Unipotent Automorphic Representations: Global Motivation

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§1. INTRODUCTION

In the paper [3], we gave a conjectural description of the discrete spectrum attached to the automorphic forms on a general reductive group. The main qualitative feature of this description was a Jordan decomposition into semisimple and unipotent constituents. This is in keeping with the dual nature of conjugacy classes and characters, and in fact, with a general parallelism between geometric objects and spectral objects that is observed in many mathematical contexts. Such a decomposition for automorphic representations would of course be parallel to the Jordan decomposition for rational conjugacy classes. It would also be analogous to the Jordan decomposition that is an essential part of the representation theory of finite algebraic groups.

The decomposition should actually apply uniformly to the automorphic representations in certain families. The families or "packets" are indexed by certain parameters which are the source of the decomposition. The quantitative side of the conjectures in [3] is a formula for the multiplicity with which a representation in any packet occurs in the

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discrete spectrum. It is a generalization of the formula for *tempered* representations which is implicit in the examples in [**15**]. In terms of the Jordan decomposition, tempered automorphic representations are semisimple. The multiplicity formula for nontempered automorphic representations contains some new signs. These are constructed out of the root numbers of certain L-functions, attached to the semisimple part of the given automorphic representation.

In this paper, we shall try to give some motivation for the conjectures. Some version of the conjectures, at least for many classical groups, ought to follow from the stable trace formula. This is certainly so in the few cases where the stable trace formula has been established [**15**], [**22**]. In general, one would need to combine the theory of endoscopy with the ordinary (or twisted) trace formula to obtain a stable trace formula. There are still a number of problems to be solved, but one can guess what the final answer will be. The purpose of this paper is to show that it is compatible with the conjectures of [**3**].

For purposes of introduction, let G be a connected, simply connected group over a number field F. We shall be interested in the spectral side of the trace formula. The essential ingredient we shall study is a certain distribution

$$I_{\mathrm{disc},t}(f)\,, \qquad\qquad f \in C_c^\infty(G(\mathbb{A})),$$

which is discrete in the parameters which describe the representations of $G(\mathbb{A})$. It is given by an explicit formula (3.1), one term of which involves the trace of f on the discrete spectrum. When the stable trace formula has been established, the payoff will be an identity

$$I_{\mathrm{disc},t}(f) = \sum_H \iota(G,H) S\hat{I}^H_{\mathrm{disc}}(f^H)\,, \tag{1.1}$$

in which H ranges over elliptic endoscopic groups, $\iota(G,H)$ is a certain constant, and $f \to S\hat{I}^H_{\mathrm{disc}}(f^H)$ is a pullback to G of a stable distribution on $H(\mathbb{A})$. (Recall that the endoscopic groups are a natural family of quasi-split groups attached to G. Recall too that a stable distribution is a special case of an invariant distribution, which arises as a natural consequence of the difference between rational conjugacy and geometric conjugacy. We refer the reader to [3, §3] for a brief discussion of these notions and of the Langlands-Shelstad transfer mapping f^H.) As a distribution on $G(\mathbb{A})$, $S\hat{I}^H_{\mathrm{disc}}(f^H)$ is not generally stable.

However, the trace of f on the discrete spectrum is also usually not stable. Endoscopic groups were actually invented by Langlands with the aim of measuring this lack of stability.

The endoscopic groups on the right hand side of (1.1) should all contribute to the multiplicity formula for representations in the discrete spectrum. However, the trace of f on the discrete spectrum is only one of several terms in the explicit formula for $I_{\mathrm{disc},t}(f)$. The other terms are the surviving remnants of Eisenstein series, and are parametrized by (conjugacy classes of) proper Levi subgroups of G. Each such term is a linear combination of distributions, which are obtained by taking the trace of a product of two operators, one being the action of f on the induced discrete spectrum, and the other being an intertwining operator that comes from Eisenstein series. These additional terms have one important function. They account for that part of the discrete spectrum of a given H which under functoriality maps into the continuous spectrum of G. However, the additional terms also contribute irrelevant information, which complicates the study of (1.1). The attempt to separate the extraneous information from the contribution of the discrete spectrum leads to combinatorial difficulties. The main point of this paper is to solve these combinatorial problems.

The results are given in §5–§8. In §5 we expand $I_{\mathrm{disc},t}(f)$ into a linear combination of irreducible characters. This hinges on the conjectures of **[3]**. However, we have only the modest goal of showing that the conjectures are compatible with (1.1), so we are free to assume them. Each coefficient in the expansion contains a certain quotient of L-functions, which comes from the global intertwining operators. If the irreducible character is tempered, this quotient should equal the parity of the pole of the L-function at $s = 1$. If the irreducible character is nontempered, however, it will have a unipotent part. When the corresponding unipotent element is not *even*, the quotient must also be expressed in terms of the order of the L-function at the center of the critical strip. The exact relation is given by Proposition 5.1, which we prove in §6. Together with Lemma 7.1, it provides the justification for the sign characters which appear in the general multiplicity formula.

In §7 we establish a parallel expansion of the right hand side of (1.1) into irreducible characters. This requires various properties from endoscopy, some known and others which are expected to hold, which we discuss in §2 and §3. The endoscopic groups H consist of the quasi-split form of G, together with groups of smaller dimension. By reasons of induction, then, the stable distributions $S\hat{I}^H_{\text{disc}}$ are uniquely determined by (1.1). However, we must derive the expansion in §7 without reference to the left hand side of (1.1). The coefficients in the expansion have to be given as certain undetermined constants, which can be regarded as "stable multiplicities", and which only later are tied precisely to the sign characters discussed above. For a parameter which contributes to the tempered discrete spectrum, the corresponding coefficient will be familiar from [15, §6,7] and [12, §12]. It is then just equal to 1, divided by the order of a certain finite group.

Our aim is to show that with the assumption of the conjectures of [**3**], the left and right hand sides of (1.1) are equal. We would thus like to establish a term by term identification of the two parallel expansions. However, this is not immediately obvious. What remains to be proved at the end of §7 is a sort of analogue for Weyl groups of the endoscopy identity (1.1). The expansion of $I_{\text{disc},t}(f)$ contains certain constants $i(x)$, which are defined if x is any connected component of a complex reductive group. The expansion for the right hand side of (1.1) is identical, except that $i(x)$ is replaced by another constant $e(x)$. In the first case, $i(x)$ is given by a finite sum over elements in the Weyl set of x. It is the analogue for Weyl groups of the left hand side of (1.1). The second constant $e(x)$ is the analogue of the right hand side of (1.1), and is given as a finite sum over the isolated conjugacy classes in x. In §8 we prove that $i(x)$ equals $e(x)$ for every component x. This establishes the term by term identification of the expansions of each side of (1.1).

At the end of §8 the reader might be wondering whether the paper has provided the global motivation claimed in the title. It is true that the identity (1.1) is weaker than the conjectural multiplicity formula (and the local conjectures on which it is based). However, the identity can still provide significant information about the discrete spectrum, for either G or its endoscopic groups. This is especially so if for one of the groups, the conjectures are known to hold. The group $GL(n)$ is such an example, thanks to recent work of Moeglin and Waldspurger [**21**]. The twisted version of (1.1), applied to $GL(n)$, will relate the discrete spectrum of many classical groups to that of $GL(n)$. In particular, it should yield some version of the multiplicity formula for the quasi-split orthogonal and symplectic groups. We shall finish the paper in §9 with an informal discussion of these questions.

Throughout the paper we shall adopt the following notational conventions. Suppose that Σ is a set on which a group Γ acts. We shall denote the set of orbits of Γ on Σ by either $\mathrm{Orb}(\Gamma, \Sigma)$ or Σ/Γ. In general, if A and B are subsets of a group Γ, we shall write

$$\mathrm{Cent}(A, B) = \{b \in B : b^{-1}ab = a, \quad \text{for all } a \in A\}$$

for the pointwise centralizer of A in B, and

$$\mathrm{Norm}(A, B) = \{b \in B : b^{-1}Ab = A\}$$

for the normalizer of A in B. Next, suppose that C is a finite union of connected components in a (nonconnected) algebraic group. Then C^+ denotes the algebraic group generated by C, and C^0 is the connected component of 1 in C^+. If s is any element in C, we set

$$C_s = \mathrm{Cent}(s, C^0) .$$

Then C_s is also an algebraic group, with identity component

$$C_s^0 = (C_s)^0 = \mathrm{Cent}(s, C^0)^0 .$$

(This differs from the notation of [**2**] and some other papers, in which the symbol C_s was reserved for the identity component of the centralizer.) We shall also write

$$Z(C) = \mathrm{Cent}(C, C^0) .$$

This group is the intersection of C^0 with the center of C^+, and is contained in $Z(C^0)$. Finally, if X is any topological space, $\pi_0(X)$ denotes the set of connected components of X.

§2. Endoscopic data

Suppose that G is a connected component of a reductive algebraic group over a number field F. Then G^+ stands for the group generated by G, and G^0 is the connected component of 1 in G^+. We shall assume that $G(F)$ is not empty. As in [**3**, §**6**], we shall also assume that G is an inner twist of a component in a quasi-split group. More precisely, we assume that there is a map

$$\eta : G \rightarrow G^* ,$$

where G^* is a component such that $(G^*)^0$ is quasi-split, and such that $G^*(F)$ contains an element which preserves some F-splitting of $(G^*)^0$ under conjugation. It is required that η extend to an isomorphism of G^+ with $(G^*)^+$ such that for any $\sigma \in \mathrm{Gal}(\bar{F}/F)$, the map

$$\eta\sigma(\eta^{-1}) : G^* \to G^*$$

is an inner automorphism by an element in $(G^*)^0$.

The standard situation is when $G^+ = G^0$. By allowing G to be a more general component, we are providing for applications of the twisted trace formula [5]. Associated to the connected component G^0 we have the L-group

$${}^LG^0 = \hat{G}^0 \rtimes W_F .$$

It is a semidirect product of a complex connected group $\hat{G}^0$ with the Weil group W_F of F. (As in [3], we follow the notation of Kottwitz [12], so that $\hat{G}^0$ stands for the identity coset of the L-group. The symbol ${}^LG^0$ can then be reserved for the full L-group of G^0.) We have not assumed that G^+ is a semidirect product of G^0 with a finite cyclic group, but this does not seem to be a serious concern. In particular, it is reasonable to define the L-group ${}^LG^+$ of G^+ simply as a semidirect product of ${}^LG^0$ by the cyclic group $\pi_0(G^+)$ of connected components in G^+. The action of $\pi_0(G^+)$ on $\hat{G}^0$ is dual to its action by outer autmorphisms on G^0. The action of $\pi_0(G^+)$ on W_F could be defined by some map of $\pi_0(G^+)$ into $H^1\big(W_F, Z(\hat{G}^0)\big)$. However, for simplicity we shall assume that $\pi_0(G^+)$ and W_F (as subgroups of ${}^LG^+$) commute. Associated to the component G we have an "L-coset"

$${}^LG = \hat{G} \rtimes W_F ,$$

in which $\hat{G}$ is a coset of $\hat{G}^0$ in a group $\hat{G}^+$ such that

$${}^LG^+ = \hat{G}^+ \rtimes W_F .$$

Notice that

$$Z(\hat{G}) = \mathrm{Cent}(\hat{G}, \hat{G}^0)$$

is in general a proper subgroup of the center

$$Z(\hat{G}^0) = \mathrm{Cent}(\hat{G}^0, \hat{G}^0)$$

of $\hat{G}^0$. We must always be careful to distinguish between these two groups. The Galois group $\Gamma = \Gamma_F$ of $\bar{F}$ over F acts on both $Z(\hat{G})$ and $Z(\hat{G}^0)$. The subgroups of Γ-invariant elements are given by

$$Z(\hat{G})^\Gamma = \mathrm{Cent}({}^L G, \hat{G}^0)$$

and

$$Z(\hat{G}^0)^\Gamma = \mathrm{Cent}({}^L G^0, \hat{G}^0)\ .$$

These too are not generally equal. Observe that

$$A_{\hat{G}} = \left(Z(\hat{G})^\Gamma\right)^0$$

is the maximal Γ-invariant torus in the center of $\hat{G}^+$. It is of course not the dual group of the maximal split torus A_G in the center of G^+. It is associated, rather, to the dual of the real vector space

$$\underline{\mathbf{a}}_G = \mathrm{Hom}\big(X^*(G)_F, \mathbf{R}\big)\ .$$

($X^*(G)_F$ denotes the module of F-rational characters on G^+.) More precisely,

$$X^*(G)_F \cong X_*(A_{\hat{G}})\ ,$$

so that the complex dual space $\underline{\mathbf{a}}^*_{G,\mathbf{C}} = X^*(G)_F \otimes \mathbf{C}$ is the Lie algebra of $A_{\hat{G}}$. We shall write

$$\kappa_G = (A_{\hat{G}^0})^{\hat{G}} = Z(\hat{G})^\Gamma \cap \left(Z(\hat{G}^0)^\Gamma\right)^0$$

for the group of fixed points of $\hat{G}$ in $A_{\hat{G}^0}$. It is a closed subgroup of $A_{\hat{G}^0}$ whose identity component equals $A_{\hat{G}}$.

The theory of endoscopy for nonconnected groups is the subject of work in progress by Kottwitz and Shelstad. As in **[3, §6]**, we shall guess at the ultimate form of some of this theory by extrapolating from the connected case. Thus, an endoscopic datum $(H, \mathcal{H}, s, \xi)$ should consist of a connected quasi-split group H over F, an extension

$$I \longrightarrow \hat{H} \longrightarrow \mathcal{H} \longrightarrow W_F \longrightarrow 1\ ,$$

a semisimple coset s in $\hat{G}/Z(\hat{G}^0)$, and an L-embedding ξ of $\mathcal{H}$ into ${}^L G^0$. The definition is similar to the one given in **[3, §3,§6]** except that s is now a coset of $Z(\hat{G}^0)$ instead of a single element in $\hat{G}$. It is required that

$$\xi(\hat{H}) = \mathrm{Cent}(s, \hat{G}^0)^0 ,$$

the connected centralizer in $\hat{G}^0$ of any element in the coset s, and that

$$s\xi(h)s^{-1} = a(w_h)\xi(h) , \qquad h \in \mathcal{H}, \tag{2.1}$$

where w_h is the image of h in W_F and $a(\cdot)$ represents a locally trivial element in $H^1\big(W_F, Z(\hat{G}^0)\big)$. In other words, $a(\cdot)$ belongs to $\ker^1\big(W_F, Z(\hat{G}^0)\big)$, the kernel of the map

$$H^1\big(W_F, Z(\hat{G}^0)\big) \longrightarrow \bigoplus_v H^1\big(W_{F_v}, Z(\hat{G}^0)\big) ,$$

in which v runs over the valuations of F. It is further required that the two extensions $\mathcal{H}$ and ${}^LH$ define the same map of W_F into $\mathrm{Out}(\hat{H})$, the group of outer automorphisms of $\hat{H}$.

Recall that an endoscopic datum is said to be *elliptic* if the set $\xi(\mathcal{H})s$ is not contained in any proper parabolic subset of ${}^LG$. Equivalently, the datum is elliptic if and only if the group

$$\xi\big(Z(\hat{H})^\Gamma\big)/\xi\big(Z(\hat{H})^\Gamma\big) \cap Z(\hat{G})^\Gamma$$

is finite, or again, if and only if $\xi(A_{\hat{H}})$ equals $A_{\hat{G}}$. Finally, two elliptic endoscopic data $(H, \mathcal{H}, s, \xi)$ and $(H', \mathcal{H}', s', \xi')$ are equivalent if there exist dual isomorphisms $\alpha : H \to H'$ and $\beta : \mathcal{H}' \to \mathcal{H}$, together with an element $g \in \hat{G}^0$ such that

$$g\xi\big(\beta(h')\big)g^{-1} = \xi'(h') , \qquad h' \in \mathcal{H}',$$

and

$$gsg^{-1} = s' .$$

Suppose that $(H, \mathcal{H}, s, \xi)$ is an elliptic endoscopic datum. We shall write $\mathrm{Aut}(H)$ for the group of elements g in $\hat{G}^0$ such that $gsg^{-1} = s$, and $g\xi(\mathcal{H})g^{-1} = \xi(\mathcal{H})$. Then $\mathrm{Aut}(H)$ is a reductive subgroup of $\hat{G}^0$. Notice that $\xi(\hat{H})Z(\hat{G}^0)^\Gamma$ is a closed subgroup of $\mathrm{Aut}(H)$. We shall need to know later that it is of finite index. Equivalently, we must establish

LEMMA 2.1. *The identity component of* $\mathrm{Aut}(H)$ *equals*

$$\xi(\hat{H})\big(Z(\hat{G}^0)^\Gamma\big)^0 = \xi(\hat{H})A_{(\hat{G}^0)} \,.$$

Let s_1 be a fixed element in the coset s, and write

$$\widetilde{C}_s = \{g \in \hat{G}^0 : s_1 g s_1^{-1} g^{-1} \in Z(\hat{G}^0)\} = \{g \in \hat{G}^0 : gsg^{-1} = s\}$$

and

$$C_s = \{g \in \hat{G}^0 : s_1 g s_1^{-1} g^{-1} = 1\} \,.$$

Then

$$g \longrightarrow s_1 g s_1^{-1} g^{-1}$$

is an injective map from $\widetilde{C}_s/C_s$ onto a closed subgroup $\hat{Z}(s)$ of $Z(\hat{G}^0)$.

LEMMA 2.2. *The subgroup*

$$\hat{Z}'(s) = \{s_1 z s_1^{-1} z^{-1} : z \in Z(\hat{G}^0)\}$$

is of finite index in $\hat{Z}(s)$.

PROOF: Suppose that g belongs to $\widetilde{C}_s$. We can write $g = g_1 z$, where g_1 belongs to the derived subgroup $\hat{G}^0_{\mathrm{der}}$ of $\hat{G}^0$ and z belongs to $Z(\hat{G}^0)$. Then

$$s_1 g s_1^{-1} g^{-1} = s_1 g_1 s_1^{-1} g_1^{-1} \cdot s_1 z s_1^{-1} z^{-1} \,.$$

In particular, both g_1, and z belong to $\widetilde{C}_s$. But the element $s_1 g_1 s_1^{-1} g_1^{-1}$ lies in G^0_{der}. The lemma follows from the fact that $\hat{G}^0_{\mathrm{der}}$ has finite center. □

PROOF OF LEMMA 2.1: According to the first condition in its definition, $\mathrm{Aut}(H)$ is contained in $\widetilde{C}_s$. Let $\mathrm{Aut}'(H)$ be the subgroup of elements $g \in \mathrm{Aut}(H)$ such that $s_1 g s_1^{-1} g^{-1}$ belongs to $\hat{Z}'(s)$. The last lemma tells us that $\mathrm{Aut}'(H)$ is of finite index in $\mathrm{Aut}(H)$.

Let g be an element in $\mathrm{Aut}'(H)$. Then we can write

$$g = g_1 z_1 \,, \qquad g_1 \in C_s \,,\ z_1 \in Z(\hat{G}^0) \,.$$

Suppose also that h is an element in $\xi(\mathcal{H})$. The second condition in the definition of $\mathrm{Aut}(H)$ implies that ghg^{-1} equals $h_1^{-1}h$, for some element $h_1 \in \xi(\hat{H})$. We can write this as

$$h z_1 h^{-1} z_1^{-1} = (h g_1 h^{-1})^{-1} h_1 g_1 \,.$$

Both h_1 and g_1 commute with s_1. It follows easily from (2.1) that hg_1h^{-1} also commutes with s_1. Therefore $hz_1h^{-1}z_1^{-1}$ commutes with s_1, and belongs to the subgroup $Z(\hat{G})$ of $Z(\hat{G}^0)$. Now

$$hz_1h^{-1}z_1^{-1} = \sigma(z_1)z_1^{-1},$$

where σ is the projection of h onto $\Gamma = \mathrm{Gal}(\bar{F}/F)$. The action of Γ on $Z(\hat{G}^0)$ factors through a finite quotient $\mathrm{Gal}(E/F)$, and this action preserves the subgroup $Z(\hat{G})$. We obtain a homomorphism

$$g \longrightarrow \sigma(z_1)z_1^{-1}$$

from $\mathrm{Aut}'(H)$ to the finite group $H^1\big(\mathrm{Gal}(E/F),\, Z(\hat{G})\big)$. Suppose that g lies in the kernel of this map. Then

$$\sigma(z_1)z_1^{-1} = \sigma(z)z^{-1}, \qquad \sigma \in \Gamma,$$

for some element $z \in Z(\hat{G})$. In other words, there is a decomposition $z_1 = zz_1'$, for elements z in $Z(\hat{G})$ and z_1' in $Z(\hat{G}^0)^\Gamma$. We can therefore write $g = g_1'z_1'$, where the element $g_1' = g_1z$ lies in the centralizer C_s. In other words, g belongs to the subgroup $C_sZ(\hat{G}^0)^\Gamma$.

It remains only to observe that $\xi(\hat{H})$ is the identity component of C_s. We obtain an embedded chain

$$\xi(\hat{H})Z(\hat{G}^0)^\Gamma \subset C_sZ(\hat{G}^0)^\Gamma \subset \mathrm{Aut}'(H) \subset \mathrm{Aut}(H)$$

of normal subgroups of finite index. Therefore $\xi(\hat{H})Z(\hat{G}^0)^\Gamma$ is of finite index in $\mathrm{Aut}(H)$, and the two groups have the same identity component. □

Let $(H, \mathcal{H}, s, \xi)$ be a fixed endoscopic datum. One is interested in the L-homomorphisms of W_F into ${}^LG$ whose image is contained in $\xi(\mathcal{H})$. (Recall that an L-homomorphism between two extensions of W_F is a homomorphism which commutes with the projection onto W_F.) One might like to be able to identify such objects with L-homomorphisms of W_F into the L-group ${}^LH$ of H. However, this is not always possible. The L-group is a semidirect product $\hat{H} \rtimes W_F$ relative to an L-action of W_F on $\hat{H}$ **[20, 1.4]**. (The action of W_F of course factors through the quotient Γ of W_F.) But the two extensions $\mathcal{H}$ and ${}^LH$ of W_F by $\hat{H}$ need not be isomorphic. In other words, there might not be an L-embedding of ${}^LH$ into ${}^LG$ which co-incides with the image of ξ. Fortunately the problem is not serious. In the case that $G = G^0$, the question can be resolved by taking a z-extension of G, as has been explained in [**20**, (4.4)]. In the general case, Shelstad has pointed out that it is necessary to work directly with extensions of the endoscopic groups H. Suppose, then, that

$$1 \longrightarrow Z_1 \longrightarrow H_1 \longrightarrow H \longrightarrow 1 \tag{2.2}$$

is a central extension of quasi-split groups over F. We shall review the question of whether there exists an L-embedding

$$\xi_1 : \mathcal{H} \longrightarrow {}^LH_1$$

which extends the canonical embedding $\hat{H} \hookrightarrow \hat{H}_1$ of dual groups.

Consider first the kernel K_F of the projection $W_F \to \Gamma_F$, a connected group. It would be no trouble to construct an embedding for the preimage $\mathcal{H}'$ of K_F in $\mathcal{H}$. For it follows easily from (2.1) that $\xi(\mathcal{H}')$ equals the subgroup $\xi(\hat{H}) \times K_F$ of ${}^LG^0$. In other words, there is a splitting $\theta : K_F \to \mathcal{H}'$ such that

$$h\theta(k)h^{-1} = \theta(w_h k w_h^{-1}) , \qquad h \in \mathcal{H},\ k \in K_F, \tag{2.3}$$

where w_h is the image of h in W_F. Now by assumption, the map of W_F into $\mathrm{Out}(\hat{H})$ defined by $\mathcal{H}$ is the same as the L-action

$$h \longrightarrow w(h) , \qquad h \in \hat{H},\ w \in W_F,$$

used to define ${}^LH$. It follows that θ can be extended to a section from W_F to $\mathcal{H}$ such that

$$\theta(w)h\theta(w)^{-1} = w(h) , \qquad h \in \hat{H},\ w \in W_F.$$

Keep in mind that it is only the restriction of θ to K_F which is a homomorphism. However, θ is uniquely determined up to multiplication by elements in the center $Z(\hat{H})$ of $\hat{H}$. Therefore

$$\theta(w_1)\theta(w_2) = b(w_1, w_2)\theta(w_1, w_2) , \qquad w_1, w_2 \in W_F,$$

where $b(w_1, w_2)$ is a 2-cocycle from W_F to $Z(\hat{H})$. By (2.3), $b(w_1, w_2)$ depends only on the images of w_1 and w_2 in Γ_F. We shall write β

for the image of b in $H^2(W_F, Z(\hat{H}_1))$, relative to the embedding of $Z(\hat{H})$ into $Z(\hat{H}_1)$. Then β is the inflation of a class in $H^2(\Gamma_F, Z(\hat{H}_1))$ which is independent of θ. Suppose that β is trivial. That is,

$$\beta(w_1, w_2) = z(w_1)w_1(z(w_2))z(w_1w_2)^{-1}, \qquad w_1, w_2 \in W_F,$$

for a function $z : W_F \to Z(\hat{H}_1)$ which is uniquely determined up to a 1-cocycle. Every element in $\mathcal{H}$ can be represented uniquely in the form

$$h\theta(w), \qquad h \in \hat{H}, \ w \in W_F,$$

and the map

$$\xi_1(h\theta(w)) = hz(w) \rtimes w \tag{2.4}$$

is then an L-embedding of $\mathcal{H}$ into ${}^LH_1$. Conversely, if an embedding ξ_1 exists, the function $z(w)$ in (2.4) will split the class β.

Assume that the embedding ξ_1 exists. Suppose also that the central subgroup Z_1 of H_1 is connected. Then we can form the L-group ${}^LZ_1 = \hat{Z}_1 \times W_F$, and there is a canonical projection ${}^LH_1 \to {}^LZ_1$. We also have an exact sequence

$$1 \longrightarrow Z(\hat{H}) \longrightarrow Z(\hat{H}_1) \longrightarrow \hat{Z}_1 \longrightarrow 1$$

of complex abelian groups. Let $z_1(w)$ be the projection of $z(w)$ onto $\hat{Z}_1$. Then z_1 is a 1-cocycle from W_F to $\hat{Z}_1$. In fact, if we agree not to distinguish between a cocycle and its corresponding cohomology class, z_1 is just the preimage of the class $b \in H^2(W_F, Z(\hat{H}))$ determined by the long exact sequence

$$\begin{aligned} \dots \to H^1(W_F, Z(\hat{H}_1)) &\to H^1(W_F, \hat{Z}_1) \\ &\to H^2(W_F, Z(\hat{H})) \to H^2(W_F, Z(\hat{H}_1)) \to \dots \end{aligned}$$

It is uniquely determined modulo the image of $H^1(W_F, Z(\hat{H}_1))$ in $H^1(W_F, \hat{Z}_1)$. The map

$$\alpha_1(w) = z_1(w) \rtimes w, \qquad w \in W_F,$$

is an L-homomorphism of W_F to ${}^LZ_1$.

Suppose that $L_F \to W_F$ is some extension of W_F. Suppose also that $\psi : L_F \to {}^L G$ is an L-homomorphism whose image is contained in $\xi(\mathcal{H})$. That is, $\psi = \xi \circ \psi_H$, for some L-homomorphism $\psi_H : L_F \to \mathcal{H}$. Then $\psi_1 = \xi_1 \circ \psi_H$ is an L-homomorphism of L_F into ${}^L H_1$. Set

$$\psi_H(t) = \gamma(t)\theta(w_t), \qquad t \in L_F,$$

where w_t is the image of t in W_F and $\gamma(t)$ belongs to $\hat{H}$. Then

$$\xi_1\big(\psi_H(t)\big) = \gamma(t)z(w_t) \times w_t, \qquad t \in L_F.$$

It follows that the composition of ψ_1 with the projection ${}^L H_1 \to {}^L Z_1$ equals α_1 (or rather, the pullback of α_1 to L_F.) Conversely, any L-homomorphism $\psi_1 : L_F \to {}^L H_1$ whose projection to ${}^L Z_1$ equals α_1 is easily seen to be of the form $\xi_1 \circ \psi_H$. We can summarize these remarks in a commutative diagram

$$\begin{tikzcd}
& & L_F \arrow[dll, "\alpha_1"'] \arrow[dl, "\psi_1"] \arrow[d, "\psi_H"] \arrow[dr, "\psi"] & \\
{}^L Z_1 & {}^L H_1 \arrow[l] & \mathcal{H} \arrow[l, "\xi_1"'] \arrow[r, "\xi"] & {}^L G
\end{tikzcd}$$

In conclusion, we want to associate pairs (H_1, ξ_1) to endoscopic data, where H_1 is a central extension (2.2) and ξ_1 is an L-embedding (2.4). We shall call such a pair a *splitting* for the endoscopic datum. We shall say that (H_1, ξ_1) is a *distinguished* splitting if, in addition, the map $H_1(\mathbb{A}) \to H(\mathbb{A})$ between adèle groups is surjective, and the central subgroup Z_1 is an induced torus. That is, Z_1 is a product of tori of the form $\mathrm{Res}_{E/F}(G_m)$. In particular, Z_1 is connected, as we assumed in the discussion above. Any endoscopic datum has a distinguished splitting. For example, the cocycle $b(w_1, w_2) \in Z(\hat{H})$ that we described above often splits. In this case, we can simply take $(H_1, \xi_1) = (H, Id)$. In general, we can always take H_1 to be a z-extension of H [**11**, §1], the existence of which is established in [**17**, **pp. 721–722**]. The first condition follows from [**11**, **Lemma 1.1(3)**], while the second is part of the definition of a z-extension. It is also part of the definition that the derived group of H_1 is simply connected. This in turn implies that $Z(\hat{H}_1)$ is a complex torus. It follows from **[17, Lemma 4]** that the class $\beta \in H^2(W_F, Z(\hat{H}_1))$ is trivial. The embedding ξ_1 therefore exists, and (H_1, ξ_1) becomes a distinguished splitting. In general, if (H_1, ξ_1) is any distinguished splitting, one needs to know that the canonical map

$$\ker^1(F, Z(\hat{H})) \longrightarrow \ker^1(F, Z(\hat{H}_1))$$

is an isomorphism. (As before, $\ker^1(F, Z(\hat{H}))$ denotes the kernel of the map

$$H^1(F, Z(\hat{H})) \longrightarrow \bigoplus_v H^1(F_v, Z(\hat{H})) .)$$

This follows from the proof of **[12, Lemma 4.3.2(a)]**. We will also use the injectivity of the map

$$H^1(F, Z(\hat{H})) \longrightarrow H^1(F, Z(\hat{H}_1)) ,$$

which is a consequence of the long exact sequence of cohomology, and the fact the group $H^0(F, \hat{Z}_1) = \pi_0(\hat{Z}_1^\Gamma)$ is trivial.

§3. The discrete part of the trace formula

We are going to study a piece of the trace formula. It consists of those distributions on the spectral side of the trace formula which are discrete with respect to the natural measure on the relevant automorphic representations. This part of the formula contains the actual trace on the discrete spectrum. It is thus the payload, the part which will eventually be used to compare automorphic representations on different groups. Of course, there are serious problems relating to the other terms in the trace formula which will have to be overcome first. Our intention in this paper is simply to see what can be learned once these other problems have been solved.

Let $\mathbb{A}$ be the adèle ring of F. We should first identify our space of test functions on $G(\mathbb{A})$, the set of $\mathbb{A}$-valued points in G. Consider the diagonalizable group $Z(G) = \mathrm{Cent}(G, G^0)$. We shall fix a closed subgroup X of the group $Z(G, \mathbb{A})$ of adèle points such that $X \cap Z(G, F)$ is closed, and such that $XZ(G, F)\backslash Z(G, \mathbb{A})$ is compact. Let χ be a character on X which is trivial on $X \cap Z(G, F)$. Then $C_c^\infty(G(\mathbb{A}), \chi)$ will denote the space of smooth functions f on $G(\mathbb{A})$, of compact support modulo X, such that

$$f(zx) = \chi(z)^{-1} f(x) , \qquad z \in \chi,\ z \in G(\mathbb{A}).$$

Let t be an arbitrary but fixed nonnegative real number. The corresponding discrete part of the trace formula is the distribution

$$I_{\mathrm{disc},t}(f), \qquad f \in C_c^\infty\big(G(\mathbb{A}),\chi\big),$$

on $C_c^\infty\big(G(\mathbb{A}),\chi\big)$ which is given by the expression

$$\text{(3.1)} \quad \sum_{\{M\}} \sum_{w \in W^G(\underline{\mathbf{a}}_M)_{\mathrm{reg}}} |\pi_0(G^+)|^{-1}|W^G(\underline{\mathbf{a}}_M)|^{-1} \\ |\det(w-1)_{\underline{\mathbf{a}}_M^G}|^{-1}\operatorname{tr}\big(M(w,0)\rho_{P,t}(0,f)\big).$$

(See **[2,** §4], **[4,** §**II.9**].) We shall describe very briefly the terms in this expression. The outer sum is over the finite set of $G^0(F)$-orbits of Levi components M of F-rational parabolic subgroups P of G^0. The inner sum is over the regular elements

$$W^G(\underline{\mathbf{a}}_M)_{\mathrm{reg}} = \{w \in W^G(\underline{\mathbf{a}}_M) : \det(w-1)_{\underline{\mathbf{a}}_M^G} \neq 0\}$$

in the Weyl set

$$W^G(\underline{\mathbf{a}}_M) = \mathrm{Norm}(A_M, G)/M$$

of (G,M). As in earlier papers, we regard the Weyl elements as operators on the real vector space

$$\underline{\mathbf{a}}_M = \mathrm{Hom}\big(X(M)_F, \mathbb{R}\big)$$

which leave invariant the kernel $\underline{\mathbf{a}}_M^G$ of the projection of $\underline{\mathbf{a}}_M$ onto $\underline{\mathbf{a}}_G$. For each M there is canonical isomorphism from

$$A_{M,\infty} = A_{M_{\mathbb{Q}}}(\mathbb{R})^0, \qquad M_{\mathbb{Q}} = \mathrm{Res}_{F/\mathbb{Q}}(M),$$

onto $\underline{\mathbf{a}}_M$. If $A_{M,\infty}^G$ denotes the preimage of $\underline{\mathbf{a}}_M^G$ in $A_{M,\infty}$, we can extend χ uniquely to a character χ_M on $X_M = A_{M,\infty}^G X$ which is trivial on $A_{M,\infty}^G$. Let $L^2_{\mathrm{disc},t}\big(M(F)\backslash M(\mathbb{A}),\chi_M^{-1}\big)$ be the subspace of $L^2\big(M(F)\backslash M(\mathbb{A}),\chi_M^{-1}\big)$ which decomposes under $M(\mathbb{A})$ as a direct sum of irreducible represention whose Archimedean infinitesimal character has norm t. Then

$$\rho_{P,t}(0) : f \to \rho_{P,t}(0,f) = \int_{X\backslash G(\mathbb{A})} f(x)\rho_{P,t}(0,x)dx$$

stands for the corresponding representation induced from $P(\mathbb{A})$ to the group $G(\mathbb{A})^+$ generated by $G(\mathbb{A})$. It acts on a Hilbert space $\mathcal{H}_{P,t}$ of χ_M^{-1}-equivariant functions on $G(\mathbb{A})^+$. Finally,

$$M(w,0): \mathcal{H}_{P,t} \longrightarrow \mathcal{H}_{P,t}, \qquad w \in W^G(\underline{\mathfrak{a}}_M)_{\mathrm{reg}},$$

is the global intertwining operator which comes from the theory of Eisenstein series. For a given conductor, $I_{\mathrm{disc},t}(f)$ is a finite linear combination of irreducible characters on $G(\mathbb{A})^+$.

There are some minor discrepancies between (3.1) and the original definition **[2, (4.3)]**. In (3.1) we have summed over the orbits $\{M\}$ instead of all Levi components which contain a given minimal one. This is why $|W^G(\underline{\mathfrak{a}}_M)|^{-1}$ appears instead of the normalizing constant $|W_0^M||W_0^G|^{-1}$ from **[2]**. The operator $\rho_{P,t}(0,f)$ here comes from a representation of $G(\mathbb{A})^+$ induced from a subgroup of the connected component $G^0(\mathbb{A})$. It is a direct sum of $|\pi_0(G^+)|$ copies of the corresponding operator from **[2]**, which comes essentially from the induced representation of $G^0(\mathbb{A})$. Hence the constant $|\pi_0(G^+)|^{-1}$ in (3.1). The difference between taking a χ-equivariant function on $G(\mathbb{A})$, as we have done here, and a function defined on the subset $G(\mathbb{A})^1$ of $G(\mathbb{A})$, as in **[2]**, is purely formal. In **[2]**, there was also the additional assumption that f was K-finite, but this was only for dealing with other terms in the trace formula.

The program for comparing trace formulas on different groups, as it is presently conceived, falls into the general framework of stabilizing the trace formula. The basic references for this problem are **[18]**, **[12]**, and **[13]**. The problem was solved completely for $G = \mathrm{SL}(2)$ in **[15]**. A general solution would include: a transfer map from functions for G to functions for endoscopic data, a stable distribution analogous to $I_{\mathrm{disc},t}$ for any quasi-split group, and an identity relating $I_{\mathrm{disc},t}$ to the corresponding stable distributions for endoscopic data. We shall discuss the transfer first, and then describe the expected properties of the other objects in the form of a hypothesis.

Suppose that $(H, \mathcal{H}, s, \xi)$ is an elliptic endoscopic datum for G. Assume also that we have fixed a distinguished splitting (H_1, ξ_1) for the endoscopic datum. As we recall from §1, ξ_1 determines an L-homomorphism $\alpha_1 : W_F \to {}^L Z_1$. Let

$$\zeta_1 : Z_1(F)\backslash Z_1(\mathbb{A}) \longrightarrow \mathbb{C}^*$$

be the character associated to α_1 by the Langlands correspondence for tori. Now, in the special case that $G = G^0$, the results **[20]** of Langlands and Shelstad imply the existence of a canonical map $f \to f^{H_1}$ from functions $f \in C_c^\infty(G(\mathbb{A})$ to functions $f^{H_1}(\gamma_{H_1})$ on suitable stable conjugacy classes in $H(\mathbb{A})$, with the property that

$$f^{H_1}(z_1\gamma_{H_1}) = \zeta_1(z_1)^{-1}f^{H_1}(\gamma_{H_1}), \qquad z_1 \in Z_1(\mathbb{A}).$$

(See also [**12**], [**13**] and [**3**].) The map must be constructed as a tensor product of the local maps $f_v \to f_v^{H_1}$, $f_v \in C_c^\infty(G(F_v))$, which are defined explicitly in [**20**]. Langlands and Shelstad expect that f^{H_1} is the set of stable orbital integrals on $H_1(\mathbb{A})$ of a function g in $C_c^\infty(H_1(\mathbb{A}), \zeta_1)$. We shall assume that this is so. In fact, we shall assume that the transfer map

$$f \longrightarrow f^{H_1}, \qquad f \in C_c^\infty(G(\mathbb{A})),$$

has been defined, and has this property, for general G.

We should actually modify the transfer mapping so that its domain is the space $C_c^\infty(G(\mathbb{A}, \chi)$ considered earlier. Lemma 4.4A of [**20**] suggests how the functions

$$f_z(x) = f(zx), \qquad z \in Z(G, \mathbb{A}),\ x \in G(\mathbb{A}),\ f \in C_c^\infty(G(\mathbb{A})),$$

should behave under the transfer map. In general, there will be a norm mapping $z \to z'$ from $Z(G,F)\backslash Z(G,\mathbb{A})$ into $Z(H,F)\backslash Z(H,\mathbb{A})$. We also have the exact sequence

$$1 \longrightarrow Z_1 \longrightarrow Z(H_1) \longrightarrow Z(H) \longrightarrow 1.$$

We can then expect a formula

$$(f_z)^{H_1}(\gamma_{H_1}) = \zeta_1(z_1)f^{H_1}(z_1\gamma_{H_1}), \tag{3.2}$$

where ζ_1 is an extension to $Z(H_1,F)\backslash Z(H_1,\mathbb{A})$ of the character on $Z_1(F)\backslash Z_1(\mathbb{A})$, and z_1 is any point in $Z(H_1,\mathbb{A})$ whose image in $Z(H,\mathbb{A})$ equals z'. Recall that χ is a character on the closed subgroup X of $Z(G,\mathbb{A})$. We shall assume that

$$\chi(z) = \chi'(z'), \qquad z \in X,$$

where χ' is a character on the image X' of X in $Z(H,\mathbb{A})$. To define the transfer mapping for functions in $C_c^\infty(G(\mathbb{A}), \chi)$, we simply multiply each side of (3.2) by $\chi(z)$, and integrate over z in $X \cap Z(G,F)\backslash X$. Let X_1 be the preimage of X' in $Z(H_1,\mathbb{A})$, and set

$$\chi_1(z_1) = \zeta_1(z_1)\chi'(z') ,$$

for any point $z_1 \in X_1$ with image z' in X'. Then χ_1 is a character on X_1, and the triple (H_1,X_1,χ_1) satisfies the conditions we imposed on (G,X,χ). In this context our assumption is that for any function $f \in C_c^\infty(G(\mathbb{A}),\chi)$ there is a function $g \in C_c^\infty(H_1(\mathbb{A}),\chi_1)$ whose stable orbital integrals are given by f^{H_1}. The function g is of course not uniquely determined by f. However, if SI is any stable distribution on $C_c^\infty(H_1(\mathbb{A}),\chi_1)$, $SI(g)$ will be uniquely determined by f. We shall therefore write

$$S\hat{I}(f^{H_1}) = SI(g) .$$

The ultimate goal is to give an expansion of $I_{\mathrm{disc},t}$ as a linear combination of stable distributions on the equivalence classes of elliptic endoscopic data $\{H\}$ for G. The coefficients will be certain constants $\iota(G,H)$, which in the case $G = G^0$ were introduced by Langlands **[18]**. (Following the usual convention of metonymy, we shall often write H in place of a full endoscopic datum $(H,\mathcal{H},s,\xi)$.) Kottwitz has established a simple formula for these constants **[12, Theorem 8.3.1]**, again when $G = G^0$. Let

$$\tau_1(G^0) = \tau(G^0)\tau(G^0_{\mathrm{sc}})^{-1}$$

be the relative Tamagawa number of G^0 **[12, §5]**. ($\tau(G^0)$ denotes the ordinary Tamagawa number of G^0, and G^0_{sc} is the simply connected cover of the derived group of G^0. Thus according to Weil's conjecture, which has been established by Kottwitz **[14]** for groups without E_8 factors, $\tau_1(G^0)$ simply equals $\tau(G^0)$.) Kottwitz' formula is then

$$\iota(G,H) = \tau_1(G^0)\tau_1(H)^{-1}|\pi_0(\mathrm{Aut}(H))|^{-1} .$$

In the general case, the constants have not yet been defined. We shall have to get by with a makeshift definition that reduces to Kottwitz' formula when $G = G^0$.

If we are given an equivalence class $\{H\}$ of elliptic endoscopic data, we shall usually assume implicitly that H is a representative of the class such that ξ is the identity. That is, $\mathcal{H}$ is an embedded subgroup of ${}^LG^0$. Then $Z(\hat{H})^\Gamma$ is a subgroup of $\hat{G}^0$ whose identity component equals $A_{\hat{G}}$. The subgroup $\kappa_G = (A_{\hat{G}^0})^{\hat{G}}$ of $\hat{G}^0$ also has $A_{\hat{G}}$ as its identity component, so that $\kappa_G \cap Z(\hat{H})^\Gamma$ is a subgroup of finite index in κ_G. For general G we shall simply define

$$(3.3)\ \iota(G,H) = \tau_1(G^0)\tau_1(H)^{-1}|\pi_0(\mathrm{Aut}(H))|^{-1}|\kappa_G/\kappa_G \cap Z(\hat{H})^\Gamma|^{-1}.$$

The fourth factor in the product on the right, which of course equals 1 when $G = G^0$, is suggested by the calculations in §7.

We can now state the hypothesis. Part of it applies to any (G, X, χ) as above, and part applies to triples (G_1, X_1, χ_1) with the restriction that G_1 is a connected quasi-split group over F.

HYPOTHESIS 3.1. *For any (G_1, X_1, χ_1) there is a stable distribution $SI^{G_1}_{\mathrm{disc},t}$ on $C^\infty_c(G_1(\mathbb{A}), \chi_1)$ with the property that for any (G, X, χ), the distribution*

$$(3.4) \qquad E_{\mathrm{disc},t}(f) = \sum_H \iota(G,H) S\hat{I}^{H_1}_{\mathrm{disc},t}(f^{H_1})$$

equals $I_{\mathrm{disc},t}(f)$. Here f stands for any function in $C^\infty_c(G(\mathbb{A}), \chi)$ and H is summed over the equivalence classes of elliptic endoscopic data for G. □

Remarks. 1. It is understood that we have fixed a distinguished splitting (H_1, ξ_1) for each H. The distribution $S\hat{I}^{H_1}_{\mathrm{disc},t}(f^{H_1})$ should then depend only on H and not on the splitting.

2. The stable distributions $SI^{G_1}_{\mathrm{disc},t}$ are uniquely determined by the condition that $E_{\mathrm{disc},t}(f)$ equals $I_{\mathrm{disc},t}(f)$. For suppose that they have been defined inductively for any group whose semisimple part has dimension less than that of G_1. Setting $G = G_1$, one simply defines

$$SI^{G_1}_{\mathrm{disc},t}(f) = I_{\mathrm{disc},t}(f) - \sum_{H \neq G_1} \iota(G,H) S\hat{I}^{H_1}_{\mathrm{disc},t}(f^{H_1}).$$

In this case, the hypothesis becomes the assertion that the right hand side is a stable distribution in f. This of course is highly nontrivial. It is likely to be resolved only by proving a similar assertion for all the other terms in the trace formula. There is a discussion of this question in the paper **[19]**.

We shall need a slightly different formula for $\iota(G,H)$ in §7. For H as above, set

$$\overline{Z}(\hat{H})^\Gamma = Z(\hat{H})^\Gamma Z(\hat{G}^0)/Z(\hat{G}^0) \cong Z(\hat{H})^\Gamma / Z(\hat{H})^\Gamma \cap Z(\hat{G})^\Gamma.$$

Since H represents an elliptic endoscopic datum, $\overline{Z}(\hat{H})^\Gamma$ is a finite (abelian) group.

LEMMA 3.2. *The constant $\iota(G,H)$ equals*

$$|\ker^1(F, Z(\hat{G}^0))|^{-1}|\pi_0(\kappa_G)|^{-1} \qquad (3.5)$$
$$|\ker^1(F, Z(\hat{H}))||\overline{Z}(\hat{H})^\Gamma|^{-1}|\mathrm{Aut}(H)/\hat{H}Z(\hat{G}^0)^\Gamma|^{-1}.$$

PROOF: The main point is the formula

$$\tau_1(G^0) = |\pi_0(Z(\hat{G}^0)^\Gamma)||\ker^1(F, Z(\hat{G}^0))|^{-1}$$

of Sansuc and Kottwitz for the relative Tamagawa number [**12, (5.1.1)**]. From this, it will be a routine matter to derive the expression (3.5) from (3.3). For Lemma 2.1 tells us that

$$|\pi_0(\mathrm{Aut}(H))|^{-1} = |\mathrm{Aut}(H)/\hat{H}Z(\hat{G}^0)^\Gamma|^{-1}|\hat{H}Z(\hat{G}^0)^\Gamma/\hat{H}A_{\hat{G}^0}|^{-1}.$$

Keeping in mind that $A_{\hat{G}^0}$ is the identity component of $Z(\hat{G}^0)^\Gamma$, we deduce that

$$\begin{aligned}
&|\hat{H}Z(\hat{G}^0)^\Gamma/\hat{H}A_{\hat{G}^0}|^{-1}\\
&= |Z(\hat{G}^0)^\Gamma/Z(\hat{G}^0)^\Gamma \cap (\hat{H}A_{\hat{G}^0})|^{-1}\\
&= |\pi_0(Z(\hat{G}^0)^\Gamma)|^{-1}|(Z(\hat{G}^0)^\Gamma \cap (\hat{H}A_{\hat{G}^0}))/A_{\hat{G}^0}|.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&|Z(\hat{G}^0)^\Gamma \cap (\hat{H}A_{\hat{G}^0}))/A_{\hat{G}^0}|\\
&= |\hat{H} \cap Z(\hat{G}^0)^\Gamma/\hat{H} \cap A_{\hat{G}^0}|\\
&= |Z(\hat{H})^\Gamma \cap Z(\hat{G})^\Gamma/Z(\hat{H})^\Gamma \cap \kappa_G|\\
&= |\pi_0(Z(\hat{H})^\Gamma \cap Z(\hat{G})^\Gamma)||\pi_0(Z(\hat{H})^\Gamma \cap \kappa_G)|^{-1}\\
&= |\pi_0(Z(\hat{H})^\Gamma)||\overline{Z}(\hat{H})^\Gamma|^{-1}|\pi_0(\kappa_G)|^{-1}|\kappa_G/Z(\hat{H})^\Gamma \cap \kappa_G|.
\end{aligned}$$

The lemma follows from the formula above for $\tau_1(G^0)$ and its analogue for $\tau_1(H)$. □

§4. THE CONJECTURAL MULTIPLICITY FORMULA

Our goal is to provide some motivation for the conjectures on non-tempered automorphic representations stated in [1] and [3]. The main global ingredient of the conjectures is a multiplicity formula for automorphic representations in the discrete spectrum. It is a generalization of similar formula for tempered automorphic representations which was implicit in the examples of [**15**] and was stated explicitly in [**12**]. We shall recall the various objects from [**3**, §8] needed to state the formula.

The automorphic representations which occur in the spectral decomposition should be attached to maps

$$\psi : L_F \times SL(2,\mathbb{C}) \rightarrow {}^L G^0 . \tag{4.1}$$

such that the projection onto $\hat{G}^0$ of the image L_F is bounded. Here L_F is hypothetical Langlands group, which we shall assume is an extension of the Weyl group W_F by a compact connected group. The maps themselves are subject to certain conditions. For example, ψ should be globally relevant, in the sense that its image must not lie in a parabolic subgroup of ${}^L G^0$ unless the corresponding parabolic subgroup of G^0 is defined over F. Another condition is designed to insure that ψ parametrizes representations of $G^0(\mathbb{A})$ which lift to $G(\mathbb{A})^+$. Let

$$S_\psi = S_\psi(G)$$

be the set of elements $s \in \hat{G}$ such that each point

$$s\psi(t')s^{-1}\psi(t')^{-1} , \qquad t' \in L_F \times SL(2,\mathbb{C}),$$

belongs to $Z(\hat{G}^0)$, and such that the class of the 1-cocycle

$$t \longrightarrow s\psi(t)s^{-1}\psi(t)^{-1} , \qquad t \in L_F ,$$

lies in the subgroup $\ker^1(L_F, Z(\hat{G}^0))$ of $H^1(L_F, Z(\hat{G}^0))$. The condition on ψ is that S_ψ be nonempty. Recall also that two parameters ψ_1 and ψ_2 are equivalent if there is an element $g \in \hat{G}^0$ such that

$$\psi_2(t,u) = g^{-1}\psi_1(t,u)ga(t) , \qquad (t,u) \in L_F \times SL(2,\mathbb{C}), \tag{4.2}$$

where $a(t)$ is a 1-cocycle of L_F in $Z(\hat{G}^0)$ whose class in $H^1(L_F, Z(\hat{G}^0))$ lies in $\ker^1(L_F, Z(\hat{G}^0))$.

Let $\Psi(G)$ denote the set of equivalence classes of maps (4.1) which satisfy the required conditions [**3**, §**8**]. Let $\Psi_0(G)$ denote the subset of (equivalence classes of) maps $\psi \in \Psi(G)$ such that the set

$$\bar{S}_\psi = \bar{S}_\psi(G) = S_\psi(G)/Z(\hat{G}^0)$$

is finite. In [4] we called these maps *elliptic.* They should parametrize automorphic representations which occur in the discrete spectrum. It will be convenient to define two other subsets of $\Psi(G)$. Let us say that ψ is ***weakly elliptic*** if the group $\bar{S}_\psi(G^0)$ (obtained by replacing G with the identity component G^0) has finite center. We shall say that ψ is ***discrete*** if it satisfies the weaker condition that the group

$$\bar{S}_\psi^+ = S_\psi^+/Z(\hat{G}^0)$$

generated by $\bar{S}_\psi$ has finite center. (Keep in mind that $\bar{S}_\psi^+$, S_ψ^+, $\bar{S}_\psi(G^0)$, $S_\psi(G^0)$, etc., are complex reductive Lie groups which are generally not connected.) Let $\Psi_0'(G)$ and $\Psi_{\mathrm{disc}}(G)$ denote the set of (equivalence classes of) maps $\psi \in \Psi(G)$ which are weakly elliptic and discrete, respectively. Then we have embeddings

$$\Psi_0(G) \subset \Psi_0'(G) \subset \Psi_{\mathrm{disc}}(G) \subset \Psi(G) .$$

Let χ be a fixed character on a subgroup X of $Z(G, \mathbb{A})$ which satisfies the conditions of §3. We may as well assume that X is contained in $Z^0(G, \mathbb{A})$, the adèle group of the identity component of $Z(G)$. There is a canonical map from ${}^LG^0$ onto the L-group ${}^LZ^0(G)$ of $Z^0(G)$. The composition of any parameter $\psi \in \Psi(G)$ with the map gives a parameter in $\Psi\big(Z^0(G)\big)$, and therefore a dual character

$$\zeta_\psi : Z^0(G, F)\backslash Z^0(G, \mathbb{A}) \longrightarrow \mathbb{C}^* .$$

We shall write $\Psi(G, \chi)$, $\Psi_0(G, \chi)$, etc., for the set of parameters ψ in $\Psi(G)$, $\Psi_0(G)$, etc., such that the character ζ_ψ coincides with χ on X.

Suppose that $\psi \in \Psi(G, \chi)$. As in [**3**, §**8**], we can form the finite set

$$\mathcal{S}_\psi = \mathcal{S}_\psi(G) = S_\psi/S_\psi^0 Z(\hat{G}^0) .$$

It is a coset of

$$\mathcal{S}_\psi(G^0) = S_\psi(G^0)/S_\psi^0 Z(\hat{G}^0)$$

in the finite group

$$\mathcal{S}_\psi^+ = \mathcal{S}_\psi(G^+) = S_\psi(G^+)/S_\psi^0 Z(\hat{G}^0) .$$

Now the local conjectures in [**3**, §**6**] assert that there is a set Π_ψ of representations attached to ψ. The elements in Π_ψ should in fact belong to $\Pi_{\text{unit}}\big(G(\mathbb{A}),\chi\big)$, the set of equivalence classes of irreducible unitary representations of $G(\mathbb{A})^+$ whose restrictions to $G^0(\mathbb{A})$ remain irreducible, and whose central character on X coincides with χ. There should also be a canonical pairing

$$< x, \pi > , \qquad\qquad x \in \mathcal{S}_\psi^+,\ \pi \in \Pi_\psi ,$$

such that the functions $x \rightarrow < x, \pi >$ are characters of nonzero finite dimensional representations of $\mathcal{S}_\psi^+$. Finally, the conjectures assert the existence of stable distributions

$$f_1 \longrightarrow f_1^{G_1}(\psi_1) , \qquad\qquad \psi_1 \in \Psi(G_1, \chi_1), \tag{4.3}$$

on $C_c^\infty\big(G_1(\mathbb{A}),\chi_1\big)$, for each (G_1, X_1, χ_1) with G_1 connected and quasi-split.

Let us recall how the distributions (4.3) are supposed to behave with respect to endoscopic data. Suppose that s is a semisimple element in $\bar{S}_\psi$. Take $\hat{H}$ to be the connected centralizer in $\hat{G}^0$ of any point in s, and set

$$\mathcal{H} = \hat{H}\psi(L_F \times SL(2,\mathbb{C})) .$$

There is obviously an injection $\hat{H} \rightarrow \mathcal{H}$ and a surjection $\mathcal{H} \rightarrow W_F$. We are assuming that the kernel of the map $L_F \rightarrow W_F$ is connected, and it follows that ψ maps both the kernel and $SL(2,\mathbb{C})$ into $\hat{H}$. Therefore

$$1 \longrightarrow \hat{H} \longrightarrow \mathcal{H} \longrightarrow W_F \longrightarrow 1$$

is a short exact sequence. We can identify $\hat{H}$, equipped with the canonical L-action of W_F induced by $\mathcal{H}$, with the dual of a well defined quasi-split group H over F. If ξ is the inclusion of $\mathcal{H}$ into ${}^LG^0$, then $(H, \mathcal{H}, s, \xi)$ is an endoscopic datum for G. It has the property that ψ equals $\xi \circ \psi_H$ for some L-homomorphism ψ_H of $L_F \times SL(2,\mathbb{C})$ into $\mathcal{H}$. Now, let (H_1, ξ_1) be any distinguished splitting for the endoscopic datum. We can construct the character χ_1 on a closed subgroup X_1 of $Z(H_1,\mathbb{A})$ as in §2, and from our remarks in §2, we see that the parameter

$$\psi_1 = \xi_1 \circ \psi_H$$

belongs to $\Psi(H_1, \chi_1)$. According to our assumptions on the transfer map $f \rightarrow f^{H_1}$, the distribution

$$f \longrightarrow f^{H_1}(\psi_1) , \qquad\qquad f \in C_c^\infty\big(G(\mathbb{A}),\chi\big),$$

makes sense. It should satisfy the formula

$$f^{H_1}(\psi_1) = \sum_{\pi \in \{\Pi_\psi\}} < \bar{s}_\psi \bar{s}, \pi > f_G(\pi) , \tag{4.4}$$

where $\bar{s}$ is the image of s in $\mathcal{S}_\psi$, s_ψ is the element

$$\psi\left(1, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$

in $S_\psi(G^0)$, and

$$f_G(\pi) = \mathrm{tr}\left(\int_{X \backslash G(\mathbb{A})} f(x)\pi(x)dx\right) .$$

As in [**3**], $\{\Pi_\psi\}$ denotes the set of orbits in Π_ψ under $\pi_0(G^+)^*$, the dual of the finite component group, which acts in the obvious way on $\Pi_{\mathrm{unit}}\big(G(\mathbb{A}), \chi\big)$. Recall that the element s_ψ was introduced in [**3, §4**] to describe the signs which occurred on the right hand side of (4.4).

The objects we have just described, namely the packets Π_ψ, the pairings $< x, \pi >$, the stable distributions (4.3), and the formula (4.4), are all consequences of the local conjectures [**3, Conjectures 6.1 and 6.2**]. The adèlic versions described here are simply restricted tensor products of the local versions in [**3**]. We shall assume their existence in what follows.

We should also recall the sign character

$$\varepsilon_\psi : \mathcal{S}_\psi^+ \longrightarrow \{\pm 1\}$$

which occurs in the conjectural multiplicity formula. Set

$$L'_F = L_F \times SL(2, \mathbb{C}) ,$$

and consider the representation

$$\tau_\psi(s, t') = \mathrm{Ad}\big(s\psi(t')\big) , \qquad s \in \bar{S}_\psi^+,\ t' \in L'_F ,$$

of $\bar{S}_\psi^+ \times L'_F$ on the Lie algebra $\underline{\hat{\mathfrak{g}}}$ of $\hat{G}$. Let

$$\tau_\psi = \bigoplus_k \tau_k = \bigoplus_k (\lambda_k \otimes \mu_k \otimes \nu_k)$$

be the decomposition of τ_ψ in which λ_k, μ_k and ν_k are irreducible (finite dimensional) representations of $\bar{S}_\psi^+$, L_F and $SL(2,\mathbf{C})$ respectively. The global L-function $L(s,\mu_k)$ will be defined as a product of local L-functions. We shall assume it has analytic continuation and satisfies the functional equation

$$L(s,\mu_k) = \varepsilon(s,\mu_k)L(1-s,\widetilde{\mu}_k) ,$$

where $\varepsilon(s,\mu_k)$ is a finite product of local root numbers. It follows from the functional equation that if μ_k is equivalent to its contragredient $\widetilde{\mu}_k$, then $\varepsilon(\frac{1}{2},\mu_k) = \pm 1$. Let us write $\hat{\underline{\mathbf{g}}}_\psi$ for the direct sum of those irreducible constituents τ_k such that (i) $\mu_k \cong \widetilde{\mu}_k$, (ii) $\varepsilon(\frac{1}{2},\mu_k) = -1$, and (iii) $\dim \nu_k$ is even. The sign character is then given by

$$\varepsilon_\psi(x) = \varepsilon_\psi^G(x) = \prod_k \det(\lambda_k(s)) , \qquad x \in \mathcal{S}_\psi^+ , \tag{4.5}$$

where the product is taken over those k such that τ_k is contained in $\hat{\underline{\mathbf{g}}}_\psi$, and s is any element in $\bar{S}_\psi^+$ which projects onto x. In other words,

$$\varepsilon_\psi^G(x) = \det\left(s, \mathrm{End}_{L_F'}(\hat{\underline{\mathbf{g}}}_\psi)\right) . \tag{4.5'}$$

We could actually have replaced the first condition in the definition of $\hat{\underline{\mathbf{g}}}_\psi$ by the stronger assertion (i′) $\tau_k \cong \widetilde{\tau}_k$. Indeed ν_k is always equal to its contragredient, and

$$\det\left(\widetilde{\lambda}_k(s)\right) = \det\lambda_k(s)^{-1} .$$

Therefore, the contribution to (4.5) of the distinct pairs $(\tau_k,\widetilde{\tau}_k)$ equals 1. It should also be noted that the condition (iii) above is not really necessary. For suppose that τ_k satisfies (i′) and (ii), but that $\dim(\nu_k)$ is odd. Then ν_k corresponds to the principal unipotent in an odd orthogonal group. Since μ_k is self-contragredient, its image must be contained in either the orthogonal or the symplectic group. We shall assume the generalization of the theorem of Fröhlich and Queyrot **[6]** which, in view of the sign $\varepsilon(\frac{1}{2},\mu_k) = -1$, implies that μ_k is actually symplectic. Finally, since the representation τ_k is self contragredient and preserves the Killing form, it must be orthogonal. For this to be so, the third representation in the tensor product must actually be symplectic. Therefore det $\lambda_k(s) = 1$, and τ_k contributes nothing to (4.5). This explains the apparent discrepancy between the present definition (4.5) and the earlier one **[3, (8.4)]**.

If ϕ is any vector in the Hilbert space $L^2\big(G^0(F)\backslash G^0(\mathbb{A}), \chi^{-1}\big)$, set

$$\big(R(y)\phi\big)(x) = \phi(\xi^{-1}xy)\ , \qquad x \in G^0(F)\backslash G^0(\mathbb{A}),$$

for any points $y \in G(\mathbb{A})^+$ and $\xi \in G^+(F)$ such that $\xi^{-1}y$ belongs to $G^0(\mathbb{A})$. This gives an extension of the regular representation to $G(\mathbb{A})^+$. For any representation $\pi \in \Pi_{\text{unit}}\big(G(\mathbb{A}), \chi\big)$, let $m_0(\pi)$ be the multiplicity with which π occurs as a discrete summand of R. Now, suppose that π belongs to a packet Π_ψ, $\psi \in \Psi(G, \chi)$. Then we have the nonnegative integer

$$m_\psi(\pi) = |S_\psi^+|^{-1} \sum_{x \in S_\psi^+} \varepsilon_\psi(x) < x, \pi > \ , \tag{4.6}$$

given explicitly in terms of the pairing. The multiplicity formula amounts to the global component of our conjecture, and will be stated formally as a hypothesis.

HYPOTHESIS 4.1. *For any representation* $\pi \in \Pi_{\text{unit}}\big(G(\mathbb{A}), \chi\big)$, *we have the multiplicity formula*

$$m_0(\pi) = \sum_{\psi \in \Psi_0(G,\chi)} m_\psi(\pi)\ . \tag{4.7}$$ □

Before discussing the conjectures, we shall collect a few simple observations for our later use. Let ψ be a fixed map in $\Psi(G)$. (We shall sometimes not distinguish between a map and its equivalence class.) Let C_ψ denote the centralizer in $\hat{G}^0$ of the image of ψ. Then $C_\psi Z(\hat{G}^0)$ is a subgroup of $S_\psi(\hat{G}^0)$. The quotient

$$\bar{C}_\psi = C_\psi Z(\hat{G}^0)/Z(\hat{G}^0)$$

is a subgroup of $\bar{S}_\psi(G^0)$. Now, the image of the cocycle

$$t \to s\psi(t)s^{-1}\psi(t^{-1})\ , \qquad t \in L_F,\ s \in S_\psi(G^0),$$

in $H^1\big(L_F, Z(\hat{G}^0)\big)$ gives a map from $S_\psi(G^0)$ into $\ker^1\big(L_F, Z(\hat{G}^0)\big)$ whose kernel is easily seen to equal $C_\psi Z(\hat{G}^0)$. We therefore obtain a continuous injection

$$S_\psi(G^0)/C_\psi Z(\hat{G}^0) \cong \bar{S}_\psi(G^0)/\bar{C}_\psi \hookrightarrow \ker^1\big(L_F, Z(\hat{G}^0)\big)\,.$$

According to Lemma 11.2.2 of [12], or rather its extension to the hypothetical group L_F, $\ker^1(L_F, Z(\hat{G}^0))$ is isomorphic to $\ker^1(F, Z(\hat{G}^0))$. In particular, $\ker^1(L_F, Z(\hat{G}^0))$ is a finite discrete group. Therefore, the connected component $\bar{S}^0_\psi$ of $\bar{S}_\psi$ maps to the identity element in $\ker^1(L_F, Z(\hat{G}^0))$. We obtain an identity

$$\bar{S}^0_\psi = \bar{C}^0_\psi \tag{4.8}$$

of connected components. In particular, if we set

$$\mathcal{C}_\psi = C_\psi Z(\hat{G}^0)/C^0_\psi Z(\hat{G}^0) = \bar{C}_\psi/\bar{C}^0_\psi ,$$

we can write the injection above as

$$\mathcal{S}_\psi(G^0)/\mathcal{C}_\psi \hookrightarrow \ker^1(L_F, Z(\hat{G}^0)) . \tag{4.9}$$

Suppose that s is a semisimple element in $\bar{S}_\psi$. According to our conventions, $\bar{S}_{\psi,s}$ denotes the centralizer of s in $\bar{S}^0_\psi$, and $\bar{S}^0_{\psi,s}$ is the connected component of 1 in $\bar{S}_{\psi,s}$. We can also take the centralizer $\bar{C}_{\psi,s}$ of s in $\bar{C}^0_\psi$, and its identity component $\bar{C}^0_{\psi,s}$. In §7 we shall use the identities $\bar{S}_{\psi,s} = \bar{C}_{\psi,s}$ and $\bar{S}^0_{\psi,s} = \bar{C}^0_{\psi,s}$. These of course follow immediately from (4.8). We shall also have occasion to consider some slightly different centralizers. Keeping in mind that s is a coset in $\hat{G}/Z(\hat{G}^0)$, we write $S_{\psi,s}$ for the centralizer in S^0_ψ of any element in the coset s. Then

$$S_{\psi,s}Z(\hat{G}^0)/Z(\hat{G}^0)$$

is a subgroup of $\bar{S}_{\psi,s}$, which by Lemma 2.2 is of finite index. In particular, we have an equality

$$\bar{S}^0_{\psi,s} = S^0_{\psi,s}Z(\hat{G}^0)/Z(\hat{G}^0) \tag{4.10}$$

of identity components. Similarly, if $C_{\psi,s}$ denotes the centralizer in C^0_ψ of any element in the coset s, we have

$$\bar{S}^0_{\psi,s} = \bar{C}^0_{\psi,s} = C^0_{\psi,s}Z(\hat{G}^0)/Z(\hat{G}^0) . \tag{4.11}$$

§5. The expansion of $I_{\mathrm{disc},t}(f)$

We have now stated two global hypotheses. As we have already noted, Hypothesis 3.1 should be a consequence of a stable trace formula. Once this is established, one could try to combine the formulas

(3.1) and (3.4) to deduce something approaching the multiplicity formula in Hypothesis 4.1. Our aims in this paper are more modest. We shall simply show that the two hypotheses are compatible. We are actually going to establish that Hypothesis 4.1, together with the local assumptions of §3, §4 and [**3**, §**7**], implies Hypothesis 3.1. More precisely, we shall show that the formula for $I_{\mathrm{disc},t}(f)$ obtained by combining Hypothesis 4.1 with (3.1) equals the formula for $E_{\mathrm{disc},t}(f)$ provided by the definition (3.4). In the process we shall gain some insight into the role of the sign characters ε_{ψ}^{G}.

In this section we shall derive a formula for $I_{\mathrm{disc},t}$ from Hypothesis 4.1. By combining (4.7) with (3.1) we will obtain an expansion for

$$I_{\mathrm{disc},t}(f)\ , \qquad\qquad f \in C_c^{\infty}\big(G(\mathbb{A}),\chi\big)\ ,$$

as a linear combination of irreducible characters. In doing this we will need to apply a local conjecture from [**3**, §**7**] for the values of normalized intertwining operators.

According to (3.1), $I_{\mathrm{disc},t}(f)$ equals the sum over $\{M\}$ and over $w \in W^G(\underline{\mathfrak{a}}_M)_{\mathrm{reg}}$, of the product of

$$|\pi_0(G^+)|^{-1}|W^G(\underline{\mathfrak{a}}_M)|^{-1}|\det(w-1)_{\underline{\mathfrak{a}}_M^G}|^{-1}$$

with

$$\mathrm{tr}\big(M(w,0)\rho_{P,t}(0,f)\big)\ . \tag{5.1}$$

Our first task is to expand (5.1) into a linear combination of irreducible characters.

For any M, and $w \in W^G(\underline{\mathfrak{a}}_M)$, we can form the component $M_w = M\cdot w$. It satisfies the same conditions as G. Now, recall that $\rho_{P,t}(0)$ is the representation of $G(\mathbb{A})^+$ obtained by parabolic induction from the action of $M(\mathbb{A})$ on

$$L^2_{\mathrm{disc},t}\big(M(F)\backslash M(\mathbb{A}),\chi_M^{-1}\big)\ . \tag{5.2}$$

This representation of $M(\mathbb{A})$ has a canonical extension to the group $M_w(\mathbb{A})^+$ generated by the coset $M_w(\mathbb{A}) = M(\mathbb{A})w$. In particular, the space (5.2) can be decomposed into a direct sum of subspaces corresponding to irreducible representations σ_w of $M_w(\mathbb{A})^+$. There is a similar decomposition

$$\mathcal{H}_{P,t} = \bigoplus_{\sigma_w}\mathcal{H}_P(\sigma_w)$$

of the induced space into subspaces which are invariant under the operator

$$M(w,0)\rho_{P,t}(0,f)\ . \tag{5.3}$$

If the restriction of σ_w to $M(\mathbb{A})$ is reducible, one sees easily that the trace of the operator (5.3) on $\mathcal{H}_P(\sigma_w)$ vanishes. Therefore, in computing the full trace (5.1), we need only consider representations σ_w which belong to the space we have denoted by $\Pi_{\mathrm{unit}}\big(M_w(\mathbb{A}),\chi_M^{-1}\big)$.

According to Hypothesis 4.1 (applied to M_w rather than G), the multiplicity with which a representation $\sigma_w \in \Pi_{\mathrm{unit}}\big(M_w(\mathbb{A}),\chi_M^{-1}\big)$ occurs in (5.2) equals

$$\sum_{\psi_w\in\Psi_0(M_w,\chi_M,t)} m_{\psi_w}(\sigma_w)\ ,$$

where $m_{\psi_w}(\sigma_w)$ is the nonnegative integer defined by (4.6). We have written $\Psi_0(M_w,\chi_M,t)$ to denote the set of parameters in $\Psi_0(M_w,\chi_M)$ whose Archimedean infinitesimal character has absolute value t. Any pair ψ_w and σ_w, with $m_{\psi_w}(\sigma_w)\neq 0$, determines a subspace of (5.2), and also a subspace of the induced space $\mathcal{H}_{P,t}$. The restriction of (5.3) to this latter subspace can be expressed in terms of the operators studied in [**3,** §7]. It equals an expression

$$m_{\psi_w}(\sigma_w)r(\psi_w)\big(R_P(\sigma_w,\psi_w)\mathcal{I}_P(\sigma,f)\big)\ , \tag{5.4}$$

whose constituents we shall describe in a moment. The trace (5.1) becomes the sum over $\psi_w \in \Psi_0(M_w,\chi_M,t)$ and $\sigma_w \in \Pi_{\psi_w}$ of the trace of the expression (5.4).

Given M and P, it is convenient to fix a dual parabolic subgroup ${}^LP = \hat{P}\rtimes W_F$ in ${}^LG^0$ with Levi component ${}^LM = \hat{M}\rtimes W_F$. The choice of P and ${}^LP$ determines an embedding of the L-group ${}^LM$ into ${}^LG^0$. It also allows us to identify $W^G(\underline{\mathfrak{a}}_M)$ with the dual Weyl set

$$\hat{W}^G(\underline{\mathfrak{a}}_M)\ =\ \mathrm{Norm}(A_{\hat{M}},\hat{G})/\hat{M}\ .$$

Returning to (5.4), we note that $\mathcal{I}_P(\sigma)$ stands for the induced representation of $G(\mathbb{A})^+$ obtained from the restriction σ of σ_w to $M(\mathbb{A})$. The operator

$$R_P(\sigma_w,\psi_w)\ =\ \bigotimes_v R_P(\sigma_{w,v},\psi_{w,v})$$

is a tensor product of local normalized intertwining operators defined in [**3, (7.4)**]. When this operator is evaluated at a smooth vector in $\mathcal{H}_{P,t}$, almost all the terms in the product reduce to 1. Finally, the scalar $r(\psi_w)$ in (5.4) is obtained from an infinite product of local normalizing functions of the form [**3, (7.2)**]. It equals

$$\lim_{\lambda\to 0}\left(L(0,\widetilde{\rho}_{P,w}\circ\phi_{\psi_w,\lambda})\varepsilon(0,\widetilde{\rho}_{P,w}\circ\phi_{\psi_w,\lambda})^{-1}L(1,\widetilde{\rho}_{P,w}\circ\phi_{\psi_w,\lambda})^{-1}\right),$$

where $\phi_{\psi_w,\lambda}$ is the twist of the global parameter

$$\phi_{\psi_w}: t \longrightarrow \psi_w\left(t,\begin{pmatrix}|t|^{\frac{1}{2}} & 0\\ 0 & |t|^{-\frac{1}{2}}\end{pmatrix}\right), \qquad t\in L_F,$$

by the vector λ in

$$\underline{\mathfrak{a}}^*_{M,\mathbb{C}} = X(M)_F\otimes\mathbb{C} \cong X_*(A_{\hat{M}})\otimes\mathbb{C},$$

and $\widetilde{\rho}_{P,w}$ is the contragredient of the adjoint representation of ${}^LM$ on

$$w^{-1}\hat{\underline{\mathfrak{n}}}_P w/w^{-1}\hat{\underline{\mathfrak{n}}}_P w\cap\hat{\underline{\mathfrak{n}}}_P .$$

Here $\hat{\underline{\mathfrak{n}}}_P$ stands for the Lie algebra of the unipotent radical of ${}^LP$. Applying the anticipated functional equation

$$L(0,\widetilde{\rho}_{P,w}\circ\phi_{\psi_w,\lambda}) = \varepsilon(0,\widetilde{\rho}_{P,w}\circ\phi_{\psi_w,\lambda})L(1,\rho_{P,w}\circ\phi_{\psi_w,\lambda}),$$

we write

$$(5.5)\qquad r(\psi_w) = \lim_{\lambda\to 0}\left(L(1,\rho_{P,w}\circ\phi_{\psi_w,\lambda})L(1,\widetilde{\rho}_{P,w}\circ\phi_{\psi_w,\lambda})^{-1}\right).$$

(See [**16, Appendix 2**].)

Having described the terms in (5.4), we go back to the expression we have obtained for (5.1). Recall [**3, §7**] that

$$R_P(\zeta\sigma_w,\psi_w) = \zeta(M_w)R_P(\sigma_w,\psi_w),$$

for any character ζ in

$$\pi_0(M_w^+)^* = \mathrm{Hom}(M_w^+/M_w^0,\mathbb{C}^*).$$

This allows us to write (5.1) as the sum over $\psi_w \in \Psi_0(M_w, \chi_M, t)$ and over the orbits $\{\sigma_w\} \in \{\Pi_{\psi_w}\}$ of $\pi_0(M_w^+)^*$ in Π_{ψ_w}, of the expression

$$m'_{\psi_w}(\sigma_w) r(\psi_w) \mathrm{tr}\big(R_P(\sigma_w, \psi_w)\mathcal{I}_P(\sigma, f)\big) .$$

where

$$m'_{\psi_w}(\sigma_w) \;=\; \sum_{\zeta \in \pi_0(M_w^+)^*} m_{\psi_w}(\zeta\sigma_w)\zeta(M_w) .$$

Applying Fourier inversion on $\pi_0(M_w^+)$ to the formula (4.6) (with G replaced by M_w), while taking into account the property (i) of the local Conjecture 6.1 in [**3**], we obtain

$$m'_{\psi_w}(\sigma_w) \;=\; |\mathcal{S}_{\psi_w}|^{-1} \sum_{u \in \mathcal{S}_{\psi_w}} \varepsilon_{\psi_w}(u) < u, \sigma_w > .$$

Therefore (5.1) equals

$$\sum_{\psi_w} \sum_{\{\sigma_w\}} |\mathcal{S}_{\psi_w}|^{-1} \sum_{u \in \mathcal{S}_{\psi_w}} \varepsilon_{\psi_w}(u) r(\psi_w) < u, \sigma_w > \mathrm{tr}\big(R_P(\sigma_w, \psi_w)\mathcal{I}_P(\sigma, f)\big) .$$

Suppose that ψ_w belongs to $\Psi_0(M_w, \chi_M, t)$. Let ψ denote the composition of ψ_w with our embedding ${}^LM \subset {}^LG^0$. We claim that ψ is well defined (as an equivalence class of parameters) in $\Psi(G, \chi, t)$. Recalling (§4) the definition of equivalent parameters, we note that it is enough to show that the map

$$\ker^1(F, Z(\hat{G}^0)) \longrightarrow \ker^1(F, Z(\hat{M})) \tag{5.6}$$

is an isomorphism. By the obvious transitivity property, we can in fact assume that M is minimal, and hence a torus. Then $Z(\hat{M})/Z(\hat{G}^0)$ is a maximal torus in an adjoint group, on which the Galois action is dual to a direct sum of permutation representations. The bijectivity of (5.6) then follows from the exact sequence

$$\begin{aligned}\pi_0\big((Z(\hat{M})/Z(\hat{G}^0))^\Gamma\big) \;&\to\; H^1(F, Z(\hat{G}^0)) \\ &\to\; H^1(F, Z(\hat{M})) \;\to\; H^1(F, Z(\hat{M})/Z(\hat{G}^0)) ,\end{aligned}$$

and its analogues for the completions of F. (See the proof of Lemma 4.3.2(a) of [**12**].) This proves the claim.

Thus, ψ_w maps to an element ψ in $\Psi(G,\chi,t)$, to which we can associate the objects $\mathcal{S}_\psi = \mathcal{S}_\psi(G)$, $\Pi_\psi = \Pi_\psi(G)$ and $\varepsilon_\psi = \varepsilon_\psi^G$ for G. The next step is to apply a conjectural formula [**3**, §**7**] for the trace of the normalized intertwining operators in terms of the pairing on $\mathcal{S}_\psi \times \Pi_\psi$. As it is stated in [**3**], the formula applies to the local intertwining operators and pairings, but the product over all valuations gives a formula for the global objects. In fact, certain constants in the local formula (namely, $c(\sigma_\chi, n_w)$, $\lambda_w(\psi_F)$ and $c(\pi_\chi, n_G)$, in the notation of [**3**, §**7**]) have the property that their products over all valuations equal 1. The global formula is therefore simpler. If ψ_M denotes the parameter ψ_w, but regarded as an element in $\Psi(M)$ rather than $\Psi(M_w)$, then the orbits $\{\sigma_w\}$ above will be in bijective correspondence with the representations $\sigma \in \Pi_{\psi_M}$ which extend to $M_w(\mathbb{A})^+$. It follows from Conjecture 7.1 of [**3**] (and also the two remarks made after the conjecture), that

$$\sum_\sigma < u, \sigma_w > \operatorname{tr}\big(R_P(\sigma_w, \psi_w)\mathcal{I}_P(\sigma, f)\big)$$

equals

$$\sum_{\pi\in\Pi_\psi} < x_u, \pi > f_G(\pi) \ ,$$

where x_u stands for the image in $\mathcal{S}_\psi$ of the point $u \in \mathcal{S}_{\psi_w}$.

We have now obtained an expansion

$$\sum_{\psi_w\in\Psi_0(M_w,\chi_M,t)} |\mathcal{S}_{\psi_w}|^{-1} \sum_{u\in\mathcal{S}_{\psi_w}} \varepsilon_{\psi_w}(u) r(\psi_w) \sum_{\pi\in\Pi_\psi} < x_u, \pi > f_G(\pi) \ ,$$

for the trace (5.1). We shall substitute this into our formula for $I_{\mathrm{disc},t}(f)$. Observe that

$$\sum_{\pi\in\Pi_\psi} < x_u, \pi > f_G(\pi) \ = \ |\pi_0(G^+)| \sum_{\pi\in\{\Pi_\psi\}} < x_u, \pi > f_G(\pi) \ .$$

Therefore, $I_{\mathrm{disc},t}(f)$ equals the triple sum over $\{M\}$, $w \in W^G(\underline{\mathfrak{a}}_M)_{\mathrm{reg}}$ and $\psi_w \in \Psi_0(M_w, \chi_M, t)$ of the product of

$$|W^G(\underline{\mathfrak{a}}_M)|^{-1} |\det(w-1)_{\underline{\mathfrak{a}}_M^G}|^{-1}$$

with

$$(5.7) \qquad |\mathcal{S}_{\psi_w}|^{-1} \sum_{u\in\mathcal{S}_{\psi_w}} \varepsilon_{\psi_w}(u) r(\psi_w) \sum_{\pi\in\{\Pi_\psi\}} < x_u, \pi > f_G(\pi) \ .$$

We propose to interchange the sum over the parameters with the sums over M and w. The outer sum will then have to be over all parameters $\psi \in \Psi(G,\chi,t)$. For any ψ there will be an M, unique up to conjugacy, such that ψ is the composition of a parameter $\psi_M \in \Psi_0(M)$ with the embedding ${}^LM \subset {}^LG^0$. The condition that ψ_M also belong to $\Psi_0(M_w)$, for a given $w \in W^G(\underline{\mathbf{a}}_M)_{\mathrm{reg}}$, is that the set $S_{\psi_M}(M_w)$ be nonempty. There is another way to state this. Recall that we have identified $W^G(\underline{\mathbf{a}}_M)$ with the dual Weyl set $\hat{W}^G(\underline{\mathbf{a}}_M)$. Then $S_{\psi_M}(M_w)$ is nonempty if and only if w belongs to the subset $W_\psi = W_\psi(G)$ of elements in $\hat{W}^G(\underline{\mathbf{a}}_M)$ which, modulo the isomorphic groups (5.6), centralize the image of ψ. It will be convenient for us to regard this subset W_ψ as the full Weyl set associated to $\bar{S}_\psi = S_\psi/Z(\hat{G}^0)$. It acts on the maximal torus

$$\bar{T}_\psi = A_{\hat{M}}Z(\hat{G}^0)/Z(\hat{G}^0)$$

of the connected component

$$\bar{S}^0_\psi = S^0_\psi Z(\hat{G}^0)/Z(\hat{G}^0) .$$

For any $w \in W_\psi$, we shall write $\det(w-1)$ for the determinant of $(w-1)$, acting on the Lie algebra of $\bar{T}_\psi$. One sees easily that

$$|\det(w-1)| = |\det(w-1)_{\underline{\mathbf{a}}_M^{G^0}}| = |\det(w-1)_{\underline{\mathbf{a}}_M^G}||\det(w-1)_{\underline{\mathbf{a}}_G^{G^0}}|^{-1} .$$

Now it is well known that $|\det(w-1)_{\underline{\mathbf{a}}_G^{G^0}}|$ equals the order of the kernel of w, acting on the dual torus

$$\left(Z(\hat{G}^0)^\Gamma\right)^0/\left(Z(\hat{G})^\Gamma\right)^0 .$$

(See **[23, II.1.7]**.) The action of w on this torus is of course independent of w, and the kernel is just the finite group of components in

$$\kappa_G = Z(\hat{G})^\Gamma \cap \left(Z(\hat{G}^0)^\Gamma\right)^0 .$$

Therefore

$$|\det(w-1)_{\underline{\mathbf{a}}_M^G}|^{-1} = |\det(w-1)|^{-1}|\pi_0(\kappa_G)|^{-1} .$$

In particular, w belongs to $W^G(\underline{\mathbf{a}}_M)_{\mathrm{reg}}$ if and only if it lies in the set

$$W_{\psi,\mathrm{reg}} = \{w \in W_\psi : \det(w-1) \neq 0\}$$

of regular elements in W_ψ. When this is so, the associated parameter in $\Psi_0(M_w)$ in fact belongs to $\Psi_0(M_w, \chi_M, t)$. We shall denote it by ψ_w, as above.

Actually, ψ_w is not uniquely determined by ψ and w. We must decide how many parameters in $\Psi_0(M_w, \chi_M, t)$ lie in the equivalence class of ψ. Keeping in mind the isomorphism (5.6), we see that two parameters ψ_w map to the same ψ if and only if they are conjugate by an element in $\hat{W}^G(\underline{\mathbf{a}}_M)$. Moreover, two such conjugates are equivalent in $\Psi_0(M_w, \chi_M, t)$ if and only if they differ by an element in $W_\psi(G^0)$. The number of ψ_w associated to ψ is therefore

$$|\hat{W}^G(\underline{\mathbf{a}}_M)||W_\psi(G^0)|^{-1} = |W^G(\underline{\mathbf{a}}_M)||W_\psi|^{-1} .$$

Thus, our interchange of summation expresses $I_{\mathrm{disc},t}(f)$ as the sum over $\psi \in \Psi(G, \chi, t)$ and $w \in W_{\psi,\mathrm{reg}}$ of the product of

$$|\pi_0(\kappa_G)|^{-1}|W_\psi|^{-1}|\det(w-1)|^{-1}$$

with (5.7).

Suppose that $\psi \in \Psi(G)$. As in the case of a local parameter, we can define the finite set

$$\begin{aligned} \mathcal{N}_\psi = \mathcal{N}_\psi(G) &= \mathrm{Norm}(\bar{T}_\psi, \bar{S}_\psi)/\bar{T}_\psi \\ &= \mathrm{Norm}(A_{\hat{M}}, S_\psi)/A_{\hat{M}}Z(\hat{G}^0) . \end{aligned}$$

Let $\mathcal{S}^1_\psi$ be the subgroup of elements in $\mathcal{N}_\psi(G^0)$ which act trivially on $\bar{T}_\psi$. This group acts freely by translation on $\mathcal{N}_\psi$, and the set of orbits can be identified canonically with W_ψ. One sees easily from the isomorphism (5.6) that

$$\mathcal{S}_{\psi_w} = \mathcal{S}_{\psi_M} \cdot w = \mathcal{S}^1_\psi \cdot w , \qquad w \in W_\psi ,$$

for M as above. We also have the Weyl group

$$\begin{aligned} W^0_\psi &= \mathrm{Norm}(\bar{T}_\psi, \bar{S}^0_\psi)/\bar{T}_\psi \\ &= \mathrm{Norm}(A_{\hat{M}}, S^0_\psi)/A_{\hat{M}}Z(\hat{G}^0) \end{aligned}$$

of the connected component $\bar{S}^0_\psi$. This too acts freely on $\mathcal{N}_\psi$, and the set of orbits can be identified canonically with $\mathcal{S}_\psi$. We obtain a commutative diagram

$$\begin{tikzcd}
& & 1 \arrow[d] & 1 \arrow[d] & \\
& & W^0_\psi \arrow[r, equal] \arrow[d] & W^0_\psi \arrow[d] & \\
1 \arrow[r] & \mathcal{S}^1_\psi \arrow[r] \arrow[d, equal] & \mathcal{N}_\psi \arrow[r] \arrow[d, shift right] & W_\psi \arrow[r] \arrow[d, shift right] & 1 \\
1 \arrow[r] & \mathcal{S}^1_\psi \arrow[r] & \mathcal{S}_\psi \arrow[r] \arrow[u, dashed, shift right] \arrow[d] & R_\psi \arrow[r] \arrow[u, dashed, shift right] \arrow[d] & 1 \\
& & 1 & 1 &
\end{tikzcd}$$

as in the local case **[3, (7.1)]**. The dotted arrows stand for splittings of short exact sequences determined by a fixed Borel subgroup of $\bar{S}^0_\psi$ containing $\bar{T}_\psi$. Similarly, one obtains a commutative diagram of groups if one replaces $\mathcal{N}_\psi$, $\mathcal{S}_\psi$, W_ψ and R_ψ by the respective finite groups $\mathcal{N}^+_\psi$, $\mathcal{S}^+_\psi$, W^+_ψ and R^+_ψ they generate. We shall write $u \to x_u$ and $u \to w_u$ for the projections of $\mathcal{N}^+_\psi$ onto $\mathcal{S}^+_\psi$ and W^+_ψ. Notice that if x is any element in $\mathcal{S}_\psi$, and $\mathcal{N}(x)$ is the corresponding orbit of W^0_ψ in $\mathcal{N}_\psi$, the second projection maps $\mathcal{N}(x)$ *bijectively* onto a subset $W(x)$ of W_ψ. We shall set

$$W(x)_{\mathrm{reg}} = W(x) \cap W_{\psi,\mathrm{reg}}$$

and

$$\mathcal{N}(x)_{\mathrm{reg}} = \{u \in \mathcal{N}(x) : w_u \in W(x)_{\mathrm{reg}}\} .$$

We apply these observations to our formula for $I_{\mathrm{disc},t}(f)$. According to the horizontal exact sequence for $\mathcal{N}_\psi$ in the diagram, the double sum over $w \in W_{\psi,\mathrm{reg}}$ and $u \in \mathcal{S}_{\psi_w} = \mathcal{S}^1_\psi w$ can be combined into a simple sum over the regular elements in $\mathcal{N}_\psi$. We shall write

$$\varepsilon^M_\psi(u) = \varepsilon_{\psi_w}(u) \tag{5.8}$$

for any point $u \in \mathcal{N}_\psi$ whose projection onto W_ψ equals w. We also set

$$r_\psi(w) = r(\psi_w) . \tag{5.9}$$

Then ε_ψ^M and r_ψ extend to well defined characters on $\mathcal{N}_\psi^+$ and W_ψ^+ respectively. The simple sum in its turn can be decomposed by the corresponding vertical exact sequence into a double sum over $x \in \mathcal{S}_\psi$ and $u \in \mathcal{N}(x)_{\mathrm{reg}}$. Observe that

$$\begin{aligned} |W_\psi||\mathcal{S}_{\psi_w}| &= |W_\psi||\mathcal{S}_\psi^1| = |\mathcal{N}_\psi| \\ &= |\mathcal{S}_\psi||W_\psi^0| \\ &= |\mathcal{S}_\psi||W(x)| \, . \end{aligned}$$

It follows that $I_{\mathrm{disc},t}(f)$ equals the sum over ψ in $\Psi(G,\chi,t)$ of the product of

$$|\pi_0(\kappa_G)|^{-1}|\mathcal{S}_\psi|^{-1}$$

with

$$\sum_{x\in\mathcal{S}_\psi} |W(x)|^{-1} \sum_{u\in\mathcal{N}(x)_{\mathrm{reg}}} \varepsilon_\psi^M(u) r_\psi(w_u)|\det(w_u-1)|^{-1} \sum_{\pi\in\{\Pi_\psi\}} < x,\pi > f_G(\pi) \, .$$

Any element $w \in W_\psi^+$ operates on $\bar{T}_\psi$. It preserves the set Σ_ψ of roots of $(\bar{S}_\psi^0, \bar{T}_\psi)$. We shall simply write $\varepsilon(w)$ for the usual sign attached to this permuation, namely the number (-1) raised to the power

$$|(-\Sigma_\psi^+) \cap (w\Sigma_\psi^+)| \, ,$$

where Σ_ψ^+ is the set of positive roots in Σ_ψ relative to some order.

PROPOSITION 5.1. *We have*

$$r_\psi(w_u) = \varepsilon(w_u)\varepsilon_\psi^G(x_u)\varepsilon_\psi^M(u)^{-1}$$

for any element $u \in \mathcal{N}_\psi^+$.

This proposition is the motivation for the introduction of the characters ε_ψ into the multiplicity formula of Hypothesis 4.1. We shall prove it in the next section. In the meantime, we can combine it with our formula for $I_{\mathrm{disc},t}(f)$.

PROPOSITION 5.2. *The distribution $I_{\mathrm{disc},t}(f)$ equals the product of $|\pi_0(\kappa_G)|^{-1}$ with*

$$\sum_{\psi\in\Psi(G,\chi,t)} \sum_{\pi\in\{\Pi_\psi\}} |\mathcal{S}_\psi|^{-1} \sum_{x\in\mathcal{S}_\psi} \varepsilon_\psi^G(x) i(x) < x, \pi > f_G(\pi) \ , \tag{5.10}$$

where

$$i(x) \ = \ |W(x)|^{-1} \sum_{w\in W(x)_{\mathrm{reg}}} \varepsilon(w)|\det(w-1)|^{-1} \ . \tag{5.11}$$

□

§6. THE SIGN CHARACTERS ε_ψ AND $\mathbf{r}_\psi$

In this section we shall pause to study the characters ε_ψ and r_ψ. Our goal is to prove Proposition 5.1. Recall that $\varepsilon_\psi = \varepsilon_\psi^G$ is the one-dimensional character (4.5) on $\mathcal{S}_\psi^+ = \mathcal{S}_\psi(G^+)$ which comes into the conjectural multiplicity formula. The function r_ψ is the one dimensional character ((5.5), (5.9)) on $W_\psi^+ = W_\psi(G^+)$ defined by the global normalizing factors. We have seen that $\mathcal{S}_\psi^+$ and W_ψ^+ are both quotients of the finite group $\mathcal{N}_\psi^+$. We can therefore identify ε_ψ and r_ψ with characters on $\mathcal{N}_\psi^+$. Proposition 5.1 can be regarded as a formula for the quotient of these two characters.

We shall begin by expressing r_ψ in terms of the orders of certain L-functions at $s = 1$. Let $\hat{\Sigma}_M$ denote the set of roots of $(\hat{G}^0, A_{\hat{M}})$. For each $\hat{\alpha} \in \hat{\Sigma}_M$ there is a representation $\rho_{\hat{\alpha}}$ of ${}^LM$ on the root space $\hat{\mathfrak{g}}_{\hat{\alpha}}$. Having already fixed the dual parabolic subgroups P and ${}^LP = \hat{P} \rtimes W_F$, we shall write $\hat{\Sigma}_P \subset \hat{\Sigma}_M$ for the set of roots of $(\hat{P}, A_{\hat{M}})$. Fix an element $w \in W_\psi^+$, and set

$$\hat{\Sigma}_{P,w} \ = \ \{\hat{\alpha} \in \hat{\Sigma}_P : \ w\hat{\alpha} \in (-\Sigma_P)\} \ .$$

Then there is a decomposition

$$\rho_{P,w} \ = \ \bigoplus_{\hat{\alpha}\in\hat{\Sigma}_{P,w}} \rho_{-\hat{\alpha}}$$

for the representation of ${}^LM$ which occurs in (5.5). Notice that the Killing form provides an isomorphism between $\rho_{-\hat{\alpha}}$ and the contragredient $\widetilde{\rho}_{\hat{\alpha}}$. The formula (5.5) becomes

$$r_\psi(w) \ = \ \lim_{\lambda\to 0} \prod_{\alpha\in\hat{\Sigma}_{P,w}} L(1-\lambda(\hat{\alpha}), \rho_{\hat{\alpha}}\circ\phi_\psi) L(1+\lambda(\hat{\alpha}), \rho_{\hat{\alpha}}\circ\phi_\psi)^{-1} \ , \tag{6.1}$$

since

$$L(1, \rho_{\hat{\alpha}} \circ \phi_{\psi,\lambda}) = L\big(1 + \lambda(\hat{\alpha}), \rho_{\hat{\alpha}} \circ \phi_{\psi}\big) .$$

We are going to show that $r_\psi(w)$ equals the character

$$\prod_{\hat{\alpha} \in \hat{\Sigma}_{P,w}} (-1)^{\mathrm{ord}_{s=1}\left(L(s,\rho_{\hat{\alpha}} \circ \phi_\psi)\right)} . \tag{6.2}$$

We claim that for every root $\hat{\alpha} \in \Sigma_{P,w}$, there is also a root $\hat{\alpha}_1 \in \hat{\Sigma}_{P,w}$ such that

$$\widetilde{\rho}_{\hat{\alpha}} \circ \psi \cong \rho_{\hat{\alpha}_1} \circ \psi .$$

To this end, observe that

$$\rho_{\hat{\alpha}} \circ \psi \cong \rho_{w\hat{\alpha}} \circ \mathrm{ad}(w) \circ \psi \cong \rho_{w\hat{\alpha}} \circ \psi .$$

The first of these isomorphisms is given by the intertwining map

$$\mathrm{Ad}(w) : \underline{\mathfrak{g}}_{\hat{\alpha}} \longrightarrow \underline{\mathfrak{g}}_{w\hat{\alpha}} ,$$

and the second follows from the fact that the image of $w \in W_\psi^+$ under the adjoint representation commutes with the image of $L_F \times SL(2, \mathbb{C})$. Now, consider the orbit

$$\mathcal{O}_w(\hat{\alpha}) = \{w^j \hat{\alpha} : j \in \mathbb{Z}\}$$

of $\hat{\alpha}$ under the cyclic group generated by w. The representations

$$\{\rho_{\hat{\beta}} \circ \psi : \hat{\beta} \in \mathcal{O}_w(\hat{\alpha})\}$$

are all equivalent, and are also equivalent to the contragredients

$$\{\widetilde{\rho}_{\hat{\beta}} \circ \psi : -\beta \in \mathcal{O}_w(\hat{\alpha})\} .$$

But after a moment's thought, we see that the intersections of $\mathcal{O}_w(\hat{\alpha})$ with $\hat{\Sigma}_{P,w}$ and $(-\hat{\Sigma}_{P,w})$ contain an equal number of roots. The claim follows. In particular, the terms in the product in (6.1) can be grouped in such a way that $\rho_{\hat{\alpha}}$ appears in the numerator as well as the denominator. This leads directly to the formula (6.2) for $r_\psi(w)$.

Recall that

$$(\rho_{\hat{\alpha}} \circ \phi_\psi)(t) = \rho_{\hat{\alpha}}\left(\psi\left(t, \begin{pmatrix} |t|^{\frac{1}{2}} & 0 \\ 0 & |t|^{-\frac{1}{2}} \end{pmatrix}\right)\right) , \qquad t \in L_F .$$

Now there is a decomposition

$$(\rho_{\hat{\alpha}} \circ \psi) = \bigoplus_{j \in J(\hat{\alpha})} (\mu_j \otimes \nu_j) ,$$

where each μ_j is an irreducible *unitary* representation of L_F and ν_j is an irreducible representation of $SL(2, \mathbb{C})$. Therefore, (6.2) can be written as a product

$$\prod_{\hat{\alpha} \in \hat{\Sigma}_{P,w}} \prod_{j \in J(\hat{\alpha})} (-1)^{\mathrm{ord}_{s=1}\left(L(s,\mu_j \otimes \nu_j)\right)} \tag{6.3}$$

where $L(s, \mu_j \times \nu_j)$ stands for the L-function of the representation

$$t \longrightarrow \mu_j(t)\nu_j \begin{pmatrix} |t|^{\frac{1}{2}} & 0 \\ 0 & |t|^{-\frac{1}{2}} \end{pmatrix} , \qquad t \in L_F ,$$

of L_F. From the discussion above we see that the contragredient acts as an involution $\mu_j \otimes \nu_j \to \widetilde{\mu}_j \otimes \widetilde{\nu}_j$ on the constituents of $\rho_{P,w}$. It is, moreover, an easy consequence of the unitarity of μ_j that

$$\overline{L(s, \mu_j \otimes \nu_j)} = L(\bar{s}, \widetilde{\mu}_j \otimes \widetilde{\nu}_j) ,$$

so that

$$\mathrm{ord}_{s=1}\big(L(s, \mu_j \otimes \nu_j)\big) = \mathrm{ord}_{s=1}\big(L(s, \widetilde{\mu}_j \otimes \widetilde{\nu}_j)\big) .$$

In particular, the contribution to (6.3) of a distinct pair of contragredient constituents cancels. The product (6.3) need only be taken over those constituents with

$$\mu_j \otimes \nu_j \cong \widetilde{\mu}_j \otimes \widetilde{\nu}_j .$$

Since any finite dimensional representation of $SL(2, \mathbb{C})$ is self contragredient, the condition is just $\mu_j \cong \widetilde{\mu}_j$.

The question then is to determine the sign

$$(-1)^{\mathrm{ord}_{s=1}\ L(s,\mu \otimes \nu)} , \tag{6.4}$$

for any irreducible representation $\mu \otimes \nu$ of $L_F \times SL(2,\mathbf{C})$ such that μ is unitary, and $\widetilde{\mu} \cong \mu$. Set $m = \deg(\mu)$ and $n = \deg(\nu)$. Then ν maps the matrix

$$\begin{pmatrix} |t|^{\frac{1}{2}} & 0 \\ 0 & |t|^{-\frac{1}{2}} \end{pmatrix}, \qquad t \in L_F,$$

to the diagonal matrix

$$\mathrm{diag}(|t|^{\frac{1}{2}(n-1)}, |t|^{\frac{1}{2}(n-3)}, \dots, |t|^{-\frac{1}{2}(n-1)})$$

in $GL(n,\mathbf{C})$. Therefore

$$L(s, \mu \otimes \nu) = \prod_{i=1}^{n} L\big(s + \frac{1}{2}(n - 2i + 1), \mu\big) .$$

We must therefore describe the order of zero or pole of $L(s,\mu)$ at any real half-integer.

Hypothesis 4.1 includes the global Langlands correspondence for $GL(m)$, which asserts that

$$L(s,\mu) = L(s,\pi)$$

for some unitary, cuspidal automorphic representation π of $GL(m,\mathbb{A})$. (See [**3**, §**2**].) Then $L(s,\mu)$ can have a real pole only if μ is the trivial one dimensional representation, in which case there is a simple pole at $s = 0$ and $s = 1$ [**7, Corollary 13.8**]. Results of Jacquet and Shalika [**8, Theorem (1.3)**], [**9, Theorem 5.3**] imply further that the only possible zero of $L(s,\mu)$ at a real half integer is at $s = \frac{1}{2}$, the center of the critical strip. The poles of $L(s,\mu)$ will contribute to (6.4) if n is odd. However, if μ is trivial and n is of odd dimension greater than 1, the poles at 0 and 1 will both contribute, and their effect will cancel. The zeros of $L(s,\mu)$ will contribute to (6.4) if n is even. From the functional equation

$$L(s,\mu) = \varepsilon(s,\mu)L(1-s,\mu) ,$$

we see that $L(s,\mu)$ has a zero at $s = \frac{1}{2}$ of even or odd order, according to whether $\varepsilon(\frac{1}{2},\mu)$ equals $+1$ or -1.

We have established

LEMMA 6.1. *If $n = \deg(\nu)$ is even, the sign (6.4) equals $\varepsilon(\frac{1}{2}, \mu)$. If n is odd, (6.4) equals 1 unless $\mu \otimes \nu$ is the trivial representation of $L_F \times SL(2, \mathbb{C})$, in which case (6.4) equals (-1).* □

If we substitute the formula of Lemma 6.1 into the product (6.3), we obtain a new expression for $r_\psi(w)$. To describe this in a convenient way, we shall define a character $\varepsilon_\psi^{G/M}$ which is closely related to the original characters ε_ψ^G and ε_ψ^M. Let $\underline{\hat{\mathfrak{m}}}$ denote the Lie algebra of $\hat{M}$, and let $\mathrm{Ad}_{\underline{\hat{\mathfrak{g}}}/\underline{\hat{\mathfrak{m}}}}$ denote the adjoint representation of ${}^L M$ on $\underline{\hat{\mathfrak{g}}}/\underline{\hat{\mathfrak{m}}}$. The group

$$\bar{N}_\psi^+ = \mathrm{Norm}(\bar{T}_\psi, \bar{S}_\psi^+) = \mathrm{Norm}(A_{\hat{M}}, S_\psi^+)/Z(\hat{G}^0)$$

also acts by the adjoint action on $\underline{\hat{\mathfrak{g}}}/\underline{\hat{\mathfrak{m}}}$, and it commutes with the composite representation $\mathrm{Ad}_{\underline{\hat{\mathfrak{g}}}/\underline{\hat{\mathfrak{m}}}} \circ \psi$ of $L'_F = L_F \times SL(2, \mathbb{C})$. Now, we have a decomposition

$$\mathrm{Ad}_{\underline{\hat{\mathfrak{g}}}/\underline{\hat{\mathfrak{m}}}} \circ \psi = \bigoplus_{\hat{\alpha} \in \hat{\Sigma}_M} \bigoplus_{j \in J(\hat{\alpha})} (\mu_j \otimes \nu_j)$$

into irreducible representations of L'_F. Let us write $(\underline{\hat{\mathfrak{g}}}/\underline{\hat{\mathfrak{m}}})_\psi$ for the direct sum of those irreducible constituents $\mu_j \otimes \nu_j$ such that (i) $\widetilde{\mu}_j \cong \mu_j$, (ii) $\varepsilon(\frac{1}{2}, \mu_j) = -1$, and (iii) $\deg(\nu_j)$ is even. Then $(\underline{\hat{\mathfrak{g}}}/\underline{\hat{\mathfrak{m}}})_\psi$ is an invariant subspace of both L'_F and $\bar{N}_\psi^+$. Define

$$\varepsilon_\psi^{G/M}(u) = \det\Big(\widetilde{u}, \mathrm{End}_{L'_F}\big((\underline{\hat{\mathfrak{g}}}/\underline{\hat{\mathfrak{m}}})_\psi\big)\Big), \qquad u \in \mathcal{N}_\psi^+, \tag{6.5}$$

where $\widetilde{u}$ is any element in $\bar{N}_\psi^+$ whose projection onto $\mathcal{N}_\psi^+ = \bar{N}_\psi^+/\bar{T}_\psi$ equals u. Observe that

$$(\underline{\hat{\mathfrak{g}}}/\underline{\hat{\mathfrak{m}}})_\psi = \bigoplus_{\hat{\alpha} \in \hat{\Sigma}_M} \underline{\hat{\mathfrak{g}}}_{\hat{\alpha},\psi},$$

where

$$\underline{\hat{\mathfrak{g}}}_{\hat{\alpha},\psi} = \underline{\hat{\mathfrak{g}}}_{\hat{\alpha}} \cap (\underline{\hat{\mathfrak{g}}}/\underline{\hat{\mathfrak{m}}})_\psi .$$

The subgroup $\mathcal{S}_\psi^1$ of $\mathcal{N}_\psi^+$ leaves invariant each of the subspaces $\underline{\hat{\mathfrak{g}}}_{\hat{\alpha},\psi}$ of $(\underline{\hat{\mathfrak{g}}}/\underline{\hat{\mathfrak{m}}})_\psi$. Since the actions of $\mathcal{S}_\psi^1$ on $\underline{\hat{\mathfrak{g}}}_{\hat{\alpha},\psi}$ and $\underline{\hat{\mathfrak{g}}}_{-\hat{\alpha},\psi}$ are contragredient, $\varepsilon_\psi^{G/M}$ is trivial on $\mathcal{S}_\psi^1$, and descends to a character on the quotient

$$\mathcal{N}_\psi^+/\mathcal{S}_\psi^1 \cong W_\psi^+ .$$

Of course the main reason for defining $\varepsilon_\psi^{G/M}$ is the formula

$$\varepsilon_\psi^G(u) = \varepsilon_\psi^{G/M}(u)\varepsilon_\psi^M(u), \qquad u \in \mathcal{N}_\psi^+, \tag{6.6}$$

which follows easily from (4.5′), (6.5) and the corresponding formula for ε_ψ^M.

To express r_ψ in terms of $\varepsilon_\psi^{G/M}$, let $\hat{\Sigma}_{M,\psi}$ be the set of roots $\hat{\alpha} \in \hat{\Sigma}_M$ such that the dimension of $\mathrm{End}_{L'_F}(\underline{\hat{\mathfrak{g}}}_{\hat{\alpha},\psi})$ is odd. It follows from properties of the determinant that

$$\varepsilon_\psi^{G/M}(w) = (-1)^{|\hat{\Sigma}_{M,\psi}\cap\hat{\Sigma}_{P,w}|}, \qquad w \in W_\psi^+.$$

This is just the contribution from the even dimensional representations ν_j to the expression for $r_\psi(w)$ given by (6.3) and Lemma 6.1. The contribution from the odd dimensional representations ν_j is simply the usual sign character $\varepsilon(w)$ attached to the group $\bar{S}_\psi^+$. Thus

$$r_\psi(w) = \varepsilon(w)\varepsilon_\psi^{G/M}(w), \qquad w \in W_\psi^+.$$

The required formula

$$r_\psi(u) = \varepsilon(w_u)\varepsilon_\psi^G(u)\varepsilon_\psi^M(u)^{-1}, \qquad u \in \mathcal{N}_\psi^+,$$

of Proposition 5.1 then follows directly from (6.6). □

The formula (6.6) can be regarded as motivation for the definition of ε_ψ^G. The introduction of this character might have seemed odd at first. However, we now have a direct connection between ε_ψ^G and the more familiar function r_ψ obtained from the normalizing factors of global intertwining operators.

§7. The expansion of $E_{\mathrm{disc,t}}(f)$

We turn now to the distribution $E_{\mathrm{disc,t}}$. It was defined in Hypothesis 3.1 as the sum

$$\sum_H \iota(G,H)S\hat{I}^{H_1}_{\mathrm{disc,t}}(f^{H_1}), \qquad f \in C_c^\infty(G(\mathbb{A}),\chi), \tag{7.1}$$

over equivalence classes of elliptic endoscopic data. We shall convert this into an expression which is parallel to the expansion (5.10) for $I_{\mathrm{disc,t}}(f)$.

Hypothesis 3.1 can be regarded as a general existence assertion. There should be a stable distribution on any quasi-split group with the property that (7.1) equals $I_{\mathrm{disc},t}(f)$ for any component G at all. Our ultimate goal is to show that this assertion is compatible with the formula (5.10) for $I_{\mathrm{disc},t}(f)$. Since the stable distributions are uniquely determined by the property, the problem is simply to show that they exist. For a given quasi-split group G_1, and a suitable character χ_1 on a subgroup X_1 of $Z(G,\mathbb{A})$, we shall try to construct the associated stable distribution $SI^{G_1}_{\mathrm{disc},t}$ in terms of the parameters $\psi_1 \in \Psi(G_1,\chi_1,t)$. Our local assumptions in §4 attach a stable distribution

$$f_1 \longrightarrow f_1^{G_1}(\psi_1), \qquad\qquad f_1 \in C_c^\infty(G_1(\mathbb{A}),\chi_1),$$

on $G_1(\mathbb{A})$ to each parameter $\psi_1 \in \Psi(G_1,\chi_1)$. Let us therefore set

$$SI^{G_1}_{\mathrm{disc},t}(f_1) = \sum_{\psi_1 \in \Psi(G_1,\chi_1,t)} SI^{G_1}_{\psi_1}(f_1), \tag{7.2}$$

where

$$SI^{G_1}_{\psi_1}(f_1) = \sigma(G_1,\psi_1) f_1^{G_1}(\psi_1),$$

for constants $\sigma(G_1,\psi_1)$ to be determined. We shall assume that the constants vanish unless ψ_1 belongs to $\Psi_0'(G_1,\chi_1,t)$, a countable subset of $\Psi(G_1,\chi_1,t)$. We shall attempt to define them so that the formula obtained by equating (7.1) with the right hand side of (5.10) is universally valid.

We fix a representative $(H,\mathcal{H},s,\xi)$, for each equivalence class of endoscopic data for G, such that $\mathcal{H}$ is a subgroup of ${}^LG^0$ and ξ is the inclusion mapping. We also fix a distinguished splitting (H_1,ξ_1) of $(H,\mathcal{H},s,\xi)$. The character χ_1 is then defined on a subgroup X_1 of $Z(H_1,\mathbb{A})$ as in §3. We begin with the formula

$$E_{\mathrm{disc},t}(f) = \sum_H \iota(G,H) \sum_{\psi_1 \in \Psi_0'(H_1,\chi_1,t)} S\widehat{I}^{H_1}_{\psi_1}(f^{H_1}) \tag{7.3}$$

obtained by applying the definition (7.2) to the groups H_1 in (7.1). Our immediate goal is to convert the double sum over H and ψ_1 to a single sum over the orbits of $\widehat{G}^0$ on a certain set. In the process, we will need to apply the formula (3.5) for the coefficients $\iota(G,H)$.

Recall that $\Psi(G)$ denotes the set of maps

$$\psi : L_F \times SL(2,\mathbf{C}) \longrightarrow {}^L G^0$$

satisfying certain conditions, and taken modulo the equivalence relation (4.2). Let us write $\widetilde{\Psi}(G)$ for the same set of parameters, but without the equivalence relation, and let $\widetilde{\Psi}(G)/\widehat{G}^0$ denote the set of $\widehat{G}^0$-orbits in $\widetilde{\Psi}(G)$. (We can also write $\widetilde{\Psi}_{\mathrm{disc}}(G)$, $\widetilde{\Psi}(G,\chi,t)$, etc., for the obvious subsets of $\widetilde{\Psi}(G)$.) We shall describe the order of the covering projection $\widetilde{\Psi}(G)/\widehat{G}^0 \longrightarrow \Psi(G)$. According to the definition (4.2), the group $\ker^1(F, Z(\widehat{G}^0))$ acts transitively on the fibres of the projection. The isotropy subgroup is just the image of $\mathcal{S}_\psi(G^0)/\mathcal{C}_\psi$ under the injection (4.9). But the finite group $\mathcal{S}_\psi(G^0)$ is bijective with the set $\mathcal{S}_\psi$. Therefore, the order of each fibre in the projection equals

$$|\ker^1(F, Z(\widehat{G}^0))|\ |\mathcal{S}_\psi|^{-1}\ |\mathcal{C}_\psi|. \tag{7.4}$$

We shall apply this remark to the quasi-split groups H_1 which occur in (7.3). We can replace the sum over $\psi_1 \in \Psi'_0(H_1, \chi_1, t)$ by the sum over $\widetilde{\Psi}'_0(H_1, \lambda_1, t)/\hat{H}_1$, provided that we divide by

$$|\ker^1(F, Z(\hat{H}_1))|\ |\mathcal{S}_{\psi_1}|^{-1}\ |\mathcal{C}_{\psi_1}|,$$

the analogue for H_1 of the integer (7.4). Since (H_1, ξ_1) is assumed to be a distinguished splitting, $\ker^1(F, Z(\hat{H}_1))$ equals $\ker^1(F, Z(\hat{H}))$. Combining this with the formula (3.5) for $\iota(G,H)$, we are able to write $E_{\mathrm{disc},t}(f)$ as the sum over H and over $\psi_1 \in \widetilde{\Psi}'_0(H_1, \chi_1, t)/\hat{H}_1$ of

$$\begin{aligned} &|\ker^1(F, Z(\widehat{G}^0))|^{-1} |\pi_0(\kappa_G)^{-1}|\bar{Z}(\hat{H})^\Gamma|^{-1}|\mathcal{S}_{\psi_1}| \\ &\qquad \times |\mathcal{C}_{\psi_1}|^{-1}|\mathrm{Aut}(H)/\hat{H}Z(\widehat{G}^0)^\Gamma|^{-1} S\hat{I}^{H_1}_{\psi_1}(f^{H_1}). \end{aligned} \tag{7.5}$$

Keep in mind that H really stands for the equivalence class of an endoscopic datum $(H, \mathcal{H}, s, \xi)$. Now, suppose that we are given a parameter $\psi_1 \in \widetilde{\Psi}(H_1, \chi_1)$. Then ψ_1 factors to an L-homomorphism from W_F into $\mathcal{H}$, which may then be composed with the embedding $\xi : \mathcal{H} \longrightarrow {}^L G^0$. In this way we obtain a parameter $\psi \in \widetilde{\Psi}(G,\chi)$. It follows from the property (2.1) of endoscopic data that the coset $s \in \widehat{G}/Z(\widehat{G}^0)$ lies in the set

$$\bar{S}_\psi = S_\psi/Z(G^0) = S_\psi(G)/Z(G^0).$$

Conversely, suppose that we are given a parameter $\psi \in \widetilde{\Psi}(G,\chi)$ and a coset $s \in \bar{S}_\psi$ consisting of semisimple elements. Then we can define an endoscopic datum $(H, \mathcal{H}, s, \xi)$ as in §4. Recall that H is the quasi-split group whose dual group is

$$\hat{H} = \mathrm{Cent}(s, \widehat{G}^0)^0,$$

equipped with the L-action induced by

$$\mathcal{H} = \hat{H}\psi(L_F \times SL(2,\mathbb{C})),$$

and ξ is the inclusion of $\mathcal{H}$ into ${}^LG^0$. The parameter ψ then factors through $\mathcal{H}$. For any distinguished splitting (H_1, ξ_1) of the endoscopic datum, we obtain the character $\chi_1 : X_1 \longrightarrow \mathbb{C}^*$ as in §3, and ψ then yields a parameter $\psi_1 \in \widetilde{\Psi}(H_1, \chi_1)$.

We have just established a correspondence between the pairs (H, ψ_1) and (ψ, s). We want the datum H to be elliptic and the parameter ψ_1 to be weakly elliptic. We ought to describe these conditions in terms of (ψ, s). Since ψ_1 factors through $\mathcal{H}$, and $\hat{H}_1$ equals $\xi_1(\hat{H})Z(\hat{H}_1)$, we have

$$C_{\psi_1}Z(\hat{H}_1)/Z(\hat{H}_1) \cong \mathrm{Cent}(\mathrm{Image}(\psi), \hat{H})Z(\hat{H})/Z(\hat{H}).$$

In other words,

$$C_{\psi_1}Z(\hat{H}_1)/Z(\hat{H}_1) \cong (C_\psi \cap \hat{H})Z(\hat{H})/Z(\hat{H}). \tag{7.6}$$

In particular, there is an isomorphism

$$C^0_{\psi_1}Z(\hat{H}_1)/Z(\hat{H}_1) \cong (C_\psi \cap \hat{H})^0 Z(\hat{H})/Z(\hat{H})$$

of the two identity components. Notice that $(C_\psi \cap \hat{H})^0$ equals $C^0_{\psi,s}$, the connected centralizer in C^0_ψ of any element in the coset s. Consequently

$$C^0_{\psi_1}Z(\hat{H}_1)/Z(\hat{H}_1) \cong C^0_{\psi,s}Z(\hat{H})/Z(\hat{H}) \cong C^0_{\psi,s}/C^0_{\psi,s} \cap Z(\hat{H})^\Gamma.$$

Thus, ψ_1 is weakly elliptic if and only if the center of $C^0_{\psi,s}/C^0_{\psi,s} \cap Z(\hat{H})^\Gamma$ is finite. Now $C^0_{\psi,s} \cap Z(\hat{H})^\Gamma$ is a central subgroup of $C^0_{\psi,s}$ which contains $A_{\hat{H}} = (Z(\hat{H})^\Gamma)^0$. Therefore, the conditions that ψ_1

be weakly elliptic and H be elliptic, taken together, are equivalent to the condition that $C^0_{\psi,s}$ has finite center modulo $A_{\widehat{G}} = (Z(\widehat{G})^\Gamma)^0$. We can describe this more simply in terms of the set

$$\bar{S}_{\psi,\mathrm{fin}} = \{s \in \bar{S}_\psi : |Z(\bar{S}^0_{\psi,s})| < \infty\}.$$

For by (4.11) we have

$$\begin{aligned} \bar{S}^0_{\psi,s} = C^0_{\psi,s} Z(\widehat{G}^0)/Z(\widehat{G}^0) &\cong C^0_{\psi,s}/C^0_{\psi,s} \cap Z(\widehat{G}^0) \\ &= C^0_{\psi,s}/C^0_{\psi,s} \cap Z(\widehat{G})^\Gamma. \end{aligned}$$

Thus, the correspondence is between elliptic pairs (H, ψ_1) and pairs (ψ, s) such that s belongs to $\bar{S}_{\psi,\mathrm{fin}}$.

The foregoing discussion will enable us to interchange the order of summation in the original double sum over H and ψ_1. Keep in mind that $(H, \mathcal{H}, s, \xi)$ stands for a representative of an equivalence class of endoscopic data for which $\mathcal{H}$ is a subgroup of ${}^L G^0$ and ξ is the inclusion mapping. The equivalence classes themselves can be identified with the $\widehat{G}^0$-orbits of such data. The stabilizer in $\widehat{G}^0$ of $(H, \mathcal{H}, s, Id)$ is the group $\mathrm{Aut}(H)$ which appears in the expression (7.5). The group $\mathrm{Aut}(H)$ in turn acts on the set of parameters $\psi \in \widetilde{\Psi}(G, \chi, t)$ such that s belongs to $\bar{S}_{\psi,\mathrm{fin}}$. The stabilizer in $\mathrm{Aut}(H)$ of a given ψ is simply the group

$$\widetilde{C}^+_{\psi,s} = \{c \in C_\psi : csc^{-1} = s\}$$

of elements in C_ψ which fix the coset s. On the other hand, we can identify the orbits $\{\psi_1\} \in \widetilde{\Psi}(H_1, \chi_1, t)/\hat{H}_1$ with the $\hat{H}$-orbits of $\{\psi\}$. This is easily seen from the injectivity of the map

$$H^1(\Gamma, Z(\hat{H})) \longrightarrow H^1(\Gamma, Z(\hat{H}_1)),$$

noted in §2, and the fact that $\hat{H}_1 = Z(\hat{H}_1)\xi_1(\hat{H})$. We can actually take $\hat{H}Z(\widehat{G}^0)^\Gamma$-orbits of $\{\psi\}$, since $Z(\widehat{G}^0)^\Gamma$ centralizes the image of ψ. But the group $\hat{H}Z(\widehat{G}^0)^\Gamma$ has finite index in $\mathrm{Aut}(H)$, by Lemma 2.1, and the stabilizer of ψ in $\hat{H}Z(\widehat{G}^0)^\Gamma$ is the subgroup

$$C^+_{\psi,s} \cap (\hat{H}Z(G^0)^\Gamma)$$

of finite index in $\widetilde{C}^+_{\psi,s}$. Therefore, we can replace the original double sum over H and ψ_1 by the sum over the $\widehat{G}^0$-orbits in the set

$$\{(\psi, s) : \psi \in \widetilde{\Psi}(G, \chi, t),\ s \in \bar{S}_{\psi,\mathrm{fin}}\},$$

if we multiply the summand (7.5) by

$$|\mathrm{Aut}(H)/\hat{H}Z(\widehat{G}^0)^\Gamma|\ |\ \widetilde{C}^+_{\psi,s}/\widetilde{C}^+_{\psi,s} \cap (\hat{H}Z(\hat{G}^0)^\Gamma)|^{-1}. \tag{7.7}$$

The stabilizer in $\widehat{G}^0$ of a given parameter $\psi \in \widetilde{\Psi}(G, \chi, t)$ is the group C_ψ. We can therefore replace the sum over $\widehat{G}^0$-orbits in $\{(\psi, s)\}$ by a double sum over $\psi \in \widetilde{\Psi}(G, \chi, t)/\widehat{G}^0$ and over the set $\mathrm{Orb}(C_\psi, \bar{S}_{\psi,\mathrm{fin}})$ of orbits of C_ψ in $\bar{S}_{\psi,\mathrm{fin}}$. Obviously, $\bar{S}_{\psi,\mathrm{fin}}$ has the same set of orbits under C_ψ as under the group

$$\bar{C}_\psi = C_\psi Z(\widehat{G}^0)/Z(\widehat{G}^0).$$

The stabilizer of s in $\bar{C}_\psi$ equals

$$\bar{C}^+_{\psi,s} = \widetilde{C}^+_{\psi,s}Z(\widehat{G}^0)/Z(\widehat{G}^0) = \mathrm{Cent}(s, \bar{C}_\psi).$$

However, we would prefer to take the orbits in $\bar{S}_{\psi,\mathrm{fin}}$ under the connected component

$$\bar{C}^0_\psi = C^0_\psi Z(\widehat{G}^0)/Z(\widehat{G}^0).$$

The $\bar{C}_\psi$-orbit of s is bijective with $\bar{C}_\psi/\bar{C}^+_{\psi,s}$, while the $\bar{C}^0_\psi$-orbit is in bijective correspondence with the quotient of $\bar{C}^0_\psi$ by the group

$$\bar{C}_{\psi,s} = \mathrm{Cent}(s, \bar{C}^0_\psi).$$

Therefore, we can indeed take the second sum over $\bar{C}^0_\psi$-orbits, provided that we multiply the summand by

$$|\bar{C}^+_{\psi,s}/\bar{C}_{\psi,s}|\ |\bar{C}_\psi/\bar{C}^0_\psi|^{-1},$$

or what is the same thing,

$$|\bar{C}^+_{\psi,s}/\bar{C}_{\psi,s}|\ |\mathcal{C}_\psi|^{-1}. \tag{7.8}$$

Finally, we can take the first sum over $\psi \in \Psi(G, \chi, t)$ instead of $\widetilde{\Psi}(G, \chi, t)/\widehat{G}^0$, if we multiply the summand by the integer (7.4). We have shown that $E_{\mathrm{disc},t}(f)$ equals the sum over $\psi \in \Psi(G, \chi, t)$ and $s \in \mathrm{Orb}(\bar{C}^0_\psi, \bar{S}_{\psi,\mathrm{fin}})$ of the expression obtained by multiplying (7.4), (7.5), (7.7) and (7.8) together. We can write this last expression as the product of

$$(7.9) \qquad |\widetilde{C}^+_{\psi,s}/\widetilde{C}^+_{\psi,s} \cap (\hat{H}Z(\widehat{G}^0)^\Gamma|^{-1}|\mathcal{C}_{\psi_1}|^{-1}|\bar{C}^+_{\psi,s}/\bar{C}_{\psi,s}|\, |\bar{Z}(\hat{H})^\Gamma|^{-1}$$

and

$$(7.10) \qquad |\pi_0(\kappa_G)|^{-1}\ |\mathcal{S}_\psi|^{-1}\ |\mathcal{S}_{\psi_1}|\ S\widehat{I}^{H_1}_{\psi_1}(f^{H_1}).$$

The term (7.9) can be simplified. We begin by writing

$$\begin{aligned} \mathcal{C}_{\psi_1} &\cong (C_\psi \cap \hat{H})Z(\hat{H})/(C_\psi \cap \hat{H})^0 Z(\hat{H}) \\ &\cong (C_\psi \cap \hat{H})/(C_\psi \cap \hat{H})^0 Z(\hat{H})^\Gamma \\ &\cong (C_\psi \cap \hat{H})Z(\widehat{G}^0)/((C_\psi \cap \hat{H})^0 Z(\hat{H})^\Gamma \cdot Z(\widehat{G}^0)). \end{aligned}$$

The first isomorphism follows from (7.6), while the second is trivial and the third is a consequence of the fact that

$$(C_\psi \cap \hat{H}) \cap Z(\widehat{G}^0) = Z(\widehat{G})^\Gamma \cap Z(\hat{H})^\Gamma \subset Z(\hat{H})^\Gamma.$$

We also observe that

$$\begin{aligned} \widetilde{C}^+_\psi/\widetilde{C}^+_{\psi,s} \cap (\hat{H}Z(\widehat{G}^0)^\Gamma) &= \widetilde{C}^+_{\psi,s}/(C_\psi \cap \hat{H})Z(\widehat{G}^0)^\Gamma \\ &\cong \widetilde{C}^+_{\psi,s}Z(\widehat{G}^0)/(C_\psi \cap \hat{H})Z(\widehat{G}^0). \end{aligned}$$

This allows us to write

$$\begin{aligned} &|\widetilde{C}^+_\psi/\widetilde{C}^+_{\psi,s} \cap (\hat{H}Z(\widehat{G}^0)^\Gamma)|^{-1}|\mathcal{C}_{\psi_1}|^{-1} \\ &\qquad = |\widetilde{C}^+_{\psi,s}Z(\widehat{G}^0)/(C_\psi \cap \hat{H})^0 Z(\hat{H})^\Gamma Z(\widehat{G}^0)|^{-1}, \end{aligned}$$

for the first two factors in the product (7.9). Let us divide both groups in the quotient on the right by $Z(\widehat{G}^0)$. The numerator becomes

$$C^+_{\psi,s}Z(G^0)/Z(G^0) = \bar{C}^+_{\psi,s},$$

and the denominator may be written

$$\begin{aligned}
&(C_\psi \cap \hat{H})^0 Z(\hat{H})^\Gamma Z(\widehat{G}^0)/Z(\widehat{G}^0) \\
&\qquad = ((C_\psi \cap \hat{H})^0 Z(\widehat{G}^0)/Z(\widehat{G}^0))(Z(\hat{H})^\Gamma Z(\widehat{G}^0)/Z(\widehat{G}^0)) \\
&\qquad = (C^0_{\psi,s} Z(\widehat{G}^0)/Z(\widehat{G}^0))\bar{Z}(\hat{H})^\Gamma \\
&\qquad = \bar{C}^0_{\psi,s}\bar{Z}(\hat{H})^\Gamma,
\end{aligned}$$

by (4.11). Therefore, (7.9) equals

$$\begin{aligned}
&|\bar{C}^+_{\psi,s}/\bar{C}^0_{\psi,s}\bar{Z}(\hat{H})^\Gamma|^{-1}|\bar{C}^+_{\psi,s}/\bar{C}_{\psi,s}|\ |\bar{Z}(\hat{H})^\Gamma|^{-1} \\
&\qquad = |\bar{C}^+_{\psi,s}/\bar{C}^0_{\psi,s}|^{-1}|\bar{C}^0_{\psi,s} \cap \bar{Z}(\hat{H})^\Gamma|^{-1}|\bar{C}^+_{\psi,s}/\bar{C}_{\psi,s}| \\
&\qquad = |\bar{C}_{\psi,s}/\bar{C}^0_{\psi,s}|^{-1}|\bar{C}^0_{\psi,s} \cap \bar{Z}(\hat{H})^\Gamma|^{-1}.
\end{aligned}$$

We noted in §4 that $\bar{C}_{\psi,s} = \bar{S}_{\psi,s}$. In particular

$$|\bar{C}_{\psi,s}/\bar{C}^0_{\psi,s}|^{-1} = |\bar{S}_{\psi,s}/\bar{S}^0_{\psi,s}|^{-1} = |\pi_0(\bar{S}_{\psi,s})|^{-1}.$$

The term (7.9) can by consequence by written as

$$|\pi_0(\bar{S}_{\psi,s})|^{-1}|\bar{S}^0_{\psi,s} \cap \bar{Z}(\hat{H})^\Gamma|^{-1}.$$

We have now established that $E_{\mathrm{disc},t}(f)$ equals the product of $|\pi_0(\kappa_G)|^{-1}$ with

$$\sum_{\psi \in \Psi(G,\chi,t)} |S_\psi|^{-1} \sum_{s \in \mathrm{Orb}(\bar{S}^0_\psi, \bar{S}_{\psi,\mathrm{fin}})} |\pi_0(\bar{S}_{\psi,s})|^{-1} \\
|S_{\psi_1}|\ |\bar{S}^0_{\psi,s} \cap \bar{Z}(\hat{H})^\Gamma|^{-1} S\widehat{I}^{H_1}_{\psi_1}(f^{H_1}).$$

By assumption,

$$S\widehat{I}^{H_1}_{\psi_1}(f^{H_1}) = \sigma(H_1,\psi_1) f^{H_1}(\psi_1).$$

Part of our local assumption in §4 is that $f^{H_1}(\psi_1)$ depends only on the image $x = \bar{s}$ of s in the set

$$\mathcal{S}_\psi = S_\psi/S^0_\psi Z(G^0) = \bar{S}_\psi/\bar{S}^0_\psi = \pi_0(\bar{S}_\psi).$$

More precisely, formula (4.4) asserts that

$$f^{H_1}(\psi_1) = \sum_{\pi\in\{\Pi_\psi\}} < \bar{s}_\psi x, \pi > f_G(\pi),$$

where $< \cdot,\cdot >$ is the global pairing on $\mathcal{S}_\psi \times \Pi_\psi$. We can therefore write $E_{\mathrm{disc},t}(f)$ as the product of $|\pi_0(\kappa_G)|^{-1}$ with

$$(7.11)\quad \sum_{\psi} \sum_{\pi\in\{\Pi_\psi\}} |\mathcal{S}_\psi|^{-1} \sum_{x\in\pi_0(\bar{S}_\psi)} \Big(\sum_{s\in\mathrm{Orb}(\bar{S}^0_\psi, x_{\mathrm{fin}})} |\pi_0(\bar{S}_{\psi,s})|^{-1}\tau(\psi,s)\Big) < \bar{s}_\psi x, \pi > f_G(\pi),$$

where

$$x_{\mathrm{fin}} = x \cap \bar{S}_{\psi,\mathrm{fin}}$$

and

$$\tau(\psi,s) = |\mathcal{S}_{\psi_1}|\; |\bar{S}^0_{\psi,s} \cap \bar{Z}(\hat{H})^\Gamma|^{-1}\sigma(H_1,\psi_1).$$

The similarity with the formula in Proposition 5.2 appears promising. We must try to define the constants $\sigma(H_1,\psi_1)$ so that the two formulas for $I_{\mathrm{disc},t}(f)$ always match.

Suppose that G_1 is an arbitrary connected quasi-split group over F, and that $\psi_1 \in \Psi(G_1)$. Then we shall set

$$(7.12)\qquad \sigma(G_1,\psi_1) = |\mathcal{S}_{\psi_1}|^{-1}\varepsilon_{\psi_1}(\bar{s}_{\psi_1})\sigma(\bar{S}^0_{\psi_1}),$$

where $\varepsilon_{\psi_1} = \varepsilon^{G_1}_{\psi_1}$ is the sign character (4.5) and $\sigma(\bar{S}^0_{\psi_1})$ is a constant, to be determined, which depends only on the isomorphism class of the complex, connected reductive group

$$\bar{S}^0_{\psi_1} = (S_{\psi_1}/Z(\widehat{G}_1))^0.$$

We also ask that this latter constant have the property that

$$(7.13)\qquad \sigma(S_1) = \sigma(S_1/Z_1)|Z_1|^{-1},$$

for any complex connected group S_1, and any subgroup Z_1 of the center of S_1. In particular, $\sigma(S_1)$ is going to have to vanish if S_1 has infinite center. This implies that $\sigma(G_1,\psi_1) = 0$ unless $\psi_1 \in \Psi'_0(G_1)$, as we would expect.

We of course want to set $G_1 = H_1$. Then

$$\begin{aligned}\bar{S}^0_{\psi_1} = \bar{C}^0_{\psi_1} &= C^0_{\psi_1} Z(\hat{H}_1)/Z(\hat{H}_1)\\ &= (C_\psi \cap \hat{H})^0 Z(\hat{H})/Z(\hat{H})\\ &= C^0_{\psi,s} Z(\hat{H})/Z(\hat{H}),\end{aligned}$$

by the formula (7.6). Since

$$C^0_{\psi,s} \cap Z(\hat{H}) = C^0_{\psi,s} \cap Z(\hat{H})^\Gamma Z(\widehat{G}^0),$$

we obtain

$$\begin{aligned}C^0_{\psi,s} Z(\hat{H})/Z(\hat{H}) &\cong C^0_{\psi,s} Z(\hat{H})^\Gamma Z(\widehat{G}^0)/Z(\hat{H})^\Gamma Z(\widehat{G}^0)\\ &\cong (C^0_{\psi,s} Z(\widehat{G})^0/Z(\widehat{G}^0))\bar{Z}(\hat{H})^\Gamma/\bar{Z}(\hat{H})^\Gamma\\ &\cong \bar{C}^0_{\psi,s}\bar{Z}(\hat{H})^\Gamma/\bar{Z}(\hat{H})^\Gamma,\end{aligned}$$

from (4.11). But $\bar{C}^0_{\psi,s} = \bar{S}^0_{\psi,s}$, so that

$$\bar{S}^0_{\psi_1} \cong \bar{S}^0_{\psi,s}\bar{Z}(\hat{H})^\Gamma/\bar{Z}(\hat{H})^\Gamma \cong \bar{S}^0_{\psi,s}/\bar{S}^0_{\psi,s} \cap \bar{Z}(\hat{H})^\Gamma.$$

It follows from the property (7.13) that

$$\sigma(\bar{S}^0_{\psi_1}) = \sigma(\bar{S}^0_{\psi,s})|\bar{S}^0_{\psi,s} \cap \bar{Z}(\hat{H})^\Gamma|.$$

Therefore,

$$\sigma(H_1, \psi_1) = |\mathcal{S}_{\psi_1}|^{-1}\varepsilon^{H_1}_{\psi_1}(\bar{s}_{\psi_1})|\bar{S}^0_{\psi,s} \cap \bar{Z}(\hat{H})^\Gamma|\sigma(\bar{S}^0_{\psi,s}),$$

so that

$$\tau(\psi, s) = \varepsilon^{H_1}_{\psi_1}(\bar{s}_{\psi_1})\sigma(\bar{S}^0_{\psi,s}).$$

LEMMA 7.1. *For H_1 and $x \in \mathcal{S}_\psi$ as in (7.11), we have*

$$\varepsilon^{H_1}_{\psi_1}(\bar{s}_{\psi_1}) = \varepsilon^G_\psi(\bar{s}_\psi x).$$

PROOF: As in §4, let

$$\tau_\psi = \bigoplus_k (\lambda_k \otimes \mu_k \otimes \nu_k)$$

be the decomposition of the representation

$$\tau_\psi : \bar{S}_\psi^+ \times L_F \times SL(2,\mathbf{C}) \longrightarrow GL(\hat{\underline{\mathbf{g}}})$$

into irreducible constituents. If I denotes the set of indices k in the direct sum, let I' denote the subset of k such that (i) $\mu_k \cong \widetilde{\mu}_k$, (ii) $\varepsilon(\frac{1}{2}, \mu_k) = -1$, and (iii) $\dim(\nu_k)$ is even. Then

$$x \in \mathcal{S}_\psi, \qquad \varepsilon_\psi^G(x) = \prod_{k \in I'} \det(\lambda_k(s)),$$

where s is any element in $\bar{S}_\psi$ which projects onto x. Notice that the element s_ψ lies in both S_ψ^+ and $SL(2,\mathbf{C})$. If k belongs to I', we obtain

$$\lambda_k(s_\psi) = \nu_k(s_\psi) = -1,$$

since $\dim \nu_k$ is even. It follows that

$$\varepsilon_\psi^G(\bar{s}_\psi) = \prod_{k \in I'} \det(\lambda_k(s_\psi)) = \prod_{k \in I'} (-1)^{\dim(\lambda_k)}.$$

Now H_1 is a central extension of the endoscopic group H attached to s. The Lie algebra of $\hat{H}$ equals the centralizer of $\mathrm{Ad}(s)$ in $\hat{\mathbf{g}}$, and the Lie algebra of $\hat{H}_1$ can be identified with the direct sum of this algebra and a central ideal. For each k, let λ_k^s be the space of s-fixed vectors for λ_k. This of course is just the intersection of the underlying space of λ_k with the Lie algebra of $\hat{H}$. Recalling the relation between ψ and ψ_1, and applying the formula for $\varepsilon_\psi^G(\bar{s}_\psi)$ to H_1, we obtain

$$\varepsilon_{\psi_1}^{H_1}(\bar{s}_{\psi_1}) = \prod_{k \in I'} (-1)^{\dim(\lambda_k^s)}.$$

Finally, we observe that the number

$$\varepsilon_\psi^G(x) = \prod_{k \in I'} \det(\lambda_k(s))$$

equals the product of all the eigenvalues, counting multiplicities, of the operators

$$\{\lambda_k(s) : \ k \in I'\}.$$

Now the contragradient $\lambda_k \longrightarrow \widetilde{\lambda}_k$ defines an involution on the representations λ_k with $k \in I'$. In particular, if ξ is an eigenvalue, not equal to ± 1, then ξ^{-1} is also an eigenvalue with the same multiplicity. Therefore,

$$\varepsilon_{\psi}^{G}(x) = (-1)^{m(-1)},$$

where $m(-1)$ is the total multiplicity of the eigenvalue (-1). By the same token,

$$\sum_{k \in I'} (\dim(\lambda_k) - \dim(\lambda_k^s)) - m(-1)$$

is an even integer. Consequently,

$$\varepsilon_{\psi}^{G}(\bar{s}_{\psi})\varepsilon_{\psi_1}^{H_1}(\bar{s}_{\psi_1}) = (-1)^{m(-1)} = \varepsilon_{\psi}^{G}(x).$$

We obtain

$$\varepsilon_{\psi}^{G}(\bar{s}_{\psi}x) = \varepsilon_{\psi}^{G}(x)\varepsilon_{\psi}^{G}(\bar{s}_{\psi}) = \varepsilon_{\psi}^{G}(x)\varepsilon_{\psi}^{G}(\bar{s}_{\psi})^{-1} = \varepsilon_{\psi_1}^{H_1}(\bar{s}_{\psi_1}),$$

as required. □

The lemma allows us to write

$$\tau(\psi, s) = \varepsilon_{\psi}^{G}(\bar{s}_{\psi}x)\sigma(\bar{S}_{\psi,s}^{0}).$$

Substituting this into (7.11), and setting

$$e(x) = \sum_{s \in \mathrm{Orb}(\bar{S}_{\psi}^{0}, x_{\mathrm{fin}})} |\pi_0(\bar{S}_{\psi,s})|^{-1}\sigma(\bar{S}_{\psi,s}^{0}), \tag{7.14}$$

we see that $E_{\mathrm{disc},t}(f)$ equals the product of $|\pi_0(\kappa_G)|^{-1}$ with

$$\sum_{\psi \in \Psi(G,\chi,t)} \sum_{\pi \in \{\Pi_{\psi}\}} |S_{\psi}|^{-1} \sum_{x \in \pi_0(\bar{S}_{\psi})} \varepsilon_{\psi}^{G}(\bar{s}_{\psi}x)e(x) < \bar{s}_{\psi}x, \pi > f_G(\pi).$$

The point $s_{\psi} \in S_{\psi}(G^0)$ belongs to the center of $S_{\psi}(G^+)$. Consequently, for any point s in the component x, the group $\bar{S}_{\psi,s}^{0}$ equals $\bar{S}_{\psi,s_{\psi}s}^{0}$. It follows that $e(x)$ equals $e(\bar{s}_{\psi}x)$. Substituting this into the formula above, and changing variables in the sum over $x \in \pi_0(\bar{S}_{\psi})$, we see that $E_{\mathrm{disc},t}(f)$ equals the product of $|\pi_0(\kappa_G)|^{-1}$ with

$$\sum_{\psi \in \Psi(G,\chi,t)} \sum_{\pi \in \{\Pi_{\psi}\}} |S_{\psi}|^{-1} \sum_{x \in \pi_0(S_{\psi})} \varepsilon_{\psi}^{G}(x)e(x) < x, \pi > f_G(\pi). \tag{7.15}$$

We have now reached the stage in §7 at which we concluded §5. Taking the two sections together, we see that Hypotheses 3.1 and 4.1 yield two parallel expansions for $I_{\mathrm{disc},t}(f)$ and $E_{\mathrm{disc},t}(f)$ into irreducible characters. Our goal is to show that these two expansions are in fact the same. The expansions are given by (5.10) and (7.15). They differ only in the coefficients $i(x)$ and $e(x)$, which are defined for any component $x \in \pi_0(\bar{S}_\psi)$ by (5.11) and (7.14). We must then show that the coefficients are equal. Recall that $e(x)$ depends on a constant $\sigma(S_1)$, which is to be defined for any complex, connected reductive group S_1 and which satisfies (7.13). We must show that $\sigma(S_1)$ can be defined for each S_1 in such a way that $i(x)$ and $e(x)$ are equal for any x. This is a property of Weyl groups which we shall establish in the next section.

§8. A combinatorial formula for Weyl groups

Suppose that S is a union of connected components in an arbitrary complex, reductive algebraic group. Then S^+ is the reductive group generated by S, and S^0 is the connected component of 1 in S^+. Recall also that we are writing S_s for the centralizer in S^0 of any element $s \in S$. This group is of course not always connected. As a slight generalization of (5.11), we set

$$i(S) = |W^0|^{-1} \sum_{w \in W_{\mathrm{reg}}} \varepsilon(w)|\det(w-1)|^{-1}, \tag{8.1}$$

where

$$W^0 = W(S^0) = \mathrm{Norm}(T, S^0)/T$$

is the Weyl group of S^0 relative to a fixed maximal torus T, and $W_{\mathrm{reg}} = W(S)_{\mathrm{reg}}$ is the set of elements w in the Weyl set

$$W = W(S) = \mathrm{Norm}(T, S)/T$$

such that $\det(w-1) \neq 0$. The determinant can be taken on the real vector space $\underline{\mathbf{a}}_T = \mathrm{Hom}(X(T), \mathbf{R})$. As in §5, $\varepsilon(w) = \pm 1$ is the parity of the number of positive roots of (S^0, T) which are mapped by w to negative roots.

As in §7, we shall write $\mathrm{Orb}(S^0, \Sigma)$ instead of Σ/S^0 for the set of orbits under conjugation by S^0 on an invariant subset Σ of S. This will prevent any confusion of orbits with cosets. The main example is when Σ equals the subset

$$S_{\mathrm{fin}} = \{s \in S : |Z(S_s^0)| < \infty\},$$

in which case the set $\mathrm{Orb}(S^0, S_{\mathrm{fin}})$ is finite.

Our object is to prove

THEOREM 8.1. *There are unique constants $\sigma(S_1)$, defined for each connected and semisimple complex group S_1, such that for any S the number*

$$e(S) = \sum_{s \in \mathrm{Orb}(S^0, S_{\mathrm{fin}})} |\pi_0(S_s)|^{-1}\sigma(S_s^0) \tag{8.2}$$

equals $i(S)$. The constants have the further property that

$$\sigma(S_1) = \sigma(S_1/Z_1)|Z_1|^{-1} \tag{8.3}$$

for any central subgroup Z_1 of S_1.

Remarks. 1. It is obviously enough to prove the theorem when S is just one connected component. We shall assume this in what follows. 2. Let us agree to write $\sigma(S_1) = 0$ if S_1 is *any* complex, connected algebraic group which is not semisimple. In particular, this constant vanishes if S_1 is a reductive group with infinite center. The equation (8.2) can then be written

$$e(S) = \sum_{s \in \mathrm{Orb}(S^0, S)} |\pi_0(S_s)|^{-1}\sigma(S_s^0) .$$

3. Theorem 8.1 is what remains to be proved of the comparison of $I_{\mathrm{disc},t}(f)$ and $E_{\mathrm{disc},t}(f)$ that we began in §5 and §7. It is interesting to observe that Theorem 8.1 is actually a miniature replica of the original problem. It is a formal analogue for Weyl groups of the problem of comparing $I_{\mathrm{disc},t}(f)$ and $E_{\mathrm{disc},t}(f)$, and indeed of many of the comparison problems, both local and global, that arise from endoscopy. I do not know whether it is part of a larger theory of endoscopy for Weyl groups, or whether results of this nature are already implicit in the representation theory of Weyl groups and finite Chevalley groups.

We shall begin the proof of Theorem 8.1 by taking note of the uniqueness of the constants $\sigma(S_1)$. For a given semisimple S_1, assume inductively that $\sigma(S_1')$ has been defined for any S_1' of dimension smaller than S_1. Then $\sigma(S_1)$ is determined by the formula,

$$\sigma(S_1)|Z(S_1)| = i(S_1) - \sum_{s \in \mathrm{Orb}(S_1^0, S_1 - Z(S_1))} |\pi_0(S_{1,s})|^{-1}\sigma(S_{1,s}^0) ,$$

which follows from the required equality of $e(S_1)$ with $i(S_1)$. In other words, the special case of (8.2) that $S = S^0 = S_1$ provides a definition of the constant $\sigma(S_1)$.

Having defined the constants $\sigma(S_1)$ we shall next establish the property (8.3). The argument is similar to part of the discussion of §7. Suppose that S is an arbitrary component, and that Z is a finite subgroup of $Z(S^0)$ which is invariant under conjugation by S. Then $\bar{S} = S/Z$ is a connected component of the reductive group $\bar{S}^+ = S^+/Z$, of which the identity component $\bar{S}^0$ equals S^0/Z.

LEMMA 8.2. *(i)* $i(S) = i(\bar{S})$.
(ii) $e(S) = e(\bar{S})$.
(iii) If $S^0 = S$, *then* $\sigma(S) = \sigma(\bar{S})|Z|^{-1}$.

PROOF: The property (i) follows easily from the definition (8.1). We shall establish the other two properties together. To this end we shall assume inductively that (ii) holds for any connected group of dimension smaller than S.

If the group

$$Z(S) = \mathrm{Cent}(S, Z(S^0))$$

is infinite, the quantities $e(\bar{S})$, $e(S)$, $\sigma(\bar{S}^0)$ and $\sigma(S^0)$ all vanish, and there is nothing to prove. We can therefore assume that $Z(S)$ is finite. This implies that the group $Z(S^+) \cap S$ is also finite. Let $\bar{s}$ be a coset in $\bar{S}_{\mathrm{fin}} = S_{\mathrm{fin}}/Z$ which does not lie in $Z(\bar{S}^+) \cap S$. Then

$$\bar{S}_{\bar{s}} = \mathrm{Cent}(\bar{s}, \bar{S}^0)$$

is a proper subgroup of $\bar{S}^0$. Since

$$\bar{S}^0_{\bar{s}} = S^0_s Z/Z = S^0_s / S^0_s \cap Z$$

for any element s in the coset $\bar{s}$, our induction assumption implies that

$$\sigma(\bar{S}^0_{\bar{s}}) = \sigma(S^0_s)|S^0_s \cap Z| .$$

Let $\widetilde{S}_{\bar{s}}$ be the normalizer in S^0 of the coset $\bar{s}$. The set of orbits in $\mathrm{Orb}(S^0, S)$ which meet $\bar{s}$ can be identified with $\mathrm{Orb}(\widetilde{S}_{\bar{s}}, \bar{s})$, a set of cardinality

$$|Z||\widetilde{S}_{\bar{s}}/S_s|^{-1} .$$

Observe that

$$\begin{aligned}
&\sum_{s\in\mathrm{Orb}(\widetilde{S}_{\bar{s}},\bar{s})} |\pi_0(S_s)|^{-1}\sigma(S_s^0)\\
&= |Z||\widetilde{S}_{\bar{s}}/S_s|^{-1}|S_s/S_s^0|^{-1}\sigma(\bar{S}_{\bar{s}}^0)|S_s^0\cap Z|^{-1}\\
&= |(\widetilde{S}_{\bar{s}}/Z)/(S_s^0 Z/Z)|^{-1}\sigma(\bar{S}_{\bar{s}}^0)\\
&= |\pi_0(\bar{S}_{\bar{s}})|^{-1}\sigma(\bar{S}_{\bar{s}}^0)\ .
\end{aligned}$$

Summing over all such $\bar{s}$, we obtain

$$e(S)-\sigma(S^0)|Z(S^+)\cap S| \;=\; e(\bar{S})-\sigma(\bar{S}^0)|Z(\bar{S}^+)\cap\bar{S}|\ .$$

If $Z(\bar{S}^+)\cap S$ is empty, it follows immediately that $e(S)$ equals $e(\bar{S})$. Suppose that $Z(\bar{S}^+)\cap S$ is not empty. Then S acts on the group S^0 by inner automorphisms, and we have

$$e(S) \;=\; e(S^0) \;=\; i(S^0) \;=\; i(S)$$

from the definitions. (The equality of $e(S^0)$ and $i(S^0)$ was part of the definition of $\sigma(S^0)$.) Similarly $e(\bar{S}) = i(\bar{S})$. The property (i) then implies that $e(S)$ equals $e(\bar{S})$ in this case as well. This is the required property (ii). Suppose that $S = S^0$. Then $Z(S^+)\cap S$ equals $Z(S)$, a group which of course is not empty. The property (iii) follows from the fact that $|Z(S)| = |Z(\bar{S})||Z|$. □

The property (iii) of the lemma is the required condition (8.3) We still have the main part of the proof of the theorem, which is to show that $e(S)$ equals $i(S)$. This of course is a problem only if S is not equal to S^0.

As a warm-up, let us verify the equality of $e(S)$ and $i(S)$ in the special case that $S^0 = T$ is a torus. Then W consists of one element w, the adjoint operation of S on T. We can assume that this element is regular. Recall **[23, II.1.7]** that

$$|\det(w-1)| \;=\; |T^w|\ ,$$

where T^w denotes the kernel of w in T. Since $\varepsilon(w) = 1$, we obtain

$$i(S) \;=\; |T^w|^{-1}\ .$$

On the other hand,

$$T^w = \mathrm{Cent}(s, T^0) = S_s ,$$

for any element $s \in S$. The regularity of w means that s belongs to S_{fin}, and that $S_s^0 = \{1\}$. Therefore $\sigma(S_s^0)$ equals 1. But the T-orbit of s equals the product of s with $\{t^{-1}w(t) : t \in T\}$, a subtorus of T. This subtorus has the same dimension as T, and must therefore equal T. In other words, the orbit of s equals S, so there is only one summand on the right hand side of (8.2). We obtain

$$e(S) = |\pi_0(S_s)|^{-1} = |T^w|^{-1} = i(S) ,$$

as required.

Now suppose that S is arbitrary. We shall use Lemma 8.2 to effect a simplification. First, observe that $i(S)$ and $e(S)$ depend only on S^0 and the set of automorphisms of S^0 induced from conjugation by S. We may therefore assume that S^+ is the semidirect product of S^0 with $\pi_0(S^+)$. Now, let S^0_{sc} be the simply-connected covering of the derived group of S^0, and let $S^0_{\mathrm{cent}} = Z(S^0)^0$ be the connected component of the center of S^0. Then

$$\widetilde{S}^0 = S^0_{\mathrm{sc}} \times S^0_{\mathrm{cent}}$$

is a finite covering group of S^0. In particular, S^0 equals $\widetilde{S}^0/Z$, where Z is the finite central subgroup of $\widetilde{S}^0$. It is then readily verified that $S = \widetilde{S}/Z$, where $\widetilde{S} = S_{\mathrm{sc}} \times S_{\mathrm{cent}}$ is a component which normalizes Z and such that the identity components $(\widetilde{S})^0$, $(S_{\mathrm{sc}})^0$ and $(S_{\mathrm{cent}})^0$ equal the respective groups $\widetilde{S}^0$, S^0_{sc} and S^0_{cent} above. Applying Lemma 8.2 and the calculaion above for tori, we obtain

$$\begin{aligned} e(S) - i(S) &= e(S_{\mathrm{sc}} \times S_{\mathrm{cent}}) - i(S_{\mathrm{sc}} \times S_{\mathrm{cent}}) \\ &= e(S_{\mathrm{sc}})e(S_{\mathrm{cent}}) - i(S_{\mathrm{sc}})i(S_{\mathrm{cent}}) \\ &= \big(e(S_{\mathrm{sc}}) - i(S_{\mathrm{sc}})\big) i(S_{\mathrm{cent}}) . \end{aligned}$$

(We have also used the fact, easily verified from the definitions, that e and i are multiplicative on products.)

It is therefore enough to show that $e(S)$ equals $i(S)$ in the special case that S^0 is semisimple and simply connected. We shall assume this from now on. If s is any semisimple element in S, the group

$$S_s = \mathrm{Cent}(s, S^0)$$

is then connected, by **[24, Theorem 8.1]**. In this case, it is part of our definition that $e(S_s)$ equals $i(S_s)$. If t is a semisimple element in S^0, the connectedness of S_t implies that the set

$$S^t = \mathrm{Cent}(t, S)$$

is either connected or empty. We can assume inductively that if $\dim(S^t) < \dim(S)$, then $e(S^t)$ equals $i(S^t)$.

LEMMA 8.3. *The required equality of $e(S)$ and $i(S)$ is equivalent to the formula*

$$\sum_{s \in \mathrm{Orb}(S^0,S)} i(S_s) = \sum_{t \in \mathrm{Orb}(S^0,S^0)} i(S^t) \,. \tag{8.4}$$

PROOF: If $s \in S$ and $t \in S^0$ are elements that commute, we write

$$S_{s,t} = \mathrm{Cent}(\{s,t\}, S^0) \,.$$

It is obvious that

$$\pi_0\big((S_s)_t\big) = \pi_0(S_{s,t}) = \pi_0\big((S_t)_s\big) \,.$$

The left hand side of (8.4) then equals

$$\begin{aligned}
&\sum_{s \in \mathrm{Orb}(S^0,S)} i(S_s) \\
&= \sum_{s \in \mathrm{Orb}(S^0,S)} e(S_s) \\
&= \sum_{s \in \mathrm{Orb}(S^0,S)} \sum_{t \in \mathrm{Orb}(S_s,S_s)} |\pi_0(S_{s,t})|^{-1} \sigma(S^0_{s,t}) \\
&= \sum_{\{(s,t) \in S \times S^0 : st = ts\}/S^0} |\pi_0(S_{s,t})|^{-1} \sigma(S^0_{s,t}) \\
&= \sum_{t \in \mathrm{Orb}(S^0,S^0)} \sum_{s \in \mathrm{Orb}(S_t,S^t)} |\pi_0(S_{s,t})|^{-1} \sigma(S^0_{s,t}) \\
&= \sum_{t \in \mathrm{Orb}(S^0,S^0)} e(S^t) \,.
\end{aligned}$$

This last expression would just be the right hand side of (8.4) if $e(S^t)$ were replaced by $i(S^t)$. But if t does not belong to $Z(S)$, $\dim(S^t)$ is smaller than $\dim(S)$, and $e(S^t)$ equals $i(S^t)$ by our induction assumption. It t belongs to $Z(S)$, S^t is just S itself. Therefore, the equality of $e(S)$ and $i(S)$ is indeed equivalent to the identity (8.4). $\square$

It remains for us to establish the formula (8.4), in which S is a component such that S^0 is semisimple and simply connected. We shall deal with each side separately. According to [**24, Theorem 7.5**], any semisimple element in S normalizes some pair (T_1, B_1) of groups, where T_1 is a maximal torus in S^0 and B_1 is a Borel subgroup of S^0 which contains T_1. Let B be a fixed Borel subgroup of S^0 which contains our fixed maximal torus T. Then any semisimple orbit of S^0 in S contains an element which normalizes T and B. The normalizer of T and B in S can be written Tw_B, where w_B is a fixed semisimple element in S which preserves some splitting of B. Let S' and T' denote the centralizers of w_B in S^0 and T. Since S^0 is simply connected, S' is a connected reductive group which contains T' as a maximal torus. In particular, we can form the usual sign character ε' on the Weyl group W' of (S', T').

LEMMA 8.4. *The left hand side of (8.4) equals the number*

$$\Delta(W', \varepsilon') = |W'|^{-1} \sum_{w \in W'_{\mathrm{reg}}} \varepsilon'(w) . \tag{8.5}$$

PROOF: Let N' denote the normalizer of T' in S'. Then

$$W' = N'/T' \cong TN'/T .$$

We claim that

$$\mathrm{Norm}(T', S^0) = TN' . \tag{8.6}$$

To see this, we shall consider the (open) chambers in the real vector spaces

$$\underline{\mathfrak{a}}_{T'} = \mathrm{Hom}(X(T'), \mathbf{R}) \subset \underline{\mathfrak{a}}_T = \mathrm{Hom}(X(T), \mathbf{R})$$

determined by the roots of S' and S^0. The fact that w_B preserves a splitting in (B, T) implies that the simple roots of $(B \cap S', T')$ are just the orbits under powers of $\mathrm{ad}(w_B)$ of the simple roots of (B, T). The corresponding positive chambers are therefore related by $\underline{\mathfrak{a}}^+_{T'} = \underline{\mathfrak{a}}_{T'} \cap \underline{\mathfrak{a}}^+_T$. By an argument of symmetry, any chamber in $\underline{\mathfrak{a}}_{T'}$ becomes the intersection of $\underline{\mathbf{a}}_{T'}$ with a uniquely determined chamber in $\underline{\mathbf{a}}_T$. Suppose that n is an element in $\mathrm{Norm}(T', S^0)$. Then n also normalizes T, since T is the centralizer of T' in S^0. The chamber $\mathrm{Ad}(n)(\underline{\mathbf{a}}_T^+)$ contains an open subset of $\underline{\mathbf{a}}_{T'}$, and therefore a chamber

$$\mathrm{Ad}(n')(\underline{\mathbf{a}}_{T'}^+) , \qquad\qquad n' \in N' ,$$

in $\underline{\mathbf{a}}_{T'}$. The map $\mathrm{Ad}(n')^{-1}\mathrm{Ad}(n)$ will then send the chambers $\underline{\mathbf{a}}_{T'}^+$ and $\underline{\mathbf{a}}_T^+$ to themselves. This justifies the claim (8.6).

We have agreed that any semisimple orbit of S^0 in S intersects Tw_B. Suppose that two elements s_1 and s in Tw_B are S^0-conjugate. Since T' is a maximal torus of both S_s and S_{s_1}, s_1 and s are conjugate by an element in the group (8.6). From this it follows that there is a canonical bijection from $\mathrm{Orb}(TN', Tw_B)$ to the semisimple elements in $\mathrm{Orb}(S^0, S)$. It is of course only semisimple orbits which are relevant to (8.4). We can therefore write the left hand side of (8.4) as

$$\begin{aligned} &\sum_{s \in \mathrm{Orb}(S^0, S)} i(S_s) \\ &= \sum_{s \in \mathrm{Orb}(TN', Tw_B)} i(S_s) \\ &= \sum_{s \in \mathrm{Orb}(TN', Tw_B)} |W(S_s)|^{-1} \sum_{w \in W(S_s)_{\mathrm{reg}}} \varepsilon_s(w) |\det(w-1)|^{-1} , \end{aligned}$$

where ε_s stands for the sign character on the Weyl group $W(S_s)$ of S_s.

If s belongs to Tw_B, S_s need not be a subgroup of S'. However, the elements in $W(S_s)$ normalize T', and are induced from the group (8.6). Therefore

$$W(S_s) = \mathrm{Cent}(s, N')/T' \cong T\mathrm{Cent}(s, N')/T .$$

In particular, $W(S_s)$ is a subgroup of W'. Thus, the simple reflections in $W(S_s)$ are also reflections in W', and since ε_s and ε' both take the value (-1) on any such reflection, we see that ε_s equals the restriction of ε' to $W(S_s)$. We can substitute this into the expression above. Our characterization of $W(S_s)$ also suggests that we should change the sum over $\mathrm{Orb}(TN', Tw_B)$ to a sum over the smaller set

$$\bar{T}_B = \mathrm{Orb}(T, Tw_B) = \{t^{-1} w_B t w_B^{-1} : t \in T\} \backslash Tw_B .$$

The expression for the left hand side of (8.4) becomes

$$|W'|^{-1} \sum_{s \in \bar{T}_B} \sum_{w \in W(S_s)_{\mathrm{reg}}} \varepsilon'(w) |\det(w-1)|^{-1} .$$

The group W' operates on $\bar{T}_B$. It is easy to check that

$$W(S_s)_{\mathrm{reg}} = \{w \in W'_{\mathrm{reg}} : w(s) = s\} .$$

The last expression can therefore be written

$$|W'|^{-1} \sum_{\{(s,w) \in \bar{T}_B \times W'_{\mathrm{reg}} : w(s)=s\}} \varepsilon'(w) |\det(w-1)|^{-1} .$$

Now T' is a finite covering of the torus

$$\{t^{-1} w_B t w_B^{-1} : t \in T\} \backslash T ,$$

and $|\det(w-1)|$ equals the number of fixed points of w in either torus. In particular, this number equals the order of the fixed point set $\bar{T}_B^w$ of w in $\bar{T}_B$. We can therefore write our expression as

$$\begin{aligned} & |W'|^{-1} \sum_{w \in W'_{\mathrm{reg}}} \sum_{s \in \bar{T}_B^w} \varepsilon'(w) |\bar{T}_B^w|^{-1} \\ & = |W'|^{-1} \sum_{w \in W'_{\mathrm{reg}}} \varepsilon'(w) \\ & = \Delta(W', \varepsilon') . \qquad \Box \end{aligned}$$

LEMMA 8.5. *The right hand side of (8.4) equals the number*

$$\Delta(W, \varepsilon) = |W^0|^{-1} \sum_{w \in W_{\mathrm{reg}}} \varepsilon(w) . \tag{8.7}$$

PROOF: Since any semisimple conjugacy class in S^0 meets T, we have a bijection from $\mathrm{Orb}(W^0, T)$ to the set of semisimple elements in $\mathrm{Orb}(S^0, S^0)$. The right hand side of (8.4) can then be written

$$\begin{aligned} & \sum_{t \in \mathrm{Orb}(S^0, S^0)} i(S^t) \\ & = \sum_{t \in \mathrm{Orb}(W^0, T)} |W(S_t)|^{-1} \sum_{w \in W(S^t)_{\mathrm{reg}}} \varepsilon^t(w) |\det(w-1)|^{-1} , \end{aligned}$$

where ε^t stands for the sign character on the Weyl set $W(S^t)$.

We need only consider elements $t \in T$ such that S^t is not empty. For any such t, T is a maximal torus in the connected group $(S^t)^0 = S_t$, and $W(S^t)$ is a subset of $W = W(S)$. We claim that ε^t is the restriction of ε to $W(S^t)$. The group $W(S_t)$ is generated by reflections which lie in $W(S^0)$. Since this group acts simply transitively on $W(S^t)$, it suffices to check that ε and ε^t coincide on one element in $W(S^t)$. Let s be a semisimple element in S^t. Then there is a conjugate

$$s_1 = gsg^{-1}, \qquad g \in S^0,$$

of s which lies in Tw_B. We can in fact choose g so that $t_1 = gtg^{-1}$ lies in the maximal torus T' of S_{s_1}. It then follows that t_1 equals $w_1(t)$ for some $w_1 \in W^0$, and that t is fixed by the element $w_1^{-1}w_Bw_1$ in $W(S)$. In other words, $w_1^{-1}w_Bw_1$ belongs to $W(S^t)$. Since this element normalizes the Borel subgroups $w_1^{-1}Bw_1$ and $w_1^{-1}Bw_1 \cap S_t$ of S^0 and S_t, we have

$$\varepsilon(w_1^{-1}w_Bw_1) = \varepsilon^t(w_1^{-1}w_Bw_1) = 1 .$$

This establishes the claim.

Since $W(S_t)$ is the centralizer of t in W^0, the right hand side of (8.4) becomes

$$|W^0|^{-1}\sum_{t\in T}\ \sum_{w\in W(S^t)_{\text{reg}}} \varepsilon(w)|\det(w-1)|^{-1} .$$

The set $W = W(S)$ operates on T, and

$$W(S^t)_{\text{reg}} = \{w \in W_{\text{reg}} : w(t) = t\} .$$

The last expression can then be written

$$\begin{aligned}
&|W^0|^{-1} \sum_{\{(t,w)\in T\times W_{\text{reg}}:w(t)=t\}} \varepsilon(w)|\det(w-1)|^{-1}\\
&= |W^0|^{-1} \sum_{w\in W_{\text{reg}}}\ \sum_{t\in T^w} \varepsilon(w)|T^w|^{-1}\\
&= |W^0|^{-1} \sum_{s\in W_{\text{reg}}} \varepsilon(w)\\
&= \Delta(W,\varepsilon) ,
\end{aligned}$$

since $|\det(w-1)|$ equals the order of the fixed point set T^w of w in T. The lemma is proved. □

LEMMA 8.6. $\Delta(W',\varepsilon') = \Delta(W,\varepsilon)$.

PROOF: The numbers $\Delta(W',\varepsilon')$ and $\Delta(W,\varepsilon)$ depend only on the Weyl set W. They are independent of the isogeny class of the underlying component S. We shall assume inductively that the required formula holds if S is replaced by a component of strictly smaller dimension.

We have the fixed Borel subgroups B and B' of S^0 and S', so we can speak of standard parabolic subgroups. Suppose that A is a standard torus in T'. In other words, A is the split component of a parabolic subgroup P' of S' which contains B'. Let M' be the Levi component of P' which contains T'. Then A equals $A_{M'} = Z(M')^0$, the connected component of the center of M'. Write

$$W'_A = W(M'/A)$$

for the Weyl group of M'/A, acting on T'/A. We can also take the centralizer M of A in S. Then M^0 is the Levi component of a standard parabolic subgroup of S^0. Write

$$W_A = W(M/A) ,$$

for the Weyl set of the component M, acting on T/A. The element w_B obviously embeds into W_A, and W'_A is just the centralizer of w_B in $W^0_A = W(M^0/A)$. If A is nontrivial, our induction hypothesis tells us that

$$\Delta(W'_A,\varepsilon'_A) = \Delta(W_A,\varepsilon_A) , \tag{8.8}$$

where ε'_A and ε_A are the sign characters on W'_A and W_A.

Suppose that w is an arbitrary element in W'. The identity component of the fixed point set $(T')^w$ is a torus in T', and equals a W'-translate

$$w_1^{-1}(A) , \qquad\qquad w_1 \in W' ,$$

of a standard torus A in T'. The element $w_1 w w_1^{-1}$ then lies in $W'_{A,\mathrm{reg}}$. It is also clear that

$$\varepsilon'(w) = \varepsilon'_A(w_1 w w_1^{-1}) .$$

Now the pair (A, w_1) is not uniquely determined by w. The number of such pairs actually equals

$$n(A)|W'_A| ,$$

where $n(A)$ is the number of chambers in $\mathrm{Hom}(X(A), \mathbf{R})$ cut out by the hyperplanes orthogonal to the roots of the corresponding standard parabolic subgroup. The elements in W' can be enumerated up to this ambiguity, however, as conjugates

$$w_1^{-1} w_A w_1 , \qquad w_A \in W'_{A,\mathrm{reg}} , \ w_1 \in W' .$$

We obtain

$$\begin{aligned} &|W'|^{-1} \sum_{w \in W'} \varepsilon'(w) \\ &= \sum_A n(A)^{-1} |W'_A|^{-1} \sum_{w_A \in W'_{A,\mathrm{reg}}} \varepsilon'_A(w_A) \\ &= \sum_A n(A)^{-1} \Delta(W'_A, \varepsilon'_A) . \end{aligned}$$

If W' is not equal to $\{1\}$, the sign character ε' is nontrivial, and the left hand side of the equation equals 0. Applying (8.8) to the right hand side, we conclude that the expression

$$\Delta(W', \varepsilon') + \sum_{A \neq \{1\}} n(A)^{-1} \Delta(W_A, \varepsilon_A) \tag{8.9}$$

vanishes if $W' \neq \{1\}$.

Now suppose that w is an arbitrary element in W. The identity component $(T^w)^0$ of the set of fixed points of w in T is a torus which commutes with any representative in S of the Weyl element w. Copying an argument from the proof of Lemma 8.5, we see that

$$(T^w)^0 = w_1^{-1}(A) ,$$

where w_1 belongs to W^0 and A is a torus in T'. In fact, we can assume that the centralizer M^0 of A in S^0 is the Levi component of a standard parabolic subgroup of S^0. This implies that A is standard torus in T'. The element $w_1 w w_1^{-1}$ then lies in $W_{A,\mathrm{reg}}$, and

$$\varepsilon(w) = \varepsilon_A(w_1 w w_1^{-1}) .$$

For a given w, how many such pairs (A, w_1) are there? We can certainly replace w_1 by a product

$$w' w_M w_1 \ , \qquad\qquad w' \in W', \ w_M \in W(M^0),$$

in which w' maps A to another standard torus in T'. However, this is the only possible ambiguity, so the number of pairs equals

$$n(A)|W(M^0)| \ = \ n(A)|W_A| \ .$$

We obtain

$$\begin{aligned} & |W^0|^{-1} \sum_{w \in W} \varepsilon(w) \\ & = \sum_A n(A)^{-1} |W_A|^{-1} \sum_{w_A \in W_{A,\mathrm{reg}}} \varepsilon_A(w_A) \\ & = \sum_A n(A)^{-1} \Delta(W_A, \varepsilon_A) \ . \end{aligned}$$

If W contains more than just the one element w_B, the left hand side of the equation equals 0. Therefore, the expression

$$\Delta(W, \varepsilon) + \sum_{A \neq \{1\}} n(A)^{-1} \Delta(W_A, \varepsilon_A) \tag{8.10}$$

vanishes if $W \neq \{w_B\}$.

The simple reflections in W' correspond to the orbits of simple roots of (B, T) under powers of $\mathrm{ad}(w_B)$. It follows that $W' = \{1\}$ if and only if $W = \{w_B\}$. In this case, both (8.9) and (8.10) are trivially equal to 1. We can therefore conclude that the expressions (8.9) and (8.10) are equal. The equality of $\Delta(W', \varepsilon')$ and $\Delta(W, \varepsilon)$ then follows. □

We have reached the end of the lemmas that make up the proof of Theorem 8.1. We obtain the general inequality of $i(S)$ with $e(S)$ immediately by combining Lemmas 8.3, 8.4, 8.5 and 8.6. The proof of Theorem 8.1 is now complete. □

§9. Concluding remarks

Theorem 8.1 tells us that the coefficients $i(x)$ and $e(x)$ in (5.10) and (7.15) are always equal. It follows that there is a term by term identification of the expansions for $I_{\mathrm{disc},t}(f)$ and $E_{\mathrm{disc},t}(f)$. We conclude that Hypothesis 3.1 is a consequence of Hypothesis 4.1 (together with the local assumptions of §3, §4 and [**3**, §**7**]). This was the task we originally set for ourselves.

We have in fact shown that the contributions to $I_{\mathrm{disc},t}(f)$ and $E_{\mathrm{disc},t}(f)$ of each parameter $\psi \in \Psi(G, \chi, t)$ are equal. Now there are some parameters for which the representation theoretic hypotheses are known. Consider the special case that G is a connected quasi-split group. Suppose that ψ is the image of a parameter $\psi_0 \in \Psi(M_0)$ for a minimal Levi subgroup M_0 of G. Since M_0 is a maximal torus in this case, ψ_0 is trivial on $SL(2, \mathbb{C})$, and is the parameter of a unitary character on $M_0(F)\backslash M_0(\mathbb{A})$. We can take Π_ψ to be the set of irreducible constituents of the corresponding induced representation of $G(\mathbb{A})$. The parameter ψ_0 factors through the quotient W_F of L_F, so there is no problem with the hypothetical Langlands group. In particular, S_ψ equals the centralizer in $\hat{G}$ of the image of W_F, and the quotient $\mathcal{S}_\psi$ is just the R-group R_ψ. The pairing on $\mathcal{S}_\psi \times \Pi_\psi$ is then determined by the global normalized intertwining operators. In fact, Conjecture 7.1 of [**3**], which we assumed in §5, is already known in this case thanks to Keys and Shahidi [**10, Theorem 5.1**]. If H_1 is associated to a point in a component $x \in \mathcal{S}_\psi$, we could just define the distribution $f \to f^{H_1}(\psi_1)$ by

$$f^{H_1}(\psi_1) = \sum_{\pi \in \Pi_\psi} < x, \pi > f_G(\pi) .$$

Then with these interpretations, the notions that went into the discussion in §5–§8 are all understood. The reader who dislikes arguments based on unproven conjectures can regard the earlier discussion as pertaining only to the parameters just described. It establishes that the contribution of these parameters to

$$E_{\mathrm{disc},t}(f) - I_{\mathrm{disc},t}(f) \tag{9.1}$$

vanishes.

This paper has concerned the conjectures in [**3**] on unipotent (and more general) automorphic representations. The long term goal is to prove them, at least in part, with the help of endoscopy and the trace formula. A first step towards the creation of a logical structure for the argument is to verify the compatibility of the notions involved and to analyze the reasons for it. This has been our emphasis, and we continue with some informal comments on the proof envisaged.

In general, Hypothesis 3.1 asserts the vanishing of the distributions (9.1). As we mentioned earlier, one should first try to deduce this from the trace formula. One would then use (9.1) to establish some version of the multiplicity formula (4.7). The formula could be assumed inductively for any proper Levi-subgroup. This would permit the application of the arguments in §5–§8 to any parameter $\psi \in \Psi(G, \chi, t)$ which is not the image of an *elliptic* parameter for an *elliptic* endoscopic datum. The contribution to (9.1) of all such parameters could then be shown to vanish. The only remaining contribution to (9.1) would come from parameters ψ such that $\bar{S}_{\psi,s}$ is finite for some element s in $\bar{S}_\psi = S_\psi/Z(\hat{G}^0)$. It is from this that we would hope to deduce some form of (4.7), again using arguments of Sections 5, 6 and 7. The sign characters ε_ψ^G would be forced on us at this stage, essentially because of Proposition 5.1.

Of course, it would not be feasible to apply the arguments of §5–§8 in precisely the way they were presented here. The correspondence from maps $W_F \to {}^LG$ to automorphic representations is much deeper than multiplicity formulas such as (4.7), and in any case, we would certainly not want to assume the existence of the Langlands group L_F. We would instead have to replace the parameters ψ by the families $\sigma = \{\sigma_v : v \notin S\}$ of conjugacy classes in ${}^LG$ attached to automorphic representations. (See [**3**, §**1**, §**8**].) For many G we can expect a bijection from $\Psi(G)$ onto the set $\Sigma(G)$ of such families. In these cases, the idea would be to define the centralizer S_ψ in terms of σ. This could probably be done by considering the set of endoscopic groups H for which σ lies in the image of the map $\Sigma(H) \to \Sigma(G)$. It is of course necessary to determine S_ψ in order to state the multiplicity formula (4.7). By definition, a parameter ψ has a Jordan decomposition $(\psi_{ss}, \psi_{\mathrm{unip}})$, where

$$\psi_{ss} : L_F \longrightarrow {}^LG$$

and

$$\psi_{\mathrm{unip}} : SL(2, \mathbf{C}) \longrightarrow S^0_{\psi_{ss}} .$$

We would describe the Jordan decomposition in terms of σ by first determining the family σ_{ss} attached to ψ_{ss}, and then describing the group $S^0_{\psi_{ss}}$ in terms of σ_{ss}. In the case of a general G, some understanding of the fibres of the map $\Psi(G) \to \Sigma(G)$ will probably be needed.

We have not said much about the local side of the conjectures. This includes the definition of the stable distributions $f_1 \to f_1^{G_1}(\psi_1)$, the construction of the packets Π_ψ and the pairing $< \cdot, \cdot >$, and the proof of the local character identity (4.4). Once the stable distributions have been defined, the packets and the pairing are determined by (4.4) (together with the maps $f \to f^{H_1}$). The essential part of the local conjecture is then the assertion that for a given ψ, certain linear combinations of the distributions

$$f \longrightarrow f^{H_1}(\psi_1), \qquad f \in C_c^\infty(G(\mathbb{A}), \chi),$$

are actually characters, as opposed to more general invariant distributions. Ideally, it would be best to deduce this locally. However, the global Hypothesis 3.1 itself carries some local information. For it ultimately implies some version of (4.7), and any such multiplicity formula tells us that certain distributions are in fact characters. I do not know how far this can be pushed. It is perhaps best to wait until Hypothesis 3.1 has actually been established.

The case in which Hypothesis 3.1 will lead to the most complete results is the example of outer twisting of $GL(n)$. The hypotheses of §4 (interpreted without reliance on the parameters $\psi \in \Psi(G)$) are now known for $GL(n)$. Moeglin and Waldspurger **[21]** have recently characterized the residual discrete spectrum for $GL(n)$ in terms of the cuspidal spectrum, and it is clear how to interpret this in terms of the Jordan decomposition **[3**, §**2]**. On the other hand, the twisted endoscopic groups for $GL(n)$ include all of the quasi-split classical groups of type B, C and D (up to isogeny). One should try to deduce the conjectural properties of the spectra of these classical groups from what is known for $GL(n)$. We will conclude with a very brief discussion of this example.

Set $G^0 = GL(n)$. If

$$J_n = \left.\begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}\right\} n,$$

then

$$\theta_n(g) = J_n^{-1}({}^t g^{-1}) J_n, \qquad g \in G^0,$$

is an outer automorphism of G^0 which leaves invariant the standard Borel subgroup. Set

$$G = G^0 \rtimes \theta_n .$$

If $\hat{\theta}_n$ denotes the same outer automorphism of $\hat{G}^0 = GL(n, \mathbf{C})$, then

$$\hat{G} = \hat{G}^0 \rtimes \hat{\theta}_n .$$

It is easy to describe the elliptic endoscopic data for G. For each integer r, with $1 \leq r \leq \frac{n}{2}$, set

$$s_r = \begin{array}{c}\\ r\left\{\begin{array}{c}\\ \\ \\ \end{array}\right. \\ \\ \\ \\ \\ \end{array}\begin{pmatrix} -1 & & & & & & 0 \\ & \ddots & & & & & \\ & & -1 & & & & \\ & & & 1 & & & \\ & & & & \ddots & & \\ & & & & & \ddots & \\ 0 & & & & & & 1 \end{pmatrix} \rtimes \hat{\theta}_n ,$$

regarded as a semisimple coset in $\hat{G}/Z(\hat{G}^0)$. Then

$$\mathrm{Cent}(s_r, \hat{G}^0) \cong \mathrm{Sp}(2r, \mathbf{C}) \times O(n-2r, \mathbf{C}) .$$

Define

$$\hat{H}_r = \mathrm{Cent}(s_r, \hat{G}^0)^0 \cong \mathrm{Sp}(2r, \mathbf{C}) \times SO(n-2r, \mathbf{C}) .$$

Let ξ_r be any L-homomorphism

$$W_F \longrightarrow \mathrm{Cent}(s_r, \hat{G}^0) \times W_F \subseteq \hat{G}^0 \times W_F .$$

This determines an endoscopic datum $(H_r, \mathcal{H}_r, s_r, \xi_r)$ whose equivalence class depends only on the map

$$\xi_r^* : \mathrm{Gal}(\bar{F}/F) \longrightarrow \pi_0(\mathrm{Cent}(s_r, \hat{G}^0)) .$$

Thus,

$$H_r \cong \begin{cases} SO(2r+1) \times SO^*(n-2r), & n \text{ even}, \\ SO(2r+1) \times \mathrm{Sp}(n-1-2r), & n \text{ odd}, \end{cases}$$

where SO^* stands for the quasi-split orthogonal group determined by ξ_r^*. When $n - 2r = 2$ and ξ_r^* is trivial, $Z(\hat{H}_r)^\Gamma$ is infinite, and the endoscopic datum is not elliptic. If we rule out this exceptional case, however, we obtain a set of representatives of elliptic endoscopic data. Observe that $\mathcal{H}_r$ can be identified with ${}^LH_r$ in each case, so there is no need to introduce the extensions that were denoted by H_1 in §2.

Suppose that

$$\psi : L_F \times SL(2, \mathbf{C}) \longrightarrow {}^LG^0$$

is a parameter in $\Psi(G^0)$. We shall identify ψ with an n-dimensional representation of the group $L_F \times SL(2, \mathbf{C})$, which can then be decomposed into a direct sum

$$\psi = \bigoplus_{k=1}^{\ell} \psi_k$$

of irreducible representations. The centralizer in $\hat{G}^0$ of the image of ψ is the group of intertwining operators. That is,

$$S_\psi(G^0) = S_\psi^0 \cong \prod_j GL(m_j, \mathbf{C}) .$$

The parameter ψ belongs to $\Psi(G)$ if and only if it is self-contragredient as a representation of $L_F \times SL(2, \mathbf{C})$. In other words, the contragredient operation acts as a permutation of order two on the irreducible constituents ψ_k. Suppose that this is the case. Then $S_\psi = S_\psi(G)$ is isomorphic to a product of components of the form

$$GL(m, \mathbf{C}) \rtimes \hat{\theta}_m$$

or

$$\big(GL(m, \mathbf{C}) \rtimes GL(m, \mathbf{C})\big) \times \hat{\tau}_m ,$$

with

$$\hat{\tau}_m(g_1, g_2) = \big(\hat{\theta}_m(g_2), \hat{\theta}_m(g_1)\big) , \qquad g_1, g_2 \in GL(m, \mathbf{C}) .$$

We are especially interested in the parameters $\psi \in \Psi(G)$ which are the images of elliptic parameters $\psi_r \in \Psi_0(H_r)$, for elliptic endoscopic data H_r. Since $A_{\hat{G}} \cong \mathbf{Z}/2\mathbf{Z}$ is finite, this means that there is an element $s \in S_\psi$ such that $S_{\psi,s}$ is finite. The condition is equivalent to

$$S_\psi \cong (\mathbf{C}^* \rtimes \hat{\theta}_1)^\ell , \tag{9.2}$$

which is to say that the irreducible constituents ψ_k of ψ are self-contragredient and mutually inequivalent. Since

$$\hat{\theta}_1(z) = z^{-1}, \qquad z \in \mathbb{C}^*,$$

we see immediately from (9.2) that there is only an orbit of S_ψ^0 in S_ψ. Therefore, ψ factors through only the one endoscopic datum H_r.

Fix an elliptic endoscopic datum H_r, and let $\psi_r \in \Psi_0(H_r)$ be a fixed elliptic parameter. The image ψ of ψ_r in $\Psi(G)$ then satisfies (9.2). For reasons of induction it is not necessary to consider a product of two classical groups, so we may assume that r equals 0 or $\frac{n}{2}$. Then H_r is either an orthogonal group or a symplectic group. To study the representations of $H_r(\mathbb{A})$ attached to ψ_r, it will be necessary to apply Hypothesis 3.1 to both G and H_r.

A missing ingredient from the local conjectures was a canonical definition of the stable distribution

$$(9.3) \qquad f_r \longrightarrow f_r^{H_r}(\psi_r), \qquad f_r \in C_c^\infty(H_r(\mathbb{A})).$$

Such a definition will be provided, at least in some cases, by the connection with G. The packet Π_ψ consists of one orbit $\{\pi_\psi\} \subseteq \Pi(G(\mathbb{A})^+)$ under the group $\pi_0(G^+)^* \cong \mathbb{Z}/2\mathbb{Z}$, and we can choose π_ψ so that

$$< \bar{s}_\psi \bar{s}, \pi_\psi > = < \bar{s}, \pi_\psi > = 1, \qquad \bar{s} \in \mathcal{S}_\psi.$$

It follows from (4.4) that

$$f^{H_r}(\psi_r) = f_G(\pi_\psi), \qquad f \in C_c^\infty(G(\mathbb{A})).$$

A similar formula holds for the corresponding stable distributions on the local groups $G(F_v)$. However, this formula may not determine (9.3) completely. The problem is that the anticipated injection

$$\{f^{H_r} : f \in C_c^\infty(G(\mathbb{A}))\} \hookrightarrow \{f_r^{H_r} : f_r \in C_c^\infty(H(\mathbb{A}))\},$$

obtained by transfer of twisted orbital integrals, could be a strict inclusion. This difficulty is tied up with the question of how many local parameters

$$\psi_r' = \bigotimes_v \psi_{r,v}', \qquad \psi_{r,v}' \in \Psi(H/F_v),$$

lift to ψ. If $\hat{H}_r$ is symplectic or odd orthogonal, the only such parameter will be ψ_r itself. However, there can be a number of ψ'_r in the even orthogonal case, and the formula then determines only a sum of distributions (9.3).

Once the distribution (9.3) has been defined (for H_r and its endoscopic groups), the packet Π_{ψ_r} and the pairing on $\mathcal{S}_{\psi_r} \times \Pi_{\psi_r}$ will be uniquely determined. Leaving aside the question of whether the required local properties of these objects can be deduced from Hypothesis 3.1, let us simply assume that the local assumptions of §3, §4 and [**3**, §**7**] hold for H_r. The next problem is to determine the stable distribution

$$SI^{H_r}_{\psi_r}(f_r) = \sigma(H_r, \psi_r) f^{H_r}_r(\psi_r) . \tag{9.4}$$

(See the notation of §7.) The distribution $f^{H_r}_r(\psi_r)$ is a local object which we are assuming is known, so it is the global constant $\sigma(H_r, \psi_r)$ which must be found. According to Hypothesis 3.1, we should take the contribution of ψ to (9.1), and set it equal to 0. I have not thought through the details, but it should just be a question of running backwards over a couple of the more trivial arguments of §5 and §7. The result will be a special case

$$\sigma(H_r, \psi_r) = |\mathcal{S}_{\psi_r}|^{-1} \varepsilon^{H_r}_{\psi_r}(\bar{s}_{\psi_r}) \tag{9.5}$$

of the general formula (7.12) we determined was compatible with Hypothesis 4.1. Observe that the sign character $\varepsilon^{H_r}_{\psi_r}(\bar{s}_{\psi_r})$ appears. It originates, through [**3, Conjecture 7.1**] and Proposition 5.1, from the normalizing factors for (nontempered) intertwining operators for $GL(n)$.

Having determined the stable distributions (9.4) (for H_r and its endoscopic groups), we can apply Hypothesis 3.1 to H_r. The contribution of ψ_r to $E_{\mathrm{disc},t}(f_r)$ can be calculated as an easy special case of the arguments in §7, or it can simply be read off from the formula (7.15), (applied to H_r instead of G). It equals

$$\sum_{\pi \in \Pi_{\psi_r}} \Big(|\mathcal{S}_{\psi_r}|^{-1} \sum_{x \in \mathcal{S}_{\psi_r}} \varepsilon^{H_r}_{\psi_r}(x) < x, \pi > \Big) \mathrm{tr}\big(\pi(f_r)\big) .$$

On the other hand, the parameter $\psi_r \in \Psi_0(H_r)$ is elliptic. Its contribution to $I_{\mathrm{disc},t}(f_r)$ equals

$$\sum_{\pi \in \Pi_{\psi_r}} m_0(\pi) \mathrm{tr}\big(\pi(f_r)\big) .$$

Identifying the coefficients in these two linear combinations of irreducible characters, we obtain the multiplicity formula

$$m_0(\pi) = |\mathcal{S}_{\psi_r}|^{-1} \sum_{x \in \mathcal{S}_{\psi_r}} \varepsilon_{\psi_r}^{H_r}(x) < x, \pi >$$

Notice that the only contribution to $m_0(\pi)$ should come from the parameter ψ_r. This suggests that the map $\Psi(H_r) \to \Sigma(H_r)$ is bijective, at least if H_r is not an even orthogonal group, the case we left ambiguous.

This discussion has been very sketchy. We have simply tried to indicate that since the spectrum of $GL(n)$ can be understood in terms of a Jordan decomposition, the same should be true for the spectrum of its endoscopic groups. The arguments of §5–§8 will be essential for this, in that they allow for the elimination of the irrelevant parameters from the study of (9.1).

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