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The L^2-Lefschetz numbers of Hecke operators

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Contents

Introduction

Suppose that G is a semisimple Lie group and that Γ is a discrete subgroup of G. We assume that Γ is an arithmetic subgroup defined by congruence conditions, and for simplicity, suppose also that G is contained in a simply connected complex group. A fundamental problem is to decompose the regular representation of G on $L^2(\Gamma\backslash G)$ into irreducible representations. In particular, if π is an irreducible representation of G, one could try to compute the multiplicity, $m_{\mathrm{disc}}(\pi)$, with which π occurs discretely in $L^2(\Gamma\backslash G)$. This is probably too much to ask in general. However, if π belongs to the integrable discrete series of G, and $\Gamma\backslash G$ is compact, there is a finite closed formula for $m_{\mathrm{disc}}(\pi)$ [13a]. We shall consider the corresponding question when $\Gamma\backslash G$ is not compact

If $\Gamma\backslash G$ is noncompact, the problem is complicated considerably by the existence of a continuous spectrum. In fact, the present state of the trace formula allows us only to answer a weaker question. The discrete series for G is a disjoint union of finite subsets $\Pi_{\mathrm{disc}}(\mu)$, parametrized by irreducible finite dimensional representations μ of G. In this paper, we shall find a formula for the sum

$$\sum_{\pi\in\Pi_{\mathrm{disc}}(\mu)} m_{\mathrm{disc}}(\pi) \tag{1}$$

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under a weak regularity assumption on the representations in $\Pi_{\mathrm{disc}}(\mu)$. The packet $\Pi_{\mathrm{disc}}(\mu)$ consists of the set of discrete series representations with the same infinitesimal character as μ. Therefore, the question of the sum of the multiplicities is quite natural from the point of view of spectral theory. More generally, we shall consider Hecke operators h on $L^2(\Gamma\backslash G)$. Any such operator commutes with the action of G. Its restriction $R_{\mathrm{disc}}(\pi, h)$ to the subspace that decomposes discretely according to π can therefore be identified with an $(m_{\mathrm{disc}}(\pi) \times m_{\mathrm{disc}}(\pi))$-matrix. We shall find a formula for

$$\sum_{\pi \in \Pi_{\mathrm{disc}}(\mu)} \mathrm{tr}\,(R_{\mathrm{disc}}(\pi, h)), \tag{2}$$

under the same regularity condition. The question is again an obvious one in the context of spectral theory.

The expressions (1) and (2) have a cohomological interpretation. Assume that Γ has no elements of finite order. Then if K is a maximal compact subgroup of G,

$$X_\Gamma = \Gamma\backslash G/K$$

is a locally symmetric Riemannian manifold which is in general not compact. Given the finite dimensional representation μ, acting on the complex vector space V_μ, one can form the locally constant sheaf

$$\mathscr{F}_\mu = V_\mu \, x_\Gamma (G/K)$$

on X_Γ. One then has the L^2-cohomology groups

$$H^q_{(2)}(X_\Gamma, \mathscr{F}_\mu),$$

with coefficients in $\mathscr{F}_\mu$. We are assuming that G has a discrete series, and the cohomology groups are known to be finite dimensional under this condition. Any Hecke operator h for Γ gives an linear map

$$H^q_{(2)}(h, \mathscr{F}_\mu)\colon H^q_{(2)}(X_\Gamma, \mathscr{F}_\mu) \to H^q_{(2)}(X_\Gamma, \mathscr{F}_\mu).$$

Consider the expressions

$$\sum_q (-1)^q \dim\,(H^q_{(2)}(X_\Gamma, \mathscr{F}_\mu)) \tag{1*}$$

and

$$\sum_q (-1)^q \,\mathrm{tr}\,(H^q_{(2)}(h, \mathscr{F}_\mu)). \tag{2*}$$

If the highest weight of μ is regular, it turns out that (1*) and (2*) are equal to the product of $(-1)^{\frac{1}{2}\dim(X_\Gamma)}$ with the respective expressions obtained from (1) and (2) by replacing μ with its contragredient $\tilde{\mu}$. We will therefore obtain formulas for (1*) and (2*). However, with this cohomological interpretation, the formulas are actually valid without the regularity assumption on μ. In particular, they will hold if μ is the trivial representation. We will thus obtain a formula for the L^2-Euler characteristic of X_Γ and, more generally, the L^2-Lefschetz number of any Hecke operator.

It is easier to state (and prove) the formulas if we work over the adèles. In the paper we shall take G to be a reductive algebraic group over $\mathbb{Q}$. For the rest of the introduction, assume that G is a semisimple, simply connected group over $\mathbb{Q}$, such that $G(\mathbb{R})$ has no compact simple factors.

The locally symmetric space X_Γ can then be recovered as a double coset space

$$X_{K_0} = G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_{\mathrm{fin}}))/K_0,$$

where K_0 is an open compact subgroup of the finite adèle group $G(\mathbb{A}_{\mathrm{fin}})$, and

$$X = G(\mathbb{R})/K_{\mathbb{R}}$$

is the associated globally symmetric space. The Hecke operators are elements in the algebra $\mathscr{H}_{K_0}$ of compactly supported functions on $G(\mathbb{A}_{\mathrm{fin}})$ which are bi-invariant under K_0. Our formula for the Lefschetz number (2*) is

$$\sum_M (-1)^{\dim(A_M)} |W_0^M| \, |W_0^G|^{-1} \sum_{\gamma \in (M(\mathbb{Q}))} \chi(M_\gamma)\, \Phi_M(\gamma, \mu)\, h_M(\gamma). \tag{3}$$

The outer sum is over the set of Levi subgroups M of G which contain a fixed minimal one. The function $\Phi_M(\gamma, \mu)$ is perhaps the most interesting ingredient in the formula. It vanishes if γ is not $\mathbb{R}$-elliptic in M, but it is built out of discrete series characters if γ is $\mathbb{R}$-elliptic in M. The function $h_M(\gamma)$ is essentially an orbital integral of h at γ. If γ is any element in $M(\mathbb{Q})$, M_γ is the centralizer of γ in M, a group which is connected if G is simply connected and γ is semisimple. Finally, $\chi(M_\gamma)$ is a simple constant which is closely related to the (classical) Euler characteristics of the locally symmetric spaces of M_γ. The sums in (3) are both finite, and the terms can be written down explicitly, at least in principle.

The formula (3) will be derived from the trace formula. We shall first show that the Lefschetz number (2*) equals

$$\operatorname{tr}(R_{\mathrm{disc}}(f_\mu h)),$$

where R_{disc} is the subrepresentation of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ which decomposes discretely, and f_μ is a certain function on $G(\mathbb{R})$. This step involves fairly familiar ideas, and will be completed early in §3 (see (3.2)). In the latter part of §3 we shall apply the trace formula. Combined with the special properties of f_μ, it will provide an expansion

$$\sum_M |W_0^M| \, |W_0^G|^{-1} \sum_{\gamma \in (M(\mathbb{Q}))_{M,S}} a^M(S, \gamma)\, I_M^G(\gamma, f_\mu)\, \hat{I}_M^M(\gamma, h_M), \tag{4}$$

for the trace of $R_{\mathrm{disc}}(f_\mu h)$ (Proposition 3.2). It will remain for us to identify the terms in (4) with those of (3).

From the results of an earlier paper [1f], the function

$$I_M^G(\gamma, f_\mu), \qquad \gamma \in M(\mathbb{R}),$$

can be evaluated in terms of characters of discrete series whenever γ is G-regular. In §4 and §5, we shall study the case of singular γ. A key step will be to show that

$$I_M^G(\gamma, f_\mu) = 0, \tag{5}$$

if γ is not semisimple. When $M = G$, this is the real analogue of a property Kottwitz has established [12] for p-adic groups. To prove it we need a general result of Harish-Chandra on unipotent orbital integrals. Harish-Chandra's theorem is unfortunately not published, and since we must use it in an essential way, we have included a proof in the appendix. Given this result, we shall establish the property (5) as part of a general formula for $I_M(\gamma, f_\mu)$ in §5 (Theorem 5.1). We will then be able to restrict the sum over γ in (4) to semisimple elements. In the earlier paper [1a], there is a simple formula for the constant $a^M(S, y)$, for any element γ in $M(\mathbb{Q})$ which is semisimple. This is all we shall need. We shall collect the various terms in §6, where they will be combined as our main formula (3) in Theorem 6.1.

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§ 1. Hecke operators and L^2-cohomology

Let G be a connected reductive algebraic group over $\mathbb{Q}$. We shall write A_G for the split component of the center of G. There are many locally symmetric spaces associated with G. First of all, fix a maximal compact subgroup $K_{\mathbb{R}}$ of $G(\mathbb{R})$, and set $K'_{\mathbb{R}} = K_{\mathbb{R}} A_G(\mathbb{R})^0$. Then

$$X = G(\mathbb{R})/K'_{\mathbb{R}}$$

is a globally symmetric space with respect to a fixed left $G(\mathbb{R})$-invariant metric. Let $\mathbb{A}_{\mathrm{fin}}$ be the ring of finite adèles of $\mathbb{Q}$, so that

$$\mathbb{A} = \mathbb{R} \times \mathbb{A}_{\mathrm{fin}}$$

is the full ring of adèles. The space

$$X \times G(\mathbb{A}_{\mathrm{fin}}) = G(\mathbb{A})/K'_{\mathbb{R}}$$

is equipped with a left $G(\mathbb{A})$-action, and a right $G(\mathbb{A}_{\mathrm{fin}})$ action. Locally symmetric spaces are attached to open compact subgroups K_0 of $G(\mathbb{A}_{\mathrm{fin}})$, through the spaces

$$\mathcal{M}_{K_0} = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_{\mathrm{fin}})/K_0.$$

It is these objects that we propose to study.

We shall work strictly over the adèles, but is would be easy to translate everything back into the language of real groups. The double coset space

$$G(\mathbb{Q})\backslash G(\mathbb{A})/G(\mathbb{R})K_0$$

is known to be finite. If

$$x_1 = 1, x_2, \ldots, x_n$$

is a set of representatives in $G(\mathbb{A}_{\mathrm{fin}})$ of the cosets, the groups

$$\Gamma_i = (G(\mathbb{Q}) \cdot x_i K_0 x_i^{-1}) \cap G(\mathbb{R}), \qquad 1 \leqq i \leqq n,$$

are arithmetic subgroups of $G(\mathbb{R})$. One sees immediately that $\mathcal{M}_{K_0}$ is the disjoint union of the spaces $\Gamma_i\backslash X$. Replacing K_0 by a subgroup of finite index if necessary, we can assume that Γ_i acts on X without fixed points. Then each $(\Gamma_i\backslash X)$ is a locally symmetric Riemannian manifold.

Define

$$\mathcal{M} = G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_{\mathrm{fin}})).$$

Then $\mathcal{M}$ is homeomorphic with

$$\varprojlim_{K_0} \mathcal{M}_{K_0},$$

a projective limit of manifolds. The advantage of considering $\mathcal{M}$ is that it has a right action under the group $G(\mathbb{A}_{\mathrm{fin}})$. Given $\mathcal{M}$, we can recover $\mathcal{M}_{K_0}$ as the space of fixed points under the open compact subgroup K_0.

We shall first recall some general properties of L^2-cohomology, after which we will be able to formulate our problem. We refer the reader to [5] and [6] for general references on L^2-cohomology, to [4] and [3] for the L^2-cohomology of locally symmetric spaces, and to [4] and [20] for relative Lie algebra cohomology.

The space $\mathcal{M}$ can obviously be equipped with a cotangent bundle $\mathcal{T}^*(\mathcal{M})$, so it makes sense to speak of differential forms on $\mathcal{M}$. More generally, we want to consider forms with values in a locally constant sheaf. Let (μ, V_μ) be a fixed finite dimensional representation of G which is irreducible and defined over $\mathbb{Q}$. Let

$$\mathscr{F}_\mu = V_\mu(\mathbb{C}) \underset{G(\mathbb{Q})}{\times} (X \times G(\mathbb{A}_{\mathrm{fin}})$$

be the corresponding locally constant sheaf on $\mathcal{M}$. A q-form with values in $\mathscr{F}_\mu$ is then a smooth section of the bundle

$$\Lambda^q \mathcal{T}^*(\mathcal{M}) \otimes \mathscr{F}_\mu.$$

which is right invariant under an open compact subgroup of $G(\mathbb{A}_{\mathrm{fin}})$. Observe that $G(\mathbb{A}_{\mathrm{fin}})$ acts by right translations on the vector space of such forms. We have already fixed a left invariant Riemannian metric on X. If we combine this with a fixed Haar measure on $G(\mathbb{A}_{\mathrm{fin}})$, we obtain a measure on $\mathcal{M}$. We also fix a Hermitian inner product on $V_\mu(\mathbb{C})$ which is invariant under $K_{\mathbb{R}}$. This allows us to speak of a square integrable form with values in $\mathcal{F}_\mu$. Let $A^q_{(2)}(\mathcal{M}, \mathcal{F}_\mu)$ be the space of q-forms ω on $\mathcal{M}$ with values in $\mathcal{F}_\mu$ which satisfy the following three conditions.

(i) ω is invariant under an open compact subgroup of $G(\mathbb{A}_{\mathrm{fin}})$.
(ii) ω is smooth.
(iii) ω and $d\omega$ are square integrable.

Then

$$A^*_{(2)}(\mathcal{M}, \mathcal{F}_\mu) = \bigoplus_{q \geqq 0} A^q_{(2)}(\mathcal{M}, \mathcal{F}_\mu)$$

is a differential complex. Its cohomology

$$H^*_{(2)}(\mathcal{M}, \mathcal{F}_\mu) = \bigoplus_{q \geqq 0} H^q_{(2)}(\mathcal{M}, \mathcal{F}_\mu)$$

can be taken as the definition of the L^2-cohomology of $\mathcal{M}$ (with values in $\mathcal{F}_\mu$). Each element $g \in G(\mathbb{A}_{\mathrm{fin}})$ acts on $A^*_{(2)}(\mathcal{M}, \mathcal{F}_\mu)$, and commutes with the differential d. We therefore obtain operators

$$H^*_{(2)}(g, \mathcal{F}_\mu) = \bigoplus_{q \geqq 0} H^q_{(2)}(g, \mathcal{F}_\mu), \qquad g \in G(\mathbb{A}_{\mathrm{fin}}), \tag{1.1}$$

on the cohomology. Thus, the L^2-cohomology provides a linear representation (1.1) of the group $G(\mathbb{A}_{\mathrm{fin}})$.

If K_0 is an open compact subgroup of $G(\mathbb{A}_{\mathrm{fin}})$, we can define the L^2-cohomology of $\mathcal{M}_{K_0}$ exactly as above. Then $H^*_{(2)}(\mathcal{M}_{K_0}, \mathcal{F}_\mu)$ equals $H^*_{(2)}, (\mathcal{M}, \mathcal{F}_\mu)^{K_0}$, the space of fixed vectors in $H^*_{(2)}(\mathcal{M}, \mathcal{F}_\mu)$ under K_0. Since the taking of cohomology commutes with direct limits, we obtain

$$H^*_{(2)}(\mathcal{M}, \mathcal{F}_\mu) = \varinjlim_{K_0} H^*_{(2)}(\mathcal{M}_{K_0}, \mathcal{F}_\mu).$$

In the language of representation theory, this asserts that the representation (1.1) is smooth.

We would also like the representation (1.1) to be admissible. In other words, we want each of the spaces $H^*_{(2)}(\mathcal{M}_{K_0}, \mathcal{F}_\mu)$ to be finite dimensional. Since this is not true in general, we must place a restriction on G. From now on, we assume that G contains a maximal torus over $\mathbb{R}$ which is anisotropic modulo A_G. We shall in fact fix such a maximal torus B such that $B(\mathbb{R})$ is contained in $K'_{\mathbb{R}}$. Borel and Casselman have shown that under this condition, all the spaces $H^*_{(2)}(\mathcal{M}_{K_0}, \mathcal{F}_\mu)$ are finite dimensional [3, Theorem 4.5]. Let $\mathcal{H}(G(\mathbb{A}_{\mathrm{fin}}))$ denote the Hecke algebra of locally constant functions of compact support on $G(\mathbb{A}_{\mathrm{fin}})$. Then for any $h \in \mathcal{H}(G(\mathbb{A}_{\mathrm{fin}}))$, the operator

$$H^q_{(2)}(h, \mathcal{F}_\mu) = \int_{G(\mathbb{A}_{\mathrm{fin}})} h(g) H^q_{(2)}(g, \mathcal{F}_\mu)\, dg$$

on $H^q_{(2)}(h, \mathscr{F}_\mu)$ is of finite rank. In particular, it has a trace. Our goal is to compute the Lefschetz number

$$\mathscr{L}_\mu(h) = \sum_q (-1)^q \operatorname{tr}(H^q_{(2)}(h, \mathscr{F}_\mu)). \tag{1.2}$$

§ 2. The spectral decomposition of cohomology

The first step is to recall the spectral decomposition of cohomology. This gives an expansion of the Lefschetz number $\mathscr{L}_\mu(h)$ into a sum of terms, which separate naturally into local and global constituents.

We can certainly regard μ as an irreducible representation of $G(\mathbb{R})$ or $G(\mathbb{C})$. In particular,

$$\mu(xz) = \xi_\mu(z)^{-1}\mu(x), \qquad z \in A_G(\mathbb{R})^0, x \in G(\mathbb{R}),$$

where ξ_μ is a quasi-character on $A_G(\mathbb{R})^0$. Let $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \xi_\mu)$ be the space of function ϕ on $G(\mathbb{Q})\backslash G(\mathbb{A})$ such that

$$\phi(zx) = \xi_\mu(z)\phi(x), \qquad z \in A_G(\mathbb{R})^0, x \in G(\mathbb{A}),$$

and which are square integrable modulo $A_G(\mathbb{R})^0$. (One can express $G(\mathbb{A})$ as a direct product of $A_G(\mathbb{R})^0$ with a normal subgroup $G(\mathbb{A})^1$ which contains $G(\mathbb{Q})$, so this latter condition makes sense). Then $G(\mathbb{A})$ acts by right translation on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \xi_\mu)$. In this paper, we shall always write $\Pi(H)$ for the set of equivalence classes of irreducible representations of a given group H. Let $\Pi(G(\mathbb{A}), \xi_\mu)$ denote the set of representations $\pi \in \Pi(G(\mathbb{A}))$ such that

$$\pi(zx) = \xi_\mu(z)\pi(x), \qquad z \in A_G(\mathbb{R})^0, x \in G(\mathbb{A}).$$

For any such π, let $m_{\mathrm{disc}}(\pi)$ denote the multiplicity with which π occurs discretely as a direct summand in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \xi_\mu)$. The nonnegative integers $m_{\mathrm{disc}}(\pi)$ are the global ingredients of the decomposition of cohomology.

There are local constituents for both the real and p-adic components of π. Set

$$\pi = \pi_{\mathbb{R}} \otimes \pi_{\mathrm{fin}},$$

where $\pi_{\mathbb{R}}$ and π_{fin} are irreducible representations of $G(\mathbb{R})$ and $G(\mathbb{A}_{\mathrm{fin}})$ respectively. If K_0 is an open compact subgroup of $G(\mathbb{A}_{\mathrm{fin}})$, let $V(\pi_{\mathrm{fin}}^{K_0})$ denote the subspace of vectors in the underlying space of π_{fin} which are fixed by K_0. This is a finite dimensional subspace which gives the contribution of π_{fin} to the cohomology. Observe that the multiplicity with which $\pi_{\mathbb{R}}$ occurs discretely in the representation of $G(\mathbb{R})$ on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K_0, \xi_\mu)$ is equal to the sum over π of the integers

$$m_{\mathrm{disc}}(\pi) \cdot \dim(V(\pi_{\mathrm{fin}}^{K_0})).$$

Let $\mathfrak{g}$ be the Lie algebra of G. (In general, let us agree to denote the Lie algebra of a group definied over $\mathbb{Q}$ by the corresponding lower case Gothic letter.) Both of $\mathfrak{g}(\mathbb{R})$ and $K'_{\mathbb{R}}$ act on the space of $K'_{\mathbb{R}}$-finite vectors of the representation $\pi_{\mathbb{R}} \otimes \mu$ of $G(\mathbb{R})$. The relative Lie algebra cohomology groups

$$H^q(\mathfrak{g}(\mathbb{R}), K'_{\mathbb{R}}; \pi_{\mathbb{R}} \otimes \mu) \tag{2.1}$$

are defined in [4, Chap. I]. They give the contribution of $\pi_{\mathbb{R}}$ to the cohomology.

Consider first the case that G is anisotropic over $\mathbb{Q}$. Then the spaces $\mathcal{M}_{K_0}$ are compact, and the groups $H^q_{(2)}$ are just ordinary de Rham cohomology. There is a well known isomorphism

$$H^q_{(2)}(\mathcal{M}_{K_0}, \mathcal{F}_\mu) \cong \bigoplus_{\pi \in \Pi(G(\mathbb{A}), \xi_\mu)} m_{\mathrm{disc}}(\pi)(H^q(\mathfrak{g}(\mathbb{R}), K'_{\mathbb{R}}; \pi_{\mathbb{R}} \otimes \mu) \otimes V(\pi_{\mathrm{fin}}^{K_0})). \tag{2.2}$$

(See [4, Chap. VII]. The coefficient $m_{\mathrm{disc}}(\pi)$ here of course stands for the multiplicity of the ensuing vector space in the direct sum.) Moreover, the isomorphism is compatible with the action of the K_0-bi-invariant functions in the Hecke algebra. Any such function operates on cohomology through its action on $V(\pi_{\mathrm{fin}}^{K_0})$. It follows that

$$H^q_{(2)}(h, \mathcal{F}_\mu) \cong \bigoplus_{\pi \in \Pi(G(\mathbb{A}), \xi_\mu)} (m_{\mathrm{disc}}(\pi) \dim H^q(\mathfrak{g}(\mathbb{R}), K'_{\mathbb{R}}; \pi_{\mathbb{R}} \otimes \mu))\, \pi_{\mathrm{fin}}(h). \tag{2.3}$$

The general case is similar. Borel and Casselman have shown [3, Theorem 4.5] that there is still a canonical isomorphism (2.2). The definition is again compatible with the Hecke algebra, and the isomorphism (2.3) continues to hold. Of course, the assumption that $G(\mathbb{R})/A_G(\mathbb{R})^0$ has a compact Cartan subgroup is essential here.

Define

$$\chi_\mu(\pi_{\mathbb{R}}) = \sum_q (-1)^q \dim H^q(\mathfrak{g}(\mathbb{R}), K'_{\mathbb{R}}; \pi_{\mathbb{R}} \otimes \mu), \tag{2.4}$$

the Euler characteristic. Then (2.3) implies

Proposition 2.1. *For any $h \in \mathcal{H}(G(\mathbb{A}_{\mathrm{fin}}))$,*

$$\mathcal{L}_\mu(h) = \sum_{\pi \in \Pi(G(\mathbb{A}), \xi_\mu)} m_{\mathrm{disc}}(\pi) \cdot \chi_\mu(\pi_{\mathbb{R}}) \cdot \mathrm{tr}\, \pi_{\mathrm{fin}}(h). \quad \square$$

Much is known about the relative Lie algebra cohomology groups (2.1). They have been completely characterized [20] if $\pi_{\mathbb{R}}$ is an arbitrary irreducible unitary representation. We are only interested in the Euler characteristic, which is considerably easier. Notice that $\chi_\mu(\pi_{\mathbb{R}})$ can be defined by (2.4) if $\pi_{\mathbb{R}}$ is any representation of $G(\mathbb{R})$ with a finite composition series. Since it is an additive function which depends only on the image of $\pi_{\mathbb{R}}$ in the Grothendieck group, χ_μ need only be computed on a basis of the Grothendieck group. Such a basis is provided by the standard representations. We recall that the standard representations are those which are parabolically induced from irreducible representations of Levi subgroups which are tempered modulo the center. A standard representation is either properly induced or it belongs to $\Pi_{\text{temp}}(G(\mathbb{R}))$, the set of representations in $\Pi(G(\mathbb{R}))$ which are tempered modulo $A_G(\mathbb{R})^0$. If $\pi_{\mathbb{R}}$ is properly induced, it follows from either [7b, §3] or [4, Theorem III.3.3] that $\chi_\mu(\pi_{\mathbb{R}})=0$. If $\pi_{\mathbb{R}}$ lies in $\Pi_{\text{temp}}(G(\mathbb{R}))$, the infinitesimal characters of $\pi_{\mathbb{R}}$ and μ will be different unless $\pi_{\mathbb{R}}$ actually belongs to $\Pi_{\text{disc}}(G(\mathbb{R}))$, the set of representations in $\Pi(G(\mathbb{R}))$ which are square integrable modulo $A_G(\mathbb{R}))^0$. It is a general fact [4, Theorem I.5.3], that the groups (2.1) all vanish if the infinitesimal characters of $\pi_{\mathbb{R}}$ and μ are different. We therefore need only be concerned with $\Pi_{\text{disc}}(G(\mathbb{R}))$.

For our purposes, it is best to discuss the discrete series in terms of the representations of a compact real form of $G(\mathbb{C})/A_G(\mathbb{C})$. To be more precise, let us fix a pair $(\bar{G}, \eta)$, where $\bar{G}$ is a reductive group over $\mathbb{R}$, and

$$\eta\colon \bar{G} \to G$$

is an isomorphism over $\mathbb{C}$ such that the automorphism $\eta^\sigma \circ \eta^{-1}$ of G is inner for σ in $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. We can use η to identify A_G with the $\mathbb{R}$-split component of the center of $\bar{G}$, and we assume that $\bar{G}(\mathbb{R})/A_G(\mathbb{R})$ is compact. Then the representations in $\Pi(\bar{G}(\mathbb{R}))$ are all finite dimensional. According to the Langlands classification [13b], the set $\Pi_{\text{disc}}(G(\mathbb{R}))$ is a disjoint union of finite subsets $\Pi_{\text{disc}}(\tau)$, which are parametrized by the irreducible representations τ in $\Pi(\bar{G}(\mathbb{R}))$. The elements in each subset $\Pi_{\text{disc}}(\tau)$ can in turn be parametrized by cosets

$$\mathscr{D}(G, B) = W(G(\mathbb{R}), B(\mathbb{R}))\backslash W(G, B),$$

where $W(G, B)$ is the Weyl group of G on B, and $W(G(\mathbb{R}), B(\mathbb{R}))$ is the subgroup of elements induced from $G(\mathbb{R})$. Now, we already have a representation μ of G. We shall identify μ with its composition by η. In particular, we shall often regard μ as an element in $\Pi(\bar{G}(\mathbb{R}))$. A similar convention applies to the contragredient

$$\tilde{\mu}(x) = \mu(x^{-1})^t, \qquad x \in G,$$

so we obtain finite subsets $\Pi_{\text{disc}}(\mu)$ and $\Pi_{\text{disc}}(\tilde{\mu})$ of $\Pi_{\text{disc}}(G(\mathbb{R}))$.

As is the usual custom, we set

$$q(G) = \tfrac{1}{2}\dim(G(\mathbb{R})/K'_{\mathbb{R}}).$$

Lemma 2.2. *Let $\pi_{\mathbb{R}}$ be any standard representation of $G(\mathbb{R})$. Then*

$$\chi_\mu(\pi_{\mathbb{R}}) = \begin{cases} (-1)^{q(G)}, & \text{if } \pi_{\mathbb{R}} \in \Pi_{\text{disc}}(\tilde{\mu}), \\ 0, & \text{otherwise}. \end{cases}$$

Proof. We have already noted that $\chi_\mu(\pi_{\mathbb{R}})=0$ unless $\pi_{\mathbb{R}}$ belongs to $\Pi_{\text{disc}}(G(\mathbb{R}))$. If $\pi_{\mathbb{R}}$ belongs to $\Pi_{\text{disc}}(G(\mathbb{R}))$, the lemma is also well known, at least when $G(\mathbb{R})$ is connected [4, Theorem II.5.3]. Of course, $G(\mathbb{R})$ is not always connected, but $K'_{\mathbb{R}}$ does meet every connected component. One can then argue as in the proof of [4, Proposition II.5.7) to establish the lemma in general. □

§ 3. Application of the trace formula

Even with a knowledge of the Euler characteristics $\chi_\mu(\pi_{\mathbb{R}})$, Proposition 2.1 does not bring us much closer to an explicit formula for $\mathscr{L}_\mu(h)$. The difficulty is the lack of information on the multiplicities $m_{\text{disc}}(\pi)$. We need to interpret $\mathscr{L}_\mu(h)$ as the trace of a certain operator on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \xi_\mu)$, which we can then hope to calculate by the trace formula.

Extending the notation of §2, we write $\Pi(G(\mathbb{R}), \xi)$, $\Pi_{\text{temp}}(G(\mathbb{R}), \xi)$, $\Pi(\bar{G}(\mathbb{R}), \xi)$, etc., for the set of representations in $\Pi(G(\mathbb{R}))$, $\Pi_{\text{temp}}(G(\mathbb{R}))$, $\Pi(\bar{G}(\mathbb{R}))$, whose central character coincides with a given quasi-character ξ on $A_G(\mathbb{R})^0$. Let us also write $\mathscr{H}_{\text{ac}}(G(\mathbb{R}), \xi)$ for the space of smooth, $K'_{\mathbb{R}}$-finite functions on $G(\mathbb{R})$ which are compactly supported modulo $A_G(\mathbb{R})^0$, and which transform under $A_G(\mathbb{R})^0$ according to ξ. If f is any function in $\mathscr{H}_{\text{ac}}(G(\mathbb{R}), \xi^{-1})$ and $\pi_{\mathbb{R}}$ belongs to $\Pi(G(\mathbb{R}), \xi)$, we can set

$$\pi_{\mathbb{R}}(f) = \int_{G(\mathbb{R})/A_G(\mathbb{R})^0} f(x)\, \pi_{\mathbb{R}}(x)\, dx.$$

The following lemma is an immediate consequence of the trace Paley-Wiener theorem of Clozel and Delorme [7a]. (See [7c, Proposition 5, Corollaire].)

Lemma 3.1. *There is function $f_\mu \in \mathscr{H}_{\text{ac}}(G(\mathbb{R}), \xi_\mu^{-1})$ such that for any $\pi_{\mathbb{R}} \in \Pi_{\text{temp}}(G(\mathbb{R}), \xi_\mu)$,*

$$\operatorname{tr} \pi_{\mathbb{R}}(f_\mu) = \begin{cases} (-1)^{q(G)}, & \text{if } \pi_{\mathbb{R}} \in \Pi_{\text{disc}}(\tilde{\mu}), \\ 0, & \text{otherwise.} \end{cases} \qquad \square$$

The function f_μ has of course been chosen to match Lemma 2.2. Since the standard characters on f_μ are just analytic continuations of tempered characters, we obtain

$$\operatorname{tr} \pi_{\mathbb{R}}(f_\mu) = \chi_\mu(\pi_{\mathbb{R}}), \tag{3.1}$$

if $\pi_{\mathbb{R}}$ is any standard representation whose central character on $A_G(\mathbb{R})^0$ equals ξ_μ. Both sides of this equation depend only on the image of $\pi_{\mathbb{R}}$ in the Grothendieck group. Since the standard representations form a basis, the Eq. (3.1) holds if $\pi_{\mathbb{R}}$ is any representation in $\Pi(G(\mathbb{R}), \xi_\mu)$. This is a result of Clozel and Delorme [7c, Théorème 3]. We have repeated their observations in order to emphasize the character formula of Lemma 3.1, which we will need later.

Now, fix the function $h \in \mathscr{H}(G(\mathbb{A}_{\text{fin}}))$, and set

$$(f_\mu h)(x) = f_\mu(x_{\mathbb{R}})\, h(x_{\text{fin}}),$$

for any point

$$x = x_{\mathbb{R}}\, x_{\text{fin}}, \qquad x_{\mathbb{R}} \in G(\mathbb{R}),\ x_{\text{fin}} \in G(\mathbb{A}_{\text{fin}}),$$

in $G(\mathbb{A})$. If

$$\pi = \pi_{\mathbb{R}} \otimes \pi_{\text{fin}}$$

is any representation in $\Pi(G(\mathbb{A}), \xi_\mu)$, we have

$$\operatorname{tr} \pi(f_\mu h) = \operatorname{tr}\Big(\int_{G(\mathbb{A})/A_G(\mathbb{R})^0} (f_\mu h)(x)\, \pi(x)\, dx \Big) = \operatorname{tr} \pi_{\mathbb{R}}(f_\mu) \cdot \operatorname{tr} \pi_{\text{fin}}(h).$$

Therefore, Proposition 2.1, combined with (3.1), yields another formula

$$\mathscr{L}_\mu(h) = \sum_{\pi \in \Pi(G(\mathbb{A}), \xi_\mu)} m_{\text{disc}}(\pi) \cdot \operatorname{tr}(f_\mu h) \tag{3.2}$$

for $\mathscr{L}_\mu(h)$.

We shall use the trace formula to evaluate the right hand side of (3.2). For the trace formula can be viewed as two different expansions for a certain linear functional I. We shall show that the spectral expansion for $I(f_\mu h)$ reduces to the right hand side of (3.2). The resulting equality with the geometric expansion will provide our explicit formula for $\mathscr{L}_\mu(h)$. In discussing the trace formula, we adopt the notation and conventions of [1e]. In particular, we fix a minimal Levi subgroup M_0 of G over $\mathbb{Q}$, and we let $\mathscr{L}$ denote the finite set of Levi subgroups which contain M_0. It is assumed that for each $M \in \mathscr{L}$, the group $M(\mathbb{R})$ is stable under the Cartan involution defined by $K_{\mathbb{R}}$. We also note that for any $M \in \mathscr{L}$, there is an associated real vector space

$$\mathfrak{a}_M = \operatorname{Hom}(X(M)_{\mathbb{Q}}, \mathbb{R}).$$

(This is at minor variance with our general use of lower case Gothic letters to denote Lie algebras over $\mathbb{Q}$.)

We propose to evaluate I at the function $f_\mu h$. Since this function is cuspidal at $v = \mathbb{R}$, Theorem 7.1(a) of [1e] tells us that the spectral expansion for $I(f_\mu h)$ takes a rather simple form. It equals

$$\sum_{t \geq 0} \sum_{\pi \in \Pi_{\text{disc}}(G, t)} a^G_{\text{disc}}(\pi) I_G(\pi, f),$$

in the notation of [1e], and by the definitions of [1e, §4], this is the same as

$$\sum_{t \geq 0} \sum_{L \in \mathscr{L}} |W_0^L| \, |W_0^G|^{-1} \sum_{s \in W^G(\mathfrak{a}_L)_{\text{reg}}} |\det(s-1)_{\mathfrak{a}_L/\mathfrak{a}_G}|^{-1} \operatorname{tr}(M_{Q|sQ}(0)\, \rho_{Q,t}(s, 0, (f_\mu h)^1)).$$

Here, $(f_\mu h)^1$ denotes the restriction of $f_\mu h$ to the subgroup

$$G(\mathbb{A})^1 = \{x \in G(\mathbb{A}) : H_G(x) = 0\},$$

and for a given $L \in \mathscr{L}$, Q is a group in $\mathscr{P}(L)$. Since G is connected, we can write

$$\operatorname{tr}(M_{Q|sQ}(0)\, \rho_{Q,t}(s, 0, (f_\mu h)^1))$$

in the more familiar notation

$$\operatorname{tr}(M_{Q|Q}(s, 0)\, \rho_{Q,t}(0, (f_\mu h)^1)), \tag{3.3}$$

where $\rho_{Q,t}(0)$ equals a direct sum

$$\bigoplus_{\sigma} \mathscr{I}_Q(\sigma) \tag{3.4}$$

of induced representations, and $M_{Q|Q}(s,0)$ is the intertwining operator corresponding to s. In (3.4), σ ranges over the irreducible subrepresentations of

$$L^2(M_Q(\mathbb{Q})A_Q(\mathbb{R})^0\backslash M_Q(\mathbb{A}))$$

whose Archimedean infinitesimal character has imaginary part with norm equal to t.

Suppose that $Q \neq G$. The expression (3.3) is a linear combination

$$\sum_{\pi \in \Pi(G(\mathbb{A})^1)} c_\pi \operatorname{tr} \pi((f_\mu h)^1)$$

of characters. In order that c_π not vanish, π must be a subrepresentation of a representation $\mathscr{I}_Q(\sigma)$, in which σ is stable under a nontrivial Weyl element s. This means that s is the restriction to $\mathfrak{a}_Q$ of an element in the complex Weyl group of G which fixes the Archimedean infinitesimal character of σ. Thus, the Archimedean infinitesimal character of π, being equal to that of σ, is fixed by a nontrivial element of the Weyl group. It is therefore singular. On the other hand, the function

$$\operatorname{tr} \pi_{\mathbb{R}}(f_\mu), \qquad \pi_{\mathbb{R}} \in \Pi(G(\mathbb{R}), \xi_\mu),$$

vanishes unless the infinitesimal character of $\pi_{\mathbb{R}}$ equals that of a discrete series. Such infinitesimal characters are necessarily regular. We therefore conclude that the expression (3.3) vanishes when $Q \neq G$.

If $L=Q=G$, the expression (3.3) is the trace of $\rho_{G,t}(0,(f_\mu h)^1)$. It follows that

$$I(f_\mu h) = \sum_{t \geq 0} \operatorname{tr} \rho_{G,t}(0,(f_\mu h)^1).$$

Our choice of f_μ implies that the summands vanish for all but finitely many t. It follows that $(f_\mu h)^1$ gives an operator of trace class on the subspace of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})^1)$ that decomposes discretely. The trace of this operator is just equal to $I(f_\mu h)$. But the operator on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})^1)$ obtained from $(f_\mu h)^1$ is clearly isomorphic to the operator on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \xi_\mu)$ obtained from $f_\mu h$. It therefore follows that

$$I(f_\mu h) = \sum_{\pi \in \Pi(G(\mathbb{A}),\ \xi_\mu)} m_{\mathrm{disc}}(\pi) \cdot \operatorname{tr} \pi(f_\mu h).$$

Combining this with (3.2), we obtain

$$\mathscr{L}_\mu(h) = I(f_\mu h). \tag{3.5}$$

The geometric expansion is given by the formula

$$I(f_\mu h)=\sum_{M\in\mathscr{L}}|W_0^M||W_0^G|^{-1}\sum_{\gamma\in(M(\mathbb{Q}))_{M,S}}a^M(S,\gamma)\,I_M(\gamma,f_\mu h),$$

expressed in the notation of [1e, Theorem 3.1]. Here, S is the union of the real valuation with a large finite set S_0 of discrete valuations. The set $(M(\mathbb{Q}))_{M,S}$ consists of (M,S)-equivalence classes $\{\gamma\}$ in $M(\mathbb{Q})$, and $f_\mu h$ is to be regarded as a function on $G(\mathbb{Q}_S)$ in evaluating the distributions $I_M(\gamma, f_\mu h)$. The invariant distributions $I_M(\gamma)=I_M^G(\gamma)$ are themselves defined and discussed in some detail in [1d, § 2]. We claim that

$$I_M(\gamma, f_\mu h)=I_M^G(\gamma, f_\mu)\,\hat{I}_M^M(\gamma, h_M), \tag{3.6}$$

where $I_M^G(\gamma)$ is the analogous distribution on $G(\mathbb{R})$, and $I_M^M(\gamma)$ is just the ordinary orbital integral on $M(\mathbb{Q}_{S_0})$. To verify this, we can assume that

$$h=\prod_{v\in S_0}h_v,\qquad h_v\in\mathscr{H}(G(F_v)).$$

Let v_2 be a fixed valuation in S_0, and write

$$h=h_1h_2,\qquad h_1=\prod_{v\neq v_2}h_v,\ h_2=h_{v_2}.$$

By the splitting formula [1d, Proposition 9.1],

$$I_M(\gamma, f_\mu h)=\sum_{L_1,L_2\in\mathscr{L}(M)}d_M^G(L_1,L_2)\,\hat{I}_M^{L_1}(\gamma,(f_\mu h_1)_{L_1})\,\hat{I}_M^{L_2}(\gamma,(h_2)_{L_2}).$$

Since $f_\mu h_1$ is cuspidal at $\mathbb{R}$, $(f_\mu h_1)_{L_1}$ vanishes unless $L_1=G$. But

$$d_M^G(G,L_2)=\begin{cases}0, & \text{if } L_2\neq M,\\ 1, & \text{if } L_2=M,\end{cases}$$

so we obtain

$$I_M(\gamma, f_\mu h)=I_M^G(\gamma, f_\mu h_1)\,\hat{I}_M^M(\gamma,(h_2)_M).$$

The required formula (3.6) follows inductively on the number of valuations in S_0.

We have established

Proposition 3.2. *For any* $h\in\mathscr{H}(G(\mathbb{A}_{\text{fin}}))$,

$$\mathscr{L}_\mu(h)=\sum_{M\in\mathscr{L}}|W_0^M||W_0^G|^{-1}\sum_{\gamma\in(M(\mathbb{Q}))_{M,S}}a^M(S,\gamma)\,I_M^G(\gamma,f_\mu)\,\hat{I}_M^M(\gamma,h_M).\quad\square$$

We now have a formula for $\mathscr{L}_\mu(h)$ in terms of orbits on G and its Levi subgroups. In order to make this explicit, we need to examine the individual terms more closely. The invariant orbital integral $\hat{I}_M^M(\gamma, h_M)$ is simple enough, and will appear with only minor modification in the final formula. However

$$I_M(\gamma, f_\mu) = I_M^G(\gamma, f_\mu)$$

is the invariant distribution attached to a weighted orbital integral, and is more complicated. We shall devote the next two paragraphs to its study.

§ 4. The functions $\Phi_M(\gamma, \tau)$

Suppose that ξ is a quasi-character on $A_G(\mathbb{R})^0$ and that f is a function in $\mathscr{H}_{ac}(G(\mathbb{R}), \xi)$. Since the contragredient $\pi \to \tilde{\pi}$ is a bijection from $\Pi(G(\mathbb{R}), \xi)$ to $\Pi(G(\mathbb{R}), \xi^{-1})$, we can form $\operatorname{tr} \tilde{\pi}(f)$ for any $\pi \in \Pi(G(\mathbb{R}), \xi)$. We shall say that f is *stable cuspidal* if the function

$$\pi \to \operatorname{tr} \tilde{\pi}(f), \qquad \pi \in \Pi_{\text{temp}}(G(\mathbb{R}), \xi),$$

is supported on the subset $\Pi_{\text{disc}}(G(\mathbb{R}), \xi)$, and is constant on the packets

$$\Pi_{\text{disc}}(\tau), \qquad \tau \in \Pi(\bar{G}(\mathbb{R}), \xi).$$

The example we have in mind, of course, is the function $f = f_\mu$ defined in the last section.

Suppose that M is a Levi subgroup in $\mathscr{L}$. If $\gamma \in M$ is any element with Jordan decomposition $\gamma = \sigma u$, we have the function

$$D^M(\gamma) = \det((1 - \operatorname{Ad}(\sigma))_{\mathfrak{m}/\mathfrak{m}_\sigma}).$$

Here, M_σ denotes the connected component of 1 in the centralizer of σ in M, and by our general convention, $\mathfrak{m}$ and $\mathfrak{m}_\sigma$ denote the Lie algebras of M and M_σ. In the case that $M_\sigma = G_\sigma$, observe that

$$D^G(\gamma) = D_M^G(\gamma)\, D^M(\gamma),$$

where

$$D_M^G(\gamma) = \det((1 - \operatorname{Ad}(\gamma))_{\mathfrak{g}/\mathfrak{m}}).$$

Suppose that M contains a maximal torus T over $\mathbb{R}$ such that $T(\mathbb{R})/A_M(\mathbb{R})^0$ is compact. Assume that $f \in \mathscr{H}_{ac}(G(\mathbb{R}), \xi)$ is stable cuspidal, and that γ belongs to $T_{\text{reg}}(\mathbb{R})$, the G-regular set in $T(\mathbb{R})$. According to Theorem 6.4 of [1 f],

$$I_M(\gamma, f) = (-1)^{\dim(A_M/A_G)} \operatorname{vol}(T(\mathbb{R})/A_M(\mathbb{R})^0)^{-1} \sum_{\pi \in \Pi_{\text{disc}}(G(\mathbb{R}), \xi)} I_M(\gamma, \pi) \cdot \operatorname{tr} \tilde{\pi}(f), \tag{4.1}$$

where

$$I_M(\gamma, \pi) = |D^G(\gamma)|^{\frac{1}{2}}\, \Theta_\pi(\gamma),$$

and Θ_π is the character of π. We would like to obtain a more general formula, in which γ is any element in $M(\mathbb{R})$.

It is sometimes convenient to write

$$\Phi_M(\gamma,f)=|D^M(\gamma)|^{-\frac{1}{2}}\, I_M(\gamma,f), \qquad \gamma\in M(\mathbb{R}). \tag{4.2}$$

In the special case that $M=G$, this is just equal to

$$\Phi_G(\gamma,f)=\int_{G_\gamma(\mathbb{R})\backslash G(\mathbb{R})} f(x^{-1}\gamma x)\,dx,$$

the ordinary unnormalized orbital integral. If τ is a representation in $\Pi(\bar{G}(\mathbb{R}),\xi)$, we shall also write

$$\operatorname{tr}\tilde{\tau}(f)=(-1)^{q(G)}\operatorname{tr}\tilde{\pi}(f), \qquad \pi\in\Pi_{\mathrm{disc}}(\tau), \tag{4.3}$$

since the number on the right depends only on τ. Finally, if γ belongs to $T_{\mathrm{reg}}(\mathbb{R})$, we set

$$\Phi_M(\gamma,\tau)=(-1)^{q(G)}\,|D_M^G(\gamma)|^{\frac{1}{2}}\sum_{\pi\in\Pi_{\mathrm{disc}}(\tau)}\Theta_\pi(\gamma). \tag{4.4}$$

The inner twist η can be used to embed $B(\mathbb{R})$ into $\bar{G}(\mathbb{R})$, and in the special case that $M=G$, one obtains the finite dimensional character

$$\Phi_G(\gamma,\tau)=\operatorname{tr}\tau(\gamma), \qquad \gamma\in B_{\mathrm{reg}}(\mathbb{R}).$$

This is a well known consequence of the character formulas for discrete series. Now, returning to the formula (4.1), we observe that

$$\begin{aligned}
&|D^M(\gamma)|^{-\frac{1}{2}}\sum_{\pi\in\Pi_{\mathrm{disc}}(G(\mathbb{R}),\xi)} I_M(\gamma,\pi)\operatorname{tr}\tilde{\pi}(f)\\
&=|D_M^G(\gamma)|^{\frac{1}{2}}\sum_{\tau\in\Pi(\bar{G}(\mathbb{R}),\xi)}\Big((-1)^{q(G)}\sum_{\pi\in\Pi_{\mathrm{disc}}(\tau)}\Theta_\pi(\gamma)\Big)\operatorname{tr}\tilde{\tau}(f)\\
&=\sum_{\tau}\Phi_M(\gamma,\tau)\operatorname{tr}\tilde{\tau}(f).
\end{aligned}$$

It follows that

$$\Phi_M(\gamma,f)=(-1)^{\dim(A_M/A_G)}\operatorname{vol}(T(\mathbb{R})/A_M(\mathbb{R})^0)^{-1}\sum_{\tau\in\Pi(\bar{G}(\mathbb{R}),\xi)}\Phi_M(\gamma,\tau)\operatorname{tr}\tilde{\tau}(f), \tag{4.5}$$

for any $\gamma\in T_{\mathrm{reg}}(\mathbb{R})$.

Before going on, we should note a property from invariant harmonic analysis which we shall require later.

Lemma 4.1. *Suppose that $\Phi(\gamma)$ is a function on $B(\mathbb{R})$ which is a finite, $W(G,B)$-invariant linear combination of quasi-characters, each of whose restriction to $A_G(\mathbb{R})^0$ equals ξ. Then there is a stable cuspidal function $f\in\mathscr{H}_{\mathrm{ac}}(G(\mathbb{R}),\xi)$ such that*

$$\Phi_G(\gamma,f)=\Phi(\gamma), \qquad \gamma\in B_{\mathrm{reg}}(\mathbb{R}).$$

Proof. Results of this nature are well known. Indeed, $\Phi(\gamma)$ can be written as a finite linear combination

$$\sum_{\tau\in\Pi(\bar{G}(\mathbb{R}),\xi)} c_\tau \operatorname{tr}(\tau(\gamma))$$

of finite dimensional characters. By the trace Paley-Wiener theorem [7a], there is a function f in $\mathscr{H}_{\mathrm{ac}}(G(\mathbb{R}),\xi)$ such that $\operatorname{tr}\tilde{\pi}(f)$ vanishes for any representation $\pi\in\Pi_{\mathrm{temp}}(G(\mathbb{R}),\xi)$ which does not belong to some $\Pi_{\mathrm{disc}}(\tau)$, and such that

$$\operatorname{tr}\tilde{\pi}(f)=(-1)^{q(G)}\operatorname{vol}(B(\mathbb{R})/A_G(\mathbb{R})^0)\,c_\tau, \qquad \pi\in\Pi_{\mathrm{disc}}(\tau),$$

for any τ. Clearly f is stable cuspidal, and

$$\operatorname{tr}\tilde{\tau}(f)=\operatorname{vol}(B(\mathbb{R})/A_G(\mathbb{R})^0)\,c_\tau, \qquad \tau\in\Pi(\bar{G}(\mathbb{R}),\xi).$$

The required formula then follows from the special case of (4.5) that $M=G$. $\square$

The function $\Phi_M(\gamma,\tau)$ will be the most interesting ingredient of our final formula. It is worth reviewing its explicit expression provided by the formulas for the characters of averaged discrete series. First, let us remind ourselves how the discrete series are parametrized. Let $Z(B)$ be the centralizer of the connected component $G(\mathbb{R})^0$ in $K_{\mathbb{R}}$. Since $G(\mathbb{R})^0$ is Zariski dense in G, $Z(B)$ is in fact equal to the intersection of $K_{\mathbb{R}}$ with the center of G. One knows [9b, Lemma 3.4] that $B(\mathbb{R})$ equals the product of its connected component $B(\mathbb{R})^0$ with $Z(B)$. Fix an order on the roots $\{\alpha\}$ of (G,B), and define ρ_B as usual to be half the sum of the positive roots. Let $\Lambda(\xi)$ denote the set of pairs

$$(\zeta,\lambda), \qquad \zeta\in Z(B)^*,\ \lambda\in\mathfrak{b}(\mathbb{C})^*,$$

such that

$$z\exp H\to\zeta(z)\,e^{(\lambda-\rho_B)(H)}, \qquad z\in Z(B),\ H\in\mathfrak{b}(\mathbb{R}),$$

is a well defined quasi-character on $B(\mathbb{R})$ whose restriction to $A_G(\mathbb{R})^0$ equals ξ. Let $\Lambda'(\xi)$ be the subset of pairs such that λ is regular. These two sets are independent of the order on the roots, and both are equipped with an action of Weyl group $W(G,B)$. The finite dimensional representations $\tau\in\Pi(\bar{G}(\mathbb{R}),\xi)$ are parametrized by the orbits of $W(G,B)$ in $\Lambda'(\xi)$, while the discrete series are parametrized by the $W(G(\mathbb{R}),B(\mathbb{R}))$-orbits in $\Lambda'(\xi)$. A packet $\Pi_{\mathrm{disc}}(\tau)$ corresponds to the partition of a given $W(G,B)$-orbit into $W(G(\mathbb{R}),B(\mathbb{R}))$-orbits. For a given $\tau\in\Pi(\bar{G}(\mathbb{R}),\xi)$, let $(\zeta,\lambda)\in\Lambda'(\xi)$ be the point in the corresponding orbit such that λ is positive on all the positive co-roots of (G,B). Then if

$$\gamma=z\exp H, \qquad z\in Z(B),\ H\in\mathfrak{b}(\mathbb{R}),$$

is a regular point in $B(\mathbb{R})$, one has

$$\operatorname{tr}\tau(\gamma)=\Phi_G(\gamma,\tau)=\Delta_B^G(H)^{-1}\,\zeta(z)\sum_{s\in W(G,B)}\varepsilon(s)\,e^{(s\lambda)(H)}. \tag{4.6}$$

Here, as usual,

$$\Delta_B^G(H) = \prod_{\alpha>0} (e^{\frac{1}{2}\alpha(H)} - e^{-\frac{1}{2}\alpha(H)}).$$

The values of the averaged discrete series characters on other maximal tori are determined by certain integer valued functions

$$\bar{c}(Q^+, R^+),$$

defined for root systems R whose Weyl group $W(R)$ contains (-1). Here R^+ is a system of positive roots for R, and Q^+ is a positive system for the set $Q = R^\vee$ of co-roots. Observe that for any R^+ and Q^+, one has corresponding positive chambers $\mathfrak{a}_{R^+}$ and $\mathfrak{a}_{Q^+}$ in the real spans of Q and R respectively. Notice also that for any root $\alpha \in R$, the set R_α of roots in R which are orthogonal to α is a root system of the same type, whose co-root system R_α equals Q_α. The functions $\bar{c}(Q^+, R^+)$ are uniquely determined by the following four properties.

(i) $\bar{c}(sQ^+, sR^+) = \bar{c}(Q^+, R^+), \quad s \in W(R)$.

(ii) The number $\bar{c}(Q^+, R^+)$ vanishes unless $\nu(X)$ negative for every $X \in \mathfrak{a}_{R^+}$ and $\nu \in \mathfrak{a}_{Q^+}$.

(iii) $\bar{c}(Q^+, R^+) + \bar{c}(s_\alpha Q^+, Q^+) = 2\bar{c}(Q^+ \cap Q_\alpha, R^+ \cap R_\alpha)$, for any reflection $s_\alpha \in W(R)$ corresponding to a root $\alpha \in R$.

(iv) If R is the empty root system $\bar{c}(Q^+, R^+) = 1$.

As above, let T be a maximal torus in M which is $\mathbb{R}$-anisotropic modulo $A_M(\mathbb{R})^0$. We take R to be the set of real roots of (G, T). The existence of the torus B means that $W(R)$ contains an element that acts as (-1). If H is a regular point in $\mathfrak{t}(\mathbb{R})$, the Lie algebra of $T(\mathbb{R})$, we shall write R_H^+ for the set of roots in R which are positive on H. Similarly, if ν is a linear function on $\mathfrak{t}(\mathbb{C})$ such that $\nu(\alpha^\vee)$ is a nonzero real number for each co-root $\alpha^\vee$ in $Q = R^\vee$, let Q_ν^+ be the set of co-roots for which these numbers are all positive. Choose an order on the roots of (G, T) which is compatible with a choice of positive restricted roots for (G, A_M). This determines a distinguished set R^+ of positive real roots, and for any regular point $H \in \mathfrak{t}(\mathbb{R})$, we obtain

$$|D_M^G(\exp H)|^{\frac{1}{2}} \Delta_T^G(H)^{-1} = \Delta_T^M(H)^{-1} \varepsilon_R(H),$$

where

$$\varepsilon_R(H) = (-1)^{|R_H^+ \cap (-R^+)|}.$$

It is best to take T from among its possible $M(\mathbb{R})$-conjugates so that $T = (T \cap B) A_M$. Then there is an element $y \in G(\mathbb{C})$, which commutes with $\mathfrak{t} \cap \mathfrak{b}$, such that $\mathrm{Ad}(y)(\mathfrak{b}(\mathbb{C})) = \mathfrak{t}(\mathbb{C})$. We shall write $\lambda \to y\lambda$ for the corresponding isomorphism from $\mathfrak{b}(\mathbb{C})^*$ onto $\mathfrak{t}(\mathbb{C})^*$. Now, suppose that $\tau \in \Pi(\bar{G}(\mathbb{R}), \xi)$, and that $(\zeta, \lambda) \in \Lambda'(\xi)$ is a point in the corresponding $W(G, B)$-orbit such that $y\lambda$ is positive on all the positive co-roots of (G, T). Then $\Phi_M(\gamma, \tau)$ vanishes for any regular point $\gamma \in T(\mathbb{R})$ unless γ is of the form

$$\gamma = z \exp H, \quad z \in Z(B),\ H \in \mathfrak{t}(\mathbb{R}), \tag{4.7}$$

in which case

$$\Phi_M(\gamma,\tau)=\Delta_T^M(H)^{-1}\,\varepsilon_R(H)\,\zeta(z)\sum_{s\in W(G,B)}\varepsilon(s)\,\bar{c}(Q^+_{ys\lambda},R^+_H)\,e^{(ys\lambda)(H)}. \tag{4.8}$$

This follows from the general formulas of Harish-Chandra [9a] for the characters of discrete series, with obvious allowances made for the possibility that $G(\mathbb{R})$ is not connected. We have followed the discussion of R. Herb [10], who treats the averaged discrete series characters in detail, and gives a simple formula for the constants $\bar{c}(Q^+, R^+)$ in terms of root systems of rank 2.

Lemma 4.2. *The function*

$$\gamma\to\Phi_M(\gamma,\tau),\qquad \gamma\in T_{\mathrm{reg}}(\mathbb{R}),$$

extends to a continuous, $W(M,T)$-invariant function on $T(\mathbb{R})$.

Proof. Shelstad has pointed out to me that the lemma follows easily from the stability of the averaged discrete series characters. We shall instead argue directly from the formula (4.8). We can assume that γ is of the form (4.7), since $\Phi_M(\gamma,\tau)$ vanishes otherwise. The Weyl group $W(M,T)$ acts on $\mathfrak{t}\cap\mathfrak{b}$, and can be regarded as a subgroup of $W(G,B)$ which commutes with y. For any element r in this subgroup, we have

$$\bar{c}(Q^+_{yrs\lambda},R^+_H)=\bar{c}(Q^+_{ys\lambda},R^+_H)=\bar{c}(Q^+_{ys\lambda},R^+_{r^{-1}H}).$$

It follows from (4.8) that the function

$$\Delta_T^M(H)\,\Phi_M(z\exp H,\tau)$$

is skew-symmetric under $W(M,T)$. Moreover, $\varepsilon_R(H)$ and $\bar{c}(Q^+_{ys\lambda},R^+_H)$ are locally constant on

$$\mathfrak{t}(\mathbb{R})'=\{H\in\mathfrak{t}(\mathbb{R})\colon \alpha(H)\neq0,\ \alpha\in R\}.$$

It follows that $\Phi_M(z\exp H,\tau)$ extends to a smooth, $W(M,T)$-invariant function on the closure of any connected of $\mathfrak{t}(\mathbb{R})'$. We need only show that the boundary values match. But $\Phi_M(z\exp H,\tau)$ is invariant under the Weyl group $W(R)$ of R. This follows directly from the definition of Φ_M in terms of characters, since every element in $W(R)$ is induced by a conjugation from $G(\mathbb{R})$. Therefore, $\Phi_M(z\exp H,\tau)$ does extend continuously across the singular hyperplanes of $A(\mathbb{R})'$. We have thus established that $\Phi_M(\gamma,\tau)$ defines a continuous function on $T(\mathbb{R})$, which is in fact invariant under both $W(M,T)$ and $W(R)$. □

From the proof of the lemma, we also obtain

Corollary 4.3. *On any connected component of the set*

$$\{\gamma\in T(\mathbb{R})\colon (\gamma)^\alpha\neq1,\ \alpha\in R\},$$

the function $\Phi_M(\gamma,\tau)$ is a finite, $W(M,T)$-invariant linear combination of quasi-characters of $T(\mathbb{R})$. □

It is convenient to extend $\Phi_M(\gamma, \tau)$ to a function on all of $M(\mathbb{R})$. If γ is any point in $M(\mathbb{R})$, we define $\Phi_M(\gamma, \tau)$ to be 0 unless γ is $M(\mathbb{R})$-conjugate to a point γ' in $T(\mathbb{R})$, in which case we set $\Phi_M(\gamma, \tau)$ equal to $\Phi_M(\gamma', \tau)$. Then $\Phi_M(\cdot, \tau)$ is an $M(\mathbb{R})$-invariant function on $M(\mathbb{R})$ which is supported on the $M(\mathbb{R})$-elliptic conjugacy classes. Furthermore, let us define the functions $\Phi_M(\gamma, \tau)$ and $\Phi_M(\gamma, f)$ for any Levi subset $M \in \mathscr{L}$ by simply setting them equal to 0 if M is not cuspidal over $\mathbb{R}$.

§ 5. Proof of Theorem 5.1

In this paragraph we shall establish a general formula for $\Phi_M(\gamma, f)$. The result will be stated in terms of a simple constant which we must first define.

Any Haar measure on $G(\mathbb{R})$ is obtained from a differential form of highest degree, which can be transferred by the inner twist η to a form on $\bar{G}(\mathbb{R})$. We obtain a uniquely determined Haar measure on $\bar{G}(\mathbb{R})$, with respect to which we define

$$v(G) = (-1)^{q(G)} \operatorname{vol}(\bar{G}(\mathbb{R})/A_G(\mathbb{R})^0)\, |\mathscr{D}(G, B)|^{-1}.$$

This constant depends only on G as a group over $\mathbb{R}$, and on a choice of Haar measure on $G(\mathbb{R})$.

Theorem 5.1. *Suppose that $f \in \mathscr{H}_{\mathrm{ac}}(G(\mathbb{R}), \xi)$ is stable cuspidal and that $\gamma \in M(\mathbb{R})$. Then*

$$\Phi_M(\gamma, f) = (-1)^{\dim(A_M/A_G)}\, v(M_\gamma)^{-1} \sum_{\tau \in \Pi(\bar{G}(\mathbb{R}), \xi)} \Phi_M(\gamma, \tau) \operatorname{tr} \bar{\tau}(f).$$

In particular, $\Phi_M(\gamma, f)$ vanishes if γ is not semisimple.

Proof. We shall need to make use of the following result on unipotent orbital integrals.

Lemma 5.2. *For any unipotent element $u \in G(\mathbb{R})$, there is a harmonic element h_u in $S(\mathfrak{b}(\mathbb{C}))$, which is homogeneous of degree*

$$\tfrac{1}{2}(\dim(G/B) - \dim(G/G_u)),$$

such that

$$\Phi_G(u, f) = \lim_{H \to 0} (\partial(h_u) F_f^B)(H).$$

Here

$$F_f^B(H) = \Delta_B^G(H) \int_{B(\mathbb{R}) \backslash G(\mathbb{R})} f(x^{-1} \exp(H) x)\, dx,$$

for any regular element H in $\mathfrak{b}(\mathbb{R})$.

This lemma is the specialization to cuspidal f of an unpublished result of Harish-Chandra. We shall give a proof in the appendix. Assuming Lemma 5.2, let us deduce the special case of the theorem in which $M = G$, and $\gamma = u$ is a unipotent element in $G(\mathbb{R})$.

Suppose first that $u \neq 1$. Since

$$F_f^B(H) = \Delta_B^G(H)\,\Phi_G(\exp h, f), \qquad H \in \mathfrak{b}_{\mathrm{reg}}(\mathbb{R}),$$

and since $\varepsilon(s)$ is just the variance of $\Delta_B^G(H)$, with respect to s, we have

$$F_f^B(sH) = \varepsilon(s)\,F_f^B(H),$$

for any $s \in W(G, B)$. Consequently,

$$(\partial(s\,h_u)\,F_f^B)(0) = \varepsilon(s)(\partial(h_u)\,F_f^B)(0), \qquad s \in W(G, B).$$

It follows from the lemma that

$$\Phi_G(u, f) = (\partial(A\,h_u)\,F_f^B)(0),$$

where

$$A\,h_u = |W(G, B)|^{-1} \sum_{s \in W(G, B)} \varepsilon(s)(s\,h_u).$$

Of course, $s\,h_u$ stands for the image of h_u under the natural action of $W(G, B)$ on $S(\mathfrak{b}(\mathbb{C}))$. We know that the harmonic elements form a finite dimensional $W(G, B)$-invariant subspace of $S(\mathfrak{b}(\mathbb{C}))$ on which the action of $W(G, B)$ is equivalent to the regular representation. Moreover, the vector

$$h_1 = \prod_{\alpha > 0} (\alpha^\vee)$$

is the harmonic element corresponding to the sign character ε of $W(G, B)$. Since $\dim(G_u) < \dim(G)$, the lemma tells us that $\deg(h_u)$ is less than $\deg(h_1)$. This implies that h_u transforms under $W(G, B)$ according to a sum of representations, none of which is equivalent to ε. Therefore $A\,h_u = 0$ and $\Phi_G(u, f)$ vanishes, as required.

Now, assume that $\gamma = u = 1$. Then

$$\Phi_G(1, f) = f(1) = \sum_{\pi \in \Pi_{\mathrm{disc}}(G(\mathbb{R}), \xi)} d\tilde{\pi}\;\mathrm{tr}(\tilde{\pi}(f)),$$

by the Plancherel formula. The formal degree $d\tilde{\pi}$ depends inversely on a choice of Haar measure on $G(\mathbb{R})$. D. Shelstad has verified from the formulas of Harish-Chandra that for compatible choices of Haar measures, the formal degrees match on groups related by inner twisting. In particular,

$$d\tilde{\pi} = \deg(\tilde{\tau})\,\mathrm{vol}(\bar{G}(\mathbb{R})/A_G(\mathbb{R})^0)^{-1}, \qquad \pi \in \Pi_{\mathrm{disc}}(\tau),$$

for any $\tau \in \Pi(\bar{G}(\mathbb{R}), \xi)$ (see [14]). Moreover,

$$\deg(\tilde{\tau}) = \deg(\tau) = \operatorname{tr}(\tau(1)) = \Phi_G(1, \tau).$$

Since

$$\sum_{\pi \in \Pi_{\mathrm{disc}}(\tau)} \operatorname{tr} \tilde{\pi}(f) = (-1)^{q(G)} |\mathscr{D}(G, B)| \operatorname{tr}(\tilde{\tau}(f)),$$

we obtain

$$\Phi_G(1, f) = v(G)^{-1} \sum_{\tau \in \Pi(\bar{G}(\mathbb{R}), \xi)} \Phi_G(1, \tau) \operatorname{tr}(\tilde{\tau}(f)),$$

which is the required formula in the special case under consideration. We shall actually combine this with a special case of (4.5). For we can write

$$\begin{aligned} &\sum_{\tau \in \Pi(\bar{G}(\mathbb{R}), \xi)} \Phi_G(1, \tau) \operatorname{tr} \tilde{\tau}(f)) \\ &\quad = \lim_{\gamma \to 1} \sum_{\tau} \Phi_G(\gamma, \tau) \operatorname{tr}(\tilde{\tau}(f)) \\ &\quad = \operatorname{vol}(B(\mathbb{R})/A_G(\mathbb{R})^0) \lim_{\gamma \to 1} \Phi_G(\gamma, f), \end{aligned}$$

where γ stands for a small regular point in $B(\mathbb{R})$. It follows that

$$I_G(1, f) = v(G)^{-1} \operatorname{vol}(B(\mathbb{R})/A_G(\mathbb{R})^0) \lim_{\gamma \to 1} \Phi_G(1, f). \tag{5.1}$$

Returning to the general case, we assume that M and $\gamma \in M(\mathbb{R})$ are arbitrary. If M is not cuspidal over $\mathbb{R}$, both sides of the required formula vanish, by definition. We can therefore assume that M contains a maximal torus T over $\mathbb{R}$ such that $T(\mathbb{R})/A_M(\mathbb{R})^0$ is compact. By the formula [1d, (2.2)], we have

$$I_M(\gamma, f) = \lim_{a \to 1} \sum_{L \in \mathscr{L}(M)} r_M^L(\gamma, a)\, I_L(a\gamma, f),$$

where a ranges over small generic points in $A_M(\mathbb{R})$. Since f is cuspidal, the descent property [1d, Corollary 8.3] implies that $I_L(a\gamma, f) = 0$ if $L \neq M$. Dividing each side by the function

$$|D^M(\gamma)|^{\frac{1}{2}} = |D^M(a\gamma)|^{\frac{1}{2}},$$

we obtain the limit formula

$$\Phi_M(\gamma, f) = \lim_{a \to 1} \Phi_M(a\gamma, f).$$

Our definitions also imply that

$$\Phi_M(\gamma, \tau) = \lim_{a \to 1} \Phi_M(a\gamma, \tau), \qquad \tau \in \Pi(\bar{G}(\mathbb{R}), \xi),$$

so it would be enough to prove the theorem with γ replaced by $a\gamma$. In other words, we can assume that $M_\gamma = G_\gamma$. Let $\gamma = \sigma u$ be the Jordan decomposition of γ. Then $G_\sigma = M_\sigma$.

Lemma 5.3. *Assuming that $M_\sigma = G_\sigma$, we can choose a finite set $\{\xi^i\}$ of quasi-characters on $A_M(\mathbb{R})^0$, and stable cuspidal functions f_σ^i in $\mathscr{H}_{\mathrm{ac}}(M_\sigma(\mathbb{R}), \xi^i)$, such that*

$$\Phi_M(\sigma\mu, f) = \sum_i \Phi_{M_\sigma}(\mu, f_\sigma^i) \tag{5.2}$$

for any μ in some $M_\sigma(\mathbb{R})$-invariant neighborhood U of 1 in $M_\sigma(\mathbb{R})$.

Proof. If μ remains within a small $M_\sigma(\mathbb{R})$-invariant neighborhood of 1 in $M_\sigma(\mathbb{R})$, the centralizer of $\sigma\mu$ is contained in that of σ. Therefore, $\Phi_M(\sigma\mu, f)$ depends on a choice of invariant measures on $M_\sigma(\mathbb{R})\backslash G(\mathbb{R})$ and $M_{\sigma\mu}(\mathbb{R})\backslash M_\sigma(\mathbb{R})$. The latter of course varies with μ. However, $\Phi_{M_\sigma}(\mu, f_\sigma^i)$ also depends on such a measure, so the formula we are trying to prove makes sense.

If σ is not $\mathbb{R}$-elliptic in M, the centralizer M_σ is contained in a proper Levi subgroup M_1 of M over $\mathbb{R}$. The same is true for $M_{\sigma\mu}$, if μ is small. It follows from the descent property [1d, Corollary 8.3] and the fact that f is cuspidal, that

$$\Phi_M(\sigma\mu, f) = |D^M(\sigma\mu)|^{-\frac{1}{2}} I_M(\sigma\mu, f) = 0.$$

The lemma follows with $f_\sigma^i = 0$ for each i. We can therefore assume that σ is $\mathbb{R}$-elliptic in M. This means that σ is $M(\mathbb{R})$-conjugate to an element in $T(\mathbb{R})$. Since $I_M(\sigma\mu, f)$ is invariant under conjugation by $M(\mathbb{R})$, we can assume that σ actually belongs to $T(\mathbb{R})$.

Let U be a small invariant neighborhood of 1 in $M_\sigma(\mathbb{R})$. We shall first establish the existence of the functions f_σ^i such that the required property holds for all M_σ-regular points μ in $U \cap T(\mathbb{R})$. The point $\sigma\mu$ is G-regular for any such μ, and we can use (4.5) to write $\Phi_M(\sigma\mu, f)$ as a sum of functions $\Phi_M(\sigma\mu, \tau)$. We must check that each function

$$\Phi_M(\sigma\mu, \tau), \qquad \mu \in U \cap T(\mathbb{R}),$$

satisfies the conditions of Lemma 4.1. Observe that $W(M_\sigma, T)$, the complex Weyl group of M_σ, is contained in the subgroup of elements in $W(M, T)$ which fix σ. Since $M_\sigma = G_\sigma$, there is no real root of (G, T) which is trivial on σ. It follows from Corollary 4.3 that for $\mu \in T(\mathbb{R})$ near 1, $\Phi_M(\sigma\mu, \tau)$ is a finite, $W(M_\sigma, T)$-invariant linear combination of quasi-characters. The conditions of Lemma 4.1 are thus satisfied. We may choose quasi-characters ξ^i on $A_M(\mathbb{R})^0$, and stable cuspidal functions f_σ^i in $\mathscr{H}_{\mathrm{ac}}(M_\sigma(\mathbb{R}), \xi^i)$, such that (5.2) holds for all points $\mu \in U \cap T(\mathbb{R})$ which are M_σ-regular.

Having proved the existence of the functions f_σ^i, we must show that (5.2) holds for all points $\mu \in U$. Suppose first that μ is an M_σ-regular point in U. If μ is elliptic in $M_\sigma(\mathbb{R})$, it is conjugate to a point in $T(\mathbb{R})$. If μ is not elliptic, the point $\sigma\mu$ is not elliptic in $M(\mathbb{R})$, and the descent properties force both sides of (5.2) to vanish. In either instance, formula (5.2) holds. Now consider the case that μ is a general point in U. According to [1d, (2.3)], we have

$$I_M(\sigma\mu, f) \overset{(M,\sigma)}{\sim} 0, \qquad \mu \in U.$$

This means that there is a function $\phi \in C_c^\infty(M(\mathbb{R}))$ such that

$$I_M(\sigma\mu, f) = I_M^M(\sigma\mu, \phi).$$

Dividing each side by $|D^M(\sigma\mu)|^{\frac{1}{2}}$, we obtain

$$\begin{aligned}\Phi_M(\sigma\mu, f) &= \int_{M_{\sigma\mu}(\mathbb{R})\backslash M(\mathbb{R})} \phi(x^{-1}\sigma\mu x)\, dx \\ &= \int_{M_\sigma(\mathbb{R})\backslash M(\mathbb{R})} \int_{M_{\sigma\mu}(\mathbb{R})\backslash M_\sigma(\mathbb{R})} \phi(y^{-1}x^{-1}\sigma\mu x y)\, dx\, dy.\end{aligned}$$

By Lemma 2.1 of [1b], we can take the integral in y over a compact set which is independent of μ. Therefore, we can choose U, and a function $\phi_\sigma \in C_c^\infty(M_\sigma(\mathbb{R}))$, such that

$$\Phi_M(\sigma\mu, f) = \int_{M_{\sigma\mu}(\mathbb{R})\backslash M_\sigma(\mathbb{R})} \phi_\sigma(x^{-1}\mu x)\, dx.$$

We have already shown that if $\mu \in U$ is $M_\sigma(\mathbb{R})$-regular, this also equals

$$\sum_i \int_{M_{\sigma\mu}(\mathbb{R})\backslash M_\sigma(\mathbb{R})} f_\sigma^i(x^{-1}\mu x)\, dx,$$

the right hand side of (5.2). Since the orbital integrals at an arbitrary point are determined by the orbital integrals at nearby regular points, the formula (5.2) holds for all $\mu \in U$. This concludes the proof of the lemma. □

We now return to the proof of the theorem. We are assuming that $\gamma = \sigma u$, with $G_\sigma = M_\sigma$. Choose stable cuspidal functions $f_\sigma^i \in \mathscr{H}_{\mathrm{ac}}(M_\sigma(\mathbb{R}), \xi^i)$ as in the last lemma. Then

$$\Phi_M(\sigma\mu, f) = \sum_i \Phi_{M_\sigma}(u, f_\sigma^i).$$

We apply to the group M_σ the special case of the theorem we have already established. If $u \neq 1$, the right hand side above equals 0, and we obtain the vanishing of $I_M(\gamma, f)$, as required. Assume then that $u = 1$. Applying the formula (5.1) to M_σ, we obtain

$$\Phi_{M_\sigma}(u, f_\sigma^i) = v(M_\sigma)^{-1} \operatorname{vol}(T(\mathbb{R})/A_M(\mathbb{R})^0) \lim_{t \to 1} \Phi_{M_\sigma}(t, f_\sigma^i),$$

where t stands for a small regular point in $T(\mathbb{R})$. Applying the formula (5.2) again, we see that

$$\Phi_M(\sigma, f) = v(M_\sigma)^{-1} \operatorname{vol}(T(\mathbb{R})/A_M(\mathbb{R})^0) \lim_{t \to 1} \Phi_M(\sigma t, f).$$

Finally, by formula (4.5), this equals

$$v(M_\sigma)^{-1}(-1)^{\dim(A_M/A_G)} \sum_{\tau \in \Pi(\bar{G}(\mathbb{R}), \xi)} \lim_{t \to 1} \Phi_M(\sigma t, \tau)) \operatorname{tr}(\tilde{\tau}(f)).$$

Since $\gamma = \sigma$, we obtain

$$(-1)^{\dim(A_M/A_G)} v(M_\gamma)^{-1} \sum_{\tau \in \Pi(\bar{G}(\mathbb{R}), \xi)} \Phi_M(\gamma, \tau) \operatorname{tr}(\tilde{\tau}(f)).$$

We have established the required formula for $I_M(\gamma, f)$ in all cases. $\square$

§ 6. The main formula

We now return to the discussion of § 3. In order to establish a formula for the Lefschetz number $\mathscr{L}_\mu(h)$, we have only to combine Proposition 3.2 with Theorem 5.1.

We observe first that

$$\prod_{v \in S} |D^M(\gamma)|_v = 1, \qquad M \in \mathscr{L},\ \gamma \in (M(\mathbb{Q}))_{M,S},$$

by the product formula. Here, S is as in Proposition 3.2, the union of the real valuation with a large finite set S_0 of discrete valuations. We can therefore multiply the corresponding term in the formula of Proposition 3.2 by

$$|D^M(\gamma)|_{\mathbb{R}}^{-\frac{1}{2}}\, |D^M(\gamma)|_{S_0}^{-\frac{1}{2}}$$

where

$$|\cdot|_{S_0} = \prod_{v \in S_0} |\cdot|_v .$$

If γ_{S_0} is any point in $M(\mathbb{Q}_{S_0})$, set

$$h_M(\gamma) = |D^M(\gamma_{S_0})|_{S_0}^{-\frac{1}{2}} \cdot \hat{I}_M^M(\gamma_{S_0}, h_M).$$

Then Proposition 3.2 yields the formula

$$\mathscr{L}_\mu(h) = \sum_{M \in \mathscr{L}} |W_0^M|\, |W_0^G|^{-1} \sum_{\gamma \in (M(\mathbb{Q}))_{M,S}} a^M(S, \gamma)\, \Phi_M(\gamma, f_\mu)\, h_M(\gamma).$$

The function f_μ belongs to $\mathscr{H}_{\mathrm{ac}}(G(\mathbb{R}), \xi_\mu^{-1})$. If τ is any representation in $\Pi(\bar{G}(\mathbb{R}), \xi_\mu^{-1})$, we have

$$\operatorname{tr}(\tilde{\tau}(f_\mu)) = \begin{cases} 1, & \text{if } \tau = \mu, \\ 0, & \text{otherwise.} \end{cases}$$

It follows from Theorem 5.1 that $\Phi_M(\gamma, f_\mu)$ equals

$$(-1)^{\dim(A_M/A_G)}\, v(M_\gamma)^{-1}\, \Phi_M(\gamma, \mu).$$

In particular, the function vanishes unless γ is semisimple. But the equivalence classes in $(M(\mathbb{Q}))_{M,S}$ which are semisimple are just $M(\mathbb{Q})$-conjugacy classes. Moreover, for any semisimple element $\gamma \in M(\mathbb{Q})$, Theorem 8.2 of [1a] asserts that

$$\begin{aligned} a^M(S, \gamma) &= |\iota^M(\gamma)|^{-1} \operatorname{vol}(M_\gamma(\mathbb{Q})\backslash M_\gamma(\mathbb{A})^1) \\ &= |\iota^M(\gamma)|^{-1} \operatorname{vol}(M_\gamma(\mathbb{Q})\, A_M(\mathbb{R})^0\backslash M_\gamma(\mathbb{A})), \end{aligned}$$

if γ is $\mathbb{Q}$-elliptic in M, and that $a^M(S, \gamma)$ vanishes otherwise. Here

$$|\iota^M(\gamma)| = |M_\gamma(\mathbb{Q})\backslash M(\mathbb{Q}, \gamma)|, \tag{6.1}$$

the number of connected components in the centralizer of γ in M which contain rational points. We also point out that if $\gamma \in M(\mathbb{Q})$ is semisimple, then

$$h_M(\gamma) = \delta_P(\gamma_{\mathrm{fin}})^{\frac{1}{2}} \int_{K_{\mathrm{fin}}} \int_{N_P(\mathbb{A}_{\mathrm{fin}})} \int_{M_\gamma(\mathbb{A}_{\mathrm{fin}})\backslash M(\mathbb{A}_{\mathrm{fin}})} h(k^{-1} m^{-1} \gamma m n k)\, dm\, dn\, dk, \tag{6.2}$$

where $P = MN_P$ is any parabolic subgroup with Levi component M, $\delta_P(\gamma_{\mathrm{fin}})$ is the modular function of P, evaluated at the image of γ in $G(\mathbb{A}_{\mathrm{fin}})$, and K_{fin} is a suitable maximal compact subgroup of $G(\mathbb{A}_{\mathrm{fin}})$. In particular, the orbital integral on M can be taken over $M(\mathbb{A}_{\mathrm{fin}})$ rather that $M(\mathbb{Q}_{S_0})$. We thus have no further need to single out the finite set S of valuations.

It remains only to collect the terms. Looking back at the formula for $v(G)$, we are lead to define

$$\chi(G) = (-1)^{q(G)} \operatorname{vol}(G(\mathbb{Q})\, A_G(\mathbb{R})^0\backslash G(\mathbb{A}))\, \operatorname{vol}(A_G(\mathbb{R})^0\backslash \bar{G}(\mathbb{R}))^{-1}\, |\mathscr{D}(G, B)|. \tag{6.3}$$

We shall also write $(M(\mathbb{Q}))$ for the set of $M(\mathbb{Q})$-conjugacy classes in $M(\mathbb{Q})$. Our main result is then

Theorem 6.1. *For any $h \in \mathscr{H}(G(\mathbb{A}_{\mathrm{fin}}))$,*

$$\mathscr{L}_\mu(h) = \sum_{M \in \mathscr{L}} (-1)^{\dim(A_M/A_G)}\, |W_0^M|\, |W_0^G|^{-1} \sum_{\gamma \in (M(\mathbb{Q}))} \chi(M_\gamma)\, |\iota^M(\gamma)|^{-1}\, \Phi_M(\gamma, \mu)\, h_M(\gamma). \quad \square$$

Remarks. 1. The sum in γ can be taken over a finite set that depends only on the support of h. This follows from a general result [1e, Theorem 3.3], or can be deduced directly from the fact that $\Phi_M(\gamma, \mu)$ vanishes uless γ is an $\mathbb{R}$-elliptic element in M. The terms $\chi(M_\gamma)$, $|\iota^M(\gamma)|^{-1}$, $\Phi_M(\gamma, \mu)$ and $h_M(\gamma)$, which are given by (6.3), (6.1), (4.8) and (6.2) respectively, can in principle be calculated explicitly. The theorem therefore provides a finite closed formula for $\mathscr{L}_\mu(h)$.

2. Let $\bar{G}$ be any inner twist of G over $\mathbb{Q}$ which is $\mathbb{R}$-anisotropic modulo A_G. Kottwitz [12] has shown that the Tamagawa number of $\bar{G}$ equals that of G, at least when G has no simple factors of type E_8. It follows that

$$\chi(G)=(-1)^{q(G)} \operatorname{vol}(\bar{G}(\mathbb{Q})\backslash\bar{G}(\mathbb{A}_{\mathrm{fin}}))\,|\mathscr{D}(G,B)|, \tag{6.4}$$

under this condition.

3. For the study of Shimura varieties, one wants to be able to replace the compact group $K_{\mathbb{R}}$ by a subgroup of finite index m. Using the fact that the actions of $G(\mathbb{R})$ and $G(\mathbb{A}_{\mathrm{fin}})$ on $G(\mathbb{Q})\backslash G(\mathbb{A})$ commute, one can show that this serves only to multiply the Lefschetz number $\mathscr{L}_\mu(h)$ by the factor m.

4. If one prefers, one can write the formula for $\mathscr{L}_\mu(h)$ as a sum over the boundary components of the Borel-Serre compactification. These are associated to standard parabolic subgroups $P=M_P N_P$ which contain a fixed minimal parabolic subgroup (defined over $\mathbb{Q}$). Indeed, the terms in the original formula depend only on the W_0^G-orbit of M. One obtains

$$\mathscr{L}_\mu(h)=\sum_P(-1)^{\dim(A_P/A_G)}\, n_P^{-1} \sum_{\gamma\in(M_P(\mathbb{Q}))} \chi(M_{P,\gamma})\,|\iota^{M_P}(\gamma)|^{-1}\, \Phi_{M_P}(\gamma,\mu)\, h_{M_P}(\gamma),$$

where n_P denotes the number of chambers in the split component A_P of P. Keep in mind that the contribution from a given P vanishes unless P is a cuspidal parabolic subgroup over $\mathbb{R}$.

5. Let K_0 be a small open compact subgroup of $G(\mathbb{A}_{\mathrm{fin}})$. Set h equal to 1_{K_0}, the characteristic function of K_0, divided by the volume of K_0 (with respect to a given Haar measure on $G(\mathbb{A}_{\mathrm{fin}})$). Then $\mathscr{L}_\mu(h)$ equals the L^2-Euler characteristic of $\mathscr{M}_{K_0}$ with coefficients in $\mathscr{F}_\mu$. Since K_0 is small, the term in the formula corresponding to $\gamma\in(M(\mathbb{Q}))$ vanishes unless $\gamma=1$. The L^2-Euler characteristic therefore equals

$$\sum_P(-1)^{\dim(A_P/A_G)}\, n_P^{-1}\, \chi(M_P)\, h_{M_P}(1)\, \Phi_{M_P}(1,\mu). \tag{6.5}$$

The leading term, that corresponding to $P=G$, equals

$$\chi(G)\operatorname{vol}(K_0)^{-1}\deg(\mu).$$

Harder [8] has shown that this equals the classical Euler characteristic of $\mathscr{M}_{K_0}$, at least when $\mu=1$. The other terms are related to Euler characteristics of the boundary components, relative to certain local systems. For $\Phi_M(1,\mu)$ is an integral linear combination of degrees of finite dimensional representations of M_P. The coefficients of these degrees are determined by the simple algebraic procedure of §4. One could ask whether the coefficients have a geometric interpretation. M. Stern [19] has established a general L^2-index formula for a Hermitian locally symmetric space, in terms of certain eta invariants and zeta functions. It would be interesting to compare the formula this provides for the L^2-Euler characteristic with the expression (6.5).

We have written this paper in the framework of L^2-cohomology, but it is clear that Theorem 6.1 gives dimension formulas for spaces of automorphic forms. For each $\pi_{\mathbb{R}} \in \Pi(G(\mathbb{R}), \xi)$, let $m_{\text{disc}}(\pi_{\mathbb{R}}, K_0)$ be the multiplicity with which $\pi_{\mathbb{R}}$ occurs discretely in the representation of $G(\mathbb{R})$ on

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K_0, \xi) = \bigoplus_{i=1}^{n} L^2(\Gamma_i \backslash G(\mathbb{R}), \xi). \tag{6.6}$$

Here K_0 is an open compact subgroup of $G(\mathbb{A}_{\text{fin}})$, and $\{\Gamma_i\}$ are the discrete subgroups of $G(\mathbb{R})$ described in §1. It h is a K_0-bi-invariant function in $\mathscr{H}(G(\mathbb{A}_{\text{fin}}))$, let $R_{\text{disc}}(\pi_{\mathbb{R}}, h)$ be the operator on the $\pi_{\mathbb{R}}$-isotypical subspace of (6.6). It can be interpreted as an $(m_{\text{disc}}(\pi_{\mathbb{R}}, K_0) \times m_{\text{disc}}(\pi_{\mathbb{R}}, K_0))$-matrix.

Corollary 6.2. *Suppose that the highest weight of the representation μ is regular. Then*

$$\sum_{\pi_{\mathbb{R}} \in \Pi_{\text{disc}}(\bar{\mu})} \operatorname{tr}(R_{\text{disc}}(\pi_{\mathbb{R}}, h))$$

equals

$$(-1)^{q(G)} \sum_{M \in \mathscr{L}} (-1)^{\dim(A_M/A_G)} |W_0^M| \, |W_0^G|^{-1} \sum_{\gamma \in (M(\mathbb{Q}))} \chi(M_\gamma) \, |\iota^M(\gamma)|^{-1} \, \Phi_M(\gamma, \mu) \, h_M(\gamma). \tag{6.7}$$

Proof. In [20] there is a classification of the unitary representations $\pi_{\mathbb{R}} \in \Pi(G(\mathbb{R}))$ such that the cohomology

$$\bigoplus_q H^q(\mathfrak{g}(\mathbb{R}), K'_{\mathbb{R}}; \pi_{\mathbb{R}} \otimes \mu)$$

does not vanish. Such representations are attached to Levi components L of certain parabolic subgroups of $G(\mathbb{C})$. It is required that the simple co-roots of L be annihilated by the highest weight of μ (relative to some ordering). Since we are assuming that the highest weight of μ is regular, L must be abelian. It follows from [20] that L is conjugate to B, and that $\Pi_{\text{disc}}(\bar{\mu})$ contains the only unitary representations with cohomology. The corollary then follows from Proposition 2.1, Lemma 2.2 and Theorem 6.1. □

There is no particular reason why the representation μ of Corollary 6.2 needs to be rational. In fact, we can replace μ by any representation $\tau \in \Pi(\bar{G}(\mathbb{R}))$ whose highest weight is regular. We obtain a formula for the sum over $\pi_{\mathbb{R}} \in \Pi_{\text{disc}}(\tilde{\tau})$ of the traces of the operators $R_{\text{disc}}(\pi_{\mathbb{R}}, h)$. If we set h equal to the unit 1_{K_0} in $\mathscr{H}_{K_0}$ we obtain an explicit formula for the sum

$$\sum_{\pi_{\mathbb{R}} \in \Pi_{\text{disc}}(\bar{\mu})} m_{\text{disc}}(\pi_{\mathbb{R}}, K_0) \tag{6.8}$$

of multiplicities.

The classical problem, of course, has been to compute the multiplicity of a single discrete series representation $\pi_{\mathbb{R}}$ of $G(\mathbb{R})$, which one usually assumes is integrable. The simplest case is when G is anisotropic over $\mathbb{Q}$. Then $G(\mathbb{Q})\backslash G(\mathbb{A})$

is compact, and the formula for $m_{\mathrm{disc}}(\pi_{\mathbb{R}}, K_0)$ is contained in [13a]. The first result for noncompact quotient was for $G=\mathrm{SL}(2)$, and appeared in Selberg's original paper [18]. There, Selberg gave a formula for the trace of certain Hecke operators for $\mathrm{SL}(2)$. More generally, if G has $\mathbb{R}$-rank one, there is a formula for $m_{\mathrm{disc}}(\pi_{\mathbb{R}}, K_0)$ in [15]. However, if G is of general rank, a formula for a single multiplicity, as opposed to the sum (6.8), will probably have to await the stabilization of the trace formula. At any rate, observe that the condition in Corollary 6.2 is weaker than the integrability of all the representations in $\Pi_{\mathrm{disc}}(\tilde{\mu})$. Such conditions were first studied by F. Williams [21], in connection with multiplicity formulas for compact quotient.

Appendix

The purpose of this appendix is to give a proof of Harish-Chandra's theorem that a unipotent orbital integral can be expressed in terms of semisimple orbital integrals. Lemma 5.2 will be a special case of this.

The main step is to establish an analogous result for the Lie algebra. We assume a familiarity with the theory of Fourier transforms on semisimple Lie algebras. The reader can consult the introduction of [9c] for a summary of the main results. Let $J=J(\mathfrak{g}(\mathbb{C}))$ be the algebra of $G(\mathbb{C})$-invariant elements in the symmetric algebra of $\mathfrak{g}(\mathbb{C})$, and let J_+ be the ideal of elements in J with zero constant term. We shall consider the space $\mathscr{D}_0$ of $G(\mathbb{R})$-invariant, tempered distributions D on $\mathfrak{g}(\mathbb{R})$ such that

$$\partial(q)\, D=0,$$

for every element $q\in J_+$. Here $\partial(q)$ is the differential operator of constant coefficients on $\mathfrak{g}(\mathbb{R})$ associated to q. We shall recall in a moment how to obtain distributions in $\mathscr{D}_0$ from either nilpotent or semisimple orbital integrals. Harish-Chandra's result can be regarded as the assertion that both these classes of examples actually exhaust the space $\mathscr{D}_0$.

Fix an nondegenerate, G-invariant bilinear form B on $\mathfrak{g}$. We can then define the Fourier transform

$$\hat{F}(Y)=\int_{\mathfrak{g}(\mathbb{R})} F(X)\, e^{-iB(X,Y)}\, dX,$$

for any function F in $\mathscr{C}(\mathfrak{g}(\mathbb{R}))$, the Schwartz space on $\mathfrak{g}(\mathbb{R})$. There is an isomorphism $q\to\hat{q}$ from J onto the algebra of $G(\mathbb{R})$-invariant polynomials on $\mathfrak{g}(\mathbb{R})$, such that

$$(\partial(q)F)\hat{}=\hat{q}\hat{F}, \qquad F\in\mathscr{C}(\mathfrak{g}(\mathbb{R})).$$

Now, suppose that $\mathfrak{o}$ is a nilpotent $G(\mathbb{R})$-orbit in $\mathfrak{g}(\mathbb{R})$. By [16, Theorem 1], there is a $G(\mathbb{R})$-invariant measure on $\mathfrak{o}$ with respect to which the integral

$$I_{\mathfrak{o}}(F)=\int_{\mathfrak{o}} F(X)\, dX, \qquad F\in\mathscr{C}(\mathfrak{g}(\mathbb{R})),$$

converges, and defines an invariant tempered distribution. Set

$$D_{\mathfrak{o}}(F) = I_{\mathfrak{o}}(\hat{F}), \qquad F \in \mathscr{C}(\mathfrak{g}(\mathbb{R})).$$

Now, suppose that q belongs to J_+. Then

$$D_{\mathfrak{o}}(\partial(q) F) = I_{\mathfrak{o}}(\hat{q} \hat{F}).$$

But $\hat{q}$ is an invariant polynomial which vanishes at 0. It is known that any such polynomial vanishes on the entire nilpotent variety. Consequently, the function $\hat{q} \hat{F}$ vanishes on $\mathfrak{o}$, so that $I_{\mathfrak{o}}(\hat{q} \hat{F}) = 0$. Thus, the distribution $D_{\mathfrak{o}}$ belongs to $\mathscr{D}_0$. Let $\mathscr{D}_0^{\mathrm{nil}}$ denote the subspace of $\mathscr{D}_0$ spanned by distributions of this form.

Suppose that T is a maximal torus of G over $\mathbb{R}$, with Lie algebra $\mathfrak{t}$. At each H in the regular set $\mathfrak{t}_{\mathrm{reg}}(\mathbb{R})$ of $\mathfrak{t}(\mathbb{R})$, one has the orbital integral

$$\phi_F^T(H) = \pi_T^G(H) \int_{T(\mathbb{R})\backslash G(\mathbb{R})} F(\mathrm{Ad}(x)^{-1} H)\, dx, \qquad F \in \mathscr{C}(\mathfrak{g}(\mathbb{R})),$$

on the Lie algebra. Here

$$\pi_T^G(H) = \prod_{\alpha > 0} \alpha(H),$$

where α ranges over the positive roots of $(\mathfrak{g}, \mathfrak{t})$ with respect to some ordering. Harish-Chandra shows that ϕ_F^T is a smooth function on $\mathfrak{t}_{\mathrm{reg}}(\mathbb{R})$, and that all the derivatives of ϕ_F^T extend continuously to the boundary of any connected component of $\mathfrak{t}_{\mathrm{reg}}(\mathbb{R})$. Let h be a harmonic element in $S(\mathfrak{t}(\mathbb{C}))$, the symmetric algebra on $\mathfrak{t}(\mathbb{C})$, and let c be a connected component of $\mathfrak{t}_{\mathrm{reg}}(\mathbb{R})$. Define an invariant tempered distribution by a limit

$$D_{c,h}(F) = \lim_{\substack{H \to 0 \\ c}} (\partial(h) \phi_F^T)(H), \qquad F \in \mathscr{C}(\mathfrak{g}(\mathbb{R})),$$

in which H approaches 0 through the regular points in c. Suppose that q belongs to J_+. Then

$$D_{c,h}(\partial(q) F) = \lim_{\substack{H \to 0 \\ c}} (\partial(h) \phi_{\hat{q}\hat{F}}^T)(H).$$

Observe that

$$\phi_{\hat{q}\hat{F}}^T(H) = \hat{q}(H)\, \phi_{\hat{F}}^T(H), \qquad H \in c.$$

The function $\hat{q}(H)$ is a Weyl invariant polynomial on $\mathfrak{t}(\mathbb{R})$ which vanishes at the origin, so $\phi_{\hat{q}\hat{F}}$ is the restriction to c of an element in the ideal of smooth functions on $\mathfrak{t}(\mathbb{R})$ generated by such polynomials. If one operates on any function in this ideal by a harmonic differential operator, one obtains another function which also vanishes at the origin. It follows that $D_{c,h}(\partial(q)F)=0$. In other words, the distribution $D_{c,h}$ belongs to $\mathscr{D}_0$. Let $\mathscr{D}_0^{ss}$ denote the subspace of $\mathscr{D}_0$ spanned by distributions of this form.

Theorem A.1. (Harish-Chandra) $\mathscr{D}_0^{\mathrm{nil}}=\mathscr{D}_0^{ss}=\mathscr{D}_0$.

Proof. We have already agreed that $\mathscr{D}_0^{\mathrm{nil}}$ and $\mathscr{D}_0^{ss}$ are subspaces of $\mathscr{D}_0$. We must therefore show that an arbitrary distribution D in $\mathscr{D}_0$ belongs to both $\mathscr{D}_0^{\mathrm{nil}}$ and $\mathscr{D}_0^{ss}$. Let I be the invariant tempered distribution which is defined by

$$I(\hat{F})=D(F), \qquad F\in\mathscr{C}(\mathfrak{g}(\mathbb{R})).$$

If q belongs to J_+, we have

$$I(\hat{q}\hat{F})=D(\partial(q)F)=0, \qquad F\in\mathscr{C}(\mathfrak{g}(\mathbb{R})).$$

It is known that the ring

$$\{\hat{q}\colon q\in J_+\}$$

generates the ideal of polynomials which vanish on the nilpotent variety in $\mathfrak{g}(\mathbb{R})$. From this, one can deduce that the ring also generates the ideal of Schwartz functions that vanish on the nilpotent variety. Thus, I is an invariant distribution which annihilates any function which vanishes on the nilpotent set. It is then possible, using the natural stratification on the nilpotent variety, to write I as a sum $\sum_{\mathfrak{o}} c_{\mathfrak{o}} I_{\mathfrak{o}}$ of nilpotent orbital integrals. We shall spare the details of the argument, since we don't need this part of the theorem for Lemma 5.2. The result, in any case, is that D equals a sum $\sum_{\mathfrak{o}} c_{\mathfrak{o}} D_{\mathfrak{o}}$, and therefore belongs to $\mathscr{D}_0^{\mathrm{nil}}$.

The main point is to show that D belongs to $\mathscr{D}_0^{ss}$. Since it is an invariant eigendistribution of J, D equals a locally integrable function on $\mathfrak{g}(\mathbb{R})$, by Harish-Chandra's fundamental theorem. We may therefore write

$$D(F)=\sum_T |W(G(\mathbb{R}),T(\mathbb{R}))|^{-1} \int_{\mathfrak{t}_{\mathrm{reg}}(\mathbb{R})} D^T(H)\,\phi_F^T(H)\,dH, \qquad F\in\mathscr{C}(\mathfrak{g}(\mathbb{R})). \quad \text{(A.1)}$$

The sum is over a set of representatives $\{T\}$ of conjugacy classes of maximal tori, with Lie algebras $\{\mathfrak{t}\}$, and each D^T stands for a locally integrable function on $\mathfrak{t}(\mathbb{R})$ which is invariant under the real Weyl group $W(G(\mathbb{R}),T(\mathbb{R}))$. The functions D^T are analytic on $\mathfrak{t}_{\mathrm{reg}}(\mathbb{R})$, and they satisfy differential equations in their own right. Let $q\to q_T$ be the Harish-Chandra isomorphism from J onto the Weyl invariant elements in the symmetric algebra on $\mathfrak{t}(\mathbb{C})$. Then

$$\partial(q_T)D^T=0, \qquad q\in J_+.$$

It follows that the restriction of D^T to any connected component of $\mathfrak{t}_{\mathrm{reg}}(\mathbb{R})$ is a harmonic polynomial. Writing n_T for the dimension of the real split component of T, we take the largest integer n such that D is supported on the closed invariant subset

$$G(n)=\{\mathrm{Ad}(x^{-1})H: x\in G(\mathbb{R}), H\in\mathfrak{t}(\mathbb{R}), n_T\geqq n\}.$$

We shall prove the required assertion by induction on n.

To make the inductive argument, we have to look at the expansions (A.1) for the invariant eigendistributions $\phi_F^T(X)$. It is easiest to use the theorem of Rossman [17], which implies that the distributions behave like characters of induced discrete series. In particular, there is a simple nonvanishing constant $\varepsilon(T)$, which depends only on T, such that the invariant distributions

$$F\to\phi_F^T(X)-\varepsilon(T)\int_{\mathfrak{t}(\mathbb{R})} e^{-iB(X,H)}\,\phi_F^T(H)\,dH,\qquad X\in\mathfrak{t}_{\mathrm{reg}}(\mathbb{R}),$$

are supported on $G(n_T+1)$. Now, suppose that $n_T=n$. Since T has minimal split component from among the tori which meet the support of D, D^T extends to a smooth function on $\mathfrak{t}(\mathbb{R})$. This follows from the jump conditions for invariant eigendistributions. (See for example [11, p. 183].) Thus, D^T defines a harmonic polynomial on all of $\mathfrak{t}(\mathbb{R})$. Therefore, there is a harmonic differential operator $\partial(h_T)$ on $\mathfrak{t}(\mathbb{R})$ such that

$$\partial(h_T)\,\varepsilon(T)\,e^{-iB(\cdot,H)}=e^{-iB(\cdot,H)}\,D^T(H),$$

for any fixed $H\in\mathfrak{t}(\mathbb{R})$. Choose a connected component c_T of $\mathfrak{t}_{\mathrm{reg}}(\mathbb{R})$ for each such T, and define

$$D_1(F)=D(F)-\sum_{\{T:n_T=n\}}|W(G(\mathbb{R}),T(\mathbb{R})|^{-1}\,D_{c_T,h_T}(F).$$

Then D_1 is also a distribution in $\mathscr{D}_0$. We claim that it is supported on $G(n+1)$. For

$$D_{c_T,h_T}(F)-\int_{\mathfrak{t}(\mathbb{R})} D^T(H)\,\phi_F^T(H)\,dH$$

equals

$$\lim_{\substack{X\to 0\\ c_T}}((\partial(h_T)\phi_F^T)(X)-\int_{\mathfrak{t}(\mathbb{R})} e^{-iB(X,H)}\,D^T(H)\,\phi_F^T(H)\,dH),$$

and by the remarks above, this distribution is supported on $G(n+1)$. Combining this with (A.1), we see that D_1 is indeed supported on $G(n+1)$. We may assume inductively that the distribution D_1 belongs to $\mathscr{D}_0^{ss}$. It then follows that D itself belongs to $\mathscr{D}_0^{ss}$. $\square$

Corollary A.2. *Suppose that* $\mathfrak{o}$ *is a nilpotent orbit in* $\mathfrak{g}(\mathbb{R})$. *Then we can choose a finite set of triplets* (T_i, c_i, h_i), *where* T_i *is a maximal torus,* c_i *is a chamber* *in* $\mathfrak{t}_i(\mathbb{R}$, *and* h_i *is a harmonic differential operator on* $T_i)$ $\mathbb{R}$ *which is homogenous of degree*

$$\tfrac{1}{2}(\dim G - \operatorname{rank} G - \dim \mathfrak{o}),$$

such that

$$I_{\mathfrak{o}}(F) = \sum_i \lim_{\substack{H \to 0 \\ c_i}} (\partial(h_i)\phi_F^{T_i})(H), \qquad F \in C_c^\infty(\mathfrak{g}(\mathbb{R})). \tag{A.2}$$

Proof. Since $\mathscr{D}_0^{\mathrm{nil}}$ equals $\mathscr{D}_0^{ss}$, it is clear from the theorem that a formula (A.2) exists. The only possible question concerns the degrees of the operators h_i. If t is a positive real number, and F belongs to $\mathscr{C}(\mathfrak{g}(\mathbb{R}))$, the function

$$F^t(X) = F(t^{-1}X), \qquad X \in \mathfrak{g}(\mathbb{R}),$$

also belongs to $\mathscr{C}(\mathfrak{g}(\mathbb{R}))$. It can therefore be substituted into (A.2). By changing variables in the integral over $\mathfrak{o}$, one finds that the resulting left hand side equals

$$(t)^{-\frac{1}{2}\dim \mathfrak{o}}\, I_{\mathfrak{o}}(F).$$

One is therefore able to discard the coefficients of t on the right which are not of the same degree. The operators $\partial(h_i)$ can consequently be chosen to be of the required degree. $\square$

To prove Lemma 5.2, we have to lift the formula (A.2) from the Lie algebra to the group. This is a standard technique of Harish-Chandra. Let $\xi(X)$ be the square root of the nonvanishing analytic function

$$\det((\exp(\tfrac{1}{2}\operatorname{ad} X) - \exp(-\tfrac{1}{2}\operatorname{ad} X))\operatorname{ad}(X)^{-1}), \qquad X \in \mathfrak{g}(\mathbb{R}),$$

such that $\xi(0) = 1$. Then $\xi(X)$ is a $G(\mathbb{R})$-invariant analytic function on $\mathfrak{g}(\mathbb{R})$ whose restriction to the nilpotent variety of $\mathfrak{g}(\mathbb{R})$ equals 1. Moreover, on the Lie algebra $\mathfrak{t}(\mathbb{R})$ of a maximal torus $T(\mathbb{R})$, one has

$$\xi(H) = \prod_{\alpha > 0} ((e^{\frac{\alpha(H)}{2}} - e^{-\frac{\alpha(H)}{2}})\,\alpha(H)^{-1}) = \Delta_T^G(H)\,\pi_T^G(H)^{-1}, \qquad H \in \mathfrak{g}(\mathbb{R}).$$

The exponential map is a diffeomorphism from a $G(\mathbb{R})$-invariant neighborhood of 0 in $\mathfrak{g}(\mathbb{R})$ to a $G(\mathbb{R})$-invariant neighborhood of 1 in $G(\mathbb{R})$. Let $\phi(X)$ be a smooth, $G(\mathbb{R})$-invariant function on $\mathfrak{g}(\mathbb{R})$, which equals 1 on a $G(\mathbb{R})$-invariant neighborhood U of 0, and which is supported in the region where the exponential map is a diffeomorphism. Then if f is any function in $C_c^\infty(G(\mathbb{R}))$, the function

$$F(X) = \phi(X)\,\xi(X) f(\exp X), \qquad X \in \mathfrak{g}(\mathbb{R}),$$

belongs to $C_c^\infty(\mathfrak{g}(\mathbb{R}))$.

Suppose that u is a unipotent element in $G(\mathbb{R})$. Let $\mathfrak{o}$ be the nilpotent orbit in $\mathfrak{g}(\mathbb{R})$ whose image under the exponential map equals the orbit of u. Then if f and F are related as above, we have

$$\begin{aligned} I_G(u,f) &= \int_{\mathfrak{o}} f(\exp X)\,dX \\ &= \int_{\mathfrak{o}} \phi(X)\,\xi(X)\,f(\exp X)\,dX \\ &= I_{\mathfrak{o}}(F). \end{aligned}$$

Moreover, if T is a maximal torus, and H is a point in $U \cap \mathfrak{t}_{\mathrm{reg}}(\mathbb{R})$, we obtain

$$\begin{aligned} F_f^T(H) &= \Delta_T^G(H) \int_{T(\mathbb{R})\backslash G(\mathbb{R})} f(x^{-1}(\exp H)x)\,dx \\ &= \pi_T^G(H) \int_{T(\mathbb{R})\backslash G(\mathbb{R})} (\phi\cdot\xi)(\mathrm{Ad}(x^{-1})H)\,f(\exp(\mathrm{Ad}(x^{-1})H))\,dx \\ &= \phi_F^T(H). \end{aligned}$$

The formula (A.2) then becomes

$$I_G(u,f) = \sum_i \lim_{\substack{H\to 0 \\ c_i}} (\partial(h_i)\,F_f^{T_i})(H). \tag{A.3}$$

We remark in passing that one could also ask for an inverse expansion of $F_f^T(H)$ in terms of unipotent orbital integrals. This is closely related to the paper [2] of Barbasch and Vogan, and would be a real analogue of the germ expansion for p-adic orbital integrals.

We can assume that B is among our set $\{T\}$ of nonconjugate tori. Lemma 5.2 concerns the case that the function $f \in C_c^\infty(G(\mathbb{R}))$ is stable cuspidal. Then F_f^T vanishes unless $T = B$. Moreover, F_f^B is a smooth function on $\mathfrak{b}(\mathbb{R})$. We can therefore take only those i on the right hand side of (A.3) for which $\mathfrak{t}_i$ equals $\mathfrak{b}$, and we may take the limit over H in $\mathfrak{b}(\mathbb{R})$ rather than just the chamber c_i. We obtain a harmonic differential operator $\partial(h_u)$ on $\mathfrak{b}(\mathbb{R})$, which is homogeneous of degree

$$\tfrac{1}{2}(\dim(G/B) - \dim(G/G_u)),$$

such that

$$\Phi_G(u,f) = I_G(u,f) = \lim_{H\to 0} (\partial(h_u)\,F_f^B)(H).$$

This was the assertion of Lemma 5.2.

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