$$

This becomes the required formula (5.2) if we relabel M' and γ' by M and γ. □

The theorem allows us to complete the induction argument begun in §4.

COROLLARY 5.3. *The distributions*

$$I_M(\gamma,f),\quad M\in\mathcal{L},\ \gamma\in M(F)\cap G_{\mathrm{reg}}(F),\ f\in\mathcal{H}(G(F)),$$

on $\mathcal{H}(G(F))$ are supported on characters.

Proof. Fix a function $f\in\mathcal{H}(G(F))$ such that $f_G=0$. We must show that $I_M(\gamma,f)$ vanishes for any M and γ. The condition on f implies that the function is cuspidal. It also implies that the right hand side of (5.2) vanishes. The identity (5.2) then tells us that $I_M(\gamma,f)=0$. □

It might be helpful to restate a version of the theorem in a more concrete form. For simplicity, assume that G is semisimple, and that

$$\tau=(M_1,\sigma,r),\quad M_1\in\mathcal{L},\ \sigma\in\Pi_2(M_1(F)),\ r\in R_{\sigma,\mathrm{reg}},$$

is an element in $T_{\mathrm{ell}}(G)$ with $Z_\tau=\{1\}$. This last condition is essentially that the cocyle η_σ of §2 splits, so we may also assume that $R(r,\sigma)$ equals $\widetilde{R}(r,\sigma)$.

COROLLARY 5.4. *Suppose that f is a function in $\mathcal{H}(G(F))$ such that for any triplet*

$$(M_1', \sigma', r'), \quad M_1' \in \mathcal{L},\, \sigma' \in \Pi_2(M_1'(F)),\, r' \in R_{\sigma'},$$

the expression

$$\mathrm{tr}(R(\sigma', r')\mathcal{I}_{P'}(\sigma', f)), \quad P' \in \mathcal{P}(M'),$$

vanishes unless (M_1', σ', r') belongs to the W_0^G-orbit of (M_1, σ, r), in which case it equals 1. Then $I_M(\gamma, f)$ equals

$$(-1)^{\dim(A_M)}|R_{\sigma,r}|^{-1}|\det(1-r)|^{-1}\sum_{\varrho\in\Pi(R_\sigma)}\mathrm{tr}(\varrho(r))|D(\gamma)|^{1/2}\Theta(\pi_\varrho^\vee, \gamma),$$

for any G-regular point γ in $M(F)_{\mathrm{ell}}$.

Proof. Though it may already be clear, let us just convince ourselves that f is cuspidal. Clearly f is a pseudo-coefficient for τ, in that

$$\Theta(\tau', f) = \begin{cases} 1, & \text{if } \tau' = \tau, \\ 0, & \text{otherwise,} \end{cases}$$

for any $\tau' \in T(G)$. Suppose that $L \in \mathcal{L}$ is a proper Levi subgroup of G. As a function on $T(L)$, f_L satisfies

$$f_L(\tau_L) = \Theta(\tau_L^G, f), \quad \tau_L \in T(L),$$

where τ_L^G is the W_0^G-orbit of τ_L. This follows directly from Proposition 2.1. Since the image of $T(L)$ in $T(G)$ is disjoint from $T_{\mathrm{ell}}(G)$, a set which contains τ, the function f_L vanishes. Therefore f is cuspidal.

We apply the theorem, taking into account the definition (3.5) of the measure on $T_{\mathrm{ell}}(G)$ and the definition (3.7) of $d(\tau)$. We obtain

$$I_M(\gamma, f) = (-1)^{\dim(A_M)}|R_{\sigma,r}|^{-1}|\det(1-r)|^{-1}\Phi_M(\tau^\vee, \gamma).$$

But the formula (2.3), applied to $\sigma^\vee$ instead of σ, gives a decomposition

$$\Theta(\tau^\vee) = \sum_{\varrho\in\Pi(R_\sigma)}\mathrm{tr}(\varrho(r))\Theta(\pi_\varrho^\vee)$$

for the virtual character $\Theta(\tau^\vee)$. Therefore

$$\Phi_M(\tau^\vee, \gamma) = |D(\gamma)|^{1/2}\Theta(\tau^\vee, \gamma) = \sum_\varrho \mathrm{tr}(\varrho(r))|D(\gamma)|^{1/2}\Theta(\pi_\varrho^\vee, \gamma).$$

The corollary follows. □

Remarks. (1) Consider the special case that $\Theta(\tau, f)$ is supported on the subset

$$\{\tau = (G, \sigma, 1) : \sigma \in \Pi_2(G(F))\}$$

of $T_{\mathrm{ell}}(G)$. The right hand side of (5.2) is then a linear combination of characters of discrete series. Under the additional assumption that F is Archimedean, Theorem 5.1 reduces to an earlier formula [7, Theorem 6.4]. This special case is also closely related to the main results in [1] and [3].

(2) It is sometimes convenient to express the right hand side of (5.2) in terms of the function

$$f_G(\tau, X) = \int_{i\mathfrak{a}^*_{G,F}} f_G(\tau_\lambda) e^{-\lambda(X)}\, d\lambda = \int_{i\mathfrak{a}^*_{G,F}} \Theta(\tau_\lambda, f) e^{-\lambda(X)}\, d\lambda, \tag{5.4}$$

for elements $\tau = (M_1, \sigma, r)$ in $T_{\mathrm{ell}}(G)$ and X in $\mathfrak{a}_{G,F}$. The assertion (5.2) of Theorem 5.1 is then equivalent to an identity

$$I_M(\gamma, f) = (-1)^{\dim(A_M/A_G)} \sum_{\tau \in T_{\mathrm{ell}}(G)/i\mathfrak{a}^*_G} |d^1(\tau)|^{-1} \Phi_M(\tau^\vee, \gamma) f_G(\tau, H_G(\gamma)), \tag{5.2*}$$

where

$$d^1(\tau) = |\widetilde{R}_{\sigma,r}|\, |\mathfrak{a}^\vee_{G,\sigma}/\mathfrak{a}^\vee_{G,F}|\, d(\tau). \tag{5.5}$$

This follows immediately from the definition (3.5) of the measure on $T_{\mathrm{ell}}(G)$.

6. Orthogonality relations

There is a simple consequence of the last theorem that deserves a separate discussion. It concerns orthogonality relations for our basic elliptic (virtual) characters. We shall show that the class functions

$$\{\Phi_G(\tau) : \tau \in T_{\mathrm{ell}}(G)\}$$

on $G(F)_{\mathrm{ell}}$, taken up to the equivalence relation defined by the action of the groups $Z_\tau \times i\mathfrak{a}^*_G$, form an orthogonal set; we shall also find an explicit formula for their norms. The result can be regarded as a generalization of Harish-Chandra's orthogonality relations for characters of discrete series.

The orthogonality relations are best motivated from the framework of §2. Recall that for any $\sigma \in \Pi_2(M(F))$, there is a unique bijection $\varrho \to \pi_\varrho$ of $\Pi(\widetilde{R}_\sigma, \chi_\sigma)$ onto $\Pi_\sigma(G(F))$ which satisfies (2.3). Equivalently, there is a unique bijection $\theta \to \Theta$ from the

χ_σ-equivariant class functions on $\widetilde{R}_\sigma$ onto a space of invariant distributions on $G(F)$ which satisfies (2.4). The identity (2.4) can be written

$$\Theta(f) = \sum_{r\in\Gamma(\widetilde{R}_\sigma)} |\widetilde{R}_{\sigma,r}|^{-1}\theta(r)\operatorname{tr}(\widetilde{R}(\sigma,r)\mathcal{I}_P(\sigma,f)), \quad f\in\mathcal{H}(G(F)),$$

where $\Gamma(\widetilde{R}_\sigma)$ denotes the set of conjugacy classes in $\widetilde{R}_\sigma$. That is,

$$\Theta(f) = \sum_{r\in\Gamma(\widetilde{R}_\sigma)} |\widetilde{R}_{\sigma,r}|^{-1}\theta(r)\Theta(\tau_r, f),$$

for $\tau_r = (M,\sigma,r)$. This ought to be viewed as a map from finite linear combinations of Dirac distributions on the space $T(G)$ to virtual characters on $G(F)$. It is the concrete expression of the isomorphism, determined by the trace Paley–Wiener theorem, from the topological dual space of $\mathcal{I}(G(F))$ onto the space of invariant distributions on $\mathcal{H}(G(F))$ which are supported on characters. In other words, the map $\theta\to\Theta$ of §2 extends to elements θ in the full dual space $\mathcal{I}'(G(F))$ of $\mathcal{I}(G(F))$. We shall be interested in the case that θ is a cuspidal test function. Then Θ will be a locally integrable function on $G(F)$. We would like to describe the inner product over $G(F)_{\mathrm{ell}}$ of two such functions in terms of the initial two functions on $T_{\mathrm{ell}}(G)$.

By a cuspidal test function θ in $\mathcal{I}'(G(F))$ we mean a function on $T(G)$ whose transpose

$$\tau \to \theta(\tau^\vee), \quad \tau\in T(G),$$

belongs to $\mathcal{I}(G(F))$, and is supported on $T_{\mathrm{ell}}(G)$. Guided by the definition (3.5) of the measure on $T_{\mathrm{ell}}(G)$, and the description above of (2.4), we define a distribution

$$\Theta(f) = \int_{T_{\mathrm{ell}}(G)} \theta(\tau)\Theta(\tau,f)\,d\tau, \quad f\in\mathcal{H}(G(F)), \tag{6.1}$$

on $\mathcal{H}(G(F))$. Then Θ can be identified with a locally integrable class function $\Theta(\gamma)$ on $G(F)$. According to our usual convention, we can form the class function

$$\Phi(\gamma) = |D(\gamma)|^{1/2}\Theta(\gamma) \tag{6.2}$$

on $G_{\mathrm{reg}}(F)$ and its restriction

$$\Phi_G(\gamma) = \int_{T_{\mathrm{ell}}(G)} \theta(\tau)\Phi_G(\tau,\gamma)\,d\tau$$

to $\Gamma_{\mathrm{ell}}(G(F))$. Following parallel notation, let us also introduce the function

$$\phi(\tau) = |d(\tau)|^{1/2}\theta(\tau). \tag{6.3}$$

THEOREM 6.1. *Suppose that θ and θ' are two cuspidal test functions in $\mathcal{I}'(G(F))$. Then the associated pairs Φ, Φ' and ϕ, ϕ' of functions, defined by* (6.2) *and* (6.3) *respectively, satisfy the inner product formula*

$$\int_{\Gamma_{\mathrm{ell}}(G(F))} \Phi(\gamma)\overline{\Phi'(\gamma)}\,d\gamma = \int_{T_{\mathrm{ell}}(G)} \phi(\tau)\overline{\phi'(\tau)}\,d\tau. \tag{6.4}$$

Proof. Notice that $\tau \to |d(\tau)|\theta(\tau^\vee)$ is also a function in $\mathcal{I}(G(F))$ which is supported on $T_{\mathrm{ell}}(G)$. The trace Paley–Wiener theorem therefore provides us with a cuspidal function $f \in \mathcal{H}(G(F))$ such that

$$f_G(\tau) = \Theta(\tau, f) = |d(\tau)|\theta(\tau^\vee), \quad \tau \in T(G).$$

We shall apply the formula (5.2) of Theorem 5.1, with $M=G$. If γ is any G-regular point in $G(F)$, we obtain

$$\begin{aligned} I_G(\gamma, f) &= \int_{T_{\mathrm{ell}}(G)} |d(\tau)|^{-1}\Phi_G(\tau^\vee, \gamma)\Theta(\tau, f)\,d\tau \\ &= \int_{T_{\mathrm{ell}}(G)} |d(\tau^\vee)|^{-1}\Theta(\tau^\vee, f)\Phi_G(\tau, \gamma)\,d\tau \\ &= \int_{T_{\mathrm{ell}}(G)} \theta(\tau)\Phi_G(\tau, \gamma)\,d\tau \\ &= \Phi_G(\gamma). \end{aligned}$$

Given θ', we define a second cuspidal function $f' \in \mathcal{H}(G(F))$ in the same way. It has the property

$$\overline{\Phi'_G(\gamma)} = \overline{I_G(\gamma, f')} = I_G(\gamma, \overline{f'}).$$

The simple version (4.15) of the local trace formula then tells us that the inner product

$$\int_{\Gamma_{\mathrm{ell}}(G(F))} \Phi_G(\gamma)\overline{\Phi'_G(\gamma)}\,d\gamma = \int_{\Gamma_{\mathrm{ell}}(G(F))} I_G(\gamma, f)I_G(\gamma, \overline{f'})\,d\gamma$$

equals

$$\int_{T_{\mathrm{ell}}(G)} |d(\tau)|^{-1}\Theta(\tau^\vee, f)\Theta(\tau, \overline{f'})\,d\tau.$$

In general, one can write

$$\Theta(\tau^\vee, f') = \mathrm{tr}(\widetilde{R}(r, \sigma^\vee)\mathcal{I}_P(\sigma^\vee, f')) = \mathrm{tr}(\widetilde{R}^\vee(r, \sigma)\mathcal{I}_P^\vee(\sigma, f')),$$

if $\tau=(M,\sigma,r)$. The term on the right stands for the character of the contragredient of the representation of $\widetilde{R}_\sigma \times G(F)$ attached to σ. Since this representation is unitary, we obtain

$$\overline{\Theta(\tau^\vee, f')} = \mathrm{tr}(R(r,\sigma)\mathcal{I}_P(\sigma,\overline{f'})) = \Theta(\tau,\overline{f'}).$$

Substituting this into the expression above, we see that the inner product on the left hand side of (6.4) equals

$$\int_{T_{\mathrm{ell}}(G)} |d(\tau)|^{-1}\Theta(\tau^\vee, f)\overline{\Theta(\tau^\vee, f')}\, d\tau.$$

Since $d(\tau)=d(\tau^\vee)$, this can be written as

$$\int_{T_{\mathrm{ell}}(G)} |d(\tau)|\theta(\tau)\overline{\theta'(\tau)}\, d\tau = \int_{T_{\mathrm{ell}}(G)} \phi(\tau)\overline{\phi'(\tau)}\, d\tau,$$

the right hand side of the required formula (6.4). □

We shall give two corollaries. The first will be the orthogonality relations for the basic elliptic (virtual) characters $\{\Theta(\tau)\}$. In the second corollary, we shall derive a dual inner product formula for the irreducible elliptic characters $\{\Theta(\pi_\varrho)\}$.

Any element $\tau=(M,\sigma,r)$ in $T(G)$ has a central character ζ_τ on $A_G(F)$. That is,

$$\Theta(\tau,\gamma a) = \zeta_\tau(a)\Theta(\tau,\gamma), \quad \gamma\in G_{\mathrm{reg}}(F),\ a\in A_G(F),$$

where ζ_τ is the restriction of the central character of σ to $A_G(F)$. Suppose that τ' is another element in $T(G)$ with the same central character. The function

$$\Phi(\tau,\gamma)\overline{\Phi(\tau',\gamma)}, \quad \gamma\in G_{\mathrm{reg}}(F),$$

is then invariant under $A_G(F)$, and can be integrated over the elliptic conjugacy classes in $G(F)/A_G(F)$. This integral can be expressed as an elliptic inner product

$$\sum_{\{T\}} |W(G(F),T(F))|^{-1} \int_{T(F)/A_G(F)} \Phi(\tau,\gamma)\overline{\Phi(\tau',\gamma)}\, d\gamma, \tag{6.5}$$

where $\{T\}$ is summed over the $G(F)$-conjugacy classes of elliptic maximal tori in G.

COROLLARY 6.2. *Suppose that τ and τ' are two elements in $T_{\mathrm{ell}}(G)$ with the same central character. Then the inner product* (6.5) *vanishes unless τ' belongs to the $(Z_\tau\times i\mathfrak{a}_G^*)$-orbit of τ in $T_{\mathrm{ell}}(G)$. However, if $\tau'=\tau=(M,\sigma,r)$, the inner product equals*

$$|R_{\sigma,\bar{r}}|\,|d(\bar{r})|,$$

where $\bar{r}$ is the image of r in R_σ.

Proof. We shall apply Theorem 6.1, with θ supported on the $(Z_\tau \times i\mathfrak{a}_G^*)$-orbit of τ and θ' supported on the $(Z_{\tau'} \times i\mathfrak{a}_G^*)$-orbit of τ'. If τ and τ' lie in different orbits, the right hand side of (6.4) vanishes. We leave the reader to check that for suitable θ and θ', the left hand side of (6.4) becomes the inner product (6.5). This implies the vanishing of the inner product.

To deal with the second assertion of the corollary, in which $\tau'=\tau=(M,\sigma,r)$, we take $\theta'=\theta$. We first substitute the formula (3.5) for the measure on $T_{\mathrm{ell}}(G)$ into the right hand side of (6.4). We obtain

$$\begin{aligned}\int_{T_{\mathrm{ell}}(G)} |\phi(\tau)|^2\, d\tau &= |d(\tau)| \int_{T_{\mathrm{ell}}(G)} |\theta(\tau)|^2\, d\tau \\ &= |d(\tau)|\, |\widetilde{R}_{\sigma,r}|^{-1} |\mathfrak{a}_{G,\sigma}^\vee/\mathfrak{a}_{G,F}^\vee|^{-1} \sum_{z\in Z_\tau/Z_\tau^0} \int_{i\mathfrak{a}_{G,F}^*} |\theta(z\tau_\lambda)|^2\, d\lambda,\end{aligned}$$

where Z_τ^0 is the stabilizer of τ in Z_τ. The integrand is independent of z, and since Z_τ^0 is the stabilizer in Z_τ of the $\widetilde{R}_\sigma$-conjugacy class of r, we have

$$|\widetilde{R}_{\sigma,r}|^{-1}|Z_\tau/Z_\tau^0| = |R_{\sigma,\bar{r}}|^{-1}.$$

The right hand side of (6.4) becomes

$$\begin{aligned}&|d(\bar{r})|\, |R_{\sigma,\bar{r}}|^{-1} |\mathfrak{a}_{G,\sigma}^\vee/\mathfrak{a}_{G,F}^\vee|^{-1} \int_{i\mathfrak{a}_{G,F}^*} |\theta(\tau_\lambda)|^2\, d\lambda \\ &\qquad = |d(\bar{r})|\, |R_{\sigma,\bar{r}}|^{-1} \int_{i\mathfrak{a}_G^*/\mathfrak{a}_{G,\sigma}^\vee} |\theta(\tau_\lambda)|^2\, d\lambda \\ &\qquad = |d(\bar{r})|\, |R_{\sigma,\bar{r}}|^{-1} |\tilde{\mathfrak{a}}_{G,F}^\vee/\mathfrak{a}_{G,\sigma}^\vee| \int_{\mathfrak{a}_{G,\sigma}} |\hat{\theta}(X)|^2\, dX,\end{aligned}$$

where

$$\hat{\theta}(X) = |\tilde{\mathfrak{a}}_{G,F}^\vee/\mathfrak{a}_{G,\sigma}^\vee|^{-1} \int_{i\mathfrak{a}_G^*/\mathfrak{a}_{G,\sigma}^\vee} \theta(\tau_\lambda) e^{\lambda(X)}\, d\lambda, \quad X \in \mathfrak{a}_{G,\sigma},$$

is the Fourier transform of θ relative to the normalized Haar measure on $i\mathfrak{a}_G^*/\mathfrak{a}_{G,\sigma}^\vee$.

On the other hand, observe that

$$\Phi(\tau_\lambda,\gamma) = \Phi(\tau,\gamma) e^{\lambda(H_G(\gamma))}, \quad \lambda \in i\mathfrak{a}_G^*/\mathfrak{a}_{G,\sigma}^\vee.$$

In particular, $\Phi(\tau,\gamma)$ vanishes unless $H_G(\gamma)$ belongs to $\mathfrak{a}_{G,\sigma}$. Consequently

$$\begin{aligned}\Phi(\gamma) &= \int_{T_{\mathrm{ell}}(G)} \theta(\tau)\Phi(\tau,\gamma)\,d\tau \\ &= |\widetilde{R}_{\sigma,r}|^{-1}|\mathfrak{a}^\vee_{G,\sigma}/\mathfrak{a}^\vee_{G,F}|^{-1} \sum_{z\in Z_\tau/Z^0_\tau} \int_{i\mathfrak{a}^*_{G,F}} \theta(z\tau_\lambda)\Phi(z\tau_\lambda,\gamma)\,d\lambda \\ &= |R_{\sigma,\bar{r}}|^{-1}|\mathfrak{a}^\vee_{G,\sigma}/\mathfrak{a}^\vee_{G,F}|^{-1}\Phi(\tau,\gamma)\int_{i\mathfrak{a}^*_{G,F}} \theta(\tau_\lambda)e^{\lambda(H_G(\gamma))}\,d\lambda \\ &= |R_{\sigma,\bar{r}}|^{-1}|\tilde{\mathfrak{a}}^\vee_{G,F}/\mathfrak{a}^\vee_{G,\sigma}|\Phi(\tau,\gamma)\hat{\theta}(H_G(\gamma)).\end{aligned}$$

The left hand side of (6.4) therefore equals

$$|R_{\sigma,\bar{r}}|^{-2}|\tilde{\mathfrak{a}}^\vee_{G,F}/\mathfrak{a}^\vee_{G,\sigma}|^2 \int_{\Gamma_{\mathrm{ell}}(G(F))} |\Phi(\tau,\gamma)|^2|\hat{\theta}(H_G(\gamma))|^2\,d\gamma.$$

Since $|\Phi(\tau,\gamma)|^2$ is invariant under translation of γ by elements in $A_G(F)$, we can write the integral in this last expression as

$$\sum_{\{T\}} |W(G(F),T(F))|^{-1} \int_{T(F)/A_G(F)} |\Phi(\tau,\gamma)|^2\eta(H_G(\gamma))\,d\gamma, \tag{6.6}$$

where

$$\eta(X) = \int_{\tilde{\mathfrak{a}}_{G,F}} |\hat{\theta}(H_G(\gamma)+Y)|^2\,dY, \quad X\in\mathfrak{a}_{G,\sigma}.$$

Identifying the expressions we have obtained for the two sides of (6.4), we see that the integral (6.6) equals a product

$$(|R_{\sigma,\bar{r}}|^2|\tilde{\mathfrak{a}}^\vee_{G,F}/\mathfrak{a}^\vee_{G,\sigma}|^{-2})\Big(|d(\bar{r})||R_{\sigma,\bar{r}}|^{-1}|\tilde{\mathfrak{a}}^\vee_{G,F}/\mathfrak{a}^\vee_{G,\sigma}|\int_{\mathfrak{a}_{G,\sigma}} |\hat{\theta}(X)|^2\,dX\Big),$$

which in turn simplifies to

$$|R_{\sigma,\bar{r}}|\,|d(\bar{r})|\,|\mathfrak{a}_{G,\sigma}/\tilde{\mathfrak{a}}_{G,F}|^{-1}\int_{\mathfrak{a}_{G,\sigma}} |\hat{\theta}(X)|^2\,dX.$$

The group $\tilde{\mathfrak{a}}_{G,F}$ of course has finite index in $\mathfrak{a}_{G,\sigma}$. We can therefore choose θ so that $\eta(X)=1$ for any point X in $\mathfrak{a}_{G,\sigma}$. The expression (6.6) then reduces to the required inner product (6.5). Moreover,

$$\int_{\mathfrak{a}_{G,\sigma}} |\hat{\theta}(X)|^2\,dX = \sum_{X\in\mathfrak{a}_{G,\sigma}/\tilde{\mathfrak{a}}_{G,F}} \eta(X) = |\mathfrak{a}_{G,\sigma}/\tilde{\mathfrak{a}}_{G,F}|.$$

The inner product (6.5) therefore equals $|R_{\sigma,\bar{r}}|\,|d(\bar{r})|$. This was the second assertion of the corollary. □

COROLLARY 6.3. *Fix* $\sigma \in \Pi_2(M(F))$, *and suppose that*

$$\{\pi_\varrho, \pi_{\varrho'} : \varrho, \varrho' \in \Pi(\widetilde{R}_\sigma, \chi_\sigma)\}$$

are two elliptic representations in $\Pi_\sigma(G(F))$. *Then the elliptic inner product*

$$\sum_{\{T\}} |W(G(F), T(F))|^{-1} \int_{T(F)/A_G(F)} |D(\gamma)| \Theta(\pi_\varrho, \gamma) \overline{\Theta(\pi_{\varrho'}, \gamma)}\, d\gamma \tag{6.7}$$

equals

$$|R_\sigma|^{-1} \sum_{r \in R_{\sigma,\mathrm{reg}}} |d(r)| \operatorname{tr}(\varrho(r)) \overline{\operatorname{tr}(\varrho'(r))}.$$

Proof. The character $\Theta(\pi_\varrho)$ is the image of the function

$$\theta(\varrho, r) = \operatorname{tr}(\varrho(r)), \quad r \in \widetilde{R}_\sigma,$$

under the correspondence of §2. It follows from (2.4) that

$$\begin{aligned} \Theta(\pi_\varrho, \gamma) &= |\widetilde{R}_\sigma|^{-1} \sum_{r \in \widetilde{R}_\sigma} \operatorname{tr}(\varrho(r)) \Theta(\tau_r, \gamma) \\ &= |R_\sigma|^{-1} \sum_{r \in R_\sigma} \operatorname{tr}(\varrho(r)) \Theta(\tau_r, \gamma) \\ &= \sum_{\gamma \in \Gamma(R_\sigma)} |R_{\sigma,r}|^{-1} \operatorname{tr}(\varrho(r)) \Theta(\tau_r, \gamma), \end{aligned}$$

for any G-regular element γ in $G(F)_{\mathrm{ell}}$. Here $\tau_r = (M, \sigma, r)$ as before, and the summands are well defined class functions on R_σ. If r belongs to the complement of $\widetilde{R}_{\sigma,\mathrm{reg}}$ in $\widetilde{R}_\sigma$, the virtual character $\Theta(\tau_r)$ is a linear combination of induced characters, and vanishes on the regular elliptic set. We may therefore take the last sum over the set $\Gamma(R_{\sigma,\mathrm{reg}})$ of R_σ-conjugacy classes in $R_{\sigma,\mathrm{reg}}$. The formula becomes

$$|D(\gamma)|^{1/2} \Theta(\pi_\varrho, \gamma) = \sum_{r \in \Gamma(R_{\sigma,\mathrm{reg}})} |R_{\sigma,r}|^{-1} \operatorname{tr}(\varrho(r)) \Phi(\tau_r, \gamma),$$

upon multiplication of each side by $|D(\gamma)|^{1/2}$.

We now substitute this formula for $|D(\gamma)|^{1/2}\Theta(\pi_\varrho, \gamma)$, and we substitute its companion for $|D(\gamma)|^{1/2}\Theta(\pi_{\varrho'}, \gamma)$, into the inner product (6.7). The result is a double sum over $r, r' \in \Gamma(R_{\sigma,\mathrm{reg}})$ of the expression obtained by multiplying

$$|R_{\sigma,r}|^{-1} |R_{\sigma,r'}|^{-1} \operatorname{tr}(\varrho(r)) \overline{\operatorname{tr}(\varrho'(r'))}$$

with the inner product

$$\sum_{\{T\}} |W(G(F),T(F))|^{-1} \int_{T(F)/A_G(F)} \Phi(\tau_r,\gamma)\overline{\Phi(\tau_{r'},\gamma)}\,d\gamma.$$

By Corollary 6.2, the last inner product vanishes unless $r=r'$, in which case it equals $|R_{\sigma,r}|\,|d(r)|$. Consequently the original inner product (6.7) equals

$$\sum_{r\in\Gamma(R_{\sigma,\mathrm{reg}})} |R_{\sigma,r}|^{-1}|d(r)|\,\mathrm{tr}(\varrho(r))\overline{\mathrm{tr}(\varrho'(r))}.$$

This in turn equals the required expression

$$|R_\sigma|^{-1} \sum_{r\in R_{\sigma,\mathrm{reg}}} |d(r)|\,\mathrm{tr}(\varrho(r))\overline{\mathrm{tr}(\varrho'(r))}. \qquad \square$$

Remarks. (1) Suppose that $\pi_\varrho\in\Pi_\sigma(G(F))$ and $\pi_{\varrho'}\in\Pi_{\sigma'}(G(F))$, where $\sigma\in\Pi_2(M(F))$ and $\sigma'\in\Pi_2(M'(F))$ have the same central character on $A_G(F)$. If (M',σ') is not W_0^G-conjugate to (M,σ), it follows easily from Corollary 6.2 that the inner product (6.7) vanishes. This special case was conjectured by Harish-Chandra [13] and was proved by Kazhdan [24, Corollary to Proposition 5.4].

(2) We are seeing a miniature dictionary between objects attached to $G(F)$ and objects attached to the groups $\widetilde{R}_\sigma$. This includes the correspondence $\pi_\varrho\leftrightarrow\varrho$ of irreducible representations, the analogy between the elliptic sets $G(F)_{\mathrm{ell}}$ and $\widetilde{R}_{\sigma,\mathrm{reg}}$, the parallel roles of the central subgroups $A_G(F)$ and Z_σ in the formulation of the last corollaries, and perhaps most striking, the analogy between the Weyl discriminant

$$D(\gamma)=\det(1-\mathrm{Ad}(\gamma))_{\mathfrak{g}/\mathfrak{g}_\gamma}$$

on $G(F)/A_G(F)$ and the function

$$d(r)=\det(1-r)_{\mathfrak{a}_M/\mathfrak{a}_G}$$

on $R_\sigma=\widetilde{R}_\sigma/Z_\sigma$. Corollary 6.3 is one of the clearest statements of this parallelism. It introduces an elliptic pairing

$$(\varrho,\varrho')_{\mathrm{ell}}=|R_\sigma|^{-1}\sum_{r\in R_{\sigma,\mathrm{reg}}} |d(r)|\,\mathrm{tr}(\varrho(r))\overline{\mathrm{tr}(\varrho'(r))}$$

between irreducible (projective) representations of the R-group. It would be interesting to investigate this pairing in the various examples [25] of nonabelian R-groups. Perhaps some version of the pairing might also play a role in the general character theory of Weyl groups.

7. The distributions ${}^cI_M(\gamma)$, $D_M(\tau_{M,X})$ and ${}^cD_M(\tau_{M,X})$

The objects $\{I_M(\gamma)\}$ are only one of several families of invariant distributions that arise from questions on harmonic analysis on $G(F)$. Some of these distributions are attached to conjugacy classes, others are associated to intertwining operators and their residues. All of them are related. The various distributions were defined and discussed in some detail in the papers [4, §§3–6] and [7]. We shall review some of them in this section, taking the opportunity to streamline a couple of the definitions.

For reasons discussed in [6, §11], it is convenient to identify objects in $\mathcal{I}(G(F))$ with functions of two variables. Thus, if ϕ belongs to the space $\mathcal{I}(G(F))$, as it is defined in §3, we set

$$\phi(\tau, X) = \int_{i\mathfrak{a}^*_{G,F}} \phi(\tau_\lambda) e^{-\lambda(X)} d\lambda,$$

for any $\tau \in T(G)$ and $X \in \mathfrak{a}_{G,F}$. This is compatible with the notation (5.4). In the earlier paper [6], $\mathcal{I}(G(F))$ was defined as a space of functions on $\Pi_{\mathrm{temp}}(G(F)) \times \mathfrak{a}_{G,F}$. The two interpretations are related by the formula

$$\phi(\tau, X) = \sum_{\varrho \in \Pi(\widetilde{R}_\sigma, \chi_\sigma)} \mathrm{tr}(\varrho^\vee(r)) \phi(\pi_\varrho, X), \quad \tau = (M, \sigma, r), \tag{7.1}$$

obtained from (2.3). The functions $\phi(\tau, X)$ in $\mathcal{I}(G(F))$ are compactly supported and smooth in X. In [6, §11] and [4, §4] we defined extensions

$$\mathcal{I}(G(F)) \subset \mathcal{I}_{\mathrm{ac}}(G(F)) \subset \widetilde{\mathcal{I}}_{\mathrm{ac}}(G(F))$$

by successively weakening the conditions of compact support and smoothness. We shall use these spaces here, and we shall regard their elements simultaneously as functions on $\Pi_{\mathrm{temp}}(G(F)) \times \mathfrak{a}_{G,F}$ and on $T(G) \times \mathfrak{a}_{G,F}$, the two being related by (7.1). In the earlier papers, we also defined extensions

$$\mathcal{H}(G(F)) \subset \mathcal{H}_{\mathrm{ac}}(G(F)) \subset \widetilde{\mathcal{H}}_{\mathrm{ac}}(G(F))$$

of the Hecke algebra, and we noted that $f \to f_G$ had a natural extension to a continuous, surjective map from any one of these latter spaces onto the corresponding space above.

Fix a group $M \in \mathcal{L}$. If f belongs to $\mathcal{H}(G(F))$, $\phi_M(f)$ is defined as the function on $\Pi_{\mathrm{temp}}(M(F)) \times \mathfrak{a}_{M,F}$ given by

$$\phi_M(f, \pi, X) = \int_{i\mathfrak{a}^*_{M,F}} \mathrm{tr}(\mathcal{R}_M(\pi_\lambda, P) \mathcal{I}_P(\pi_\lambda, f)) e^{-\lambda(X)} \, d\lambda.$$

More generally, one can form the function

$$\phi_{M,\mu}(f,\pi,X)=\int_{\mu+i\mathfrak{a}^*_{M,F}} \operatorname{tr}(\mathcal{R}_M(\pi_\lambda,P)\mathcal{I}_P(\pi_\lambda,f))e^{-\lambda(X)}\,d\lambda \tag{7.2}$$

for any point $\mu\in\mathfrak{a}^*_M$ for which the integrand is regular. Then $\phi_{M,\mu}$ extends to a continuous map from $\widetilde{\mathcal{H}}_{\mathrm{ac}}(G(F))$ to $\widetilde{\mathcal{I}}_{\mathrm{ac}}(M(F))$ which sends $\mathcal{H}_{\mathrm{ac}}(G(F))$ into $\mathcal{I}_{\mathrm{ac}}(M(F))$. (See [6, §11], [4, §4].) Bear in mind that the image of $\mathcal{H}(G(F))$ under $\phi_{M,\mu}$, or in particular under the map ϕ_M, is not generally contained in $\mathcal{I}(M(F))$. If f lies in $\mathcal{H}(G(F))$, $\phi_M(f,\pi,X)$ need not be compactly supported in X. However, one can define another map which preserves the property of compact support at the expense of smoothness.

For each group $Q\in\mathcal{F}(M)$, let $\mu_Q\in\mathfrak{a}^*_Q$ be a fixed point in the chamber $(\mathfrak{a}^*_Q)^+$ associated to Q which is very far from any of the walls. Then if X lies in $\mathfrak{a}_M$, we can set

$$\mu(X)=\mu_Q,$$

where $Q\in\mathcal{F}(M)$ is the unique group such that X lies in $\mathfrak{a}^+_Q$. For any $f\in\widetilde{\mathcal{H}}_{\mathrm{ac}}(G(F))$, we define a function ${}^c\phi_M(f)$ on $\Pi_{\mathrm{temp}}(M(F))\times\mathfrak{a}_{M,F}$ by setting

$${}^c\phi_M(f,\pi,X)=\phi_{M,\mu(X)}(f,\pi,X). \tag{7.3}$$

This definition is slightly different from the one on p. 341 of [4]. However, it is a simple matter to show that ${}^c\phi_M$ maps $\widetilde{\mathcal{H}}_{\mathrm{ac}}(G(F))$ continuously to $\widetilde{\mathcal{I}}_{\mathrm{ac}}(M(F))$ as in [4, §4]. Moreover, the compact support property [4, Lemma 4.2] remains valid, a fact we will leave the reader to verify.

The distributions $I_M(\gamma)$ are defined inductively as linear functionals on either of the spaces $\mathcal{H}_{\mathrm{ac}}(G(F))$ or $\widetilde{\mathcal{H}}_{\mathrm{ac}}(G(F))$, by the formula

$$J_M(\gamma,f)=\sum_{L\in\mathcal{L}(M)}\hat{I}^L_M(\gamma,\phi_L(f)). \tag{7.4}$$

To know that the definition works, one has to realize that the weighted orbital integral $J_M(\gamma,f)$ makes sense for f in $\mathcal{H}_{\mathrm{ac}}(G(F))$ on $\widetilde{\mathcal{H}}_{\mathrm{ac}}(G(F))$ as well as for functions in the original space $\mathcal{H}(G(F))$. We remark that the reformulation (5.2*) of Theorem 5.1 remains valid if f is any function in $\widetilde{\mathcal{H}}_{\mathrm{ac}}(G(F))$.

One difficulty with the distribution $I_M(\gamma)$ is that it does not preserve the property of compact support. If f lies in $\mathcal{H}(G(F))$, and γ is confined to a maximal torus in $M(F)$, $\gamma\to I_M(\gamma,f)$ generally has unbounded support. This is due to the presence of the map ϕ_L in the definition (7.4). The problem can be rectified by replacing ϕ_L by ${}^c\phi_L$. We obtain invariant distributions

$${}^cI_M(\gamma)={}^cI^G_M(\gamma),\quad \gamma\in M(F)\cap G_{\mathrm{reg}}(F),$$

defined inductively by

$$J_M(\gamma,f)=\sum_{L\in\mathcal{L}(M)} {}^c\hat{I}^L_M(\gamma,{}^c\phi_L(f)) \tag{7.5}$$

for any function f in $\widetilde{\mathcal{H}}_{\rm ac}(G(F))$. If f lies in $\mathcal{H}(G(F))$ and γ remains in a maximal torus, $\gamma\rightarrow{}^cI_M(\gamma,f)$ does have bounded support [4, Lemma 4.4].

Recall also that the relations among the various objects are expressed in terms of invariant maps $\{\theta_M=\theta^G_M\}$ and $\{{}^c\theta_M={}^c\theta^G_M\}$ from $\widetilde{\mathcal{H}}_{\rm ac}(G(F))$ to $\widetilde{\mathcal{I}}_{\rm ac}(M(F))$. The maps are defined inductively by formulas

$${}^c\phi_M(f)=\sum_{L\in\mathcal{L}(M)} \hat{\theta}^L_M(\phi_L(f)) \tag{7.6}$$

and

$$\phi_M(f)=\sum_{L\in\mathcal{L}(M)} {}^c\hat{\theta}^L_M({}^c\phi_L(f)). \tag{7.7}$$

The other relations are given by

$${}^cI_M(\gamma,f)=\sum_{L\in\mathcal{L}(M)} \hat{I}^L_M(\gamma,{}^c\theta_L(f)), \tag{7.8}$$

$$I_M(\gamma,f)=\sum_{L\in\mathcal{L}(M)} {}^c\hat{I}^L_M(\gamma,\theta_L(f)), \tag{7.9}$$

and

$$\sum_{L\in\mathcal{L}(M)} {}^c\hat{\theta}^L_M(\theta_L(f))=\sum_{L\in\mathcal{L}(M)} \hat{\theta}^L_M({}^c\theta_L(f))=\begin{cases} f_G, & \text{if } M=G,\\ 0, & \text{otherwise,}\end{cases} \tag{7.10}$$

all being valid for any f in $\widetilde{\mathcal{H}}_{\rm ac}(G(F))$. (See [4, §4].) The notion of having support on characters is relevant to invariant maps as well as distributions. Theorem 6.1 of [4] establishes this property for ${}^cI_M(\gamma)$, θ_M and ${}^c\theta_M$, thereby justifying the notation in the formulas above. The result relies on the fact that the original distributions $I_M(\gamma)$ are supported on characters, which we of course have established in present local context in Corollary 5.3.

There are other invariant distributions, which are supported on characters, that come from residues. We refer the reader to [7, §1] for the definition of the residue

$$\operatorname*{Res}_{\Omega,\Lambda\to\Lambda_0}\psi(\Lambda)=\operatorname*{Res}_{\Omega}\psi(\Lambda_0)$$

of a meromorphic function ψ on $\mathfrak{a}^*_{M,\mathbf{C}}$ with respect to a residue datum Ω for (G,M). The operation is essentially just a sequence of iterated residues of ψ at Λ_0. Applied to the weighted characters, it yields an invariant distribution

$$f\rightarrow\operatorname*{Res}_{\Omega,\Lambda\to\Lambda_0} a_\Lambda(\mathcal{R}_M(\pi_\Lambda,P)\mathcal{I}_P(\pi_\Lambda,f))$$

on $\mathcal{H}(G(F))$ for any representation $\pi \in \Pi_{\mathrm{temp}}(M(F))$ and any analytic function a_Λ on $\mathfrak{a}^*_{M,\mathbf{C}}$ [6, Lemma 8.1]. One can then establish that this distribution is supported on characters [7, Theorem 5.2].

Residues arise from changes of contour. Fix $\pi \in \Pi_{\mathrm{temp}}(M(F))$, and take $a_\Lambda = e^{-\Lambda(X)}$ for a point X in $\mathfrak{a}_{M,F}$. Consider two other objects, a point $\mu \in \mathfrak{a}^*_M$ in general position, and a family

$$\mathcal{N} = \{\nu_L : L \in \mathcal{L}(M)\},$$

where for each L, ν_L is a point in $\mathfrak{a}^*_L$ in general position. Then for each L there is a finite set $R_L(\mu, \mathcal{N}_L)$ of residue data for (L, M), which depends only on the family

$$\mathcal{N}_L = \{\nu_{L_1} : L_1 \in \mathcal{L}(M), L_1 \subset L\},$$

with the property that

$$\int_{\mu + i\mathfrak{a}^*_{M,F}} \psi(\Lambda)\, d\Lambda = \sum_{\substack{L \in \mathcal{L}(M) \\ L \subset L'}} \int_{\nu_L + i\mathfrak{a}^*_{L,F}} \Big(\sum_{\Omega \in R_L(\mu, \mathcal{N}_L)} \operatorname*{Res}_{\Omega} \psi(\Lambda_\Omega + \lambda) \Big)\, d\lambda,$$

for any group $L' \in \mathcal{L}(M)$, and for

$$\psi(\Lambda) = e^{-\Lambda(X)} \operatorname{tr}(\mathcal{R}^{L'}_M(\pi_\Lambda, P) \mathcal{I}_P(\pi_\Lambda, f)), \quad f \in \mathcal{H}(G(F)).$$

This is a simple consequence of the usual residue theorem [6, Proposition 10.1]. The residue data are determined in a straightforward geometric fashion from μ, $\mathcal{N}$ and the singular hyperplanes of the function ψ. We set $D^{G,\mathcal{N}}_{M,\mu}(\pi, X, f)$ equal to

$$\int_{i\mathfrak{a}^*_{G,F}} \Big(\sum_{\Omega \in R_G(\mu, \mathcal{N})} \operatorname*{Res}_{\Omega, \Lambda \to \Lambda_\Omega + \lambda} (e^{-\Lambda(X)} \operatorname{tr}(\mathcal{R}_M(\pi_\Lambda, P) \mathcal{I}_P(\pi_\Lambda, f))) \Big)\, d\lambda.$$

Then $D^{G,\mathcal{N}}_{M,\mu}(\pi, X)$ is an invariant distribution on $\mathcal{H}(G(F))$ which is supported on characters.

There are two cases of particular interest. Fix a small point $\varepsilon \in \mathfrak{a}^*_M$ in general position. We also have the point $\mu(X) \in \mathfrak{a}^*_M$ for any element $X \in \mathfrak{a}_{M,F}$. Set

$$\mathcal{N}(X, \varepsilon) = \{\nu_L = \mu(X_L) + \varepsilon_L : L \in \mathcal{L}(M)\},$$

where X_L and ε_L denote the respective projections of X and ε onto $\mathfrak{a}_{L,F}$ and $\mathfrak{a}^*_L$. With this notation, we define

$$D_M(\pi, X, f) = D^G_M(\pi, X, f) = D^{G,\mathcal{N}(0,\varepsilon)}_{M,\mu(X)+\varepsilon}(\pi, X, f) \tag{7.11}$$

and

$$^{c}D_M(\pi,X,f)={}^{c}D_M^G(\pi,X,f)=D_{M,\varepsilon}^{G,\mathcal{N}(X,\varepsilon)}(\pi,X,f). \tag{7.12}$$

We obtain two more families $\{D_M(\pi,X)\}$ and $\{{}^cD_M(\pi,X)\}$ of invariant distributions which are supported on characters. The first is defined by the residue scheme of the real Paley–Wiener theorem and the spectral decomposition of Eisenstein series. The second is defined by the "inverse" residue scheme. There is no need to include ε in the notation; as a matter of fact, the next lemma implies that the values of the distributions at cuspidal functions are independent of ε. Observe that the value of $D_M(\pi,X)$ or ${}^cD_M(\pi,X)$ at any function f in $\mathcal{H}(G(F))$ depends only on the restriction of f to

$$G(F)^{X_G}=\{x\in G(F):H_G(x)=X_G\}.$$

It depends in fact only on the derivatives of the function

$$\Lambda\to\mathcal{I}_P(\pi_\Lambda,f^{X_G})=\int_{G(F)^{X_G}} f(x)\mathcal{I}_P(\pi_\Lambda,f)\,dx,\quad \Lambda\in\mathfrak{a}^*_{M,\mathbf{C}},$$

up to a certain order, at a finite set of points. In particular, $D_M(\pi,X)$ and ${}^cD_M(\pi,X)$ can be regarded as distributions on $\widetilde{\mathcal{H}}_{\mathrm{ac}}(G(F))$. Incidentally, $D_M(\pi,X)$ is almost the same as the distribution defined (and denoted by $D_M^{\varepsilon}(\pi,X)$) in [7, §6]. The minor discrepancy, as with the map ${}^c\phi_M$ above, is due to our use here of the point $\mu(X)$.

There are of course similar distributions indexed by elements in $T(M)$ instead of $\Pi_{\mathrm{temp}}(M(F))$. The connection is given by the analogue of (7.1). That is,

$$D_M(\tau_M,X,f)=\sum_{\varrho\in\Pi(\widetilde{R}_\sigma^M,\chi_\sigma)}\operatorname{tr}(\varrho^\vee(r))D_M(\pi_\varrho,X,f)$$

and

$$^{c}D_M(\tau_M,X,f)=\sum_{\varrho\in\Pi(\widetilde{R}_\sigma^M,\chi_\sigma)}\operatorname{tr}(\varrho^\vee(r))\,{}^cD_M(\pi_\varrho,X,f),$$

for any element $\tau_M=(M_1,\sigma,r)$ in $T(M)$.

LEMMA 7.1. *Suppose that f is a cuspidal function in $\widetilde{\mathcal{H}}_{\mathrm{ac}}(G(F))$. Then $\theta_M(f)$ and ${}^c\theta_M(f)$ are cuspidal functions in $\widetilde{\mathcal{I}}_{\mathrm{ac}}(M(F))$. Furthermore,*

$$D_M(\tau_M,X,f)=\theta_M(f,\tau_M,X) \tag{7.13}$$

and

$$^{c}D_M(\tau_M,X,f)={}^c\theta_M(f,\tau_M,X) \tag{7.14}$$

for any $\tau_M \in T(M)$ *and* $X \in \mathfrak{a}_{M,F}$.

Proof. The part of the lemma that pertains to $\theta_M(f)$ was established in [7, §6]. Since the definitions are slightly different here, we must persuade ourselves that the results continue to hold. The essential step is to verify the analogue of the formula (4.9) in [4, Lemma 4.7]. In the present context, the assertion is that

$$\theta_M(h,\pi,X) = I_{M,\mu(X)+\varepsilon}(\pi,X,h), \tag{7.15}$$

for any $h \in \widetilde{\mathcal{H}}_{\mathrm{ac}}(G(F))$, $\pi \in \Pi_{\mathrm{temp}}(M(F))$ and $X \in \mathfrak{a}_{M,F}$. Here

$$I_{M,\mu}(\pi,X,h) = I_M(\pi_\mu,X,h)e^{-\mu(X)}, \quad \mu \in \mathfrak{a}_M^*,$$

is the invariant distribution which is supported on characters [4, Theorem 6.1], and which is defined inductively in [4, §3] by a formula

$$\phi_{M,\mu}(h,\pi,X) = \sum_{L \in \mathcal{L}(M)} \hat{I}^L_{M,\mu}(\pi,X,\phi_L(h)).$$

Taking $\mu = \mu(X)+\varepsilon$, we can write the left hand side of this formula as

$$\phi_{M,\mu(X)+\varepsilon}(h,\pi,X) = \phi_{M,\mu(X)}(h,\pi,X) = {}^c\phi_M(h,\pi,X) = \sum_{L \in \mathcal{L}(M)} \hat{\theta}^L_M(\phi_L(f),\pi,X),$$

by deforming the contour of integration in (7.2), and then turning to the definitions (7.3) and (7.6). Assuming (7.15) inductively, we have

$$\hat{\theta}^L_M(\phi_L(h),\pi,X) = \hat{I}^L_{M,\mu(X)+\varepsilon}(\pi,X,\phi_L(h))$$

for any $L \neq G$. The required formula (7.15) is then the identity of terms with $L = G$.

Apply (7.15) with h equal to the cuspidal function f. If $\pi \in \Pi_{\mathrm{temp}}(M(F))$ is properly induced, the descent formula [4, Corollary 8.5] tells us that $I_{M,\mu(X)+\varepsilon}(\pi,X,f)$ vanishes. Therefore $\theta_M(f,\pi,X)$ also vanishes. It follows that $\theta_M(f)$ is a cuspidal function. Combining this with (7.10), we see inductively that the function ${}^c\theta_M(f)$ is also cuspidal.

It remains to establish (7.13) and (7.14). At first glance, this might seem to be a direct consequence of the two pairs (7.6), (7.7) and (7.11), (7.12) of parallel definitions. However, we must be careful. For example, if ϱ is a nontempered representation

$$\mathcal{I}_R(\sigma_\Lambda), \quad R \in \mathcal{F}^L(M_0),\ \sigma \in \Pi_{\mathrm{temp}}(M_R(F)),\ \Lambda \in \mathfrak{a}^*_{R,\mathbf{C}},$$

of $L(F)$, the function $\phi_L(f,\varrho,X)$, defined by analytic continuation from its values for purely imaginary Λ, is not generally equal to to the integral

$$\int_{i\mathfrak{a}^*_{L,F}} \mathrm{tr}(\mathcal{R}_L(\varrho_\lambda,Q)\mathcal{I}_Q(\varrho_\lambda,f))e^{-\lambda(X)}\,d\lambda, \quad Q \in \mathcal{P}(L).$$

The identity (7.13) is actually false if f is not required to be cuspidal.

It will be enough to prove the identities with τ_M replaced by any representation $\pi \in \Pi_{\mathrm{temp}}(M(F))$. There is also no harm in assuming that $M \neq G$. We shall use the general reduction formula of [7], which for any μ and $\mathcal{N}$, relates $I_{M,\mu}(\pi, X, f)$ with the residues $D_{M,\mu}^{L,\mathcal{N}_L}(\pi, X)$. Applied to the cuspidal function f, the formula is

$$I_{M,\mu}(\pi, X, f) = \sum_{L \in \mathcal{L}(M)} \widehat{D}_{M,\mu}^{L,\mathcal{N}_L}(\pi, X, I_{L,\nu_L}(f)), \tag{7.16}$$

in the notation of [7, Corollary 6.1]. $I_{L,\nu_L}(f)$ really stands for the function

$$(\Lambda, \varrho, X) \to I_{L,\nu_L}(\varrho, X_L, f) e^{-\Lambda(X)}$$

of three variables, $\Lambda \in \mathfrak{a}_{M,\mathbf{C}}^*$, ϱ a standard representation of $L(F)$ as above, and $X \in \mathfrak{a}_{M,F}$. (See [7, (2.5) and the discussion preceding Corollary 6.1].) If ϱ is properly induced, the descent formula [4, Corollary 8.5] implies that $I_{L,\nu_L}(\varrho, X_L, f)$ vanishes, as explained at the beginning of [7, §6]. Since π is tempered, $D_{M,\mu}^{L,\mathcal{N}_L}(\pi)$ is supported at those ϱ with unitary central character (as in the proof of [7, Corollary 6.2]), so it suffices to consider only the case that ϱ is tempered. To obtain (7.13), we take $\mathcal{N} = \mathcal{N}(0, \varepsilon)$ and $\mu = \mu(X) + \varepsilon$. Following the proof of [7, Corollary 6.2], we see that

$$I_{L,\nu_L}(\varrho, X_L, f) = I_{L,\varepsilon}(\varrho, X_L, f) = 0, \quad \varrho \in \Pi_{\mathrm{temp}}(L(F)),\ L \neq G,$$

by [4, Lemma 4.5]. In particular, the summands with $L \neq G$ on the right hand side of the identity (7.16) vanish. The summand with $L = G$ equals $D_M(\pi, X, f)$, while by (7.15) the left hand side of (7.16) equals $\theta(f, \pi, X)$. The identity reduces simply to (7.13). To establish (7.14) we take $\mathcal{N} = \mathcal{N}(X, \varepsilon)$ and $\mu = \varepsilon$. Then it is the left hand side of (7.16) which vanishes, while (7.15) tells us that

$$I_{L,\nu_L}(\varrho, X_L, f) = I_{L,\mu(X_L)+\varepsilon_L}(\varrho, X_L, f) = \theta_L(f, \varrho, X_L),$$

for tempered ϱ. If we assume inductively that (7.14) holds when G is replaced by a proper parabolic subgroup L, the corresponding summand on the right hand side of (7.16) equals ${}^c\hat{\theta}_M^L(\theta_L(f), \pi, X)$. The summand with $L = G$ equals ${}^cD_M(\pi, X, f)$. Therefore (7.16) reduces to

$${}^cD_M(\pi, X, f) + \sum_{L \neq G} {}^c\hat{\theta}_M^L(\theta_L(f), \pi, X) = 0.$$

We combine this with the equation

$${}^c\theta_M(\pi, X, f) + \sum_{L \neq G} {}^c\hat{\theta}_M^L(\theta_L(f), \pi, X) = 0$$

obtained from (7.10). The obvious conclusion is the required formula (7.14). This completes the proof of the lemma. □

8. Truncated characters

The invariant distribution ${}^cI_M(\gamma, f)$ is a companion to $I_M(\gamma, f)$. It is given by an inductive definition (7.5) which is obviously parallel to that (7.4) of $I_M(\gamma, f)$. As we recalled in §7, the property that distinguishes ${}^cI_M(\gamma, f)$ from $I_M(\gamma, f)$ is its compact support in γ. Now for cuspidal f, we have an expansion (5.2) for $I_M(\gamma, f)$ in terms of elliptic tempered characters of $G(F)$. Is there a similar expansion for ${}^cI_M(\gamma, f)$? In particular, can one be more precise about the support of ${}^cI_M(\gamma, f)$ if f is cuspidal? The answer to these questions is yes. We shall derive an expansion for ${}^cI_M(\gamma, f)$ in terms of "compact traces" of elliptic tempered characters which is parallel to the expansion for $I_M(\gamma, f)$.

Set

$$M(F)^G = \{\gamma \in M(F) : H_M(\gamma) \in \mathfrak{a}_G\}, \quad M \in \mathcal{L}.$$

Then if Θ is a finite linear combination of irreducible characters on $G(F)$, and γ is a G-regular point in $M(F)$, we define the truncated character

$${}^c\Phi_M(\gamma) = \begin{cases} \Phi_M(\gamma), & \text{if } \gamma \in M(F)^G, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, we have

$${}^c\Phi_M(\tau^\vee, \gamma) = \begin{cases} |D(\gamma)|^{1/2}\Theta(\tau^\vee, \gamma), & \text{if } \gamma \in M(F)_{\mathrm{ell}} \cap M(F)^G, \\ 0, & \text{otherwise,} \end{cases} \tag{8.1}$$

for any $\tau \in T_{\mathrm{ell}}(G)$. Truncated characters play an important role in p-adic harmonic analysis, as for example in Clozel's proof of the Howe conjecture. The Archimedean case is simpler. Then ${}^c\Phi_M(\tau^\vee, \gamma)$ can be ignored if $M \neq G$, since it vanishes for γ in an open dense subset of $M(F)$, while ${}^c\Phi_M(\tau^\vee, \gamma)$ equals $\Phi_M(\tau^\vee, \gamma)$ if $M = G$.

THEOREM 8.1. *Suppose that f is a cuspidal function in $\mathcal{H}(G(F))$. Then*

$${}^cI_M(\gamma, f) = (-1)^{\dim(A_M/A_G)} \int_{T_{\mathrm{ell}}(G)} |d(\tau)|^{-1}\, {}^c\Phi_M(\tau^\vee, \gamma)\Theta(\tau, f)\, d\tau, \tag{8.2}$$

for any group $M \in \mathcal{L}$ and any G-regular point γ in $M(F)$.

Proof. We noted at the end of §5 that the assertion (5.2) of Theorem 5.1 had an equivalent formulation (5.2*). The same remark applies here. The formula (8.1) has an equivalent version

$${}^cI_M(\gamma, f) = (-1)^{\dim(A_M/A_G)} \sum_{\tau \in T_{\mathrm{ell}}(G)/i\mathfrak{a}_G^*} |d^1(\tau)|^{-1}\, {}^c\Phi_M(\tau^\vee, \gamma) f_G(\tau, H_G(\gamma)), \tag{8.2*}$$

in which f is permitted to lie in the larger space $\widetilde{\mathcal{H}}_{\mathrm{ac}}(G(F))$. We shall prove it in this form.

As in the proof of Theorem 5.1, we can easily dispense with the case that γ does not belong to $M(F)_{\mathrm{ell}}$. The right hand side of (8.2*) vanishes by definition in this case. To show that the left hand side ${}^cI_M(\gamma, f)$ also vanishes, we write

$$ {}^cI_M(\gamma, f) = \sum_{L \in \mathcal{L}(M)} \hat{I}_M^L(\gamma, {}^c\theta_L(f)), $$

as in (7.8). By Lemma 7.1 the function ${}^c\theta_L(f) \in \widetilde{\mathcal{I}}_{\mathrm{ac}}(L(F))$ is cuspidal. We can therefore apply the descent formula [4, Corollary 8.5] to conclude that

$$ \hat{I}_M^L(\gamma, {}^c\theta_L(f)) = 0. $$

Thus ${}^cI_M(\gamma, f)$ vanishes, and the formula (8.2*) holds for γ outside of $M(F)_{\mathrm{ell}}$.

To deal with the elliptic points in $M(F)$, we shall compare the two expressions (5.2*) and (7.9) for $I_M(\gamma, f)$. The first one expresses $I_M(\gamma, f)$ as a function

$$ F_1(\gamma) = (-1)^{\dim(A_M/A_G)} \sum_{\tau \in T_{\mathrm{ell}}(G)/i\mathfrak{a}_G^*} |d^1(\tau)|^{-1} \Phi_M(\tau^\vee, \gamma) f_G(\tau, H_G(\gamma)), $$

while the second allows us to write $I_M(\gamma, f)$ in the form

$$ {}^cI_M(\gamma, f) + \sum_{\substack{L \in \mathcal{L}(M) \\ L \neq G}} {}^c\hat{I}_M^L(\gamma, \theta_L(f)). $$

In other words,

$$ {}^cI_M(\gamma, f) = F_1(\gamma) - \sum_{\substack{L \in \mathcal{L}(M) \\ L \neq G}} {}^c\hat{I}_M^L(\gamma, \theta_L(f)). $$

Assume inductively that (8.2*) holds whenever G is replaced by a Levi subgroup $L \subsetneq G$. By Lemma 7.1 the function $\theta_L(f) \in \widetilde{\mathcal{I}}_{\mathrm{ac}}(L(F))$ is cuspidal. We can therefore use the induction assumption to write

$$ {}^c\hat{I}_M^L(\gamma, \theta_L(f)) = (-1)^{\dim(A_M/A_L)} \sum_{\tau_L \in T_{\mathrm{ell}}(L)/i\mathfrak{a}_L^*} |d^1(\tau_L)|^{-1}\, {}^c\Phi_M^L(\tau_L^\vee, \gamma) \theta_L(f, \tau_L, H_L(\gamma)), $$

for any $L \subsetneq G$; we are using the notation

$$ {}^c\Phi_M^L(\tau_L^\vee, \gamma) = {}^c\Phi_M(\tau_L^\vee, \gamma) $$

here to emphasize the role of L. Lemma 7.1 also asserts that

$$ \theta_L(f, \tau_L, H_L(\gamma)) = D_L(\tau_L, H_L(\gamma), f). $$

It thus follows that ${}^cI_M(\gamma, f)$ equals the difference between $F_1(\gamma)$ and the function

$$F_2(\gamma) = \sum_{\substack{L\in\mathcal{L}(M)\\ L\neq G}} (-1)^{\dim(A_M/A_L)} \sum_{\tau_L\in T_{\mathrm{ell}}(L)/i\mathfrak{a}_L^*} |d^1(\tau_L)|^{-1}\, {}^c\Phi_M^L(\tau_L^\vee, \gamma) D_L(\tau_L, H_L(\gamma), f).$$

Given the G-regular point $\gamma\in M(F)_{\mathrm{ell}}$, we form the vector $H_M(\gamma)$ in $\mathfrak{a}_M$. This vector belongs to a chamber $\mathfrak{a}_Q^+$, for a uniquely determined parabolic subgroup $Q\in\mathcal{F}(M)$. We shall consider separately the cases that $Q=G$ and $Q\neq G$.

If $Q=G$, γ lies in $M(F)^G$. Then

$$\Phi_M(\tau^\vee, \gamma) = {}^c\Phi_M(\tau^\vee, \gamma), \quad \tau\in T_{\mathrm{ell}}(G),$$

from which it follows that $F_1(\gamma)$ equals the right hand side of the formula (8.2*). The vectors $H_L(\gamma)$ which occur in the expression above for $F_2(\gamma)$ are each equal to $H_M(\gamma)$, a point in $\mathfrak{a}_G$. The change of contour involved in the definition of the distributions $D_L(\tau_L, H_L(\gamma), f)$ is therefore trivial, and $D_L(\tau_L, H_L(\gamma), f)$ vanishes for any $L\subsetneq G$. Consequently $F_2(\gamma)$ vanishes. We have obtained an identity

$${}^cI_M(\gamma, f) = F_1(\gamma) - F_2(\gamma) = F_1(\gamma),$$

which is just the required formula (8.2*) in the case under consideration.

Suppose finally that $Q\neq G$. Then γ does not lie in $M(F)^G$, so the functions ${}^c\Phi_M(\tau^\vee, \gamma)$ on the right hand side of (8.2*) are all equal to 0. To complete the proof of the theorem, we must show that ${}^cI_M(\gamma, f)$ vanishes. Suppose that a is a point in the semigroup

$$\Gamma(Q) = \{a\in A_Q(F) : H_Q(a)\in \overline{\mathfrak{a}_Q^+}\cap\mathfrak{a}_M^G\}.$$

Then γa is still a G-regular point in $M(F)_{\mathrm{ell}}$, and the vector

$$H_M(\gamma a) = H_M(\gamma) + H_Q(a)$$

stays within the chamber $\mathfrak{a}_Q^+$. We shall study the identity

$${}^cI_M(\gamma a, f) = F_1(\gamma a) - F_2(\gamma a) \tag{8.3}$$

as a function of a.

The left hand side of (8.3) is a compactly supported function of a. Since we are taking f in $\mathcal{H}(G(F))$, this follows from [4, Lemma 4.4]. Our remaining task is to establish that the right hand side of (8.3) behaves in the opposite way. We shall show that the function on the right is $\Gamma(Q)$-finite. This means that its $\Gamma(Q)$-translates span a finite dimensional space of functions of a, or equivalently, that the right hand side of (8.3) is a finite sum

$$\sum_k p_k(H_Q(a))\zeta_k(a), \tag{8.4}$$

where $\{\zeta_k\}$ are quasi-characters on $A_Q(F)$ and $\{p_k\}$ are polynomial functions on $\mathfrak{a}_Q$.

To this end, we observe that the first function $F_1(\gamma a)$ on the right hand side of (8.3) is a finite linear combination of virtual characters

$$\Phi_M(\tau^\vee, \gamma a), \quad \tau \in T_{\mathrm{ell}}(G)/i\mathfrak{a}_G^*.$$

Each such virtual character can in turn be written as a finite linear combination of irreducible tempered characters

$$\Phi_M(\pi, \gamma a) = |D(\gamma a)|^{1/2}\Theta(\pi, \gamma a).$$

If F is Archimedean, $\gamma\Gamma(Q)$ is contained in a connected component of $T(F)\cap G_{\mathrm{reg}}(F)$, where T is a maximal torus in G. In this case one concludes from Harish-Chandra's theory of characters on real groups, and the differential equations they satisfy, that each $\Phi_M(\pi, \gamma a)$ is a $\Gamma(Q)$-finite function of a. If F is p-adic, we can use Casselman's theorem [15, Theorem 5.2]. This result provides a formula

$$\Phi_M(\pi, \gamma a) = \Phi_M(\pi_Q, \gamma a)$$

for $\Phi_M(\pi)$ in terms of the normalized Jacquet module

$$\pi_Q = (\delta_{\bar{Q}})^{-1/2}\pi_{N_{\bar{Q}}} = (\delta_Q)^{1/2}\pi_{N_{\bar{Q}}}$$

attached to the group $\bar{Q}\in\mathcal{P}(M_Q)$ opposite to Q. Since π_Q has finitely many composition factors, each $\Phi_M(\pi, \gamma a)$ is a $\Gamma(Q)$-finite function in the p-adic case as well. It follows that $F_1(\gamma a)$ is a $\Gamma(Q)$-finite function of a. To deal with the second function $F_2(\gamma a)$, observe that if L is not contained in M_Q, the vector $H_M(\gamma a)$ does not lie in $\mathfrak{a}_L$, and the truncated characters ${}^c\Phi_M^L(\tau_L^\vee, \gamma)$ vanish. It follows that

$$F_2(\gamma a) = \sum_{\substack{L\in\mathcal{L}(M)\\ L\subset M_Q}} (-1)^{\dim(A_M/A_L)} \sum_{\tau_L} |d^1(\tau_L)|^{-1}\, {}^c\Phi_M^L(\tau_L^\vee, \gamma a) D_L(\tau_L, H_L(\gamma a), f),$$

where τ_L can in fact be summed over a finite subset of $T_{\mathrm{ell}}(L)/i\mathfrak{a}_L^*$. If τ_L equals (M_1, σ, r), the function

$$a \to {}^c\Phi_M^L(\tau_L^\vee, \gamma a), \quad a \in \Gamma(Q),$$

is just a multiple of the central character of σ. Moreover, $D_L(\tau_L, H_L(\gamma a), f)$ is a finite linear combination of residues, taken with respect to the complex variable $\lambda \in \mathfrak{a}^*_{L,\mathbf{C}}$, of functions

$$\mathrm{tr}(\mathcal{R}_L(\pi_\lambda, P)\mathcal{I}_P(\pi_\lambda, f))e^{-\lambda(H_L(\gamma)+H_L(a))}, \quad \pi \in \Pi_\sigma(L(F)).$$

As a function of a, any such residue is obviously the product of an unramified quasi-character with a polynomial in $H_L(a)$. It follows that $F_2(\gamma a)$ is also a $\Gamma(Q)$-finite function of a.

We have just established that

$$F_1(\gamma a) - F_2(\gamma a), \quad a \in \Gamma(Q),$$

the right hand side of (8.3), is $\Gamma(Q)$-finite. It is therefore of the general form (8.4). We have also observed that the same function, as the left hand side of (8.3), is compactly supported in $a \in \Gamma(Q)$. The two properties are mutually exclusive. They force each side of (8.3) to vanish. In particular, ${}^cI_M(\gamma a, f)$ equals 0 for each $a \in \Gamma(Q)$. Taking $a=1$, we obtain the required property that ${}^cI_M(\gamma, f)$ vanishes. This establishes the formula (8.2*) in the final case that $Q \neq G$. The proof of the theorem is complete. □

Observe that the theorem gives a precise description of the support of ${}^cI_M(\gamma, f)$. This has an application to the maps ${}^c\phi_M(f)$, which may play a role in the study of p-adic orbital integrals. We know that for any $f \in \mathcal{H}(G(F))$, the functions $X \rightarrow {}^c\phi_M(f, \tau_M, X)$ are compactly supported. We shall show that with a (noninvariant) condition on f, stronger than cuspidality, the support has a description like that of the theorem.

A function $f \in \mathcal{H}(G(F))$ is cuspidal if and only if the invariant orbital integrals

$$J_G(\gamma, f) = I_G(\gamma, f), \quad \gamma \in G_{\mathrm{reg}}(F),$$

vanish for γ in the complement of the elliptic set $G(F)_{\mathrm{ell}}$. We shall use the uninspired phrase *cuspidal with respect to weighted orbital integrals* for the stronger condition that all of the weighted orbital integrals

$$J_M(\gamma, f), \quad M \in \mathcal{L}, \ \gamma \in M(F) \cap G_{\mathrm{reg}}(F),$$

vanish except in the case that M equals G and γ lies in $G(F)_{\mathrm{ell}}$. (The term *very cuspidal* has been taken [30], and denotes a slightly different property.) Observe, for example, that any function which is supported on $G(F)_{\mathrm{ell}}$ is cuspidal with respect to weighted orbital integrals. Since we also require that the function lie in $\mathcal{H}(G(F))$, this example pertains essentially to the p-adic case. It can be used to study the invariant orbital integrals of spherical functions.

COROLLARY 8.2. *Suppose that $f\in\mathcal{H}(G(F))$ is cuspidal with respect to orbital integrals, and that $M\in\mathcal{L}$. Then the function ${}^c\phi_M(f)$ is cuspidal, and its values*

$${}^c\phi_M(f,\tau_M,X),\quad \tau_M\in T(M),\ X\in\mathfrak{a}_{M,F},$$

vanish for elements X in the complement of $\mathfrak{a}_G\cap\mathfrak{a}_{M,F}$.

Proof. We shall combine the theorem with the inductive definition

$$J_M(\gamma,f)=\sum_{L\in\mathcal{L}(M)}{}^c\hat{I}_M^L(\gamma,{}^c\phi_L(f)),\quad \gamma\in M(F)\cap G_{\mathrm{reg}}(F),$$

of ${}^cI_M(\gamma)$. If $M=G$, ${}^c\phi_M(f)$ equals f_G, and the assertions are obvious. We can therefore assume that $M\neq G$. Then the left hand side of the formula vanishes by assumption. Moreover, ${}^cI_M^M(\gamma)$ equals $I_M^M(\gamma)$, and is simply the invariant orbital integral on $M(F)$. We obtain

$$\hat{I}_M^M(\gamma,{}^c\phi_M(f))=-\sum_{\substack{L\in\mathcal{L}(M)\\ L\neq M}}{}^c\hat{I}_M^L(\gamma,{}^c\phi_L(f)),\tag{8.5}$$

for any G-regular element γ in $M(F)$.

Assume inductively that the assertions of the corollary are true if M is replaced by any group $L\in\mathcal{L}(M)$ with $L\neq M$. Then the functions ${}^c\phi_L(f)$ on the right hand side of (8.5) are cuspidal. Applying the version (8.2*) of the theorem, with G replaced by L, we write the right hand side of (8.5) as

$$\sum_{L\neq M}(-1)^{\dim(A_M/A_L)+1}\sum_{\tau_L\in T_{\mathrm{ell}}(L)/i\mathfrak{a}_L^*}|d^1(\tau_L)|^{-1}\,{}^c\Phi_M^L(\tau_L^\vee,\gamma){}^c\phi_L(f,\tau_L,H_L(\gamma)).\tag{8.6}$$

Suppose that γ does not lie in $M(F)_{\mathrm{ell}}$. Then the functions ${}^c\Phi_M^L(\tau_L^\vee,\gamma)$ vanish by definition, and the right hand side of (8.5) equals 0. We have taken γ in $M(F)\cap G_{\mathrm{reg}}(F)$, but this set is dense in $M_{\mathrm{reg}}(F)$. It follows that

$$\hat{I}_M^M(\gamma,{}^c\phi_M(f))=0$$

for any γ in the complement of $M(F)_{\mathrm{ell}}$ in $M_{\mathrm{reg}}(F)$. Consequently, ${}^c\phi_M(f)$ is a cuspidal function in $\widetilde{\mathcal{I}}_{\mathrm{ac}}(M(F))$, as required.

To deal with the second assertion of the corollary, we shall again use the fact that $\hat{I}_M^M(\gamma,{}^c\phi_M(f))$ equals (8.6). Fix a point X in the complement of $\mathfrak{a}_G$ in $\mathfrak{a}_{M,F}$. We claim that $\hat{I}_M^M(\gamma,{}^c\phi_M(f))=0$ for any point γ in $M(F)\cap G_{\mathrm{reg}}(F)$ with $H_M(\gamma)=X$. Suppose this is not so. Then one of the summands in (8.6) is nonzero. From the nonvanishing of

${}^c\Phi^L_M(\tau^\vee_L,\gamma)$ we infer that X lies in $\mathfrak{a}_L\cap\mathfrak{a}_{M,F}$. In particular, X is also equal to $H_L(\gamma)$. From the nonvanishing of the second function

$$ {}^c\phi_L(f,\tau_L,H_L(\gamma)) = {}^c\phi_L(f,\tau_L,X) $$

and our induction hypothesis, we conclude that X actually lies in $\mathfrak{a}_G\cap\mathfrak{a}_{M,F}$. This contradicts the assumption on X. The claim is therefore valid. It remains for us simply to evaluate a distribution $\widehat{\Theta}(\tau_M)$, $\tau_M\in T(M)$, on the cuspidal function ${}^c\phi_M(f)$. It follows easily from the formula (1.3) that

$$ {}^c\phi_M(f,\tau_M,X) = \int \Phi_M(\tau_M,\gamma)\hat{I}^M_M(\gamma,{}^c\phi_M(f))\,d\gamma, $$

where $d\gamma$ is the appropriate measure on the set

$$ \{\gamma\in\Gamma_{\mathrm{ell}}(M(F)) : H_M(\gamma)=X\}. $$

We have just seen that the integrand vanishes on the G-regular classes, a set whose complement has measure 0. It follows that ${}^c\phi_M(f,\tau_M,X)=0$ for any $\tau_M\in T(M)$. This establishes the second assertion of the corollary. □

9. Characters and residues

We shall finish the paper with a simple application of Theorems 5.1 and 8.1. We shall establish a relation between elliptic tempered characters and residues of intertwining operators. The characters will take their usual form as the normalized functions $\{\Phi_M(\tau^\vee,\gamma)\}$ and their truncated analogues $\{{}^c\Phi_M(\tau^\vee,\gamma)\}$. The residues will come in through the two families $\{D_M(\tau_M,X)\}$ and $\{{}^cD_M(\tau_M,X)\}$ of distributions discussed in §7. We shall in fact prove two identities, in which the two functions associated to characters are related separately to the two families of distributions.

The invariant distributions

$$ \{D_M(\tau_M,X), {}^cD_M(\tau_M,X) : \tau_M\in T_{\mathrm{ell}}(M), X\in\mathfrak{a}_{M,F}\} $$

are supported on characters. Their values at cuspidal functions $f\in\mathcal{H}(G(F))$ can therefore be written as linear combinations of distributions $f_G(\tau,X_G)$. We obtain expressions

$$ D_M(\tau_M,X,f) = \sum_{\tau\in T_{\mathrm{ell}}(G)/i\mathfrak{a}^*_G} D_M(\tau_M,X,\tau)f_G(\tau,X_G) \tag{9.1} $$

and

$$^cD_M(\tau_M, X, f) = \sum_{\tau \in T_{\mathrm{ell}}(G)/i\mathfrak{a}_G^*} {}^cD_M(\tau_M, X, \tau) f_G(\tau, X_G), \tag{9.2}$$

with uniquely determined coefficients $\{D_M(\tau_M, X, \tau)\}$ and $\{{}^cD_M(\tau_M, X, \tau)\}$ such that

$$D_M(\tau_M, X, z\tau_\lambda) = \chi_\tau(z) D_M(\tau_M, X, \tau) e^{-\lambda(X_G)}$$

and

$$^cD_M(\tau_M, X, z\tau_\lambda) = \chi_\tau(z) {}^cD_M(\tau_M, X, \tau) e^{-\lambda(X_G)},$$

for any $z \in Z_\tau$ and $\lambda \in i\mathfrak{a}_G^*$.

These coefficients are curious objects. If we work backwards through their definitions in terms of residues, keeping in mind that the residues define invariant distributions, we see that they are related to subquotients of induced representations. For example, if $\tau_M = \sigma$ and $\tau = \pi$ are square integrable representations (in $\Pi_2(M(F))$ and $\Pi_2(G(F))$ respectively), and X is constrained to lie in any of the convex cones in the relevant decomposition of $\mathfrak{a}_M$, $D_M(\tau_M, X, \tau)$ and ${}^cD_M(\tau_M, X, \tau)$ are finite linear combinations of functions

$$p_\Lambda(X) e^{-\Lambda(X)}, \quad \Lambda \in \mathfrak{a}_{M,\mathbf{C}}^*, \ p_\Lambda(X) \in \mathbf{C}[X], \tag{9.3}$$

taken over points Λ at which the induced representation $\mathcal{I}_P(\sigma_\Lambda)$ is reducible. More precisely, $\mathcal{I}_P(\sigma_\Lambda)$ should have irreducible constituents whose expansions in terms of standard representations contain π. The residue operations will assign polynomials to these constituents, the sum of which equals $p_\Lambda(X)$.

As before we shall write

$$\Phi_M^L(\tau_L^\vee, \gamma) = \Phi_M(\tau_L^\vee, \gamma)$$

and

$$^c\Phi_M^L(\tau_L^\vee, \gamma) = {}^c\Phi_M(\tau_L^\vee, \gamma), \quad \tau_L \in T(L), \ \gamma \in M(F) \cap G_{\mathrm{reg}}(F),$$

for the normalized characters on a Levi subgroup $L \in \mathcal{L}(M)$.

THEOREM 9.1. *Suppose that τ belongs to $T_{\mathrm{ell}}(G)$, that M is a group in $\mathcal{L}$, and that γ is a G-regular point in $M(F)$. Then*

$$\Phi_M(\tau^\vee, \gamma) = \sum_L \sum_{\tau_L} d_L(\tau_L, \tau) {}^c\Phi_M^L(\tau_L^\vee, \gamma) D_L(\tau_L, H_L(\gamma), \tau) \tag{9.4}$$

and

$$^c\Phi_M(\tau^\vee, \gamma) = \sum_L \sum_{\tau_L} d_L(\tau_L, \tau) \Phi_M^L(\tau_L^\vee, \gamma) {}^cD_L(\tau_L, H_L(\gamma), \tau), \tag{9.5}$$

where L and τ_L are summed over $\mathcal{L}(M)$ and $T_{\mathrm{ell}}(L)/i\mathfrak{a}_L^*$ respectively, and

$$d_L(\tau_L,\tau)=(-1)^{\dim(A_L/A_G)}|d^1(\tau_L)|^{-1}|d^1(\tau)|.$$

Proof. Suppose that f is a cuspidal function in $\mathcal{H}(G(F))$. To derive the first formula, we start with an expression

$$\sum_{\tau_G\in T_{\mathrm{ell}}(G)/i\mathfrak{a}_G^*} |d^1(\tau_G)|^{-1}\Phi_M(\tau_G^\vee,\gamma)f_G(\tau_G,H_G(\gamma)), \tag{9.6}$$

obtained from the version (5.2*) of Theorem 5.1. We shall repeat some of the formal manipulations of the proof of Theorem 8.1. According to (5.2*), the expression (9.6) equals the product of $(-1)^{\dim(A_M/A_G)}$ with $I_M(\gamma,f)$. This can in turn be written

$$\sum_{L\in\mathcal{L}(M)} (-1)^{\dim(A_M/A_G)}\,{}^c\hat{I}_M^L(\gamma,\theta_L(f)),$$

by (7.9). But $\theta_L(f)$ is cuspidal by Lemma 7.1. We can therefore apply the version (8.2*) of Theorem 8.1, with G replaced by L, to the summand ${}^c\hat{I}_M^L(\gamma,\theta_L(f))$. The whole expression becomes

$$\sum_{L\in\mathcal{L}(M)} (-1)^{\dim(A_L/A_G)} \sum_{\tau_L\in T_{\mathrm{ell}}(L)/i\mathfrak{a}_L^*} |d^1(\tau_L)|^{-1}\,{}^c\Phi_M^L(\tau_L^\vee,\gamma)\theta_L(f,\tau_L,H_L(\gamma)).$$

The last step is to make the substitution

$$\begin{aligned}\theta_L(f,\tau_L,H_L(\gamma)) &= D_L(\tau_L,H_L(\gamma),f)\\ &= \sum_{\tau_G\in T_{\mathrm{ell}}(G)/i\mathfrak{a}_G^*} D_L(\tau_L,H_L(\gamma),\tau_G)f_G(\tau_G,H_G(\gamma)),\end{aligned}$$

which comes from Lemma 7.1 and the definition (9.1). With this substitution, the expression becomes a linear combination of functionals $f_G(\tau_G,H_G(\gamma))$. The coefficient of $f_G(\tau,H_G(\gamma))$ is in fact the product of $|d^1(\tau)|^{-1}$ with the right hand side of (9.4). The same coefficient in the original expression is simply the product of $|d^1(\tau)|^{-1}$ with the left hand side of (9.4). If we take f to be a pseudocoefficient of τ, the identity (9.4) follows.

To derive (9.5), we perform similar manipulations on the dual expression

$$\sum_{\tau_G\in T_{\mathrm{ell}}(G)/i\mathfrak{a}_G^*} |d^1(\tau_G)|^{-1}\,{}^c\Phi_M(\tau_G^\vee,\gamma)f_G(\tau_G,H_G(\gamma)). \tag{9.7}$$

Applying the formulas (8.2*) and (7.8) in turn, we reduce the expression to

$$\sum_{L\in\mathcal{L}(M)} (-1)^{\dim(A_M/A_G)}\hat{I}_M^L(\gamma,{}^c\theta_L(f)).$$

The formula (5.2*) for $\hat{I}^L_M(\gamma)$ then expands the expression to

$$\sum_{L\in\mathcal{L}(M)}(-1)^{\dim(A_L/A_G)}\sum_{\tau_L\in T_{\mathrm{ell}}(L)/i\mathfrak{a}^*_L}|d^1(\tau_L)|^{-1}\Phi^L_M(\tau^\vee_L,\gamma)^c\theta_L(f,\tau_L,H_L(\gamma)).$$

Finally, with the substitution

$${}^c\theta_L(f,\tau_L,H_L(\gamma))=\sum_{\tau_G\in T_{\mathrm{ell}}(G)/i\mathfrak{a}^*_G}{}^cD_L(\tau_L,H_L(\gamma),\tau_G)f_G(\tau_G,H_G(\gamma)),$$

obtained from Lemma 7.1 and the definition (9.2), we arrive at a linear combination of functionals $f_G(\tau_G,H_G(\gamma))$. Comparing coefficients of $f_G(\tau,H_G(\gamma))$ in the initial and final expressions, as before, we obtain the required identity (9.5). □

Remarks. (1) The sum over L in (9.4) is a bit misleading. If f is Archimedean, the terms with $L\neq M$ vanish almost everywhere, and can be ignored. These terms also vanish for p-adic F if $H_M(\gamma)$ does not lie on any of the singular hyperplanes in $\mathfrak{a}_M$. The identity (9.5), on the other hand, is a kind of inversion formula, and here the sum over L is an essential ingredient.

(2) The formula (9.4) gives an interpretation for the restriction to $M(F)_{\mathrm{ell}}$ of a (virtual) character $\Theta(\tau^\vee)$ on $G(F)$. There is another well known such interpretation. In the p-adic case it is Casselman's theorem [15] for characters in terms of Jacquet modules. For real groups it is Osborne's conjecture, proved by Hecht and Schmid [23], for characters in terms of $\mathfrak{n}_P$-homology modules. The residues $D_M(\tau_M,X,\tau)$, which account for the character exponents here, are not so explicit. Still, it would be interesting to try to relate them directly to Jacquet modules or $\mathfrak{n}_P$-homology modules.

We conclude with a few informal comments on a possible application of the results in §§7–9. Fix a cuspidal function $f\in\mathcal{H}(G(F))$. Suppose also that π is a representation in $\Pi_{\mathrm{temp}}(M(F))$, and that $a(\lambda)$ belongs to the Paley–Wiener space on $i\mathfrak{a}^*_{M,F}$. For some purposes it would be useful to understand the integral

$$\int_{i\mathfrak{a}^*_{M,F}}a(\lambda)\,\mathrm{tr}(\mathcal{R}_M(\pi_\lambda,P)\mathcal{I}_P(\pi_\lambda,f))\,d\lambda \tag{9.8}$$

as a linear functional in $a(\lambda)$. This integral appears, for example, in Waldspurger's formula [38, §II, Théorème] for orbital integrals of spherical functions on $GL(n)$. Similar integrals have been studied by Laumon [33] and [34] in work on the cohomology with compact support of Shimura varieties.

One can apply Fourier inversion to $a(\lambda)$. The problem becomes that of understanding

$$\int_{i\mathfrak{a}^*_{M,F}}e^{-\lambda(X)}\,\mathrm{tr}(\mathcal{R}_M(\pi_\lambda,P)\mathcal{I}_P(\pi_\lambda,f))\,d\lambda=\phi_M(f,\pi,X)$$

as a function of $X \in \mathfrak{a}_{M,F}$. In view of (7.1), this is in turn equivalent to studying the function

$$X \to \phi_M(f, \tau_M, X),$$

for a fixed element $\tau_M \in T(M)$. Suppose that f is actually cuspidal with respect to weighted orbital integrals. Then one can exploit the expansion

$$\phi_M(f, \tau_M, X) = \sum_{L \in \mathcal{L}(M)} {}^c\hat{\theta}_M^L({}^c\phi_L(f), \tau_M, X)$$

given by (7.7). Indeed, ${}^c\phi_L(f)$ is a cuspidal function by Corollary 8.2, and one can apply Lemma 7.1. This yields an identity

$${}^c\hat{\theta}_M^L({}^c\phi_L(f), \tau_M, X) = {}^c\widehat{D}_M^L(\tau_M, X, {}^c\phi_L(f)),$$

which in fact vanishes unless τ_M lies in $T_{\mathrm{ell}}(M)$. It follows from the definition (9.2) that

$$\phi_M(f, \tau_M, X) = \sum_{L \in \mathcal{L}(M)} \sum_{\tau_L \in T_{\mathrm{ell}}(L)/i\mathfrak{a}_L^*} {}^cD_M^L(\tau_M, X, \tau_L)\, {}^c\phi_L(f, \tau_L, X_L). \tag{9.9}$$

The problem then reduces to studying ${}^c\phi_L(f, \tau_L, X_L)$ and ${}^cD_M^L(\tau_M, X, \tau_L)$ separately as functions of X.

The qualitative behaviour of the right hand side of (9.9) is pretty clear. By Corollary 8.2, ${}^c\phi_L(f, \tau_L, X_L)$ is supported on the set of $X \in \mathfrak{a}_{M,F}$ which lie in $\mathfrak{a}_M^L \oplus \mathfrak{a}_G$. On the other hand, if X remains in the appropriate chamber, ${}^cD_M^L(\tau_M, X, \tau_L)$ is a linear combination of functions (9.3). One could hope to obtain more information about these functions from the inversion formula (9.5).

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