TOWARDS A LOCAL TRACE FORMULA

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1. Suppose that G is a connected, reductive algebraic group over a local field F. We assume that F is of characteristic 0. We can form the Hilbert space $L^2(G(F))$ of functions on $G(F)$ which are square integrable with respect to the Haar measure. The regular representation

$$(R(y_1, y_2)\phi)(x) = \phi(y_1^{-1}xy_2), \qquad \phi \in L^2(G(F)), \qquad x, y_1, y_2 \in G(F),$$

is then a unitary representation of $G(F) \times G(F)$ on $L^2(G(F))$. Kazhdan has suggested that there should be a local trace formula attached to R which is analogous to the global trace formula for automorphic forms. The purpose of this note is to discuss how one might go about proving such an identity, and to describe the ultimate form the identity is likely to take.

To see the analogy with automorphic forms more clearly, consider the diagonal embedding of F into the ring

$$A_F = F \oplus F.$$

The group $G(A_F)$ of A_F-valued points in G is just $G(F) \times G(F)$. The group $G(F)$ embeds into $G(A_F)$ as the diagonal subgroup. Observe that we can map $L^2(G(F))$ isomorphically onto $L^2(G(F)\backslash G(A_F))$ by sending any $\phi \in L^2(G(F))$ to the function

$$(g_1, g_2) \to \phi(g_1^{-1}g_2), \qquad (g_1, g_2) \in G(F)\backslash G(A_F).$$

In this way, the representation R becomes equivalent to the regular representation of $G(A_F)$ on $L^2(G(F)\backslash G(A_F))$.

As is well known, R may be interpreted as a representation of the convolution algebra

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$$C_c^\infty(G(A_F)) = C_c^\infty(G(F) \times G(F)).$$

Consider a function in this algebra of the form

$$f(y_1, y_2) = f_1(y_1)f_2(y_2), \qquad y_1, y_2 \in G(F),$$

for functions f_1 and f_2 in $C_c^\infty(G(F))$. Then $R(f)$ is an operator on $L^2(G(F))$ which maps a function ϕ to the function

$$\begin{aligned}(R(f)\phi)(x) &= \int_{G(A_F)} f(g)(R(g)\phi)(x)dg \\ &= \int_{G(F)}\int_{G(F)} f_1(u)f_2(y)\phi(u^{-1}xy)dudy \\ &= \int_{G(F)}\left(\int_{G(F)} f_1(xu)f_2(uy)du\right)\phi(y)dy.\end{aligned}$$

Thus, $R(f)$ is an integral operator with kernel

$$K(x, y) = \int_{G(F)} f_1(xu)f_2(uy)du, \qquad x, y \in G(F). \tag{1.1}$$

In the 1970's, Harish-Chandra studied the values of this kernel on the diagonal. He introduced a certain truncation of the resulting function, which he used in the case of p-adic F to show that the restriction of $R(f)$ to the space of cusp forms has finite rank [3(b)]. Harish-Chandra's truncation remains somewhat mysterious, and it is not clear what role it might play in the local trace formula.

Implicit in Kazhdan's suggestion is that one should play off (1.1) with the formula for $K(x, y)$ given by Eisenstein integrals. The identity obtained by equating these two formulas for $K(x, x)$ could then be taken as the starting point. The function $K(x, x)$ will not be integrable unless $G(F)$ is compact. However, one can always multiply each expression for $K(x, x)$ by the characteristic function of a large compact set. This is the most naive form of truncation, but, perhaps surprisingly, it appears that it will be possible to compute the resulting integrals. One obtains the weighted orbital integrals and weighted characters which are the local terms in the global trace formula [1(h)]. It has always seemed strange that these local objects should have occurred only in a global context. It now appears that they do have a genuine local interpretation, as objects which arise naturally from a problem in local harmonic analysis.

2. As an example, consider the case that G is anisotropic over F. Then $G(F)$ is a compact group. The function $K(x, x)$ is of course smooth, and is therefore integrable. Changing variables, and applying the Weyl integration formula, we obtain

$$\int_{G(F)} K(x, x)dx = \int_{G(F)} \int_{G(F)} f_1(xu)f_2(ux)dudx$$

$$= \int_{G(F)} \int_{G(F)} f_1(u)f_2(x^{-1}ux)dudx$$

$$= \int_{G(F)} \left(\int_{(G(F))} \int_{G(F)} |D(\gamma)| f_1(x_1^{-1}\gamma x_1)f_2(x^{-1}x_1^{-1}\gamma x_1 x)dx_1 d\gamma\right)dx,$$

where $(G(F))$ stands for the set of regular conjugacy classes in $G(F)$, $d\gamma$ is the measure on $(G(F))$ induced from appropriate Haar measures on maximal tori of $G(F)$, and $D(\gamma)$ is the Weyl discriminant. Thus

$$D(\gamma) = det(1 - Ad(\gamma))_{\mathfrak{g}/\mathfrak{g}_\gamma},$$

where $\mathfrak{g}$ is the Lie algebra of G, and $\mathfrak{g}_\gamma$ is the Lie algebra of the centralizer of γ in G. Since $G(F)$ is compact, we can take the integral over x inside the integrals over x_1 and γ. Changing variables from x to $x_2 = x_1 x$, we obtain

$$\int_{G(F)} K(x, x)dx = \int_{(G(F))} J_G(\gamma, f)d\gamma,$$

where

$$J_G(\gamma, f) = |D(\gamma)| \int_{G(F)} f_1(x_1^{-1}\gamma x_1)dx_1 \int_{G(F)} f_2(x_2^{-1}\gamma x_2)dx_2.$$

We have expressed the integral of $K(x, x)$ in terms of products of orbital integrals.

On the other hand, the integral of $K(x, x)$ equals the trace of $R(f)$. This operator decomposes into a direct sum over the irreducible constituents of R. As motivation for the noncompact case, let us describe the decomposition formally in terms of the function $K(x, x)$.

If (σ, V_σ) is an irreducible representation of $G(F)$, let $HS(V_\sigma)$ be the Hilbert space of Hilbert-Schmidt operators on V_σ. (This of course is just a finite dimensional matrix algebra in the present case, since $G(F)$ is compact.) The L^2-direct sum

$$\bigoplus_\sigma HS(V_\sigma),$$

taken over the set of equivalence classes of σ, and relative to the inner product

$$\Big(\bigoplus_\sigma S_\sigma, \bigoplus_\sigma S'_\sigma\Big) = \sum_\sigma (S_\sigma, S'_\sigma)\deg(\sigma) = \sum_\sigma \operatorname{tr}(S_\sigma(S'_\sigma)^*)\deg(\sigma),$$

is a Hilbert space which supports a unitary representation

$$\bigoplus_\sigma S_\sigma \to \bigoplus_\sigma(\sigma(y_2)S_\sigma\sigma(y_1)^{-1}), \qquad y_1, y_2 \in G(F),$$

of $G(F) \times G(F)$. The map

$$\bigoplus_\sigma S_\sigma \to \sum_\sigma \operatorname{tr}(\sigma(x)S_\sigma)\deg(\sigma), \qquad x \in G(F), \tag{2.1}$$

is then an isometric isomorphism from $\oplus_\sigma HS(V_\sigma)$ onto $L^2(G(F))$ which intertwines the two representations of $G(F) \times G(F)$. For each σ, let $\mathcal{B}_\sigma$ be an orthonormal basis of $HS(V_\sigma)$. By pulling back the operator $R(f)$ to $\oplus_\sigma$ HS(V_σ), we obtain a second formula

$$\sum_\sigma \sum_{S\in\mathcal{B}_\sigma} \operatorname{tr}(\sigma(x)\sigma(f_2)S\sigma(\tilde{f}_1))\overline{\operatorname{tr}(\sigma(y)S)}\deg(\sigma)$$

for the kernel $K(x, y)$. We are writing $\tilde{f}_1$ here for the function $x_1 \to f_1(x_1^{-1})$. In particular, $\operatorname{tr}(\sigma(\tilde{f}_1))$ equals $\operatorname{tr}(\tilde{\sigma}(f_1))$, where $\tilde{\sigma}$ stands for the contragradient of σ. Since f is smooth, the terms in the series are rapidly decreasing. We obtain

$$\begin{aligned}\int_{G(F)} K(x,x)dx &= \sum_{\sigma}\sum_{S\in\mathcal{B}_\sigma}\int_{G(F)} \operatorname{tr}(\sigma(x)\sigma(f_2)S\sigma(\tilde{f}_1))\overline{\operatorname{tr}(\sigma(x)S)}dx \deg(\sigma)\\ &= \sum_{\sigma}\sum_{S\in\mathcal{B}_\sigma} \operatorname{tr}(\sigma(f_2)S\sigma(\tilde{f}_1)S^*)\\ &= \sum_{\sigma} \operatorname{tr}(\sigma(\tilde{f}_1))\operatorname{tr}(\sigma(f_2))\\ &= \sum_{\pi=(\tilde{\sigma},\sigma)} J_G(\pi,f),\end{aligned}$$

where

$$J_G(\pi,f) = \operatorname{tr}(\tilde{\sigma}(f_1))\operatorname{tr}(\sigma(f_2)).$$

We have expressed the integral of $K(x, x)$ in a second way, in terms of products of characters.

The local trace formula for compact groups is then the identity

$$\int_{(G(F))} J_G(\gamma,f)d\gamma = \sum_{\pi=(\tilde{\sigma},\sigma)} J_G(\pi,f), \tag{2.2}$$

which is thus a simple consequence of the Weyl integration formula and the Peter-Weyl theorem.

3. Now, suppose that G is a general reductive group. Again there are two parallel formulas for $K(x, x)$. One is a geometric expansion in terms of regular semisimple conjugacy classes of $G(F)$. The other is a spectral expansion in terms of irreducible tempered representations of $G(F)$. We shall describe each of these in turn.

The geometric expansion is again a consequence of the Weyl integration formula. Rather than writing this simply in terms of conjugacy classes in $G(F)$, we prefer to keep track of the elliptic conjugacy classes in Levi subgroups of G. Let M_0 be a fixed minimal Levi subgroup of G over F, and let $\mathcal{L}$ stand for the (finite) set of Levi subgroups of G which contain M_0. For any $M \in \mathcal{L}$, we have the split component A_M of the center of M, and the (restricted) Weyl group W_0^M of (M, A_{M_0}). We shall write $M(F)_{ell}$ for the set of elements in $M(F)$ whose centralizer in G is a torus which is F-anisotropic modulo A_M. Then $(M(F)_{ell})$ will denote the set of $M(F)$-conjugacy classes in $M(F)_{ell}$. Any strongly G-regular element in $G(F)$ is $G(F)$-conjugate to a class in one of the sets $(M(F)_{ell})$. This class is unique modulo conjugation by the Weyl group W_0^G of G. Applying the Weyl integration formula to the right hand side of the identity

$$K(x, x) = \int_{G(F)} f_1(u) f_2(x^{-1}ux)\,du$$

obtained from (1.1), we arrive at the expression

$$(3.1) \quad \sum_{M \in \mathcal{L}} |W_0^M| \, |W_0^G|^{-1} \int_{(M(F)_{ell})} |D(\gamma)| \cdot \int_{A_M(F)\backslash G(F)} f_1(x_1^{-1}\gamma x_1) f_2(x^{-1}x_1^{-1}\gamma x_1 x)\,dx_1\,d\gamma$$

for $K(x, x)$. As before, $D(\gamma)$ is the Weyl discriminant, and $d\gamma$ stands for the measure on $(M(F)_{ell})$ induced from Haar measures on maximal tori of $M(F)$. This is the geometric expansion of $K(x, x)$.

The spectral expansion is a consequence of Harish-Chandra's Plancherel formula [3(a)], [3(b)]. Suppose that $M \in \mathcal{L}$. Let $M(F)^1$ be the kernel of the usual homomorphism H_M from $M(F)$ to

$$\mathfrak{a}_M = \mathrm{Hom}(X(M)_F, \mathbf{R}).$$

We shall write $\Pi_2(M(F)^1)$ for the set of equivalence classes of irreducible square integrable representations of $M(F)^1$. Observe that a representation $\sigma \in \Pi_2(M(F)^1)$ may be identified with an orbit

$$\sigma_\lambda(m) = \sigma_0(m) e^{\lambda(H_M(m))}, \qquad \lambda \in i\mathfrak{a}_M^*, \qquad m \in M(F),$$

of irreducible representations of $M(F)$ under the action of $i\mathfrak{a}_M^*$. If P is a fixed parabolic subgroup with Levi component M, we can then form the induced representation $\mathcal{I}_P(\sigma_\lambda)$ of $G(F)$. For each such σ, we fix a suitable orthonormal basis $\mathcal{B}_\sigma$ of the space of Hilbert-Schmidt operators acting on the underlying space of $\mathcal{I}_P(\sigma_0)$. The Plancherel formula provides an analogue of the map (2.1). This in turn leads to a second formula

$$(3.2)\quad \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\sigma\in\Pi_2(M(F)^1)} \sum_{S\in\mathcal{B}_\sigma} \int_{i\mathfrak{a}_M^*/i\mathfrak{a}_M^\vee} \operatorname{tr}(\mathcal{I}_P(\sigma_\lambda, x)\mathcal{I}_P(\sigma_\lambda, f_2) \cdot S\mathcal{I}_P(\sigma_\lambda, \tilde{f}_1)) \cdot \overline{\operatorname{tr}(\mathcal{I}_P(\sigma_\lambda, x)S)} m(\sigma_\lambda)\, d\lambda$$

for the kernel $K(x, x)$. Here

$$m(\sigma_\lambda) = d_\sigma \mu(\sigma_\lambda)$$

is the Plancherel density, given by the product of the formal degree of σ with Harish-Chandra's μ-function, while

$$\mathfrak{a}_M^\vee = \operatorname{Hom}(H_M(M(F)), 2\pi\mathbf{Z}).$$

(The subgroup $\mathfrak{a}_M^\vee \subseteq \mathfrak{a}_M^*$ is a lattice if F is a p-adic field, and is trivial if F is Archimedean.) This is the spectral expansion of $K(x, x)$.

The question becomes how one might integrate (3.1) and (3.2) over x in $A_G(F)\backslash G(F)$. It can be shown that the terms in (3.1) and (3.2) corresponding to $M = G$ are both integrable. However, none of the other terms turn out to be integrable. This can be seen most clearly in (3.1). For suppose that for some proper M,

$$\int_{A_G(F)\backslash G(F)} \int_{(M(F)_{ell})} \int_{A_M(F)\backslash G(F)} |D(\gamma)| f_1(x_1^{-1}\gamma x_1) f_2(x^{-1}x_1^{-1}\gamma x_1 x)\, dx_1\, d\gamma\, dx$$

converged (as a triple integral). Taking the integral over x inside the other integrals, and making a change of variables, we would obtain

$$\int_{(M(F)_{ell})} |D(\gamma)| \int_{A_M(F)\backslash G(F)} f_1(x_1^{-1}\gamma x_1)\, dx_1 \int_{A_G(F)\backslash G(F)} f_2(x_2^{-1}\gamma x_2)\, dx_2.$$

Since the integrand in x_2 is left-invariant under $A_M(F)$, and $A_G(F)\backslash A_M(F)$ has infinite volume, we reach a contradiction. It will therefore be necessary to truncate (3.1) and (3.2) in some fashion before the integration can be attempted.

4. Let K be a fixed maximal compact subgroup of $G(F)$. We assume that K is in good position relative to M_0, and that K corresponds to a special vertex if F is p-adic. Then

$$G(F) = KA_{M_0}(F)K.$$

Let T be a point in $\mathfrak{a}_{M_0}$ which is highly regular, in the sense that the infimum

$$d(T) = \inf_{\alpha} |\alpha(T)|,$$

taken over the roots α of (G, A_{M_0}), is large. We then define $u(x, T)$ to be the characteristic function of the set of points

$$x = k_1 h k_2, \qquad k_1, k_2 \in K, \qquad h \in A_G(F)\backslash A_{M_0}(F),$$

in $A_G(F)\backslash G(F)$ such that $H_{M_0}(h)$ lies in the convex hull

$$H_{cx}(\{sT : s \in W_0^G\}/\mathfrak{a}_G).$$

(For each M, there is a canonical decomposition $\mathfrak{a}_M = \mathfrak{a}_M^G \oplus \mathfrak{a}_G$, so in particular, $\mathfrak{a}_G$ can be regarded as a subspace of $\mathfrak{a}_{M_0}$. We are writing $H_{cx}(S/\mathfrak{a}_G)$ here for the convex hull of the projection onto $\mathfrak{a}_{M_0}/\mathfrak{a}_G$ of a subset S of $\mathfrak{a}_{M_0}$.) The function $u(x, T)$ is $A_G(F)$-invariant. It can be regarded as the characteristic function of a large compact subset of $A_G(F)\backslash G(F)$.

The product of $u(x, T)$ with each of the expressions (3.1) and (3.2) is integrable over $A_G(F)\backslash G(F)$. The problem is to make sense of the integrals. There are several questions here, but we can see that the essential computational step would be as follows. For fixed elements $\gamma \in (M(F)_{ell})$ and $\sigma \in \Pi_2(M(F)^1)$, find asymptotic formulas (as $d(T)$ becomes large) for the integrals

$$(4.1) \qquad \int_{A_G(F)\backslash G(F)} \int_{A_M(F)\backslash G(F)} f_1(x_1^{-1}\gamma x_1) f_2(x^{-1}x_1^{-1}\gamma x_1 x) u(x, T)\, dx_1\, dx,$$

and

$$(4.2) \qquad \int_{A_G(F)\backslash G(F)} \operatorname{tr}(\mathcal{I}_P(\sigma_\lambda, x)S)\overline{\operatorname{tr}(\mathcal{I}_P(\sigma_{\lambda'}, x)S')}u(x, T)\, dx.$$

In (4.2), S and S' are K-finite Hilbert-Schmidt operators on the space of $\mathcal{I}_P(\sigma_0)$, and λ and λ' are points in $i\mathfrak{a}_M^*$ whose projections onto $i\mathfrak{a}_G^*$ are equal.

5. We shall look at (4.1) and (4.2) separately. The discussion will be slightly simpler if we do not have to deal with lattices in the spaces $\mathfrak{a}_M$. Let us therefore assume until further notice that the field F is Archimedean. The maps H_M are then surjective, and the subgroups $\mathfrak{a}_M^\vee$ are trivial.

Consider first the expression (4.1). The integrals over x and x_1 are both over compact sets, so we may interchange their order. The expression becomes

$$\int_{A_M(F)\backslash G(F)} \int_{A_G(F)\backslash G(F)} f_1(x_1^{-1}\gamma x_1) f_2(x^{-1}x_1^{-1}\gamma x_1 x) u(x, T)\, dx\, dx_1$$

$$= \int_{A_M(F)\backslash G(F)} \int_{A_G(F)\backslash G(F)} f_1(x_1^{-1}\gamma x_1) f_2(x_2^{-1}\gamma x_2) u(x_1^{-1}x_2, T)\, dx_2\, dx_1$$

$$= \int_{A_M(F)\backslash G(F)} \int_{A_M(F)\backslash G(F)} f_1(x_1^{-1}\gamma x_1) f_2(x_2^{-1}\gamma x_2) u(x_1, x_2, T)\, dx_2\, dx_1$$

where

$$u(x_1, x_2, T) = \int_{A_G(F)\backslash A_M(F)} u(x_1^{-1}a x_2, T)\, da.$$

Since the centralizer of γ in $G(F)$ is compact modulo $A_M(F)$, the integrals over x_1 and x_2 in the last expression may be taken over compact sets. In particular, T may be taken to be highly regular in a sense which is uniform in x_1 and x_2.

The next lemma is the main point. Let $\mathcal{P}(M)$ be the (finite) set of parabolic subgroups $P = MN_P$ of G with Levi component M. For any such P, and any point

$$x = nmk, \qquad n \in N_P(F), \qquad m \in M(F), \qquad k \in K,$$

in $G(F)$, we set

$$H_P(x) = H_M(m),$$

as usual. We also write T_P for the projection onto $\mathfrak{a}_M$ of any Weyl translate

$$sT, \qquad s \in W_0^G,$$

such that sT lies in the positive chamber of some minimal parabolic subgroup $P_0 \in \mathcal{P}(M_0)$, with $P_0 \subset P$.

LEMMA 5.1. *Assume that $M \neq G$. Then there is a subset $S(x_1, x_2, T)$ of $A_G(F)\backslash A_M(F)$ with the following properties.*

(i) *$vol(S(x_1, x_2, T)) \leq C(x_1, x_2)e^{-\epsilon d(T)}$, for a locally bounded function $C(x_1, x_2)$ and a positive constant ϵ.*

(ii) *If a lies in the complement of $S(x_1, x_2, T)$, then $u(x_1^{-1}ax_2, T)$ equals 1 if and only if $H_M(a)$ lies in*

$$H_{cx}(\{T_P + H_P(x_1) - H_{\bar{P}}(x_2): P \in \mathcal{P}(M)\}/\mathfrak{a}_G). \tag{5.1}$$

This lemma is a generalization of [1(g), Lemma 3]. In the p-adic case treated in [1(g)], $S(x_1, x_2, T)$ is actually empty for $d(T)$ large.

The lemma allows us to relate the weight factor $u(x_1, x_2, T)$ to the volume of a convex hull. Let $\bar{v}_M(x_1, x_2, T)$ be the volume in $\mathfrak{a}_M/\mathfrak{a}_G$ of the set (5.1). Since H_M defines a proper map of $A_M(F)$ onto $\mathfrak{a}_M$, we can choose the Haar measures on these groups so that $u(x_1, x_2, T)$ is asymptotic to $\bar{v}_M(x_1, x_2, T)$. The original expression (4.1) will then equal the integral

$$\int_{A_M(F)\backslash G(F)} \int_{A_M(F)\backslash G(F)} f_1(x_1^{-1}\gamma x_1)f_2(x_2^{-1}\gamma x_2)\bar{v}_M(x_1, x_2, T)dx_2dx_1, \tag{5.2}$$

modulo a function of T which is $O(e^{-\epsilon d(T)})$. The function $\bar{v}_M(x_1, x_2, T)$ is, incidentally, a polynomial in T. Its constant term equals

$$\bar{v}_M(x_1, x_2) = (-1)^{\dim(A_M/A_G)}v_M(x_1, x_2),$$

where

$$v_M(x_1, x_2) = \mathrm{vol}(H_{cx}(\{H_{\bar{P}}(x_1) - H_P(x_2): P \in \mathcal{P}(M)\}/\underline{a}_G)). \tag{5.3}$$

6. Now consider the second expression (4.2). It is convenient to write

$$\mathrm{tr}(\mathcal{I}_P(\sigma_\lambda, k_1xk_2)S) = E(x, \psi_S, \lambda)_{(k_1,k_2)}, \qquad k_1, k_2 \in K,$$

in Harish-Chandra's notation for Eisenstein integrals. (See [3(a), Section 7] or [1(c), Section I.3-I.4].) Here, ψ_S is a $(K \cap M(F))$-spherical function from $M(F)$ to a K-finite space V_K of functions on $K \times K$, and $E(x, \psi_S, \lambda)$ is the Eisenstein integral with values in V_K. Then (4.2) becomes an integral

$$\int_{A_G(F)\backslash G(F)} (E(x, \psi_S, \lambda), E(x, \psi_{S'}, \lambda'))u(x, T)dx \tag{6.1}$$

of inner products in V_K.

In the special case of K-bi-invariant functions on $GL(n, F)$ (and with F a p-adic field), Waldspurger has found an exact formula for (6.1) in [5(a)]. It is valid whenever $d(T)$ is large, and is given in terms of Harish-Chandra's c-functions. Remarkably, it is the exact analogue of Langlands' formula ([4], [1(d)]) for the inner product of truncated cuspidal Eisenstein series. It is likely that Waldspurger's techniques will carry over to the general case of (6.1), yielding an asymptotic formula analogous to the asymptotic formula [1(d)] for arbitrary Eisenstein series. Alternatively, recent ideas of Casselman on inner products of distributions apply to (6.1), and will perhaps lead to a slick proof of the same inner product formula. In any case, the result will be an expression

$$\sum_{P_1} \sum_{s,s' \in W(\mathfrak{a}_M, \mathfrak{a}_{M_1})} (c(s, \lambda)\psi_S, c(s', \lambda')\psi_{S'})e^{(s\lambda - s'\lambda')(T_{P_1})}\theta_{P_1}(s\lambda - s'\lambda')^{-1} \tag{6.2}$$

a reader familiar with Eisenstein series will recognize. The outer sum is over parabolic subgroups $P_1 = M_1 N_{P_1}$ which contain a fixed minimal parabolic subgroup P_0, which is in turn contained in the original group $P = MN_P$. The inner sum is over the set of isomorphisms from $\mathfrak{a}_M$ onto $\mathfrak{a}_{M_1}$ which are the restrictions of elements in W_0^G. The function $c(s, \lambda)$ is of course Harish-Chandra's c-function. Finally,

$$\theta_{P_1}(s\lambda - s'\lambda') = \mathrm{vol}(\mathfrak{a}_{M_1}^G/\mathbf{Z}(\Delta_{P_1}^\vee))^{-1} \prod_{a \in \Delta_{P_1}} (s\lambda - s'\lambda')(\alpha^\vee),$$

where Δ_{P_1} is the set of simple roots of (P_1, A_{M_1}). In the case of Archimedean F that we are considering, the relation between (6.2) and (4.2) will be asymptotic. The difference between the two expressions should be bounded by

$$\rho(\lambda, \lambda')\|S\|\,\|S'\|e^{-\epsilon d(T)},$$

where $\rho(\lambda, \lambda')$ is a locally bounded function on $i\mathfrak{a}_M^* \times i\mathfrak{a}_M^*$, and ϵ is a positive constant.

7. The source of the trace formula is to be the identity between the geometric and spectral expansions (3.1) and (3.2). We are proposing to truncate these expressions simply by multiplying them by the characteristic function $u(x, T)$. The resulting integrals over x in $A_G(F)\backslash G(F)$ will of course be equal. We would like to obtain explicit formulas for the integrals by substituting the expressions (5.2) and (6.2).

On the geometric side there is an immediate question of uniformity. The asymptotic approximation (5.2) of (4.1) is only valid for fixed γ. However, γ is to be integrated over all regular elements in (3.1). I have not investigated whether there is an estimate which will take care of the elements γ in (3.1) which approach the singular set. If this is not possible, we may require a second kind of truncation, the sole purpose of which is to handle such questions of uniformity. This was the case for the global trace formula.

On the spectral side, we must compute the contribution of (6.2) to (3.2). This entails changing (6.2) by replacing S and S' with $\mathcal{I}_P(\sigma_\lambda, f_2)S\mathcal{I}_P(\sigma_\lambda, \tilde{f}_1)$ and S, respectively. We would then take the limit as λ' approached λ, and finally integrate the product of the resulting expression with $m(\sigma_\lambda)$ over $\lambda \in i\mathfrak{a}_M^*$. The combinatorics of this procedure are similar to the case of Eisenstein series [1(f)], and have been carried out by Waldspurger [5(a)], at least in a special case. Again, I do not know whether there will be a serious problem in general concerning the uniformity in λ of the asymptotic approximation (6.2). However, the analogous problem has been solved for Eisenstein series [1(e)], where it is presumably more difficult.

The end result would be an explicit trace formula which we can now describe. On the geometric side will be the distributions

$$(7.1)\quad J_M(\gamma, f)$$

$$= |D(\gamma)| \int_{A_M(F)\backslash G(F)} \int_{A_M(F)\backslash G(F)} f_1(x_1^{-1}\gamma x_1)f_2(x_2^{-1}\gamma x_2)v_M(x_1, x_2)dx_1dx_2,$$

where γ belongs to $M(F)_{ell}$ and $v_M(x_1, x_2)$ is the volume (5.3). The terms on the spectral side require a little more description.

Consider the subgroup

$$G(A_F)^1 = \{(y_1, y_2) \in G(F) \times G(F) \colon H_G(y_1) = H_G(y_2)\}$$

of $G(A_F) = G(F) \times G(F)$. We shall write $\Pi(G)$ for the set of (equivalence classes of) representations of this subgroup obtained by restricting irreducible constituents $\tilde{\pi}_1 \otimes \pi_2$ of the induced representations

$$\mathcal{I}_P(\tilde{\sigma}_{-\lambda} \otimes \sigma_\lambda) = \mathcal{I}_{\bar{P}}(\tilde{\sigma}_{-\lambda}) \otimes \mathcal{I}_P(\sigma_\lambda), \qquad M \in \mathcal{L}, \quad \sigma \in \Pi_2(M(F)^1), \quad \lambda \in i\mathfrak{a}_M^*,$$

of $G(A_F)$. (This notation is motivated by (5.3), which leads us to identify P with the parabolic subgroup $\bar{P} \times P$ of $G \times G$, instead of $P \times P$.) Associated to these induced representations, we have normalized intertwining operators

$$R(w, \tilde{\sigma}_{-\lambda} \otimes \sigma_\lambda) = R(w, \tilde{\sigma}_{-\lambda}) \otimes R(w, \sigma_\lambda), \qquad w \in W(\mathfrak{a}_M, \mathfrak{a}_M),$$

from $\mathcal{I}_P(\tilde{\sigma}_{-\lambda} \otimes \sigma_\lambda)$ to $\mathcal{I}_P(w(\tilde{\sigma}_{-\lambda}) \otimes w(\sigma_\lambda))$. These are independent of how w is represented in the normalizer of $M(F)$. It is really not $\Pi(G)$ that we want, however, but the subset $\Pi_{disc}(G)$ of such representations in which $w(\sigma_\lambda) = \sigma_\lambda$ for some element w in

$$W(\mathfrak{a}_M)_{reg} = \{w \in W(\mathfrak{a}_M, \mathfrak{a}_M) \colon \det(w - 1)_{\mathfrak{a}_M^G} \neq 0\}.$$

Now, in analogy with automorphic forms, we write

$$I_{disc}(f) = I_{disc}^G(f)$$

for the expression obtained by taking the sum over $M \in \mathcal{L}$, $w \in W(\mathfrak{a}_M)_{reg}$, $\sigma \in \Pi_2(M(F)^1)$, and $\lambda \in i\mathfrak{a}_M^*/i\mathfrak{a}_G^*$, of the product of

$$|W_0^M| \, |W_0^G|^{-1} \, |\det(w - 1)_{\mathfrak{a}_M^G}|^{-1} \epsilon_{\sigma_\lambda}(w)$$

with

$$\int_{i\mathfrak{a}_G^*} \operatorname{tr}(R(w, \tilde{\sigma}_{-(\lambda+\mu)} \otimes \sigma_{\lambda+\mu}) \mathcal{I}_P(\tilde{\sigma}_{-(\lambda+\mu)} \otimes \sigma_{\lambda+\mu}, f)) d\mu. \tag{7.2}$$

Here $\epsilon_{\sigma_\lambda}(w)$ is a certain sign character which is peculiar to the local setting, and is defined in terms of the zeros of the Plancherel density. Observe that the factor (7.2) depends only on the restriction f^1 of f to $G(A_F)^1$. This factor actually vanishes unless $w(\sigma_\lambda) = \sigma_\lambda$, a condition which can be satisfied for only finitely many λ. (In the present case of Archimedean F, the condition can be satisfied for at most one λ. We have written the formula as sum over λ so that the p-adic analogue will be more transparent.) We can therefore write

$$I_{disc}(f) = \sum_{\pi \in \Pi_{disc}(G)} a^G_{disc}(\pi) \operatorname{tr}\pi(f^1), \tag{7.3}$$

a linear combination of irreducible characters in $\Pi_{disc}(G)$. The complex numbers

$$a^M_{disc}(\pi), \qquad M \in \mathcal{L}, \qquad \pi \in \Pi_{disc}(M),$$

can be defined in this way for all Levi subgroups, and will appear as coefficients on the spectral side.

Suppose that $M \in \mathcal{L}$. We shall write $\Pi_{temp}(M(A_F)^1)$ for the set of equivalence classes of irreducible tempered representations of $M(A_F)^1$. Each representation $\pi \in \Pi_{temp}(M(A_F)^1)$ can be identified with an orbit

$$\pi_\lambda = \tilde{\pi}_{1,-\lambda} \otimes \pi_{2,\lambda}, \qquad \lambda \in i\mathfrak{a}^*_M,$$

of irreducible representations of $M(A_F) = M(F) \times M(F)$ under the action of $i\mathfrak{a}^*_M$. For any such π, and any $P \in \mathcal{P}(M)$, we can form the induced representations

$$\mathcal{I}_P(\pi_\lambda, f) = \mathcal{I}_{\overline{P}}(\tilde{\pi}_{1,-\lambda}, f_1) \otimes \mathcal{I}_P(\pi_{2,\lambda}, f_2), \qquad \lambda \in i\mathfrak{a}^*_M.$$

We also have the standard unnormalized intertwining operators

$$J_{P'|P}(\pi_\lambda) = J_{\overline{P}'|\overline{P}}(\tilde{\pi}_{1,-\lambda}) \otimes J_{P'|P}(\pi_{2,\lambda}), \qquad P' \in \mathcal{P}(M),$$

from $\mathcal{I}_P(\pi_\lambda)$ to $\mathcal{I}_{P'}(\pi_\lambda)$, each of which can be written as a product of a rational function

$$r_{P'|P}(\pi_\lambda) = r_{\overline{P}'|\overline{P}}(\tilde{\pi}_{1,-\lambda}) r_{P'|P}(\pi_{2,\lambda})$$

with a normalized intertwining operator

$$R_{P'|P}(\pi_\lambda) = R_{\bar{P}'|\bar{P}}(\bar{\pi}_{1,-\lambda}) \otimes R_{P'|P}(\pi_{2,\lambda})$$

[1(j), Theorem 2.1]. It is easy to show that the limit

$$\mathcal{J}_M(\pi_\lambda, P) = \lim_{\nu\to 0} \sum_{P'\in\mathcal{P}(M)} (J_{P'|P}(\pi_\lambda)^{-1} J_{P'|P}(\pi_{\lambda+\nu}))\theta_{P'}(\nu)^{-1}$$

exists [1(b), Lemma 6.2]. In general, the function

$$J_M(\pi_\lambda, f) = \operatorname{tr}(\mathcal{J}_M(\pi_\lambda, P)\mathcal{I}_P(\pi_\lambda, f))$$

will have singularities in λ. However, it can be shown that if π belongs to $\Pi_{disc}(M)$, then $J_M(\pi_\lambda, f)$ is a Schwartz function $\lambda \in i\mathfrak{a}_M^*$. The distributions

$$j_M(\pi, f) = \int_{i\mathfrak{a}_M^*} J_M(\pi_\lambda, f)\,d\lambda, \qquad \pi \in \Pi_{disc}(M), \tag{7.4}$$

will be the remaining terms on the spectral side.

The local trace formula will be the identity of distributions

$$\sum_{M\in\mathcal{L}} |W_0^M|\,|W_0^G|^{-1}(-1)^{dim(A_M/A_G)} \int_{(M(F)_{ell})} J_M(\gamma, f)\,d\gamma \tag{7.5}$$

and

$$\sum_{M\in\mathcal{L}} |W_0^M|\,|W_0^G|^{-1}(-1)^{dim(A_M/A_G)} \sum_{\pi\in\Pi_{disc}(M)} a_{disc}^M(\pi) j_M(\pi, f). \tag{7.6}$$

We have described it for the function $f = f_1 \times f_2$. However, the distributions $J_M(\gamma, f)$ and $j_M(\pi, f)$ make sense for any function f in $C_c^\infty(G(A_F))$, and the identity would hold in this generality. Of course there are still some analytic questions to be answered, so the identity must remain conjectural for the present. (Incidentally, the notation in (7.6) is slightly at odds with that used in connection with automorphic forms. In the papers [1(h)–1(k)] we defined $J_M(\pi_\lambda, f)$ in terms of the normalized intertwining operators $R_{P'|P}(\pi_\lambda)$ instead of the unnormalized operators $J_{P'|P}(\pi_\lambda)$ used here. Moreover, we denoted the corresponding integral (7.4) simply by $J_M(\pi, f)$.)

8. A distribution I on $G(A_F)$ is said to be *invariant* if it is left unchanged under conjugation. That is,

$$I(f^g) = I(f), \qquad f \in C_c^\infty(G(A_F)), \qquad g \in G(A_F),$$

where

$$f^g(g_1) = f(gg_1g^{-1}), \qquad g, g_1 \in G(A_F).$$

The distributions $J_M(\gamma)$ and $j_M(\pi)$, defined by (7.1) and (7.4), are not invariant if $M \neq G$. Following the methods of the global trace formula [1(h)], [1(i)], we shall sketch how the local identity we have just described could be converted into an *invariant* local trace formula.

It is best to restrict our attention to the Hecke algebra $\mathcal{H}(G(A_F))$ of functions in $C_c^\infty(G(A_F))$ which are left and right finite under the maximal compact subgroup $K \times K$. The results of Clozel and Delorme [2] allow one to characterize the topological vector space $\mathcal{I}(G(A_F)^1)$ of functions on $\Pi_{temp}(G(A_F)^1)$ of the form

$$f_G^1(\pi) = \mathrm{tr}(\pi(f^1)), \qquad \pi \in \Pi_{temp}(G(A_F)^1), \qquad f \in \mathcal{H}(G(A_F)).$$

Suppose that θ is a continuous linear map from $\mathcal{H}(G(A_F))$ to a topological vector space $\mathcal{V}$. We can assume that $\theta(f)$ depends only on f^1. Then θ is said to be *supported on characters* if $\theta(f)$ depends only on the function f_G^1. When θ has this property, there is a unique continuous map

$$\hat{\theta}: \mathcal{I}(G(A_F)^1) \to \mathcal{V}$$

such that $\hat{\theta}(f_G^1) = \theta(f)$ for all f.

Suppose that $M \in \mathcal{L}$. Given a representation $\pi \in \Pi_{temp}(M(A_F)^1)$, we can form the limit

$$\mathcal{R}_M(\pi_\lambda, P) = \lim_{\nu \to 0} \sum_{P' \in \mathcal{P}(M)} R_{P'|P}(\pi_\lambda)^{-1} R_{P'|P}(\pi_{\lambda+\nu})\theta_{P'}(\nu)^{-1}.$$

Then for any $f \in \mathcal{H}(G(A_F))$, we define $\phi_M(f)$ to be the function on $\Pi_{temp}(M(A_F)^1)$ whose value at π equals

$$\phi_M(f, \pi) = \int_{i\mathfrak{a}_M^*} \mathrm{tr}(\mathcal{R}_M(\pi_\lambda, P)\mathcal{I}_P(\pi_\lambda, f))\,d\lambda.$$

us, ϕ_M is a transform from functions on $G(A_F)$ to functions on $\Pi_{temp}(M(A_F)^1)$. Indeed, one can show that ϕ_M is a continuous linear mapping from $\mathcal{H}(G(A_F))$ into $\mathcal{I}(M(A_F)^1)$. (See [1(j), Theorem 12.1].)

Consider first an element $\gamma \in M(F)_{ell}$. It is not hard to show that the variance of ϕ_M under conjugation is the same as that of the distribution $J_M(\gamma, f)$. The extent to which these objects differ can therefore be measured by invariant distributions. For each $\gamma \in M(F)_{ell}$, one can define an invariant distribution

$$I_M(\gamma, f) = I_M^G(\gamma, f), \qquad f \in \mathcal{H}(G(A_F)),$$

which is supported on characters, and which satisfies the inductive formula

$$J_M(\gamma, f) = \sum_{L \in \mathcal{L}(M)} \hat{I}_M^L(\gamma, \phi_L(f)). \tag{8.1}$$

Here $\mathcal{L}(M)$ denotes the set of Levi subgroups which contain M. (See [1(h), Section 2].)

Next suppose that π is a representation in $\Pi_{disc}(M)$. The variance of ϕ_M under conjugation also matches that of $j_M(\pi, f)$. It follows without difficulty that there is an invariant distribution

$$i_M(\pi, f) = i_M^G(\pi, f), \qquad f \in G(A_F),$$

which is supported on characters, and satisfies the inductive formula

$$j_M(\pi, f) = \sum_{L \in \mathcal{L}(M)} \hat{i}_M^L(\pi, \phi_L(f)). \tag{8.2}$$

This distribution can also be defined directly. The limit

$$r_M(\pi_\lambda, P) = \lim_{\nu \to 0} \sum_{P' \in \mathcal{P}(M)} r_{P'|P}(\pi_\lambda)^{-1} r_{P'|P}(\pi_{\lambda+\nu}) \theta_{P'}(\nu)^{-1} \tag{8.3}$$

provides a rational function of λ whose poles do not meet $i\mathfrak{a}_M^*$. It is then an easy consequence of [1(b), Corollary 6.5] that

$$i_M(\pi, f) = \int_{i\mathfrak{a}_M^*} r_M(\pi_\lambda, P) \operatorname{tr}(\mathcal{I}_P(\pi_\lambda, f)) d\lambda. \tag{8.4}$$

The following proposition gives the final invariant local trace formula. As in the global case, it is a formal consequence of the definitions (8.1) and (8.2).

PROPOSITION 8.1. *The identity of the noninvariant expressions* (7.5) *and* (7.6) *implies that the invariant expressions*

$$\sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{dim(A_M/A_G)} \int_{(M(F)_{ell})} I_M(\gamma, f)d\gamma, \tag{8.5}$$

and

$$\sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{dim(A_M/A_G)} \sum_{\pi\in\Pi_{disc}(M)} a_{disc}^M(\pi)i_M(\pi, f) \tag{8.6}$$

are also equal.

Proof. Write $J(f)$ for the two equal quantities (7.5) and (7.6). We shall set $I(f) = I^G(f)$ equal to the expression (8.5). Substituting (8.1) into (7.5), and then applying the definition of I^L, we obtain

$$\begin{aligned} J(f) &= \sum_{M\in\mathcal{L}} \sum_{L\in\mathcal{L}(M)} |W_0^M||W_0^G|^{-1}(-1)^{dim(A_M/A_G)} \int_{(M(F)_{ell})} \hat{I}_M^L(\gamma, \phi_L(f))d\gamma \\ &= \sum_{L\in\mathcal{L}} |W_0^L||W_0^G|^{-1}(-1)^{dim(A_L/A_G)}\hat{I}^L(\phi_L(f)). \end{aligned}$$

Similarly, if $i(f) = i^G(f)$ denotes the expression (8.6), we can write

$$J(f) = \sum_{L\in\mathcal{L}} |W_0^L||W_0^G|^{-1}(-1)^{dim(A_L/A_G)}\hat{i}^L(\phi_L(f)),$$

for (7.6). We are trying to show that I^G equals i^G. We may assume inductively that

$$\hat{I}^L(\phi_L(f)) = \hat{i}^L(\phi_L(f)),$$

for any $L \subsetneq G$. The corresponding terms in the two expansions of $J(f)$ therefore cancel. All that remains is the required equality of the two distributions

$$\hat{I}^G(\phi_G(f)) = \hat{I}^G(f_G^1) = I^G(f)$$

and

$$\hat{i}^G(\phi_G(f)) = \hat{i}^G(f_G^1) = i^G(f). \qquad \square$$

9. There are two special cases of our putative local trace formula which have already been established. Before discussing these, we shall first say a word about p-adic groups. From Section 5 through Section 8, we were assuming that F was Archimedean. Now, take F to be a p-adic field. Then $H_M(A_M(F))$ is only a lattice in $\mathfrak{a}_M$. The volume $\bar{v}_M(x_1, x_2, T)$ in (5.2) must be replaced by the number of lattice points in a convex hull. There will be an analogous change in the asymptotic formula (6.2) for (4.2). However, these difficulties are not serious, and may be handled by the methods of [1(g), Sections 4-5]. We can expect that the changes caused by replacing $\mathfrak{a}_M$ by a lattice will run parallel on the geometric and spectral sides. The discrepancies should cancel, leaving intact the identity of (7.5) and (7.6). The invariant identity of (8.5) and (8.6) would continue to hold, and the definitions (7.1), (7.4), (8.1) and (8.2) would remain the same, except with the domain of integration in (7.4) changed to $i\mathfrak{a}_M^*/i\mathfrak{a}_M^\vee$.

The first special case comes from the noninvariant identity of (7.5) with (7.6). Take G to be the general linear group $GL(n)$, and F to be a p-adic field. Choose f_2 to be supported on $G(F)_{ell}$. Then the terms with $M \neq G$ in (7.5) vanish, and the expression reduces to an average

(9.1)

$$\int_{(G(F)_{ell})} \left(|D(\gamma)| \int_{A_G(F)\backslash G(F)} f_1(x_1^{-1}\gamma x_1)dx_1 \cdot \int_{A_G(F)\backslash G(F)} f_2(x_2^{-1}\gamma x_2)dx_2 \right) d\gamma$$

of products of elliptic orbital integrals. The spectral expression (7.6) also simplifies. Suppose that $\pi \in \Pi_{disc}(M)$ is the restriction of $\tilde{\pi}_1 \otimes \pi_2$ to $M(A_F)^1$. Applying a general splitting property [1(h), Corollary 7.4] to the operators $\mathcal{J}_M(\pi_\lambda, P)$, and using the fact that f_2 is supported on the elliptic set, we can show that

$$j_M(\pi, f) = \int_{i\mathfrak{a}_M^*/i\mathfrak{a}_M^\vee} \tilde{f}_{1,M}(\pi_{1,\lambda}) \mathrm{tr}(\mathcal{R}_M(\pi_{2,\lambda}, P)\mathcal{J}_P(\pi_{2,\lambda}, f_2))d\lambda,$$

where

$$\tilde{f}_{1,M}(\pi_{1,\lambda}) = \mathrm{tr}(\mathcal{I}_{\bar{P}}(\tilde{\pi}_{1,-\lambda}, f_1)).$$

We are in the case of $GL(n)$, in which induced tempered representations are all irreducible. Therefore, π_1 equals π_2, and is induced from a discrete series. Applying a general descent property [1(h), Corollary 7.2] to the operator $\mathcal{R}_M(\pi_{2,\lambda}, P)$, we conclude that $j_M(\pi, f)$ vanishes unless π_1 is actually a discrete series. Now, specialize f_2 to a K-bi-invariant function on $G(F)$. Then $\tilde{f}_{1,M}(\pi_{1,\lambda})$ will vanish unless π_1 is unramified. In other words, M equals M_0, and the restriction of $\pi_1 = \pi_2$ to $M_0(F)^1$ is the trivial representation τ. Since

$$|W_0^{M_0}|\,|W_0^G|^{-1}(-1)^{dim(A_{M_0}/A_G)} = (-1)^{n-1}(n!)^{-1},$$

the expression (7.6) becomes

$$(9.2) \qquad (-1)^{n-1}(n!)^{-1} \int_{i\mathfrak{a}^*_{M_0}/i\mathfrak{a}^\vee_{M_0}} \tilde{f}_{1,M_0}(\tau_\lambda)\mathrm{tr}(\mathcal{R}_{M_0}(\tau_\lambda, P_0)\mathcal{I}_{P_0}(\tau_\lambda, f_2))d\lambda.$$

The identity of (9.1) with (9.2) is due to Waldspurger, and is the main result of [5(a)]. In another paper [5(b)], Waldspurger uses this identity in a remarkable way to establish some cases of Rogawski's conjecture on Shalika germs for p-adic orbital integrals.

For the second special case, we take F to be either real or p-adic. We allow G to be any connected group, except we assume for simplicity that the split component A_G is trivial. Take f_1 to be a pseudo-coefficient of a discrete series representation $\pi_1 \in \Pi_2(G(F))$. That is, if π_1' is any irreducible tempered representation of $G(F)$, $\mathrm{tr}(\tilde{\pi}_1'(f_1))$ equals 1 or 0, according to whether π_1' is equivalent to π_1 or not. Then the expression (8.6) equals $\mathrm{tr}(\pi_1(f_2))$. By the splitting and descent formulas [1(h), Proposition 9.1 and Corollary 8.3], the expression (8.5) equals

$$\sum_{M\in\mathcal{L}} |W_0^M|\,|W_0^G|^{-1}(-1)^{dim(A_M/A_G)} \int_{(M(F)_{ell})} I_M(\gamma, f_1)I_G(\gamma, f_2)d\gamma.$$

The function f_2 is supposed to belong to the Hecke algebra on $G(F)$. However, it is clear by density that this simpler form of the invariant trace formula holds for any $f_2 \in C_c^\infty(G(F))$. Fix a group M and an element $\gamma_1 \in$

$M(F)_{ell}$, and let f_2 approach the Dirac delta measure on $G(F)$ at γ_1. Then $\operatorname{tr}(\pi_1(f_2))$ approaches $\Theta_{\pi_1}(\gamma_1)$, the value of the character of π_1 at γ_1. By the Weyl integration formula, the function

$$|D(\gamma)|^{1/2} I_G(\gamma, f_2) = |D(\gamma)| \int_{A_M(F)\backslash G(F)} f_2(x_2^{-1}\gamma x_2)dx_2, \qquad \gamma \in (M(F)_{ell}),$$

approaches the Dirac measure on $(M(F)_{ell})$ at the conjugacy class of γ_1 in $M(F)$. Taking into account the different W_0^G-orbits of γ_1 which occur in (8.5), we obtain

$$I_M(\gamma_1, f_1) = (-1)^{dim(A_M)} |D(\gamma_1)|^{1/2} \Theta_\pi(\gamma_1). \tag{9.3}$$

For real F, this is essentially Theorem 6.4 of [1(k)].

Suppose that f_1 is actually a matrix coefficient of π_1. If π_1 is not supercuspidal, this presents the technical problem of extending the distributions $I_M(\gamma)$ to the Schwartz space. Leaving this question aside, we see that the function $\phi_L(f_1)$ will vanish for any $L \neq G$. This implies that

$$I_M(\gamma_1, f_1) = |D(\gamma_1)|^{1/2} \int_{A_M(F)\backslash G(F)} f_1(x_1^{-1}\gamma_1 x_1) v_M(x_1) dx_1,$$

where $v_M(x_1) = v_M(x_1, 1)$. The formula (9.3) becomes

$$\int_{A_M(F)\backslash G(F)} f_1(x_1^{-1}\gamma_1 x_1) v_M(x_1) dx_1 = (-1)^{dim(A_M)} \Theta_{\pi_1}(\gamma_1). \tag{9.4}$$

In this form the identity is the main result of [1(a)], when F is real, and of [1(g)], when F is p-adic and π_1 is supercuspidal. (The author of these papers seems to have had some trouble distinguishing between a representation and its contragradient. In [1(a)], $\Theta_\omega(h)$ should be replaced by $\Theta_{\tilde{\omega}}(h)$, while $\Theta_\pi(\gamma)$ should be replaced by $\Theta_{\tilde{\pi}}(\gamma)$ in [1(g)].) If F is p-adic and π_1 is special, the formulas (9.3) and (9.4) have not been established. The local trace formula would be a natural way to prove them.

10. We shall conclude with some brief general remarks. As in the global case, the local trace formula should be a special case of a local twisted trace formula. For this, we would allow G to be any connected component of a (nonconnected) reductive group over F. Let G^0 be the identity component of the group generated by G. We would then define

$$(R(y_1, y_2)\phi)(x) = \phi(y_1^{-1}xy_2), \qquad x \in G^0(F), \qquad y_1, y_2 \in G(F),$$

for any function $\phi \in L^2(G^0(F))$. This provides a canonical extension of the regular representation of $G^0(A_F)$ to the group generated by $G(A_F)$. In this setting, the definitions (7.1), (7.4), (8.1) and (8.2), the identity between (7.5) and (7.6), and the identity between (8.5) and (8.6), should all remain valid.

As we mentioned in Section 1, there are strong similarities between the local and global trace formulas. The reader can compare the invariant local formula with the invariant global formula (3) in the introduction of [1(h)]. The local formula is actually less complicated. One reason for this is that the geometric terms are parametrized only by semisimple elements. In the global formula, there are also terms on the geometric side parametrized by unipotent classes in the discrete subgroup. These account for the coefficients $a^M(\gamma) = a^M(S, \gamma)$ in [1(h), (3)]. The spectral analogue of a semisimple class is a tempered representation. The only spectral terms in the local formula come from tempered representations. In the global formula there are also terms coming from nontempered representations in the discrete spectrum. These are responsible for the extra local terms $I_M(\pi, f)$ which occur on the spectral side of the global formula.

Nevertheless, the basic local ingredients of the global formula are the invariant distributions $I_M(\gamma, f)$. In the global setting, f stands for a function on an adèle group, whereas in the definition (8.1), f is a function on the product of $G(F)$ with itself. However, the splitting formula [1(h), Proposition 9.1] can be applied in both situations. In each case, it allows one to write the distributions on the product of groups in terms of similar distributions on a single group $G(F)$. Now, there are believed to be relations between the values these distributions assume on different groups (that is, on groups related by endoscopy). Such identities would in turn provide a stable (global) trace formula, and would lead to reciprocity laws between automorphic forms on different groups. A natural question, which was perhaps behind Kazhdan's original suggestion, is whether one can use the local trace formula to establish these identities.

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