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A LOCAL TRACE FORMULA

by James ARTHUR*

CONTENTS

Introduction

Suppose for a moment that H is a locally compact group and that Γ is a discrete subgroup of H. The problem of the spectral decomposition of $L^2(\Gamma\backslash H)$ has long been known to be both difficult and important. In the 1950's, Selberg introduced a trace formula in certain special cases, and emphasized the importance of obtaining a more general formula. The first goal of Selberg's program was to construct the continuous spectrum. One would then be able to restrict certain operators to the complement of the continuous spectrum, namely the discrete spectrum. The second goal was to construct an explicit expression for the traces of these operators on the discrete spectrum.

The most fruitful setting for the problem has been the case of a reductive Lie group, and an arithmetic congruence subgroup. This is actually equivalent to taking $H = G(\mathbf{A}_F)$ and $\Gamma = G(F)$, where G is a reductive algebraic group over a number field F. The group of F-rational points is then a discrete subgroup of the group of F-adélic points, and the quotient has finite invariant volume, modulo the center. In this setting, Selberg's program has been carried out ([12], [26], [28]). The trace formula

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takes the form of two different expansions of a certain distribution on $G(\mathbf{A}_F)$. One expansion is in terms of irreducible representations—spectral data—and includes the trace on the discrete spectrum. The other expansion is in terms of conjugacy classes—geometric data—and includes invariant orbital integrals.

The trace formula promises to yield deep information about the discrete spectrum of $L^2(\Gamma\backslash H)$. However, there are serious problems still to be solved before this goal can be fully realized. Most of the problems are local in nature, and involve the distributions on the geometric side. Besides invariant orbital integrals on conjugacy classes, these terms include more exotic objects, weighted orbital integrals, not previously encountered in local harmonic analysis. It is believed that there are striking relationships between orbital integrals—both invariant and weighted—on different groups. It is these identities which must be established before the trace formula can be fully exploited.

A few years ago, Kazhdan suggested that there should also be a different kind of trace formula, attached to a real or p-adic group. Suppose now that F is a local field of characteristic 0, and that G is a connected, reductive algebraic group over F. Consider the regular representation

$$(R(y_1, y_2)\,\varphi)\,(x) = \varphi(y_1^{-1}\,xy_2), \quad \varphi \in L^2(G(F)), \ x, y_1, y_2 \in G(F),$$

of $G(F) \times G(F)$ on the Hilbert space $L^2(G(F))$. The spectral decomposition in this case is less deep. It is given by Harish-Chandra's Plancherel formula, which provides a rather explicit decomposition

$$R = R_{\mathrm{disc}} \oplus R_{\mathrm{cont}}$$

of R into subrepresentations with purely discrete or continuous spectrum. Consider a smooth, compactly supported function on $G(F) \times G(F)$ of the form

$$f(y_1, y_2) = f_1(y_1)\,f_2(y_2), \quad y_1, y_2 \in G(F).$$

Then

$$R(f) = \int_{G(F)} \int_{G(F)} f_1(y_1)\,f_2(y_2)\,R(y_1, y_2)\,dy_1\,dy_2$$

is a bounded linear operator on $L^2(G(F))$ which has a decomposition

$$R(f) = R_{\mathrm{disc}}(f) \oplus R_{\mathrm{cont}}(f).$$

Kazhdan's proposal amounts to finding an explicit formula for the trace of the operator $R_{\mathrm{disc}}(f)$. The purpose of this paper is to establish such a formula.

The operator $R(f)$ maps any function $\varphi \in L^2(G(F))$ to the function

$$\begin{aligned}(R(f)\,\varphi)\,(x) &= \int_{G(F)} \int_{G(F)} f_1(u)\,f_2(y)\,\varphi(u^{-1}\,xy)\,du\,dy \\ &= \int_{G(F)} \int_{G(F)} f_1(xu)\,f_2(uy)\,\varphi(y)\,du\,dy \\ &= \int_{G(F)} K(x, y)\,\varphi(y)\,dy,\end{aligned}$$

where

$$K(x,y) = \int_{G(F)} f_1(xu) f_2(uy)\, du = \int_{G(F)} f_1(u) f_2(x^{-1} uy)\, du.$$

Therefore, $R(f)$ is an integral operator with smooth kernel $K(x,y)$. Our goal is to convert this simple formula for the kernel into an expression for the trace of $R_{\mathrm{disc}}(f)$.

If $G(F)$ is compact, the problem is easy. For the Peter-Weyl theorem tells us that $R_{\mathrm{disc}} = R$. Moreover, the trace of an integral operator with smooth kernel on a compact (real or p-adic) manifold is just the integral over the diagonal. Therefore

$$\int_{G(F)} \int_{G(F)} f_1(u) f_2(x^{-1} ux)\, du\, dx = \mathrm{tr}(R_{\mathrm{disc}}(f))$$

in this case. The Weyl integration formula provides an expansion of the left hand side in terms of conjugacy classes $\{ \gamma \}$ of $G(F)$. Similarly, the Peter-Weyl theorem gives an expansion of the right hand side in terms of irreducible representations $\{ \sigma \}$ of $G(F)$. The result is

$$\int J_G(\gamma, f)\, d\gamma = \sum_{\pi = (\sigma^\vee, \sigma)} J_G(\pi, f),$$

where

$$J_G(\gamma, f) = | D(\gamma) | \int_{G(F)} f_1(x_1^{-1} \gamma x_1)\, dx_1 \int_{G(F)} f_2(x_2^{-1} \gamma x_2)\, dx_2$$

and

$$J_G(\pi, f) = \mathrm{tr}(\sigma^\vee(f_1))\, \mathrm{tr}(\sigma(f_2)).$$

The measure $d\gamma$ is supported on the regular conjugacy classes, and comes in the usual way from the normalized Haar measure on the maximal torus that centralizes γ, while

$$D(\gamma) = \det(1 - \mathrm{Ad}(\gamma))_{\mathfrak{g}/\mathfrak{g}_\gamma}$$

is the Weyl discriminant.

The real problem comes when $G(F)$ is not compact. In this case, Harish-Chandra's Plancherel formula tells us that there is a continuous spectrum. Then $R(f)$ is not of trace class and $K(x, x)$ is not integrable, even modulo the split component $A_G(F)$ of the center of $G(F)$. This circumstance complicates matters considerably.

Our initial strategy will be as follows. We shall multiply $K(x, x)$ by the characteristic function $u(x, T)$ of a large compact subset of $G(F)/A_G(F)$. This allows us to define a distribution

$$K^T(f) = \int_{G(F)/A_G(F)} K(x, x)\, u(x, T)\, dx$$

in which the truncation parameter T can vary over lattice points in a chamber $\mathfrak{a}_0^+$ of a real vector space. The Weyl integration formula is of course still valid for a noncompact group. It leads directly to an expansion of

$$K(x, x) = \int_{G(F)} f_1(xu) f_2(ux)\, du = \int_{G(F)} f_1(u) f_2(x^{-1} ux)\, du$$

in terms of regular semisimple conjugacy classes of G(F). Harish-Chandra's Plancherel formula gives a parallel expansion of $K(x, x)$ in terms of irreducible tempered representations of G(F). Together, they provide two expansions

$$(1) \qquad \sum_{M} |W_0^M| \, |W_0^G|^{-1} \int_{\Gamma_{\mathrm{ell}}(M(F))} K^T(\gamma, f)\, d\gamma$$

and

$$(2) \qquad \sum_{M} |W_0^M| \, |W_0^G|^{-1} \int_{\Pi_2(M(F))} K^T(\sigma, f)\, d\sigma$$

for $K^T(f)$.

The identity of (1) and (2) can be regarded as a preliminary version of the local trace formula. The steps required to derive it are formal, and will be taken in § 2 and § 3. However, the terms in (1) and (2) are themselves too formal to be of much use. For one thing, it is hard to see how the distributions $K^T(\gamma,f)$ and $K^T(\sigma,f)$ behave as functions of T. Our main task will be to obtain expansions of (1) and (2) which are more concrete, and which do not depend on the parameter T. Along the way, we shall have to solve a number of analytic and combinatorial problems. Rather than trying to describe these in any detail here, we shall just give a brief summary of the overall process.

We study the geometric expansion (1) in § 4-§ 6. This leads to some understanding of $K^T(f)$ as a function of T. We shall show that as T approaches infinity, $K^T(f)$ is asymptotic to another distribution $J^T(f)$ which is more manageable (Proposition 4.5). We shall then see (Proposition 6.1) that $J^T(f)$ is of the form

$$(3) \qquad \sum_{k=0}^{N} p_k(T,f)\, e^{\xi_k(T)},$$

where $\xi_0 = 0, \xi_1, \ldots, \xi_N$ are distinct points in the dual space $i\mathfrak{a}_0^*$, and where each $p_k(T,f)$ is a polynomial in T. In particular, the constant term

$$\tilde{J}(f) = p_0(0,f)$$

can be defined, and is uniquely determined by the original function $K^T(f)$. The discussion in § 4-§ 6 also provides a rather explicit formula for $\tilde{J}(f)$. The result (Proposition 6.1) is a geometric expansion of $\tilde{J}(f)$ in terms of weighted orbital integrals on semisimple conjugacy classes.

We study the spectral expansion (2) of $K^T(f)$ in § 7-§ 11. A key step is an asymptotic formula (Theorem 8.1) for the truncated inner product of matrix coefficients of induced representations. This generalizes an inner product formula of Waldspurger for spherical functions on the p-adic general linear group. We shall use the inner product formula to construct a function from spectral data which is asymptotic to $K^T(f)$ and which is of the general form (3). Since these two properties characterize a function of T uniquely, we will be able to conclude that the function actually equals $J^T(f)$ (Lemma 11.1). The corresponding constant term then provides a second formula for $\tilde{J}(f)$ (Corollary 11.2). In the end, we obtain a spectral expansion of $\tilde{J}(f)$ in terms of weighted characters of induced representations (Proposition 11.3).

The local trace formula is a straightforward consequence of the two expansions of $\tilde{J}(f)$. It will be established in § 12. The final result (Theorem 12.2) is an identity between two expressions

$$\sum_M |W_0^M| \, |W_0^G|^{-1} (-1)^{\dim(A_M/A_G)} \int_{\Gamma_{\mathrm{ell}}(M)} J_M(\gamma, f)\, d\gamma \tag{4}$$

and

$$\sum_M |W_0^M| \, |W_0^G|^{-1} (-1)^{\dim(A_M/A_G)} \int_{\Pi_{\mathrm{disc}}(M)} a^M(\pi)\, J_M(\pi, f)\, d\pi \tag{5}$$

which are remarkably similar to the geometric and spectral sides [12, (3.2) and (3.3)] of the global trace formula. We refer the reader to § 12 for precise descriptions of the terms in these expressions. It is enough to say here that the weighted orbital integrals $J_M(\gamma, f)$ and the weighted characters $J_M(\pi, f)$ are essentially the local terms that occur in the global trace formula. As the main constituents of (4) and (5), they now have a purely local interpretation. They have arisen naturally as the solution to a problem in local harmonic analysis.

An obvious question at this point is whether the local trace formula can be used to establish the local identities required for a comparison of global trace formulas. Such identities actually make sense only for invariant distributions. The terms $J_M(\gamma, f)$ and $J_M(\pi, f)$ in (4) and (5) are in fact not invariant if $M \neq G$. However, it would not be difficult to establish an invariant version of the local trace formula from the identity of (4) and (5). (See [13, § 8] and [14, § 1-2].) It is likely that this invariant local trace formula will have significant applications to the study of the invariant distributions that occur in the invariant global trace formula.

In keeping the introduction within bounds, we have not given much of a description of the techniques in the main body (§ 4-§ 11) of the paper. These techniques have been discussed elsewhere, in an earlier provisional paper [13], and to some extent in [14, § 2]. We also refer the reader to [9], where some of the methods of § 4-§ 6 were applied to a special case of the problem.

This paper owes a good deal to the work of Waldspurger [31]. In addition to establishing a truncated inner product formula, Waldspurger solved a number of combinatorial problems related to p-adic groups, and established a connection between weighted characters and local harmonic analysis. I have also profited from conversations with Casselman and Kazhdan. Finally, I would like to thank the referee for a number of helpful comments.

1. The group G(F)

Let G be a connected, reductive algebraic group over a local field F of characteristic 0. Our concern is the harmonic analysis of the locally compact group G(F). To this end, we fix a maximal compact subgroup K of G(F), which is hyperspecial [30] if F is a p-adic field.

We first recall some of the usual objects attached to G. The split component of the center of G is denoted by A_G. We also have the real vector space

$$\mathfrak{a}_G = \mathrm{Hom}(X(G)_F, \mathbf{R})$$

obtained from the module $X(G)_F$ of F-rational characters on G. There is a canonical homomorphism

$$H_G : G(F) \to \mathfrak{a}_G$$

defined by

$$e^{\langle H_G(x), \chi\rangle} = |\,\chi(x)\,|, \quad x \in G(F),\ \chi \in X(G)_F,$$

where $|\cdot|$ is the normalized valuation on F. The kernel of H_G is denoted by $G(F)^1$. Set

$$\mathfrak{a}_{G,F} = H_G(G(F))$$

and

$$\tilde{\mathfrak{a}}_{G,F} = H_G(A_G(F)).$$

If F is Archimedean, $\tilde{\mathfrak{a}}_{G,F} = \mathfrak{a}_{G,F} = \mathfrak{a}_G$. However, if F is a p-adic field, $\tilde{\mathfrak{a}}_{G,F}$ and $\mathfrak{a}_{G,F}$ are lattices in $\mathfrak{a}_G$.

We also fix an F-rational Levi component M_0 of some minimal parabolic subgroup of G defined over F. We assume K and $A_{M_0}(F)$ are in good relative position, in the sense that the Lie algebras of K and $M_0(F)$ are orthogonal relative to the Killing form if F is Archimedean, and the vertex of K in the Bruhat-Tits building lies in the apartment of M_0 if F is p-adic. Then

$$G(F) = KM_0(F)\,K.$$

Suppose that P is a parabolic subgroup of G which is defined over F and contains M_0. Then

$$G(F) = P(F)\,K = M_P(F)\,N_P(F)\,K,$$

where N_P is the unipotent radical of P, and M_P is the unique Levi component of P which contains M_0. Both N_P and M_P are defined over F. In particular, M_P satisfies the same conditions as G, and we can define the split torus $A_P = A_{M_P}$ and the real vector space $\mathfrak{a}_P = \mathfrak{a}_{M_P}$. Associated to P we have the subsets

$$\Delta_P \subset \Sigma_P^r \subset \Sigma_P$$

of roots of (P, A_P), Δ_P being the simple roots, Σ_P^r the reduced roots, and Σ_P the set of all roots. As usual, we regard these roots as linear functions on $\mathfrak{a}_P$. If Q is a parabolic subgroup of G which contains P, let Δ_P^Q denote the simple roots of the parabolic subgroup $P \cap M_Q$ of M_Q. Then Δ_P^Q is a subset of Δ_P.

By a *Levi subgroup* of G, we mean a group M which contains M_0, and which is the Levi component of a parabolic subgroup of G which is defined over F. For any such M, set $K_M = M(F) \cap K$. Then the triplet (M, K_M, M_0) satisfies the same hypotheses as (G, K, M_0). We write $\mathscr{P}(M)$ for the set of parabolic subgroups P over F with $M_P = M$. We also let $\mathscr{F}(M)$ and $\mathscr{L}(M)$ denote the finite sets of parabolic subgroups and Levi subgroups of G which contain M. In the special case that $M = M_0$, we will generally write $\mathscr{F} = \mathscr{F}(M_0)$ and $\mathscr{L} = \mathscr{L}(M_0)$. We shall also write $A_0 = A_{M_0}$, $\mathfrak{a}_0 = \mathfrak{a}_{M_0}$ and $H_0 = H_{M_0}$.

We shall have to keep some account of Haar measures. Unless otherwise stated, the Haar measure on a compact group will be normalized to have total volume 1. We also fix Haar measures on each of the spaces $\mathfrak{a}_M$, for Levi subgroups $M \in \mathscr{L}$. We can then take the corresponding dual measures on the spaces $i\mathfrak{a}_M^*$. If F is a p-adic field, $\tilde{\mathfrak{a}}_{M,F} = H_M(A_M(F))$ is a lattice in $\mathfrak{a}_M$. In this case, we normalize the measure on $\mathfrak{a}_M$ so that the volume of the quotient $\mathfrak{a}_M/\tilde{\mathfrak{a}}_{M,F}$ equals 1. Then the volume of the quotient

$$i\mathfrak{a}_M^*/\mathrm{Hom}(\tilde{\mathfrak{a}}_{M,F}, 2\pi i\mathbf{Z})$$

with respect to the dual measure is also equal to 1. In particular, the measure on $i\mathfrak{a}_M^*$ is the one fixed by Harish-Chandra in [23, § 2]. It assigns the quotient

$$i\mathfrak{a}_{M,F}^* = i\mathfrak{a}_M^*/\mathrm{Hom}(\mathfrak{a}_{M,F}, 2\pi i\mathbf{Z})$$

a volume equal to the index $|\, \mathfrak{a}_{M,F}/\tilde{\mathfrak{a}}_{M,F}\,|$. In general, the kernel of H_M in $A_M(F)$ is compact, and therefore has a canonical Haar measure. Since the group $H_M(A_M(F))$ is either discrete or equal to $\mathfrak{a}_M$, it also has an assigned Haar measure. The two measures determine a unique Haar measure on $A_M(F)$. Similarly, any choice of Haar measure on $M(F)$, together with the Haar measure on $H_M(M(F))$, determines a unique Haar measure on the kernel $M(F)^1$ of H_M.

There is a canonical way to compare a Haar measure on $G(F)$ with one on any Levi subgroup $M(F)$. Suppose that $P \in \mathscr{P}(M)$. Then we have the modular function

$$\delta_P(mn) = e^{2\rho_P(H_M(m))}, \quad m \in M(F),\ n \in N_P(F),$$

on $P(F)$, where $2\rho_P$ is the usual sum of roots (with multiplicity) of (P, A_P). Let $\overline{P} \in \mathscr{P}(M)$ be the parabolic subgroup which is opposite to P. Then the number

$$\gamma(P) = \int_{N_P(F)} e^{2\rho_{\overline{P}}(H_{\overline{P}}(n))}\, dn$$

is finite, and depends linearly on the choice of Haar measure dn on $N_P(F)$. Consequently, $\gamma(P)^{-1}\, dn$ is a canonical Haar measure on $N_P(F)$. If dm is a Haar measure on $M(F)$, a standard integration formula of Harish-Chandra asserts the existence of a Haar measure dx on $G(F)$ such that

$$(1.1)\qquad \int_{G(F)} f(x)\,dx = (\gamma(P)\,\gamma(\bar{P}))^{-1}\int_{N_P(F)}\int_{M(F)}\int_{N_{\bar{P}}(F)} f(nm\bar{n})\,\delta_P(m)^{-1}\,d\bar{n}\,dm\,dn,$$

for any function $f \in C_c(G(F))$. We shall say that the measures dx and dm are *compatible.* It is in fact not hard to show that dx is independent of the choice of $P \in \mathscr{P}(M)$. Moreover, compatability has the obvious transitivity property relative to Levi subgroups of M.

How do compatible measures behave in other integration formulas? If $P \in \mathscr{P}(M)$, set

$$x = m_P(x)\,n_P(x)\,k_P(x),\quad m_P(x) \in M(F),\ n_P(x) \in N_P(F),\ k_P(x) \in K,$$

and

$$H_P(x) = H_M(m_P(x)),$$

for any point $x \in G(F)$. Then the right hand side of (1.1) can be written

$$\begin{aligned}
&(\gamma(P)\,\gamma(\bar{P}))^{-1}\int_{N_P(F)}\int_{M_P(F)}\int_{N_{\bar{P}}(F)} f(nmm_P(\bar{n})\,n_P(\bar{n})\,k_P(\bar{n}))\,\delta_P(m)^{-1}\,d\bar{n}\,dm\,dn\\
&= (\gamma(P)\,\gamma(\bar{P}))^{-1}\int_{N_P(F)}\int_{M_P(F)}\int_{N_{\bar{P}}(F)} f(nmk_P(\bar{n}))\,e^{2\rho_P(H_P(\bar{n}))}\,\delta_P(m)^{-1}\,d\bar{n}\,dm\,dn\\
&= (\gamma(P)\,\gamma(\bar{P}))^{-1}\int_{N_P(F)}\int_{M(F)}\int_{N_{\bar{P}}(F)} f(mnk_P(\bar{n}))\,e^{2\rho_P(H_P(\bar{n}))}\,d\bar{n}\,dm\,dn.
\end{aligned}$$

It is known that

$$\int_K \varphi(k)\,dk = \gamma(\bar{P})^{-1}\int_{N_{\bar{P}}(F)} \varphi(k_P(\bar{n}))\,e^{2\rho_P(H_P(\bar{n}))}\,d\bar{n},$$

where φ is any function in $C(K \cap P(F)\backslash K)$, and dk is the normalized Haar measure on K. (To obtain the constant $\gamma(\bar{P})^{-1}$, simply set $\varphi = 1$.) Substituting into (1.1), we see that

$$(1.2)\qquad \int_{G(F)} f(x)\,dx = \gamma(P)^{-1}\int_K\int_{M(F)}\int_{N_P(F)} f(mnk)\,dn\,dm\,dk,$$

for any $f \in C_c(G(F))$, and for compatible measures dx and dm on $G(F)$ and $M(F)$.

Suppose that $P_0 \in \mathscr{P}(M_0)$ is a minimal parabolic subgroup. Every element in $M_0(F)$ can be conjugated under K into the set

$$M_0(F)^+_{P_0} = \{\, m \in M_0(F) : \alpha(H_0(m)) \geqslant 0,\ \alpha \in \Delta_{P_0} \}.$$

In particular,

$$G(F) = K.M_0(F)^+_{P_0}.K.$$

Suppose that dx and dm are compatible Haar measures on $G(F)$ and $M_0(F)$. Then there is a nonnegative function D_{P_0} on $M_0(F)$, which is supported on $M_0(F)^+_{P_0}$ and is invariant under $M_0(F)^1$, such that

$$\int_{G(F)} f(x)\, dx = \int_K \int_K \int_{M_0(F)} D_{P_0}(m) f(k_1 m k_2)\, dm\, dk_2\, dk_1, \tag{1.3}$$

for any $f \in C_c(G(F))$. If F is Archimedean, it is well known that

$$D_{P_0}(m) = c_0 \prod_\alpha | e^{\alpha(H_M(m))} - e^{-\alpha(H_M(m))} |, \quad m \in M_0(F)^+_{P_0},$$

where α ranges over the roots of (P_0, A_0), repeated according to multiplicity, and c_0 is an absolute constant. It is less well known that c_0 is actually equal to 1. This amusing fact was observed by Harish-Chandra, but was never explicitly published as far as I know. In the p-adic case

$$\begin{aligned} D_{P_0}(m) &= \mathrm{vol}_G(KmK)\, \mathrm{vol}_{M_0}(K_{M_0})^{-1} \\ &= | K/K \cap m^{-1} Km |\, \mathrm{vol}_G(K)\, \mathrm{vol}_{M_0}(K_{M_0})^{-1}, \quad m \in M_0(F)^+_{P_0}, \end{aligned}$$

where vol_G denote the volume in $G(F)$. What the two cases have in common is a simple asymptotic formula for $D_{P_0}(m)$. To state it, we let $\|\cdot\|$ be the Euclidean norm attached to a fixed inner product on $\mathfrak{a}_0$ which is invariant under the Weyl group W_0^G of (G, A_0).

Lemma **1.1.** — *Suppose that* P *is a parabolic subgroup which contains* P_0, *and that* $\delta > 0$. *Then there are positive constants* C *and* ε *such that*

$$| D_{P_0}(m)^{1/2} - \delta_P(m)^{1/2} D_{P_0 \cap M_P}(m)^{1/2} | \leqslant C\, \delta_{P_0}(m)^{1/2} e^{-\varepsilon \| H_0(m) \|},$$

for all points m in $M_0(F)^+_{P_0}$ *such that* $\alpha(H_0(m)) \geqslant \delta \| H_0(m) \|$ *for every root* α *in* $\Delta_{P_0} - \Delta^P_{P_0}$.

For real groups, the lemma is simply the assertion that the constant c_0 above equals 1. This can be extracted from the proof [24, p. 381] of the formula (1.3). We leave the reader to battle with the different choices of Haar measures on the various, groups. If F is p-adic, the required asymptotic formula is actually an exact formula. When G is simply connected, it is just the formula given by [27, Proposition 3.2.15]. In this case, we leave the reader to relate $\gamma(P_0)$ with the number denoted $Q(q^{-1})$ in [27], and also to extend the lemma to groups which are not simply connected.

The following estimate is a straightforward consequence of the lemma.

Corollary **1.2.** — *There is a positive constant* C *such that*

$$D_{P_0}(m)^{1/2} \leqslant C\, \delta_{P_0}(m)^{1/2}, \quad m \in M_{P_0}(F)^+_{P_0}. \quad \square$$

We should add some comments concerning the spaces $\mathfrak{a}_M$. These will be useful in the p-adic case for dealing with lattices related to $\mathfrak{a}_{M,F}$ and $\tilde{\mathfrak{a}}_{M,F}$. First, let us agree to set

$$V_1^\vee = \mathrm{Hom}(V_1, 2\pi i \mathbf{Z}) \subseteq i V^*,$$

if V_1 is any closed subgroup of a real vector space V. This is simply to have uniform notation in the example that $V = \mathfrak{a}_M$ and V_1 equals either $\mathfrak{a}_{M,F}$ or $\tilde{\mathfrak{a}}_{M,F}$. If F is p-adic, V_1 is a lattice in this example, and $V_1^\vee$ is the dual lattice in iV^*. In the Archimedean case $V_1 = V$ and $V_1^\vee = \{0\}$. In each case, the group $iV_1^* = iV^*/V_1^\vee$ is identified with the unitary dual of V_1 under the pairing

$$(v, \lambda) \to e^{\lambda(v)}, \quad v \in V, \ \lambda \in iV_1^*.$$

The embeddings

$$A_G(F) \subset A_M(F) \subset M(F) \subset G(F)$$

give rise to a commutative diagram

$$\begin{array}{ccc} \mathfrak{a}_{M,F} & \overset{h_{MG}}{\twoheadrightarrow} & \mathfrak{a}_{G,F} \\ \cup & & \cup \\ \tilde{\mathfrak{a}}_{M,F} & \overset{\tilde{h}_{MG}}{\hookleftarrow} & \tilde{\mathfrak{a}}_{G,F} \end{array}$$

The surjectivity of h_{MG} follows, for example, from the property $G(F) = KM_0(F)K$ and the fact that K lies in the kernel of the map H_G. The injectivity of $\tilde{h}_{MG}$ is a consequence of the fact that A_G is a subtorus of A_M. We can also form the dual diagram

$$\begin{array}{ccc} \mathfrak{a}^\vee_{M,F} & \overset{h^\vee_{MG}}{\hookleftarrow} & \mathfrak{a}^\vee_{G,F} \\ \cap & & \cap \\ \tilde{\mathfrak{a}}^\vee_{M,F} & \overset{\tilde{h}^\vee_{MG}}{\twoheadrightarrow} & \tilde{\mathfrak{a}}^\vee_{G,F}, \end{array}$$

in which $h^\vee_{MG}$ is injective and $\tilde{h}^\vee_{MG}$ is surjective. These four maps have extensions

$$h_{MG} : \mathfrak{a}_M \twoheadrightarrow \mathfrak{a}_G,$$
$$\tilde{h}_{MG} : \mathfrak{a}_G \hookrightarrow \mathfrak{a}_M,$$
$$h^\vee_{MG} : i\mathfrak{a}_G^* \hookrightarrow i\mathfrak{a}_M^*,$$
$$\tilde{h}^\vee_{MG} : i\mathfrak{a}_M^* \twoheadrightarrow i\mathfrak{a}_G^*,$$

as linear maps between real vector spaces. We shall identify $\mathfrak{a}_G$ and $i\mathfrak{a}_G^*$ with their respective images in $\mathfrak{a}_M$ and $i\mathfrak{a}_M^*$. Let $\mathfrak{a}_M^G$ be the kernel of h_{MG} in $\mathfrak{a}_M$. Then $\mathfrak{a}_M = \mathfrak{a}_M^G \oplus \mathfrak{a}_G$. We shall often simply identify $\mathfrak{a}_M^G$ with the quotient $\mathfrak{a}_M/\mathfrak{a}_G$. Similarly, $i\mathfrak{a}_M^*$ equals $(i\mathfrak{a}_M^*)^G \oplus i\mathfrak{a}_G^*$, where $(i\mathfrak{a}_M^*)^G$ is the kernel of $\tilde{h}^\vee_{MG}$ in $i\mathfrak{a}_M^*$. Again, we shall often identify $(i\mathfrak{a}_M^*)^G$ with $i\mathfrak{a}_M^*/i\mathfrak{a}_G^*$.

The decomposition $\mathfrak{a}_M = \mathfrak{a}_M^G \oplus \mathfrak{a}_G$ is orthogonal relative to the restriction to $\mathfrak{a}_M$ of the W_0^G-invariant inner product on $\mathfrak{a}_0$. It is a consequence of the definitions that the map h_{MG} is identified with the orthogonal projection of $\mathfrak{a}_M$ onto $\mathfrak{a}_G$. In particular, $\mathfrak{a}_{G,F}$ is just the projection of $\mathfrak{a}_{M,F}$ onto $\mathfrak{a}_G$. On the other hand, we have

$$(1.4) \qquad \tilde{\mathfrak{a}}_{G,F} = \mathfrak{a}_G \cap \tilde{\mathfrak{a}}_{M,F}.$$

This follows easily from the fact that A_G is a subtorus of A_M. Similar assertions apply to the dual spaces. The inner product on $\mathfrak{a}_0$ determines a positive definite bilinear form on $i\mathfrak{a}_M^*$, relative to which $(i\mathfrak{a}_M^*)^G$ and $i\mathfrak{a}_G^*$ are orthogonal. The map $\widetilde{h}_{MG}^\vee$ is the orthogonal projection of $i\mathfrak{a}_M^*$ onto $i\mathfrak{a}_G^*$, and

$$\mathfrak{a}_{G,F}^\vee = i\mathfrak{a}_G^* \cap \mathfrak{a}_{M,F}^\vee. \tag{1.5}$$

We have fixed Haar measures on the spaces $\mathfrak{a}_M$ and $\mathfrak{a}_G$. These define a Haar measure on the orthogonal complement $\mathfrak{a}_M^G \cong \mathfrak{a}_M/\mathfrak{a}_G$. If F is a p-adic field, it follows easily from (1.4) that the volume of the quotient $\mathfrak{a}_M/\widetilde{\mathfrak{a}}_{M,F} + \mathfrak{a}_G$ equals 1. In general, we take the dual Haar measure on the real vector space

$$i\mathfrak{a}_M^*/i\mathfrak{a}_G^* = i(\mathfrak{a}_M^G)^* \cong i(\mathfrak{a}_M/\mathfrak{a}_G)^* = (i\mathfrak{a}_M^*)^G.$$

The quotient $i\mathfrak{a}_M^*/\widetilde{\mathfrak{a}}_{M,F}^\vee$ can of course be identified with the group of unramified characters of $A_M(F)$.

Lemma **1.3.** — *Suppose that χ is an unramified character on $A_M(F)$ which is trivial on $A_G(F)$. Then there is an element $\mu \in (i\mathfrak{a}_M^*)^G$ such that*

$$\chi(a) = e^{\mu(H_M(a))}, \quad a \in A_M(F).$$

Proof. — By assumption,

$$\chi(a) = e^{\mu_1(H_M(a))}, \quad a \in A_M(F),$$

for some linear function $\mu_1 \in i\mathfrak{a}_M^*$ which maps $\widetilde{\mathfrak{a}}_{G,F}$ into $2\pi i\mathbf{Z}$. In particular, the restriction of μ to $\widetilde{\mathfrak{a}}_{G,F}$ lies in $2\pi i\widetilde{\mathfrak{a}}_{G,F}^\vee$. Recall that the map $\widetilde{h}_{MG}^\vee$ sends $\widetilde{\mathfrak{a}}_{M,F}^\vee$ surjectively onto $\widetilde{\mathfrak{a}}_{G,F}^\vee$. We can therefore find an element μ_0 in $i\mathfrak{a}_M^*$ which maps $\widetilde{\mathfrak{a}}_{M,F}$ to $2\pi i\mathbf{Z}$, and such that the difference

$$\mu = \mu_1 - \mu_0$$

is trivial on $\widetilde{\mathfrak{a}}_{G,F}^\vee$. The element μ then lies in $(i\mathfrak{a}_M^*)^G$, and satisfies the conditions of the lemma. □

2. Two expansions of the kernel

We turn now to our primary object of study, the regular representation R of $G(F) \times G(F)$ on $L^2(G(F))$. Recall that

$$(R(y_1, y_2)\,\varphi)(x) = \varphi(y_1^{-1} x y_2), \quad x, y_1, y_2 \in G(F),$$

for any function $\varphi \in L^2(G(F))$. It is convenient to use notation which emphasizes the analogy with automorphic forms. In particular, we shall write

$$G(A_F) = G(F) \times G(F)$$

for the group of points in G with values in the ring

$$A_F = F \oplus F.$$

Then G(F) embeds diagonally as a subgroup of $G(A_F)$. The map which sends any $\varphi \in L^2(G(F))$ to the function

$$(x_1, x_2) \to \varphi(x_1^{-1} x_2), \quad (x_1, x_2) \in G(F)\backslash G(A_F),$$

is an isomorphism of $L^2(G(F))$ onto $L^2(G(F)\backslash G(A_F))$ which intertwines R with the regular representation of $G(A_F)$ on $L^2(G(F)\backslash G(A_F))$.

Let $\mathscr{H}(G(F))$ be the Hecke algebra of smooth, compactly supported functions on G(F) which are left and right K-finite. Similarly, let $\mathscr{H}(G(A_F))$ be the Hecke algebra on $G(A_F)$ relative to the maximal compact subgroup $K \times K$. We fix a function in $\mathscr{H}(G(A_F))$ of the form

$$f(y_1, y_2) = f_1(y_1) f_2(y_2), \quad y_1, y_2 \in G(F),$$

for functions f_1 and f_2 in $\mathscr{H}(G(F))$. We then form the linear operator

$$R(f) = \int_{G(A_F)} f(y)\, R(y)\, dy$$

on $L^2(G(F))$. As we observed in the introduction, $R(f)$ is an integral operator on $L^2(G(F))$ with integral kernel

$$K(x, y) = \int_{G(F)} f_1(xu) f_2(uy)\, du, \quad x, y \in G(F). \tag{2.1}$$

We propose to study this kernel as a function on the diagonal. There are two fundamental formulas in harmonic analysis that lead to parallel expansions of $K(x, y)$. The Weyl integration formula provides an expansion into geometric data, while the Plancherel formula leads to an expansion into spectral data.

We shall first recall a version of the Weyl integration formula that is suitable for our purposes. Let $\Gamma_{\mathrm{ell}}(G(F))$ denote the set of conjugacy classes $\{\gamma\}$ in G(F) such that the centralizer of γ in G(F) is compact modulo $A_G(F)$. We need only be concerned with the intersection of $\Gamma_{\mathrm{ell}}(G(F))$ with $G_{\mathrm{reg}}(F)$, the set of G-regular elements in G(F). Recall that we have fixed a Haar measure on $A_G(F)$. It determines a canonical measure $d\gamma$ on the set $\Gamma_{\mathrm{ell}}(G(F))$, which vanishes on the complement of $G_{\mathrm{reg}}(F)$ in $\Gamma_{\mathrm{ell}}(G(F))$, and such that

$$\int_{\Gamma_{\mathrm{ell}}(G(F))} \varphi(\gamma)\, d\gamma = \sum_{\{T\}} |W(G(F), T(F))|^{-1} \int_{T(F)} \varphi(t)\, dt,$$

for any continuous function φ of compact support on $\Gamma_{\mathrm{ell}}(G(F)) \cap G_{\mathrm{reg}}(F)$. Here, $\{T\}$ is a set of representatives of G(F)-conjugacy classes of maximal tori in G over F with $T(F)/A_G(F)$ compact, $W(G(F), T(F))$ is the Weyl group of $(G(F), T(F))$, and dt is the Haar measure on T(F) determined by the Haar measure on $A_G(F)$ and the normalized Haar measure on the compact group $T(F)/A_G(F)$. Now an arbitrary G-regular conjugacy class in G(F) is the image of a class $\{\gamma\} \in \Gamma_{\mathrm{ell}}(M(F))$, for some Levi subgroup M which contains M_0. The pair $(M, \{\gamma\})$ is uniquely determined only up to the action of the Weyl group W_0^G of (G, A_0), so the number of such pairs equals $|W_0^G||W_0^M|^{-1}$. The Weyl integration formula can then be written

$$(2.2) \qquad \int_{G(F)} h(x)\,dx = \sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \int_{\Gamma_{\mathrm{ell}}(M(F))} |D(\gamma)| \left(\int_{A_M(F)\backslash G(F)} h(x^{-1}\gamma x)\,dx \right) d\gamma,$$

where h is any function in $C_c^\infty(G(F))$, and

$$D(\gamma) = \det(1 - \mathrm{Ad}(\gamma))_{\mathfrak{g}/\mathfrak{g}_\gamma}$$

is the Weyl discriminant. Each side of the formula depends on a choice of Haar measure on $G(F)$.

Returning to the kernel (2.1), we change variables in the integral over u. This gives a formula

$$K(x, x) = \int_{G(F)} f_1(u) f_2(x^{-1}ux)\,du$$

which reflects the behaviour of f_1 and f_2 on conjugacy classes. The Weyl integration formula then gives the expression

$$(2.3) \qquad \sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \int_{\Gamma_{\mathrm{ell}}(M(F))} |D(\gamma)| \left(\int_{A_M(F)\backslash G(F)} f_1(x_1^{-1}\gamma x_1) f_2(x^{-1}x_1^{-1}\gamma x_1 x)\,dx_1 \right) d\gamma$$

for $K(x, x)$. This the geometric expansion of the kernel.

Before recalling the Plancherel formula, we should mention a point concerning irreducible representations. Let us write $\Pi_{\mathrm{temp}}(H)$ for the set of (equivalence classes of) irreducible tempered representations of any suitable group H. There is a locally free action of $i\mathfrak{a}_G^*$ on $\Pi_{\mathrm{temp}}(G(F))$ defined by

$$\pi_\lambda(x) = \pi(x)\, e^{\lambda(H_G(x))}, \quad x \in G(F),$$

for any $\pi \in \Pi_{\mathrm{temp}}(G(F))$ and $\lambda \in i\mathfrak{a}_G^*$. Let $\bar{\pi}$ denote the restriction of π to the normal subgroup

$$G(F)^1 = \{ x \in G(F) : H_G(x) = 0 \}.$$

If F is Archimedean, $\bar{\pi}$ is irreducible, and $\{\pi_\lambda\} \leftrightarrow \bar{\pi}$ is a bijective correspondence between the $i\mathfrak{a}_G^*$-orbits in $\Pi_{\mathrm{temp}}(G(F))$ and $\Pi_{\mathrm{temp}}(G(F)^1)$. However, if F is p-adic, the representation $\bar{\pi}$ could be reducible. In this case, the action of $G(F)$ by conjugation on $\Pi_{\mathrm{temp}}(G(F)^1)$ is nontrivial, and the set $\{\bar{\pi}\}$ of irreducible constituents of $\bar{\pi}$ is a $G(F)$-orbit. In general, we shall write $\mathfrak{a}_{G,\pi}^\vee$ for the stabilizer of π in $i\mathfrak{a}_G^*$. Then

$$\mathfrak{a}_{G,F}^\vee \subset \mathfrak{a}_{G,\pi}^\vee \subset \tilde{\mathfrak{a}}_{G,F}^\vee,$$

and the group

$$i\mathfrak{a}_{G,\pi}^* = i\mathfrak{a}_G^* / \mathfrak{a}_{G,\pi}^\vee$$

acts simply transitively on the orbit $\{\pi_\lambda\}$.

These comments of course apply if G is replaced by any Levi subgroup $M \in \mathscr{L}$. For a given $\sigma \in \Pi_{\mathrm{temp}}(M(F))$ and $\lambda \in i\mathfrak{a}_M^*$, we write

$$\mathscr{I}_P(\sigma_\lambda, x), \quad P \in \mathscr{P}(M), \ x \in G(F),$$

as usual for the corresponding parabolically induced representation of G(F). It acts on a Hilbert space $\mathscr{H}_P(\sigma)$, of vector valued functions on K, which is independent of λ. Let $\Pi_2(M(F))$ be the subset of representations in $\Pi_{\mathrm{temp}}(M(F))$ which are square integrable modulo $A_M(F)$. We shall write $\{\Pi_2(M(F))\}$ or $\Pi_2(M(F))/i\mathfrak{a}_M^*$ for the set of $i\mathfrak{a}_M^*$-orbits in $\Pi_2(M(F))$. These orbits are of course the connected components in the natural topology on $\Pi_2(M(F))$. Now we have fixed a measure $d\lambda$ on $i\mathfrak{a}_M^*$. This determines a measure on either of the two quotient spaces

$$i\mathfrak{a}_{M,F}^* = i\mathfrak{a}_M^*/\mathfrak{a}_{M,F}^\vee \rightarrow i\mathfrak{a}_M^*/\mathfrak{a}_{M,\sigma}^\vee = i\mathfrak{a}_{M,\sigma}^*.$$

Following Harish-Chandra [23, § 2], we define a measure $d\sigma$ on $\Pi_2(M(F))$ by setting

$$\begin{aligned}\int_{\Pi_2(M(F))} \varphi(\sigma)\, d\sigma &= \sum_{\sigma \in \{\Pi_2(M(F))\}} \int_{i\mathfrak{a}_{M,\sigma}^*} \varphi(\sigma_\lambda)\, d\lambda \\ &= \sum_{\sigma \in \{\Pi_2(M(F))\}} |\, \mathfrak{a}_{M,\sigma}^\vee/\mathfrak{a}_{M,F}^\vee \,|^{-1} \int_{i\mathfrak{a}_{M,F}^*} \varphi(\sigma_\lambda)\, d\lambda,\end{aligned} \tag{2.4}$$

for any function $\varphi \in C_c(M(F))$. Harish-Chandra's Plancherel formula [22], [23] can then be written

$$h(1) = \sum_{M \in \mathscr{L}} |\, W_0^M \,|\, |\, W_0^G \,|^{-1} \int_{\Pi_2(M(F))} m(\sigma)\, \mathrm{tr}(\mathscr{I}_P(\sigma, h))\, d\sigma, \tag{2.5}$$

where h is any function in $C_c^\infty(G(F))$, $m(\sigma)$ is the Plancherel density, and P is any group in $\mathscr{P}(M)$. We will eventually need the precise description of $m(\sigma)$ in terms of Harish-Chandra's μ-functions and the formal degree of σ (and the various constants which occur in Harish-Chandra's formula). For the moment, however, we need only know that $m(\sigma)$ is a smooth, tempered function on $\Pi_2(M(F))$ which depends inversely on a choice of Haar measure on G(F) and the choice of Haar measure on $i\mathfrak{a}_M^*$.

Returning again to the kernel (2.1), we set

$$h(v) = \int_{G(F)} f_1(xu) f_2(uvx)\, du, \quad v \in G(F).$$

Then h is a function in $C_c^\infty(G(F))$ such that

$$h(1) = K(x, x).$$

If

$$f_1^\vee(x_1) = f_1(x_1^{-1}),$$

we can also write

$$\mathscr{I}_P(\sigma, h) = AB^*,$$

where

$$A = \mathscr{I}_P(\sigma, f_1^\vee)\, \mathscr{I}_P(\sigma, x)\, \mathscr{I}_P(\sigma, f_2)$$

and

$$B = \mathscr{I}_P(\sigma, x).$$

For each P and σ, let $\mathscr{B}_P(\sigma)$ be a fixed K-finite orthonormal basis of the Hilbert space of Hilbert-Schmidt operators on $\mathscr{H}_P(\sigma)$. Then

$$\operatorname{tr}(\mathscr{I}_P(\sigma, h)) = \operatorname{tr}(AB^*) = \sum_{S \in \mathscr{B}_P(\sigma)} (A, S^*)\, \overline{(B, S^*)}.$$

Therefore

$$\begin{aligned}\operatorname{tr}(\mathscr{I}_P(\sigma, h)) &= \sum_S \operatorname{tr}(\mathscr{I}_P(\sigma, f_1^\vee)\, \mathscr{I}_P(\sigma, x)\, \mathscr{I}_P(\sigma, f_2)\, S)\, \overline{\operatorname{tr}(\mathscr{I}_P(\sigma, x)\, S)} \\ &= \sum_S \operatorname{tr}(\mathscr{I}_P(\sigma, x)\, S(f))\, \overline{\operatorname{tr}(\mathscr{I}_P(\sigma, x)\, S)},\end{aligned}$$

where

$$S(f) = \mathscr{I}_P(\sigma, f_2)\, S \mathscr{I}_P(\sigma, f_1^\vee).$$

The Plancherel formula applied to h then becomes the expression

$$\sum_{M \in \mathscr{L}} |W_0^M|\, |W_0^G|^{-1} \int_{\Pi_2(M(F))} m(\sigma) \Big(\sum_{S \in \mathscr{B}_P(\sigma)} \operatorname{tr}(\mathscr{I}_P(\sigma, x)\, S(f))\, \overline{\operatorname{tr}(\mathscr{I}_P(\sigma, x)\, S)} \Big)\, d\sigma \tag{2.6}$$

for $K(x, x)$. This is the spectral expansion of the kernel.

3. Truncation

We have described two different expansions (2.3) and (2.6) for $K(x, x)$, the value of the kernel of $R(f)$ on the diagonal. It is not hard to show that the terms with $M = G$ in each expansion are integrable over x in $G(F)/A_G(F)$. In fact, if G is semisimple, the integral of the term with $M = G$ in (2.6) is just the trace of $R_{\mathrm{disc}}(f)$. If we were able to integrate the other terms, the resulting identity could serve as the local trace formula. However, the terms with $M \neq G$ in each expression are not integrable. We must truncate these functions in such a way so that they can all be integrated.

The truncation procedure is simpler than what has been used for the global trace formula [3]. It is perhaps surprising that it works out so well. We shall simply multiply each term by the characteristic function of a large compact subset of $G(F)/A_G(F)$. To describe this we must first recall some simple notions [1, § 3], [6, § 3] which are relevant to the parameter of truncation.

Fix a Levi subgroup $M \in \mathscr{L}$. A set

$$\mathscr{Y} = \mathscr{Y}_M = \{ Y_P : P \in \mathscr{P}(M) \}$$

of points in $\mathfrak{a}_M$, indexed by the finite set $\mathscr{P}(M)$, is said to be a (G, M)-*orthogonal set* if it satisfies the following property. For any pair P and P′ of adjacent groups in $\mathscr{P}(M)$, whose chambers in $\mathfrak{a}_M$ share the wall determined by the simple root α in $\Delta_P \cap (-\Delta_{P'})$, we have

$$Y_P - Y_{P'} = r_{P,P'}\,\alpha^\vee,$$

for a real number $r_{P,P'}$. Recall that $\alpha^\vee$ is the "co-root" associated to the simple root $\alpha \in \Delta_P$ [2, § 1]. We say that the orthogonal set is *positive* if each of the numbers $r_{P,P'}$ is nonnegative. This is the case, for example, if the number

$$d(\mathcal{Y}) = \inf_{\{\alpha \in \Delta_P : P \in \mathcal{P}(M)\}} \alpha(Y_P) \tag{3.1}$$

is positive. Given $\mathcal{Y}$, we can form the (M_Q, M)-orthogonal set

$$\mathcal{Y}_M^Q = \{ Y_{P \cap M_Q} = Y_P : P \in \mathcal{P}(M),\ P \subset Q \}$$

for any group $Q \in \mathcal{F}(M)$. If L belongs to $\mathcal{L}(M)$ and Q is a group in $\mathcal{P}(L)$, we define Y_Q to be the projection onto $\mathfrak{a}_L$ of any point Y_P, with $P \in \mathcal{P}(M)$ and $P \subset Q$. Then Y_Q is independent of P, and

$$\mathcal{Y}_L = \{ Y_Q : Q \in \mathcal{P}(L) \}$$

is a (G, L)-orthogonal set. In general, we shall write $S_M(\mathcal{Y})$ for the convex hull in $\mathfrak{a}_M/\mathfrak{a}_G$ of a (G, M)-orthogonal set $\mathcal{Y}$.

One example is the set

$$\{ -H_P(x) : P \in \mathcal{P}(M) \},$$

defined for any point $x \in G(F)$. This is a positive (G, M)-orthogonal set [1, Lemma 3.6], which is a familiar ingredient of the global trace formula. It will play a parallel role here in the final description of the terms on the geometric side.

The truncation process depends on an even simpler orthogonal set. Suppose that T is any point in $\mathfrak{a}_0$. If P_0 is any group in $\mathcal{P}(M_0)$, let T_{P_0} be the unique W_0^G-translate of T which lies in the closure of the chamber $\mathfrak{a}_{P_0}^+$. Then

$$\{ T_{P_0} : P_0 \in \mathcal{P}(M_0) \}$$

is a positive (G, M_0)-orthogonal set, which we shall denote simply by T. We shall assume that T is highly regular, in the sense that its distance from any of the singular hyperplanes in $\mathfrak{a}_0$ is large. In other words, the number

$$d(T) = \inf\{ \alpha(T_{P_0}) : \alpha \in \Delta_{P_0},\ P_0 \in \mathcal{P}(M_0) \}$$

is suitably large. Keep in mind that we can also form the (G, M)-orthogonal set

$$T_M = \{ T_P : P \in \mathcal{P}(M) \}, \quad M \in \mathcal{L},$$

as above, where T_P is the projection onto $\mathfrak{a}_M$ of any point T_{P_0}, P_0 being any group in $\mathcal{P}(M_0)$ which is contained in P. For future reference, we note that

$$d(T) \leqslant \alpha(T_P), \quad P \in \mathcal{P}(M),\ \alpha \in \Delta_P. \tag{3.2}$$

Indeed, if $P_0 \subset P$ as above, and α is the projection onto $\mathfrak{a}_M^*$ of the root $\alpha_0 \in \Delta_{P_0} - \Delta_{P_0}^P$, then

$$\alpha_0(T_{P_0}) \leqslant \alpha(T_{P_0}) = \alpha(T_P).$$

This follows from the elementary properties of roots.

We fix the point $T \in \mathfrak{a}_0$ with $d(T)$ large. If F is a p-adic field, we also assume that T belongs to the lattice $\mathfrak{a}_{M_0, F}$. The truncation will be based on the decomposition $G(F) = KM_0(F)K$. We define $u(x, T)$ to be the characteristic function of the set of points

$$x = k_1 m k_2, \quad m \in A_G(F)\backslash M_0(F), \ k_1, k_2 \in K,$$

in $A_G(F)\backslash G(F)$ such that $H_0(m)$ lies in the convex hull $S_{M_0}(T)$. Since $S_{M_0}(T)$ is a large compact subset of $\mathfrak{a}_M/\mathfrak{a}_G$, $u(x, T)$ is the characteristic function of a large compact subset of $A_G(F)\backslash G(F)$. In particular, the integral

$$K^T(f) = \int_{A_G(F)\backslash G(F)} K(x, x)\, u(x, T)\, dx \tag{3.3}$$

converges.

The distribution $K^T(f)$ will eventually lead to a local trace formula. One begins with the two expansions of $K^T(f)$ obtained by substituting the formulas (2.3) and (2.6) for $K(x, x)$ into (3.3). The geometric expansion is

$$K^T(f) = \sum_{M \in \mathscr{L}} |W_0^M|\, |W_0^G|^{-1} \int_{\Gamma_{\mathrm{ell}}(M(F))} K^T(\gamma, f)\, d\gamma, \tag{3.4}$$

where

$$K^T(\gamma, f) = |D(\gamma)| \int_{A_G(F)\backslash G(F)} \int_{A_M(F)\backslash G(F)} f_1(x_1^{-1}\gamma x_1) f_2(x^{-1}x_1^{-1}\gamma x_1 x)\, u(x, T)\, dx_1\, dx.$$

In this last expression, it is convenient to take the integral over x inside the integral over x_1, and then to replace x by $x_1^{-1}x$. The resulting integral over $A_G(F)\backslash G(F)$ can then be expressed as a double integral over $a \in A_G(F)\backslash A_M(F)$ and $x_2 \in A_M(F)\backslash G(F)$. Since a commutes with γ, we obtain

$$K^T(\gamma, f) = |D(\gamma)| \int_{A_M(F)\backslash G(F)} \int_{A_M(F)\backslash G(F)} f_1(x_1^{-1}\gamma x_1) f_2(x_2^{-1}\gamma x_2)\, u_M(x_1, x_2, T)\, dx_1\, dx_2, \tag{3.5}$$

where

$$u_M(x_1, x_2, T) = \int_{A_G(F)\backslash A_M(F)} u(x_1^{-1} a x_2, T)\, da.$$

The spectral expansion is just

$$K^T(f) = \sum_{M \in \mathscr{L}} |W_0^M|\, |W_0^G|^{-1} \int_{\Pi_2(M(F))} K^T(\sigma, f)\, d\sigma, \tag{3.6}$$

where

$$K^T(\sigma, f) = m(\sigma) \int_{A_G(F)\backslash G(F)} \Big(\sum_{S \in \mathscr{B}_P(\sigma)} \operatorname{tr}(\mathscr{I}_P(\sigma, x)\, S(f))\, \overline{\operatorname{tr}(\mathscr{I}_P(\sigma, x)\, S)} \Big)\, u(x, T)\, dx. \tag{3.7}$$

The geometric and spectral expansions (3.4) and (3.6) of $K^T(f)$ are not very useful as they stand. It is not even clear how they behave with respect to the truncation parameter T. The rest of the paper will be devoted to deriving new expressions from (3.4) and (3.6) which are more tractible.

We shall study the geometric expansion in § 4-6 and the spectral expansion in § 7-11. In the remainder of this section, we shall discuss various characteristic functions which are related to (G, M)-orthogonal sets. This is partly a review, similar results being familiar from the global trace formula. We shall use the results here in our study of both the geometric and spectral terms.

Let $\mathcal{Y}$ be a fixed (G, M)-orthogonal set. We first recall the decomposition of the characteristic function of $S_M(\mathcal{Y})$, valid when $\mathcal{Y}$ is positive, which comes from Langlands' combinatorial lemma [1, § 2-3]. Suppose that Λ is a point in $\mathfrak{a}^*_{M,\mathbf{C}}$ whose real part $\Lambda_R \in \mathfrak{a}^*_M$ is in general position. If $P \in \mathcal{P}(M)$, set

$$\Delta_P^\Lambda = \{ \alpha \in \Delta_P : \Lambda_R(\alpha^\vee) < 0 \}.$$

Let φ_P^Λ denote the characteristic function of the set of $H \in \mathfrak{a}_M$ such that $\varpi_\alpha(H) > 0$ for each $\alpha \in \Delta_P^\Lambda$, and $\varpi_\alpha(H) \leqslant 0$ for each α in the complement of Δ_P^Λ in Δ_P. Here

$$\hat{\Delta}_P = \{ \varpi_\alpha : \alpha \in \Delta_P \}$$

is the basis of $(\mathfrak{a}_M^G)^*$ which is dual to $\{ \alpha^\vee : \alpha \in \Delta_P \}$. We define

$$\sigma_M(H, \mathcal{Y}) = \sum_{P \in \mathcal{P}(M)} (-1)^{|\Delta_P^\Lambda|} \varphi_P^\Lambda(H - Y_P) \tag{3.8}$$

for any point $H \in \mathfrak{a}_M/\mathfrak{a}_G$. Then the function $\sigma_M(\cdot, \mathcal{Y})$ vanishes on the complement of $S_M(\mathcal{Y})$, and in particular, is compactly supported. Moreover, if $\mathcal{Y}$ is positive, $\sigma_M(\cdot, \mathcal{Y})$ is actually equal to the characteristic function of $S_M(\mathcal{Y})$. (See Lemma 3.2 and Corollary 3.3 of [1] for the special case that H lies on the complement of a finite set of hyperplanes. The general case follows the same way from the stronger version [2, Lemma 6.3] of Langlands' combinatorial lemma.) The decomposition (3.8) will be useful later on for interpreting the integrals on the geometric side.

If Q is a group in $\mathcal{F}(M)$, we shall often write $S_M^Q(\mathcal{Y}) = S_M(\mathcal{Y}_M^Q)$ and $\sigma_M^Q(H, \mathcal{Y}) = \sigma_M(H, \mathcal{Y}_M^Q)$. We shall also write τ_Q for the characteristic function in $\mathfrak{a}_0$ of

$$\{ H \in \mathfrak{a}_0 : \alpha(H) > 0, \alpha \in \Delta_Q \}.$$

A second consequence of Langlands' combinatorial lemma is a formula

$$\sum_{Q \in \mathcal{F}(M)} \sigma_M^Q(H, \mathcal{Y})\, \tau_Q(H - Y_Q) = 1 \tag{3.9}$$

for any point $H \in \mathfrak{a}_M$. In particular, if $\mathcal{Y}$ is positive, the summands in (3.9) are characteristic functions, and we obtain a partition of $\mathfrak{a}_M$ into a finite disjoint union of subsets. To prove (3.9), simply substitute the definition (3.8) of $\sigma_M^Q(H, \mathcal{Y})$ into the left hand side. It then follows from [2, Lemma 6.3] that the resulting double sum over P and Q equals 1.

Assume now that the number $d(\mathcal{Y})$ is positive. This means that each Y_P belongs to the positive chamber

$$\mathfrak{a}_P^+ = \{ H \in \mathfrak{a}_P : \tau_P(H) = 1 \}.$$

Then the intersection of $S_M(\mathcal{Y})$ with $\mathfrak{a}_P^+$ has a simple description. For convenience, we include a proof of this property.

Lemma **3.1.** — *For any group* $P \in \mathcal{P}(M)$ *we have*

$$S_M(\mathcal{Y}) \cap \mathfrak{a}_P^+ = \{ H \in \mathfrak{a}_P^+ : \varpi(H - Y_P) \leqslant 0, \varpi \in \hat{\Delta}_P \}.$$

Proof. — According to [1, Lemma 3.2], $S_M(\mathcal{Y})$ is just the intersection over $Q \in \mathcal{P}(M)$ of the sets

$$S_Q = \{ H \in \mathfrak{a}_M : \varpi(H - Y_Q) \leqslant 0, \varpi \in \hat{\Delta}_Q \}.$$

In particular, $S_M(\mathcal{Y}) \cap \mathfrak{a}_P^+$ is contained in the required set $S_P \cap \mathfrak{a}_P^+$. We must show conversely that $S_P \cap \mathfrak{a}_P^+$ is contained in each of the sets S_Q.

Fix a point $H \in S_P \cap \mathfrak{a}_P^+$. Suppose that H belongs to a set S_Q, and that $Q_1 \in \mathcal{P}(M)$ is a group which is adjacent to Q. Assume that the root β in $\Delta_{Q_1} \cap (-\Delta_Q)$ is negative on $\mathfrak{a}_P^+$. We shall show that H also belongs to S_{Q_1}. Observe that $\hat{\Delta}_{Q_1}$ is the union of $\hat{\Delta}_{Q_1} \cap \hat{\Delta}_Q$ with the weight $\varpi_\beta \in \hat{\Delta}_{Q_1}$ corresponding to β. If ϖ belongs to $\hat{\Delta}_{Q_1} \cap \hat{\Delta}_Q$, we have

$$\varpi(H - Y_{Q_1}) = \varpi(H - Y_Q - r_{Q_1,Q}\, \beta^\vee) = \varpi(H - Y_Q) \leqslant 0.$$

Therefore, if we write

$$H - Y_{Q_1} = -\sum_{\alpha \in \Delta_{Q_1}} t_\alpha \alpha^\vee + Z, \quad t_\alpha \in \mathbf{R},\ Z \in \mathfrak{a}_G,$$

we see that each of the numbers t_α, $\alpha \neq \beta$, is nonnegative. On the other hand

$$\beta(H - Y_{Q_1}) = \beta(H) - \beta(Y_{Q_1}) \leqslant 0,$$

since $Y_{Q_1} \in \mathfrak{a}_{Q_1}^+$, $H \in \mathfrak{a}_P^+$, and β is negative on $\mathfrak{a}_P^+$. Consequently

$$-\beta(\beta^\vee)\, t_\beta = \beta(H - Y_{Q_1}) + \sum_{\alpha \neq \beta} t_\alpha\, \beta(\alpha^\vee) \leqslant 0,$$

so that t_β is also nonnegative. It follows that $\varpi(H - Y_{Q_1}) \leqslant 0$ for every $\varpi \in \hat{\Delta}_{Q_1}$. In other words, H belongs to S_{Q_1}. This completes the argument. For since H lies in S_P by assumption, it belongs to all the sets S_Q, and therefore to $S_M(\mathcal{Y})$. □

If P belongs to $\mathcal{P}(M)$, we shall write φ_P for the characteristic function in $\mathfrak{a}_0$ of

$$\{ H \in \mathfrak{a}_0 : \varpi(H) \leqslant 0, \varpi \in \hat{\Delta}_P \}.$$

Set

$$\tau_P(H, Y_P) = \tau_P(H)\, \varphi_P(H - Y_P), \quad H \in \mathfrak{a}_0,$$

for any point Y_P in $\mathfrak{a}_P^+$. If Y_P is a member of an orthogonal set $\mathcal{Y}$ as in the last lemma, $\tau_P(\cdot, Y_P)$ is just the characteristic function in $\mathfrak{a}_M$ of the set $S_M(\mathcal{Y}) \cap \mathfrak{a}_P^+$. Given any group $Q \in \mathscr{F}(M)$ which contains P, we write $\varphi_P^Q = \varphi_{M_Q \cap P}$ and $\tau_P^Q = \tau_{M_Q \cap P}$ for the functions corresponding to the parabolic subgroup $M_Q \cap P \supset P$ of M_Q. We continue to write Y_Q for the projection of Y_P onto $\mathfrak{a}_Q$.

Lemma **3.2.** — *Fix* $P \in \mathscr{P}(M)$, *and suppose that* Y_P *and* Y'_P *are points in* $\mathfrak{a}_P^+$. *Then*

$$\sum_{Q \supset P} \tau_P^Q(H, Y_P)\, \tau_Q(H - Y_Q) = \tau_P(H), \tag{3.10}$$

$$\sum_{Q \supset P} \tau_P^Q(H, Y_P)\, \tau_Q(H - Y_Q, Y'_Q) = \tau_P(H, Y_P + Y'_P), \tag{3.11}$$

and

$$\sum_{Q \supset P} \varphi_P^Q(H)\, \tau_Q(H, Y_Q) = \varphi_P(H - Y_Q), \tag{3.12}$$

for any point $H \in \mathfrak{a}_0$.

Proof. — We can assume that Y_P belongs to a (G, M)-orthogonal set $\mathscr{Y} = \{ Y_{P_1} : P_1 \in \mathscr{P}(M) \}$ for which each Y_{P_1} lies in $\mathfrak{a}_{P_1}^+$. We can also assume that H lies in $\mathfrak{a}_M$. Suppose for a moment that H is actually a regular point. Then for any fixed $Q \in \mathscr{F}(M)$, H belongs to some open chamber $\mathfrak{a}_{P_1 \cap M_Q}^+$. Applying the last lemma, with G replaced by M_Q, we see that

$$\sigma_M^Q(H, \mathscr{Y}) = \sum_{\{P_1 \in \mathscr{P}(M) : P_1 \subset Q\}} \tau_{P_1}^Q(H, Y_{P_1}).$$

It follows from (3.9) that

$$\sum_{P_1 \in \mathscr{P}(M)} \sum_{Q \supset P_1} \tau_{P_1}^Q(H, Y_{P_1})\, \tau_Q(H - Y_Q) = 1. \tag{3.13}$$

Now, we claim that each function

$$H \to \tau_P^Q(H, Y_P)\, \tau_Q(H - Y_Q), \quad H \in \mathfrak{a}_M,\ Q \supset P,$$

is supported on $\mathfrak{a}_P^+$. To see this, observe that if the function equals 1, we can write

$$H - Y_P = \sum_{\varpi \in \hat{\Delta}_Q} c_\varpi \varpi^\vee - \sum_{\alpha \in \Delta_P^Q} c_\alpha \alpha^\vee + Z,$$

for nonnegative real numbers $\{ c_\varpi \}$ and $\{ c_\alpha \}$, and for $Z \in \mathfrak{a}_G$. Let β be a root in $\Delta_P - \Delta_P^Q$. Then

$$\beta(H - Y_P) = c_{\varpi_\beta} - \sum_{\alpha \in \Delta_P^Q} c_\alpha\, \beta(\alpha^\vee) \geqslant 0.$$

Consequently

$$\beta(H) = \beta(H - Y_P) + \beta(Y_P) > 0.$$

On the other hand, if α is a root in Δ_P^Q, $\alpha(H)$ is positive, since $\tau_P^Q(H, Y_P) = 1$. Therefore H belongs to $\mathfrak{a}_P^+$, as claimed. The formula (3.10) then follows by multiplying each side of (3.13) with the characteristic function $\tau_P(H)$.

To establish the second formula (3.11) we multiply each side of (3.10) by $\varphi_P(H - Y_P - Y'_P)$. The right hand side then coincides with the right hand side of (3.11). The summands on the corresponding left hand sides are related by the inequality

$$\begin{aligned}&\tau_P^Q(H, Y_P)\,\tau_Q(H - Y_Q)\,\varphi_P(H - Y_P - Y'_P)\\ &\quad\leqslant \tau_P^Q(H, Y_P)\,\tau_Q(H - Y_Q)\,\varphi_Q(H - Y_Q - Y_{Q'}) = \tau_P^Q(H, Y_P)\,\tau_Q(H - Y_Q, Y'_Q).\end{aligned}$$

We must show that this inequality of characteristic functions is actually an equality. Suppose that H is any point such that the characteristic function on the right equals 1. We need to prove that $\varphi_P(H - Y_P - Y'_P)$ equals 1. Since $\varphi_Q(H - Y_Q - Y'_Q) = 1$, this amounts to showing that $\varpi(H - Y_Q - Y'_Q) \leqslant 0$ for every weight ϖ in $\hat{\Delta}_P - \hat{\Delta}_Q$. Take such a ϖ, and write $\varpi = \varpi_1 + \varpi_2$, where ϖ_2 belongs to $\mathfrak{a}_Q^*$ and ϖ_1 belongs to the orthogonal complement $(\mathfrak{a}_P^*)^Q$ of $\mathfrak{a}_Q^*$ in $\mathfrak{a}_P^*$. Since ϖ belongs to the closure in $\mathfrak{a}_P^*$ of the chamber corresponding to P, ϖ_1 and ϖ_2 lie in the respective closures of the corresponding chambers in $(\mathfrak{a}_P^*)^Q$ and $\mathfrak{a}_Q^*$. Given that

$$\tau_P^Q(H, Y_P) = 1 = \varphi_Q(H - Y_Q - Y'_Q),$$

we can then assert that $\varpi_1(H - Y_P) \leqslant 0$ and $\varpi_2(H - Y_Q - Y'_Q) \leqslant 0$. It follows that

$$\varpi(H - Y_Q - Y'_Q) = \varpi_1(H - Y_P) - \varpi_1(Y'_P) + \varpi_2(H - Y_Q - Y'_Q) \leqslant 0,$$

as required. This gives (3.11).

To prove the third formula, replace H by $H + Y_P$ in (3.11). Then in the resulting formula

$$\sum_{Q \supset P} \tau_P^Q(H + Y_P, Y_P)\,\tau_Q(H, Y'_Q) = \tau_P(H + Y_P, Y_P + Y'_P),$$

let Y_P approach infinity, in the sense that its distance from all the walls of $\mathfrak{a}_P^+$ becomes large. In the limit we obtain

$$\sum_{Q \supset P} \varphi_P^Q(H)\,\tau_Q(H, Y'_Q) = \varphi_P(H - Y'_P),$$

which is just (3.12). □

Remark. — Given $Y_P \in \mathfrak{a}_P^+$, it is convenient to write

$$\bar{\tau}_P(H, Y_P) = \bar{\tau}_P(H)\,\varphi_P(H - Y_P), \quad H \in \mathfrak{a}_0,$$

for the characteristic function of

$$\{\, H \in \mathfrak{a}_0 : \alpha(H) \geqslant 0, \alpha \in \Delta_P;\ \varpi(H - Y_P) \leqslant 0, \varpi \in \hat{\Delta}_P \,\}.$$

From (3.11) one can easily derive the parallel formula

$$\sum_{Q \supset P} \bar{\tau}_P^Q(H, Y_P)\,\tau_Q(H - Y_P, Y'_Q) = \bar{\tau}_P(H, Y_P + Y'_P). \tag{3.14}$$

4. The geometric side

We turn first to the geometric expansion of $K^T(f)$. This expansion is not in a satisfactory state. For example, the terms corresponding to $M = G$ ought to reduce simply to invariant orbital integrals of f. This is not the case, for as they stand, these terms contain the weight factor $u_G(x_1, x_2, T)$. In the next two sections we shall show that $K^T(f)$ is asymptotic in T to another function $J^T(f)$, which is defined by a better behaved geometric expansion. This second expansion will be formally similar to the first, except that $u_M(x_1, x_2, T)$ will be replaced by a different weight factor $v_M(x_1, x_2, T)$. The new factor has better properties, and is closer to what occurs in the global trace formula. In particular $v_G(x_1, x_2, T)$ will be identically equal to 1; the terms corresponding to $M = G$ in the expansion of $J^T(f)$ will then be invariant orbital integrals.

In order to keep track of estimates, we must make use of a height function $\|\cdot\|$ on G(F). Recall [2, § 1] that this is determined by a finite dimensional representation $\Lambda_0 : G \to GL(V_0)$ of G over F, and a basis $\{v_1, \ldots, v_n\}$ of $V_0(F)$. The height of any vector

$$v = \sum_{i=1}^{n} \xi_i v_i, \quad \xi_i \in F,$$

in $V_0(F)$ is defined to be

$$\| v \| = \begin{cases} (\Sigma_i |\xi_i|^2)^{1/2}, & \text{if F is Archimedean,} \\ \max_i |\xi_i|, & \text{otherwise.} \end{cases}$$

The basis of $V_0(F)$ determines a basis of the vector space of endomorphisms of $V_0(F)$, which in turn provides a height function.

$$\| x \| = \| \Lambda_0(x) \|, \quad x \in G(F),$$

on G(F). We can assume that $\| x \| \geqslant 1$, and that

$$\| xy \| \leqslant \| x \| \, \| y \|, \quad x, y \in G(F). \tag{4.1}$$

We fix the height function $\| x \|$ on G(F) in such a way that if $\| \Lambda(x) \|$ is another height function, attached to a second rational representation Λ, then

$$\| \Lambda(x) \| \leqslant c_\Lambda \| x \|^{N_\Lambda}, \quad x \in G(F), \tag{4.2}$$

for positive constants c_Λ and N_Λ. We can also choose constants c and N so that

$$\| x^{-1} \| \leqslant c \| x \|^N, \quad x \in G(F). \tag{4.3}$$

If $P \in \mathscr{F}$ is a parabolic subgroup, we have agreed to write

$$x = m_P(x)\, n_P(x)\, k_P(x), \quad m_P(x) \in M_P(F),\ n_P(x) \in N_P(F),\ k_P(x) \in K, \tag{4.4}$$

for the components of a point $x \in G(F)$ relative to the decomposition

$$G(F) = M_P(F)\, N_P(F)\, K.$$

It is not difficult to show that c and N may be chosen so that

$$(4.5) \qquad \| m_{\mathrm{P}}(x) \| + \| n_{\mathrm{P}}(x) \| \leqslant c \| x \|^{\mathrm{N}}, \quad x \in \mathrm{G(F)}.$$

Suppose that S is a maximal torus of G over F. We will need some general estimates on the size of elements x in $\mathrm{G(F)}$ and γ in $\mathrm{S}_{\mathrm{reg}}(\mathrm{F}) = \mathrm{S(F)} \cap \mathrm{G}_{\mathrm{reg}}(\mathrm{F})$, in terms of the size of the corresponding conjugates $x^{-1}\gamma x$.

Lemma **4.1**. — *Suppose that the maximal torus* S *is* F*-anisotropic modulo* $\mathrm{A_G}$. *Then one can choose an element* $\gamma_0 \in \mathrm{S}_{\mathrm{reg}}(\mathrm{F})$ *and constants* c_0 *and* N_0 *such that*

$$\| x \| \leqslant c_0 \| x^{-1} \gamma_0 x \|^{\mathrm{N}_0}$$

for every point $x \in \mathrm{G(F)}^1$.

Proof. — Let $\mathrm{P}_0 = \mathrm{N}_0 \mathrm{M}_0$ be a fixed minimal parabolic subgroup. Then $\overline{\mathrm{N}}_0(\mathrm{F})\, \mathrm{M}_0(\mathrm{F})\, \mathrm{N}_0(\mathrm{F})$ is an open subset of $\mathrm{G(F)}$ which contains 1. (As usual, $\overline{\mathrm{N}}_0$ denotes the unipotent radical of the parabolic subgroup $\overline{\mathrm{P}}_0 \in \mathscr{P}(\mathrm{M}_0)$ opposite to P_0.) Observe that the set of points $\gamma \in \mathrm{S(F)}$, such that $k^{-1}\gamma k$ belongs to $\overline{\mathrm{N}}_0(\mathrm{F})\, \mathrm{M}_0(\mathrm{F})\, \mathrm{N}_0(\mathrm{F})$ for each $k \in \mathrm{K}$, contains an open neighbourhood of 1 in $\mathrm{S(F)}$. We take γ_0 to be any G-regular point in this set.

For any $k \in \mathrm{K}$, let $\mathrm{Y}(k^{-1}\gamma_0 k)$ be a vector in the Lie algebra $\overline{\mathfrak{n}}_0(\mathrm{F})$ of $\overline{\mathrm{N}}_0(\mathrm{F})$ such that $k^{-1}\gamma_0 k$ belongs to $\exp(\mathrm{Y}(k^{-1}\gamma_0 k))\, \mathrm{M}_0(\mathrm{F})\, \mathrm{N}_0(\mathrm{F})$. This vector of course has a decomposition

$$\mathrm{Y}(k^{-1}\gamma_0 k) = \sum_{\beta \in \Sigma_{\mathrm{P}_0}} \mathrm{Y}_\beta(k^{-1}\gamma_0 k), \quad \mathrm{Y}_\beta(k^{-1}\gamma_0 k) \in \overline{\mathfrak{n}}_\beta(\mathrm{F}),$$

relative to the root spaces $\overline{\mathfrak{n}}_\beta$ of $\overline{\mathfrak{n}}_0$. For any simple root $\alpha \in \Delta_{\mathrm{P}_0}$, let $\Sigma_{\mathrm{P}_0}(\alpha)$ be the set of roots $\beta \in \Sigma_{\mathrm{P}_0}$ which contain α in their simple root decomposition. Then the vector

$$\mathrm{Y}^\alpha(k^{-1}\gamma_0 k) = \sum_{\beta \in \Sigma_{\mathrm{P}_0}(\alpha)} \mathrm{Y}_\beta(k^{-1}\gamma_0 k)$$

cannot vanish. Otherwise $k^{-1}\gamma_0 k$ would belong to the maximal parabolic subgroup determined by α, contradicting the fact that $k^{-1}\gamma_0 k$ is a G-regular element in an anisotropic torus. Choose a height function $\|\cdot\|$ on the vector space $\overline{\mathfrak{n}}_0(\mathrm{F})$, relative to a basis which is compatible with the root space decomposition. Then $\| \mathrm{Y}^\alpha(k^{-1}\gamma_0 k) \|$ is a continuous nonvanishing function on the compact group K. Since the function is determined by the components $\| \mathrm{Y}_\beta(k^{-1}\gamma_0 k) \|$ with $\beta \in \Sigma_{\mathrm{P}_0}(\alpha)$, we can choose a positive constant ε_0 such that

$$\max_{\beta \in \Sigma_{\mathrm{P}_0}(\alpha)} \| \mathrm{Y}_\beta(k^{-1}\gamma_0 k) \| \geqslant \varepsilon_0,$$

for all $k \in \mathrm{K}$, and each $\alpha \in \Delta_{\mathrm{P}_0}$.

Now suppose that x is a variable point in $\mathrm{G(F)}^1$. Given positive functions $a(x)$ and $b(x)$ of x, we shall use the notation $a(x) \prec b(x)$ or $b(x) \succ a(x)$ to indicate that

$$a(x) \leqslant c\, b(x)^{\mathrm{N}}, \quad x \in \mathrm{G(F)}^1,$$

for positive constants c and N. We can write

$$x = kak^+, \quad k \in K, \ a \in A_0(F) \cap G(F)^1, \ k^+ \in K^+,$$

where $H_0(a)$ lies in the closure of $\mathfrak{a}_{P_0}^+$ and $K^+ \subseteq G(F)^1$ is a fixed compact set such that

$$(A_0(F) \cap G(F)^1)\, K^+ = (M_0(F) \cap G(F)^1)\, K.$$

It then follows from (4.1) and the definition of $\|\cdot\|$ that

$$\| x \| \prec \max_{\alpha \in \Delta_{P_0}} e^{\alpha(H_0(a))}.$$

It remains for us to examine $\| x^{-1} \gamma_0 x \|$ as a function of k, a and k^+.

In the notation above, we have

$$k^{-1} \gamma_0 k = \exp(Y(k^{-1} \gamma_0 k))\, m_0 n_0,$$

for elements $m_0 \in M_0(F)$ and $n_0 \in N_0(F)$. The points $a^{-1} m_0 a$ and $a^{-1} n_0 a$ remain bounded, since a acts on $N_0(F)$ by contraction. Combining this with (4.1), we obtain

$$\begin{aligned} \| x^{-1} \gamma_0 x \| &\geqslant (\| k^+ \| \, \| (k^+)^{-1} \|)^{-1} \| a^{-1} k^{-1} \gamma_0 k a \| \\ &\succ \| a^{-1} \exp(Y(k^{-1} \gamma_0 k))\, a \|. \end{aligned}$$

Since exp is a bijective polynomial mapping of $\bar{\mathfrak{n}}_0$ onto $\bar{N}_0$, there are positive constants c and N such that

$$\| Y \| \leqslant c \, \| \exp(Y) \|^N, \quad Y \in \bar{\mathfrak{n}}_0(F).$$

It follows that

$$\| a^{-1} \exp(Y(k^{-1} \gamma_0 k))\, a \| \succ \| \mathrm{Ad}(a^{-1})\, Y(k^{-1} \gamma_0 k) \|.$$

But if β belongs to $\Sigma_{P_0}(\alpha)$, for some $\alpha \in \Delta_{P_0}$, we have

$$\begin{aligned} \| \mathrm{Ad}(a^{-1})\, Y(k^{-1} \gamma_0 k) \| &\geqslant \| \mathrm{Ad}(a^{-1})\, Y_\beta(k^{-1} \gamma_0 k) \| \\ &\geqslant e^{\alpha(H_0(a))} \| Y_\beta(k^{-1} \gamma_0 k) \|, \end{aligned}$$

since $\alpha(H_0(a)) \leqslant \beta(H_0(a))$. The supremum over β and α then gives us

$$\| \mathrm{Ad}(a^{-1})\, Y(k^{-1} \gamma_0 k) \| \geqslant \varepsilon_0 \max_{\alpha \in \Delta_{P_0}} (e^{(H_0(a))}) \succ \| x \|.$$

Putting all these inequalities together, we end up with the relation

$$\| x \| \prec \| x^{-1} \gamma_0 x \|.$$

In other words, there are constants c_0 and N_0 such that

$$\| x \| \leqslant c_0 \, \| x^{-1} \gamma_0 x \|^{N_0}, \quad x \in G(F)^1. \quad \square$$

Lemma 4.1 will be required only for the proof of the next result, in which S is an arbitrary maximal torus over F.

Lemma **4.2.** — *There is a positive integer k, with the property that for any given compact subset Ω of $G(F)$ there is a constant c_Ω such that*

$$\inf_{s \in S(F)} \| sx \| \leqslant c_\Omega \, | D(\gamma) |^{-k},$$

for all points $\gamma \in S_{reg}(F)$ and $x \in G(F)$ with $x^{-1} \gamma x \in \Omega$.

Proof. — Suppose that F' is a finite extension of F. One can extend the absolute value $|\cdot|$ from F to F', and the height function $\|\cdot\|$ from $G(F)$ to $G(F')$. We claim that there are constants c and N such that

$$\inf_{s \in S(F)} \| sx \| \leqslant c \inf_{s' \in S'(F)} \| s' x \|^N, \quad x \in G(F). \tag{4.6}$$

To establish (4.6), suppose first that S is F-anisotropic modulo A_G. Choose $\gamma_0 \in S_{reg}(F)$ as in Lemma 4.1. Since $G(F)^1 A_G(F)$ is of finite index in $G(F)$, we can use Lemma 4.1 in conjunction with the property (4.1) to show that

$$\inf_{s \in S(F)} \| sx \| \leqslant c_0 \, \| x^{-1} \gamma_0 x \|^{N_0}, \quad x \in G(F),$$

for constants c_0 and N_0. But (4.1) and (4.3) tell us that there are constants c_1 and N_1 such that

$$\| x^{-1} \gamma_0 x \| = \| (s' x)^{-1} \gamma_0 (s' x) \| \leqslant c_1 \, \| s' x \|^{N_1},$$

for any elements $x \in G(F)$ and $s' \in S(F')$. This establishes (4.6) in case S is anisotropic. In the general case, we can always replace S by a $G(F)$-conjugate. We may therefore assume that S is contained in a fixed Levi subgroup $M \in \mathscr{L}$, and that S is F-anisotropic modulo A_M. Take any parabolic subgroup $P \in \mathscr{P}(M)$. Then

$$\begin{aligned}\inf_{s \in S(F)} \| sx \| &= \inf_s \| s m_P(x) \, n_P(x) \, k_P(x) \| \\ &\leqslant (\inf_s \| s m_P(x) \|) \, \| n_P(x) \| \, \| k_P(x) \|,\end{aligned}$$

by (4.1). Applying the anisotropic case we have just established, we obtain

$$\begin{aligned}\inf_{s \in S(F)} \| sx \| &\leqslant c_2 (\inf_{s' \in S(F')} \| s' m_P(x) \|^{N_2}) \, \| n_P(x) \| \\ &= c_2 \inf_{s'} \| m_P(s' x) \|^{N_2} \, \| n_P(s' x) \|,\end{aligned}$$

for constants c_2 and N_2. The general estimate (4.6) then follow from the property (4.5).

In view of (4.6), it is sufficient to prove the lemma for a given S with F replaced by F'. We can of course choose the extension so that S splits over F'. It is therefore sufficient to prove the lemma in the special case that S splits over F.

Assuming that S splits over F, we take $B = SN$ to be a Borel subgroup of G over F which contains S. Since $G(F)$ equals $S(F)\,N(F)\,K$, and K is compact, it will be enough to verify the condition of the lemma for points $x = n$ in the unipotent radical $N(F)$. Set

$$n^{-1} \gamma n = \gamma \nu, \quad \nu \in N(F),$$

for any points $\gamma \in S_{reg}(F)$ and $n \in N(F)$. For any given γ, $n \to \nu$ is a bijection of $N(F)$ to itself. Consider the inverse $n = n(\gamma, \nu)$ as a function of both γ and ν. There is a positive integer k such that the map

$$(\gamma, \nu) \to D(\gamma)^k n(\gamma, \nu)$$

is defined by an F-rational morphism between the algebraic varieties $S \times N$ and N. This is a consequence of the proof of Lemma 10 of [17]. (See also [8, p. 237].) But if $n^{-1}\gamma n$ lies in the compact set Ω, γ and ν must lie in compact subsets Ω_S and Ω_N of $S(F)$ and $N(F)$ which depend only on Ω. It follows that

$$\| n(\gamma, \nu) \| \leqslant c_\Omega \, | D(\gamma) |^{-k},$$

for a constant c_Ω that depends only on Ω. This completes the proof of the lemma. □

Fix a Levi subgroup $M \in \mathscr{L}$. If γ is a G-regular point in $M(F)$, we recall (3.5) that $K^T(\gamma, f)$ equals

$$| D(\gamma) | \int_{A_M(F)\backslash G(F)} \int_{A_M(F)\backslash G(F)} f_1(x_1^{-1}\gamma x_1) f_2(x_2^{-1}\gamma x_2)\, u_M(x_1, x_2, T)\, dx_1\, dx_2,$$

where

$$u_M(x_1, x_2, T) = \int_{A_G(F)\backslash A_M(F)} u(x_1^{-1} a x_2, T)\, da.$$

We would like to be able to replace u_M by a different weight function.

If $x_1, x_2 \in G(F)$, set

$$Y_P(x_1, x_2, T) = T_P + H_P(x_1) - H_{\bar{P}}(x_2), \quad P \in \mathscr{P}(M).$$

The points

$$\mathscr{Y}_M(x_1, x_2, T) = \{ Y_P(x_1, x_2, T) : P \in \mathscr{P}(M) \}$$

form a (G, M)-orthogonal set, which is positive if $d(T)$ is large relative to x_1 and x_2. The second weight function is given by the integral

$$v_M(x_1, x_2, T) = \int_{A_G(F)\backslash A_M(F)} \sigma_M(H_M(a), \mathscr{Y}_M(x_1, x_2, T))\, da.$$

We then define $J^T(\gamma, f)$ to be the corresponding weighted orbital integral

$$| D(\gamma) | \int_{A_M(F)\backslash G(F)} \int_{A_M(F)\backslash G(F)} f_1(x_1^{-1}\gamma x_1) f_2(x_2^{-1}\gamma x_2)\, v_M(x_1, x_2, T)\, dx_1\, dx_2.$$

Let S be a fixed maximal torus in M which is F-anisotropic modulo A_M. Our aim is to show that the integrals of $K^T(\gamma, f)$ and $J^T(\gamma, f)$ over $\gamma \in S_{reg}(F)$ differ by a function which approaches 0 as $d(T)$ approaches infinity. We shall first deal with integrals over domains

$$S(\varepsilon, T) = \{ x \in S_{reg}(F) : | D(\gamma) | \leqslant e^{-\varepsilon \| T \|} \}, \tag{4.7}$$

which for large T lie near the singular set.

Lemma **4.3.** — *Given* $\varepsilon > 0$, *we can choose a constant* c *such that for any* T,

$$\int_{S(\varepsilon, T)} (|K^T(\gamma, f)| + |J^T(\gamma, f)|)\, d\gamma \leqslant c e^{-\frac{\varepsilon \| T \|}{2}}.$$

Proof. — Take points $x_1, x_2 \in G(F)$ and $a \in A_M(F)$. If we set

$$x_1^{-1} a x_2 = k_1 h k_2, \quad k_1, k_2 \in K, \ h \in M_0(F),$$

we see easily that

$$\log \| x_1^{-1} a x_2 \| \leqslant c_0(\| H_0(h) \| + 1),$$

for some constant c_0. It then follows from the definition of $u(\cdot, T)$ that there is a constant c_1 such that

$$\inf_{z \in A_G(F)} (\log \| x_1^{-1} z a x_2 \|) \leqslant c_1(\| T \| + 1),$$

whenever $u(x_1^{-1} a x_2, T)$ does not vanish. In fact, by the properties (4.1) and (4.3), we can choose c_1 so that

$$\inf_{z \in A_G(F)} (\log \| z a \|) \leqslant c_1(\| T \| + \log \| x_1 \| + \log \| x_2 \|).$$

The integral over a in $A_G(F) \backslash A_M(F)$ then yields an inequality

(4.8) $$u_M(x_1, x_2, T) \leqslant c_2(\| T \| + \log \| x_1 \| + \log \| x_2 \|)^{d_2},$$

for positive constants c_2 and d_2.

Now assume that

$$f_1(x_1^{-1} \gamma x_1) f_2(x_2^{-1} \gamma x_2) \neq 0.$$

Taking $\Omega \subseteq G(F)$ to be any compact set which contains the support of f_1 and f_2, we obtain

$$\inf_{s \in S(F)} \| s x_i \| \leqslant c_\Omega | D(\gamma) |^{-k}, \quad i = 1, 2,$$

from Lemma 4.2. We shall combine this with the estimate (4.8). Notice that (4.8) remains valid if either x_1 or x_2 is replaced by a left translate from $A_M(F)$. Since $S(F)/A_M(F)$ is compact, we obtain a constant c'_Ω such that

$$u_M(x_1, x_2, T) \leqslant c'_\Omega(\| T \| + \log(| D(\gamma) |^{-k}))^{d_2}.$$

The next step is to apply a fundamental theorem ([15, Theorem 2], [18, Theorem 14]) of Harish-Chandra on orbital integrals. This result yields a constant c_3 such that

$$| D(\gamma) |^{1/2} \int_{A_M(F) \backslash G(F)} | f_i(x_i^{-1} \gamma x_i) | \, dx_i \leqslant c_3, \quad i = 1, 2,$$

for all $\gamma \in S_{\mathrm{reg}}(F)$. Putting these two estimates into the definition of $K^T(\gamma, f)$, we obtain the inequality

$$| K^T(\gamma, f) | \leqslant c'_\Omega c_3^2(\| T \| + \log(| D(\gamma) |^{-k}))^{d_2}.$$

It is not hard to show that any power of $\log |D(\gamma)|$ is integrable over any bounded subset B of $S_{reg}(F)$. In fact, there is a constant c_B such that

$$\int_{B \cap S(\varepsilon, T)} (\| T \| + \log(| D(\gamma) |^{-k}))^{d_2} \, d\gamma \leqslant c_B \, e^{-\frac{\varepsilon \| T \|}{2}}.$$

This is an exercise in elementary real or p-adic analysis which we leave to the reader. Taking

$$B = \{ \gamma \in S_{reg}(F) : K^T(\gamma, f) \neq 0 \},$$

we see finally that

$$\int_{S(\varepsilon, T)} | K^T(\gamma, f) | \, d\gamma \leqslant c e^{-\frac{\varepsilon \| T \|}{2}},$$

for some constant c. This establishes half of the lemma.

The proof of the other half of the lemma is similar. It is a simple consequence of (4.5) that the points $H_P(x)$, $P \in \mathscr{P}(M)$, can be bounded in terms of x. In fact there are positive constants c_1 and d_1 such that

$$\| H_P(x)) \| \leqslant c_1(1 + \log \| x \|)^{d_1}, \quad P \in \mathscr{P}(M), \ x \in G(F).$$

It follows from the definition of $v_M(x_1, x_2, T)$ that

$$v_M(x_1, x_2, T) \leqslant c_2(\| T \| + \log \| x_1 \| + \log \| x_2 \|)^{d_2},$$

for constants c_2 and d_2. This, of course, is the analogue of (4.8) for the second weight function. We can consequently estimate $J^T(\gamma, f)$ by arguing exactly as above. We obtain a constant c such that

$$\int_{S(\varepsilon, T)} | J^T(\gamma, f) | \, d\gamma \leqslant c e^{-\frac{\varepsilon \| T \|}{2}}.$$

The lemma follows. □

We must now compare the integrals of $K^T(\gamma, f)$ and $J^T(\gamma, f)$ over γ in the complement of $S(\varepsilon, T)$ in $S(F)$. The essential step is to estimate the difference of the two weight functions. This is summarized in the next lemma, which is one of the main technical results of the paper. We shall postpone its proof until § 5.

Lemma **4.4.** — *Suppose that* $\delta > 0$. *Then there are positive numbers* C, ε_1 *and* ε_2 *such that*

$$| u_M(x_1, x_2, T) - v_M(x_1, x_2, T) | \leqslant C e^{-\varepsilon_1 \| T \|},$$

for all T *with* $d(T) \geqslant \delta \| T \|$, *and all* x_1 *and* x_2 *in the set*

$$\{ x \in G(F) : \| x \| \leqslant e^{\varepsilon_2 \| T \|} \}. \tag{4.9}$$

Granting Lemma 4.4, let us compare the two integrals. Let ε be an arbitrary but fixed positive number. We shall apply Lemma 4.2 to points γ in $S(F) - S(\varepsilon, T)$, with Ω any compact subset of G(F) which contains the support of f_1 and f_2. Combined with the definition of $S(\varepsilon, T)$, the lemma asserts that if γ belongs to $S(F) - S(\varepsilon, T)$, and x_1 and x_2 are such that

$$f_1(x_1^{-1}\gamma x_1) f_2(x_2^{-1}\gamma x_2) \neq 0,$$

then

$$\inf_{s \in S(F)} \| s x_i \| \leqslant c \,|\, D(\gamma) \,|^{-k} \leqslant C e^{k\varepsilon \| T \|}, \quad i = 1, 2,$$

for constants c and k. The constants are, of course, independent of ε. We choose ε so that $k\varepsilon$ is smaller than the constant ε_2 given by Lemma 4.4. Then, translating x_1 and x_2 by elements in $S(F)$ if necessary, and taking $\| T \|$ to be sufficiently large, we can assume that

$$\| x_i \| \leqslant e^{\varepsilon_2 \| T \|}, \quad i = 1, 2.$$

It follows from Lemma 4.4 that

$$| u_M(x_1, x_2, T) - v_M(x_1, x_2, T) | \leqslant C e^{-\varepsilon_1 \| T \|}.$$

Recalling the definition of $K^T(\gamma, f)$ and $J^T(\gamma, f)$, we see finally that

$$(4.10) \qquad \int_{S(F) - S(\varepsilon, T)} | K^T(\gamma, f) - J^T(\gamma, f) | \, d\gamma \leqslant C_1 \, e^{-\varepsilon_1 \| T \|},$$

where C_1 equals the constant

$$C \int_{S_{\mathrm{reg}}(F)} | D(\gamma) | \left(\int_{A_M(F) \backslash G(F)} | f_1(x_1^{-1}\gamma x_1) | \, dx_1 \int_{A_M(F) \backslash G(F)} | f_2(x_2^{-1}\gamma x_2) | \, dx_2 \right) d\gamma.$$

The finiteness of C_1 follows from the theorem of Harish-Chandra ([15, Theorem 2], [18, Theorem 14]).

We have shown that for any $\delta > 0$, there are positive constants ε, ε_1 and C_1 such that (4.10) holds for all T with $d(T) \geqslant \delta \| T \|$. When we combine this with Lemma 4.3, we obtain a similar estimate for the integral over γ in the entire set $S_{\mathrm{reg}}(F)$. The conclusion is that there are positive constants C' and

$$\varepsilon' = \min\left(\varepsilon_1, \frac{\varepsilon}{2}\right)$$

such that

$$(4.11) \qquad \int_{S_{\mathrm{reg}}(F)} | K^T(\gamma, f) - J^T(\gamma, f) | \, d\gamma \leqslant C' \, e^{-\varepsilon' \| T \|},$$

for all T with $d(T) \geqslant \delta \| T \|$.

Define

$$(4.12) \qquad J^T(f) = \sum_{M \in \mathscr{L}} | W_0^M | \, | W_0^G |^{-1} \int_{\Gamma_{\mathrm{ell}}(M(F))} J^T(\gamma, f) \, d\gamma.$$

There are, of course, only finitely many $M \in \mathscr{L}$, and finitely many conjugacy classes of (anisotropic) tori S in any M. The estimate (4.11) therefore leads immediately to the following proposition, which gives the promised asymptotic formula for $K^T(f)$.

Proposition **4.5.** — *Suppose that* $\delta > 0$. *Then there are positive numbers* C *and* ε *such that*

$$|K^T(f) - J^T(f)| \leqslant Ce^{-\varepsilon\|T\|}, \tag{4.13}$$

for all T *with* $d(T) \geqslant \delta \|T\|$. □

5. Proof of the main geometric lemma

In the last section we established an asymptotic formula (4.13) which will eventually lead to an explicit expression for the geometric side. However, we put aside the proof of Lemma 4.4, which was the essential step in the process. We shall now give the proof. It is an extension of the derivation of Lemma 3 of [9], which was the main ingredient of that paper.

We fix the positive number δ, and we assume from now on that $d(T) \geqslant \delta \|T\|$. We also fix the Levi subgroup $M \in \mathscr{L}$. Lemma 4.4 is trivial if $\|T\|$ remains bounded. It therefore suffices to prove the lemma for $\|T\|$ sufficiently large. We must select a positive number ε_2 such that if

$$\|x_i\| \leqslant e^{\varepsilon_2\|T\|}, \quad i = 1, 2, \tag{5.1}$$

then

$$|u_M(x_1, x_2, T) - v_M(x_1, x_2, T)| \leqslant Ce^{-\varepsilon_1\|T\|},$$

for positive constants C and ε_1.

Recall that $u_M(x_1, x_2, T)$ and $v_M(x_1, x_2, T)$ are the integrals over a in $A_M(F)/A_G(F)$ of two compactly supported functions $u(x_1^{-1}ax_2, T)$ and $\sigma_M(H_M(a), \mathscr{Y}_M(x_1, x_2, T))$. We shall decompose each of these integrals into a finite sum over the groups $Q \in \mathscr{F}(M)$. Indeed, for any $\varepsilon > 0$, one can partition $A_M(F)/A_G(F)$ into a disjoint union over $Q \in \mathscr{F}(M)$ of the sets

$$A_M(Q, \varepsilon) = \{a \in A_M(F)/A_G(F) : \sigma_M^Q(H_M(a), \varepsilon T)\, \tau_Q(H_M(a) - \varepsilon T_Q) = 1\}.$$

This is just the formula (3.9) applied to the positive (G, M)-orthogonal set $\{\varepsilon T_P : P \in \mathscr{P}(M)\}$. We shall fix the positive number ε, which we take to be small in a sense that depends only on G and δ. We also fix a group $Q \in \mathscr{F}(M)$, and then take $a \in A_M(F)/A_G(F)$ to be any fixed point in the set $A_M(Q, \varepsilon)$. Since the vectors εT_Q are to be large, this means that a^{-1} will act on $N_Q(F)$ by contraction.

Most of our efforts will be devoted to the function $u(x_1^{-1}ax_2, T)$ whose integral gives the first weight factor. If x_1 and x_2 belong to G(F), we can write

$$x_1^{-1}ax_2 = k_Q(x_1)^{-1}\, n_Q(x_1)^{-1}\, m_Q(x_1)^{-1}\, am_{\bar{Q}}(x_2)\, n_{\bar{Q}}(x_2)\, k_{\bar{Q}}(x_2).$$

The idea is to move $n_Q(x_1)^{-1}$ to the right under conjugation, in order to take advantage of the contracting property of the element a^{-1}. The vector $\log(n_Q(x_1)^{-1})$ lies in the Lie algebra $\mathfrak{n}_Q(F)$ of $N_Q(F)$, and its conjugate

$$X' = \mathrm{Ad}(m_Q(x_1)^{-1}\, am_{\bar{Q}}(x_2)\, n_{\bar{Q}}(x_2))^{-1}(\log(n_Q(x_1)^{-1}))$$

belongs to the Lie algebra $\mathfrak{g}(F)$ of $G(F)$. Similarly, we shall move $n_{\bar{Q}}(x_2)$ to the left under conjugation to exploit the contracting property of a. We form the vector $\log(n_{\bar{Q}}(x_2)^{-1})$ in $\mathfrak{n}_{\bar{Q}}(F)$, and its conjugate

$$X = \mathrm{Ad}(m_Q(x_1)^{-1}\, a m_{\bar{Q}}(x_2))\, (\log(n_{\bar{Q}}(x_2)^{-1})).$$

Then we have

$$x_1^{-1}\, a x_2 = k_Q(x_1)^{-1}\, \xi^{-1}\, m_Q(x_1)^{-1}\, a m_{\bar{Q}}(x_2)\, \xi'\, k_{\bar{Q}}(x_2),$$

where $\xi = \exp(X)$ and $\xi' = \exp(X')$. Fix a height function $\|\cdot\|$ on the vector space $\mathfrak{g}(F)$. We shall estimate the heights $\|X\|$ and $\|X'\|$.

The vector X' is defined as the value of

$$\mathrm{Ad}(n_{\bar{Q}}(x_2)^{-1}) \circ \mathrm{Ad}(m_{\bar{Q}}(x_2)^{-1}) \circ \mathrm{Ad}(a^{-1}) \circ \mathrm{Ad}(m_Q(x_1)),$$

a composition of four operators on $\mathfrak{g}(F)$, at the vector $\log(n_Q(x_1)^{-1})$. We have to estimate the heights of these objects, all of which are determined by the height function on $\mathfrak{g}(F)$. Since exp is a bijective polynomial map of $\mathfrak{n}_Q(F)$ onto $N_Q(F)$, $\|\log(n_Q(x_1)^{-1})\|$ is bounded by some polynomial in $\|n_Q(x_1)\|$. By (4.5), this is bounded in turn by a polynomial in $\|x_1\|$, or equivalently since $\|x_1\| \geqslant 1$, by a constant multiple of a power of $\|x_1\|$. Observe next that the height of the operator $\mathrm{Ad}(n_{\bar{Q}}(x_2)^{-1})$ is bounded by a constant multiple of a power of $\|x_2\|$. This follows from the properties (4.2), (4.3) and (4.5). Similar remarks apply to the contributions of $\mathrm{Ad}(m_{\bar{Q}}(x_2)^{-1})$ and $\mathrm{Ad}(m_Q(x_1))$. The final contribution is that of $\mathrm{Ad}(a^{-1})$, and is bounded by the height of the restriction of this operator to $\mathfrak{n}_Q(F)$. This is of course where we exploit the contraction property of $\mathrm{Ad}(a^{-1})$. Since a belongs to $A_M(Q, \varepsilon)$, $H_M(a)$ can be written as the sum of a vector in $\mathfrak{a}_Q^+$ with a convex linear combination of points

$$\{\varepsilon T_P : P \in \mathscr{P}(M), P \subset Q\}.$$

It follows that if α is any root for the action of A_M on $\mathfrak{n}_Q$, then

$$\alpha(H_M(a)) \geqslant \varepsilon \inf_{P \subset Q} \alpha(T_P) \geqslant \varepsilon\, d(T) \geqslant \varepsilon\, \delta\, \|T\|,$$

by (3.2). Therefore the height of the operator $\mathrm{Ad}(a)^{-1}$ on $\mathfrak{n}_Q(F)$ is bounded by a constant multiple of $e^{-\varepsilon\delta\|T\|}$. Putting these contributions together, we see that

$$\|X'\| \leqslant c'\, e^{-\varepsilon\delta\|T\|}(\|x_1\| + \|x_2\|)^{n'},$$

for constants c' and n'.

We estimate $\|X\|$ the same way. Taking into account the fact that $\mathrm{Ad}(a)$ acts on $\mathfrak{n}_{\bar{Q}}(F)$ by contraction, we find that

$$\|X\| \leqslant c e^{-\varepsilon\delta\|T\|}(\|x_1\| + \|x_2\|)^{n},$$

for constants c and n. In particular, if x_1 and x_2 satisfy (5.1), we have an inequality

$$\|X\| + \|X'\| \leqslant C e^{-\varepsilon'\|T\|}, \tag{5.2}$$

where

$$\varepsilon' = \varepsilon\,\delta - \varepsilon_2\, n,$$

and C and n are fixed constants. Once we have chosen ε we are free to take ε_2 to be as small as we like. This allows us to assume that the number ε' is positive.

It follows from the definitions that

$$u(x_1^{-1}\, ax_2, \mathrm{T}) = u(\xi^{-1}\, m_{\mathrm{Q}}(x_1)^{-1}\, am_{\overline{\mathrm{Q}}}(x_2)\, \xi', \mathrm{T}).$$

Fix a minimal parabolic subgroup $\mathrm{P}_0 \in \mathscr{P}(\mathrm{M}_0)$ which is contained in Q. Then there is a point

$$h = h_{\mathrm{Q}}(a, x_1, x_2)$$

in $\mathrm{M}_0(\mathrm{F})$, with $\mathrm{H}_0(h)$ in the closure $\overline{\mathfrak{a}_{\mathrm{P}_0}^+}$ of $\mathfrak{a}_{\mathrm{P}_0}^+$, such that

$$m_{\mathrm{Q}}(x_1)^{-1}\, am_{\overline{\mathrm{Q}}}(x_2) = k^{-1}\, hk', \quad k, k' \in \mathrm{K} \cap \mathrm{M}_{\mathrm{Q}}(\mathrm{F}). \tag{5.3}$$

The inequality (5.2) guarantees that for large $\|\mathrm{T}\|$, the points $\xi = \exp(\mathrm{X})$ and $\xi' = \exp(\mathrm{X}')$ are very close to 1. For this reason, it makes sense to study the function

$$u(m_{\mathrm{Q}}(x_1)^{-1}\, am_{\overline{\mathrm{Q}}}(x_2), \mathrm{T}) = u(h, \mathrm{T}).$$

Write

$$\mathrm{H}_0(h) = \mathrm{H}_0^{\mathrm{Q}}(h) + \mathrm{H}_{\mathrm{Q}}(h)$$

for the decomposition of $\mathrm{H}_0(h)$ relative to the direct sum $\mathfrak{a}_0 = \mathfrak{a}_{\mathrm{M}_0}^{\mathrm{Q}} \oplus \mathfrak{a}_{\mathrm{Q}}$. Observe that

$$\mathrm{H}_{\mathrm{Q}}(h) = -\,\mathrm{H}_{\mathrm{Q}}(x_1) + \mathrm{H}_{\mathrm{Q}}(a) + \mathrm{H}_{\overline{\mathrm{Q}}}(x_2). \tag{5.4}$$

As for the other point $\mathrm{H}_0^{\mathrm{Q}}(h)$, we claim that there is a constant c_{Q} such that

$$\|\mathrm{H}_0^{\mathrm{Q}}(h)\| \leqslant c_{\mathrm{Q}}(1 + \|\mathrm{H}_0^{\mathrm{Q}}(a)\| + \log\|x_1\| + \log\|x_2\|), \tag{5.5}$$

for all points h, a, x_1 and x_2 related as in (5.3). To see this, choose $r > 0$ so that the product of $\mathrm{A}_{\mathrm{Q}}(\mathrm{F})$ with

$$\mathrm{M}_{\mathrm{Q}}(\mathrm{F})^r = \{ m \in \mathrm{M}_{\mathrm{Q}}(\mathrm{F}) : \|\mathrm{H}_{\mathrm{Q}}(m)\| \leqslant r \}$$

equals $\mathrm{M}_{\mathrm{Q}}(\mathrm{F})$. It is clear that for any $m \in \mathrm{M}_{\mathrm{Q}}(\mathrm{F})$, and any decomposition

$$m = m_r\, a, \quad m_r \in \mathrm{M}_{\mathrm{Q}}(\mathrm{F})^r, \ a \in \mathrm{A}_{\mathrm{Q}}(\mathrm{F}),$$

the height $\|m_r\|$ is bounded by a fixed constant multiple of $\|m\|$. It is also easy to check that if m belongs to the subgroup $\mathrm{M}_0(\mathrm{F})$, then

$$\|\mathrm{H}_0^{\mathrm{Q}}(m)\| \leqslant c_1(1 + \|\log(m_r)\|) \leqslant c_1'(1 + \|\mathrm{H}_0^{\mathrm{Q}}(m)\|),$$

for constants c_1 and c_1' which depend only on r. Applying the first half of this estimate to the element h in (5.3) (with r replaced by $3r$), we obtain

$$\begin{aligned}\|\mathrm{H}_0^{\mathrm{Q}}(h)\| &\leqslant c_1(1 + \log\|km_{\mathrm{Q}}(x_1)_r^{-1}\, a_r\, m_{\overline{\mathrm{Q}}}(x_2)_r\, (k')^{-1}\|) \\ &\leqslant c_2(1 + \log\|m_{\mathrm{Q}}(x_1)^{-1}\| + \log\|a_r\| + \log\|m_{\overline{\mathrm{Q}}}(x_2)\|),\end{aligned}$$

for some constant c_2. By the second half of the estimate, and also the properties (4.3) and (4.5), this is bounded by a constant multiple of

$$(1 + \log \| x_1 \| + \| H_0^Q(a) \| + \log \| x_2 \|).$$

The claim follows.

We are assuming that x_1 and x_2 satisfy the inequality (5.1) and that a lies in $A_M(Q, \varepsilon)$. The first condition leads to

$$\log \| x_1 \| + \log \| x_2 \| \leqslant 2\varepsilon_2 \| T \| \leqslant 2\varepsilon_2\, \delta^{-1}\, d(T).$$

The second condition implies that $H_0^Q(a)$ lies in the convex hull in $\mathfrak{a}_M/\mathfrak{a}_Q$ of the points

$$\{ \varepsilon T_P : P \in \mathscr{P}(M), P \subset Q \},$$

and since

$$\| T_P \| \leqslant \| T \| \leqslant \delta^{-1}\, d(T)$$

for any such P, we obtain the inequality

$$(5.6) \qquad \| H_0^Q(a) \| \leqslant \varepsilon\, \delta^{-1}\, d(T).$$

Combining these remarks with the inequality (5.5), we find that

$$(5.7) \qquad \| H_0^Q(h) \| \leqslant \delta_Q\, d(T),$$

where

$$\delta_Q = c_Q(d(T)^{-1} + \varepsilon\, \delta^{-1} + 2\varepsilon_2\, \delta^{-1}).$$

We are free to make $d(T)$ as large as we want, and we can choose ε and ε_2 to be small. We can therefore assume that the positive number δ_Q is small. It follows from [9, Lemma 1] that the characteristic function $u(h, T)$ equals 1 if and only if the vector

$$H_Q(h) = - H_Q(x_1) + H_Q(a) + H_{\overline{Q}}(x_2)$$

lies in the convex hull $S_{M_Q}(T)$.

There is a similar way to characterize the function whose integral gives the second weight factor.

Lemma **5.1.** — *Assume that x_1 and x_2 satisfy* (5.1) *for a small positive number ε_2. Then $\mathscr{Y}_M(x_1, x_2, T)$ is a positive orthogonal set, and the characteristic function*

$$\sigma_M(H_M(a), \mathscr{Y}_M(x_1, x_2, T)), \quad a \in A_M(Q, \varepsilon),$$

equals 1 *if and only if the vector*

$$H_Q(h) = - H_Q(x_1) + H_Q(a) + H_{\overline{Q}}(x_2)$$

lies in the convex hull $S_{M_Q}(T)$.

Proof. — It follows from (4.5) and (5.1) that if α is any root in Δ_P, for some $P \in \mathscr{P}(M)$, then

$$\begin{aligned} | \alpha(H_P(x_1) - H_{\overline{P}}(x_2)) | &\leqslant c(\log \| x_1 \| + \log \| x_2 \|) \\ &\leqslant 2c\varepsilon_2\, \delta^{-1}\, d(T), \end{aligned}$$

for some constant c. This estimate leads directly to the inequality

$$(5.8) \qquad d(\mathscr{Y}_M(x_1, x_2, T)) \geqslant (1 - 2c\varepsilon_2\, \delta^{-1})\, d(T)$$

for the (G, M)-orthogonal set

$$\mathscr{Y}_M(x_1, x_2, T) = \{ H_P(x_1) - H_{\overline{P}}(x_2) + T_P : P \in \mathscr{P}(M) \}.$$

Since ε_2 is very small, $2c\varepsilon_2\, \delta^{-1}$ is close to 0, and the number $d(\mathscr{Y}_M(x_1, x_2, T))$ is positive. This implies that $\mathscr{Y}_M(x_1, x_2, T)$ is a positive orthogonal set, as required. In particular, $\sigma_M(\cdot, \mathscr{Y}_M(x_1, x_2, T))$ is the characteristic function in $\mathfrak{a}_M/\mathfrak{a}_G$ of the convex hull $S_M(\mathscr{Y}_M(x_1, x_2, T))$.

We have also chosen ε so that $\varepsilon\, \delta^{-1}$ is small. Let a be any point in $A_M(Q, \varepsilon)$. It follows from (5.6) and (5.8) that the projection of $H_M(a)$ onto $\mathfrak{a}_M^Q$ is small relative to $d(\mathscr{Y}(x_1, x_2, T))$. Appealing to [9, Lemma 1] as above, we deduce that $H_M(a)$ belongs to $S_M(\mathscr{Y}_M(x_1, x_2, T))$ if and only if $H_Q(a)$ lies in the convex hull $S_{M_Q}(\mathscr{Y}_{M_Q}(x_1, x_2, T))$. We can also apply Lemma 3.1, since $d(\mathscr{Y}_M(x_1, x_2, T))$ is positive. It tells us that the intersection of $\mathfrak{a}_Q^+$ with $S_{M_Q}(\mathscr{Y}_{M_Q})$ is the set

$$\{ H \in \mathfrak{a}_Q^+ : \varpi(H - Y_Q) \leqslant 0,\ \varpi \in \hat{\Delta}_Q \}.$$

Since a belongs to $A_M(Q, \varepsilon)$, $H_Q(a)$ lies in the chamber $\mathfrak{a}_Q^+$. Moreover

$$H_Q(a) - Y_Q(x_1, x_2, T) = H_Q(a) - H_Q(x_1) + H_{\overline{Q}}(x_2) - T_Q = H_Q(h) - T_Q.$$

We can assume that ε_2 is small relative to ε, so our assumptions on a, x_1 and x_2 imply that $H_Q(h)$ also lies in $\mathfrak{a}_Q^+$. We conclude that $H_Q(a)$ belongs to $S_{M_Q}(\mathscr{Y}_{M_Q}(x_1, x_2, T))$ if and only if $H_Q(h)$ belongs to $S_{M_Q}(T)$. This gives the second assertion of the lemma. □

If we combine the discussion preceding the lemma with the lemma itself, we see that

$$(5.9) \qquad u(h, T) = \sigma_M(H_M(a), \mathscr{Y}_M(x_1, x_2, T)),$$

for any point $a \in A_M(Q, \varepsilon)$. However, it is not $u(h, T)$ that we want. Instead, we must consider the original function

$$\begin{aligned} u(x_1^{-1}\, a x_2, T) &= u(\xi^{-1}\, m_Q(x_1)^{-1}\, a m_{\overline{Q}}(x_2)\, \xi', T) \\ &= u(\xi^{-1}\, k^{-1}\, h k'\, \xi', T) \\ &= u(\zeta^{-1}\, h\zeta', T), \end{aligned}$$

where $\zeta = k\xi k^{-1}$ and $\zeta' = k'\, \xi'(k')^{-1}$. It follows from (5.2) that if ε_2 is small relative to ε, we can make ζ and ζ' approach 0 as $\|T\|$ approaches infinity. In particular, if F is a p-adic field, ζ and ζ' both belong to K when $\|T\|$ is large. In this case we have

$$u(x_1^{-1}\, a x_2, T) = u(h, T),$$

for any $a \in A_M(Q, \varepsilon)$. Combining this with (5.9), we see that

$$u(x_1^{-1}\, a x_2, T) = \sigma_M(H_M(a), \mathscr{Y}_M(x_1, x_2, T)).$$

Neither function in the last identity depends on Q, so the identity holds for any point a in $A_M(F)/A_G(F)$. Integrating over a, we see that the weight factors $u_M(x_1, x_2, T)$ and $v_M(x_1, x_2, T)$ are actually equal. This establishes a strong version of Lemma 4.4 for p-adic F. The required asymptotic formula holds as an exact formula for all T, x_1 and x_2 satisfying the given conditions.

It remains to establish Lemma 4.4 for Archimedean F. We can always reduce the problem from $\mathbf{C}$ to $\mathbf{R}$ by reduction of scalars. We shall therefore assume for the rest of § 5 that F is equal to $\mathbf{R}$. Then we can identify $\mathfrak{a}_0$ with the Lie algebra of $A_0(\mathbf{R})$, and

$$G(\mathbf{R}) = K \exp(\overline{\mathfrak{a}_{P_0}^+})\, K.$$

We can also take the height function on $\mathfrak{g}(\mathbf{R})$ to be a positive definite quadratic form given by

$$\| X \|^2 = - B(X, \theta X), \quad X \in \mathfrak{g}(\mathbf{R}),$$

where B is a $G(\mathbf{R})$-invariant form, and θ is the Cartan involution with respect to K.

Lemma **5.2.** — *Suppose that* X *and* X' *are points in* $\mathfrak{g}(\mathbf{R})$, *and that* H_0 *and* H_1 *are points in* $\overline{\mathfrak{a}_{P_0}^+}$ *with the property that*

$$\exp(X)^{-1} \exp(H_0) \exp(X') = k_1 \exp(H_1)\, k_1', \quad k_1, k_1' \in K.$$

Then

$$\| H_1 - H_0 \| \leqslant \| X \| + \| X' \|.$$

Proof. — It is enough to prove the lemma when H_0 is a point in the open chamber $\mathfrak{a}_{P_0}^+$. When H_0 is fixed, H_1 becomes a well defined function of X and X', and we write

$$f_\lambda(X, X') = \lambda(H_1)$$

for any linear functional $\lambda \in \mathfrak{a}_0^*$. Then $f_\lambda(0, 0)$ equals $\lambda(H_0)$. Observe that $f_\lambda(tX, tX')$ is a continuous, piecewise smooth function of $t \in \mathbf{R}$. It follows that

$$\begin{aligned} |\lambda(H_1 - H_0)| &= |f_\lambda(X, X') - f_\lambda(0, 0)| \\ &\leqslant \sup_{0<t<1} \left| \frac{d}{dt} f_\lambda(tX, tX') \right| \\ &\leqslant \sup_t \left| \frac{d}{ds} f_\lambda((t+s)\, X, tX')_{s=0} \right| \\ &\qquad + \sup_t \left| \frac{d}{ds} f_\lambda(tX, (t+s)\, X')_{s=0} \right|. \end{aligned}$$

We shall estimate the two derivatives in this last expression.

Set $h_t = \exp(H_t)$, where H_t is the point in $\overline{\mathfrak{a}_{P_0}^+}$ such that

$$\exp(tX)^{-1} \exp(H_0) \exp(tX') = k_t^{-1} \exp(H_t)\, k_t',$$

for points k_t and k'_t in K. Then $h_0 = \exp H_0$, and

$$\exp(tX)^{-1} h_0 \exp((t+s) X') = k_t^{-1} h_t \exp(sX'_t) k'_t,$$

where $X'_t = \mathrm{Ad}(k'_t) X'$. Given that H_0 belongs to $\mathfrak{a}_{P_0}^+$, one shows easily that H_t also belongs to the open chamber $\mathfrak{a}_{P_0}^+$ if t is in general position. This in turn implies [16, Lemma 21] that there is a decomposition

$$X'_t = H(X'_t) - \mathrm{Ad}(h_t)^{-1} C(X'_t) + C'(X'_t),$$

where $C(X'_t)$ and $C'(X'_t)$ belong to the Lie algebra of K, and $H(X'_t)$ belongs to the Lie algebra $\mathfrak{a}_0$ of $A_0(\mathbf{R})$. The vector $H(X'_t)$ is just the orthogonal projection of X'_t onto the subalgebra $\mathfrak{a}_0$ of $\mathfrak{g}(\mathbf{R})$. Consequently

$$\| H(X'_t) \| \leqslant \| X'_t \| = \| X' \|.$$

However, if F is any smooth function on a neighbourhood in $G(\mathbf{R})$ of the point

$$\exp(tX)^{-1} h_0 \exp(tX') = k_t^{-1} h_t k'_t,$$

the derivative

$$\frac{d}{ds} F(k_t^{-1} h_t \exp(sX'_t) k'_t)_{s=0}$$

is the sum of three partial derivatives, at the points $h_t \in A_0(\mathbf{R})$, $k_t \in K$ and $k'_t \in K$, relative to the respective right invariant vector fields attached to $H(X'_t)$, $C(X'_t)$ and $C'(X'_t)$. In particular, if

$$F(k \exp(H) k') = \lambda(H), \quad H \in \mathfrak{a}_{P_0}^+, \ k, k' \in K,$$

the obvious extension of the function

$$F(k_t^{-1} h_t k'_t) = \lambda(H_0(h_t)) = \lambda(H_t) = f_\lambda(tX, tX'),$$

the derivative above simply equals $\lambda(H(X'_t))$. It follows that

$$\frac{d}{ds} f_\lambda(tX, (t+s) X')_{s=0} = \lambda(H(X'_t)).$$

This derivative is bounded in absolute value by $\| \lambda \| \| X' \|$. Arguing in a similar manner, we also see that

$$\left| \frac{d}{ds} f_\lambda((t+s) X, tX')_{s=0} \right| \leqslant \| \lambda \| \| X \|.$$

Combining these estimates with the discussion above, we obtain

$$\| H_1 - H_0 \| \leqslant \| X \| + \| X' \|. \quad \square$$

We shall now finish the proof of Lemma 4.4 for $F = \mathbf{R}$. The first weight factor $u_M(x_1, x_2, T)$ equals the sum over $Q \in \mathscr{F}(M)$ of the integrals

$$\int_{A_M(Q,\varepsilon)} u(x_1^{-1} a x_2, T) \, da, \tag{5.10}$$

while the second weight function $v_M(x_1, x_2, T)$ is the sum over Q of

$$(5.11) \qquad \int_{A_M(Q,\varepsilon)} \sigma_M(H_M(a), \mathscr{Y}_M(x_1, x_2, T))\, da.$$

We must estimate the difference between the two summands (5.10) and (5.11).

Fix $Q \in \mathscr{F}(M)$ and $a \in A_M(Q, \varepsilon)$. Then we can write

$$u(x_1^{-1} a x_2, T) = u(\zeta^{-1} h \zeta', T)$$

as before, where $h = h_Q(a, x_1, x_2)$ as in (5.3), and

$$\zeta = k\xi k^{-1} = \exp(\mathrm{Ad}(k)\, X)$$

and

$$\zeta' = k'\, \xi'(k')^{-1} = \exp(\mathrm{Ad}(k')\, X'),$$

in the notation above. Let

$$h_1 = h_{1,Q}(a, x_1, x_2)$$

be the point in $A_0(\mathbf{R})$, with $H_0(h_1) \in \overline{\mathfrak{a}_{P_0}^+}$, such that

$$\zeta^{-1} h \zeta' = k_1 h_1 k_1', \quad k_1, k_1' \in K.$$

Then by Lemma 5.2

$$\| H_0(h_1) - H_0(h) \| \leqslant \| \mathrm{Ad}(k)\, X \| + \| \mathrm{Ad}(k')\, X' \| = \| X \| + \| X' \|.$$

We are assuming that x_1 and x_2 satisfy (5.1). It then follows from (5.2) that

$$(5.12) \qquad \| H_0(h_1) - H_0(h) \| \leqslant C e^{-\varepsilon' \|T\|}.$$

In particular, the point $H_0^Q(h_1)$ can be made very close to $H_0^Q(h)$. In view of (5.7), we can therefore assume that

$$\| H_0^Q(h_1) \| \leqslant \delta_Q'\, d(T),$$

where δ_Q' is a small positive number. It follows from [9, Lemma 1] that the characteristic function

$$u(h_1, T) = u(x_1^{-1} a x_2, T)$$

equals 1 if and only if the vector $H_Q(h_1)$ lies in $S_{M_Q}(T)$. We can certainly write

$$H_Q(h_1) = H_Q(a) - H_Q(x_1) + H_{\bar{Q}}(x_2) + (H_Q(h_1) - H_Q(h)).$$

We therefore conclude that the integrand in (5.10) equals 1 or 0, according to whether or not $H_Q(a)$ lies in the translated polytope

$$(5.13) \qquad (H_Q(x_1) - H_{\bar{Q}}(x_2) - (H_Q(h_1) - H_Q(h))) + S_{M_Q}(T).$$

To deal with the other expression (5.11), we simply apply Lemma 5.1. It tells us that the integrand in (5.11) equals 1 or 0, according to whether or not $H_Q(a)$ lies in the translated polytope

$$(5.14) \qquad (H_Q(x_1) - H_{\bar{Q}}(x_2)) + S_{M_Q}(T).$$

Consider now the difference between (5.10) and (5.11). According to the definition of $A_M(Q, \varepsilon)$, the integral in either (5.10) or (5.11) can be changed under the transformation

$$a \to H_0^Q(a) \oplus H_Q(a)$$

to a double integral over the product of $S_M^Q(\varepsilon T)$ with the affine chamber $(\varepsilon T_Q + \mathfrak{a}_Q^+)/\mathfrak{a}_G$ in $\mathfrak{a}_Q/\mathfrak{a}_G$. Let $\Delta_{M_Q}(T)$ be the subset of points in $\mathfrak{a}_Q/\mathfrak{a}_G$ which lie in either the complement of (5.13) in (5.14) or the complement of (5.14) in (5.13). Since the set (5.14) is the translate of the polytope (5.13) by the vector $(H_Q(h_1) - H_Q(h))$, we can use (5.12) to estimate the volume of $\Delta_{M_Q}(T)$. The volume of any facet of $S_{M_Q}(T)$ is bounded by a polynomial in $\| T \|$, from which it follows easily that

$$\mathrm{vol}(\Delta_{M_Q}(T)) \leqslant p_1(\| T \|)\, e^{-\varepsilon' \| T \|},$$

for some polynomial p_1. Consequently, the difference between (5.10) and (5.11) is bounded in absolute value by

$$\mathrm{vol}(S_M^Q(T))\, p_1(\| T \|)\, e^{-\varepsilon' \| T \|}.$$

But $\mathrm{vol}(S_M^Q(T))$ is also bounded by a polynomial in $\| T \|$. Therefore the difference between (5.10) and (5.11) is bounded in absolute value by $C_Q\, e^{-\varepsilon_1 \| T \|}$, where C_Q is some constant and $\varepsilon_1 = \frac{\varepsilon'}{2}$. Taking the sum over $Q \in \mathscr{F}(M)$, we see that

$$| u_M(x_1, x_2, T) - v_M(x_1, x_2, T) | \leqslant C e^{-\varepsilon_1 \| T \|},$$

with $C = \sum_Q C_Q$. This is the required estimate.

We have just established Lemma 4.4 for the remaining case $F = \mathbf{R}$. The lemma therefore holds in general. Since Lemma 4.4 represented the unproved portion of Proposition 4.5, we have also completely proved the asymptotic formula (4.13) for the geometric side. □

6. The function $J^T(f)$

We have shown that the truncated integral $K^T(f)$ is asymptotic to a function $J^T(f)$ defined by a manageable geometric expansion. We shall later see that $K^T(f)$ is also asymptotic to a function defined by a parallel spectral expansion. In fact, we will want to identify this second function with $J^T(f)$. To do so, we will need to have a good understanding of $J^T(f)$ as a function of T.

Recall (4.12) that $J^T(f)$ is defined in terms of the distributions

$$J^T(\gamma, f) = | D(\gamma) | \int_{A_M(F)\backslash G(F)} \int_{A_M(F)\backslash G(F)} f_1(x_1^{-1} \gamma x_1) f_2(x_2^{-1} \gamma x_2)\, v_M(x_1, x_2, T)\, dx_1\, dx_2,$$

with $\gamma \in \Gamma_{\mathrm{ell}}(M(F))$ and

$$v_M(x_1, x_2, T) = \int_{A_G(F)\backslash A_M(F)} \sigma_M(H_M(a), \mathscr{Y}_M(x_1, x_2, T))\, da.$$

We shall study $v_M(x_1, x_2, T)$ as a function of T. If F is Archimedean, the integral over $A_G(F)\backslash A_M(F)$ can be transformed into an integral over $\mathfrak{a}_M/\mathfrak{a}_G$. It then follows easily from [4, Lemma 6.3] that $v_M(x_1, x_2, T)$ is a polynomial in T. However, if F is p-adic, the integral can be replaced only by a sum over the lattice $\tilde{\mathfrak{a}}_{M,F}/\tilde{\mathfrak{a}}_{G,F}$. Instead of a volume of a convex hull, we are faced with having to count lattice points.

Assume for the time being that F is a p-adic field. Then T belongs to the lattice $\mathfrak{a}_{M_0,F}$ in $\mathfrak{a}_{M_0}$, and the points T_P, $P \in \mathscr{P}(M)$, all belong to the lattice $\mathfrak{a}_{M,F}$ in $\mathfrak{a}_M$. The kernel of the surjective map

$$H_M : A_G(F)\backslash A_M(F) \to \tilde{\mathfrak{a}}_{M,F}/\tilde{\mathfrak{a}}_{G,F}$$

is a compact group which has volume 1. Therefore, as we mentioned above, we can write

$$v_M(x_1, x_2, T) = \sum_{X \in \tilde{\mathfrak{a}}_{M,F}/\tilde{\mathfrak{a}}_{G,F}} \sigma_M(X, \mathscr{Y}_M(x_1, x_2, T)).$$

It will be convenient to write

$$\tilde{\mathscr{L}}_M = \tilde{\mathfrak{a}}_{M,F} + \mathfrak{a}_G/\mathfrak{a}_G$$

and

$$\mathscr{L}_M = \mathfrak{a}_{M,F} + \mathfrak{a}_G/\mathfrak{a}_G$$

for the lattices in $\mathfrak{a}_M/\mathfrak{a}_G$ obtained by projecting $\tilde{\mathfrak{a}}_{M,F}$ and $\mathfrak{a}_{M,F}$ onto the quotient. We can also form the dual lattices

$$\mathscr{L}^\vee = \mathrm{Hom}(\mathscr{L}, 2\pi i\mathbf{Z}), \quad \mathscr{L} = \tilde{\mathscr{L}}_M, \mathscr{L}_M,$$

in $(i\mathfrak{a}_M^*)^G$. Since

$$\tilde{\mathfrak{a}}_{M,F}/\tilde{\mathfrak{a}}_{G,F} = \tilde{\mathfrak{a}}_{M,F}/\tilde{\mathfrak{a}}_{M,F} \cap \mathfrak{a}_G \cong \tilde{\mathfrak{a}}_{M,F} + \mathfrak{a}_G/\mathfrak{a}_G = \tilde{\mathscr{L}}_M$$

by (1.4), we can take the sum above over $X \in \tilde{\mathscr{L}}_M$. However, for future comparison purposes, it would be preferable to take a sum over $\mathscr{L}_M$. We can certainly do this, provided that we also take a sum over $\tilde{\mathscr{L}}_M^\vee/\mathscr{L}_M^\vee$, a finite quotient which is identified with the character group of $\mathscr{L}_M/\tilde{\mathscr{L}}_M$ under the pairing

$$e^{\nu(X)}, \quad \nu \in \tilde{\mathscr{L}}_M^\vee/\mathscr{L}_M^\vee, \ X \in \mathscr{L}_M/\tilde{\mathscr{L}}_M.$$

Thus, $v_M(x_1, x_2, T)$ equals

$$|\mathscr{L}_M/\tilde{\mathscr{L}}_M|^{-1} \sum_{\nu \in \tilde{\mathscr{L}}_M^\vee/\mathscr{L}_M^\vee} \sum_{X \in \mathscr{L}_M} \sigma_M(X, \mathscr{Y}_M(x_1, x_2, T))\, e^{\nu(X)}.$$

Let Λ be a small point in $(\mathfrak{a}_M/\mathfrak{a}_G)^*_{\mathbf{C}}$ in general position. According to the definition (3.8),

$$\begin{aligned}\sigma_M(X, \mathscr{Y}_M(x_1, x_2, T)) &= \sum_{P \in \mathscr{P}(M)} (-1)^{|\Delta_P^\Lambda|}\, \varphi_P^\Lambda(X - Y_P) \\ &= \lim_{\Lambda \to 0} \sum_{P \in \mathscr{P}(M)} (-1)^{|\Delta_P^\Lambda|}\, \varphi_P^\Lambda(X - Y_P)\, e^{\Lambda(X)},\end{aligned}$$

where

$$Y_P = Y_P(x_1, x_2, T) = T_P + H_P(x_1) - H_{\overline{P}}(x_2).$$

We observe for future reference that Y_P belongs to the lattice $\mathfrak{a}_{M,F}$, and that $\varphi_P^\Delta(X - Y_P)$ depends only on the image of Y_P in $\mathscr{L}_M$. It follows from the definition of φ_P^Δ that the function $e^{\Lambda(X)}$ is rapidly decreasing on the support of $\varphi_P^\Delta(X - Y_P)$. In particular, the product of these two functions is summable over X in $\mathscr{L}_M$. Therefore, $v_M(x_1, x_2, T)$ equals the expression obtained from

$$|\mathscr{L}_M/\widetilde{\mathscr{L}}_M|^{-1} \sum_{X \in \mathscr{L}_M} (-1)^{|\Delta_P^\Delta|} \varphi_P^\Delta(X - Y_P)\, e^{(\Lambda+\nu)(X)} \tag{6.1}$$

by first summing over $P \in \mathscr{P}(M)$, then taking the limit as Λ approaches 0, and finally summing over $\nu \in \widetilde{\mathscr{L}}_M^\vee/\mathscr{L}_M^\vee$.

If k is any positive number, we shall write

$$\mu_{\alpha,k} = k \log(q_F)\, \alpha^\vee, \quad \alpha \in \Delta_P,$$

where q_F is the order of the residue class field of F. The additive subgroup

$$\mathscr{L}_{M,k} = k \log(q_F)\, \mathbf{Z}(\Delta_P^\vee) = \{ \sum_{\alpha \in \Delta_P} n_\alpha \mu_{\alpha,k} : n_\alpha \in \mathbf{Z} \}$$

of $\mathfrak{a}_M$ is a lattice in $\mathfrak{a}_M^G$ which is independent of P. Having agreed earlier to identify $\mathfrak{a}_M^G$ with $\mathfrak{a}_M/\mathfrak{a}_G$, we shall also regard $\mathscr{L}_{M,k}$ as a lattice in $\mathfrak{a}_M/\mathfrak{a}_G$. It is easy to see [9, p. 12] that if k is a suitably large positive integer, $\mathscr{L}_{M,k}$ is a sublattice of $\widetilde{\mathscr{L}}_M$. We choose such a k, valid for all M, for once and for all. We then write (6.1) as

$$|\mathscr{L}_M/\widetilde{\mathscr{L}}_M|^{-1} \sum_{X \in \mathscr{L}_M/\mathscr{L}_{M,k}} \sum_{X' \in \mathscr{L}_{M,k}} (-1)^{|\Delta_P^\Delta|} \varphi_P^\Delta(X' + X - Y_P)\, e^{(\Lambda+\nu)(X'+X)}.$$

As in [9, § 4], we shall evaluate the sum over X′ as a multiple geometric series.

If Y is any point in $\mathscr{L}_M$ and X belongs to $\mathscr{L}_M/\mathscr{L}_{M,k}$, let $X_P(Y)$ be the representative of X in $\mathscr{L}_M$ such that

$$X_P(Y) - Y = \sum_{\alpha \in \Delta_P} r_\alpha \mu_{\alpha,k}, \tag{6.2}$$

for real numbers r_α with $-1 < r_\alpha \leqslant 0$. Set

$$\begin{aligned} X_P^\Delta(Y) &= X_P(Y) + \sum_{\alpha \in \Delta_P^\Delta} \mu_{\alpha,k} \\ &= Y + \sum_{\alpha \in \Delta_P^\Delta} (1 + r_\alpha)\, \mu_{\alpha,k} + \sum_{\alpha \in \Delta_P - \Delta_P^\Delta} r_\alpha \mu_{\alpha,k}. \end{aligned}$$

Then $X_P^\Delta(Y)$ is also a representative of X in $\mathscr{L}_M$, and we can set

$$\varphi_P^\Delta(X' + X - Y_P) = \varphi_P^\Delta(X' + X_P^\Delta(Y_P) - Y_P)$$

in the sum above. The set of points $X' \in \mathscr{L}_{M,k}$ for which this characteristic function equals 1 is just the set

$$\{ \sum_{\alpha \in \Delta_P^\Delta} n_\alpha \mu_{\alpha,k} - \sum_{\alpha \in \Delta_P - \Delta_P^\Delta} n_\alpha \mu_{\alpha,k} \},$$

in which each n_α ranges over the nonnegative integers. Therefore

$$(-1)^{|\Delta_P^\Lambda|} \sum_{X' \in \mathscr{L}_{M,k}} \varphi_P^\Lambda(X' + X - Y_P)\, e^{(\Lambda+\nu)(X'+X)}$$
$$= (-1)^{|\Delta_P^\Lambda|} \sum_{X' \in \mathscr{L}_{M,k}} \varphi_P^\Lambda(X' + X_P^\Lambda(Y_P) - Y_P)\, e^{(\Lambda+\nu)(X'+X_P^\Lambda(Y_P))}$$
$$= (-1)^{|\Delta_P^\Lambda|}\, e^{(\Lambda+\nu)(X_P^\Lambda(Y_P))} \prod_{\alpha \in \Delta_P^\Lambda} (1 - e^{(\Lambda+\nu)(\mu_{\alpha,k})})^{-1} \prod_{\alpha \in \Delta_P - \Delta_P^\Lambda} (1 - e^{-(\Lambda+\nu)(\mu_{\alpha,k})})^{-1}$$
$$= e^{(\Lambda+\nu)(X_P(Y_P))} \prod_{\alpha \in \Delta_P} (1 - e^{-(\Lambda+\nu)(\mu_{\alpha,k})})^{-1}.$$

(The notation $X_P(Y_P)$ is an unfortunate consequence of using the subscript P for the map $X \to X_P$ as well as the set $\{Y_P\}$. We hope that the meaning is clear.) Set

$$\theta_{P,k}(\lambda) = \mathrm{vol}(\mathfrak{a}_M^G/\mathscr{L}_{M,k})^{-1} \prod_{\alpha \in \Delta_P} (1 - e^{-\lambda(\mu_{\alpha,k})}) \tag{6.3}$$

for any point $\lambda \in \mathfrak{a}_{M,\mathbf{C}}^*$. Recall that we have fixed the Haar measure on the space $\mathfrak{a}_M^G \cong \mathfrak{a}_M/\mathfrak{a}_G$. It has the property that the quotient of $\mathfrak{a}_M/\mathfrak{a}_G$ by the lattice $\widetilde{\mathscr{L}}_M$ has volume 1. This means that

$$|\mathscr{L}_M/\widetilde{\mathscr{L}}_M|^{-1} \prod_{\alpha \in \Delta_P} (1 - e^{-(\Lambda+\nu)(\mu_{\alpha,k})})^{-1} = |\mathscr{L}_M/\mathscr{L}_{M,k}|^{-1}\, \theta_{P,k}(\Lambda+\nu)^{-1}.$$

Therefore (6.1) equals

$$|\mathscr{L}_M/\mathscr{L}_{M,k}|^{-1} \sum_{X \in \mathscr{L}_M/\mathscr{L}_{M,k}} e^{(\Lambda+\nu)(X_P(Y_P))}\, \theta_{P,k}(\Lambda+\nu)^{-1}.$$

Now, set $X_P = X_P(0)$. Then if $Y = Y_P$ in the expression (6.2), we have

$$X_P(Y_P) = Y_P + \sum_{\alpha \in \Delta_P} r_\alpha\, \mu_{\alpha,k} = Y_P + (X - Y_P)_P.$$

Replacing X by $X - Y_P$ in the sum over $\mathscr{L}_M/\mathscr{L}_{M,k}$ above, we see that (6.1) equals

$$|\mathscr{L}_M/\mathscr{L}_{M,k}|^{-1} \sum_{X \in \mathscr{L}_M/\mathscr{L}_{M,k}} e^{(\Lambda+\nu)(X_P+Y_P)}\, \theta_{P,k}(\Lambda+\nu)^{-1}.$$

We have established that $v_M(x_1, x_2, T)$ equals

$$\sum_{\nu \in \widetilde{\mathscr{L}}_M^\vee/\mathscr{L}_M^\vee} \lim_{\Lambda \to 0} \Big(\sum_{P \in \mathscr{P}(M)} |\mathscr{L}_M/\mathscr{L}_{M,k}|^{-1} \sum_{X \in \mathscr{L}_M/\mathscr{L}_{M,k}} e^{(\Lambda+\nu)(X_P+Y_P)}\, \theta_{P,k}(\Lambda+\nu)^{-1}\Big), \tag{6.4}$$

where

$$Y_P = T_P + H_P(x_1) - H_{\bar{P}}(x_2).$$

The discussion above implies that the function in the brackets is analytic at $\Lambda = 0$, so the limit does exist. To analyze $v_M(x_1, x_2, T)$ as a function of T, replace Λ by $z\Lambda$, $z \in \mathbf{C}$, and then take the Laurent expansion at $z = 0$ of the expression in the brackets. (See [31, p. 315].) The constant term of the Laurent expansion is a finite sum of functions

$$q_{P,\nu}(T_P)\, e^{\nu(T_P)}, \quad \nu \in \widetilde{\mathscr{L}}_M^\vee/\mathscr{L}_M^\vee,\ P \in \mathscr{P}(M),$$

where $q_{P,\nu}$ is a polynomial function on $\mathfrak{a}_M$. These functions depend only on the image of T in $\mathfrak{a}_0/\mathfrak{a}_G$, so we shall assume that T lies in the lattice

$$\mathscr{L}_0 = \mathscr{L}_{M_0} = \mathfrak{a}_{M_0,F} + \mathfrak{a}_G/\mathfrak{a}_G.$$

For any P, the map $T \to T_P$ sends $\mathscr{L}_0$ surjectively onto the intersection of $\mathscr{L}_M$ with the closure $\overline{\mathfrak{a}_P^+}$ of the chamber associated to P. We may as well restrict T to lie in the intersection of $\mathscr{L}_0$ with the suitably regular points in some fixed chamber $\mathfrak{a}_0^+$ of $\mathfrak{a}_0/\mathfrak{a}_G$. Then T_P ranges over the suitably regular points in $\mathscr{L}_M \cap \mathfrak{a}_P^+$. It follows that

$$v_M(x_1, x_2, T) = \sum_{\xi \in \left(\frac{1}{N}\mathscr{L}_0^\vee\right)/\mathscr{L}_0^\vee} q_\xi(T)\, e^{\xi(T)}, \tag{6.5}$$

where N is a positive integer which can be chosen independently of M, $q_\xi(T)$ is a polynomial in T, and

$$\mathscr{L}_0^\vee = \mathrm{Hom}(\mathscr{L}_0, 2\pi i\mathbf{Z})$$

as above.

The coefficients $q_\xi(T)$ in the decomposition (6.5) are obviously uniquely determined. In particular, the " constant term " $q_0(0)$ is a well defined function of $v_M(x_1, x_2, T)$. To obtain an explicit formula for $q_0(0)$, take the summand corresponding to $\nu = 0$ in the expression (6.4) for $v_M(x_1, x_2, T)$, and then set $T = 0$. The result is

$$\tilde{v}_M(x_1, x_2) = \lim_{\Lambda \to 0} \sum_{P \in \mathscr{P}(M)} |\mathscr{L}_M/\mathscr{L}_{M,k}|^{-1} \sum_{X \in \mathscr{L}_M/\mathscr{L}_{M,k}} e^{\Lambda(X_P + H_P(x_1) - H_{\bar{P}}(x_2))}\, \theta_{P,k}(\Lambda)^{-1}. \tag{6.6}$$

We substitute the formula (6.4) we have obtained for $v_M(x_1, x_2, T)$ into the expression for $J^T(\gamma, f)$. The integral over γ, whose convergence we treated in § 4, then provides a description of $J^T(f)$ as a function of T. Indeed, the decomposition (6.5) of $v_M(x_1, x_2, T)$ gives us a similar decomposition for $J^T(f)$. In particular, we can define the " constant term " $\tilde{J}(f)$ of $J^T(f)$. Moreover, $\tilde{J}(f)$ can be written in terms of the function $\tilde{v}_M(x_1, x_2)$ defined by (6.6).

In summary, we have obtained

Proposition **6.1.** — *There is a decomposition*

$$J^T(f) = \sum_{\xi \in \left(\frac{1}{N}\mathscr{L}_0^\vee\right)/\mathscr{L}_0^\vee} p_\xi(T, f)\, e^{\xi(T)}, \quad T \in \mathscr{L}_0 \cap \mathfrak{a}_0^+,$$

where N *is a fixed positive integer, and* $p_\xi(T, f)$ *is a polynomial in* T. *Moreover, the constant term*

$$\tilde{J}(f) = p_0(0, f)$$

of $J^T(f)$ *is given by*

$$\tilde{J}(f) = \sum_{M \in \mathscr{L}} |W_0^M|\, |W_0^G|^{-1} \int_{\Gamma_{\mathrm{ell}}(M(F))} \tilde{J}_M(\gamma, f)\, d\gamma,$$

where

$$\tilde{J}_M(\gamma, f) = |D(\gamma)| \int_{A_M(F)\backslash G(F)} \int_{A_M(F)\backslash G(F)} f_1(x_1^{-1}\gamma x_1) f_2(x_2^{-1}\gamma x_2)\, \tilde{v}_M(x_1, x_2)\, dx_1\, dx_2. \quad \square$$

Remarks. — 1. Proposition 6.1 was proved if F is a p-adic field, but with minor modifications it holds in general. If F is Archimedean, the groups $\mathscr{L}_M$ and $\widetilde{\mathscr{L}}_M$ are both equal to $\mathfrak{a}_M/\mathfrak{a}_G$. In this case, we simply set $\mathscr{L}_{M,k} = \mathfrak{a}_M/\mathfrak{a}_G$, and we take $\theta_{P,k}(\lambda)$ to be the usual function

$$(6.7) \qquad \theta_P(\lambda) = \mathrm{vol}(\mathfrak{a}_M^G/\mathbf{Z}(\Delta_P^\vee))^{-1} \prod_{\alpha \in \Delta_P} \lambda(\alpha^\vee).$$

The definition (6.6) and the formula in the proposition make sense in this context. With these interpretations, Proposition 6.1 is valid for all F.

2. As a function of T, the original distribution $K^T(f)$ is invariant under the Weyl group W_0^G. It follows easily from Propositions 4.5 and 6.1 that $J^T(f)$ is also W_0^G-invariant. This implies that the constant term $\tilde{J}(f) = p_0(0,f)$ is independent of the chamber $\mathfrak{a}_0^+$.

7. Eisenstein integrals and c-functions

In studying the spectral expansion (3.6) of $K^T(f)$, it will be convenient to formulate some of the problems in terms of Eisenstein integrals and c-functions. These objects are of course an essential part of Harish-Chandra's Plancherel theorem. We shall review some of their basic properties.

If π is an admissible tempered representation of G(F), $\mathscr{A}_\pi(G)$ will stand for the space of functions on G(F) spanned by K-finite matrix coefficients of π. Let $\mathscr{A}_{\mathrm{temp}}(G)$ be the sum over all such π of these spaces, and set $\mathscr{A}_2(G)$ equal to the subspace obtained by taking only those $\pi \in \Pi_2(G(F))$. Suppose that τ is a unitary, two-sided representation of K on a finite dimensional Hilbert space V. Then $\mathscr{A}_{\mathrm{temp}}(G, \tau)$ will denote the space of functions $f \in \mathscr{A}_{\mathrm{temp}}(G) \otimes V$ such that

$$f(k_1 x k_2) = \tau(k_1) f(x) \tau(k_2), \quad x \in G(F),\ k_1, k_2 \in K.$$

The subspaces $\mathscr{A}_\pi(G, \tau)$ and $\mathscr{A}_2(G, \tau)$ of $\mathscr{A}_{\mathrm{temp}}(G, \tau)$ are defined in the same way. If π belongs to $\Pi_2(G(F))$, the inner product

$$(\psi', \psi) = \int_{A_G(F)\backslash G(F)} (\psi'(x), \psi(x))\, dx, \quad \psi', \psi \in \mathscr{A}_\pi(G, \tau),$$

is defined, and we can form the corresponding norm $\|\psi\| = (\psi, \psi)^{1/2}$.

Suppose that $M \in \mathscr{L}$ is a fixed Levi subgroup. Then τ_M denotes the restriction of τ to $K_M = K \cap M(F)$. If f is a function in $\mathscr{A}_{\mathrm{temp}}(G, \tau)$ and $P \in \mathscr{P}(M)$, we shall write $C^P f$ for the weak constant term of f ([20, § 21], [23, § 3]). It is the uniquely determined function in $\mathscr{A}_{\mathrm{temp}}(M, \tau_M)$ such that

$$\delta_P(ma)^{1/2} f(ma) - (C^P f)(ma), \quad m \in M(F),\ a \in A_M(F),$$

is asymptotic to 0 as a approaches infinity along the chamber of P.

If P' and P are groups in $\mathscr{P}(M)$, let $\tau_{P'|P}$ denote the subrepresentation of τ_M on the invariant subspace

$$V_{P'|P} = \{ v \in V : \tau(n')\, v\tau(n) = v,\ n' \in N_{P'}(F) \cap K,\ n \in N_P(F) \cap K \}$$

of V. (See [19, § 11]. This definition is only significant in the p-adic case, for if F is Archimedean, $\tau_{P'|P}$ equals τ_M.) Since $\tau_{P'|P}$ is a two-sided representation of K_M, we can form the corresponding spaces $\mathscr{A}_{\mathrm{temp}}(M, \tau_{P'|P})$, $\mathscr{A}_2(M, \tau_{P'|P})$, etc., of spherical functions on M(F). The Eisenstein integral, which depends on a parameter $\lambda \in i\mathfrak{a}_M^*$, maps functions $\psi \in \mathscr{A}_2(M, \tau_{P|P})$ to functions

$$E_P(\psi, \lambda) : x \to E_P(x, \psi, \lambda), \quad x \in G(F),$$

in $\mathscr{A}_{\mathrm{temp}}(G, \tau)$. It is defined by

$$E_P(x, \psi, \lambda) = \int_K \tau(k)^{-1}\, \psi_P(kx)\, e^{(\lambda + \rho_P)(H_P(kx))}\, dk,$$

where

$$\psi_P(nmk) = \psi(m)\, \tau(k), \quad n \in N_P(F),\ m \in M(F),\ k \in K.$$

For fixed x and ψ, the Eisenstein integral extends to an entire function of $\lambda \in \mathfrak{a}_{M,\mathbf{C}}^*$.

Recall that the c-functions are defined by weak constant terms of Eisenstein integrals. Suppose that

$$P_1 = M_1 N_{P_1}, \quad M_1 \in \mathscr{L},$$

is a parabolic subgroup which is associated to P. This means that the Levi components M and M_1 are conjugate, or equivalently, that the set $W(\mathfrak{a}_M, \mathfrak{a}_{M_1})$ of all possible isomorphisms of $\mathfrak{a}_M$ onto $\mathfrak{a}_{M_1}$ obtained by restricting elements in W_0^G to $\mathfrak{a}_M$, is nonempty. Then the weak constant term

$$(C^{P_1} E_P)(\psi, \lambda) : m_1 \to (C^{P_1} E_P)(m_1, \psi, \lambda), \quad m_1 \in M_1(F),$$

equals

$$\sum_{s \in W(\mathfrak{a}_M, \mathfrak{a}_{M_1})} (c_{P_1|P}(s, \lambda)\, \psi)(m_1)\, e^{(s\lambda)(H_{M_1}(m_1))},$$

where each $c_{P_1|P}(s, \lambda)$ extends to a meromorphic function of $\lambda \in \mathfrak{a}_{M,\mathbf{C}}^*$ with values in the space of linear maps from $\mathscr{A}_2(M, \tau_{P|P})$ to $\mathscr{A}_2(M_1, \tau_{P_1|\bar{P}_1})$. Harish-Chandra has established a number of functional equations relating Eisenstein integrals, c-functions and the auxiliary c-functions

$$c^0_{P_1|P}(s, \lambda) = c_{P_1|P}(s, \lambda)\, c_{P|P}(1, \lambda)^{-1}$$

and

$${}^0c_{P_1|P}(s, \lambda) = c_{P_1|P_1}(1, s\lambda)^{-1}\, c_{P_1|P}(s, \lambda).$$

These have been summarized in [7, § I.2].

Next we review some of the properties of intertwining operators, and in particular, their connection with c-functions. Suppose that Γ is a finite set of classes of irreducible representations of K. Let V_Γ be the finite dimensional subspace of functions

$$\alpha : (k_1, k_2) \to \alpha_{k_1, k_2}$$

in $L^2(K \times K)$ which transform in each variable according to representations in Γ. Then there is a two-sided representation τ_Γ on V_Γ defined by

$$(\tau_\Gamma(k_1') \, \alpha \tau_\Gamma(k_2'))_{k_1, k_2} = \alpha_{k_1 k_1', k_2' k_2}.$$

Suppose that σ is a representation in $\Pi_2(M(F))$. For each $P \in \mathscr{P}(M)$, the induced representations $\mathscr{I}_P(\sigma_\lambda)$, $\lambda \in \mathfrak{a}_{M,\mathbf{C}}^*$, all act on the Hilbert space $\mathscr{H}_P(\sigma)$. Let $\mathscr{H}_P(\sigma)_\Gamma$ be the subspace of vectors in $\mathscr{H}_P(\sigma)$ which transform under K according to representations in Γ. Harish-Chandra has defined an isomorphism $T \to \psi_T$ from $\mathrm{End}(\mathscr{H}_P(\sigma)_\Gamma)$ onto $\mathscr{A}_\sigma(M, (\tau_\Gamma)_{P|P})$ ([22, § 7]). The map has the properties that

$$(\psi_S, \psi_T) = d_\sigma^{-1} \, \mathrm{tr}(ST^*), \quad S, T \in \mathrm{End}(\mathscr{H}_P(\sigma)_\Gamma), \tag{7.1}$$

and

$$E_P(x, \psi_T, \lambda)_{k_1, k_2} = \mathrm{tr}(\mathscr{I}_P(\sigma_\lambda, k_1 x k_2) \, T), \tag{7.2}$$

where d_σ is the formal degree of σ. It is through this map that the c-functions are related to intertwining operators.

The unnormalized intertwining operators

$$J_{P'|P}(\sigma_\lambda) : \mathscr{H}_P(\sigma) \to \mathscr{H}_{P'}(\sigma), \quad P, P' \in \mathscr{P}(M),$$

are defined by integrals over $N_{P'}(F) \cap N_P(F) \backslash N_{P'}(F)$ [11, § 1]. We do not fix invariant measures on these spaces, but we can use the constants

$$\gamma(P) = \int_{N_P(F)} e^{2\rho_{\bar{P}}(H_{\bar{P}}(n))} \, dn$$

as in § 1 to take care of the indeterminacy in the choice of measures. The normalized intertwining operators

$$R_{P'|P}(\sigma_\lambda) : \mathscr{H}_P(\sigma) \to \mathscr{H}_{P'}(\sigma)$$

are better behaved [11, Theorem 2.1]. They are related to the original operators through a product

$$J_{P'|P}(\sigma_\lambda) = r_{P'|P}(\sigma_\lambda) \, R_{P'|P}(\sigma_\lambda), \tag{7.3}$$

in which

$$r_{P'|P}(\sigma_\lambda) = \prod_{\beta \in \Sigma_{P'}^r \cap \Sigma_{\bar{P}}^r} r_\beta(\sigma_\lambda)$$

is a meromorphic scalar valued function composed of normalizing factors for maximal parabolic subgroups. In this paper, we will not need to deal explicitly with the normalized operators, but we will use the factors $r_{P'|P}(\sigma_\lambda)$ to keep track of the singularities of the operators $J_{P'|P}(\sigma_\lambda)$ and the c-functions. Recall that

$$\mu_{P'|P}(\sigma_\lambda) = (J_{P|P'}(\sigma_\lambda)\, J_{P'|P}(\sigma_\lambda))^{-1}$$

is a scalar valued meromorphic function of λ which is analytic on $i\mathfrak{a}_M^*$. The case that $P' = \overline{P}$ is particularly significant. The function $\mu_{\overline{P}|P}(\sigma_\lambda)$ depends inversely on a choice of Haar measure on $N_{\overline{P}}(F) \times N_P(F)$, but

$$\mu(\sigma_\lambda) = \gamma(\overline{P})\,\gamma(P)\,\mu_{\overline{P}|P}(\sigma_\lambda) \tag{7.4}$$

is not only independent of any choice of measures, but is also independent of the choice of $P \in \mathscr{P}(M)$. This is essentially Harish-Chandra's μ-function. It is a consequence of the properties of the operators $R_{P'|P}(\sigma_\lambda)$ that

$$\mu(\sigma_\lambda)^{-1} = (\gamma(\overline{P})\,\gamma(P))^{-1}\, r_{P|\overline{P}}(\sigma_\lambda)\, r_{\overline{P}|P}(\sigma_\lambda). \tag{7.5}$$

The relationship between intertwining operators and c-functions can be described as follows. Suppose that $P_1 \in \mathscr{P}(M_1)$ is another parabolic subgroup, and that $s \in W(\mathfrak{a}_M, \mathfrak{a}_{M_1})$. Then

$$c_{P_1|P}(s, \lambda)\, \psi_T = \gamma(s^{-1}\,\overline{P}_1)^{-1}\, s\psi_{T_1}, \tag{7.6}$$

where

$$T_1 = J_{s^{-1}\overline{P}_1|P}(\sigma_\lambda)\, TJ_{P|s^{-1}P_1}(\sigma_\lambda)$$

and

$$(s\psi)\,(m_1) = w_s\,\psi(w_s^{-1}\, m_1\, w_s)\, w_s^{-1}, \quad m_1 \in M_1(F),$$

in which w_s is a representative of s in K. (See [7, (I.2.4)] and the formulas at the end of [7, § I.3], which were quoted from Harish-Chandra's formula [22, Corollary 18.1] for real groups. The p-adic case follows in a similar way from [19, Theorem 23].) There are similar formulas for the ${}^0c$ and c^0 functions. For example, it will be convenient to link the ${}^0c$-functions with the operators

$$R_{P_1|P}(s, \sigma_\lambda) = w_s\, R_{s^{-1}P_1|P}(\sigma_\lambda)$$

(which depend on the representative w_s of s). The relationship is

$${}^0c_{P_1|P}(s, \lambda)\, \psi_T = \psi_{T'}, \tag{7.7}$$

where

$$T' = R_{P_1|P}(s, \sigma_\lambda)\, TR_{P_1|P}(s, \sigma_\lambda)^{-1}.$$

(See [22, Lemma 18.1], [7, (I.2.15)].) Let $c_{P_1|P}(s, \lambda)_\sigma$ denote the restriction of $c_{P_1|P}(s, \lambda)$ to $\mathscr{A}_\sigma(M, (\tau_\Gamma)_{P|P})$. The formula

$$\mu(\sigma_\lambda)^{-1} = c_{P_1|P}(s, \lambda)_\sigma^*\, c_{P_1|P}(s, \lambda)_\sigma, \quad \lambda \in i\mathfrak{a}_M^*, \tag{7.8}$$

is a straightforward consequence of (7.6) and the definition (7.4) of the μ-function.

We shall state a precise form of the relationship between the Plancherel density $m(\sigma_\lambda)$, and the functions d_σ and $\mu(\sigma_\lambda)$. It can be read directly off Harish-Chandra's explicit Plancherel formula [22, Theorem 27.3], [23, § 16]. We first choose compatible Haar measures on $G(F)$ and $M(F)$, in the sense of § 1. Recall also that in § 1 we fixed dual Haar measures on $i\mathfrak{a}_M^*$ and $A_M(F)$. The measure on $A_M(F)$, together with the Haar measure on $M(F)$, then determines a Haar measure on $M(F)/A_M(F)$. The function $m(\sigma_\lambda)$ depends inversely on both the choice of measure on $i\mathfrak{a}_M^*$ and the choice of measure on $G(F)$. With our conventions, this reduces simply to an inverse dependence on a choice of Haar measure on $M(F)/A_M(F)$. Since d_σ has the same property, the quotient of $m(\sigma_\lambda)$ and d_σ is independent of any choice of Haar measure. The relationship is then just

$$m(\sigma_\lambda) = d_\sigma\, \mu(\sigma_\lambda). \tag{7.9}$$

Harish-Chandra's formula looks slightly more complicated, but this is due to different normalizations of μ-functions ([22, Lemma 13.4], [23, Theorem 4]), different normalizations of Haar measures on $G(F)$ and $M(F)$ ([20, Lemma 7.1], [23, § 5]) and the fact that he takes a sum over $G(F)$-conjugacy classes of groups M instead of the full set $\mathscr{L}$. This accounts for the constants $c(G/A)^{-2}$, $\gamma(G/A)^{-1}$ and $|w(G/A)|^{-1}$ in his formula rather than the quotient $|W_0^M|\,|W_0^G|^{-1}$ in (2.4).

We will need to deal with the weak constant term of an Eisenstein integral along an arbitrary parabolic. For this, it is convenient to work with standard parabolic subgroups. Accordingly, assume for the rest of the section that $P_0 \in \mathscr{P}(M_0)$ is a fixed minimal parabolic subgroup. We assume also that $P \in \mathscr{P}(M)$ contains P_0, and in addition that Q is some other parabolic subgroup which contains P_0. Since P is uniquely determined by its standard Levi component M, we shall often write

$$E(x, \psi, \lambda) = E_P(x, \psi, \lambda).$$

Similarly, if $P_1 \in \mathscr{P}(M_1)$ also contains P_0, and s belongs to $W(\mathfrak{a}_M, \mathfrak{a}_{M_1})$, we can set

$$c(s, \lambda) = c_{P_1|P}(s, \lambda).$$

Finally, if Q happens to contain both P and P_1, and $r \in W(\mathfrak{a}_M, \mathfrak{a}_{M_1})$ leaves $\mathfrak{a}_Q$ pointwise fixed, we can write

$$E^Q(m_Q, \psi, \lambda) = E_{P\cap M_Q}(m_Q, \psi, \lambda), \quad m_Q \in M_Q(F),$$

and

$$c^Q(r, \lambda) = c_{P_1\cap M_Q|P\cap M_Q}(r, \lambda)$$

for the corresponding objects on M_Q. In general, let $W(\mathfrak{a}_P; Q)$ denote the set of elements s in

$$\bigcup_{P_1 \supset P_0} W(\mathfrak{a}_P, \mathfrak{a}_{P_1})$$

such that the space $\mathfrak{a}_{P_1} = s\mathfrak{a}_P$ contains $\mathfrak{a}_Q$, and such that $s^{-1}(\alpha)$ is a root of (P, A_P) for every root $\alpha \in \Delta_{P_1}^Q$. The weak constant term of E along Q is then given by

$$(7.10) \qquad (C^Q E)(m_Q, \psi, \lambda) = \sum_{s \in W(\mathfrak{a}_P;\, Q)} E^Q(m_Q, c^Q(1, s\lambda)^{-1} c(s, \lambda)\psi, s\lambda),$$

for any $m_Q \in M_Q(F)$, $\psi \in \mathscr{A}_2(M, \tau_{P|P})$ and $\lambda \in i\mathfrak{a}_M^*$. This formula is easily established by checking that for any group P_1, with $P_0 \subset P_1 \subset Q$, the cuspidal component of $(C^{P_1} E)(x, \psi, \lambda)$ matches that of the weak constant term along $P_1 \cap M_Q$ of the right hand side of (7.10).

Let us recall Harish-Chandra's quantitative asymptotic relation between E and $C^Q E$. While we are at it, we shall also state the standard estimate for an Eisenstein integral. Let

$$\Xi_G(x) = \int_K e^{\rho_{P_0}(H_{P_0}(kx))}\, dk$$

be the elementary spherical function. We also set

$$N(\lambda) = \begin{cases} 1 + \|\lambda\|, & \text{if F is Archimedean,} \\ 1, & \text{otherwise,} \end{cases}$$

for any point $\lambda \in i\mathfrak{a}_M^*$.

Lemma **7.1.** — *a) Fix a representation* $\sigma \in \Pi_2(M(F))$ *and a positive number* δ. *Then there are positive constants* C, k *and* ε *such that*

$$\|\delta_Q(m)^{1/2} E(m, \psi, \lambda) - (C^Q E)(m, \psi, \lambda)\| \leqslant CN(\lambda)^k \|\psi\| \,\Xi_{M_Q}(m)\, e^{-\varepsilon \|H_0(m)\|},$$

for all $\lambda \in i\mathfrak{a}_M^*$, $\psi \in \mathscr{A}_2(M, \tau_{P|P})$, *and all* $m \in M_0(F)_{P_0}^+$ *such that*

$$\alpha(H_0(m)) \geqslant \delta \|H_0(m)\|, \quad \alpha \in \Delta_{P_0} - \Delta_{P_0}^Q.$$

b) There is a positive constant C *such that*

$$\|E(m, \psi, \lambda)\| \leqslant C \|\psi\| \,\Xi_G(m),$$

for all $\lambda \in i\mathfrak{a}_M^*$, $\psi \in \mathscr{A}_2(M, \tau_{P|P})$ *and* $m \in M_0(F)$.

Proof. — Part *a)* is included in Harish-Chandra's asymptotic estimates for functions in $\mathscr{A}_{\text{temp}}(G, \tau)$. For Archimedean F it follows directly from [21, Lemma 14.5]. For p-adic F there is unfortunately less of Harish-Chandra's work in print, but in this case, *a)* can be deduced from [18, Theorem 7] and the theory [23, § 3] of the weak constant term. Part *b)* follows directly from the definitions of E and Ξ_G, and the fact that the pointwise values of ψ can be bounded in terms of $\|\psi\|$. □

We shall actually apply the asymptotic relation in terms of the function D_{P_0} discussed in § 1.

Corollary **7.2.** — *Given* $\delta > 0$ *we can choose positive constants* C, k *and* ε *such that*

$$\| D_{P_0}(m)^{1/2} E(m, \psi, \lambda) - D_{P_0 \cap M_Q}(m)^{1/2}(C^Q E)(m, \psi, \lambda)\|$$

is bounded by

$$CN(\lambda)^k \|\psi\| e^{-\varepsilon \| H_0(m) \|},$$

for all λ, ψ, *and* m *as in part a) of the lemma.*

Proof. — Multiply the estimate in Lemma 7.1 *a)* by $D_{P_0 \cap M_Q}(m)^{1/2}$. By Lemma 1.1 and Lemma 7.1 *b)*,

$$\| D_{P_0}(m)^{1/2} E(m, \psi, \lambda) - \delta_Q(m)^{1/2} D_{P_0 \cap M_Q}(m)^{1/2} E(m, \psi, \lambda)\|$$

is bounded by

$$C_1 \delta_{P_0}(m)^{1/2} e^{-\varepsilon_1 \| H_0(m)\|} \|\psi\| \Xi_G(m),$$

for positive constants C_1 and ε_1. We can also apply Harish-Chandra's estimate

$$\delta_{P_0}(m)^{1/2} \Xi_G(m) \leqslant C_2(1 + \| H_0(m)\|)^{d_2} \tag{7.11}$$

for the elementary spherical function ([16, Theorem 3], [19, Theorem 25]). The original function can therefore be bounded by an expression of the form

$$C \|\psi\| e^{-\varepsilon \| H_0(m)\|}.$$

Moreover, applying Corollary 1.2 and (7.11) to M_Q, we obtain an estimate

$$D_{P_0 \cap M_Q}(m)^{1/2} \Xi_{M_Q}(m) \leqslant C' \delta_{P_0 \cap M_Q}(m)^{1/2} \Xi_{M_Q}(m) \leqslant C' C'_2(1 + \| H_0(m)\|)^{d'_2}. \tag{7.12}$$

The corollary follows. □

It will be necessary to control the weak constant term (7.10) in future induction arguments. Consider an element $s \in W(\mathfrak{a}_P; Q)$. It follows from (7.8) that the linear map

$$\mu(\sigma_\lambda)^{1/2} c(s, \lambda)_\sigma$$

is unitary. Similarly, if μ^Q denotes the μ-function relative to M_Q instead of G, the linear map

$$\mu^Q(s\sigma_\lambda)^{1/2} c^Q(1, s\lambda)_{s\sigma}$$

is also unitary. Consequently

$$\| c^Q(1, s\lambda)^{-1} c(s, \lambda) \psi \| = \mu^Q(s\sigma_\lambda)^{1/2} \mu(\sigma_\lambda)^{-1/2} \|\psi\|, \tag{7.13}$$

for any $\psi \in \mathscr{A}_\sigma(M, \tau_{P|P})$. As a function of λ, the μ-function $\mu^Q(s\sigma_\lambda)$ is analytic and of polynomial growth [22, Theorem 25.1], [23, § 6]. Therefore, the left hand side of (7.13) is bounded by

$$C\mu(\sigma_\lambda)^{-1/2} N(\lambda)^k \|\psi\|,$$

for constants C and k. Applying this to (7.10) in conjunction with Lemma 7.1 *b)*, and taking into account the estimate (7.12), we obtain constants C, k and d such that

$$(7.14) \qquad D_{P_0 \cap M_Q}(m)^{1/2} \| (C^Q E)(m, \psi, \lambda) \| \leqslant C \mu(\sigma_\lambda)^{-1/2} N(\lambda)^k \| \psi \| (1 + \| H_0(m) \|)^d,$$

for all $\lambda \in i\mathfrak{a}_M^*$, $\psi \in \mathscr{A}_\sigma(M, \tau_{P|P})$ and $m \in M_0(F)$.

Actually, $(C^Q E)(m, \psi, \lambda)$ is an analytic function of $\lambda \in i\mathfrak{a}_M^*$, and one could remove the factor $\mu(\sigma_\lambda)^{-1/2}$ from (7.14) with some further argument. However, we shall only require the estimate in its present form.

8. Inner products

For the next two sections M and M′ will be fixed Levi subgroups in $\mathscr{L}$. It is convenient to assume that they are both standard with respect to a fixed minimal parabolic subgroup $P_0 \in \mathscr{P}(M_0)$. In other words, there are (uniquely determined) parabolic subgroups $P \in \mathscr{P}(M)$ and $P' \in \mathscr{P}(M')$ which contain P_0. We shall also fix a two sided unitary representation τ of K on a finite dimensional Hilbert space. We can then form the Eisenstein integrals

$$E(x, \psi, \lambda) = E_P(x, \psi, \lambda), \quad \lambda \in i\mathfrak{a}_M^*, \ \psi \in \mathscr{A}_2(M, \tau_{P|P}),$$

and

$$E(x, \psi', \lambda') = E_{P'}(x, \psi', \lambda'), \quad \lambda' \in i\mathfrak{a}_{M'}^*, \ \psi' \in \mathscr{A}_2(M', \tau_{P'|P'}),$$

as in § 7. These functions are not square integrable in x. However, if we multiply them each by the characteristic function $u(x, T)$, we can form their inner product over any of the sets

$$G(F)^Z = \{ x \in G(F) : H_G(x) = Z \}, \quad Z \in \mathfrak{a}_{G,F}.$$

Our goal is to establish an asymptotic formula for this inner product. For real groups of rank 1, such formulas have been proved by techniques from the spectral theory of ordinary second order differential operators. Waldspurger [31] has used completely different methods to establish a truncated inner product formula for p-adic spherical functions on $GL(n)$. We shall use Waldspurger's techniques to prove a general result which can be regarded as a local analogue of the inner product formula [3], [5] for truncated Eisenstein series.

Define

$$(8.1) \qquad \Omega_{P_0}^T(\lambda', \lambda, \psi', \psi)^Z = \int_{G(F)^Z} (E(x, \psi', \lambda'), E(x, \psi, \lambda))\, u(x, T)\, dx.$$

It is clear that $\Omega_{P_0}^T(\lambda', \lambda, \psi', \psi)^Z$ extends to an entire function of $(\lambda', -\bar{\lambda})$ in $\mathfrak{a}_{M',\mathbf{C}}^* \times \mathfrak{a}_{M,\mathbf{C}}^*$. However, we shall be mainly concerned with its values for imaginary (λ', λ). In particular, it is only for these values that the asymptotic formula will hold. We shall state the formula in this section, and prove it in the next one.

In order to state the asymptotic formula, we must make some preliminary observations. These remarks are really only of interest if F is p-adic, but there is no need to assume this explicitly. Our discussion at this point pertains only to a general Levi subgroup $M \in \mathscr{L}$, and does not rely on the minimal parabolic subgroup P_0. We shall describe some inner products on $\mathscr{A}_2(M, \tau_M)$.

Let Ψ' and Ψ be fixed functions in $\mathscr{A}_2(M, \tau_M)$. For any X in $\mathfrak{a}_{M,F}$, we can certainly take the inner product

$$(\Psi', \Psi)^X = \int_{M(F)^X} (\Psi'(m), \Psi(m))\, dm.$$

However, it is useful to introduce another bilinear form that depends on the point T and a group $P \in \mathscr{P}(M)$. For a given point $\Lambda \in \mathfrak{a}^*_{M,C}$, the function

$$\Psi'_\Lambda(m) = \Psi'(m)\, e^{\Lambda(H_M(m))}, \quad m \in M(F),$$

also belongs to $\mathscr{A}_2(M, \tau_M)$. We shall assume for the moment that the real part of $\Lambda(\alpha^\vee)$ is positive for every root $\alpha \in \Delta_P$. Then φ^Λ_P equals φ_P, the characteristic function of

$$\{ H \in \mathfrak{a}_M : \varpi_\alpha(H) \leqslant 0,\ \alpha \in \Delta_P \}.$$

The bilinear form is defined by

$$r^T_P(\Psi'_\Lambda, \Psi)^Z = \int_{M(F) \cap G(F)^Z} (\Psi'_\Lambda(m), \Psi(m))\, \varphi_P(H_M(m) - T_P)\, dm.$$

Equivalently, we have

$$r^T_P(\Psi'_\Lambda, \Psi)^Z = \int_{\mathfrak{a}^Z_{M,F}} (\Psi', \Psi)^X\, \varphi_P(X - T_P)\, e^{\Lambda(X)}\, dX, \tag{8.2}$$

where

$$\mathfrak{a}^Z_{M,F} = \{ X \in \mathfrak{a}_{M,F} : h_{MG}(X) = Z \}.$$

It is clear from our condition on Λ that the integral in (8.2) converges absolutely. Moreover, it is not hard to show that $r^T_P(\Psi'_\Lambda, \Psi)^Z$ has analytic continuation as a meromorphic function of $\Lambda \in \mathfrak{a}^*_{M,C}$. We shall prove this fact in detail, in order to introduce some auxiliary notions we will need later.

Observe that

$$r^T_P(\Psi'_{\Lambda+\zeta}, \Psi)^Z = e^{\zeta(Z)}\, r^T_P(\Psi'_\Lambda, \Psi)^Z,$$

for any point ζ in $\mathfrak{a}^*_{G,C}$. It is therefore enough to prove the analytic continuation if Λ is replaced by any $\Lambda + \zeta$. In particular, we can assume that Λ belongs to $(\mathfrak{a}^*_{M,C})^G$. There is also no loss in generality in assuming that

$$(\Psi'(mz), \Psi(mz)) = (\Psi'(m), \Psi(m)), \quad m \in M(F),\ z \in A_G(F). \tag{8.3}$$

Taken together these two assumptions imply that

$$r^T_P(\Psi'_\Lambda, \Psi)^{Z+\tilde{Z}} = r^T_P(\Psi'_\Lambda, \Psi)^Z,$$

for any point $\tilde{Z}$ in $\tilde{\mathfrak{a}}_{G,F}$. In particular, the expression

$$r_P^T(\Psi', \Psi)_\Lambda = \sum_{Z \in \mathfrak{a}_{G,F}/\tilde{\mathfrak{a}}_{G,F}} r_P^T(\Psi'_\Lambda, \Psi)^Z$$

is well defined. If ζ belongs to $\tilde{\mathfrak{a}}_{G,F}^\vee$, the pair (Ψ'_ζ, Ψ) also satisfies (8.3), and we can form the function $r_P^T(\Psi'_\zeta, \Psi)_\Lambda$. We obtain an inversion formula

$$r_P^T(\Psi'_\Lambda, \Psi)^Z = |\,\mathfrak{a}_{G,F}/\tilde{\mathfrak{a}}_{G,F}\,|^{-1} \sum_\zeta r_P^T(\Psi'_\zeta, \Psi)_\Lambda\, e^{-\zeta(Z)},$$

where the sum is taken over ζ in $\tilde{\mathfrak{a}}_{G,F}^\vee/\mathfrak{a}_{G,F}^\vee$. It is therefore enough to show that $r_P^T(\Psi', \Psi)_\Lambda$ extends to a meromorphic function of Λ in $(\mathfrak{a}_{M,C}^*)^G$.

Obviously $\mathscr{A}_2(M, \tau_M)$ is a direct sum of eigenspaces under the action of the compact abelian group $A_M(F) \cap M(F)^1$. Since different eigenspaces are orthogonal under $r_P^T(\cdot, \cdot)_\Lambda$, we may assume that Ψ' and Ψ lie in the same eigenspace. We can also assume that Ψ' and Ψ are eigenvectors under the action of $A_M(F)$. This means that

$$(8.4) \qquad (\Psi'(ma), \Psi(ma)) = (\Psi'(m), \Psi(m))\, e^{\mu(H_M(a))}, \quad m \in M(F),\ a \in A_M(F),$$

for some point $\mu \in i\mathfrak{a}_M^*$. We can then form the inner product

$$\begin{aligned}(\Psi', \Psi)_\mu &= (\Psi'_{-\mu}, \Psi)\\ &= \int_{M(F)/A_M(F)} (\Psi'(m), \Psi(m))\, e^{-\mu(H_M(m))}\, dm\\ &= \sum_{X \in \mathfrak{a}_{M,F}/\tilde{\mathfrak{a}}_{M,F}} (\Psi', \Psi)^X e^{-\mu(X)}.\end{aligned}$$

The point μ is not unique. However, (8.3) does tell us that the restriction of μ to $\mathfrak{a}_G$ lies in $\tilde{\mathfrak{a}}_{G,F}^\vee$. It follows from Lemma 1.3 that μ can be chosen to lie in $(i\mathfrak{a}_M^*)^G$.

The expression for $r_P^T(\Psi', \Psi)_\Lambda$ can be evaluated explicitly. Assume for the moment that F is a p-adic field. It follows from (8.2) that

$$(8.5) \qquad r_P^T(\Psi', \Psi)_\Lambda = \sum_{X \in \mathfrak{a}_{M,F}/\tilde{\mathfrak{a}}_{G,F}} (\Psi', \Psi)^X \varphi_P(X - T_P)\, e^{\Lambda(X)}.$$

We fix a number $\ell = (k')^{-1}$ for all time, where k' is a suitably large positive integer. We can then form the lattice

$$\mathscr{L}_{M,\ell} = \{ \sum_{\alpha \in \Delta_P} n_\alpha \mu_{\alpha,\ell} : n_\alpha \in \mathbf{Z} \}$$

in $\mathfrak{a}_M/\mathfrak{a}_G$, as in § 6. Since k' is large, we may assume that $\mathscr{L}_{M,\ell}$ contains both $\mathscr{L}_M = \mathfrak{a}_{M,F} + \mathfrak{a}_G/\mathfrak{a}_G$ and $\tilde{\mathscr{L}}_M = \tilde{\mathfrak{a}}_{M,F} + \mathfrak{a}_G/\mathfrak{a}_G$ as sublattices. The pairing

$$e^{\nu(X)}, \quad \nu \in \tilde{\mathscr{L}}_M^\vee/\mathscr{L}_{M,\ell}^\vee,\ X \in \mathscr{L}_{M,\ell}/\tilde{\mathscr{L}}_M,$$

identifies $\tilde{\mathscr{L}}_M^\vee/\mathscr{L}_{M,\ell}^\vee$ with the dual group of $\mathscr{L}_{M,\ell}/\tilde{\mathscr{L}}_M$. Choose a point $\mu \in (i\mathfrak{a}_M^*)^G$ such that (8.4) holds. Since

$$(\Psi', \Psi)^{X+\tilde{X}} = (\Psi', \Psi)^X e^{\mu(\tilde{X})}$$

for any $\tilde{X} \in \tilde{\mathscr{L}}_M$, we can write

$$r_P^T(\Psi', \Psi)_\Lambda = \sum_{X \in \mathfrak{a}_{M,F}/\tilde{\mathfrak{a}}_{M,F}} \sum_{\tilde{X} \in \tilde{\mathscr{L}}_M} (\Psi', \Psi)^X \varphi_P(X + \tilde{X} - T_P)\, e^{\Lambda(X)}\, e^{(\Lambda+\mu)(\tilde{X})}$$

$$= \sum_X (\Psi', \Psi)^X e^{\Lambda(X)} \,|\, \tilde{\mathscr{L}}_M^\vee / \mathscr{L}_{M,\ell}^\vee \,|^{-1} \sum_{\nu \in \tilde{\mathscr{L}}_M^\vee/\mathscr{L}_{M,\ell}^\vee} \sum_{\tilde{X} \in \mathscr{L}_{M,\ell}} \varphi_P(X + \tilde{X} - T_P)\, e^{(\Lambda+\mu+\nu)(\tilde{X})},$$

by applying Fourier inversion to the finite abelian group $\mathscr{L}_{M,\ell}/\tilde{\mathscr{L}}_M$. The sum over $\tilde{X}$ is easily expressed as a multiple geometric series as in § 6. Using the fact that $\Lambda + \mu + \nu$ belongs to $(\mathfrak{a}_{M,\mathbf{C}}^*)^G$, we write

$$\sum_{\tilde{X} \in \mathscr{L}_{M,\ell}} \varphi_P(X + \tilde{X} - T_P)\, e^{(\Lambda+\mu+\nu)(\tilde{X})}$$

$$= \sum_{\tilde{X} \in \mathscr{L}_{M,\ell}} \varphi_P(\tilde{X})\, e^{(\Lambda+\mu+\nu)(\tilde{X})}\, e^{(\Lambda+\mu+\nu)(T_P - X)}$$

$$= \operatorname{vol}(\mathfrak{a}_M^G/\mathscr{L}_{M,\ell})^{-1}\, \theta_{P,\ell}(\Lambda + \mu + \nu)^{-1}\, e^{(\Lambda+\mu+\nu)(T_P)}\, e^{-\Lambda(X)}\, e^{-(\mu+\nu)(X)},$$

where

$$\theta_{P,\ell}(\lambda) = \operatorname{vol}(\mathfrak{a}_M^G/\mathscr{L}_{M,\ell})^{-1} \prod_{\alpha \in \Delta_P} (1 - e^{-\lambda(\mu_{\alpha,\ell})}), \quad \lambda \in \mathfrak{a}_{M,\mathbf{C}}^*.$$

Since

$$|\, \tilde{\mathscr{L}}_M^\vee/\mathscr{L}_{M,\ell}^\vee \,|^{-1} \operatorname{vol}(\mathfrak{a}_M^G/\mathscr{L}_{M,\ell})^{-1} = \operatorname{vol}(\mathfrak{a}_M^G/\tilde{\mathscr{L}}_M)^{-1} = 1,$$

and

$$\sum_{X \in \mathfrak{a}_{M,F}/\tilde{\mathfrak{a}}_{M,F}} e^{-(\mu+\nu)(X)} (\Psi', \Psi)^X = (\Psi', \Psi)_{\mu+\nu},$$

we obtain

$$(8.6) \qquad r_P^T(\Psi', \Psi)_\Lambda = \sum_{\nu \in \tilde{\mathscr{L}}_M^\vee/\mathscr{L}_{M,\ell}^\vee} (\Psi', \Psi)_{\mu+\nu}\, e^{(\Lambda+\mu+\nu)(T_P)}\, \theta_{P,\ell}(\Lambda + \mu + \nu)^{-1}.$$

If F is Archimedean, the argument is similar but simpler. In this case $\mathscr{L}_M$, $\tilde{\mathscr{L}}_M$ and $\mathscr{L}_{M,\ell}$ are all just equal to $\mathfrak{a}_M/\mathfrak{a}_G$. If we agree to set $\theta_{P,\ell} = \theta_P$ for Archimedean F, as in § 6, the formula (8.6) will remain in force.

Before going on, we observe that (8.6) simplifies slightly if $\Psi' \in \mathscr{A}_{\pi'}(M, \tau_M)$ and $\Psi \in \mathscr{A}_\pi(M, \tau_M)$, for fixed representations π' and π in $\Pi_2(M(F))$. Let $\mathscr{E}^G(\pi', \pi)$ denote the set of points $\nu \in (i\mathfrak{a}_M^*)^G$ such that π' is equivalent to π_ν. Then the group

$$\mathscr{L}_\pi^\vee = \{ v \in \tilde{\mathscr{L}}_M^\vee : \pi_\nu \cong \pi \},$$

which lies between $\mathscr{L}_M^\vee$ and $\tilde{\mathscr{L}}_M^\vee$, acts simply transitively on $\mathscr{E}^G(\pi', \pi)$. Observe that the inner product $(\Psi', \Psi)_{\mu+\nu}$ in (8.6) vanishes unless $\mu + \nu$ belongs to $\mathscr{E}^G(\pi', \pi)$. Rewriting $\mu + \nu$ as ν, we obtain

$$(8.7) \qquad r_P^T(\Psi', \Psi)_\Lambda = \sum_{\nu \in \mathscr{E}^G(\pi', \pi)/\mathscr{L}_{M,\ell}^\vee} (\Psi', \Psi)_\nu\, e^{(\Lambda+\nu)(T_P)}\, \theta_{P,\ell}(\Lambda + \nu)^{-1},$$

in the special case that $\Psi' \in \mathscr{A}_{\pi'}(M, \tau_M)$ and $\Psi \in \mathscr{A}_\pi(M, \tau_M)$.

The formula (8.6) implies that $r_P^T(\Psi', \Psi)_\Lambda$ extends to a meromorphic function of Λ in $(\mathfrak{a}_{M,\mathbf{C}}^*)^G$. This implies the assertion we set out to check, namely that each $r_T^P(\Psi'_\Lambda, \Psi)^Z$ can be analytically continued to a meromorphic function of $\Lambda \in \mathfrak{a}_{M,\mathbf{C}}^*$. The assertion is valid for arbitrary functions Ψ' and Ψ in $\mathscr{A}_2(M, \tau_M)$. In fact it is convenient to extend the various bilinear forms above to the full space $\mathscr{A}(M, \tau_M)$ by defining them to be zero on the complement of $\mathscr{A}_2(M, \tau_M)$. (The existence of a vector space complement of $\mathscr{A}_2(M, \tau_M)$ in $\mathscr{A}(M, \tau_M)$ is a consequence of Harish-Chandra's theory of the constant term.) The analytic continuation remains in force. For general Ψ' and Ψ we shall also write

$$r_P^T(\Psi', \Psi) = r_P^T(\Psi', \Psi)_0$$

and

$$r_P^T(\Psi', \Psi)^Z = r_P^T(\Psi'_0, \Psi)^Z$$

if the functions are analytic at $\Lambda = 0$.

We now return to our discussion at the beginning of the section. Then $P' \in \mathscr{P}(M')$ and $P \in \mathscr{P}(M)$ are standard parabolic subgroups, $\psi' \in \mathscr{A}_2(M', \tau_{P'|P'})$ and $\psi \in \mathscr{A}_2(M, \tau_{P|P})$ are fixed functions, and $E(\psi', \lambda')$, $\lambda' \in i\mathfrak{a}_{M'}^*$, and $E(\psi, \lambda)$, $\lambda \in i\mathfrak{a}_M^*$, are the corresponding Eisenstein integrals. For each standard parabolic subgroup $P_1 \in \mathscr{P}(M_1)$, we can form the weak constant terms $(C^{P_1} E)(\psi', \lambda')$ and $(C^{P_1} E)(\psi, \lambda)$. We obtain functions in $\mathscr{A}_{\text{temp}}(M_1, \tau_{P_1|\bar{P}_1})$. Define

$$\omega_{P_0}^T(\lambda', \lambda, \psi', \psi)^Z = \sum_{P_1 \supset P_0} r_{P_1}^T((C^{P_1} E)(\psi', \lambda'), (C^{P_1} E)(\psi, \lambda))^Z. \tag{8.8}$$

This function is to be our asymptotic approximation of $\Omega_{P_0}^T(\lambda', \lambda, \psi', \psi)^Z$. Observe that a summand corresponding to P_1 will be nonzero only if the cuspidal components of both $(C^{P_1} E)(\psi', \lambda')$ and $(C^{P_1} E)(\psi, \lambda)$ are nonzero. This means that P_1 is associated to both P and P', and in particular, that P is associated to P'. We should also point out that the summands on the right are only defined for λ' and λ in general position. However, the sum does extend to a meromorphic function of $(\lambda', -\bar{\lambda})$ in $\mathfrak{a}_{M',\mathbf{C}}^* \times \mathfrak{a}_{M,\mathbf{C}}^*$.

We want to establish an asymptotic relationship which is uniform in (λ', λ). It can be shown that $\omega_{P_0}^T(\lambda', \lambda, \psi', \psi)^Z$ is actually an analytic function of (λ', λ) in $i\mathfrak{a}_{M'}^* \times i\mathfrak{a}_M^*$, and one could probably establish an asymptotic formula for all such points. However, to avoid burdening the reader with the required extra generality [21, § 8], we shall be content to prove a slightly weaker result that allows for the singularities of the c-functions. This is in fact quite natural. As in Harish-Chandra's estimates for wave packets, we will ultimately rely on the Plancherel density to cancel the singularities.

To state the asymptotic formula, it is convenient to fix representations $\sigma' \in \Pi_2(M'(F))$ and $\sigma \in \Pi_2(M(F))$, and to assume that the vectors ψ' and ψ lie in $\mathscr{A}_{\sigma'}(M', \tau_{P'|P'})$ and $\mathscr{A}_\sigma(M, \tau_{P|P})$ respectively. This of course entails no loss of generality. We may as well also assume that the truncation parameter $T \in \mathfrak{a}_{M_0,F}$ lies in the chamber $\mathfrak{a}_{P_0}^+$ corresponding to P_0. Recalling the functions $\mu(\sigma_\lambda)$ and $N(\lambda)$ from § 7, we introduce the quantity

$$N_k(\lambda', \lambda, \psi', \psi) = \mu(\sigma'_{\lambda'})^{-1/2} \mu(\sigma_\lambda)^{-1/2} N(\lambda')^k N(\lambda)^k \| \psi' \| \| \psi \|, \tag{8.9}$$

with k any positive integer, in order to describe the error in the asymptotic estimate. Let us also write

$$\begin{aligned}\mathscr{F}(\sigma', \sigma) &= \{(\lambda' \lambda) \in i\mathfrak{a}_{M'}^* \times i\mathfrak{a}_M^* : \mu(\sigma'_{\lambda'})\, \mu(\sigma_\lambda) \neq 0\} \\ &= \{(\lambda', \lambda) : N_k(\lambda', \lambda, \psi', \psi) \neq 0\}.\end{aligned}$$

The asymptotic inner product formula can then be stated as follows.

Theorem **8.1.** — *Suppose that* $\delta > 0$. *Then there are positive constants* C, k *and* ε *such that*

$$|\,\Omega_{P_0}^T(\lambda', \lambda, \psi', \psi)^Z - \omega_{P_0}^T(\lambda', \lambda, \psi', \psi)^Z\,| \leqslant CN_k(\lambda', \lambda, \psi', \psi)\, e^{-\varepsilon \|T\|},$$

for all $(\lambda', \lambda) \in \mathscr{F}(\sigma', \sigma)$, $Z \in \mathfrak{a}_{G,F}$, $\psi' \in \mathscr{A}_{\sigma'}(M', \tau_{P'|P'})$ *and* $\psi \in \mathscr{A}_\sigma(M, \tau_{P|P})$, *and all* $T \in \mathfrak{a}_{M_0,F} \cap \mathfrak{a}_{P_0}^+$ *with* $d(T) \geqslant \delta \,\|\, T\,\|$.

We shall prove Theorem 8.1 in the next section. Notice that the estimate of the theorem implies that the meromorphic function $\omega_{P_0}^T(\lambda', \lambda, \psi', \psi)^Z$ is analytic for all (λ', λ) in $\mathscr{F}(\sigma', \sigma)$. This follows directly from the continuity of $N_k(\lambda', \lambda, \psi', \psi)$ on $\mathscr{F}(\sigma', \sigma)$ and the fact that $\Omega_{P_0}^T(\lambda', \lambda, \psi', \psi)^Z$ is analytic everywhere.

Suppose that the central characters of σ' and σ coincide on $A_G(F)$. Then (8.3) will hold, and if $\lambda' - \lambda$ belongs to the space $(i\mathfrak{a}_M^*)^G$, we can define functions

$$\Omega_{P_0}^T(\lambda', \lambda, \psi', \psi) = \int_{A_G(F)\backslash G(F)} (E(x, \psi', \lambda'), E(x, \psi, \lambda))\, u(x, T)\, dx$$

and

$$\omega_{P_0}^T(\lambda', \lambda, \psi', \psi) = \sum_{P_1 \supset P_0} r_{P_1}^T((C^{P_1} E)(\psi', \lambda'), (C^{P_1} E)(\psi, \lambda)).$$

The following corollary is an immediate consequence of the theorem.

Corollary **8.2.** — *Given* $\delta > 0$, *we can choose the constants of the theorem so that*

$$|\,\Omega_{P_0}^T(\lambda', \lambda, \psi', \psi) - \omega_{P_0}^T(\lambda', \lambda, \psi', \psi)\,| \leqslant CN_k(\lambda', \lambda, \psi', \psi)\, e^{-\varepsilon \|T\|},$$

for all (λ', λ), ψ', ψ *and* T *as in the theorem, with the additional condition that* $\lambda' - \lambda$ *belongs to* $(i\mathfrak{a}_M^*)^G$. □

We conclude this section with some comments on the asymptotic expression $\omega_{P_0}^T(\lambda', \lambda, \psi', \psi)$ in the corollary. Substituting the c-functions into the constant terms, we first write $\omega_{P_0}^T(\lambda', \lambda, \psi', \psi)$ as

$$\sum_{P_1 \supset P_0} \sum_{s' \in W(\mathfrak{a}_{P'}, \mathfrak{a}_{P_1})} \sum_{s \in W(\mathfrak{a}_P, \mathfrak{a}_{P_1})} r_{P_1}^T(\Psi'_{s'}, \Psi_s),$$

where

$$\Psi'_{s'}(m_1) = (c(s', \lambda')\, \psi')\,(m_1)\, e^{(s'\lambda')(H_{M_1}(m_1))}$$

and

$$\Psi_s(m_1) = (c(s, \lambda)\, \psi)\,(m_1)\, e^{(s\lambda)(H_M(m_1))}.$$

We are assuming that the representations $\sigma' \in \Pi_2(M'(F))$ and $\sigma \in \Pi_2(M(F))$ are fixed and that $\psi' \in \mathscr{A}_{\sigma'}(M', \tau_{P'|P'})$ and $\psi \in \mathscr{A}_\sigma(M, \tau_{P|P})$. Consequently, the functions $\Psi'_{s'}$

and Ψ_s belong to $\mathscr{A}_{s'\sigma'_{\lambda'}}(M_1, \tau_{P_1|P_1})$ and $\mathscr{A}_{s\sigma_\lambda}(M_1, \tau_{P_1|P_1})$ respectively. Applying the formula (8.7) (with P, π', π replaced by P_1, $s'\sigma'_{\lambda'}$, $s\sigma_\lambda$, and with $\Lambda = 0$), we obtain a more explicit expression for $\omega^T_{P_0}(\lambda', \lambda, \psi', \psi)$. The result is

$$\omega^T_{P_0}(\lambda', \lambda, \psi', \psi) = \sum_{P_1 \supset P_0} \sum_{s', s} \sum_{\Lambda_1} (\Psi'_{s'}, \Psi_s)_{\Lambda_1} e^{\Lambda_1(T_{P_1})} \theta_{P_1, t}(\Lambda_1)^{-1}, \tag{8.10}$$

where the sums are taken over $s' \in W(\mathfrak{a}_{M'}, \mathfrak{a}_{M_1})$, $s \in W(\mathfrak{a}_M, \mathfrak{a}_{M_1})$ and $\Lambda_1 \in \mathscr{E}^G(s'\sigma'_{\lambda'}, s\sigma_\lambda)/\mathscr{L}^\vee_{M,t}$. Observe that

$$\mathscr{E}^G(s'\sigma'_{\lambda'}, s\sigma_\lambda) = s'\lambda' - s\lambda + \mathscr{E}^G(s'\sigma', s\sigma),$$

and if $\Lambda_1 = s'\lambda' - s\lambda + \nu_1$, for $\nu_1 \in \mathscr{E}^G(s'\sigma', s\sigma)$, then

$$(\Psi'_{s'}, \Psi_s)_{\Lambda_1} = (c(s', \lambda')\psi', c(s, \lambda)\psi)_{\nu_1}.$$

With this substitution, the right hand side of (8.10) looks more like the asymptotic formula [5, p. 36] for the inner product of truncated Eisenstein series.

If $F = \mathbf{R}$, much of this discussion is superfluous. For $M(\mathbf{R})$ is the direct product of $M(\mathbf{R})^1$ with $A_M(\mathbf{R})^0$, and we can take σ' and σ to be representations of $M'(\mathbf{R})/A_{M'}(\mathbf{R})^0$ and $M(\mathbf{R})/A_M(\mathbf{R})^0$ respectively. The set $\mathscr{E}^G(s'\sigma'_{\lambda'}, s\sigma_\lambda)$ is either empty or contains the one point $s'\lambda' - s\lambda$. The formula (8.10) becomes

$$\omega^T_{P_0}(\lambda', \lambda, \psi', \psi) = \sum_{P_1 \supset P_0} \sum_{s', s} (c(s', \lambda')\psi', c(s, \lambda)\psi) e^{(s'\lambda' - s\lambda)(T_{P_1})} \theta_{P_1}(s'\lambda' - s\lambda)^{-1}.$$

This is an exact analogue of the asymptotic formula for Eisenstein series.

9. Proof of the inner product formula

The purpose of this section is to prove Theorem 8.1. In the special case that G is a torus, notice that

$$\Omega^T_{P_0}(\lambda', \lambda, \psi', \psi)^Z = (\psi', \psi)^Z = \omega^T_{P_0}(\lambda', \lambda, \psi', \psi)^Z,$$

so there is nothing to prove. In general, we shall assume inductively that the theorem holds if G is replaced by any proper Levi subgroup.

The key step in the proof of the theorem will be to show that the difference

$$\Delta^T_{P_0}(\lambda', \lambda, \psi', \psi)^Z = \Omega^T_{P_0}(\lambda', \lambda, \psi', \psi)^Z - \omega^T_{P_0}(\lambda', \lambda, \psi', \psi)^Z \tag{9.1}$$

has a limit as T approaches infinity. This idea is due to Waldspurger, who worked with p-adic spherical functions on GL(n) [31, Proposition II.4]. The existence of the limit will be an easy consequence of the following weaker version of Theorem 8.1.

Lemma **9.1.** — *For any positive numbers* δ *and* r, *we can choose positive constants* C, k *and* ε *such that*

$$|\Delta^{T+S}_{P_0}(\lambda', \lambda, \psi', \psi)^Z - \Delta^T_{P_0}(\lambda', \lambda, \psi', \psi)^Z| \leqslant CN_k(\lambda', \lambda, \psi', \psi) e^{-\varepsilon\|T\|},$$

for all (λ', λ), Z, ψ', ψ *and* T *as in the statement of Theorem* 8.1, *and all points* $S \in \mathfrak{a}_{M_0, F} \cap \mathfrak{a}^+_{P_0}$ *with* $\|S\| \leqslant r\|T\|$.

Proof. — Assume that T and S satisfy the conditions of the lemma. For any point

$$x = k_1 m k_2, \quad k_1, k_2 \in \mathrm{K}, \ m \in \mathrm{M}_0(\mathrm{F})^+_{\mathrm{P}_0},$$

in G(F), we have

$$u(x, \mathrm{T}+\mathrm{S}) = \bar{\tau}_{\mathrm{P}_0}(\mathrm{H}_0(m), \mathrm{T}+\mathrm{S})$$

by Lemma 3.1. It follows from the integration formula (1.3) that $\Omega^{\mathrm{T}+\mathrm{S}}_{\mathrm{P}_0}(\lambda', \lambda, \psi', \psi)^{\mathrm{Z}}$ equals

$$\int_{\mathrm{M}_0(\mathrm{F}) \cap \mathrm{G}(\mathrm{F})^{\mathrm{Z}}} \mathrm{D}_{\mathrm{P}_0}(m)\, \bar{\tau}_{\mathrm{P}_0}(\mathrm{H}_0(m), \mathrm{T}+\mathrm{S})\, (\mathrm{E}(m, \psi', \lambda'), \mathrm{E}(m, \psi, \lambda))\, dm. \tag{9.2}$$

Into this expression we substitute the expansion (3.14) for $\bar{\tau}_{\mathrm{P}_0}(\cdot, \mathrm{T}+\mathrm{S})$. We obtain the sum over parabolic subgroups Q which contain P_0, and the integral over m in $\mathrm{M}_0(\mathrm{F}) \cap \mathrm{G}(\mathrm{F})^{\mathrm{Z}}$, of the product of

$$\mathrm{D}_{\mathrm{P}_0}(m)\, (\mathrm{E}(m, \psi', \lambda'), \mathrm{E}(m, \psi, \lambda)) \tag{9.3}$$

with

$$\bar{\tau}^{\mathrm{Q}}_{\mathrm{P}_0}(\mathrm{H}_0(m), \mathrm{T})\, \tau_{\mathrm{Q}}(\mathrm{H}_{\mathrm{Q}}(m) - \mathrm{T}_{\mathrm{Q}}, \mathrm{S}_{\mathrm{Q}}). \tag{9.4}$$

Fix $\mathrm{Q} \supset \mathrm{P}_0$ for the moment, and consider points $m \in \mathrm{M}_0(\mathrm{F}) \cap \mathrm{G}(\mathrm{F})^{\mathrm{Z}}$ such that the function (9.4) does not vanish. For any such m, we can write

$$\mathrm{H}_0(m) = \mathrm{T} - \sum_{\beta \in \Delta^{\mathrm{Q}}_{\mathrm{P}_0}} c_\beta \beta^\vee + \sum_{\varpi \in \hat{\Delta}_{\mathrm{Q}}} d_\varpi \varpi^\vee + \mathrm{Z},$$

where $\{\beta^\vee\}$ are "co-roots", $\{\varpi^\vee\}$ are "co-weights", and $\{c_\beta\}$ and $\{d_\varpi\}$ are non-negative real numbers. This follows directly from the definition of the two functions in the product (9.4). Let α be any root in $\Delta_{\mathrm{P}_0} - \Delta^{\mathrm{Q}}_{\mathrm{P}_0}$. Since $\alpha(\beta^\vee) \leqslant 0$ for each $\beta \in \Delta^{\mathrm{Q}}_{\mathrm{P}_0}$, we have

$$\alpha(\mathrm{H}_0(m)) \geqslant \alpha(\mathrm{T}) + d_{\varpi_\alpha} \geqslant \alpha(\mathrm{T}) \geqslant d(\mathrm{T}) \geqslant \delta \,\|\mathrm{T}\|,$$

with $\varpi_\alpha \in \hat{\Delta}_{\mathrm{Q}}$ being the weight corresponding to α. Furthermore, $\bar{\tau}_{\mathrm{P}_0}(\mathrm{H}_0(m), \mathrm{T}+\mathrm{S})$ equals 1, and this implies that

$$\|\mathrm{H}_0(m)\| \leqslant \|\mathrm{S}+\mathrm{T}\| \leqslant (1+r)\,\|\mathrm{T}\|.$$

Consequently,

$$\alpha(\mathrm{H}_0(m)) \geqslant \delta_1 \,\|\mathrm{H}_0(m)\|,$$

where $\delta_1 = \delta(1+r)^{-1}$. We may therefore apply Corollary 7.2, which tells us that

$$\|\mathrm{D}_{\mathrm{P}_0}(m)^{1/2}\, \mathrm{E}(m, \psi, \lambda) - \mathrm{D}_{\mathrm{P}_0 \cap \mathrm{M}_{\mathrm{Q}}}(m)^{1/2}\, (\mathrm{C}^{\mathrm{Q}}\, \mathrm{E})\, (m, \psi, \lambda)\|$$

is bounded by a function of the form

$$\mathrm{C}\mathrm{N}(\lambda)^k \,\|\psi\|\, e^{-\varepsilon \|\mathrm{H}_0(m)\|}.$$

Applied to both (ψ, λ) and (ψ', λ'), this estimate enables us to replace (9.3) with a similar expression

$$(9.5) \qquad D_{P_0 \cap M_Q}(m) \, ((C^Q E)(m, \psi', \lambda'), (C^Q E)(m, \psi, \lambda))$$

built out of weak constant terms. For we can use the inequality (7.14) (applied to both Q and G) to take care of the resulting cross-terms. The difference between (9.3) and (9.5) is then bounded in absolute value by a function of the form

$$CN_k(\lambda' \lambda, \psi', \psi) \, (1 + \| H_0(m) \|)^d \, e^{-\varepsilon \| H_0(m) \|}.$$

For any Q, let $W^{T, Q}(m)$ denote the product of (9.5) with $\hat{\tau}^Q_{P_0}(H_0(m), T)$. If $Q = G$, the expressions (9.3) and (9.5) are equal, and the integral of $W^{T, Q}(m)$ is precisely the contribution of Q to the original expression (9.2). Suppose that $Q \neq G$. Then we can choose a root $\alpha \in \Delta_{P_0} - \Delta^Q_{P_0}$, and as we have seen, $\alpha(H_0(m)) \geqslant \delta \| T \|$ for any m such that (9.4) does not vanish. It follows without difficulty that the integral over $M_0(F) \cap G(F)^Z$ of the product of (9.4) with a function

$$(1 + \| H_0(m) \|)^d \, e^{-\varepsilon \| H_0(m) \|}$$

is bounded by $C_1 \, e^{-\varepsilon_1 \| T \|}$, for positive constants C_1 and ε_1. We have thus shown that the difference between (9.2) and

$$\sum_{Q \supset P_0} \int_{M_0(F) \cap G(F)^Z} \tau_Q(H_Q(m) - T_Q, S_Q) \, W^{T, Q}(m) \, dm$$

is bounded in absolute value by an expression

$$(9.6) \qquad CN_k(\lambda', \lambda, \psi', \psi) \, e^{-\varepsilon \| T \|}$$

of the required form.

Let us decompose the integral

$$\int_{M_0(F) \cap G(F)^Z} \tau_Q(H_Q(m) - T_Q, S_Q) \, W^{T, Q}(m) \, dm$$

into a double integral

$$\int_{\mathfrak{a}^Z_{M_Q, F}} \left(\int_{M_0(F) \cap M_Q(F)^X} W^{T, Q}(m) \, dm \right) \tau_Q(X - T_Q, S_Q) \, dX.$$

Here, we recall that

$$\mathfrak{a}^Z_{M_Q, F} = \{ X \in \mathfrak{a}_{M_Q, F} : h_{M_Q G}(X) = Z \}.$$

Substituting the formula (7.10) for $C^Q E$ into (9.5), and then using (1.3) to change variables, we find that

$$\int_{M_0(F) \cap M_Q(F)^X} W^{T, Q}(m) \, dm$$

equals

$$\sum_{s',s} \Omega^{\mathrm{T}}_{\mathrm{P}_0\cap \mathrm{M}_\mathrm{Q}}(s'\lambda', s\lambda, c^{\mathrm{Q}}(1, s'\lambda')^{-1} c(s', \lambda')\psi', c^{\mathrm{Q}}(1, s\lambda)^{-1} c(s, \lambda)\psi)^{\mathrm{X}}, \tag{9.7}$$

the sum being taken over $s' \in \mathrm{W}(\mathfrak{a}_{\mathrm{P}'}; \mathrm{Q})$ and $s \in \mathrm{W}(\mathfrak{a}_\mathrm{P}; \mathrm{Q})$. Suppose that $\mathrm{Q} \neq \mathrm{G}$. Then by our induction hypothesis, Theorem 8.1 holds for M_Q. This will allow us to replace the function $\Omega^{\mathrm{T}}_{\mathrm{P}_0\cap\mathrm{M}_\mathrm{Q}}$ in (9.7) by the corresponding function $\omega^{\mathrm{T}}_{\mathrm{P}_0\cap\mathrm{M}_\mathrm{Q}}$. The difference between (9.7) and the new expression will in fact be bounded in absolute value by the sum over s' and s of the product of

$$\mu^{\mathrm{Q}}(s\sigma_\lambda)^{-1/2} \| c^{\mathrm{Q}}(1, s\lambda)^{-1} c(s, \lambda)\psi \|, \tag{9.8}$$

$$\mu^{\mathrm{Q}}(s'\sigma'_{\lambda'})^{-1/2} \| c^{\mathrm{Q}}(1, s'\lambda')^{-1} c(s', \lambda')\psi' \|, \tag{9.8'}$$

and a function of the form

$$\mathrm{C}(\mathrm{N}(\lambda)\,\mathrm{N}(\lambda'))^k e^{-\varepsilon\|\mathrm{T}\|}.$$

According to the relation (7.13), the product of (9.8) and (9.8′) equals

$$\mu(\sigma_\lambda)^{-1/2} \|\psi\| \mu(\sigma'_{\lambda'})^{-1/2} \|\psi'\|.$$

Thus, for any $\mathrm{Q} \neq \mathrm{G}$, the difference between (9.7) and

$$\sum_{s',s} \omega^{\mathrm{T}}_{\mathrm{P}_0\cap \mathrm{M}_\mathrm{Q}}(s'\lambda', s\lambda, c^{\mathrm{Q}}(1, s'\lambda')^{-1} c(s', \lambda')\psi', c^{\mathrm{Q}}(1, s\lambda)^{-1} c(s, \lambda)\psi)^{\mathrm{X}} \tag{9.9}$$

is bounded in absolute value by a function of the form (9.6). If $\mathrm{Q} = \mathrm{G}$, we cannot apply the induction hypothesis. However, in this case the difference between (9.7) and (9.9) is just the function $\Delta^{\mathrm{T}}_{\mathrm{P}_0}(\lambda', \lambda, \psi', \psi)^{\mathrm{Z}}$. Putting together everything we have shown so far, we conclude that the difference between

$$\Omega^{\mathrm{T}+\mathrm{S}}_{\mathrm{P}_0}(\lambda', \lambda, \psi', \psi)^{\mathrm{Z}} - \Delta^{\mathrm{T}}_{\mathrm{P}_0}(\lambda'\lambda, \psi', \psi)^{\mathrm{Z}}, \tag{9.10}$$

and the function obtained by taking the sum over $\mathrm{Q} \supset \mathrm{P}_0$ and the integral over $\mathrm{X} \in \mathfrak{a}^{\mathrm{Z}}_{\mathrm{M}_\mathrm{Q},\mathrm{F}}$ of the product of $\tau_\mathrm{Q}(\mathrm{X} - \mathrm{T}_\mathrm{Q}, \mathrm{S}_\mathrm{Q})$ with (9.9), is bounded in absolute value by a function of the form (9.6).

The expression (9.9) can be simplified. Applying the definition (8.8) to M_Q, we first write the expression as

$$\sum_{s',s}\sum_{\mathrm{R}_1} r^{\mathrm{T}}_{\mathrm{R}_1}((\mathrm{C}^{\mathrm{R}_1}\mathrm{E}^{\mathrm{Q}})(c^{\mathrm{Q}}(1, s'\lambda')^{-1} c(s', \lambda')\psi', s'\lambda'), (\mathrm{C}^{\mathrm{R}_1}\mathrm{E}^{\mathrm{Q}})(c^{\mathrm{Q}}(1, s\lambda)^{-1} c(s, \lambda)\psi, s\lambda))^{\mathrm{X}}.$$

The inner sum is over the parabolic subgroups R_1 of M_Q which contain $\mathrm{P}_0 \cap \mathrm{M}_\mathrm{Q}$. Next we apply the formula (7.10) for $\mathrm{C}^{\mathrm{Q}}\,\mathrm{E}$. This absorbs the sum over s', s, and we obtain

$$\sum_{\mathrm{R}_1} r^{\mathrm{T}}_{\mathrm{R}_1}((\mathrm{C}^{\mathrm{R}_1}\mathrm{C}^{\mathrm{Q}}\,\mathrm{E})(\psi', \lambda'), (\mathrm{C}^{\mathrm{R}_1}\mathrm{C}^{\mathrm{Q}}\,\mathrm{E})(\psi, \lambda))^{\mathrm{X}}.$$

Finally, the transitivity property of the weak constant term implies that $\mathrm{C}^{\mathrm{R}_1}\mathrm{C}^{\mathrm{Q}} = \mathrm{C}^{\mathrm{P}_1}$, where $\mathrm{P}_1 = \mathrm{Q}(\mathrm{R}_1)$ is the unique parabolic subgroup with $\mathrm{P}_1 \subset \mathrm{Q}$ and $\mathrm{P}_1 \cap \mathrm{M}_\mathrm{Q} = \mathrm{R}_1$. It follows easily that (9.9) equals the sum over $\{\mathrm{P}_1 : \mathrm{P}_0 \subset \mathrm{P}_1 \subset \mathrm{Q}\}$ of

$$r^{\mathrm{T}}_{\mathrm{P}_1\cap\mathrm{M}_\mathrm{Q}}((\mathrm{C}^{\mathrm{P}_1}\,\mathrm{E})(\psi', \lambda'), (\mathrm{C}^{\mathrm{P}_1}\,\mathrm{E})(\psi, \lambda))^{\mathrm{X}}. \tag{9.11}$$

To get back the function which is asymptotic to (9.10) we must multiply (9.11) with $\tau_Q(X - T_Q, S_Q)$, and then take the integral over $X \in \mathfrak{a}^Z_{M_Q, F}$, and the sum over P_1 and Q. This becomes the sum over $P_1 \supset P_0$ of

$$\text{(9.12)} \qquad \sum_{Q \supset P_1} \int_{\mathfrak{a}^Z_{M_Q, F}} \tau_Q(X - T_Q, S_Q)\, r^T_{M_Q \cap P_1}(\Psi', \Psi)^X\, dX,$$

where $\Psi' = (C^{P_1} E)\,(\psi', \lambda')$ and $\Psi = (C^{P_1} E)\,(\psi, \lambda)$. Our final task is to evaluate (9.12).

Let us temporarily replace Ψ' by Ψ'_Λ, where Λ is a point in $\mathfrak{a}^*_{M, C}$ such that the real part of $\Lambda(\alpha^\vee)$ is large for every root $\alpha \in \Delta_{P_1}$. We can then make use of the formula

$$r^T_{P_1 \cap M_Q}(\Psi'_\Lambda, \Psi)^X = \int_{\mathfrak{a}^X_{M, F}} (\Psi', \Psi)^{X_1}\, \varphi_{P_1 \cap M_Q}(X_1 - T_{P_1})\, e^{\Lambda(X_1)}\, dX_1,$$

obtained by applying (8.2) to M_Q. Substituting this into (9.12), we combine the resulting double integral over $X_1 \in \mathfrak{a}^X_{M, F}$ and $X \in \mathfrak{a}^Z_{M_Q, F}$ into a single convergent integral over $X \in \mathfrak{a}^Z_{M, F}$. We obtain

$$\begin{aligned} \int_{\mathfrak{a}^Z_{M, F}} (\Psi', \Psi)^X \{ \sum_{Q \supset P_1} \varphi_{P_1 \cap M_Q}(X - T_{P_1})\, \tau_Q(X - T_Q, S_Q)\, e^{\Lambda(X)} \}\, dX & \\ = \int_{\mathfrak{a}^Z_{M, F}} (\Psi', \Psi)^X\, \varphi_{P_1}(X - (T_{P_1} + S_{P_1}))\, e^{\Lambda(X)}\, dX & \\ = r^{T+S}_{P_1}(\Psi'_\Lambda, \Psi)^Z. & \end{aligned}$$

The last steps follow from (8.2), (3.12) and the fact that $\varphi_{P_1 \cap M_Q} = \varphi^Q_{P_1}$. But $r^{T+S}_{P_1}(\Psi'_\Lambda, \Psi)^Z$ extends to a meromorphic function of $\Lambda \in \mathfrak{a}^*_{M, C}$ whose value at $\Lambda = 0$ equals

$$r^{T+S}_{P_1}(\Psi', \Psi)^Z = r^{T+S}_{P_1}((C^{P_1} E)\,(\psi', \lambda'), (C^{P_1} E)\,(\psi, \lambda))^Z.$$

This then equals the original expression (9.12).

It follows from the definition (8.8) and what we have just proved that the required sum of (9.12) over P_1 equals $\omega^{T+S}_{P_0}(\lambda', \lambda, \psi', \psi)^Z$. This is our asymptotic approximation of (9.10). More precisely, the difference between (9.10) and $\omega^{T+S}_{P_0}(\lambda', \lambda, \psi', \psi)^Z$ is bounded in absolute value by a function of the form (9.6). However, the difference is nothing more than the original given expression

$$\Delta^{T+S}_{P_0}(\lambda', \lambda, \psi', \psi)^Z - \Delta^T_{P_0}(\lambda', \lambda, \psi', \psi)^Z.$$

The proof of Lemma 9.1 is therefore complete. □

Let us write $\lim\limits_{\substack{T \to \infty \\ \delta}}$ to denote a limit as $\|T\|$ approaches infinity, with T ranging over the set of T for which $d(T) \geqslant \delta \|T\|$.

Lemma **9.2.** — *The limit*

$$\Delta_{P_0}(\lambda', \lambda, \psi', \psi)^Z = \lim_{\substack{T \to \infty \\ \delta}} \Delta^T_{P_0}(\lambda', \lambda, \psi', \psi)^Z$$

exists, uniformly for (λ', λ) *in compact subsets of* $\mathscr{F}(\sigma', \sigma)$. *In fact there are positive constants* C, k *and* ε *such that*

$$| \Delta_{P_0}(\lambda', \lambda, \psi', \psi)^Z - \Delta^T_{P_0}(\lambda', \lambda, \psi', \psi)^Z | \leqslant C N_k(\lambda', \lambda, \psi', \psi)\, e^{-\varepsilon \| T \|},$$

for all (λ', λ), Z, ψ', ψ *and* T *as in the statement of Theorem* 8.1.

Proof. — Fix $T_1 \in \mathfrak{a}_{M_0, F} \cap \mathfrak{a}^+_{P_0}$ with $d(T_1) \geqslant \delta \| T_1 \|$. Set $T_n = nT_1$. Applying the last lemma with $T = T_n$ and $S = T_1$, we see that

$$| \Delta^{T_{n+1}}_{P_0}(\lambda', \lambda, \psi', \psi)^Z - \Delta^{T_n}_{P_0}(\lambda', \lambda, \psi', \psi)^Z | \leqslant C N_k(\lambda', \lambda, \psi', \psi)\, e^{-\varepsilon \| T_n \|},$$

for positive constants C, k and ε. Since $e^{-\varepsilon \| T_n \|}$ equals $(e^{-\varepsilon \| T_1 \|})^n$, the limit

$$\begin{aligned} \Delta_{P_0}(\lambda', \lambda, \psi', \psi)^Z &= \lim_{n \to \infty} \Delta^{T_{n+1}}_{P_0}(\lambda', \lambda, \psi', \psi)^Z \\ &= \Delta^{T_1}_{P_0}(\lambda', \lambda, \psi', \psi)^Z \\ &\quad + \sum_{k=1}^{\infty} (\Delta^{T_{k+1}}_{P_0}(\lambda', \lambda, \psi', \psi)^Z - \Delta^{T_k}_{P_0}(\lambda', \lambda, \psi', \psi)^Z) \end{aligned}$$

exists, and

$$| \Delta_{P_0}(\lambda', \lambda, \psi', \psi)^Z - \Delta^{T_n}_{P_0}(\lambda', \lambda, \psi', \psi)^Z | \leqslant C_1 N_k(\lambda', \lambda, \psi', \psi)\, e^{-\varepsilon \| T_n \|}, \quad n \geqslant 0,$$

where $C_1 = C(1 - e^{-\varepsilon \| T_1 \|})^{-1}$. Suppose that T is any point with $d(T) \geqslant \delta \| T \|$. We may assume that $\| T \| \geqslant \| T_1 \|$. Choose a positive integer n such that T_n belongs to $T + \mathfrak{a}^+_{P_0}$. It is easy to check that $\| T_n \| \geqslant \| T \|$, and we can also arrange to have $\| T_n - T \| \leqslant r \| T \|$, for a positive constant r which depends only on G and δ. We can therefore apply Lemma 9.1, with $S = T_n - T$. Combined with the inequality above, it yields an estimate

$$| \Delta_{P_0}(\lambda', \lambda, \psi', \psi)^Z - \Delta^T_{P_0}(\lambda', \lambda, \psi', \psi)^Z | \leqslant C N_k(\lambda', \lambda, \psi', \psi)\, e^{-\varepsilon \| T \|},$$

of the desired form. In particular,

$$\Delta_{P_0}(\lambda', \lambda, \psi', \psi)^Z = \lim_{\substack{T \to \infty \\ \delta}} \Delta^T_{P_0}(\lambda', \lambda, \psi', \psi)^Z.$$

The uniform convergence in (λ', λ) follows directly from the estimate. □

Proof of Theorem 8.1. — Suppose that the limit $\Delta_{P_0}(\lambda', \lambda, \psi', \psi)^Z$ vanishes for all (λ', λ) in an open dense subset $\mathscr{F}_*(\sigma', \sigma)$ of $\mathscr{F}(\sigma', \sigma)$. Then the estimate provided by Lemma 9.2 reduces to the required estimate for

$$\Delta^T_{P_0}(\lambda', \lambda, \psi', \psi)^Z = \Omega^T_{P_0}(\lambda', \lambda, \psi', \psi)^Z - \omega^T_{P_0}(\lambda', \lambda, \psi', \psi)^Z,$$

whenever (λ', λ) belongs to $\mathscr{F}_*(\sigma', \sigma)$. Recall that $\Omega^T_{P_0}(\lambda', \lambda, \psi', \psi)^Z$ is an analytic function of (λ', λ). Since the majorizing function $N_k(\lambda', \lambda, \psi', \psi)$ is locally bounded on $\mathscr{F}(\sigma', \sigma)$, the meromorphic function $\omega_{P_0}^T(\lambda', \lambda, \psi', \psi)^Z$ must itself be locally bounded on $\mathscr{F}(\sigma', \sigma)$. This means that $\omega_{P_0}^T(\lambda', \lambda, \psi', \psi)^Z$ is actually analytic on $\mathscr{F}(\sigma', \sigma)$. The required estimate will then hold on all of $\mathscr{F}(\sigma', \sigma)$, which is what we want to prove. Our remaining task, then, is to show that $\Delta_{P_0}(\lambda', \lambda, \psi', \psi)^Z = 0$ on an open dense set.

Observe that if ζ is any point in $i\mathfrak{a}_{G,F}^*$, the expressions $\Delta_{P_0}(\lambda', \lambda, \psi'_\zeta, \psi)^Z$, $\Delta_{P_0}(\lambda', \lambda, \psi', \psi_{-\zeta})^Z$, $\Delta_{P_0}(\lambda', \lambda - \zeta, \psi', \psi)^Z$ and $\Delta_{P_0}(\lambda' + \zeta, \lambda, \psi', \psi)^Z$ are each equal to

$$e^{\zeta(Z)}\, \Delta_{P_0}(\lambda', \lambda, \psi', \psi)^Z.$$

Replacing σ' and σ by representations of the form σ'_ζ and $\sigma_{-\zeta}$ if necessary, we can assume that the central characters of σ' and σ are trivial on $A_G(F)$. Replacing λ by $\lambda - \zeta$ if necessary, we can also assume that λ belongs to the space

$$(i\mathfrak{a}_{M,F}^*)^G = (i\mathfrak{a}_M^*)^G/(i\mathfrak{a}_M^*)^G \cap \mathfrak{a}_{M,F}^\vee \cong (i\mathfrak{a}_M^*)^G + \mathfrak{a}_{M,F}^\vee/\mathfrak{a}_{M,F}^\vee.$$

Similarly, we can assume that λ' belongs to $(i\mathfrak{a}_{M',F}^*)^G$. In particular, we can form the function

$$\Delta_{P_0}^T(\lambda', \lambda, \psi', \psi) = \Omega_{P_0}^T(\lambda', \lambda, \psi', \psi) - \omega_{P_0}^T(\lambda', \lambda, \psi', \psi)$$

considered in Corollary 8.2. Now

$$\Delta_{P_0}^T(\lambda', \lambda, \psi', \psi) = \sum_{Z \in \mathfrak{a}_{G,F}/\tilde{\mathfrak{a}}_{G,F}} \Delta_{P_0}^T(\lambda', \lambda, \psi', \psi)^Z.$$

It follows from Lemma 9.2 that the limit

$$\Delta_{P_0}(\lambda', \lambda, \psi', \psi) = \lim_{\substack{T \to \infty \\ \delta}} \Delta_{P_0}^T(\lambda', \lambda, \psi', \psi)$$

exists uniformly for (λ', λ) in compact subsets of $\mathscr{F}(\sigma', \sigma)$. On the other hand, we have an inversion formula

$$\Delta_{P_0}(\lambda', \lambda, \psi', \psi)^Z = |\, \mathfrak{a}_{G,F}/\tilde{\mathfrak{a}}_{G,F}\,|^{-1} \sum_\zeta \Delta_{P_0}(\lambda', \lambda, \psi'_\zeta, \psi)\, e^{-\zeta(Z)},$$

where the sum is over the dual finite group $\tilde{\mathfrak{a}}_{G,F}^\vee/\mathfrak{a}_{G,F}^\vee$. It is therefore enough to show that $\Delta_{P_0}(\lambda', \lambda, \psi', \psi)$ vanishes for all (λ', λ) on an open dense subset of

$$\mathscr{F}(\sigma', \sigma)^G = ((i\mathfrak{a}_{M',F}^*)^G \times (i\mathfrak{a}_{M,F}^*)^G) \cap \mathscr{F}(\sigma', \sigma).$$

The reason for this simple reduction is that (8.10) provides us with a formula for $\omega_{P_0}^T(\lambda', \lambda, \psi', \psi)$. We find that $\omega_{P_0}^T(\lambda', \lambda, \psi', \psi)$ equals the sum over $P_1 \supset P_0$, $s' \in W(\mathfrak{a}_{M'}, \mathfrak{a}_{M_1})$, $s \in W(\mathfrak{a}_M, \mathfrak{a}_{M_1})$ and $\nu_1 \in \mathscr{E}^G(s'\sigma', s\sigma)/\mathscr{L}_{M,\ell}^\vee$ of the functions

$$(9.13) \qquad (c(s', \lambda')\,\psi', c(s, \lambda)\,\psi)_{\nu_1}\, e^{(s'\lambda' - s\lambda + \nu_1)(T_{P_1})}\, \theta_{P_1,\ell}(s'\lambda' - s\lambda + \nu_1)^{-1}.$$

Observe that as a function of $(\lambda', \lambda) \in \mathscr{F}(\sigma', \sigma)^G$, each summand (9.13) has finitely many singular hypersurfaces, which are independent of T. Let $\mathscr{F}_\ell(\sigma', \sigma)^G$ be the complement in $\mathscr{F}(\sigma', \sigma)^G$ of these hypersurfaces. Then $\Delta_{P_0}(\lambda', \lambda, \psi', \psi)$ is a uniform limit of continuous functions on $\mathscr{F}_{\ell}(\sigma', \sigma)^{G}$, and is itself a continuous function of (λ', λ) in $\mathscr{F}_{\ell}(\sigma', \sigma)^{G}$. Suppose that

$$a : (i\mathfrak{a}^{*}_{M,F})^{G} \to \mathscr{A}_{\sigma}(M, \tau_{P|P})$$

and

$$a' : (i\mathfrak{a}^{*}_{M',F})^{G} \to \mathscr{A}_{\sigma'}(M', \tau_{P'|P'})$$

are smooth, compactly supported functions such that $a'(\lambda') \otimes a(\lambda)$ is supported on the open subset $\mathscr{F}_{\ell}(\sigma', \sigma)^{G}$ of $(i\mathfrak{a}^{*}_{M',F})^{G} \times (i\mathfrak{a}^{*}_{M,F})^{G}$. We would like to prove that the integral

$$\Delta_{P_0}(a', a) = \int_{(i\mathfrak{a}^{*}_{M',F})^{G}} \int_{(i\mathfrak{a}^{*}_{M,F})^{G}} \Delta_{P_0}(\lambda', \lambda, a'(\lambda'), a(\lambda))\, \mu(\sigma'_{\lambda'})\, \mu(\sigma_{\lambda})\, d\lambda\, d\lambda'$$

vanishes. It is clear that

$$\Delta_{P_0}(a', a) = \lim_{\substack{T \to \infty \\ \delta}} \Delta^{T}_{P_0}(a', a),$$

where $\Delta^{T}_{P_0}(a', a)$ is the difference between

$$\Omega^{T}_{P_0}(a', a) = \iint \Omega^{T}_{P_0}(\lambda', \lambda, a'(\lambda'), a(\lambda))\, \mu(\sigma'_{\lambda'})\, \mu(\sigma_{\lambda})\, d\lambda\, d\lambda'$$

and

$$\omega^{T}_{P_0}(a', a) = \iint \omega^{T}_{P_0}(\lambda', \lambda, a'(\lambda'), a(\lambda))\, \mu(\sigma'_{\lambda'})\, \mu(\sigma_{\lambda})\, d\lambda\, d\lambda'.$$

We shall show that the limit of each of these functions is zero.

To deal with $\Omega^{T}_{P_0}(a', a)$ we will use a version of the Plancherel formula on $G(F)/A_G(F)$. If

$$n^{G}(\sigma) = |\, W(\mathfrak{a}_M) \,|\, |\, \mathscr{L}^{\vee}_{\sigma}/\mathscr{L}^{\vee}_{M} \,|,$$

the function

$$E_a(x) = n^{G}(\sigma)^{-1} \int_{(i\mathfrak{a}^{*}_{M,F})^{G}} E(x, a(\lambda), \lambda)\, \mu(\sigma_{\lambda})\, d\lambda$$

is a τ-spherical Schwartz function on $G(F)/A_G(F)$. One then has the formula

$$\int_{G(F)/A_G(F)} (E_{a'}(x), E_a(x))\, dx = n^{G}(\sigma)^{-1} \int_{(i\mathfrak{a}^{*}_{M,F})^{G}} (a'_M(\lambda), a(\lambda))\, \mu(\sigma_{\lambda})\, d\lambda, \tag{9.14}$$

where

$$a'_M : (i\mathfrak{a}^{*}_{M,F})^{G} \to \mathscr{A}_{\sigma}(M, \tau_{P|P})$$

is a function obtained by symmetrizing a' with respect to both $W(\mathfrak{a}_M)$ and $\mathscr{L}^{\vee}_{\sigma}$. More precisely, $a'_M(\lambda)$ is defined to be the function

$$n^{G}(\sigma)^{-1} \sum_{s \in W(\mathfrak{a}_M, \mathfrak{a}_{M'})} \sum_{\nu \in \mathscr{E}^{G}(s\sigma, \sigma)/\mathscr{L}^{\vee}_{M}} ({}^{0}c(s, \lambda + \nu)^{-1}\, a'(s\lambda + s\nu))_{-\nu}$$

in $\mathscr{A}_\sigma(\mathrm{M}, \tau_{\mathrm{P}|\mathrm{P}})$. Using the fact that the wave packets E_a lie in the Schwartz space ([21, Theorem 19.2], [23, Theorem 6]), one deduces the formula (9.14) from the original Plancherel formula (2.5), the relations (7.7) and (7.9), and the properties (7.1) and (7.2). In the present context, our support condition on a' and a insures that the function

$$(({}^0c(s, \lambda + \nu)^{-1}\, a'(s\lambda + s\nu))_{-\nu},\, a(\lambda)) = ({}^0c(s, \lambda + \nu)^{-1}\, a'(s\lambda + s\nu),\, a(\lambda))_\nu$$

vanishes for each s and ν. This means that $(a'_{\mathrm{M}}(\lambda), a(\lambda))$ equals 0 for every λ, so the right hand side of (9.14) in fact vanishes. On the other hand, applying the definition (8.1) and changing orders of integration, we see that $\Omega^{\mathrm{T}}_{\mathrm{P}_0}(a', a)$ is just the product of $n^{\mathrm{G}}(\sigma)^2$ with

$$\int_{\mathrm{G}(\mathrm{F})/\mathrm{A}_{\mathrm{G}}(\mathrm{F})} (\mathrm{E}_{a'}(x), \mathrm{E}_a(x))\, u(x, \mathrm{T})\, dx.$$

As T approaches infinity, this approaches the left hand side of (9.14). Since both sides of (9.14) vanish, we obtain

$$\lim_{\substack{\mathrm{T}\to\infty\\ \delta}} \Omega^{\mathrm{T}}_{\mathrm{P}_0}(a', a) = 0.$$

Next, consider $\omega^{\mathrm{T}}_{\mathrm{P}_0}(a', a)$. The contribution of the summand (9.13) to $\omega^{\mathrm{T}}_{\mathrm{P}_0}(a', a)$ equals the integral over (λ', λ) of the product of $e^{(s'\lambda' - s\lambda + \nu_1)(\mathrm{T}_{\mathrm{P}_1})}$ with the function

$$(9.15)\qquad (c(s', \lambda')\, a'(\lambda'),\, c(s, \lambda)\, a(\lambda))_{\nu_1}\, \theta_{\mathrm{P}_1, t}(s'\lambda' - s\lambda + \nu_1)^{-1}\, \mu(\sigma'_{\lambda'})\, \mu(\sigma_\lambda).$$

The support condition on (a', a) implies that (9.15) is a smooth, compactly supported function of (λ', λ) in $(i\mathfrak{a}^*_{\mathrm{M}'\mathrm{F}})^{\mathrm{G}} \times (i\mathfrak{a}^*_{\mathrm{M},\mathrm{F}})^{\mathrm{G}}$. The contribution to $\omega^{\mathrm{T}}_{\mathrm{P}_0}(a', a)$ is therefore a Schwartz function of

$$((s')^{-1}\,\mathrm{T}_{\mathrm{P}_1},\, -s^{-1}\,\mathrm{T}_{\mathrm{P}_1}).$$

It approaches 0 as $\|\mathrm{T}\|$ approaches infinity. Since $\omega^{\mathrm{T}}_{\mathrm{P}_0}(a', a)$ is a finite sum of such contributions, we obtain

$$\lim_{\substack{\mathrm{T}\to\infty\\ \delta}} \omega^{\mathrm{T}}_{\mathrm{P}_0}(a', a) = 0.$$

We have established that $\Delta_{\mathrm{P}_0}(a', a) = 0$. Since $\Delta_{\mathrm{P}_0}(\lambda', \lambda, \psi', \psi)$ is a continuous function on $\mathscr{F}_t(a', a)^{\mathrm{G}}$, and (a', a) is an arbitrary test function, this proves that $\Delta_{\mathrm{P}_0}(\lambda', \lambda, \psi', \psi)$ vanishes on the open dense subset $\mathscr{F}_t(\sigma', \sigma)^{\mathrm{G}}$ of $\mathscr{F}(\sigma', \sigma)^{\mathrm{G}}$. We have attained our goal. As we noted earlier, the estimate required for Theorem 8.1 follows, completing the proof of the theorem. □

10. The spectral side

In this section we shall use the asymptotic inner product formula to investigate the spectral expansion of $\mathrm{K}^{\mathrm{T}}(f)$. This will eventually lead to a spectral expansion of the distribution $\mathrm{J}^{\mathrm{T}}(f)$ and of its constant term $\tilde{\mathrm{J}}(f)$. The final expansion will be the result of various operations applied to an expression obtained by substituting the inner product formula into the original expansion (3.6) of $K^T(f)$. For real groups, the process is quite similar to the combinatorial manipulations that were used to deal with the spectral side of the global trace formula [6]. The idea of applying such techniques to local harmonic analysis is due to Waldspurger [31], who carried out the procedure for the general linear group of a p-adic field. At this stage, p-adic groups are more difficult than real groups, for they contain combinatorial difficulties not encountered in the global trace formula over a number field.

As we left it in § 3, the spectral expansion of $K^T(f)$ was

$$\sum_{M \in \mathscr{L}} |W_0^M| \, |W_0^G|^{-1} \int_{\Pi_2(M(F))} K^T(\sigma, f) \, d\sigma.$$

Recall that $\{\Pi_2(M(F))\}$ stands for the set of orbits of $i\mathfrak{a}_M^*$, or equivalently the quotient $i\mathfrak{a}_{M,F}^* = i\mathfrak{a}_M^*/\mathfrak{a}_{M,F}^\vee$, in $\Pi_2(M(F))$. Since the stabilizer in $i\mathfrak{a}_{M,F}^*$ of any σ is $\mathfrak{a}_{M,\sigma}^\vee/\mathfrak{a}_{M,F}^\vee$, we have

$$K^T(f) = \sum_M |W_0^M| \, |W_0^G|^{-1} \sum_{\sigma \in \{\Pi_2(M(F))\}} |\mathfrak{a}_{M,\sigma}^\vee/\mathfrak{a}_{M,F}^\vee|^{-1} \int_{i\mathfrak{a}_{M,F}^*} K^T(\sigma_\lambda, f) \, d\lambda.$$

Since $m(\sigma_\lambda)$ equals $d_\sigma \, \mu(\sigma_\lambda)$ by (7.9), we shall write $K^T(\sigma_\lambda, f)$ as

$$\mu(\sigma_\lambda) \int_{A_G(F)\backslash G(F)} \sum_{S \in \mathscr{B}_P(\sigma)} \operatorname{tr}(\mathscr{I}_P(\sigma_\lambda, x) \, S_\lambda(f)) \, \overline{\operatorname{tr}(\mathscr{I}_P(\sigma_\lambda, x) \, S)} \, u(x, T) \, dx,$$

where

$$S_\lambda(f) = d_\sigma \, \mathscr{I}_P(\sigma_\lambda, f_2) \, S \mathscr{I}_P(\sigma_\lambda, f_1^\vee).$$

We can certainly take the integral over $\mathscr{A}_G(F)\backslash G(F)$ inside the sum over S. We can also replace x by $k_1 x k_2$, and then integrate over k_1 and k_2 in K. The formula (7.2) then gives an interpretation in terms of Eisenstein integrals. Since $u(x, T)$ is left and right K-invariant, the double integral over K leads to a (pointwise) inner product of Eisenstein integrals. We obtain

$$\begin{aligned} &\int_{A_G(F)\backslash G(F)} \operatorname{tr}(\mathscr{I}_P(\sigma_\lambda, x) \, S_\lambda(f)) \, \overline{\operatorname{tr}(\mathscr{I}_P(\sigma_\lambda, x) \, S)} \, u(x, T) \, dx \\ &\qquad = \int_{A_G(F)\backslash G(F)} (E_P(x, \psi_{S_\lambda(f)}, \lambda), E_P(x, \psi_S, \lambda)) \, u(x, T) \, dx \\ &\qquad = \Omega_{P_0}^T(\lambda, \lambda, \psi_{S_\lambda(f)}, \psi_S), \end{aligned}$$

in the notation of § 8. Here $P_0 \in \mathscr{P}(M_0)$ depends on M, and is any minimal parabolic subgroup for which M is standard. We see that $K^T(f)$ equals

$$\sum_M |W_0^M| \, |W_0^G|^{-1} \sum_\sigma \sum_S |\mathfrak{a}_{M,\sigma}^\vee/\mathfrak{a}_{M,F}^\vee|^{-1} \int_{i\mathfrak{a}_{M,F}^*} \Omega_{P_0}^T(\lambda, \lambda, \psi_{S_\lambda(f)}, \psi_S) \, \mu(\sigma_\lambda) \, d\lambda,$$

where the sums are over $M \in \mathscr{L}$, $\sigma \in \{\Pi_2(M(F))\}$ and $S \in \mathscr{B}_P(\sigma)$.

Motivated by Corollary 8.2, we set $k^T(f)$ equal to

$$(10.1)\qquad \sum_M |W_0^M|\,|W_0^G|^{-1}\sum_\sigma\sum_S |\mathfrak{a}^\vee_{M,\sigma}/\mathfrak{a}^\vee_{M,F}|^{-1}\int_{i\mathfrak{a}^*_{M,F}} \omega^T_{P_0}(\lambda,\lambda,\psi_{S_\lambda(f)},\psi_S)\,\mu(\sigma_\lambda)\,d\lambda.$$

By our assumptions on f and $\mathscr{B}_P(\sigma)$, the summand will vanish for all but finitely many σ and S. To apply the estimate in Corollary 8.2, we note that

$$\begin{aligned} N_k(\lambda,\lambda,\psi_{S_\lambda(f)},\psi_S)\,\mu(\sigma_\lambda) &= N(\lambda)^{2k}\,\|\psi_{S_\lambda(f)}\|\,\|\psi_S\| \\ &= N(\lambda)^{2k}\,d_\sigma^{-1}\,\|S_\lambda(f)\|_2\,\|S\|_2, \end{aligned}$$

by (7.1). This is bounded by

$$N(\lambda)^{2k}\,\|\mathscr{I}_P(\sigma_\lambda,f_2)\|_\infty\,\|S\|_2\,\|\mathscr{I}_P(\sigma_\lambda,f_1^\vee)\|_\infty\,\|S\|_2,$$

a rapidly decreasing function of $\lambda\in i\mathfrak{a}^*_{M,F}$. Corollary 8.2 then tells us that the integrand in (10.1) is analytic and rapidly decreasing on $i\mathfrak{a}^*_{M,F}$. The estimate of Corollary 8.2 also leads directly to

Lemma **10.1**. — *There are positive constants* C *and* ε *such that*

$$(10.2)\qquad |K^T(f)-k^T(f)|\leqslant Ce^{-\varepsilon\|T\|},$$

for all T *with* $d(T)\geqslant\delta\,\|T\|$. □

Since the integrand in (10.1) is a limiting value of

$$\omega^T_{P_0}(\lambda',\lambda,\psi_{S_\lambda(f)},\psi_S)\,\mu(\sigma_\lambda),$$

for points $\lambda',\lambda\in i\mathfrak{a}^*_{M,F}$ in general position, we shall investigate the expression provided by (8.10) for this latter function. It equals

$$(10.3)\qquad \sum_{P_1}\sum_s\sum_{s'}\sum_{\nu_1}(c(s',\lambda')\,\psi_{S_\lambda(f)},c(s,\lambda)\,\psi_S)_{\nu_1}\,e^{\Lambda_1(T_{P_1})}\,\theta_{P_1,t}(\Lambda_1)^{-1}\,\mu(\sigma_\lambda),$$

where the sums are over $P_1\supset P_0$, $s\in W(\mathfrak{a}_M,\mathfrak{a}_{M_1})$, $s'\in W(\mathfrak{a}_M,\mathfrak{a}_{M_1})$, and $\nu_1\in\mathscr{E}^G(s'\sigma,s\sigma)/\mathscr{L}^\vee_{M,t}$, and where $\Lambda_1=(s'\lambda'-s\lambda)+\nu_1$. Following [6], we make a change of variables

$$s'=st,\quad t\in W(\mathfrak{a}_M),$$

in the sum over s'. Setting $Q=s^{-1}P_1$, we can then replace the sum over P_1 and s by a sum over the groups $Q\in\mathscr{P}(M)$. It follows from the definitions that

$$e^{\Lambda_1(T_{P_1})}\,\theta_{P_1,t}(\Lambda_1)^{-1}=e^{\Lambda(T_Q)}\,\theta_{Q,t}(\Lambda)^{-1},$$

where $\Lambda=s^{-1}\Lambda_1$. Moreover,

$$c(st,\lambda')=c_{P_1|P}(st,\lambda')=sc_{Q|P}(t,\lambda')$$

and

$$c(s,\lambda)=c_{P_1|P}(s,\lambda)=sc_{Q|P}(1,\lambda).$$

(See [7 (I.2.12)].) Since the operator

$$s:\mathscr{A}_2(M,\tau_M)\to\mathscr{A}_2(M_1,\tau_{M_1})$$

is an isometry between inner products $(\cdot,\cdot)_{s^{-1}\nu_1}$ and $(\cdot,\cdot)_{\nu_1}$, the expression (10.3) becomes

$$\sum_{t\in W(\mathfrak{a}_M)}\ \sum_{Q\in\mathscr{P}(M)}\ \sum_{\nu}(c_{Q|P}(t,\lambda')\,\psi_{S_{\lambda'(f)}},\,c_{Q|P}(1,\lambda)\,\psi_S)_\nu\,e^{\Lambda(T_Q)}\,\theta_{Q,t}(\Lambda)^{-1}\,\mu(\sigma_\lambda),$$

where ν is summed over $\mathscr{E}^G(t\sigma,\sigma)/\mathscr{L}^\vee_{M,t}$ and Λ equals $t\lambda'-\lambda+\nu$.

Harish-Chandra has established the functional equation

$$c_{Q|P}(t,\lambda') = c_{Q|P}(1,t\lambda')\,{}^0c_{P|P}(t,\lambda').$$

(See [19, Theorem 21], [22, Corollary 17.2], [7, (I.2.9)].) For simplicity we shall write

$$c_{Q|P}(t\lambda') = c_{Q|P}(1,t\lambda'),$$

and also

$${}^0c(t,\lambda') = {}^0c_{P|P}(t,\lambda')$$

if the group P is understood. Suppose that ν belongs to $\mathscr{E}^G(t\sigma,\sigma)$. Given vectors $\varphi'\in\mathscr{A}_{t\sigma}(M,\tau_{P|P})$ and $\varphi\in\mathscr{A}_\sigma(M,\tau_{P|P})$, the reader can check that

$$\begin{aligned}(\varphi',c_{Q|P}(\lambda)\,\varphi)_\nu &= (\varphi',c_{Q|P}(1,\lambda)_\sigma\,\varphi)_\nu\\ &= (c_{Q|P}(1,\lambda-\nu)^*_{t\sigma}\,\varphi',\varphi)_\nu.\end{aligned}$$

Moreover, since $t\sigma\cong\sigma_\nu$, we can use (7.8) to write

$$\mu(\sigma_\lambda) = \mu((t\sigma)_{\lambda-\nu}) = c_{Q|P}(1,\lambda-\nu)^{-1}_{t\sigma}(c_{Q|P}(1,\lambda-\nu)^*_{t\sigma})^{-1}.$$

It follows that (10.3) equals the sum over $t\in W(\mathfrak{a}_M)$ of

$$\sum_{Q\in\mathscr{P}(M)}\ \sum_{\nu}(c_{Q|P}(\lambda-\nu)^{-1}\,c_{Q|P}(t\lambda')\,{}^0c(t,\lambda')\,\psi_{S_{\lambda'(f)}},\psi_S)_\nu\,e^{\Lambda(T_Q)}\,\theta_{Q,t}(\Lambda)^{-1},\tag{10.4}$$

where ν is summed over $\mathscr{E}^G(t\sigma,\sigma)/\mathscr{L}^\vee_{M,t}$ and Λ equals $t\lambda'-\lambda+\nu$. To retrieve the original expression (10.1) for $k^T(f)$, we must set λ' equal to λ in the sum over $t\in W(\mathfrak{a}_M)$ of (10.4), integrate the resulting expression over $\lambda\in i\mathfrak{a}^*_{M,F}$, and then sum the product of this with

$$|W^M_0|\,|W^G_0|^{-1}\,|\mathfrak{a}^\vee_{M,\sigma}/\mathfrak{a}^\vee_{M,F}|^{-1}$$

over S, σ and M.

If F is Archimedean, $\mathscr{L}^\vee_{M,t}=\{0\}$, and the sum over ν in (10.4) contains at most one term. However, if F is p-adic, this sum is over a more complicated finite set. To deal with the associated combinatorial problems, we turn to another idea of Waldspurger.

Lemma **10.2.** — *If* F *is a p-adic field we can choose a family of functions*

$$\{u_P\in C^\infty_c(i\mathfrak{a}^*_P/i\mathfrak{a}^*_G) : P\in\mathscr{F}\}$$

with the following properties :

(i) $$\sum_{\nu\in\mathscr{L}^\vee_{M_P,t}} u_P(\Lambda+\nu) = 1,\quad \Lambda\in i\mathfrak{a}^*_P/i\mathfrak{a}^*_G,$$

(ii) *For each* P, *the function*

$$u_{P,\ell}(\Lambda) = u_P(\Lambda)\,\theta_P(\Lambda)\,\theta_{P,\ell}(\Lambda)^{-1}, \quad \Lambda \in i\mathfrak{a}_P^*/i\mathfrak{a}_G^*, \tag{10.5}$$

is smooth on $i\mathfrak{a}_P^*/i\mathfrak{a}_G^*$.

(iii) *If* P *contains* P_1, u_P *and* $u_{P,\ell}$ *are the restrictions of* u_{P_1} *and* $u_{P_1,\ell}$ *to* $i\mathfrak{a}_P^*$.

Proof. — The fact that F is p-adic means that for each P, $\mathscr{L}^\vee_{M_P,\ell}$ is a lattice in $i\mathfrak{a}^*_{M_P}/i\mathfrak{a}^*_G$. Suppose first that $P_0 \in \mathscr{P}(M_0)$ is a minimal parabolic subgroup. We shall assume that u_{P_0} is supported on the set

$$\left\{\Lambda \in i\mathfrak{a}_0^*/i\mathfrak{a}_G^* : |\Lambda(\mu_{\alpha,\ell})| \leqslant \frac{3\pi}{2},\ \alpha \in \Delta_{P_0}\right\}. \tag{10.6}$$

Glancing back at the definitions (6.3) and (6.7) of $\theta_{P,\ell}$ and θ_P, we see that the only singularities of $u_{P_0}(\Lambda)\,\theta_{P_0,\ell}(\Lambda)^{-1}$ will be hyperplanes through the origin, and that these will be cancelled by the zero sets of θ_{P_0}. Therefore if u_{P_0} is smooth, the function $u_{P_0,\ell}$ will also be smooth. On the other hand, the region of support for u_{P_0} contains a fundamental domain for the lattice $\mathscr{L}^\vee_{M_0,\ell}$. It is then easy to arrange that

$$\sum_{\nu \in \mathscr{L}^\vee_{M_0,\ell}} u_{P_0}(\Lambda + \nu) = 1, \quad \Lambda \in i\mathfrak{a}_0^*.$$

(See [31, Lemme II.7.1].) If

$$P_0' = sP_0, \quad s \in W_0^G,$$

is any other minimal parabolic subgroup, define

$$u_{P_0'}(\Lambda) = u_{P_0}(s^{-1}\Lambda).$$

Then conditions (i) and (ii) are also valid for $u_{P_0'}$.

Suppose that $P \in \mathscr{F}$ is an arbitrary parabolic subgroup. Choose a minimal parabolic subgroup P_0 which is contained in P, and set u_P equal to the restriction of u_{P_0} to $i\mathfrak{a}_P^*$. This function is independent of P_0. Suppose that P contains a group P_1. Recall that any root $\alpha \in \Delta_P$ is the restriction to $\mathfrak{a}_P$ of a unique root $\alpha_1 \in \Delta_{P_1} - \Delta^P_{P_1}$, and that $\alpha^\vee$ is the projection of $\alpha_1^\vee$ onto $\mathfrak{a}_P$. In particular, $\mu_{\alpha,\ell}$ is the projection of $\mu_{\alpha_1,\ell}$ onto $\mathfrak{a}_P$, and $\mathscr{L}^\vee_{M_P,\ell}$ is the subgroup of characters in $\mathscr{L}^\vee_{M_{P_1},\ell}$ which are trivial on the elements $\mu_{\beta,\ell}$, $\beta \in \Delta^P_{P_1}$. It follows from the definitions (6.3) and (6.7) that the function $\theta_P\,\theta^{-1}_{P,\ell}$ is the restriction of $\theta_{P_1}\,\theta^{-1}_{P_1,\ell}$ to $i\mathfrak{a}_P^*$. Moreover, it is obvious that u_P is the restriction of u_{P_1} to $i\mathfrak{a}_P^*$. Consequently, $u_{P,\ell}$ equals the restriction to $i\mathfrak{a}_P^*$ of $u_{P_1,\ell}$. This is the third condition.

Assume now that $P_1 = P_0$ is minimal. Since $u_{P_0,\ell}$ is smooth and compactly supported, the same is true of the function $u_{P,\ell}$. This is the second condition. It remains to establish the first condition. Suppose that Λ is a point $i\mathfrak{a}_P^*$. If ν is any element in the complement of $\mathscr{L}^\vee_{M_P,\ell}$ in $\mathscr{L}^\vee_{M_0,\ell}$, there is a root $\alpha \in \Delta^P_{P_0}$ such that $\nu(\mu_{\alpha,\ell})$ is a nonzero integral multiple of $2\pi i$, so that $|(\Lambda + \nu)(\mu_{\alpha,\ell})|$ is bounded below by 2π. In other words, $\Lambda + \nu$ lies outside the support of u_{P_0}. We obtain

$$\sum_{\nu \in \mathscr{L}^\vee_{M_P,\ell}} u_P(\Lambda + \nu) = \sum_{\nu \in \mathscr{L}^\vee_{M_P,\ell}} u_{P_0}(\Lambda + \nu)$$
$$= \sum_{\nu \in \mathscr{L}^\vee_{M_0,\ell}} u_{P_0}(\Lambda + \nu) = 1,$$

the required first condition on u_P. □

If F is Archimedean, we simply set $u_P = 1$. Then $u_{P,\ell}$ is also equal to 1, and the three conditions of the lemma are trivially true.

We return to the expression (10.4). It depends on an element $t \in W(\mathfrak{a}_M)$, which will be fixed until further notice. The expression equals

$$\sum_{Q \in \mathscr{P}(M)} \sum_{\nu \in \mathscr{E}^G(t\sigma,\sigma)/\mathscr{L}^\vee_{M,\ell}} \Delta_Q(\nu)\, e^{\Lambda(T_Q)}\, \theta_{Q,\ell}(\Lambda)^{-1},$$

where

$$\Lambda = \Lambda(\lambda', \lambda, \nu) = t\lambda' - \lambda + \nu,$$

and

$$\Delta_Q(\nu) = (c_{Q|P}(\lambda - \nu)^{-1}\, c_{Q|P}(t\lambda')\, {}^0c(t, \lambda')\, \psi_{S_\lambda(f)}, \psi_S)_\nu.$$

Applying the last lemma, we write (10.4) as

$$\sum_Q \sum_\nu \Delta_Q(\nu)\, e^{\Lambda(T_Q)}\, \theta_{Q,\ell}(\Lambda)^{-1} \sum_{\xi \in \mathscr{L}^\vee_{M,\ell}} u_Q(\Lambda + \xi),$$

an expression which also equals

$$\sum_Q \sum_\nu \sum_\xi \Delta_Q(\nu + \xi)\, e^{(\Lambda+\xi)(T_Q)}\, \theta_{Q,\ell}(\Lambda + \xi)^{-1}\, u_Q(\Lambda + \xi),$$

in view of the definitions of Δ_Q and $\theta_{Q,\ell}$, and the fact that T_Q belongs to $\mathfrak{a}_{M,F}$. Since

$$\Lambda + \xi = t\lambda' - \lambda + (\nu + \xi),$$

we can combine the double sum over (ν, ξ) into a single sum over $\mathscr{E}^G(t\sigma, \sigma)$. It follows from the definition (10.5) that (10.4) equals

$$\sum_{Q \in \mathscr{P}(M)} \sum_{\nu \in \mathscr{E}^G(t\sigma,\sigma)} \Delta_Q(\nu)\, e^{\Lambda(T_Q)}\, u_{Q,\ell}(\Lambda)\, \theta_Q(\Lambda)^{-1}.$$

Define a Levi subgroup $L \in \mathscr{L}(M)$ by setting

$$\mathfrak{a}_L = \{ H \in \mathfrak{a}_M : tH = H \}.$$

From now on we will take $\lambda' = \lambda + \zeta$, where ζ is restricted to lie in the subspace $i\mathfrak{a}_L^*$. Then $t\zeta = \zeta$, and

$$\Lambda = t\lambda - \lambda + \zeta + \nu.$$

We shall write λ_L for the projection of λ onto $i\mathfrak{a}_L^*$, relative to the canonical decomposition $i\mathfrak{a}_M^* = (i\mathfrak{a}_M^*)^L \oplus i\mathfrak{a}_L^*$. Then the map

$$(10.7) \qquad (\lambda, \zeta, \nu) \to (\Lambda, \lambda_L, \nu), \qquad \Lambda = t\lambda' - \lambda + \nu, \qquad \lambda' = \lambda + \zeta,$$

is a bijection of $i\mathfrak{a}_M^* \times i\mathfrak{a}_L^* \times \mathscr{E}^G(t\sigma, \sigma)$ onto itself. In particular, the points λ and $\lambda' = \lambda + \zeta$ are uniquely determined by Λ, λ_L and ν. Define

$$D_Q(\Lambda, \lambda_L, \nu) = \Delta_Q(\Lambda, \lambda_L, \nu)\, u_{Q,t}(\Lambda), \tag{10.8}$$

where

$$\Delta_Q(\Lambda, \lambda_L, \nu) = \Delta_Q(\nu) = (c_{Q|P}(\lambda - \nu)^{-1}\, c_{Q|P}(t\lambda')\, {}^0c(t, \lambda')\, \psi_{S_\lambda(f)}, \psi_S)_\nu$$

as above. It follows easily from the definition that for fixed ν, $\Delta_Q(\cdot, \cdot, \nu)$ extends to a meromorphic function on $\mathfrak{a}_{M,C}^* \times \mathfrak{a}_{L,C}^*$ which is analytic for points in $i\mathfrak{a}_M^* \times i\mathfrak{a}_L^*$ in general position. In particular, $D_Q(\Lambda, \lambda_L, \nu)$ is well defined for generic (Λ, λ_L). However, in contrast to the situation for Eisenstein series [6, p. 1298], $D_Q(\cdot, \cdot, \nu)$ is not a smooth function on $i\mathfrak{a}_M^* \times i\mathfrak{a}_L^*$. We note for future reference that

$$\Delta_Q(\Lambda + \nu_M, \lambda_L, \nu + \nu_M) = \Delta_Q(\Lambda, \lambda_L, \nu) \tag{10.9}$$

for any point $\nu_M \in \mathscr{L}_M^\vee$. This is an immediate consequence of the definitions of Λ and the inner product $(\cdot, \cdot)_\nu$. We shall also write

$$C_Q(\Lambda, T) = e^{\Lambda(T_Q)}. \tag{10.10}$$

In this notation, the formula for (10.4) becomes

$$\sum_{\nu \in \mathscr{E}^G(t\sigma, \sigma)} \sum_{Q \in \mathscr{P}(M)} C_Q(\Lambda, T)\, D_Q(\Lambda, \lambda_L, \nu)\, \theta_Q(\Lambda)^{-1},$$

where $\Lambda = t\lambda' - \lambda + \nu$ is understood to be a variable that depends on ν.

We would like to apply the notions introduced in [4, § 6] to the sum over Q. We begin by observing that

$$\{ C_Q(\Lambda, T) : Q \in \mathscr{P}(M) \}$$

is a (G, M)-family of functions of $\Lambda \in i\mathfrak{a}_M^*$, in the sense of [4, p. 36]. This is an immediate consequence of the definition of T_Q.

Lemma **10.3.** — *a) For any $\Lambda \in i\mathfrak{a}_M^*$, the set of points $\lambda_L \in i\mathfrak{a}_L^*$ such that $D_Q(\cdot, \cdot, \nu)$ is regular at (Λ, λ_L) is an open dense subset of $i\mathfrak{a}_L^*$.*

b) $D_Q(\Lambda, \lambda_L, \nu)$ can be regarded as a smooth function of $\Lambda \in i\mathfrak{a}_M^*$ with values in a topological vector space of meromorphic functions of λ_L.

c) The set

$$\{ D_Q(\Lambda, \lambda_L, \nu) : Q \in \mathscr{P}(M) \}$$

is a (G, M)-family of functions of $\Lambda \in i\mathfrak{a}_M^*$.

Proof. — We begin by investigating the regularity of D_Q. According to (7.6),

$$c_{Q|P}(\lambda)\, \psi_T = \gamma(\overline{Q})^{-1}\, \psi_{J_{\overline{Q}|P}(\sigma_\lambda) T J_{P|Q}(\sigma_\lambda)},$$

for any $T \in \operatorname{End}(\mathscr{H}_P(\sigma)_\Gamma)$ and $\lambda \in i\mathfrak{a}_M^*$. If we divide $c_{Q|P}(\lambda)$ by the normalizing factor $r_{\overline{Q}|Q}(\sigma_\lambda)$, the J-functions on the right become normalized intertwining operators, which are both regular and invertible for any $\lambda \in i\mathfrak{a}_M^*$. The function ${}^0c(t, \lambda)$ takes values which are unitary operators on $\mathscr{H}_P(\sigma)_\Gamma$, and is therefore already regular for all $\lambda \in i\mathfrak{a}_M^*$. It follows from the definition (10.7) and (10.8) that $D_Q(\Lambda, \lambda_L, \nu)$ is regular, up to singularities in the normalizing factors. More precisely, if

$$r_Q(\Lambda, \lambda_L, \nu) = r_{\overline{Q}|Q}((t\sigma)_{\lambda-\nu})^{-1}\, r_{\overline{Q}|Q}((t\sigma)_{t\lambda'}),$$

the function

$$d_Q(\Lambda, \lambda_L, \nu) = r_Q(\Lambda, \lambda_L, \nu)^{-1}\, D_Q(\Lambda, \lambda_L, \nu)$$

is regular for all $\Lambda \in i\mathfrak{a}_M^*$ and $\lambda_L \in i\mathfrak{a}_L^*$. It is enough to prove the assertion *a)* with $D_Q(\cdot, \cdot, \nu)$ replaced by $r_Q(\cdot, \cdot, \nu)$.

Observe that

$$r_{\overline{Q}|Q}((t\sigma)_{\lambda-\nu})^{-1} = r_{\overline{Q}|Q}(\sigma_\lambda)^{-1} = \prod_{\beta \in \Sigma_{\overline{Q}}^r} r_\beta(\sigma_\lambda)^{-1},$$

since ν belongs to $\mathscr{E}^G(t\sigma, \sigma)$. Moreover,

$$r_{\overline{Q}|Q}((t\sigma)_{t\lambda'}) = \prod_{\beta \in \Sigma_{\overline{Q}}^r} r_{t^{-1}\beta}(\sigma_{\lambda'}),$$

since the rank one normalizing factors can be chosen so that

$$r_\beta((t\sigma)_{t\lambda'}) = r_{t^{-1}\beta}(\sigma_{\lambda'}).$$

It follows that

$$r_Q(\Lambda, \lambda_L, \nu) = \prod_{\beta \in \Sigma_{\overline{Q}}^r} r_\beta(\sigma_\lambda)^{-1}\, r_{t^{-1}\beta}(\sigma_{\lambda'}). \tag{10.11}$$

Suppose that β belongs to the subset $\Sigma_{\overline{Q}\cap L}^r$ of roots in $\Sigma_{\overline{Q}}^r$ which vanish on $\mathfrak{a}_L$. Then $\alpha = t^{-1}\beta$ also vanishes on $\mathfrak{a}_L$. Since λ and λ' have the same projection onto $(i\mathfrak{a}_M^*)^L$, $r_\alpha(\sigma_{\lambda'})$ equals $r_\alpha(\sigma_\lambda)$. If α remains in $\Sigma_{\overline{Q}\cap L}^r$, the function $r_\alpha(\sigma_{\lambda'})$ will then cancel the term $r_\alpha(\sigma_\lambda)^{-1}$ in the product (10.11). It follows that $r_Q(\Lambda, \lambda_L, \nu)$ can be written

$$\prod_\alpha r_{-\alpha}(\sigma_\lambda)^{-1}\, r_\alpha(\sigma_\lambda) \,.\, \prod_\beta r_\beta(\sigma_\lambda)^{-1}\, r_{t^{-1}\beta}(\sigma_{\lambda'}),$$

where the products are taken over $\{\alpha \in \Sigma_{Q\cap L}^r : t\alpha \in \Sigma_{\overline{Q}\cap L}^r\}$ and $\{\beta \in \Sigma_{\overline{Q}}^r - \Sigma_{\overline{Q}\cap L}^r\}$.

For any root α, $r_\alpha(\sigma_\lambda)$ is a meromorphic function of $\lambda(\sigma^\vee)$ which does not vanish for any imaginary λ. If $r_\alpha(\sigma_\lambda)$ has a pole at a point $\lambda_0 \in i\mathfrak{a}_M^*$, the representation σ_{λ_0} will be fixed by the simple reflection w_α. (This is a consequence of Corollary 5.4.2.2 of [29] and its analogue for real groups.) In this case

$$r_{-\alpha}(\sigma_{\lambda_0+\mu}) = r_{-\alpha}(w_\alpha(\sigma_{\lambda_0-\mu})) = r_\alpha(\sigma_{\lambda_0-\mu}), \quad \mu \in i\mathfrak{a}_M^*,$$

so the quotient of this function by $r_\alpha(\sigma_{\lambda_0+\mu})$ is analytic at $\mu = 0$. We have established that

$$r_\alpha(\sigma_\lambda)^{-1}\, r_{-\alpha}(\sigma_\lambda) \tag{10.12}$$

is an analytic function of $\lambda \in i\mathfrak{a}_M^*$. Therefore, the product over α above contributes no singularities to $r_Q(\Lambda, \lambda_L, \nu)$. Now, for the purpose of proving *a)*, we can translate λ_L

by a generic point in $i\mathfrak{a}_L^*$. This has the effect of translating both λ and λ' by the same generic point. Since β and $t^{-1}\beta$ both have nontrivial restriction to $\mathfrak{a}_L$, for any root β in $\Sigma_{\bar{Q}}^r - \Sigma_{\bar{Q}\cap L}^r$, we can arrange that the singularities in corresponding product over β avoid the given point Λ. This proves *a)*.

The assertion *b)* is essentially a restatement of *a)*. The function $u_{Q,\ell}$ in (10.8) depends only on Λ. Consequently, $D_Q(\Lambda, \lambda_L, \nu)$ extends to a meromorphic function of λ_L, which by *a)* is well defined for any point $\Lambda \in i\mathfrak{a}_M^*$. We leave the reader to formulate a definition of the topological vector space in which D_Q takes values.

Finally, we must establish that $\{D_Q\}$ is a (G, M)-family. Although it is not an important point, the definition of a (G, M)-family in [4] requires that the functions $D_Q(\Lambda) = D_Q(\Lambda, \lambda_L, \nu)$ be smooth in Λ. This is the reason for the interpretation *b)*. The essential condition on a (G, M)-family concerns the compatibility of functions attached to adjacent groups Q and Q′ in $\mathscr{P}(M)$. Let α be the unique simple root of (Q, A_M) which is not a root of (Q', A_M). Then the hyperplane

$$i\mathfrak{a}_{M_\alpha}^* = \{\Lambda \in i\mathfrak{a}_M^* : \Lambda(\alpha^\vee) = 0\}$$

is generated by the common wall of the chambers of Q and Q′ in $i\mathfrak{a}_M^*$. We must show that if Λ belongs to $i\mathfrak{a}_{M_\alpha}^*$, then $D_{Q'}(\Lambda)$ equals $D_Q(\Lambda)$.

Suppose that $Q_\alpha \in \mathscr{P}(M_\alpha)$ is the parabolic subgroup generated by Q and Q′. According to Lemma 10.1, $u_{Q_\alpha,\ell}$ equals the restriction of both $u_{Q',\ell}$ and $u_{Q,\ell}$ to $i\mathfrak{a}_{M_\alpha}^*$. In particular, $u_{Q',\ell}(\Lambda) = u_{Q,\ell}(\Lambda)$ for any $\Lambda \in i\mathfrak{a}_{M_\alpha}^*$. To deal with the other factor in the definition of $D_{Q'}$, we first write

$$c_{Q'|P}(\lambda-\nu)^{-1}\,c_{Q'|P}(t\lambda') = c_{Q|P}(\lambda-\nu)^{-1}\,c_{Q'|Q}^0(\lambda-\nu)^{-1}\,c_{Q'|Q}^0(t\lambda')\,c_{Q|P}(t\lambda'),$$

by the functional equation [7, (I.2.9)]. Now it is easy to check that

$$c_{Q'|Q}^0(\lambda) = J_{Q'|Q}^\ell(\lambda)\,J_{\bar{Q}|\bar{Q}'}^r(\lambda)^{-1},$$

where J^ℓ and J^r are the standard left and right intertwining integrals defined in [7, § I.2]. (The formula [7, (I.2.14)] was transcribed incorrectly.) In particular, $c_{Q'|Q}^0(\lambda)$ depends only on the projection of λ onto $i\mathfrak{a}_M^*/i\mathfrak{a}_{M_\alpha}^*$. Since $t\lambda'$ equals $(\lambda-\nu)+\Lambda$, we find that $c_{Q'|Q}^0(t\lambda')$ equals $c_{Q'|Q}^0(\lambda-\nu)$ if $\Lambda \in i\mathfrak{a}_{M_\alpha}^*$. The required assertion

$$D_{Q'}(\Lambda) = D_Q(\Lambda), \quad \Lambda \in i\mathfrak{a}_{M_\alpha}^*,$$

follows. □

Corollary **10.4.** — *Suppose that* R *is a group in* $\mathscr{F}(M)$. *Then the limit*

$$D_M^R(t\lambda - \lambda + \nu, \lambda_L, \nu) = \lim_{\Lambda \to (t\lambda - \lambda + \nu)} \sum_{\{Q \in \mathscr{P}(M) : Q \subset R\}} D_Q(\Lambda, \lambda_L, \nu)\,\theta_Q(\Lambda)^{-1}$$

extends to a smooth function of $\lambda \in i\mathfrak{a}_M^*$. *If* F *is Archimedean, the function belongs to the Schwartz space on* $i\mathfrak{a}_M^*$.

Proof. — We have

$$D_Q(\Lambda, \lambda_L, \nu) = r_Q(\Lambda, \lambda_L, \nu)\, d_Q(\Lambda, \lambda_L, \nu)$$

in the notation of the proof of the lemma. Since $\{r_Q\}$ and $\{d_Q\}$ are themselves (G, M)-families of functions of Λ, we can apply the splitting formula [10, Corollary 7.4] to the product $\{D_Q\}$. We see that $D_M^R(t\lambda - \lambda + \nu, \lambda_L, \nu)$ equals

$$\sum_{L', L''} d_M^{M_R}(L', L'')\, r_M^{Q'}(t\lambda - \lambda + \nu, \lambda_L, \nu)\, d_M^{Q''}(t\lambda - \lambda + \nu, \lambda_L, \nu),$$

in the notation of [10, § 7]. The sum is over groups $L', L'' \in \mathscr{L}(M)$ which are contained in the Levi component M_R of R. Now $d_Q(\Lambda, \lambda_L, \nu)$ is constructed from normalized intertwining operators, which are analytic functions of $\lambda, \lambda' \in i\mathfrak{a}_M^*$. In the Archimedean case, these functions are also rational in λ, λ' [11, Theorem 2.1 (R_6)]. It follows from [4, Lemma 6.2] that $d_M^{Q''}(t\lambda - \lambda + \nu, \lambda_L, \lambda)$ is a smooth function of $\lambda \in i\mathfrak{a}_M^*$, which in the Archimedean case can be written as a finite sum of products of rational functions with matrix coefficients of the operator $S_\lambda(f)$. Since the matrix coefficients of $S_\lambda(f)$ are Schwartz functions, it will therefore be enough for us to show that $r_M^{Q'}(t\lambda - \lambda + \nu, \lambda_L, \nu)$ is a smooth function of $\lambda \in i\mathfrak{a}_M^*$, any derivative of which is slowly increasing.

Recall that a group $P \in \mathscr{P}(M)$ has been fixed. Set

$$\tilde{r}_Q(\Lambda, \lambda_L, \nu) = r_Q(\Lambda, \lambda_L, \nu)\, r_P(\Lambda, \lambda_L, \nu)^{-1}.$$

Then

$$r_M^{Q'}(t\lambda - \lambda + \nu, \lambda_L, \nu) = r_P(t\lambda - \lambda + \nu, \lambda_L, \nu)\, \tilde{r}_M^{Q'}(t\lambda - \lambda + \nu, \lambda_L, \nu).$$

We see directly from (10.11) that

$$\tilde{r}_Q(\Lambda, \lambda_L, \nu) = \prod_{\beta \in \Sigma_Q^r \cap \Sigma_P^r} (r_\beta(\sigma_\lambda)\, r_{-\beta}(\sigma_\lambda)^{-1})^{-1} (r_{t^{-1}\beta}(\sigma_{\lambda'})\, r_{-t^{-1}\beta}(\sigma_{\lambda'})^{-1}).$$

Having established the analyticity of each function (10.12), we know the product on the right is an analytic function of $(\lambda, \lambda') \in i\mathfrak{a}_M^* \times i\mathfrak{a}_M^*$. It follows from [4, Lemma 6.2] that

$$\tilde{r}_M^{Q'}(t\lambda - \lambda + \nu, \lambda_L, \nu) = \lim_{\Lambda \to (t\lambda - \lambda + \nu)} \tilde{r}_M^{Q'}(\Lambda, \lambda_L, \nu)$$

is an analytic function of $\lambda \in i\mathfrak{a}_M^*$. On the other hand, setting $\lambda' = \lambda$ in (10.11), we obtain

$$r_P(t\lambda - \lambda + \nu, \lambda_L, \nu) = \prod_{\beta \in \Sigma_P^r} r_\beta(\sigma_\lambda)^{-1}\, r_{t^{-1}\beta}(\sigma_\lambda)$$

$$= \prod_{\{\beta \in \Sigma_P^r : t\beta \in \Sigma_{\bar{P}}^r\}} r_{-\beta}(\sigma_\lambda)^{-1}\, r_\beta(\sigma_\lambda).$$

This too is a product of functions (10.12), and so is analytic at any $\lambda \in i\mathfrak{a}_M^*$. Thus, $r_M^{Q'}(t\lambda - \lambda + \nu, \lambda_L, \nu)$ is an analytic function of $\lambda \in i\mathfrak{a}_M^*$.

If F is Archimedean, the normalizing factors are given explicitly in terms of gamma-functions [11, § 3]. From this, one verifies that derivatives of (10.12) are slowing increasing.

The argument above then confirms that any derivative in λ of $r_M^{Q'}(t\lambda - \lambda + \nu, \lambda_L, \nu)$ is also slowly increasing. The proof of the corollary is complete. □

We return to our study of $k^T(f)$. We have written (10.4) as the sum over $\nu \in \mathscr{E}^G(t\sigma, \sigma)$ of the expression

$$\sum_{Q \in \mathscr{P}(M)} C_Q(\Lambda, T)\, D_Q(\Lambda, \lambda_L, \nu)\, \theta_Q(\Lambda)^{-1} \tag{10.13}$$

built out of a product of (G, M)-families. There are two splitting formulas that can be brought to bear on this sum over Q. The last corollary suggests that we turn to the second one [10, Corollary 7.4], instead of the formula [4, Lemma 6.3] that was used for Eisenstein series in [6, Lemma 2.1]. This result allows us to express the sum (10.13) as

$$\sum_{L_1, L_2 \in \mathscr{L}(M)} d_M^G(L_1, L_2)\, C_M^{Q_1}(\Lambda, T)\, D_M^{Q_2}(\Lambda, \lambda_L, \nu),$$

in the notation introduced in [10, § 7] and used above in the proof of Corollary 10.4. In particular, $d_M^G(L_1, L_2)$ is the constant defined in [10, p. 356], and

$$(L_1, L_2) \to (Q_1, Q_2) \in \mathscr{P}(L_1) \times \mathscr{P}(L_2)$$

is the retraction defined on [10, p. 357-358]. (The reader can also consult the proof of Corollary 11.2 below for a precise description of the retraction.)

It follows from Lemma 10.3 and [4, Lemma 6.2] that (10.13) can be regarded as a smooth function of Λ with values in a space of meromorphic functions of λ_L. Therefore the limit of (10.13), as λ' approaches a generic point λ, exists. In particular, the original limit of the sum over t of (10.4) may be brought inside the sum over t and also the sum over $\nu \in \mathscr{E}^G(t\sigma, \sigma)$. The limit of (10.13) becomes

$$\sum_{L_1, L_2 \in \mathscr{L}(M)} d_M^G(L_1, L_2)\, C_M^{Q_1}(t\lambda - \lambda + \nu, T)\, D_M^{Q_2}(t\lambda - \lambda + \nu, \lambda_L, \nu). \tag{10.14}$$

The last corollary tells us that the summands in (10.14) extend to smooth functions of $\lambda \in i\mathfrak{a}_M^*$. If F is Archimedean, $\mathscr{E}^G(t\sigma, \sigma)$ consists of at most one element, and (10.14) is rapidly decreasing in λ. If F is p-adic, $i\mathfrak{a}_{M,F}^*$ is compact, and ν can be taken to lie in a finite subset of $\mathscr{E}^G(t\sigma, \sigma)$. In particular, the sum and integral over ν and λ, required to construct $k^T(f)$ from (10.14), are absolutely convergent. They can both be taken inside the sums over t, L_1 and L_2.

We summarize what we have obtained so far.

Lemma **10.5**. — *The distribution* $k^T(f)$ *equals the sum over* $M \in \mathscr{L}$, $\sigma \in \{\Pi_2(M(F))\}$, $S \in \mathscr{B}_P(\sigma)$ *and* $t \in W(\mathfrak{a}_M)$, *of the product of*

$$|W_0^M|\,|W_0^G|^{-1}\,|\mathfrak{a}_{M,\sigma}^\vee/\mathfrak{a}_{M,F}^\vee|^{-1}$$

with

$$\sum_{L_1, L_2} d_M^G(L_1, L_2) \int_{i\mathfrak{a}_{M,F}^*} \sum_{\nu} C_M^{Q_1}(t\lambda - \lambda + \nu, T)\, D_M^{Q_2}(t\lambda - \lambda + \nu, \lambda_L, \nu)\, d\lambda, \tag{10.15}$$

where (L_1, L_2) *and* ν *are summed over* $\mathscr{L}(M) \times \mathscr{L}(M)$ *and* $\mathscr{E}^G(t\sigma, \sigma)$ *respectively.* □

11. The spectral side, continued

We proceed with the discussion initiated in the last section. We shall investigate the formula for $k^T(f)$ provided by Lemma 10.5, with particular regard for its dependence on T. We continue to combine the techniques [6, § 2-4] of the global trace formula with methods of Waldspurger [31] for dealing with p-adic groups.

Consider the integral over $i\mathfrak{a}^*_{M,F} = i\mathfrak{a}^*_M/\mathfrak{a}^\vee_{M,F}$ in the expression (10.15). According to (1.5), $\mathfrak{a}^\vee_{L,F}$ equals $i\mathfrak{a}^*_L \cap \mathfrak{a}^\vee_{M,F}$. The integral can therefore be decomposed into a double integral over λ^L in $i\mathfrak{a}^*_M/\mathfrak{a}^\vee_{M,F} + i\mathfrak{a}^*_L$ and λ_L in $i\mathfrak{a}^*_{L,F} = i\mathfrak{a}^*_L/\mathfrak{a}^\vee_{L,F}$. Observe that $t\lambda - \lambda$ depends only on the image of λ in $i\mathfrak{a}^*_M/\mathfrak{a}^\vee_{M,F} + i\mathfrak{a}^*_L$. In fact, if

$$\mathfrak{a}^\vee_{M,t} = (t-1)\,\mathfrak{a}^\vee_{M,F} = \{ t\nu - \nu : \nu \in \mathfrak{a}^\vee_{M,F} \},$$

the map

$$\lambda^L \to \mu = t\lambda^L - \lambda^L$$

is a diffeomorphism of $i\mathfrak{a}^*_M/\mathfrak{a}^\vee_{M,F} + i\mathfrak{a}^*_L$ onto $(i\mathfrak{a}^*_M)^L/\mathfrak{a}^\vee_{M,t}$ whose Jacobian determinant (relative to our chosen measures on these spaces) equals $|\det(t-1)_{\mathfrak{a}^L_M}|$. Therefore (10.15) can be written as the product of $|\det(t-1)_{\mathfrak{a}^L_M}|^{-1}$ with

$$\sum_{L_1, L_2} d^G_M(L_1, L_2) \left\{ \int_{i\mathfrak{a}^*_{L,F}} d\lambda_L \int d\mu \sum_\nu C^{Q_1}_M(\mu+\nu, T)\, D^{Q_2}_M(\mu+\nu, \lambda_L, \nu) \right\}, \tag{11.1}$$

where μ is integrated over $(i\mathfrak{a}^*_M)^L/\mathfrak{a}^\vee_{M,t}$ and ν is summed over $\mathscr{E}^G(t\sigma, \sigma)$. It follows from the definition (10.8) that $D^{Q_2}_M(\cdot, \lambda_L, \nu)$ depends only on the images of λ_L and ν in the quotients $i\mathfrak{a}^*_{L,F} = i\mathfrak{a}^*_L/\mathfrak{a}^\vee_{L,F}$ and $\mathscr{E}^G(t\sigma, \sigma)/\mathfrak{a}^\vee_{M,t}$. (Indeed, $\mathfrak{a}^\vee_{M,t}$ is contained in $\mathfrak{a}^\vee_{M,F}$, and from the definition in § 8 of the inner product $(\cdot, \cdot)_\nu$, it is clear that $D^{Q_2}_M$ is invariant under translation of the third variable by elements in $\mathfrak{a}^\vee_{M,F}$.) We can therefore write the expression in the brackets of (11.1) as

$$\sum_{\nu \in \mathscr{E}^G(t\sigma, \sigma)/\mathfrak{a}^\vee_{M,t}} \int_{(i\mathfrak{a}^*_M)^L} C^{Q_1}_M(\mu+\nu, T) \left(\int_{i\mathfrak{a}^*_{L,F}} D^{Q_2}_M(\mu+\nu, \lambda_L, \nu)\, d\lambda_L \right) d\mu.$$

Recall that $\sigma^{Q_1}_M(\cdot, T)$ stands for the characteristic function in $\mathfrak{a}_M/\mathfrak{a}_{L_1}$ of the convex hull of

$$T^{Q_1}_M = \{ T_Q : Q \in \mathscr{P}(M), Q \subset Q_1 \}.$$

We can write

$$C^{Q_1}_M(\mu+\nu, T) = \int_{T_{Q_1} + \mathfrak{a}^{L_1}_M} \sigma^{Q_1}_M(H, T)\, e^{(\mu+\nu)(H)}\, dH.$$

(See [1, pp. 219-220], [6, (3.1)].) Our expression (11.1) becomes

$$\sum_{\nu \in \mathscr{E}^G(t\sigma, \sigma)/\mathfrak{a}^\vee_{M,t}} \sum_{L_1 \in \mathscr{L}(M)} \int_{T_{Q_1} + \mathfrak{a}^{L_1}_M} \sigma^{Q_1}_M(H, T)\, \Phi_{L_1}(H, \nu)\, dH,$$

where $\Phi_{L_1}(H, \nu)$ equals

$$\sum_{L_2 \in \mathscr{L}(M)} d_M^G(L_1, L_2) \int_{(i\mathfrak{a}_M^*)^L} \left(\int_{i\mathfrak{a}_{L,F}^*} D_M^{Q_2}(\mu + \nu, \lambda_L, \nu)\, d\lambda_L \right) e^{(\mu+\nu)(H)}\, d\mu,$$

for any point $H \in \mathfrak{a}_M$. The integral

$$\int_{i\mathfrak{a}_{L,F}^*} D_M^{Q_2}(\mu + \nu, \lambda_L, \nu)\, d\lambda_L$$

converges absolutely, and the resulting function of $\mu \in (i\mathfrak{a}_M^*)^L$ belongs to a Schwartz space. Therefore if

$$H = H_M^L + H_L, \quad H_M^L \in \mathfrak{a}_M^L, \; H_L \in \mathfrak{a}_L,$$

the function $\Phi_{L_1}(H, \nu)$ is the product of $e^{\nu(H_L)}$ with a Schwartz function of H_M^L. We should also point out that the sum over ν above can be taken over a finite set of representatives of orbits in $\mathscr{E}^G(t\sigma, \sigma)/\mathfrak{a}_{M,t}^\vee$. This is trivial if F is Archimedean, and in the p-adic case it follows from the definition of $\Phi_{L_1}(H, \nu)$, the compact support of the function $u_{Q,t}(\cdot)$ in (10.8), and the fact that the lattice $(\mathfrak{a}_{M,t}^\vee + \mathfrak{a}_{L,F}^\vee)$ has only finitely many orbits in $\mathscr{E}^G(t\sigma, \sigma)$.

We assume that $d(T) \geqslant \delta \,\|T\|$ for some fixed $\delta > 0$. We must study the function

$$\int_{T_{Q_1} + \mathfrak{a}_M^{L_1}} \sigma_M^{Q_1}(H, T)\, \Phi_{L_1}(H, \nu)\, dH \tag{11.2}$$

as $\|T\|$ approaches infinity. There are two cases. If L_1 does not contain L, we argue as on p. 1306 of [6], and conclude that the absolute value of (11.2) is $O(\|T\|^{-n})$ for any n. If L_1 does contain L, we argue as on p. 1307 of [6]. In this case, (11.2) differs from the function

$$\int_{T_{Q_1} + \mathfrak{a}_L^{L_1}} \sigma_L^{Q_1}(H_L, T) \left(\int_{\mathfrak{a}_M^L} \Phi_{L_1}(H_M^L + H_L, \nu)\, dH_M^L \right) dH_L$$

by an expression whose absolute value is $O(\|T\|^{-n})$ for any n. Before summarizing these conclusions, we shall describe the last function slightly differently. Let ν_L denote the projection of any element $\nu \in \mathscr{E}^G(t\sigma, \sigma)$ onto $i\mathfrak{a}_L^*$. This depends only on the $\mathfrak{a}_{M,t}^\vee$-orbit of ν. Combining Fourier inversion on $\mathfrak{a}_M^L$ with the definition of Φ_{L_1}, we see that

$$\int_{\mathfrak{a}_M^L} \Phi_{L_1}(H_M^L + H_L, \nu)\, dH_M^L$$

equals

$$\sum_{L_2 \in \mathscr{L}(M)} d_M^G(L_1, L_2)\, e^{\nu_L(H_L)} \int_{i\mathfrak{a}_{L,F}^*} D_M^{Q_2}(\nu_L, \lambda_L, \nu)\, d\lambda_L.$$

But if L_1 contains L,

$$\int_{T_{Q_1} + \mathfrak{a}_L^{L_1}} \sigma_L^{Q_1}(H_L, T)\, e^{\nu_L(H_L)}\, dH_L = C_L^{Q_1}(\nu_L, T),$$

so that the function which is asymptotic to (11.2) can be written as

$$\sum_{L_2 \in \mathscr{L}(M)} d_M^G(L_1, L_2)\, C_L^{Q_1}(\nu_L, T) \int_{i\mathfrak{a}_{L,F}^*} D_M^{Q_2}(\nu_L, \lambda_L, \nu)\, d\lambda_L.$$

We have reached the following conclusion. There is a constant c_n for each n, such that the difference between (11.1) and the expression

$$\sum_{\nu \in \mathscr{E}^G(t\sigma, \sigma)/\mathfrak{a}_{M,t}^\vee} \sum_{\substack{L_1 \in \mathscr{L}(L) \\ L_2 \in \mathscr{L}(M)}} d_M^G(L_1, L_2)\, C_L^{Q_1}(\nu_L, T) \int_{i\mathfrak{a}_{L,F}^*} D_M^{Q_2}(\nu_L, \lambda_L, \nu)\, d\lambda_L \tag{11.3}$$

is bounded in absolute value by $c_n \, \| T \|^{-n}$.

We now have an expression from which we can reconstruct the distribution $J^T(f)$ obtained earlier from the geometric side. To account for the dependence of the group L on the element $t \in W(\mathfrak{a}_M)$, we shall write $W(\mathfrak{a}_M)$ as the disjoint union over $L \in \mathscr{L}(M)$ of the sets

$$W(\mathfrak{a}_M^L)_{\text{reg}} = \{ t \in W(\mathfrak{a}_M^L) : \det(t - 1)_{\mathfrak{a}_M^L} \neq 0 \}.$$

Lemma **11.1.** — *The distribution* $J^T(f)$ *equals the sum over* $M \in \mathscr{L}$, $\sigma \in \{ \Pi_2(M(F)) \}$, $S \in \mathscr{B}_P(\sigma)$, $L \in \mathscr{L}(M)$ *and* $t \in W(\mathfrak{a}_M^L)_{\text{reg}}$ *of the product of*

$$| W_0^M | \, | W_0^G |^{-1} \, | \det(t - 1)_{\mathfrak{a}_M^L} |^{-1} \, | \mathfrak{a}_{M,\sigma}^\vee / \mathfrak{a}_{M,F}^\vee |^{-1} \tag{11.4}$$

with the expression (11.3).

Proof. — Write $J_{\text{spec}}^T(f)$ for the given sum. We must show that $J_{\text{spec}}^T(f)$ equals $J^T(f)$. Combining Lemmas 10.1 and 10.5 with the discussion above, we see that there is a constant c_n for every positive integer n, with the property that

$$| K^T(f) - J_{\text{spec}}^T(f) | \leqslant c_n \, \| T \|^{-n}$$

whenever $d(T) \geqslant \delta \, \| T \|$. But $J^T(f)$ bears a similar relationship with $K^T(f)$, by Proposition 4.5. We can therefore choose c_n such that

$$| J^T(f) - J_{\text{spec}}^T(f) | \leqslant c_n \, \| T \|^{-n}$$

whenever $d(T) \geqslant \delta \, \| T \|$. According to Proposition 6.1, $J^T(f)$ is of the general form

$$\sum_{\xi \in \left(\frac{1}{N} \mathscr{L}_0^\vee\right)/\mathscr{L}_0^\vee} p_\xi(T, f)\, e^{\xi(T)}, \tag{11.5}$$

where N is a positive integer and $p_\xi(T, f)$ is a function whose restriction to each chamber $\mathscr{L}_0 \cap \mathfrak{a}_0^+$ is a polynomial in T. If we can show that $J_{\text{spec}}^T(f)$ is also of the form (11.5), we will be done. For a nonzero expression (11.5) cannot be rapidly decreasing as T ranges over the points in any chamber.

By definition [4, § 6]

$$C_L^{Q_1}(\nu_L, T) = \lim_{\zeta \to 0} \sum_{\{R \in \mathscr{P}(L) : R \subset Q_1\}} e^{(\zeta + \nu_L)(T_R)}\, \theta_R(\zeta + \nu_L)^{-1},$$

where ζ is a small point in $i\mathfrak{a}_L^*$ in general position. Taking the Laurent expansion of the function

$$e^{(z\zeta+\nu_L)(T_R)}\,\theta_R(z\zeta+\nu_L)^{-1},\quad z\in\mathbf{C},$$

about $z=0$ as in [31, p. 315], we see that

$$C_L^{Q_1}(\nu_L, T) = \sum_{\{R\in\mathcal{P}(L):R\subset Q_1\}} q_{R,\nu_L}(T_R)\, e^{\nu_L(T_R)},$$

where $\{q_{R,\nu_L}(T_R)\}$ are polynomials in T_R. This in turn is of the form

$$\sum_{\xi\in\left(\frac{1}{N}\mathcal{L}_0^\vee\right)/\mathcal{L}_0^\vee} e^{\xi(T)}\, q_\xi(T), \tag{11.6}$$

where N is a positive integer which can be chosen independently of ν, L and M, and each $q_\xi(T)$ is a polynomial in $T\in\mathcal{L}_0\cap\mathfrak{a}_0^+$. We note for future use that $q_0(T)$ vanishes identically unless ν_L belongs to the subgroup $\mathcal{L}_L^\vee$ of $\mathcal{L}_M^\vee$. Indeed, $e^{\nu_L(T_R)}$ equals 1 for all $T\in\mathcal{L}_0$ precisely when ν_L lies in $\mathcal{L}_L^\vee$. Suppose that this is actually the case. Then $q_0(T)$ equals $C_L^{Q_1}(\nu_L, T)$, and is a homogeneous polynomial of total degree equal to $\dim(A_L/A_{L_1})$. In particular, $q_0(0)$ vanishes unless $L_1 = L$, in which case it equals 1. We shall use this in the proof of the corollary below.

To complete the proof of the lemma, we need only observe that if the expression (11.6) is substituted back into (11.3), the result is just a finite linear combination of such expressions. The same is then true of $J_{\mathrm{spec}}^T(f)$, which for any given f is just a finite linear combination of functions (11.3). In other words, $J_{\mathrm{spec}}^T(f)$ is of the form (11.5), and consequently equals $J^T(f)$. □

It is not $J^T(f)$ that we ultimately want, but rather the constant term $\tilde{J}(f) = p_0(0,f)$ in the expansion (11.5) of $J^T(f)$.

Corollary **11.2.** — *The distribution* $\tilde{J}(f)$ *equals the sum over* $M\in\mathcal{L}$, $\sigma\in\{\Pi_2(M(F))\}$, $S\in\mathcal{B}_P(\sigma)$, $L\in\mathcal{L}(M)$ *and* $t\in W(\mathfrak{a}_M^L)_{\mathrm{reg}}$ *of the product of* (11.4) *with the expression*

$$\sum_{\nu^L\in\mathcal{E}^L(t\sigma,\sigma)/\mathfrak{a}_{M,t}^\vee}\ \sum_{\nu_L\in\mathcal{L}_L^\vee}\int_{i\mathfrak{a}_{L,F}^*} D_L(\nu_L, \lambda_L, \nu^L+\nu_L)\, d\lambda_L. \tag{11.7}$$

Proof. — We have seen that as a function of T, (11.3) is of the general form (11.5). According to remarks in the proof of the lemma, the constant term of any summand in (11.3) vanishes except when $L_1 = L$ and the projection of ν onto $i\mathfrak{a}_L^*$ lies in $\mathcal{L}_L^\vee$. It follows from the lemma itself that $\tilde{J}(f)$ equals the sum over M, σ, S, L and t of the product of (11.4) with the expression

$$\sum_{\nu}\ \sum_{L_2\in\mathcal{L}(M)} d_M^G(L, L_2)\int_{i\mathfrak{a}_{L,F}^*} D_M^{Q_2}(\nu_L, \lambda_L, \nu)\, d\lambda_L, \tag{11.8}$$

in which ν is summed over the set

$$\{\nu\in\mathcal{E}^G(t\sigma,\sigma)/\mathfrak{a}_{M,t}^\vee : \nu_L\in\mathcal{L}_L^\vee\}.$$

It remains to show that (11.8) equals (11.7).

If ν is an element in $\mathscr{E}^G(t\sigma, \sigma)$ such that ν_L belongs to $\mathscr{L}_L^\vee$, the element $\nu - \nu_L$ belongs to the set

$$\mathscr{E}^G(t\sigma, \sigma) \cap (i\mathfrak{a}_M^*)^L = \mathscr{E}^L(t\sigma, \sigma).$$

The sum over ν in (11.8) can therefore be written as a double sum over (ν^L, ν_L) as in (11.7).

To treat the sum over L_2 in (11.8), we should examine the retraction

$$(L_1, L_2) \to (Q_1, Q_2) \in \mathscr{P}(L_1) \times \mathscr{P}(L_2)$$

which has been implicit in our recent discussion. The retraction is defined by an arbitrary point X in $\mathfrak{a}_M^G$ in general position. For a given pair (L_1, L_2), assume that the constant $d_M^G(L_1, L_2)$ does not vanish. Then $\mathfrak{a}_M^G$ is the direct sum of $\mathfrak{a}_{L_1}^G$ and $\mathfrak{a}_{L_2}^G$, and we can write

$$X = X_1 - X_2, \quad X_i \in \mathfrak{a}_{L_i}^G, \quad i = 1, 2.$$

The groups $Q_i \in \mathscr{P}(L_i)$ are determined by the condition that X_i belongs to the chamber $\mathfrak{a}_{Q_i}^+$. (See [10, § 7].) We have set $L_1 = L$ in (11.8). Then if

$$X = X_L - X_M^L, \quad X_L \in \mathfrak{a}_L^G, \; X_M^L \in \mathfrak{a}_M^L,$$

our point X_2 is the projection of X_M^L onto $\mathfrak{a}_{L_2}^G$, relative to the decomposition $\mathfrak{a}_M^G = \mathfrak{a}_L^G \oplus \mathfrak{a}_{L_2}^G$. In other words, the retraction $L_2 \to Q_2$ is determined by a point X_M^L in general position in $\mathfrak{a}_M^L$. This was the set-up for the descent formula [10, Corollary 7.2], which tells us that

$$\sum_{L_2 \in \mathscr{L}(M)} d_M^G(L, L_2)\, D_M^{Q_2}(\nu_L, \lambda_L, \nu^L + \nu_L) = D_L(\nu_L, \lambda_L, \nu^L + \nu_L).$$

If we substitute this into (11.8) we obtain (11.7), and complete the proof of the corollary. □

We want to interpret Corollary 11.2 as an elementary identity involving induced representations and intertwining operators. However, we shall first derive an expression that will be easier to compare with the earlier geometric formula for $\tilde{J}(f)$.

Consider the expression (11.7). The double sum-integral over ν_L and λ_L is absolutely convergent, so its order may be reversed. According to the definition (10.8),

$$\begin{aligned}
&\sum_{\nu_L \in \mathscr{L}_L^\vee} D_L(\nu_L, \lambda_L, \nu^L + \nu_L) \\
&\quad = \sum_{\nu_L} \lim_{\zeta \to 0} \sum_{R \in \mathscr{P}(L)} D_R(\zeta + \nu_L, \lambda_L, \nu^L + \nu_L)\, \theta_R(\zeta + \nu_L)^{-1} \\
&\quad = \lim_{\zeta \to 0} \sum_{R \in \mathscr{P}(L)} \sum_{\nu_L} \Delta_{Q_R}(\zeta + \nu_L, \lambda_L, \nu^L + \nu_L)\, u_{Q_R, t}(\zeta + \nu_L)\, \theta_R(\zeta + \nu_L)^{-1},
\end{aligned}$$

where ζ stands for a small generic point in $i\mathfrak{a}_L^*$, and $Q = Q_R$ stands for any group in $\mathscr{P}(M)$ which is contained in R. We are making implicit use of Lemma 10.3 *a)*, which tells us that $\Delta_Q(\zeta + \nu_L, \lambda_L, \nu^L + \nu_L)$ is well defined for a generic point λ_L. By (10.9),

$$\Delta_Q(\zeta + \nu_L, \lambda_L, \nu^L + \nu_L) = \Delta_Q(\zeta, \lambda_L, \nu^L), \quad \nu_L \in \mathscr{L}_L^\vee,$$

since $\mathscr{L}_{\mathrm{L}}^{\vee}$ is a subgroup of $\mathscr{L}_{\mathrm{M}}^{\vee}$. Moreover, we deduce that

$$\begin{aligned}
&\sum_{\nu_{\mathrm{L}} \in \mathscr{L}_{\mathrm{L}}^{\vee}} u_{\mathrm{Q_R},\ell}(\zeta+\nu_{\mathrm{L}})\,\theta_{\mathrm{R}}(\zeta+\nu_{\mathrm{L}})^{-1}\\
&\qquad = \sum_{\nu_{\mathrm{L}} \in \mathscr{L}_{\mathrm{L}}^{\vee}} u_{\mathrm{R},\ell}(\zeta+\nu_{\mathrm{L}})\,\theta_{\mathrm{R}}(\zeta+\nu_{\mathrm{L}})^{-1}\\
&\qquad = \sum_{\nu_{\mathrm{L}} \in \mathscr{L}_{\mathrm{L}}^{\vee}} u_{\mathrm{R}}(\zeta+\nu_{\mathrm{L}})\,\theta_{\mathrm{R},\ell}(\zeta+\nu_{\mathrm{L}})^{-1}\\
&\qquad = \sum_{\nu_{\mathrm{L}} \in \mathscr{L}_{\mathrm{L}}^{\vee}/\mathscr{L}_{\mathrm{L},\ell}^{\vee}} \Big(\sum_{\nu_{\mathrm{L},\ell} \in \mathscr{L}_{\mathrm{L},\ell}^{\vee}} u_{\mathrm{R}}(\zeta+\nu_{\mathrm{L}}+\nu_{\mathrm{L},\ell})\Big)\,\theta_{\mathrm{R},\ell}(\zeta+\nu_{\mathrm{L}})^{-1}\\
&\qquad = \sum_{\nu_{\mathrm{L}} \in \mathscr{L}_{\mathrm{L}}^{\vee}/\mathscr{L}_{\mathrm{L},\ell}^{\vee}} \theta_{\mathrm{R},\ell}(\zeta+\nu_{\mathrm{L}})^{-1},
\end{aligned}$$

from the conditions of Lemma 10.2, and the fact that $\theta_{\mathrm{R},\ell}$ is invariant under $\mathscr{L}_{\mathrm{L},\ell}^{\vee}$. It follows that (11.7) equals the sum over $\nu^{\mathrm{L}} \in \mathscr{E}^{\mathrm{L}}(t\sigma,\sigma)/\mathfrak{a}_{\mathrm{M},t}^{\vee}$ and the integral over $\lambda_{\mathrm{L}} \in i\mathfrak{a}_{\mathrm{L,F}}^{*}$ of

$$(11.9) \qquad \lim_{\zeta\to 0} \sum_{\mathrm{R}\in\mathscr{P}(\mathrm{L})} \Delta_{\mathrm{Q_R}}(\zeta,\lambda_{\mathrm{L}},\nu^{\mathrm{L}}) \Big(\sum_{\nu_{\mathrm{L}} \in \mathscr{L}_{\mathrm{L}}^{\vee}/\mathscr{L}_{\mathrm{L},\ell}^{\vee}} \theta_{\mathrm{R},\ell}(\zeta+\nu_{\mathrm{L}})^{-1}\Big).$$

It remains for us to express the sum in brackets in (11.9) in terms of the function $\theta_{\mathrm{R},k}$, where k is a large positive integer such that

$$\mathscr{L}_{\mathrm{L},k} \subset \mathscr{L}_{\mathrm{L}} \subset \mathscr{L}_{\mathrm{L},\ell},$$

for each L. If X belongs to $\mathscr{L}_{\mathrm{L},\ell}/\mathscr{L}_{\mathrm{L},k}$, and R is in $\mathscr{P}(\mathrm{L})$, we write $\mathrm{X_R}$ as in § 6 for the representative of X in $\mathscr{L}_{\mathrm{L},\ell}$ such that

$$\mathrm{X_R} = \sum_{\alpha\in\Delta_{\mathrm{R}}} r_\alpha \mu_{\alpha,k}, \qquad -1 < r_\alpha \leqslant 0.$$

Applying the identity

$$(1-t)^{-1} = (1-t^{\mathrm{N}})^{-1}(1+t+\ldots+t^{\mathrm{N}-1}),$$

with $\mathrm{N} = k\ell^{-1}$ and $t = e^{-(\zeta+\nu_{\mathrm{L}})(\mu_{\alpha,\ell})}$, to the definition (6.3) of $\theta_{\mathrm{R},k}$ and $\theta_{\mathrm{R},\ell}$, we obtain

$$\begin{aligned}
&\sum_{\nu_{\mathrm{L}} \in \mathscr{L}_{\mathrm{L}}^{\vee}/\mathscr{L}_{\mathrm{L},\ell}^{\vee}} \theta_{\mathrm{R},\ell}(\zeta+\nu_{\mathrm{L}})^{-1}\\
&\qquad = \sum_{\nu_{\mathrm{L}}} |\mathscr{L}_{\mathrm{L},\ell}/\mathscr{L}_{\mathrm{L},k}|^{-1}\,\theta_{\mathrm{R},k}(\zeta+\nu_{\mathrm{L}})^{-1}\Big(\sum_{\mathrm{X}\in\mathscr{L}_{\mathrm{L},\ell}/\mathscr{L}_{\mathrm{L},k}} e^{(\zeta+\nu_{\mathrm{L}})(\mathrm{X_R})}\Big)\\
&\qquad = |\mathscr{L}_{\mathrm{L},\ell}/\mathscr{L}_{\mathrm{L},k}|^{-1} \sum_{\nu_{\mathrm{L}} \in \mathscr{L}_{\mathrm{L}}^{\vee}/\mathscr{L}_{\mathrm{L},\ell}^{\vee}} \;\sum_{\mathrm{X}\in\mathscr{L}_{\mathrm{L},\ell}/\mathscr{L}_{\mathrm{L},k}} e^{\zeta(\mathrm{X_R})}\,\theta_{\mathrm{R},k}(\zeta)^{-1}\,e^{\nu_{\mathrm{L}}(\mathrm{X})}.
\end{aligned}$$

The last equation follows from the invariance of the function $\theta_{\mathrm{R},k}$ under the translation by $\mathscr{L}_{\mathrm{L},k}^{\vee}$, a group which contains $\mathscr{L}_{\mathrm{L}}^{\vee}$. This in turn equals

$$|\mathscr{L}_{\mathrm{L}}/\mathscr{L}_{\mathrm{L},k}|^{-1}\Big(\sum_{\mathrm{X}\in\mathscr{L}_{\mathrm{L}}/\mathscr{L}_{\mathrm{L},k}} e^{\zeta(\mathrm{X_R})}\Big)\,\theta_{\mathrm{R},k}(\zeta)^{-1},$$

by Fourier inversion on the finite abelian group $\mathscr{L}_{\mathrm{L},\ell}/\mathscr{L}_{\mathrm{L}}$.

We pause to restate Corollary 11.2 in terms of the new expression we have obtained from (11.7). It will be convenient to take the sum over S inside all the other operations.

This is certainly permissible, since $\Delta_{Q_R}(\zeta, \lambda_L, \nu^L)$ vanishes for all but finitely many S. Then $\tilde{J}(f)$ equals the sum over $M \in \mathscr{L}$, $\sigma \in \{\Pi_2(M(F))\}$, $L \in \mathscr{L}(M)$, $t \in W(\mathfrak{a}_M^L)_{\mathrm{reg}}$, $\nu^L \in \mathscr{E}^L(t\sigma, \sigma)/\mathfrak{a}_{M,t}^\vee$ and the integral over $\lambda_L \in i\mathfrak{a}_{L,F}^*$ of the product of (11.4) with

$$(11.10) \qquad \lim_{\zeta \to 0} \sum_{R \in \mathscr{P}(L)} \Big(\sum_{S \in \mathscr{B}_P(\sigma)} \Delta_{Q_R}(\zeta, \lambda_L, \nu^L) \Big) \Big(|\mathscr{L}_L/\mathscr{L}_{L,k}|^{-1} \sum_{X \in \mathscr{L}_L/\mathscr{L}_{L,k}} e^{\zeta(X_R)} \theta_{R,k}(\zeta)^{-1} \Big).$$

This begins to resemble the geometric formula for $\tilde{J}(f)$ of § 6.

The representation theoretic objects are of course wrapped up in $\Delta_{Q_R}(\zeta, \lambda_L, \nu)$. Fix t and ν^L, and let μ be the uniquely determined point in $(i\mathfrak{a}_M^*)^L$ such that $\nu^L = \mu - t\mu$. We shall also take a variable point $\xi \in (i\mathfrak{a}_M^*)^L$ in general position which approaches 0. Fixing the other two points in the triplet $(\zeta, \lambda_L, \nu^L)$ as well (with the proviso that λ_L be in general position), we set $\lambda = \xi + \mu + \lambda_L$, and $\lambda' = \lambda + \zeta$. Then the point

$$\Lambda = t\lambda' - \lambda + \nu^L = t\xi - \xi + \zeta$$

approaches ζ as ξ approaches 0. It follows from the definition that $\Delta_Q(\zeta, \lambda_L, \nu^L)$ is the limit as ξ approaches 0 of

$$(c_{Q|P}(\lambda - \nu^L)^{-1} c_{Q|P}(t\lambda')\, {}^0c(t, \lambda')\, \psi_{S_\lambda(f)}, \psi_S)_\nu.$$

Set $\bar{\sigma} = \sigma_\mu$. Since ν^L belongs to $\mathscr{E}^L(t\sigma, \sigma)$, we have

$$t\bar{\sigma} = (t\sigma)_{t\mu} = \sigma_{\nu^L + t\mu} = \bar{\sigma}.$$

Let A_μ be the map which sends any $\psi \in \mathscr{A}_\sigma(M, \tau_{P|P})$ to the function

$$\psi_\mu(m) = \psi(m)\, e^{\mu(H_M(m))}, \quad m \in M(F),$$

in $\mathscr{A}_{\bar{\sigma}}(M, \tau_{P|P})$. Then if T belongs to $\mathrm{End}(\mathscr{H}_P(\sigma)_\Gamma)$ for some Γ, and $\bar{T}$ is the corresponding operator in $\mathrm{End}(\mathscr{H}_P(\bar{\sigma})_\Gamma)$, $A_\mu(\psi_T)$ equals $\psi_{\bar{T}}$. It follows from the definitions of the c-functions, and the invariance of ${}^0c(t, \cdot)$ under translation by $i\mathfrak{a}_L^*$, that

$$\begin{aligned}
&(c_{Q|P}(\lambda - \nu^L)^{-1} c_{Q|P}(t\lambda')\, {}^0c(t, \lambda')\, \psi_{S_\lambda(f)}, \psi_S)_\nu \\
&\quad = (A_{t\mu}(c_{Q|P}(\lambda - \nu^L)^{-1} c_{Q|P}(t\lambda')\, {}^0c(t, \lambda')\, \psi_{S_\lambda(f)}), A_\mu(\psi_S))_0 \\
&\quad = (c_{Q|P}(\lambda - \nu^L - t\mu)^{-1} c_{Q|P}(t\lambda' - t\mu)\, {}^0c(t, \lambda' - \mu)\, A_\mu(\psi_{S_\lambda(f)}), A_\mu(\psi_S))_0 \\
&\quad = (c_{Q|P}(\xi + \lambda_L)^{-1} c_{Q|P}(t\xi + \lambda_L + \zeta)\, {}^0c(t, \xi)\, \psi_{\overline{S_\lambda(f)}}, \psi_{\bar{S}})_0.
\end{aligned}$$

To this last expression we apply the formulas (7.6) and (7.7) for c-functions, and also the inner product formula (7.1). Since

$$\overline{S_\lambda(f)} = d_{\bar{\sigma}}\, \mathscr{I}_P(\bar{\sigma}_{\lambda-\mu}, f_2)\, \bar{S}\mathscr{I}_P(\bar{\sigma}_{\lambda-\mu}, f_1^\vee),$$

we find that

$$\begin{aligned}
\Delta_Q(\zeta, \lambda_L, \nu^L) &= \lim_{\xi \to 0} (d_{\bar{\sigma}}(\psi_{T_2(\xi)\bar{S}T_1(\xi)}, \psi_{\bar{S}})_0) \\
&= \lim_{\xi \to 0} (\mathrm{tr}(T_2(\xi)\, \bar{S}T_1(\xi)\, \bar{S}^*)),
\end{aligned}$$

where

$$T_1(\xi) = \mathscr{I}_P(\bar{\sigma}_{\xi+\lambda_L}, f_1^\vee)\, R(t, \bar{\sigma}_\xi)^{-1} J_{P|Q}(\bar{\sigma}_{t\xi+\lambda_L+\zeta})\, J_{P|Q}(\bar{\sigma}_{\xi+\lambda_L})^{-1}$$

and

$$T_2(\xi) = J_{\bar{Q}|P}(\bar{\sigma}_{\xi+\lambda_L})^{-1} J_{\bar{Q}|P}(\bar{\sigma}_{t\xi+\lambda_L+\zeta})\, R(t, \bar{\sigma}_\xi)\, \mathscr{I}_P(\bar{\sigma}_{\xi+\lambda_L}, f_2).$$

This formula becomes slightly more tractible when we take the required sum over $S \in \mathscr{B}_P(\sigma)$. For $\{\bar{S} : S \in \mathscr{B}_P(\sigma)\}$ is an orthonormal basis of the space of Hilbert-Schmidt operators on $\mathscr{H}_P(\bar{\sigma})$. Therefore

$$\sum_{S \in \mathscr{B}_P(\sigma)} \operatorname{tr}(T_2(\xi)\, \bar{S} T_1(\xi)\, \bar{S}^*)$$

equals the trace of the endomorphism $T \to T_2(\xi)\, T T_1(\xi)$ on the space of Hilbert-Schmidt operators, or what is the same thing, the product of the traces of $T_1(\xi)$ and $T_2(\xi)$. We obtain

$$\sum_{S \in \mathscr{B}_P(\sigma)} \Delta_Q(\zeta, \lambda_L, \nu^L) = \lim_{\xi \to 0} (\operatorname{tr}(T_1(\xi)) \operatorname{tr}(T_2(\xi))),$$

for the operators $T_1(\xi)$ and $T_2(\xi)$ on $\mathscr{H}_P(\bar{\sigma})$ defined above.

It is not hard to compute the limit as ξ approaches 0. The group $P \in \mathscr{P}(M)$ is fixed, so that $\Pi = P \cap L$ is a fixed parabolic subgroup of L. Remember that $Q = Q_R$ stands for any group in $\mathscr{P}(M)$ which is contained in R. We take Q to be $R(\Pi)$, the unique such group whose intersection with L equals Π. Then the only singular hyperplanes which separate the chambers of P and Q in $\mathfrak{a}_M$ correspond to roots which do not vanish on $\mathfrak{a}_L$. It follows that $J_{P|Q}(\bar{\sigma}_{\lambda_L})$ is well defined and analytic at the generic point $\lambda_L \in i\mathfrak{a}_L^*$. Taking the limits of each of the four operators in the product $T_1(\xi)$, we obtain

$$\lim_{\xi \to 0} T_1(\xi) = \mathscr{I}_P(\bar{\sigma}_{\lambda_L}, f_1^\vee)\, R(t, \bar{\sigma})^{-1} J_{P|R(\Pi)}(\bar{\sigma}_{\lambda_L + \zeta})\, J_{P|R(\Pi)}(\bar{\sigma}_{\lambda_L})^{-1}.$$

To deal with $T_2(\xi)$, we note that $\bar{Q} = \bar{R}(\bar{\Pi})$, and that

$$d(\bar{Q}, P) = d(\bar{R}(\bar{\Pi}), \bar{R}(\Pi)) + d(\bar{R}(\Pi), P).$$

(We write $d(\bar{Q}, P)$ for the number of singular hyperplanes which separate the chambers of $\bar{Q}$ and P.) It follows that

$$J_{\bar{Q}|P}(\cdot) = J_{\bar{R}(\bar{\Pi})|\bar{R}(\Pi)}(\cdot)\, J_{\bar{R}(\Pi)|P}(\cdot).$$

Therefore

$$\lim_{\xi \to 0} T_2(\xi) = J_{\bar{R}(\Pi)|P}(\bar{\sigma}_{\lambda_L})^{-1} . \varepsilon_{\bar{\sigma}}(t) . J_{\bar{R}(\Pi)|P}(\bar{\sigma}_{\lambda_L + \zeta})\, R(t, \bar{\sigma})\, \mathscr{I}_P(\bar{\sigma}_{\lambda_L}, f_2),$$

where

$$\varepsilon_{\bar{\sigma}}(t) = \lim_{\xi \to 0} (J_{\bar{R}(\bar{\Pi})|\bar{R}(\Pi)}(\bar{\sigma}_\xi)^{-1} J_{\bar{R}(\bar{\Pi})|\bar{R}(\Pi)}(\bar{\sigma}_{t\xi})).$$

We are using the fact that $J_{\bar{R}(\bar{\Pi})|\bar{R}(\Pi)}(\bar{\sigma}_\eta)$ depends only on the projection of the point $\eta \in i\mathfrak{a}_M^*$ onto $i\mathfrak{a}_M^*/i\mathfrak{a}_L^*$.

The operator $\varepsilon_{\bar{\sigma}}(t)$ is actually a scalar. To evaluate it, write

$$J_*(\bar{\sigma}_\eta) = r_*(\bar{\sigma}_\eta)\, R_*(\bar{\sigma}_\eta).$$

The normalized intertwining operator, being regular at $\eta = 0$, contributes nothing to the limit. Therefore, $\varepsilon_{\bar{\sigma}}(t)$ equals

$$\lim_{\xi \to 0} (r_{\bar{R}(\bar{\Pi})|\bar{R}(\Pi)}(\bar{\sigma}_\xi)^{-1} r_{\bar{R}(\bar{\Pi})|\bar{R}(\Pi)}(\bar{\sigma}_{t\xi})) = \lim_{\xi \to 0} \prod_{\beta \in \Sigma^r_{\bar{\Pi}}} r_\beta(\bar{\sigma}_\xi)^{-1} r_\beta(\bar{\sigma}_{t\xi}).$$

Let $\Sigma^r_{\overline{\Pi},\bar\sigma}$ denote the set of roots $\beta \in \Sigma^r_{\overline{\Pi}}$ such that the function $r_\beta(\bar\sigma_\xi)$ has a simple pole at $\xi = 0$. If β belongs to the complement of $\Sigma^r_{\overline{\Pi},\bar\sigma}$ in $\Sigma^r_{\overline{\Pi}}$, $r_\beta(\bar\sigma_\xi)$ is analytic and nonzero at $\xi = 0$, and so contributes nothing to the last limit. (This follows from the fact that the rank one μ-function $\mu_\beta(\bar\sigma_\xi) = |r_\beta(\bar\sigma_\xi)|^{-2}$ is either analytic and nonzero at $\xi = 0$, or has a zero of order 2.) We obtain

$$\varepsilon_{\bar\sigma}(t) = \lim_{\xi\to 0} \prod_{\beta\in\Sigma^r_{\overline{\Pi},\bar\sigma}} r_\beta(\bar\sigma_\xi)^{-1} r_\beta(\bar\sigma_{t\xi}) = \lim_{\xi\to 0} \prod_{\beta\in\Sigma^r_{\overline{\Pi},\bar\sigma}} \xi(\beta^\vee)\,(t\xi)\,(\beta^\vee)^{-1}.$$

Because $r_\beta(\bar\sigma_{t\xi})$ equals $r_{t^{-1}\beta}(\bar\sigma_\xi)$, we see that t^{-1} maps $\Sigma^r_{\overline{\Pi},\bar\sigma}$ into the union of $\Sigma^r_{\overline{\Pi},\bar\sigma}$ with $\Sigma^r_{\Pi,\bar\sigma}$. It follows that $\varepsilon_{\bar\sigma}(t)$ equals (-1) raised to the power

$$|\, t(\Sigma^r_{\Pi,\bar\sigma}) \cap \Sigma^r_{\overline{\Pi},\bar\sigma}\,|.$$

We have established that

$$\sum_{S\in\mathscr{P}_P(\sigma)} \Delta_{Q_R}(\zeta, \lambda_L, \nu^L)$$

equals the product of $\varepsilon_{\bar\sigma}(t)$ with the traces of each of the operators

$$\mathscr{I}_P(\bar\sigma_{\lambda_L}, f_1^\vee)\, R(t,\bar\sigma)^{-1}\, J_{P|R(\Pi)}(\bar\sigma_{\lambda_L+\zeta})\, J_{P|R(\Pi)}(\bar\sigma_{\lambda_L})^{-1}$$

and

$$J_{\overline{R}(\Pi)|P}(\bar\sigma_{\lambda_L})^{-1}\, J_{\overline{R}(\Pi)|P}(\bar\sigma_{\lambda_L+\zeta})\, R(t,\bar\sigma)\, \mathscr{I}_P(\bar\sigma_{\lambda_L}, f_2).$$

If we substitute this expression into (11.10), we obtain an elementary but rather complicated formula for $\tilde{J}(f)$ in terms of representation theoretic data. Before stating the formula, we shall describe how to combine the required sums over σ and ν^L.

In general, for any $t \in W(\mathfrak{a}^L_M)$ we shall write $\Pi_2(M(F))^t$ for the set of representations in $\Pi_2(M(F))$ which are fixed by t. We also write $\Pi_2(M(F))^t/i\mathfrak{a}^*_L$ for the set of orbits in $\Pi_2(M(F))^t$ under the action of $i\mathfrak{a}^*_L$. Fix $t \in W(\mathfrak{a}^L_M)_{\text{reg}}$ as above. For a given $\sigma \in \Pi_2(M(F))$, we have associated a representation $\bar\sigma = \sigma_\mu$ in $\Pi_2(M(F))^t$ to each point $\nu^L \in \mathscr{E}^L(t\sigma,\sigma)$. Conversely, if $\bar\sigma = \sigma_\mu$ is a representation in $\Pi_2(M(F))^t$, the point $\nu^L = \mu - t\mu$ belongs to $\mathscr{E}^L(t\sigma,\sigma)$. Now the original sum over $\nu^L \in \mathscr{E}^L(t\sigma,\sigma)$ was taken only modulo the action of $\mathfrak{a}^\vee_{M,t}$. The isotropy group $\mathfrak{a}^\vee_{M,t}$ is isomorphic under the map $\mu \to \nu^L = \mu - t\mu$ to the group

$$(\mathfrak{a}^\vee_{M,F} + i\mathfrak{a}^*_L)/i\mathfrak{a}^*_L \cong \mathfrak{a}^\vee_{M,F}/\mathfrak{a}^\vee_{M,F} \cap i\mathfrak{a}^*_L = \mathfrak{a}^\vee_{M,F}/\mathfrak{a}^\vee_{L,F}.$$

On the other hand, two representations σ_μ and $\sigma_{\mu'}$ define the same object in $\Pi_2(M(F))^t/i\mathfrak{a}^*_L$ if and only if $\mu - \mu'$ belongs to the group

$$(\mathfrak{a}^\vee_{M,\sigma} + i\mathfrak{a}^*_L)/i\mathfrak{a}^*_L \cong \mathfrak{a}^\vee_{M,\sigma}/\mathfrak{a}^\vee_{M,\sigma} \cap i\mathfrak{a}^*_L = \mathfrak{a}^\vee_{M,\sigma}/\mathfrak{a}^\vee_{L,\sigma}.$$

We are supposed to sum the product of (11.4) and (11.10) over $\nu^L \in \mathscr{E}^L(t\sigma,\sigma)/\mathfrak{a}^\vee_{M,t}$. We see that we can convert this to a sum over $\bar\sigma \in \Pi_2(M(F))^t/i\mathfrak{a}^*_L$, provided we multiply the summand by

$$|(\mathfrak{a}^\vee_{M,\sigma}/\mathfrak{a}^\vee_{L,\sigma})/(\mathfrak{a}^\vee_{M,F}/\mathfrak{a}^\vee_{L,F})| = |\mathfrak{a}^\vee_{M,\sigma}/\mathfrak{a}^\vee_{M,F}|\,|\mathfrak{a}^\vee_{L,\sigma}/\mathfrak{a}^\vee_{L,F}|^{-1}.$$

Observe that the product of (11.4) with this number equals

$$|W_0^M|\,|W_0^G|^{-1}\,|\det(t-1)_{\mathfrak{a}_M^L}|^{-1}\,|\mathfrak{a}_{L,\sigma}^\vee/\mathfrak{a}_{L,F}^\vee|^{-1}.$$

We have at last obtained a reasonably explicit spectral expansion for $\tilde{J}(f)$. In stating it, we shall write σ instead of $\bar{\sigma}$ for a representation in $\Pi_2(M(F))^t$ and λ instead of λ_L for a point in $i\mathfrak{a}_L^*$. For each such σ and λ, set $\tilde{J}_L(\sigma_\lambda, t, f)$ equal to

$$\lim_{\zeta\to 0} \sum_{R\in\mathscr{P}(L)} \tau_{1,R}(\zeta)\,\tau_{2,\bar{R}}(\zeta)\,\Big(|\mathscr{L}_L/\mathscr{L}_{L,k}|^{-1} \sum_{X\in\mathscr{L}_L/\mathscr{L}_{L,k}} e^{\zeta(X_R)}\,\theta_{R,k}(\zeta)^{-1}\Big), \tag{11.11}$$

where

$$\tau_{1,R}(\zeta) = \operatorname{tr}(\mathscr{I}_P(\sigma_\lambda, f_1^\vee)\,R(t,\sigma)^{-1}\,J_{P|R(\Pi)}(\sigma_{\lambda+\zeta})\,J_{P|R(\Pi)}(\sigma_\lambda)^{-1})$$

and

$$\tau_{2,\bar{R}}(\zeta) = \operatorname{tr}(J_{\bar{R}(\Pi)|P}(\sigma_\lambda)^{-1}\,J_{\bar{R}(\Pi)|P}(\sigma_{\lambda+\zeta})\,R(t,\sigma)\,\mathscr{I}_P(\sigma_\lambda, f_2)),$$

for any point $\zeta \in i\mathfrak{a}_L^*$. The discussion above can then be summarized as

Proposition **11.3.** — *The distribution* $\tilde{J}(f)$ *equals*

$$\sum_M\sum_L\sum_t\sum_\sigma |W_0^M|\,|W_0^G|^{-1}\,|\det(t-1)_{\mathfrak{a}_M^L}|^{-1}\varepsilon_\sigma(t)\,|\mathfrak{a}_{L,\sigma}^\vee/\mathfrak{a}_{L,F}^\vee|^{-1}\int_{i\mathfrak{a}_{L,F}^*} \tilde{J}_L(\sigma_\lambda, t, f)\,d\lambda, \tag{11.12}$$

with the sums being taken over $M \in \mathscr{L}$, $L \in \mathscr{L}(M)$, $t \in W(\mathfrak{a}_M^L)_{\text{reg}}$ *and* $\sigma \in \Pi_2(M(F))^t/i\mathfrak{a}_L^*$. □

12. The local trace formula

We are at last in a position to establish our local trace formula. We shall first describe the objects which go into the final formula. We shall then derive the formula from the two expansions (Propositions 6.1 and 11.3) for the distribution $\tilde{J}(f)$.

If $x = (x_1, x_2)$ belongs to $G(A_F) = G(F) \times G(F)$, and $M \in \mathscr{L}$ is a Levi subgroup, the functions

$$v_P(\Lambda, x) = e^{-\Lambda(H_P(x_2) - H_{\bar{P}}(x_1))}, \quad \Lambda \in i\mathfrak{a}_M^*,\ P \in \mathscr{P}(M), \tag{12.1}$$

form a (G, M)-family. The limit

$$v_M(x) = \lim_{\Lambda\to 0} \sum_{P\in\mathscr{P}(M)} v_P(\Lambda, x)\,\theta_P(\Lambda)^{-1}$$

then exists [4, Lemma 6.2], and in fact equals the volume in $\mathfrak{a}_M/\mathfrak{a}_G$ of the convex hull of the points

$$\{-H_P(x_2) + H_{\bar{P}}(x_1) : P \in \mathscr{P}(M)\}.$$

The function $v_M(x)$ is invariant under left translation by $M(A_F) = M(F) \times M(F)$. If γ is a G-regular element in M(F), embedded diagonally in $M(A_F)$, we can define the weighted orbital integral

$$J_M(\gamma, f) = |D(\gamma)| \int_{A_M(A_F)\backslash G(A_F)} f(x^{-1}\gamma x)\,v_M(x)\,dx, \tag{12.2}$$

which of course also equals

$$|D(\gamma)|\int_{A_M(F)\backslash G(F)}\int_{A_M(F)\backslash G(F)} f_1(x_1^{-1}\gamma x_1)f_2(x_2^{-1}\gamma x_2)\, v_M(x_1, x_2)\, dx_1\, dx_2.$$

Let us write $\Gamma_{\mathrm{ell}}(M)$ simply for the set of conjugacy classes in $M(A_F)$ of the form (γ, γ), where γ is an F-elliptic conjugacy class in $M(F)$. The distributions $J_M(\gamma, f)$, taken at the G-regular elements in $\Gamma_{\mathrm{ell}}(M)$, will be the main ingredients on the geometric side of the formula.

To describe the terms on the other side, we shall first define a distribution that can be called the "discrete part" of the spectral side of the formula. Suppose that σ is a representation in $\Pi_2(M(F))$, for some $M \in \mathscr{L}$. We shall write $\sigma^\vee$ for the contragredient of σ. In § 11 we introduced a function ε_σ, which may be regarded as a sign character on the group

$$W_\sigma = \{ t \in W(\mathfrak{a}_M) : t\sigma \cong \sigma \}.$$

Then

$$\varepsilon_\sigma(t) = (-1)^{|t(\Sigma^r_{P,\sigma}) \cap \Sigma^r_{\bar{P},\sigma}|}, \quad t \in W_\sigma, \tag{12.3}$$

where $P \in \mathscr{P}(M)$ is a parabolic subgroup, and $\Sigma^r_{P,\sigma}$ is the set of reduced roots $\beta \in \Sigma^r_P$ whose normalizing factor $r_\beta(\sigma_\xi)$ has a pole at $\xi = 0$. If t belongs to W_σ, we can form the (normalized) intertwining operator

$$R(t, \sigma^\vee \otimes \sigma) = R(t, \sigma^\vee) \otimes R(t, \sigma)$$

from the induced representation

$$\mathscr{I}_P(\sigma^\vee \otimes \sigma) = \mathscr{I}_P(\sigma^\vee) \otimes \mathscr{I}_P(\sigma)$$

to itself. This operator is independent of the representative of t in the normalizer of $M(F)$ in $G(F)$. Moreover, it depends only on the orbit of σ in $\Pi_2(M(F))^t/i\mathfrak{a}^*_G$. The distribution

$$\int_{i\mathfrak{a}^*_{G,F}} \tilde{J}_G(\sigma_\lambda, t, f)\, d\lambda = \int_{i\mathfrak{a}^*_{G,F}} \operatorname{tr}(R(t, \sigma^\vee \otimes \sigma)\, \mathscr{I}_P(\sigma^\vee_{-\lambda} \otimes \sigma_\lambda, f))\, d\lambda$$

also depends only on the orbit of σ, and in addition, depends only on the restriction f^1 of f to the subgroup

$$G(A_F)^1 = \{(y_1, y_2) \in G(A_F) : H_G(y_1) = H_G(y_2)\}.$$

In analogy with automorphic forms, we define

$$I_{\mathrm{disc}}(f) = I^G_{\mathrm{disc}}(f)$$

to be the expression

$$\sum_{M, t, \sigma} |W^M_0|\, |W^G_0|^{-1}\, |\det(t-1)_{\mathfrak{a}^G_M}|^{-1}\, \varepsilon_\sigma(t)\, |\mathfrak{a}^\vee_{G,\sigma}/\mathfrak{a}^\vee_{G,F}|^{-1} \int_{i\mathfrak{a}^*_{G,F}} \tilde{J}_G(\sigma_\lambda, t, f)\, d\lambda, \tag{12.4}$$

where the sums are taken over $M \in \mathscr{L}$, $t \in W(\mathfrak{a}^G_M)_{\mathrm{reg}}$ and $\sigma \in \Pi_2(M(F))^t/i\mathfrak{a}^*_G$. Regarded as a distribution in f^1, $I_{\mathrm{disc}}(f)$ is a finite linear combination of irreducible characters. We will use the coefficients to describe the general terms on the spectral side.

Let $\Pi_{\mathrm{disc}}(G)$ denote the set of equivalence classes of irreducible representations $\pi = \pi_1^\vee \otimes \pi_2$ of $G(A_F)$ which are constituents of induced representations

$$\mathscr{I}_P(\sigma^\vee \otimes \sigma) = \mathscr{I}_P(\sigma^\vee) \otimes \mathscr{I}_P(\sigma), \quad P \in \mathscr{P}(M),$$

in which σ is a representation in $\Pi_2(M(F))^t$ for some element $t \in W(\mathfrak{a}_M^G)_{\mathrm{reg}}$. We write $\Pi_{\mathrm{disc}}(G)/i\mathfrak{a}_G^*$ for the set of orbits in $\Pi_{\mathrm{disc}}(G)$ under the action

$$\pi \to \pi_\lambda = \pi_{1,-\lambda}^\vee \otimes \pi_{2,\lambda}, \quad \lambda \in i\mathfrak{a}_G^*,$$

of $i\mathfrak{a}_G^*$. It is known that σ is uniquely determined by π, up to conjugation by W_0^G. We can therefore define a measure $d\pi$ on $\Pi_{\mathrm{disc}}(G)$ by setting

$$\int_{\Pi_{\mathrm{disc}}(G)} \varphi(\pi)\, d\pi = \sum_{\pi \in \Pi_{\mathrm{disc}}(G)/i\mathfrak{a}_G^*} |\mathfrak{a}_{G,\sigma}^\vee / \mathfrak{a}_{G,F}^\vee|^{-1} \int_{i\mathfrak{a}_{G,F}^*} \varphi(\pi_\lambda)\, d\lambda, \qquad \varphi \in C_c(\Pi_{\mathrm{disc}}(G)). \tag{12.5}$$

It follows from the definitions that we can write

$$I_{\mathrm{disc}}(f) = \int_{\Pi_{\mathrm{disc}}(G)} a_{\mathrm{disc}}^G(\pi)\, \mathrm{tr}(\pi(f))\, d\pi, \tag{12.6}$$

where each $a_{\mathrm{disc}}^G(\pi)$ is a uniquely determined complex number that depends only on the $i\mathfrak{a}_G^*$-orbit of π. The numbers

$$a_{\mathrm{disc}}^M(\pi), \quad M \in \mathscr{L},\ \pi \in \Pi_{\mathrm{disc}}(M)/i\mathfrak{a}_M^*,$$

can be defined in this way for all Levi subgroups. They will appear as the general coefficients on the spectral side.

Suppose that $M \in \mathscr{L}$ is a general Levi subgroup, and that $\pi = \pi_1^\vee \otimes \pi_2$ is a representation in $\Pi_{\mathrm{disc}}(M)$. For any $P \in \mathscr{P}(M)$ and $\lambda \in i\mathfrak{a}_M^*$, we can form the induced representation

$$\mathscr{I}_P(\pi_\lambda, f) = \mathscr{I}_P(\pi_{1,-\lambda}^\vee, f_1) \otimes \mathscr{I}_P(\pi_{2,\lambda}, f_2),$$

of the Hecke algebra $\mathscr{H}(G(A_F))$, and the standard unnormalized intertwining operators

$$J_{Q|P}(\pi_\lambda) = J_{\overline{Q}|P}(\pi_{1,-\lambda}^\vee) \otimes J_{Q|P}(\pi_{2,\lambda}), \quad Q \in \mathscr{P}(M).$$

Observe that $J_{Q|P}(\pi_\lambda)$ maps $\mathscr{H}_P(\pi_1^\vee) \otimes \mathscr{H}_P(\pi_2)$ to $\mathscr{H}_{\overline{Q}}(\pi_1^\vee) \otimes \mathscr{H}_Q(\pi_2)$. However, the operators

$$\mathscr{J}_Q(\Lambda, \pi_\lambda, P) = J_{Q|P}(\pi_\lambda)^{-1} J_{Q|P}(\pi_{\lambda+\Lambda}), \quad \Lambda \in i\mathfrak{a}_M^*, \tag{12.7}$$

map $\mathscr{H}_P(\pi) = \mathscr{H}_P(\pi_1^\vee) \otimes \mathscr{H}_P(\pi_2)$ to itself. In fact, the set

$$\{\mathscr{J}_Q(\Lambda, \pi_\lambda, P) : Q \in \mathscr{P}(M)\}$$

can be regarded as a (G, M)-family of functions of $\Lambda \in i\mathfrak{a}_M^*$ with values in the space of (operator-valued) meromorphic functions of λ. In particular, the limit

$$\mathscr{J}_M(\pi_\lambda, P) = \lim_{\Lambda \to 0} \sum_{Q \in \mathscr{P}(M)} \mathscr{J}_Q(\Lambda, \pi_\lambda, P)\, \theta_Q(\Lambda)^{-1}$$

exist, and is a meromorphic function of λ.

Lemma **12.1.** — *The matrix coefficients of the operator* $\mathscr{J}_M(\pi_\lambda, P)$ *are analytic functions of* $\lambda \in i\mathfrak{a}^*_{M,F}$ *whose derivatives are slowly increasing.*

Proof. — Consider the special case that

$$\pi = \sigma^\vee \otimes \sigma, \quad \sigma \in \Pi_2(M(F)).$$

The assertion of the lemma then becomes a special case of Corollary 10.4, in which $L = M$ and $t = 1$. For arbitrary π, the lemma can be proved either directly from this special case, or by mimicking the relevant part of the proof of Corollary 10.4. □

Writing $\mathscr{J}_M(\pi, P)$ for the value of $\mathscr{J}_M(\pi_\lambda, P)$ at $\lambda = 0$, we define

$$J_M(\pi, f) = \operatorname{tr}(\mathscr{J}_M(\pi, P)\, \mathscr{I}_P(\pi, f)). \tag{12.8}$$

The distributions $J_M(\pi, f)$, which depend only on equivalence classes of representations $\pi \in \Pi_{\mathrm{disc}}(M)$, will be the main ingredients of the spectral side of the formula.

We can now state the local trace formula.

Theorem **12.2.** — *For any function* $f \in \mathscr{H}(G(A_F))$, *the expression*

$$\sum_{M \in \mathscr{L}} |W_0^M|\, |W_0^G|^{-1} (-1)^{\dim(A_M/A_G)} \int_{\Gamma_{\mathrm{ell}}(M)} J_M(\gamma, f)\, d\gamma \tag{12.9}$$

equals

$$\sum_{M \in \mathscr{L}} |W_0^M|\, |W_0^G|^{-1} (-1)^{\dim(A_M/A_G)} \int_{\Pi_{\mathrm{disc}}(M)} a^M_{\mathrm{disc}}(\pi)\, J_M(\pi, f)\, d\pi. \tag{12.10}$$

Proof. — We should first observe that the integrals in the two expressions are absolutely convergent. For the integrand in (12.9) is a locally integrable function of compact support (cf. Lemma 4.3). In the case of (12.10), the K-finiteness of f implies that the expression is a finite linear combination of integrals

$$\int_{i\mathfrak{a}^*_{M,F}} J_M(\pi_\lambda, f)\, d\lambda, \quad \pi \in \Pi_{\mathrm{disc}}(M).$$

But it follows easily from Lemma 12.1 that $J_M(\pi_\lambda, f)$ is a Schwartz function of $\lambda \in i\mathfrak{a}^*_{M,F}$. Next we note that it is sufficient to take

$$f(x) = f_1(x_1) f_2(x_2), \quad f_i \in \mathscr{H}(G(F)),$$

as before. Given our control over the convergence of the integrals, this follows from a standard approximation argument.

We make the induction assumption that the theorem holds if G is replaced by any proper Levi subgroup. We shall write $J_{\mathrm{geom}}(f) = J^G_{\mathrm{geom}}(f)$ and $J_{\mathrm{spec}}(f) = J^G_{\mathrm{spec}}(f)$ for the respective expressions (12.9) and (12.10). Our aim is to convert the geometric and spectral expressions of $\tilde{J}(f)$ into two parallel linear combinations of distributions $\{J^{M_Q}_{\mathrm{geom}}(f_Q)\}$ and $\{J^{M_Q}_{\mathrm{spec}}(f_Q)\}$, in which Q ranges over the groups in $\mathscr{F}$, and

$$f_Q(m) = \delta_Q(m)^{1/2} \int_{K \times K} \int_{N_Q(A_F)} f(k^{-1} m n k)\, dn\, dk, \quad m \in M_Q(A_F).$$

We will then be able to exploit the induction hypothesis, in the form

$$J^{M_Q}_{\mathrm{geom}}(f_Q) = J^{M_Q}_{\mathrm{spec}}(f_Q), \qquad Q \neq G. \tag{12.11}$$

Consider first the geometric side. According to Proposition 6.1,

$$\tilde{J}(f) = \sum_M |W_0^M| \, |W_0^G|^{-1} \int_{\Gamma_{\mathrm{ell}}(M)} \tilde{J}_M(\gamma, f)\, d\gamma,$$

where

$$\tilde{J}_M(\gamma, f) = |D(\gamma)| \int_{A_M(\mathbf{A}_F)\backslash G(\mathbf{A}_F)} f(x^{-1}\gamma x)\, \tilde{v}_M(x)\, dx.$$

Motivated by the definition (6.6) of $\tilde{v}_M(x)$, we set

$$c_{\overline{P}}(\Lambda) = |\mathscr{L}_M/\mathscr{L}_{M,k}|^{-1} \sum_{X \in \mathscr{L}_M/\mathscr{L}_{M,k}} e^{\Lambda(X_P)}\, \theta_{P,k}(\Lambda)^{-1}\, \theta_P(\Lambda), \tag{12.12}$$

for $P \in \mathscr{P}(M)$ and $\Lambda \in i\mathfrak{a}_M^*$. Then

$$\begin{aligned}\tilde{v}_M(x) &= \lim_{\Lambda \to 0} \sum_{P \in \mathscr{P}(M)} v_{\overline{P}}(\Lambda, x)\, c_{\overline{P}}(\Lambda)\, \theta_P(\Lambda)^{-1} \\ &= (-1)^{\dim(A_M/A_G)} \lim_{\Lambda \to 0} \sum_{P \in \mathscr{P}(M)} v_P(\Lambda, x)\, c_P(\Lambda)\, \theta_P(\Lambda)^{-1},\end{aligned}$$

since $\theta_P(\Lambda)$ equals $(-1)^{\dim(A_M/A_G)}\, \theta_{\overline{P}}(\Lambda)$. It is clear that $c_P(\Lambda)$ is smooth for Λ near 0. Moreover, it is easy to check from the definitions of X_P, $\theta_{P,k}$ and θ_P that $\{c_P(\Lambda) : P \in \mathscr{P}(M)\}$ is a (G, M)-family of functions of Λ, in a neighbourhood of 0 in $i\mathfrak{a}_M^*$. It follows from the product formula [4, Lemma 6.3] that

$$\tilde{v}_M(x) = (-1)^{\dim(A_M/A_G)} \sum_{Q \in \mathscr{F}(M)} v_M^Q(x)\, c'_Q,$$

in the notation of [4, § 6]. When we substitute this back in the integral above, and apply the usual change of variable formula, we obtain a term

$$|D(\gamma)| \int_{A_M(\mathbf{A}_F)\backslash G(\mathbf{A}_F)} f(x^{-1}\gamma x)\, v_M^Q(x)\, dx = J_M^{M_Q}(\gamma, f_Q).$$

This is the analogue for M_Q of the distribution $J_M(\gamma) = J_M^G(\gamma)$. Therefore the original sum over M, taken inside the sum over Q, yields the distribution $J^{M_Q}_{\mathrm{geom}}(f_Q)$. It follows that

$$\tilde{J}(f) = \sum_{Q \in \mathscr{F}} |W_0^{M_Q}| \, |W_0^G|^{-1} (-1)^{\dim(A_Q/A_G)} J^{M_Q}_{\mathrm{geom}}(f_Q)\, c'_Q. \tag{12.13}$$

We turn next to the spectral side. By Proposition 11.3, $\tilde{J}(f)$ may be written as the sum over $L \in \mathscr{L}$ of the product of $|W_0^L| \, |W_0^G|^{-1}$ with

$$\sum_{M, t, \sigma} |W_0^M| \, |W_0^L|^{-1} |\det(t-1)_{\mathfrak{a}_M^L}|^{-1} \varepsilon_\sigma(t)\, |\mathfrak{a}^\vee_{L,\sigma}/\mathfrak{a}^\vee_{L,F}|^{-1} \int_{i\mathfrak{a}^*_{L,F}} \tilde{J}_L(\sigma_\lambda, t, f)\, d\lambda, \tag{12.14}$$

where the triple sum is over $\{M \in \mathscr{L} : M \subset L\}$, $t \in W(\mathfrak{a}_M^L)_{\mathrm{reg}}$ and $\sigma \in \Pi_2(M(F))^t/i\mathfrak{a}_L^*$. The distribution $\tilde{J}_L(\sigma_\lambda, t, f)$ is defined by the expression (11.11), in which we can recognize the formula (12.12) for the function $c_{\overline{R}}(\cdot)$. Substituting for this formula in (11.11), we obtain

$$\begin{aligned}\tilde{J}_L(\sigma_\lambda, t, f) &= \lim_{\zeta\to 0} \sum_{R\in\mathscr{P}(L)} \tau_{1,R}(\zeta)\, \tau_{2,\overline{R}}(\zeta)\, c_{\overline{R}}(\zeta)\, \theta_R(\zeta)^{-1} \\ &= (-1)^{\dim(A_L/A_G)} \lim_{\zeta\to 0} \sum_{R\in\mathscr{P}(L)} \tau_{1,\overline{R}}(\zeta)\, \tau_{2,R}(\zeta)\, c_R(\zeta)\, \theta_R(\zeta)^{-1}.\end{aligned}$$

We must examine the numbers $\tau_{1,\overline{R}}(\zeta)$ and $\tau_{2,R}(\zeta)$, which were defined prior to the statement of Proposition 11.3.

Writing $A^\vee$ for the transpose of an operator A, one sees from the definitions of induced representations and intertwining operators that

$$\begin{aligned}\tau_{1,\overline{R}}(\zeta) &= \mathrm{tr}(\mathscr{I}_P(\sigma_\lambda, f_1^\vee)\, R(t,\sigma)^{-1} J_{P|\overline{R}(\Pi)}(\sigma_{\lambda+\zeta})\, J_{P|\overline{R}(\Pi)}(\sigma_\lambda)^{-1}) \\ &= \mathrm{tr}(J^\vee_{P|\overline{R}(\Pi)}(\sigma_\lambda)^{-1} J^\vee_{P|\overline{R}(\Pi)}(\sigma_{\lambda+\zeta})\, R^\vee(t,\sigma)^{-1} \mathscr{I}^\vee_P(\sigma_\lambda, f_1^\vee)) \\ &= \mathrm{tr}(J_{\overline{R}(\Pi)|P}(\sigma^\vee_{-\lambda})^{-1} J_{\overline{R}(\Pi)|P}(\sigma^\vee_{-(\lambda+\zeta)})\, R(t,\sigma^\vee)\, \mathscr{I}_P(\sigma^\vee_{-\lambda}, f_1)).\end{aligned}$$

If we set

$$J_{R|P}(\sigma^\vee_{-\lambda}\otimes\sigma_\lambda) = J_{\overline{R}(\Pi)|P}(\sigma^\vee_{-\lambda}) \otimes J_{R(\Pi)|P}(\sigma_\lambda), \quad R\in\mathscr{P}(L),$$

we can then write the product $\tau_{1,\overline{R}}(\zeta)\,\tau_{2,R}(\zeta)$ as

$$\mathrm{tr}(J_{R|P}(\sigma^\vee_{-\lambda}\otimes\sigma_\lambda)^{-1} J_{R|P}(\sigma^\vee_{-(\lambda+\zeta)}\otimes\sigma_{\lambda+\zeta})\, R(t,\sigma^\vee\otimes\sigma)\, \mathscr{I}_P(\sigma^\vee_{-\lambda}\otimes\sigma_\lambda, f)),$$

in the notation above. Recall that P is any group in $\mathscr{P}(M)$ such that $P\cap L = \Pi$. We claim that $\tilde{J}_L(\sigma_\lambda, t, f)$ is independent of which such P is chosen. To see this, write the unnormalized operator

$$J_{R|P}(\sigma^\vee_{-\lambda}\otimes\sigma_\lambda)^{-1} J_{R|P}(\sigma^\vee_{-(\lambda+\zeta)}\otimes\sigma_{\lambda+\zeta})$$

in the expression above as the product of a normalized operator

$$R_{R|P}(\sigma^\vee_{-\lambda}\otimes\sigma_\lambda)^{-1} R_{R|P}(\sigma^\vee_{-(\lambda+\zeta)}\otimes\sigma_{\lambda+\zeta}) \tag{12.15}$$

and the scalar

$$\begin{aligned}& r_{\overline{R}(\Pi)|P}(\sigma^\vee_{-\lambda})^{-1} r_{\overline{R}(\Pi)|P}(\sigma^\vee_{-(\lambda+\zeta)})\, r_{R(\Pi)|P}(\sigma_\lambda)^{-1} r_{R(\Pi)|P}(\sigma_{\lambda+\zeta}) \\ &\qquad = r_{P|\overline{R}(\Pi)}(\sigma_\lambda)^{-1} r_{P|\overline{R}(\Pi)}(\sigma_{\lambda+\zeta})\, r_{R(\Pi)|P}(\sigma_\lambda)^{-1} r_{R(\Pi)|P}(\sigma_{\lambda+\zeta}).\end{aligned}$$

The property $P\cap L = \Pi$ implies that

$$d(R(\Pi), P) + d(P, \overline{R}(\Pi)) = d(R(\Pi), \overline{R}(\Pi)).$$

The scalar therefore equals

$$r_{R(\Pi)|\overline{R}(\Pi)}(\sigma_\lambda)^{-1} r_{R(\Pi)|\overline{R}(\Pi)}(\sigma_{\lambda+\zeta}),$$

and is independent of P. On the other hand, the contribution to $\tilde{J}_L(\sigma_\lambda, t, f)$ of the term (12.15) is also independent of P. This follows from the multiplicative properties of normalized intertwining operators, as on [4, p. 44]. Having justified the claim, we are free to choose $P = S(\Pi)$, where S is a fixed group in $\mathscr{P}(L)$.

For any irreducible subrepresentation $\pi = \pi_1^\vee \otimes \pi_2$ of the induced representation $\mathscr{I}_\Pi(\sigma^\vee \otimes \sigma)$ of $L(A_F)$, there is a canonical embedding of $\mathscr{H}_S(\pi)$ into $\mathscr{H}_P(\sigma^\vee \otimes \sigma)$. The restriction of $\mathscr{I}_P(\sigma_{-\lambda}^\vee \otimes \sigma_\lambda, f)$ to this subspace equals $\mathscr{I}_S(\pi_\lambda, f)$, while the restriction of

$$J_{R|P}(\sigma_{-\lambda}^\vee \otimes \sigma_\lambda)^{-1} J_{R|P}(\sigma_{-(\lambda+\zeta)}^\vee \otimes \sigma_{\lambda+\zeta})$$

to $\mathscr{H}_S(\pi)$ equals the operator

$$J_{R|S}(\pi_\lambda)^{-1} J_{R|S}(\pi_{\lambda+\zeta}) = \mathscr{I}_R(\zeta, \pi_\lambda, S).$$

With these observations, we can apply the definition (12.4) of I_{disc}^L to the expression (12.14). The two parallel expansions in (M, t, σ) become parallel expansions in the elements $\pi \in \Pi_{\mathrm{disc}}(L)/i\mathfrak{a}_L^*$. Combining the formulas we have obtained for $\tilde{J}_L(\sigma_\lambda, t, f)$ and $\tau_{1,\overline{R}}(\zeta)\, \tau_{2,R}(\zeta)$, we see that (12.14) equals

$$\int_{\Pi_{\mathrm{disc}}(L)} a_{\mathrm{disc}}^L(\pi)\, \tilde{J}_L(\pi, f)\, d\pi,$$

where

$$\tilde{J}_L(\pi, f) = (-1)^{\dim(A_L/A_G)} \lim_{\zeta \to 0} \sum_{R \in \mathscr{P}(L)} \mathrm{tr}(\mathscr{I}_R(\zeta, \pi, S)\, \mathscr{I}_S(\pi, f))\, c_R(\zeta)\, \theta_R(\zeta)^{-1}.$$

Thus,

$$\tilde{J}(f) = \sum_{L \in \mathscr{L}} |W_0^L|\, |W_0^G|^{-1} \int_{\Pi_{\mathrm{disc}}(L)} a_{\mathrm{disc}}^L(\pi)\, \tilde{J}_L(\pi, f)\, d\pi.$$

Now, we can write

$$\tilde{J}_L(\pi, f) = (-1)^{\dim(A_L/A_G)} \sum_{Q \in \mathscr{F}(L)} \mathrm{tr}(\mathscr{I}_L^Q(\pi, S)\, \mathscr{I}_S(\pi, f))\, c_Q',$$

by the product formula [4, Lemma 6.3]. Moreover, a standard argument [4, Lemma 7.1] gives us

$$\mathrm{tr}(\mathscr{I}_L^Q(\pi, S)\, \mathscr{I}_S(\pi, f)) = J_L^{M_Q}(\pi, f_Q),$$

the analogue for M_Q of the distribution $J_L(\pi) = J_L^G(\pi)$. Take the sum over Q outside the sum over L. After making the substitution

$$J_{\mathrm{spec}}^{M_Q}(f_Q) = \sum_{\{L : L \subset Q\}} |W_0^L|\, |W_0^{M_Q}|^{-1} (-1)^{\dim(A_L/A_Q)} \int_{\Pi_{\mathrm{disc}}(L)} a_{\mathrm{disc}}^L(\pi)\, J_L^{M_Q}(\pi, f_Q)\, d\pi,$$

we are left with the formula

$$\tilde{J}(f) = \sum_{Q \in \mathscr{F}} |W_0^{M_Q}|\, |W_0^G|^{-1} (-1)^{\dim(A_Q/A_G)} J_{\mathrm{spec}}^{M_Q}(f_Q)\, c_Q'. \tag{12.16}$$

The theorem follows from the identity of right hand sides of (12.13) and (12.16). By our induction hypothesis (12.11), the terms corresponding to $Q \neq G$ are pairwise equal. This leaves the two terms with $Q = G$, which are just $J_{\mathrm{geom}}(f)$ and $J_{\mathrm{spec}}(f)$. The equality of these distributions was what we had to prove. □

Remarks. — 1. The local trace formula is thus the identity between the two distributions (12.9) and (12.10). This should be compared with the global trace formula. The global trace formula of course applies to a function on an adèle group, rather than a real or p-adic group. It is nevertheless an identity of two distributions [12, (3.2) and (3.3)], defined by geometric and spectral expansions which are remarkably similar to (12.9) and (12.10). The local ingredients of these expansions are distributions on the adèle group which are analogues of (12.2) and (12.8). (The notation in (12.8) is slightly different from that used in the global trace formula. The distribution $J_M(\pi, f)$ here has been defined in terms of unnormalized intertwining operators, whereas its global counterpart in [12, (3.3)] was defined in terms of normalized operators.)

2. The local trace formula is actually not as difficult as the global trace formula. One reason for this is that the geometric terms are parameterized only by semisimple conjugacy classes. In the global formula, the geometric terms are parametrized by unipotent as well as semisimple classes. A similar phenomenon occurs on the spectral side. The terms in the local formula are parametrized by tempered representations, while in the global formula there are also terms coming from non-tempered representations in the discrete spectrum.

3. The sign $\varepsilon_\sigma(t)$, which occurs in the definition (12.4) and (12.6) of the coefficients $a^G_{\mathrm{disc}}(\pi)$, is an interesting object. It has a simple description in terms of the R-group of σ. Recall [25] that there is a decomposition

$$W_\sigma = W'_\sigma \rtimes R_\sigma$$

of the stabilizer of σ in $W(\mathfrak{a}_M)$ into a semi-direct product. The group W'_σ is the Weyl group of a root system, namely, the reduced roots whose corresponding rank one Plancherel density vanishes at σ. The R-group R_σ is the subgroup of elements in W_σ which preserve a positive chamber for this root system. It follows from the definition (12.3) that ε_σ is the pull-back to W_σ of the usual sign character of the Weyl group W'_σ.

4. The distributions (12.2) and (12.8) are not invariant. However, it is not hard to derive an invariant local trace formula from the identity of (12.9) and (12.10). One ends up with an identity of two expansions which are identical to (12.9) and (12.10), except that $J_M(\gamma, f)$ and $J_M(\pi, f)$ are replaced by invariant distributions. The process, which is similar to that used in the global trace formula, is described in [13, § 8]. (See also [14, § 1-2].)

5. The formula is likely to have a number of applications to local harmonic analysis. Three such examples, all based on a natural approximation argument, have been sketched in [14, § 3]. We hope to investigate further applications in another paper.

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