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Intertwining operators and residues II. Invariant distributions

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Introduction

Suppose that G is a reductive algebraic group over a field F of characteristic 0. In the text we shall usually take F to be a general local field, but for purposes of illustration let us assume in the introduction that F is isomorphic to $\mathbb{R}$. In the paper [1(e)] we introduced the weighted characters

$$J_M(\pi_\lambda, f), \quad \pi \in \Pi(M(F)), \lambda \in \mathfrak{a}^*_{M,\mathbb{C}}, f \in \mathscr{H}(G(F)).$$

These objects are like ordinary induced characters

$$\operatorname{tr}(\mathscr{I}_P(\pi_\lambda, f)), \quad P \in \mathscr{P}(M),$$

except that one first composes $\mathscr{I}_P(\pi_\lambda, f)$ with another operator on the space of $\mathscr{I}_P(\pi_\lambda, f)$. This new operator is the logarithmic derivative of the standard intertwining operator in the case of real rank one, and in general has poles in λ. One of the aims of [1(e)] was to investigate the iterated residues

$$\operatorname*{Res}_{\Omega}(J_M(\pi_\lambda, f)). \tag{1}$$

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(We refer the reader to the introduction of [1(e)] for a general discussion as well as a description of the notation used here and below.) If the number of iterated residues is at least equal to $\dim(\mathfrak{a}_M/\mathfrak{a}_G)$, the expression (1) is an invariant distribution in f. What is its connection with other natural invariant distributions on $\mathscr{H}(G(F))$?

In [1(f)] we studied two families

$$I_M(\pi, X, f), \quad \pi \in \Pi(M(F)), X \in \mathfrak{a}_M, \tag{2}$$

and

$$I_M(\gamma, f), \quad \gamma \in M(F), \tag{3}$$

of invariant distributions on $\mathscr{H}(G(F))$. These arise naturally as the local terms of the spectral and geometric sides of the invariant trace formula. It is important to be able to compare the two kinds of terms. In [1(f), §4–5] we gave a rather abstract procedure for doing this, which seems to be sufficient for the applications of the trace formula. Still, it would be interesting to find a more direct connection between the distributions (2) and (3).

In this paper we shall show that the three families of invariant distributions (1), (2) and (3) are all closely related. The distributions (2) were constructed by a formal procedure from the contour integrals

$$J_M(\pi, X, f) = \int_{i\mathfrak{a}_M^*} J_M(\pi_\lambda, f) e^{-\lambda(X)} d\lambda, \quad X \in \mathfrak{a}_M.$$

Deformations of contours inevitably produce residues, so it is not surprising that (2) and (1) should be related. The distributions (3) were constructed by the same formal procedure from the weighted orbital integrals.

$$J_M(\gamma, f), \quad \gamma \in M(F),$$

discussed in [1(d)]. If γ is restricted to lie in a Cartan subgroup $T(F)$ of $M(F)$, then $J_M(\gamma, f)$ is compactly supported in γ. However, it happens that $I_M(\gamma, f)$ is not compactly supported in γ. The residues (1) turn out to be the reason. In the end, it turns out that the distributions (1), the distributions (2) and the asymptotic behaviour of the distributions (3) can all be systematically computed from each other. We shall in fact see that everything may be computed from sufficient information about any one of the three families in the special case of rank 1.

In §1 we shall recall briefly how the three families of distributions are defined. The residues (1) are distinguished by the fact that they are supported on a finite set of representations induced from π. We shall call such distributions π-discrete. In §2 and §3 we shall establish some general properties of π-discrete distributions. Chief among these is Proposition 2.2, which pertains to the normalizing factors for representations induced from Levi subgroups $L \supset M$. The proposition asserts that the normalizing factors have nice properties when they act on a π-discrete distribution on L. This is a generalization of [1(e), Proposition 5.2].

In §4 we shall study the residues in earnest. Recall that

$$\lambda \to J_M(\pi_\lambda, f), \quad \lambda \in \mathfrak{a}^*_{M,\mathbb{C}},$$

is meromorphic, with poles across finitely many hyperplanes. If the residues did not exist, the function

$$J_{M,\mu}(\pi, X, f) \;=\; J_M(\pi_\mu, X, f)e^{-\mu(X)} \;=\; \int_{\mu + i\mathfrak{a}^*_M} J_M(\pi_\lambda, f)e^{-\lambda(X)}\,d\lambda$$

would be independent of $\mu \in \mathfrak{a}^*_M$. As it is, $J_{M,\mu}(\pi, X, f)$ is locally constant on the complement of a finite set of affine hyperplanes. A similar assertion applies to the associated invariant distribution

$$I_{M,\mu}(\pi, X, f) \;=\; I_M(\pi_\mu, X, f)e^{-\mu(X)}.$$

The problem is to compute the jumps of these functions as one moves between different affine chambers. Our main result is Theorem 4.1, which provides an expansion for $I_{M,\mu}(\pi, X, f)$ in terms of the residues (1), the normalizing factors for intertwining operators, and the functions

$$I_{L,\nu_L}(\varrho, h_L(X), f), \quad L \in \mathscr{L}(M),\ \varrho \in \Sigma(L(F)).$$

Here, ν_L is an arbitrary point in general position in $\mathfrak{a}^*_L$. In particular, $\nu = \nu_M$ is an arbitrary point in $\mathfrak{a}^*_M$. Restated as Corollary 4.2, the theorem gives a recursion formula for the difference

$$I_{M,\mu}(\pi, X, f) - I_{M,\nu}(\pi, X, f).$$

Theorem 4.1 can be regarded as a dual version of the various expansions for weighted orbital integrals and their associated invariant distributions.

It is necessary to show that the invariant distributions (1) defined by residues depend only on the characters of f. We will be able to establish this from Theorem 4.1, and the analogous property for the distributions (2), which was proved in [1(f)] and [1(g)]. The proof is actually inductive, the initial induction assumption appearing in §1. Having established Theorem 4.1, we will then be able to complete the argument in §5.

In §6 we shall look at Theorem 4.1 in the special case that f is cuspidal. The formula simplifies considerably. If additional constraints are imposed on π and $\{\nu_L\}$, the expansion for $I_{M,\mu}(\pi, X, f)$ reduces to just one term (Corollary 6.2). The distribution becomes simply a finite sum of residues (1). This has implications for the asymptotic structure of $I_M(\gamma, f)$ (Lemma 6.6). On the other hand, if f is a pseudo-coefficient for a discrete series representation, we shall show that $I_M(\gamma, f)$ equals the value at γ of the discrete series character (Theorem 6.4). (This formula is a variant of the main result of [1(a)], and will be used in another paper on the traces of Hecke operators.) We shall combine the two formulas in Theorem 6.5. The result is a curious identity between the characters of discrete series and residues of intertwining operators. The formula is reminiscent of Osborne's conjecture. However, it attaches to *every* character exponent induced representations which contain the given discrete series as a composition factor.

§1. Residues

Let G be a reductive algebraic group over a field F, of characteristic 0. In this article we shall impose two conditions which were not in the preceding paper [1(e)]. We shall assume that G is connected, and that F is a local field. For we want to study invariant distributions that rely on the trace Paley–Wiener theorem, and this has been established in general only for connected groups. The second condition, that on F, is essentially for convenience. We write v for the (normalized) valuation on F.

We shall adopt the notations and conventions of [1(e)], often without further comment. In particular,

$$\mathfrak{a}_{G,v} = H_G(G(F))$$

is a closed subgroup of

$$\mathfrak{a}_G = \mathrm{Hom}(X(G)_F, \mathbb{R}).$$

The two groups are equal if v is Archimedean, but if v is discrete, $\mathfrak{a}_{G,v}$ is a lattice in $\mathfrak{a}_G$. The unitary dual of $\mathfrak{a}_{G,v}$ is isomorphic to

$$i\mathfrak{a}^*_{G,v} = i\mathfrak{a}^*_G / i\mathfrak{a}^\vee_{G,v}$$

where

$$\mathfrak{a}_{G,v}^{\vee} = \mathrm{Hom}(\mathfrak{a}_{G,v}, \mathbb{Z}).$$

We are interested in the Hecke algebra $\mathscr{H}(G(F))$ of functions on $G(F)$ which are left and right finite with respect to a suitable fixed maximal compact subgroup K of $G(F)$. We also have the larger space $\mathscr{H}_{\mathrm{ac}}(G(F))$, introduced in [1(e), §11], as well as corresponding spaces $\mathscr{I}(G(F))$ and $\mathscr{I}_{\mathrm{ac}}(G(F))$ of functions on $\Pi_{\mathrm{temp}}(G(F)) \times \mathfrak{a}_{G,v}$. These are related by a continuous surjective map $f \to f_G$ from $\mathscr{H}_{\mathrm{ac}}(G(F))$ onto $\mathscr{I}_{\mathrm{ac}}(G(F))$, which maps $\mathscr{H}(G(F))$ onto $\mathscr{I}(G(F))$. As in [1(e)], we will sometimes regard an element ϕ in the smaller space $\mathscr{I}(G(F))$ as a function of just one variable in $\Pi_{\mathrm{temp}}(G(F))$. The two interpretations are related by a Fourier transform

$$\phi(\pi, X) = \int_{i\mathfrak{a}_{G,v}^*} \phi(\pi_\lambda) e^{-\lambda(X)}\, d\lambda, \quad (\pi, X) \in \Pi_{\mathrm{temp}}(G(F)) \times \mathfrak{a}_{G,v}.$$

Thus, if f belongs to $\mathscr{H}(G(F))$, we can either write

$$f_G(\pi_\lambda) = \mathrm{tr}(\pi_\lambda(f))$$

or

$$f_G(\pi, X) = \mathrm{tr}\,\pi(f^X) = \mathrm{tr}\left(\int_{G(F)^X} f(x)\pi(x)\, dx\right),$$

where f^X stands for the restriction of f to

$$G(F)^X = \{x \in G(F) : H_G(x) = X\}.$$

Suppose that I is a continuous linear functional or "distribution" on $\mathscr{H}_{\mathrm{ac}}(G(F))$, which is invariant. We say that I is supported on characters if $I(f) = 0$ for every function f such that f_G vanishes. If this is so, there is a unique "distribution" $\hat{I}$ on $\mathscr{I}_{\mathrm{ac}}(G(F))$ such that

$$I(f) = \hat{I}(f_G), \quad f \in \mathscr{H}_{\mathrm{ac}}(G(F)).$$

The symbol M always stands for a Levi component of some parabolic subgroup of G over F which is in good relative position with respect to K. That is, K must be admissible relative to M, in the sense of §1 of [1(b)]. As always $\mathscr{L}(M)$ denotes the finite set of Levi subgroups which contain M. In the paper [1(f)] we introduced two families

$$I_M(\gamma, f) = I_M^G(\gamma, f), \quad \gamma \in M(F), f \in \mathscr{H}_{\mathrm{ac}}(G(F)),$$

and

$$I_M(\pi, X, f) \;=\; I_M^G(\pi, X, f), \quad \pi \in \Pi(M(F)),\ X \in \mathfrak{a}_{M,v},\ f \in \mathscr{H}_{\mathrm{ac}}(G(F)),$$

of invariant distributions on $\mathscr{H}_{\mathrm{ac}}(G(F))$ which were eventually shown to be supported on characters ([1(f), Theorem 6.1], [1(g), Theorem 5.1]). They are characterized by formulas

$$J_M(\gamma, f) \;=\; \sum_{L \in \mathscr{L}(M)} \hat{I}_M^L(\gamma, \phi_L(f)) \tag{1.1}$$

and

$$J_M(\pi, X, f) \;=\; \sum_{L \in \mathscr{L}(M)} \hat{I}_M^L(\pi, X, \phi_L(f)), \tag{1.2}$$

in which $J_M(\gamma, f)$ is a weighted orbital integral [1(d), §6], $J_M(\pi, X, f)$ is a weighted character [1(e), §7], and

$$\phi_L\colon \mathscr{H}_{\mathrm{ac}}(G(F)) \to \mathscr{I}_{\mathrm{ac}}(L(F))$$

is the map defined in §12 of [1(e)]. The two families are closely related. Roughly speaking, $\{I_M(\pi, X)\}$ measures the obstruction to $\{I_M(\gamma)\}$ being compactly supported in γ. In fact, there is an asymptotic expansion for $I_M(\gamma, f)$ in terms of certain maps

$$\theta_L\colon \tilde{\mathscr{H}}_{\mathrm{ac}}(G(F)) \to \tilde{\mathscr{I}}_{\mathrm{ac}}(L(F)), \quad L \in \mathscr{L}(M),$$

and these maps are completely determined by the distributions $\{I_L(\pi, X, f)\}$. (See [1(f), (4.11), Lemma 4.1, and (4.9)].) Thus, the second family of distributions determines the asymptotic behaviour of the first.

For the second family of distributions, it is sometimes appropriate to take a standard representation $\varrho \in \Sigma(M(F))$ instead of the irreducible π. (See [1(e), §5]. Recall that a standard representation is induced from a representation which is tempered modulo the center, and may be reducible.) One defines distributions $J_M(\varrho, X, f)$ and $I_M(\varrho, X, f)$ in a similar manner. The two cases are related by a formula

$$I_M(\pi, X, f) \;=\; \sum_{P \in \mathscr{P}(M)} \omega_P \sum_{L \in \mathscr{L}(M)} r_{M,\varepsilon_P}^L(\pi, X, I_L(f)), \tag{1.3}$$

where

$$r_{M,\varepsilon_P}^L(\pi, X, I_L(f))$$

equals

$$\int_{\varepsilon_P + i\mathfrak{a}^*_{M,v}/i\mathfrak{a}^*_{L,v}} \sum_{\varrho \in \Sigma(M(F))} r^L_M(\pi_\lambda, \varrho_\lambda) I_L(\varrho^L_\lambda, h_L(X), f) e^{-\lambda(X)} \, d\lambda.$$

(See [1(f), (3.2)].) The notation here follows [1(e)] and [1(f)]. In particular, ε_P stands for a small point in general position in the chamber $(\mathfrak{a}^*_P)^+$, and

$$\omega_P = \operatorname{vol}(\mathfrak{a}^+_P \cap B) \operatorname{vol}(B)^{-1},$$

where B is a ball in $\mathfrak{a}_M$, centered at the origin. The function $r^L_M(\pi_\lambda, \varrho_\lambda)$ is obtained from the ratios of the normalizing factors for π_λ and ϱ_λ.

Our ultimate goal is to show how to compute $I_M(\pi, X, f)$ in terms of residues. Fix an element $L \in \mathscr{L}(M)$. A residue datum Ω for (L, M) is a pair $(\mathscr{E}_\Omega, \Lambda_\Omega)$, where

$$\mathscr{E}_\Omega = (E_1, \ldots, E_r)$$

is an orthogonal basis of $(\mathfrak{a}^L_M)^*$ and Λ_Ω is a point in $(\mathfrak{a}^L_M)^*_{\mathbf{C}}$. It is required that there be an embedded sequence

$$M = M_0 \subset M_1 \subset \cdots \subset M_r = L$$

of elements in $\mathscr{L}(M)$ such that

$$\mathfrak{a}_{M_i} = \{H \in \mathfrak{a}_{M_{i-1}} : E_i(H) = 0\}, \quad 1 \leqslant i \leqslant r.$$

(See [1(e), §8].) Given such an Ω, as well as a meromorphic function $\psi(\Lambda)$ on $\mathfrak{a}^*_{M,\mathbf{C}}$ and a point $\Lambda_0 \in \Lambda_\Omega + \mathfrak{a}^*_{L,\mathbf{C}}$ in general position, define

$$\operatorname*{Res}_{\Omega, \Lambda \to \Lambda_0} \psi(\Lambda) = (2\pi i)^{-r} \int_{\Gamma_r} \cdots \int_{\Gamma_1} \psi(\Lambda_0 + z_1 E_1 + \cdots + z_r E_r) \, dz_1 \ldots dz_r.$$

As in [1(e), §8], $\Gamma_1, \ldots, \Gamma_r$ are small positively oriented circles about the origin in the complex plane such that for each i, the radius of Γ_i is much smaller than that of Γ_{i+1}. It is this condition on the radii which allows us to express an iterated residue as an iterated contour integral in r complex variables.

We are interested in the case that

$$\psi(\Lambda) = a_\Lambda J^L_M(\pi_\Lambda, g), \quad \pi \in \Pi(M(F)), g \in \mathscr{H}(L(F)),$$

where a_Λ is an analytic function. Recall that

$$J_M^L(\pi_\Lambda, g) = \mathrm{tr}(\mathscr{R}_M^L(\pi_\Lambda, R_0)\mathscr{I}_{R_0}(\pi_\Lambda, g)),$$

where $\mathscr{I}_{R_0}(\pi_\Lambda)$ is the representation induced from a parabolic subgroup R_0, and $\mathscr{R}_M^L(\pi_\Lambda, R_0)$ is an operator on the underlying space $\mathscr{V}_{R_0}(\pi)$ which is obtained from normalized intertwining operators ([1(e), §6]). It is the Fourier transform (in Λ) of $J_M^L(\pi_\Lambda, g)$ which equals $J_M^L(\pi, X, g)$. According to Lemma 8.1 of [1(e)], the distribution

$$\operatorname*{Res}_{\Omega,\Lambda\to\Lambda_0}(a_\Lambda J_M^L(\pi_\Lambda, g)), \quad g \in \mathscr{H}(L(F)), \tag{1.4}$$

is invariant. We would like to know that it is supported on characters. Instead of trying to show this directly, we shall make an induction hypothesis. We assume that for any $L \neq G$, and for any π and a_Λ, the distribution (1.4) is supported on characters. In §5 we shall complete the induction argument by showing that the same thing is true if $L = G$.

§2. π-discrete distributions

For the next several sections, the Levi subgroup M and the representation $\pi \in \Pi(M(F))$ will be fixed. We would like to relate $I_M(\pi, X, f)$ with the residues (1.4) of the distributions $J_M^L(\pi_\lambda)$. However, we shall not actually discuss the residues in detail until §4. The purpose of this section is to introduce a general family of distributions of which the residues are typical examples.

It is best to take functions in $\mathscr{H}(G(F))$ which also depend analytically on a parameter $\Lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$. Let $\mathscr{H}(\mathfrak{a}_M, G(F))$ denote the space of functions

$$F: (\mathfrak{a}_{M,\mathbb{C}}^*/i\mathfrak{a}_{M,v}^\vee) \times G(F) \to \mathbb{C}$$

such that

$$X \to \int_{i\mathfrak{a}_{M,v}^*} F(\Lambda, x)e^{-\Lambda(X)}\,d\Lambda, \quad X \in \mathfrak{a}_{M,v}, \ x \in G(F),$$

is a smooth, compactly supported function on $\mathfrak{a}_{M,v}$ with values in $\mathscr{H}(G(F))$. In other words,

$$F(\Lambda): x \to F(\Lambda, x)$$

is a Paley–Wiener function of Λ with values in $\mathscr{H}(G(F))_\Gamma$, for some finite subset Γ of $\Pi(K)$. (The reader is asked to tolerate notation in which F stands for both a function and a field.) Similarly, we can define $\mathscr{I}(\mathfrak{a}_M, G(F))$ to be the space of functions

$$\Phi\colon (\mathfrak{a}^*_{M,\mathbb{C}}/i\mathfrak{a}^\vee_{M,v}) \times \Pi_{\mathrm{temp}}(G(F)) \to \mathbb{C}$$

such that for some finite $\Gamma \subset \Pi(K)$,

$$\Phi(\Lambda)\colon \pi \to \Phi(\Lambda, \pi)$$

is a Paley–Wiener function of Λ with values in $\mathscr{I}(G(F))_\Gamma$. As always, any such function is analytic in π; as a function in the various continuous co-ordinates of $\Pi_{\mathrm{temp}}(G(F))$, Φ extends analytically to the entire complex domain. It can therefore be defined for each standard representation $\varrho \in \Sigma(G(F))$. Both of these new spaces are algebras, and the elementary notions from invariant harmonic analysis [1(f), §1] extend formally to this setting. In particular,

$$F \to F_G(\Lambda, \pi) \quad = \quad \mathrm{tr}\, \pi(F(\Lambda))$$

is a continuous surjective map from $\mathscr{H}(\mathfrak{a}_M, G(F))$ onto $\mathscr{I}(\mathfrak{a}_M, G(F))$.

Now, suppose that π is a general representation in $\Pi(M(F))$. Let

$$D \quad = \quad D(\pi)\colon F \to D(\pi, F),$$

be a distribution (i.e., continuous linear functional) on $\mathscr{H}(\mathfrak{a}_M, G(F))$ which is supported on characters. Then there is a unique distribution $\hat{D} = \hat{D}(\pi)$ on $\mathscr{I}(\mathfrak{a}_M, G(F))$ such that

$$D(\pi, F) \quad = \quad \hat{D}(\pi, F_G), \quad F \in \mathscr{H}(\mathfrak{a}_M, G(F)).$$

We shall say that D is *π-discrete* if, as well as being supported on characters, $D(\pi, F)$ factors to a distribution on the space

$$\{\mathscr{I}_P(\pi_\Lambda, F(\Lambda))\colon P \in \mathscr{P}(M),\ \Lambda \in \mathfrak{a}^*_{M,\mathbb{C}}/i\mathfrak{a}^\vee_{M,v}\}$$

which is supported at finitely many points $\Lambda_1, \ldots, \Lambda_t$. Of course, a distribution on a space of analytic functions does not in general have support. However, if a_Λ is a function which is analytic in a neighbourhood of Λ_0 in $\mathfrak{a}^*_{M,\mathbb{C}}$, we shall write

$$d^n_{M,\Lambda\to\Lambda_0} a_\Lambda$$

for the Taylor polynomial of a_Λ at $\Lambda = \Lambda_0$ of total degree n. Then the condition on D is that $D(\pi, F)$ depends only on

$$\bigoplus_{i=1}^{t} d^n_{M,\Lambda\to\Lambda_i}\mathscr{I}_P(\pi_\Lambda, F(\Lambda)),$$

for some fixed integer n.

For a simple example, take a differential operator $\Delta = \Delta_\Lambda$ on $\mathfrak{a}^*_{M,\mathbf{C}}$ and a fixed point $\Lambda_0 \in \mathfrak{a}^*_{M,\mathbf{C}}$. Then

$$F \to \lim_{\Lambda\to\Lambda_0} \Delta_\Lambda \operatorname{tr}(\mathscr{I}_P(\pi_\Lambda, F(\Lambda))), \quad F \in \mathscr{H}(\mathfrak{a}_M, G(F)),$$

is a π-discrete distribution. More interesting examples are provided by the residues. Suppose for a moment that G is replaced by a group $L \in \mathscr{L}(M)$, with $L \neq G$, and that Ω is a residue datum for (L, M). Then the residue

$$\operatorname*{Res}_{\Omega,\Lambda\to\Lambda_\Omega} J^L_M(\pi_\Lambda, F(\Lambda)), \quad F \in \mathscr{H}(\mathfrak{a}_M, L(F)),$$

is supported on characters. This follows from the induction hypothesis of §1. The distribution is obviously supported at a finite set of points. It is therefore a π-discrete distribution on $\mathscr{H}(\mathfrak{a}_M, L(F))$.

LEMMA 2.1: *Suppose that D is a π-discrete distribution on $\mathscr{H}(\mathfrak{a}_M, G(F))$. Then there is an n such that the value*

$$\hat{D}(\pi, \Phi), \quad \Phi \in \mathscr{I}(\mathfrak{a}_M, G(F)),$$

depends only on an expression

$$\bigoplus_{i=1}^{t}\bigoplus_{j=1}^{s} d^n_{M,\Lambda\to\Lambda_i} \mathrm{d}^n_{M_j,\eta\to 0}\Phi(\Lambda, \varrho^G_{j,\eta}),$$

for fixed points $\Lambda_i \in \mathfrak{a}^*_{M,\mathbf{C}}$, Levi subgroups M_j of G and standard representations $\varrho_j \in \Sigma(M_j(F))$.

Proof: The finite support property of D concerns the operators $\mathscr{I}_P(\pi_\Lambda, F(\Lambda))$, not their traces. But we are also given that $D(\pi, F)$ depends only on F_G. We must convert this abstract assertion into a finite support property in the function

$$F_G(\Lambda, \varrho) = \operatorname{tr}\varrho(F(\Lambda)), \quad \varrho \in \Sigma(G(F)).$$

If F is Archimedean, let $\mathscr{Z}(G)$ be the center of the universal enveloping algebra. If F is non-Archimedean, we take $\mathscr{Z}(G)$ to be the Bernstein center [2]. In either case, $\mathscr{Z}(G)$ acts on $\mathscr{H}(G(F))$, so it also acts on $\mathscr{H}(\mathfrak{a}_M, G(F))$ through the second factor. From the definition of π-discrete we see that D is annihilated by an ideal of finite codimension in $\mathscr{Z}(G)$. The lemma will then follow from a straightforward infinitesimal character argument. We leave the details to the reader. □

For any invariant distribution on $\mathscr{I}(\mathfrak{a}_M, G(F))$ which is supported on a finite set of points, the space of test functions can be enlarged. Let us write $\mathscr{I}^+(\mathfrak{a}_M, G(F))$ for the space of functions Φ, defined almost everywhere on

$$(\mathfrak{a}^*_{M,\mathbb{C}}/i\mathfrak{a}_{M,v}) \times \Sigma(G(F)),$$

which satisfy the following condition. For any Levi subgroup M_1 of G, and any $\varrho \in \Sigma(M_1(F))$,

$$(\Lambda, \Lambda_1) \to \Phi(\Lambda, \varrho^G_{\Lambda_1}), \quad (\Lambda, \Lambda_1) \in \mathfrak{a}^*_{M,\mathbb{C}} \times \mathfrak{a}^*_{M_1,\mathbb{C}},$$

is a meromorphic function whose poles lie along hyperplanes of the form

$$\Lambda(X) + \Lambda_1(X_1) = c, \quad c \in \mathbb{C},$$

if F is Archimedean, and

$$q^{-(\Lambda(X)+\Lambda_1(X_1))} = c, \quad c \in \mathbb{C},$$

if F is non-Archimedean with residual order q. Here (X, X_1) is a vector in $(\mathfrak{a}_{M,v} \oplus \mathfrak{a}_{M_1,v})$ which we assume has nonzero projection onto the diagonally embedded subgroup $\mathfrak{a}_{G,v}$. If $\Phi \in \mathscr{I}^+(\mathfrak{a}_M, G(F))$ and $\lambda \in \mathfrak{a}^*_{G,\mathbb{C}}$, the function

$$\Phi_\lambda(\Lambda, \varrho) = \Phi(\Lambda + \lambda, \varrho_\lambda), \quad \Lambda \in \mathfrak{a}^*_{M,\mathbb{C}}, \varrho \in \Sigma(G(F)),$$

also belongs to $\mathscr{I}^+(\mathfrak{a}_M, G(F))$. Notice that for almost all λ, the singularities of Φ_λ will not meet a given finite set of points (Λ, ϱ). This is a consequence of the condition on (X, X_1) above.

Suppose that $D = D(\pi)$ is a π-discrete distribution on $\mathscr{H}(\mathfrak{a}_M, G(F))$ and that Φ belongs to $\mathscr{I}^+(\mathfrak{a}_M, G(F))$. The last lemma implies that $\hat{D}(\pi, \Phi_\lambda)$ is defined whenever $\lambda \in \mathfrak{a}^*_{G,\mathbb{C}}$ is in general position. Moreover, $\hat{D}(\pi, \Phi_\lambda)$ is a meromorphic function of λ. Set

$$(\hat{D} \cdot \Phi^\vee_\lambda)(\pi, \Psi) = \hat{D}(\pi, \Phi_\lambda \Psi), \quad \Psi \in \mathscr{I}(\mathfrak{a}_M, G(F)),$$

for λ in general position. Then $\hat{D} \cdot \Phi_\lambda^\vee$ is another distribution on $\mathscr{I}(\mathfrak{a}_M, G(F))$. We shall write

$$\mathscr{D}(\pi) = \mathscr{D}^G(\pi)$$

for the space of distributions obtained in this way from all such choices of D and Φ. Observe that if D is fixed, and Φ and λ vary, then $\{\hat{D} \cdot \Phi_\lambda^\vee\}$ is a finite dimensional subspace of $\mathscr{D}(\pi)$. Any distribution $\delta = \delta(\pi)$ in $\mathscr{D}(\pi)$ is supported at a finite set of points, and if Φ belongs to $\mathscr{I}^+(\mathfrak{a}_M, G(F))$,

$$\lambda \to \delta(\pi, \Phi_\lambda) \quad \lambda \in \mathfrak{a}_{G,\mathbf{C}}^*,$$

is defined as a meromorphic function. Obviously, $\Phi_\lambda^\vee$ can be made to act on any distribution in $\mathscr{D}(\pi)$. Therefore

$$\Phi \to \Phi_\lambda^\vee$$

may be interpreted as a homomorphism from the algebra $\mathscr{I}^+(\mathfrak{a}_M, G(F))$ to the algebra of meromorphic functions of λ with values in the space of endomorphisms of $\mathscr{D}(\pi)$.

The main purpose of this discussion is to accommodate the normalizing factors for induced representations discussed in [1(e)]. Assume that we have fixed normalizing factors

$$r_{P'|P}(\pi_\lambda), \quad P, P' \in \mathscr{P}(M),$$

(for all possible choices of M and π) which satisfy the conditions of [1(e), Theorem 2.1]. Suppose that L is an element in $\mathscr{L}(M)$. If Q, Q' belong to $\mathscr{P}(L)$, the normalizing factors $r_{Q'|Q}(\pi_\Lambda^L)$ are defined. We also have normalizing factors $r_{Q'|Q}(\varrho_\lambda)$ for each $\varrho \in \Sigma(L(F))$. If $\lambda \in \mathfrak{a}_{L,\mathbf{C}}^*$, set

$$\tilde{r}_{Q'|Q,\lambda} : (\Lambda, \varrho) \to \tilde{r}_{Q'|Q}(\pi_{\Lambda+\lambda}^L, \varrho_\lambda), \quad (\Lambda, \varrho) \in \mathfrak{a}_{M,\mathbf{C}}^* \times \Sigma(L(F)),$$

where

$$\tilde{r}_{Q'|Q}(\pi_{\Lambda+\lambda}^L, \varrho_\lambda) = r_{Q'|Q}(\pi_{\Lambda+\lambda}^L)^{-1} r_{Q'|Q}(\varrho_\lambda),$$

as in [1(e)]. The earlier definitions are of course valid if G is replaced by L, and we see easily that $\tilde{r}_{Q'|Q,\lambda}$ is a function in $\mathscr{I}^+(\mathfrak{a}_M, L(F))$. At this point, we have imposed no condition of block equivalence on π_λ^L and ϱ; the usual transitivity property [1(e), Proposition 5.2] consequently fails for $\tilde{r}_{Q'|Q,\lambda}$.

However, let us set

$$r_{Q'|Q,\lambda} = \tilde{r}^{\vee}_{Q'|Q,\lambda},$$

so that $r_{Q'|Q,\lambda}$ is an endomorphism of $\mathscr{D}^L(\pi)$. The next proposition, which is our justification of the constructions above, asserts that $r_{Q'|Q,\lambda}$ does have the transitivity property.

For any root α of (G, A_M), set $q_{v,\alpha}(\lambda) = \lambda(\alpha^\vee)$ if F is Archimedean, and put $q_{v,\alpha}(\lambda) = q_v^{-\lambda(\alpha^\vee)}$ if F is non-Archimedean of residual order q_v.

PROPOSITION 2.2: *We have*

$$r_{Q''|Q,\lambda} = r_{Q''|Q',\lambda} \cdot r_{Q'|Q,\lambda}, \quad Q, Q', Q'' \in \mathscr{P}(L). \tag{2.1}$$

Moreover, $r_{Q'|Q,\lambda}$ *is as rational function of the variables* $\{q_{v,\alpha}(\lambda)\}$ *with values in the space of endomorphisms of* $\mathscr{D}^L(\pi)$.

REMARK: Consider the special case in which $L = M$ and $D(\pi)$ equals the character of π. That is,

$$D(\pi, F) = \operatorname{tr} \pi(F(0)), \quad F \in \mathscr{H}(\mathfrak{a}_M, M(F)).$$

Then $D(\pi)$ is a π-discrete distribution on $\mathscr{H}(\mathfrak{a}_M, M(F))$ whose Fourier transform equals

$$\hat{D}(\pi, \Phi) = \sum_{\varrho \in \Sigma(M(F))} \Delta(\pi, \varrho)\Phi(0, \varrho), \quad \Phi \in \mathscr{I}(\mathfrak{a}_M, M(F)),$$

the formal decomposition into standard characters. (See §5 of [1(e)].) It is obvious that

$$(\hat{D} \cdot r_{Q'|Q,\lambda})(\pi, \Phi) = \sum_{\varrho} \Delta(\pi, \varrho)\tilde{r}_{Q'|Q}(\pi_\lambda, \varrho_\lambda)\Phi(0, \varrho), \quad Q, Q' \in \mathscr{P}(M).$$

The proposition in this case is essentially equivalent to Proposition 5.2 of [1(e)].

We shall reduce the proof of Proposition 2.2 to a second assertion. If σ is a representation which belongs to either $\Pi(L(F))$ or $\Sigma(L(F))$, set

$$\mu_L(\sigma_\lambda) = (r_{Q|\bar{Q}}(\sigma_\lambda)r_{\bar{Q}|Q}(\sigma_\lambda))^{-1}, \quad Q \in \mathscr{P}(\mathrm{L}), \lambda \in \mathfrak{a}^*_{L,\mathbb{C}}.$$

Since the normalized intertwining operators

$$R_{Q'|Q}(\sigma_\lambda) = r_{Q'|Q}(\sigma_\lambda)^{-1} J_{Q'|Q}(\sigma_\lambda)$$

satisfy

$$R_{Q|\bar{Q}}(\sigma_\lambda) R_{\bar{Q}|Q}(\sigma_\lambda) = 1,$$

the operator

$$J_{Q|\bar{Q}}(\sigma_\lambda) J_{\bar{Q}|Q}(\sigma_\lambda)$$

is equal to the product of $\mu_L(\sigma_\lambda)^{-1}$ with the identity operator. Thus, $\mu_L(\sigma_\lambda)$ is just the usual μ-function. It is independent of $Q \in \mathscr{P}(L)$. Corollary 5.3 of [1(e)] asserts that if $\varrho \in \Sigma(L(F))$ contains σ as a composition factor, then $\mu_L(\sigma_\lambda)$ equals $\mu_L(\varrho_\lambda)$.

LEMMA 2.3: *Suppose that $D = D(\pi)$ is a π-discrete distribution on $\mathscr{H}(\mathfrak{a}_M, L(F))$, and that Φ belongs to $\mathscr{I}^+(\mathfrak{a}_M, L(F))$. Set*

$$\Phi_\lambda^1(\Lambda, \varrho) = \mu_L(\pi_{\Lambda+\lambda}^L)^{-1} \Phi_\lambda(\Lambda, \varrho)$$

and

$$\Phi_\lambda^2(\Lambda, \varrho) = \mu_L(\varrho_\lambda)^{-1} \Phi_\lambda(\Lambda, \varrho).$$

Then

$$\hat{D}(\pi, \Phi_\lambda^1) = \hat{D}(\pi, \Phi_\lambda^2).$$

This lemma is the main step in the proof of Proposition 2.2. It will be a consequence of some general properties of (unnormalized) intertwining operators which we shall review in the next section. We shall postpone the proof of the lemma until then.

Assuming Lemma 2.3, let us establish the proposition. For $\lambda \in \mathfrak{a}_{L,\mathbb{C}}^*$ in general position, the function

$$\tilde{\mu}_{L,\lambda} : (\Lambda, \varrho) \to \mu_L(\pi_{\Lambda+\lambda}^L)^{-1} \mu_L(\varrho_\lambda), \quad (\Lambda, \varrho) \in \mathfrak{a}_{M,\mathbb{C}}^* \times \Sigma(L(F)),$$

belongs to $\mathscr{I}^+(\mathfrak{a}_M, L(F))$. Choose an arbitrary function Ψ in $\mathscr{I}^+(\mathfrak{a}_M, L(F))$, and set

$$\Phi_\lambda(\Lambda, \varrho) = \mu_L(\varrho_\lambda) \Psi_\lambda(\Lambda, \varrho).$$

Then

$$\tilde{\mu}_{L,\lambda}\Psi_\lambda = (\Phi^1_\lambda - \Phi^2_\lambda) + \Psi_\lambda,$$

in the notation of Lemma 2.3. It then follows from the lemma that

$$D(\pi, \tilde{\mu}_{L,\lambda}\Psi_\lambda) = D(\pi, \Psi_\lambda).$$

Consequently, $\tilde{\mu}^\vee_{L,\lambda}$ equals 1. But

$$\tilde{\mu}_{L,\lambda} \cdot \tilde{r}_{Q|\bar{Q},\lambda} \cdot \tilde{r}_{\bar{Q}|Q,\lambda} = 1.$$

We have therefore established that

$$r_{Q|\bar{Q},\lambda} \cdot r_{\bar{Q}|Q,\lambda} = 1. \tag{2.2}$$

Given (2.2), we can now prove Proposition 2.2 in exactly the same way as the relevant portion of Theorem 2.1 of [1(e)]. For example, to establish (2.1), we make use of the decomposition

$$r_{Q'|Q,\lambda} = \prod_{\beta \in \Sigma^r_{Q'} \cap \Sigma^r_{\bar{Q}}} r_{\beta,\lambda} \tag{2.3}$$

which is the analogue of [1(e), (2.1)]. The formula (2.2), with G replaced by a group L_β of which L is a maximal Levi subgroup, implies that

$$r_{-\beta,\lambda} \cdot r_{\beta,\lambda} = 1.$$

The relation (2.1) then follows. The rationality assertion of Proposition 2.2 is trivial if F is non-Archimedean, for the normalizing factors are themselves rational in this case. If F is Archimedean, the normalizing factors are constructed from gamma functions. The functions $r_{P'|P,\lambda}$ therefore satisfy an analogue of the estimate (3.8) in [1(e)]. As in §3 of [1(e)], the rationality assertion then follows from the multiplicative property (2.1). This completes the proof of Proposition 2.2. □

Fix $Q_0 \in \mathscr{P}(L)$. Then

$$r_{Q,\lambda}(\zeta, Q_0) = r^{-1}_{Q|Q_0,\lambda} \cdot r_{Q|Q_0,\lambda+\zeta}, \quad Q \in \mathscr{P}(L), \zeta \in i\mathfrak{a}^*_L,$$

is a (G, L) family. As usual [1(b), §6], we can define

$$r^G_{L,\lambda} = \lim_{\zeta \to 0} \sum_{Q \in \mathscr{P}(L)} r_{Q,\lambda}(\zeta, Q_0)\theta_Q(\zeta)^{-1}. \tag{2.4}$$

It follows from Proposition 2.2 that $r^G_{L,\lambda}$ is a rational function of the variables $\{q_{v,\alpha}(\lambda)\}$ with values in $\mathrm{End}(\mathscr{D}^L(\pi))$. It is independent of Q_0. Suppose that $L_1 \in \mathscr{L}(L)$, and that Q_1 belongs to $\mathscr{P}(L_1)$. Then

$$r^{Q_1}_{L,\lambda} = \lim_{\zeta\to 0} \sum_{\{Q\in\mathscr{P}(L):Q\subset Q_1\}} r_{Q,\lambda}(\zeta, Q_0)\theta_Q(\zeta)^{-1}$$

depends only on L_1 and not on Q_1. In fact, $r^{Q_1}_{L,\lambda}$ equals the function $r^{L_1}_{L,\lambda}$ defined by (2.4), but with G replaced by L_1.

Suppose that $D = D(\pi)$ is a π-discrete distribution on $\mathscr{H}(\mathfrak{a}_M, L(F))$ and that $L_1 \in \mathscr{L}(L)$, as above. In practice, we shall want to consider $(\hat{D}\cdot r^{L_1}_{L,\lambda})$ as a distribution on $\mathscr{I}(L_1(F))$ which also depends on a point $X \in \mathfrak{a}_{M,v}$. As a matter of fact, $\mathscr{I}(L_1(F))$ has in the past been regarded as a space of functions on $\Sigma(L_1(F)) \times \mathfrak{a}_{L_1,v}$, so this should be reflected in the notation. If $\phi \in \mathscr{I}(L_1(F))$ and $X \in \mathfrak{a}_{M,v}$, and if $\nu_L \in \mathfrak{a}^*_L$ is a point in general position, define

$$(\hat{D}\cdot r^{L_1}_{L,\nu_L})(\pi, X, \phi) = \int_{\nu_L + i\mathfrak{a}^*_{L,v}/i\mathfrak{a}^*_{L_1,v}} (\hat{D}\cdot r^{L_1}_{L,\lambda})(\pi, \phi^X_\lambda)\,d\lambda, \tag{2.5}$$

where

$$\phi^X_\lambda(\Lambda, \varrho) = \phi^X(\Lambda + \lambda, \varrho_\lambda) = e^{-(\Lambda+\lambda)(X)}\phi(\varrho^{L_1}_\lambda, h_{L_1}(X)),$$

$$(\Lambda, \varrho) \in \mathfrak{a}^*_{M,\mathbb{C}} \times \Sigma(L(F)).$$

The convergence of this integral follows from the second assertion of Proposition 2.2. More generally, we can take any ϕ that behaves well on the support of the given distribution. By Lemma 2.1, we can assume that

$$\hat{D}(\pi, \Phi), \quad \Phi \in \mathscr{I}(\mathfrak{a}_M, L(F)),$$

depends only on a function

$$\bigoplus_i \bigoplus_j d^n_{M,\Lambda\to\Lambda_i} d^n_{M_j,\eta\to 0}\Phi(\Lambda, \varrho^L_{j,\eta}).$$

Then the definition of (2.5) makes sense if ϕ is any function on $\Sigma(L_1(F)) \times \mathfrak{a}_{L_1,v}$ such that the restricted function

$$\bigoplus_j d^n_{M_j,\eta\to 0}\phi(\varrho^{L_1}_{j,\eta+\lambda}, Y), \quad \lambda \in \nu_L + i\mathfrak{a}^*_L,\ Y \in \mathfrak{a}_{L_1,v},$$

is the same as that derived from some function in $\mathcal{I}(L_1(F))$. For example, if $f \in \mathcal{H}(G(F))$, we could take ϕ to be the function

$$I_{L_1}(f)\colon (\sigma, Y) \;=\; I_{L_1}(\sigma, Y, f), \quad \sigma \in \Sigma(L_1(F)),\ Y \in \mathfrak{a}_{L_1,v}. \tag{2.6}$$

This function has the required behaviour if $v_L \in \mathfrak{a}_L^*$ is in general position, and the associated function in (2.5) is

$$I^X_{L_1,\lambda}(f)\colon (\Lambda, \varrho) \to \mathrm{e}^{-(\Lambda+\lambda)(X)} I_{L_1}(\varrho_\lambda^{L_1}, h_{L_1}(X, f)). \tag{2.7}$$

In the special case that $L = M$ and $D(\pi)$ is the character of π, we shall usually write

$$r^{L_1}_{M,\lambda}(\pi, \Phi) \;=\; (\hat{D}\cdot r^{L_1}_{M,\lambda})(\pi, \Phi), \quad \Phi \in \mathcal{I}(\mathfrak{a}_M, M(F)).$$

This is equal to

$$\sum_{\varrho\in\Sigma(M(F))} r^{L_1}_M(\pi_\lambda, \varrho_\lambda)\Phi_\lambda(0, \varrho).$$

(By definition [1(e)], $r^{L_1}_M(\pi_\lambda, \varrho_\lambda)$ is the number obtained in the usual way from the (G, M)-family

$$r_P(\zeta, \pi_\lambda, \varrho_\lambda, P_0) \;=\; \Delta(\pi, \varrho)\tilde{r}_{P|P_0}(\pi_\lambda, \varrho_\lambda)^{-1}\tilde{r}_{P|P_0}(\pi_{\lambda+\zeta}, \varrho_{\lambda+\zeta}),$$
$$P \in \mathcal{P}(M),\ \zeta \in i\mathfrak{a}^*_{M,v}.)$$

If $v \in \mathfrak{a}_M^*$ is in general position,

$$\begin{aligned} &r^{L_1}_{M,v}(\pi, X, I_{L_1}(f)) \\ &\quad= \int_{v+i\mathfrak{a}^*_{M,v}/i\mathfrak{a}^*_{L_1,v}} (r^{L_1}_{M,\lambda})(\pi, I^X_{L_1,\lambda}(f))\,\mathrm{d}\lambda \\ &\quad= \int \sum_{\varrho\in\Sigma(M(F))} r^{L_1}_M(\pi_\lambda, \varrho_\lambda) I_{L_1}(\varrho^{L_1}_\lambda, h_{L_1}(X), f)\mathrm{e}^{-\lambda(X)}\,\mathrm{d}\lambda. \end{aligned}$$

The notation is compatible with that of (1.3).

§3. Admissible families of operators

The purpose of this section is to prove Lemma 2.3 and a related result (Lemma 3.1) which will be needed in §4. We shall recapitulate some formal notions, introduced in §8 of [1(e)], of which the lemmas will be easy consequences.

Let L be a fixed Levi subgroup of G. Suppose that for each integer j, $1 \leqslant j \leqslant s$, we are given a Levi subgroup M_j of L, a parabolic subgroup $R_j \in \mathscr{P}^L(M_j)$, and a standard representation $\varrho_j \in \Sigma(M_j(F))$. If $n \geqslant 0$ and $g \in \mathscr{H}(L(F))$, set

$$\sigma(g) = \bigoplus_{j=1}^{s} d^n_{M_j, \eta \to 0}(\mathscr{I}_{R_j}(\varrho_{j,\eta}, g)). \tag{3.1}$$

Then σ is a representation of $\mathscr{H}(L(F))$. It acts on a direct sum of spaces of polynomials with values in $\mathscr{V}_{R_j}(\varrho_j)$. The induced representation

$$\mathscr{I}_Q(\sigma, f), \quad Q \in \mathscr{P}(L), f \in \mathscr{H}(G(F)),$$

can be identified with

$$\bigoplus_{j=1}^{s} d^n_{M_j, \eta \to 0}(\mathscr{I}_{Q(R_j)}(\varrho_{j,\eta}, f)).$$

(Recall that $P_j = Q(R_j)$ is the group in $\mathscr{P}(M_j)$ such that $P_j \subset Q$ and $P_j \cap L = R_j$.)

Fix groups Q, $Q' \in \mathscr{P}(L)$. Suppose that

$$A = \{A(\varrho) : \mathscr{V}_Q(\varrho) \to \mathscr{V}_{Q'}(\varrho), \varrho \in \Sigma(L(F))\}$$

is a family of linear operators which depends meromorphically on ϱ. In other words, any K-finite matrix coefficient of $A(\varrho)$ is meromorphic in the natural complex coordinates of $\Sigma(L(F))$. We assume that the singularities of $A(\varrho)$ are such that the function

$$A_\lambda(\varrho) = A(\varrho_\lambda), \quad \varrho \in \Sigma(L(F)),$$

is analytic at any predetermined finite set of points ϱ whenever λ is a point in $\mathfrak{a}^*_{L,\mathbb{C}}$ in general position. Then if σ is as in (3.1) and λ is in general position, the operator

$$A_\lambda(\sigma) = \bigoplus_{j=1}^{s} d^n_{M_j, \eta \to 0} A(\mathscr{I}_{R_j}(\varrho_{j,\eta+\lambda}))$$

from $\mathscr{V}_Q(\sigma)$ to $\mathscr{V}_{Q'}(\sigma)$ is defined. Recall that $\mathscr{V}_Q(\sigma)$ can be identified with a space of functions from K to the space on which σ acts. We shall say that the family is *admissible* if for every such σ, $A_\lambda(\sigma)$ is represented by a K-finite kernel with values in the algebra

$$\{\sigma(g) : g \in \mathscr{H}(L(F))\}.$$

This is a definite restriction on the family A. It implies that for every self intertwining operator of a representation σ as in (3.1), there will be a corresponding relation among the operators $\{A(\varrho)\}$.

The most obvious admissible families comes from functions in $\mathscr{H}(G(F))$. Choose $f \in \mathscr{H}(G(F))$. Then the operator

$$\mathscr{I}_Q(\varrho, f) : \mathscr{V}_Q(\varrho) \to \mathscr{V}_Q(\varrho), \quad \varrho \in \Sigma(L(F)),$$

is represented by a kernel

$$\varrho(f_{Q,k_1,k_2}), \quad k_1, k_2 \in K,$$

where f_{Q,k_1,k_2} denotes the function

$$m \to \delta_Q(m)^{1/2} \int_{N_Q(F)} f(k_1^{-1} mnk_2)\, \mathrm{d}n, \quad m \in L(F),$$

in $\mathscr{H}(L(F))$. Therefore

$$A = \{\mathscr{I}_Q(\varrho, f)\}$$

is an admissible family. If D is any invariant distribution on $\mathscr{H}(L(F))$, we can define the induced distribution D^G on $\mathscr{H}(G(F))$ by

$$D^G(f) = \int_K D(f_{Q,k,k})\, \mathrm{d}k.$$

There is a formal reciprocity identity

$$D^G(f) = \hat{D}(f_L), \quad f \in \mathscr{H}(G(F)).$$

Now, suppose that $D = D(\pi)$ is a π-discrete distribution on $\mathscr{H}(\mathfrak{a}_M, L(F))$. As in §2, π denotes a representation in $\Pi(M(F))$, for a fixed Levi subgroup M of L. We shall show how to define the induced distribution $D^G = D^G(\pi)$ on any admissible family. Actually, the domain of D consists of functions which also depend on Λ, so we take

$$A = \{A(\Lambda, \varrho) : \mathscr{V}_Q(\varrho) \to \mathscr{V}_{Q'}(\varrho)\}$$

to be an admissible family of operators that depend meromorphically on a parameter Λ in $\mathfrak{a}^*_{M,\mathbb{C}}/i\mathfrak{a}^\vee_{M,\mathbb{C}}$ as well as ϱ. Again we want the function

$$A_\lambda(\Lambda, \varrho) = A(\Lambda + \lambda, \varrho_\lambda)$$

to be analytic at any predetermined finite set of points (Λ, ϱ) whenever $\lambda \in \mathfrak{a}^*_{L,\mathbb{C}}$ is in general position. To take care of this, let us assume that the singularities of $A_\lambda(\Lambda, \varrho)$ have constraints like those we imposed in §2 on the singularities of a function in $\mathscr{I}^+(\mathfrak{a}_M, L(F))$. Choose an integer $n \geqslant 0$ and points $\Lambda_1, \ldots, \Lambda_t$ in $\mathfrak{a}^*_{M,\mathbb{C}}$ such that the value

$$D(\pi, F), \quad F \in \mathscr{H}(\mathfrak{a}_M, L(F)),$$

depends only on the operator

$$\tau(F) = \bigoplus_{i=1}^{t} d^n_{M,\Lambda\to\Lambda_i}(\mathscr{I}_R(\pi_\Lambda, F(\Lambda))). \tag{3.2}$$

We can regard τ as a representation of $\mathscr{H}(\mathfrak{a}_M, L(F))$. Now π can be represented as a subquotient of a standard representation. Therefore, τ is a subquotient of a representation like (3.1), but with the appropriate dependence on Λ. (See (3.4) below.) It follows from the admissibility of A that for fixed λ in general position,

$$A_\lambda(\tau) = \bigoplus_{i=1}^{t} d^n_{M,\Lambda\to\Lambda_i} A_\lambda(\Lambda, \mathscr{I}_R(\pi_\Lambda))$$

is uniquely defined as an operator from $\mathscr{V}_Q(\tau)$ to $\mathscr{V}_{Q'}(\tau)$. Indeed, $A_\lambda(\tau)$ is represented by a kernel

$$\tau(F_{k_1,k_2}), \quad k_1, k_2 \in K,$$

where F_{k_1,k_2} is a function in $\mathscr{H}(a_M, L(F))$, which is K-finite in (k_1, k_2), and such that

$$F_{l_1k_1,l_2k_2}(\Lambda, m) = F_{k_1,k_2}(\Lambda, l_1^{-1}ml_2), \quad m \in L(F),\, l_1, l_2 \in K \cap L(F).$$

The induced distribution is then defined by

$$D^G(\pi, A_\lambda) = \int_K D(\pi, F_{k,k})\,\mathrm{d}k. \tag{3.3}$$

It depends only on $A_\lambda(\tau)$.

In analogy with the map $f \to f_L$, let us define

$$A_L(\Lambda, \varrho) = \operatorname{tr}(A(\Lambda, \varrho)),$$

for a given admissible family $\{A(\Lambda, \varrho)\}$. Then A_L is a function in $\mathscr{I}^+(\mathfrak{a}_M, L(F))$. Clearly, we have

$$A_{L,\lambda}(\Lambda, \varrho) = \operatorname{tr}(A_\lambda(\Lambda, \varrho)), \quad \lambda \in \mathfrak{a}^*_{L,\mathbb{C}}.$$

LEMMA 3.1: $D^G(\pi, A_\lambda) = \hat{D}(\pi, A_{L,\lambda})$.

Proof: We have agreed that

$$D(\pi, F), \quad F \in \mathscr{H}(\mathfrak{a}_M, L(F)),$$

depends only on the operator $\tau(F)$ defined by (3.2). Moreover, by Lemma 2.1, we can choose $\{(M_j, R_j, \varrho_j)\}$ as in (3.1) such that

$$\hat{D}(\pi, \Phi), \quad \Phi \in \mathscr{I}(\mathfrak{a}_M, L(F)),$$

depends only on the vector

$$\Phi(\sigma) = \bigoplus_{i=1}^{t} \bigoplus_{j=1}^{s} d^n_{M,\Lambda\to\Lambda_i} d^n_{M_j,\eta\to 0} \Phi(\Lambda, \varrho^L_{j,\eta}).$$

Here σ stands for the representation

$$\sigma(F) = \bigoplus_{ij} d^n_{M,\Lambda\to\Lambda_i} d^n_{M_j,\eta\to 0} (\mathscr{I}_{R_j}(\varrho_{j,\eta}, F(\Lambda))) \tag{3.4}$$

of the algebra $\mathscr{H}(\mathfrak{a}_M, L(F))$. Notice also that the map $\Phi \to \Phi(\sigma)$ can be regarded as a finite dimensional representation of the algebra $\mathscr{I}(\mathfrak{a}_M, L(F))$.

The admissibility of A means that $A(\Lambda, \varrho)$ can be represented locally (i.e., infinitesimally) by a good kernel. Having chosen τ and σ, we can always find another representation of the general form (3.4) which contains both τ and σ as subquotients. We can therefore find a K-finite function

$$F_{k_1,k_2}, \quad k_1, k_2 \in K,$$

from $K \times K$ to the algebra $\mathscr{H}(\mathfrak{a}_M, L(F))$ which represents the kernel of the operator A_λ at both τ and σ. Here λ is a fixed point in $\mathfrak{a}^*_{L,\mathbb{C}}$ which is in general position (relative to τ and σ). Then

$$D^G(\pi, A_\lambda) = \int_K D(\pi, F_{k,k})\,\mathrm{d}k = \hat{D}(\pi, \Phi'),$$

where

$$\Phi'(\Lambda, \varrho) = \int_K \operatorname{tr} \varrho(F_{k,k}(\Lambda))\mathrm{d}k, \quad (\Lambda, \varrho) \in \mathfrak{a}^*_{M,\mathrm{C}} \times \Sigma(L(F)).$$

If F_{k_1,k_2} represented the kernel of A_λ everywhere, we would have

$$A_{L,\lambda}(\Lambda, \varrho) = \operatorname{tr}(A_\lambda(\Lambda, \varrho)) = \Phi'(\Lambda, \varrho)$$

for all (Λ, ϱ). This need not be so, of course, but F_{k_1,k_2} does represent the kernel at σ. Therefore

$$A_{L,\lambda}(\sigma) = \Phi'(\sigma).$$

Since the value of $\hat{D}(\pi)$ at Φ' depends only on $\Phi'(\sigma)$, we have

$$\hat{D}(\pi, \Phi') = \hat{D}(\pi, A_{L,\lambda}).$$

The lemma follows. □

The next lemma is the main reason for the definitions of this section. Its proof is an immediate consequence of the discussion of §8 of [1(e)].

LEMMA 3.2: *Suppose that* Γ *is a finite subset of* $\Pi(K)$. *Then the unnormalized intertwining operators*

$$J_{Q'|Q}(\varrho_\lambda): \mathscr{V}_Q(\varrho)_\Gamma \to \mathscr{V}_{Q'}(\varrho)_\Gamma, \quad \varrho \in \Sigma(L(F)),$$

form an admissible family. □

We can now prove Lemma 2.3. Let Γ be a finite subset of $\Pi(K)$ and let $E_Q(\varrho)_\Gamma$ be the projection of $\mathscr{V}_Q(\varrho)$ onto $\mathscr{V}_Q(\varrho)_\Gamma$. The first step is to prove that

$$\mathrm{A}(\varrho) = \mu_L(\varrho)^{-1} E_Q(\varrho)_\Gamma, \quad \varrho \in \Sigma(L(F)), \tag{3.5}$$

is an admissible family of operators. This is not a trivial assertion, for it implies a linear relation among (the derivatives of) Plancherel densities for every self intertwining operator of a representation of $\mathscr{H}(L(F))$ of the form (3.1). However,

$$A_\lambda(\varrho) \;=\; \mu_L(\varrho_\lambda)^{-1}E_Q(\varrho)_\Gamma$$

equals the restriction of the operator

$$J_{Q|\bar{Q}}(\varrho_\lambda)J_{\bar{Q}|Q}(\varrho_\lambda)$$

to $\mathscr{V}_Q(\varrho)_\Gamma$. Since admissibility is preserved under composition, Lemma 3.2 tells us that (3.5) is indeed an admissible family.

In Lemma 2.3, we are provided with a π-discrete distribution $D = D(\pi)$. Choose a representation τ of $\mathscr{H}(\mathfrak{a}_M, L(F))$ as in (3.2) such that

$$D(\pi, F), \quad F \in \mathscr{H}(\mathfrak{a}_M, L(F)),$$

depends only on the operator $\tau(F)$. Similarly, choose σ as in (3.4) such that

$$\hat{D}(\pi, \Phi), \quad \Phi \in \mathscr{H}(\mathfrak{a}_M, L(F)),$$

depends only on $\Phi(\sigma)$. It will be good enough to prove Lemma 2.3 with Φ_λ replaced by an arbitrary function $\Phi \in \mathscr{I}(\mathfrak{a}_M, L(F))$. Fix such a Φ, and choose a function $F \in \mathscr{H}(\mathfrak{a}_M, L(F))$ with $F_G = \Phi$. Fix $\lambda \in \mathfrak{a}^*_{L,\mathbf{C}}$ in general position, and define

$$\Phi^1(\Lambda, \varrho) \;=\; \mu_L(\pi^L_{L+\lambda})^{-1}\Phi(\Lambda, \varrho),$$

$$\Phi^2(\Lambda, \varrho) \;=\; \mu_L(\varrho_\lambda)^{-1}\Phi(\Lambda, \varrho),$$

and

$$F^1(\Lambda, m) \;=\; \mu_L(\pi^L_{\Lambda+\lambda})^{-1}F(\Lambda, m), \quad m \in L(F).$$

Then F^1_G equals Φ^1. The admissibility of the family (3.5) means that the inverse of the μ-function is an infinitesimal multiplier at τ and σ. In other words, there is a function $\tilde{F} \in \mathscr{H}(\mathfrak{a}_M, L(F))$ such that

$$\tau(\tilde{F}) \;=\; \tau(F^1)$$

and

$$\tilde{F}_G(\sigma) \;=\; \Phi^2(\sigma).$$

It follows from our conditions on τ and σ that

$$\hat{D}(\pi, \Phi^1) = D(\pi, F^1) = D(\pi, \tilde{F}),$$

and

$$\hat{D}(\pi, \Phi^2) = \hat{D}(\pi, \tilde{F}_G) = D(\pi, \tilde{F}).$$

We have thus established

$$\hat{D}(\pi, \Phi^1) = \hat{D}(\pi, \Phi^2),$$

the required formula of Lemma 2.3. □

§4. The main formula

As in the last two sections, π is a fixed representation in $\Pi(M(F))$. Suppose that $\mu \in \mathfrak{a}_M^*$ is a point in general position. Our goal is to evaluate the distribution

$$I_{M,\mu}(\pi, X, f) = I_M(\pi_\mu, X, f)e^{-\mu(X)}, \quad f \in \mathscr{H}(G(F)),$$

in terms of residues and the functions $r_{L,\lambda}^{L_1}$ obtained from the normalizing factors.

Suppose that we are given a set

$$\mathscr{N} = \mathscr{N}_G = \{\nu_L : L \in \mathscr{L}(M)\},$$

where each ν_L is a point in general position in $\mathfrak{a}_L^*$. For example, if $\nu \in \mathfrak{a}_M^*$ is any point in general position, let $\mathscr{N} = \mathscr{N}(\nu)$ be the collection in which ν_L is the projection of ν onto $\mathfrak{a}_L^*$. For any given $\mathscr{N}$, we shall try to express $I_{M,\mu}(\pi, X, f)$ in terms of the distributions

$$I_{L,\nu_L}(\varrho, h_L(X), f)), \quad L \in \mathscr{L}(M), \varrho \in \Sigma(L(F)).$$

We begin by working with the noninvariant distribution

$$J_{M,\mu}(\pi, X, f) = J_M(\pi_\mu, X, f)e^{-\mu(X)}, \quad f \in \mathscr{H}(G(F)).$$

By definition,

$$J_{M,\mu}(\pi, X, f) = \int_{\mu + i\mathfrak{a}_{M,v}^*} J_M(\pi_\Lambda, f)e^{-\Lambda(X)}\, d\Lambda.$$

We shall use the residue scheme of §10 of [1(e)] to change the contour. According to Proposition 10.1 of [1(e)], there is associated to each $L \in \mathscr{L}(M)$ a finite collection

$$R_L = R_L(\mu, \mathscr{N}_L)$$

of residue data for (L, M) such that $J_{M,\mu}(\pi, X, f)$ equals

$$\sum_{L \in \mathscr{L}(M)} \int_{\nu_L + i\mathfrak{a}^*_{L,v}} \sum_{\Omega \in R_L} \operatorname*{Res}_{\Omega, \Lambda \to \Lambda_\Omega + \lambda} (J_M(\pi_\Lambda, f) e^{-\Lambda(X)})\, d\lambda.$$

As the notation suggests, the collection R_L depends only on the set

$$\mathscr{N}_L = \{\nu_{L'} : L' \in \mathscr{L}^L(M)\}.$$

Recall that if $F \in \mathscr{H}(\mathfrak{a}_M, L(F))$, we can regard

$$F(\Lambda): m \to F(\Lambda, m), \quad \Lambda \in (\mathfrak{a}^*_{M,\mathbf{C}}/i\mathfrak{a}^\vee_{M,v}),\ m \in L(F),$$

as a function ofΛ with values in $\mathscr{H}(L(F))$. We define

$$D^{L,\mathscr{N}_L}_{M,\mu}(\pi, F) = \sum_{\Omega \in R_L(\mu, \mathscr{N}_L)} \operatorname*{Res}_{\Omega, \Lambda \to \Lambda_\Omega} J^L_M(\pi_\Lambda, F(\Lambda)). \tag{4.1}$$

In the special case that $\mathscr{N} = \mathscr{N}(\nu)$ as above, we will usually write

$$R_L(\mu, \nu) = R_L(\mu, \mathscr{N}_L)$$

and

$$D^{L,\nu}_{M,\mu}(\pi) = D^{L,\mathscr{N}_L}_{M,\mu}(\pi).$$

In general, $D^{L,\mathscr{N}_L}_{M,\mu}(\pi)$ is a distribution on $\mathscr{H}(\mathfrak{a}_M, L(F))$ which supported at a finite set of points (in the sense of §2).

If $L \neq G$, our induction hypothesis implies that the distribution $D^{L,\mathscr{N}_L}_{M,\mu}$ is supported on characters. It is therefore π-discrete. The constructions of §2 provide additional distributions

$$(\hat{D}^{L,\mathscr{N}_L}_{M,\mu} \cdot r^{L_1}_{L,\lambda})(\pi), \quad L_1 \in \mathscr{L}(L),\ \lambda \in \mathfrak{a}^*_{L,\mathbf{C}},$$

on $\mathscr{I}(\mathfrak{a}_M, L(F))$. We shall employ the notation (2.5), by which we can write

$$(\hat{D}^{L,\mathscr{N}_L}_{M,\mu} \cdot r^{L_1}_{L,\nu_L})(\pi, X, \phi),$$

where ϕ is the function

$$I_{L_1}(f)\colon (\sigma, Y) \to I_{L_1}(\sigma, Y, f), \quad \sigma \in \Sigma(L_1(F)),\ Y \in \mathfrak{a}_{L_1,v}.$$

If $L = G$, we do not yet know that the distribution

$$D_{M,\mu}^{G,\mathcal{N}_G} = D_{M,\mu}^{\mathcal{N}}$$

is supported on characters. This will be established in §5. In the meantime, we shall indulge in a harmless abuse of notation for the sake of a uniform formula. We shall write

$$(\hat{D}_{M,\mu}^{G,\mathcal{N}_G} \cdot r_{G,\nu_G}^{G})(\pi, X, I_G(f)) = (\hat{D}_{M,\mu}^{\mathcal{N}})(\pi, X, f_G)$$

when we really mean

$$D_{M,\mu}^{\mathcal{N}}(\pi, e^{-(\cdot)(X)}f(\cdot)) = \sum_{\Omega \in R_G} \operatorname*{Res}_{\Omega, \Lambda \to \Lambda_\Omega} (J_M(\pi_\Lambda, f^{h_G(X)})e^{-\Lambda(X)}).$$

If $f \in \mathcal{H}(G(F))$, this equals

$$\int_{i\mathfrak{a}_{G,v}^*} \sum_{\Omega \in R_G} \operatorname*{Res}_{\Omega, \Lambda \to \Lambda_\Omega + \lambda} (J_M(\pi_\Lambda, f)e^{-\Lambda(X)})\, d\lambda.$$

The next theorem gives the main reduction formula.

THEOREM 4.1: *For any function $f \in \mathcal{H}_{\mathrm{ac}}(G(F))$, we have*

$$I_{M,\mu}(\pi, X, f) = \sum_{L_1 \supset L \supset M} (\hat{D}_{M,\mu}^{L,\mathcal{N}_L} \cdot r_{L,\nu_L}^{L_1})(\pi, X, I_{L_1}(f)).$$

Proof: The main step is to prove an analogous formula for $J_{M,\mu}(\pi, X, f)$, with $f \in \mathcal{H}(G(F))$. We have already noted that this distribution equals the sum over $L \in \mathcal{L}(M)$ and the integral over $\lambda \in \nu_L + i\mathfrak{a}_{L,v}^*$ of the expression

$$\sum_{\Omega \in R_L} \operatorname*{Res}_{\Omega, \Lambda \to \Lambda_\Omega + \lambda} (J_M(\pi_\Lambda, f)e^{-\Lambda(X)}). \tag{4.2}$$

By Proposition 9.1 of [1(e)], the expression (4.2) equals

$$\sum_{\Omega \in R_L} \operatorname*{Res}_{\Omega, \Lambda \to \Lambda_\Omega + \lambda} (e^{-\Lambda(X)} \operatorname{tr}(\mathcal{I}_{P_0}(\pi_\Lambda, f)\mathcal{R}_L(\pi_\Lambda, P_0)\Gamma_\Omega(\pi_\Lambda, P_0))).$$

Here, $\Gamma_\Omega(\pi_\Lambda, P_0)$ is the meromorphic function Λ with values in the space of operators on $\mathscr{V}_{P_0}(\pi)$ which was defined at the beginning of §9 of [1(e)], and P_0 is any element in $\mathscr{P}(M)$. We can assume that $P_0 = Q_0(R)$, for fixed elements $Q_0 \in \mathscr{P}(L)$ and $R \in P^L(M)$. Now $\mathscr{R}_L(\pi_\Lambda, P_0)$ is the operator

$$\lim_{\zeta \to 0} \sum_{Q \in \mathscr{P}(L)} \mathscr{R}_Q(\zeta, \pi_\Lambda, P_0)\theta_Q(\zeta)^{-1},$$

obtained from the (G, L)-family

$$\mathscr{R}_Q(\zeta, \pi_\Lambda, P_0) \quad = \quad R_{Q(R)|Q_0(R)}(\pi_\Lambda)^{-1} R_{Q(R)|Q_0(R)}(\pi_{\Lambda+\zeta}), \quad Q \in \mathscr{P}(L), \zeta \in i\mathfrak{a}_L^*.$$

It is analytic for Λ near any of the points $\Lambda_\Omega + \lambda$, as long as $\lambda \in \mathfrak{a}_{L,\mathbb{C}}^*$ is in general position. We can therefore take the limit in ζ outside the residue operator. Consequently, (4.2) equals the limit at $\zeta = 0$ of the sum over $Q \in \mathscr{P}(L)$ of the product of $\theta_Q(\zeta)^{-1}$ with

$$\sum_{\Omega \in R_L} \operatorname*{Res}_{\Omega, \Lambda \to \Lambda_\Omega + \lambda} (e^{-\Lambda(X)} \operatorname{tr}(\mathscr{I}_{P_0}(\pi_\Lambda, f)\mathscr{R}_Q(\zeta, \pi_\Lambda, P_0)\Gamma_\Omega(\pi_\Lambda, P_0))). \tag{4.3}$$

Assume that $L \neq G$. We are going to apply Lemma 3.1, with

$$D(\pi) \quad = \quad D_{M,\mu}^{L,\mathscr{N}_L}(\pi).$$

Let

$$A \quad = \quad \{A(\Lambda, \varrho) : \mathscr{V}_{Q_0}(\varrho) \to \mathscr{V}_{Q_0}(\varrho)\}$$

be as in Lemma 3.1, an admissible family of operators that depend meromorphically on Λ. If $\lambda \in \mathfrak{a}_{L,\mathbb{C}}^*$ is in general position, the operator

$$A_\lambda(\Lambda, \mathscr{I}_R(\pi_\Lambda)) \quad = \quad A(\Lambda + \lambda, \mathscr{I}_R(\pi_{\Lambda+\lambda}))$$

on $\mathscr{V}_{P_0}(\pi)$ is uniquely defined and analytic for Λ in a neighbourhood of each of the points Λ_Ω in $\mathfrak{a}_{M,\mathbb{C}}^*$. Consider the expression

$$\sum_{\Omega \in R_L} \operatorname*{Res}_{\Omega, \Lambda \to \Lambda_\Omega + \lambda} \operatorname{tr}(A(\Lambda, \mathscr{I}_R(\pi_\Lambda))\Gamma_\Omega(\pi_\Lambda, P_0)). \tag{4.4}$$

The operator $\Gamma_\Omega(\pi_\Lambda, P_0)$ acts on the space

$$\mathscr{V}_{P_0}(\pi) \quad = \quad \mathscr{V}_{Q_0}(\mathscr{V}_R(\pi))$$

entirely through the fibre, by means of the operator

$$\Gamma_\Omega(\pi_\Lambda, R): \mathscr{V}_R(\pi) \to \mathscr{V}_R(\pi).$$

But Proposition 9.1 of [1(e)] implies that

$$\sum_{\Omega \in R_L} \operatorname*{Res}_{\Omega, \Lambda \to \Lambda_\Omega} \operatorname{tr}(\mathscr{I}_R(\pi_\Lambda, F(\Lambda))\Gamma_\Omega(\pi_\Lambda, R)) = D_{M,\mu}^{L,\mathcal{N}_L}(\pi, F),$$

for any $F \in \mathscr{H}(\mathfrak{a}_M, L(F))$. Choose a function $F_{k_1,k_2} \in \mathscr{H}(\mathfrak{a}_M, L(F))$ to represent the kernel of A at $(\Lambda, \mathscr{I}_R(\pi_\Lambda))$ (up to sufficiently high infinitesimal order). Then (4.4) equals

$$\begin{aligned} &\int_K \sum_{\Omega \in R_L} \operatorname*{Res}_{\Omega, \Lambda \to \Lambda_\Omega + \lambda} \operatorname{tr}(\mathscr{I}_R(\pi_\Lambda, F_{k,k}(\Lambda))\Gamma_\Omega(\pi_\Lambda, R))\, dk \\ &= \int_K D_{M,\mu}^{L,\mathcal{N}_L}(\pi, F_{k,k})\, dk \\ &= (D_{M,\mu}^{L,\mathcal{N}_L})^G(\pi, A_\lambda), \end{aligned}$$

according to the definition (3.3). From Lemma 3.1 we then obtain the equality of (4.4) with

$$\hat{D}_{M,\mu}^{L,\mathcal{N}_L}(\pi, \Phi_\lambda),$$

where

$$\Phi_\lambda(\Lambda, \varrho) = A_{L,\lambda}(\Lambda, \varrho) = \operatorname{tr}(A_\lambda(\Lambda, \varrho)).$$

We shall apply this last formula to (4.3). Since

$$r_{Q(R)|Q_0(R)}(\pi_\Lambda) = r_{Q|Q_0}(\pi_\Lambda^L),$$

the operator $\mathscr{R}_Q(\zeta, \pi_\Lambda, P_0)$ equals

$$(r_{Q|Q_0}(\pi_\Lambda^L)^{-1} r_{Q|Q_0}(\pi_{\Lambda+\zeta}^L))^{-1} J_{Q(R)|Q_0(R)}(\pi_\Lambda)^{-1} J_{Q(R)|Q_0(R)}(\pi_{\Lambda+\zeta}).$$

Define

$$A(\Lambda, \varrho) = a_\Lambda A(\varrho),$$

where

$$a_\Lambda = e^{-\Lambda(X)}(r_{Q|Q_0}(\pi_\Lambda^L)^{-1} r_{Q|Q_0}(\pi_{\Lambda+\zeta}^L))^{-1}$$

and

$$A(\varrho) = \mathscr{I}_{Q_0}(\varrho, f) J_{Q|Q_0}(\varrho)^{-1} J_{Q|Q_0}(\varrho_\zeta).$$

By Lemma 3.2, $\{A(\varrho)\}$ is an admissible family of operators. Moreover, by the transitivity of induction,

$$A(\Lambda, \mathscr{I}_R(\pi_\Lambda)) = a_\Lambda \mathscr{I}_{P_0}(\pi_\Lambda, f) J_{Q(R)|Q_0(R)}(\pi_\Lambda)^{-1} J_{Q(R)|Q_0(R)}(\pi_{\Lambda+\zeta}).$$

It follows that the expression (4.3) equals (4.4). Thus, (4.3) is equal to

$$\hat{D}_{M,\mu}^{L,\mathscr{N}_L}(\pi, \Phi_\lambda),$$

where

$$\Phi_\lambda(\Lambda, \varrho) = a_{\Lambda+\lambda} \operatorname{tr}(A(\varrho_\lambda)).$$

Observe that

$$\begin{aligned}
&(r_{Q|Q_0}(\varrho_\lambda)^{-1} r_{Q|Q_0}(\varrho_{\lambda+\zeta}))^{-1} \operatorname{tr}(A(\varrho_\lambda)) \\
&\quad = \operatorname{tr}(\mathscr{I}_{Q_0}(\varrho_\lambda, f) R_{Q|Q_0}(\varrho_\lambda)^{-1} R_{Q|Q_0}(\varrho_{\lambda+\zeta})) \\
&\quad = \operatorname{tr}(\mathscr{I}_{Q_0}(\varrho_\lambda, f) \mathscr{R}_Q(\zeta, \varrho_\lambda, Q_0)).
\end{aligned}$$

Therefore,

$$\Phi_\lambda(\Lambda, \varrho) = \tilde{r}_{Q|Q_0}(\pi_{\Lambda+\lambda}^L, \varrho_\lambda)^{-1} \tilde{r}_{Q|Q_0}(\pi_{\Lambda+\lambda+\zeta}^L, \varrho_{\lambda+\zeta}) \Psi_{Q,\lambda}(\zeta, \Lambda, \varrho, Q_0),$$

where

$$\Psi_{Q,\lambda}(\zeta, \Lambda, \varrho, Q_0) = e^{-(\Lambda+\lambda)(X)} \operatorname{tr}(\mathscr{I}_{Q_0}(\varrho_\lambda, f) \mathscr{R}_Q(\zeta, \varrho_\lambda, Q_0)).$$

To obtain (4.2), we must multiply the formula we have just obtained for (4.3) by $\theta_Q(\zeta)^{-1}$, sum over $Q \in \mathscr{P}(L)$, and then take the limit as ζ approaches 0. However, let us first write

$$\hat{D}_{M,\mu}^{L,\mathscr{N}_L}(\pi, \Phi_\lambda) = (\hat{D}_{M,\mu}^{L,\mathscr{N}_L} \cdot \Phi_\lambda^\vee)(\pi, 1),$$

where 1 stands for the constant function of (Λ, ϱ). Note that

$$\begin{aligned}\Phi_\lambda^\vee &= (\tilde{r}_{Q|Q_0,\lambda}^{-1}\tilde{r}_{Q|Q_0,\lambda+\zeta}\Psi_{Q,\lambda}(\zeta, Q_0))^\vee \\ &= (\tilde{r}_{Q|Q_0,\lambda}^\vee)^{-1}\cdot \tilde{r}_{Q|Q_0,\lambda+\zeta}^\vee \cdot \Psi_{Q,\lambda}^\vee(\zeta, Q_0) \\ &= r_{Q|Q_0,\lambda}^{-1}\cdot r_{Q|Q_0,\lambda+\zeta}\cdot \Psi_{Q,\lambda}^\vee(\zeta, Q_0) \\ &= r_{Q,\lambda}(\zeta, Q_0)\cdot \Psi_{Q,\lambda}^\vee(\zeta, Q_0),\end{aligned}$$

in the notation of §2. This is a product of (G, L)-families, and we may apply Corollary 6.5 of [1(b)]. Consequently,

$$\lim_{\zeta\to 0}\sum_{Q\in\mathcal{P}(L)}(r_{Q,\lambda}(\zeta, Q_0)\cdot \Psi_{Q,\lambda}^\vee(\zeta, Q_0))\theta_Q(\zeta)^{-1}$$

is equal to

$$\sum_{L_1\in\mathcal{L}(L)} r_{L,\lambda}^{L_1}\cdot \Psi_{L_1,\lambda}^\vee .$$

As with $r_{L,\lambda}^{L_1}$ we have suppressed Q_0 in the notation $\Psi_{L_1,\lambda}^\vee$. Indeed,

$$\Psi_{L_1,\lambda}(\Lambda, \varrho) = \mathrm{e}^{-(\Lambda+\lambda)(X)}\operatorname{tr}(\mathcal{R}_{L_1}(\varrho_\lambda, Q_0)\mathcal{I}_{Q_0}(\varrho_\lambda, f)),$$

and by formula (7.8) of [1(b)], this equals

$$\mathrm{e}^{-(\Lambda+\lambda)(X)}J_{L_1}(\varrho_\lambda^{L_1}, f).$$

In particular, $\Psi_{L_1,\lambda}$ is independent of Q_0. Since

$$(\hat{D}_{M,\mu}^{L,\mathcal{N}_L}\cdot r_{L,\lambda}^{L_1}\cdot \Psi_{L_1,\lambda}^\vee)(\pi, 1) = (\hat{D}_{M,\mu}^{L,\mathcal{N}_L}\cdot r_{L,\lambda}^{L_1})(\pi, \Psi_{L_1,\lambda}),$$

we can therefore rewrite (4.2) as an expression

$$\sum_{L_1\in\mathcal{L}(L)}(\hat{D}_{M,\mu}^{L,\mathcal{N}_L}\cdot r_{L,\lambda}^{L_1})(\pi, \Psi_{L_1,\lambda}), \tag{4.5}$$

in which

$$\Psi_{L_1,\lambda}(\Lambda, \varrho) = \mathrm{e}^{-(\Lambda+\lambda)(X)}J_{L_1}(\varrho_\lambda^{L_1}, f).$$

We have shown that the original expression (4.2) equals (4.5) for any $L \neq G$. According to our convention above, the same equality is trivially true when $L = G$. Therefore, the original distribution $J_{M,\mu}(\pi, X, f)$ is equal to the sum over $L \in \mathscr{L}(M)$ and the integral over λ in $(\nu_L + i\mathfrak{a}^*_{L,v})$ of (4.5). Take the integral inside the sum over L_1 which appears in (4.5). Then for a given L_1, replace λ by

$$\lambda + \eta, \quad \eta \in i\mathfrak{a}^*_{L_1,v},$$

and integrate first over η. Note that

$$r^{L_1}_{L,\lambda+\eta} = r^{L_1}_{L,\lambda}.$$

Also

$$\begin{aligned} &\int_{i\mathfrak{a}^*_{L_1,v}} \Psi_{L_1,\lambda+\eta}(\Lambda, \varrho)\, d\eta \\ &= e^{-(\Lambda+\lambda)(X)} \int_{i\mathfrak{a}^*_{L_1,v}} e^{-\eta(X)} J_{L_1}(\varrho^{L_1}_{\lambda+\eta}, f)\, d\eta \\ &= e^{-(\Lambda+\lambda)(X)} J_{L_1}(\varrho^{L_1}_{\lambda}, h_{L_1}(X), f) \\ &= J^X_{L_1,\lambda}(\Lambda, \varrho), \end{aligned}$$

where

$$J_{L_1}(f)\colon (\sigma, Y) \to J_{L_1}(\sigma, Y, f), \quad \sigma \in \Sigma(L_1(F)),\ Y \in \mathfrak{a}_{L_1,v}.$$

It follows that

$$\int_{\nu_L + i\mathfrak{a}^*_{L,v}} (\hat{D}^{L,\mathcal{N}_L}_{M,\mu} \cdot r^{L_1}_{L,\lambda})(\pi, \Psi_{L_1,\lambda})\, d\lambda$$

is equal to

$$(\hat{D}^{L,\mathcal{N}_L}_{M,\mu} \cdot r^{L_1}_{L,\nu_L})(\pi, X, J_{L_1}(f)),$$

in the notation of (2.5). Putting these facts together, we see at last that

$$J_{M,\mu}(\pi, X, f) = \sum_{L_1 \supset L \supset M} (\hat{D}^{L,\mathcal{N}_L}_{M,\mu} \cdot r^{L_1}_{L,\nu_L})(\pi, X, J_{L_1}(f)).$$

We have established the analogue for $J_{M,\mu}(\pi, X, f)$ of the required formula. At this point, f is just a function in $\mathscr{H}(G(F))$. However, both sides of the formula depend only on the restriction of f to $G(F)^{h_G(X)}$. Since the restriction of any function in $\mathscr{H}_{\mathrm{ac}}(G(F))$ to this set coincides with that of a function in $\mathscr{H}(G(F))$, the formula remains valid if f belongs to $\mathscr{H}_{\mathrm{ac}}(G(F))$.

We assume inductively that the required formula for $I_{M,\mu}(\pi, X, f)$ holds if G is replaced by a proper Levi subgroup $L' \in \mathscr{L}(M)$. The case of G will be a consequence of the formula for $J_{M,\mu}(\pi, X, f)$ we have just proved. For it follows from the definition that

$$J_{M,\mu}(\pi, X, f) = \sum_{L' \in \mathscr{L}(M)} \hat{I}^{L'}_{M,\mu}(\pi, X, \phi_{L'}(f)).$$

The special case that $\mu = 0$ (and $M = L_1$) also implies that

$$J_{L_1}(f) = \sum_{L' \in \mathscr{L}(L_1)} \hat{I}^{L'}_{L_1}(\phi_{L'}(f)).$$

After substituting these two identities into the formula above, we apply the induction assumption to the terms with $L' \neq G$. We are left only with the terms corresponding to $L' = G$, which give

$$\hat{I}^G_{M,\mu}(\pi, X, \phi_G(f)) = \sum_{L_1 \supset L \supset M} (\hat{D}^{L,\mathscr{N}_L}_{M,\mu}, r^{L_1}_{L,\nu_L})(\pi, X, \hat{I}^G_{L_1}(\phi_G(f))).$$

Since

$$\hat{I}^G_{M,\mu}(\pi, X, \phi_G(f)) = I_{M,\mu}(\pi, X, f)$$

and

$$\hat{I}^G_{L_1}(\phi_G(f)) = I_{L_1}(f),$$

we obtain the required formula. $\square$

COROLLARY 4.2. *Set $\nu = \nu_M$. Then the difference*

$$I_{M,\mu}(\pi, X, f) - I_{M,\nu}(\pi, X, f) \tag{4.6}$$

equals

$$\sum_{\{L_1, L \in \mathscr{L}(M): L_1 \supset L \supsetneq M\}} (\hat{D}^{L,\mathscr{N}_L}_{M,\mu} \cdot r^{L_1}_{L,\nu_L})(\pi, X, I_{L_1}(f)).$$

Proof: The theorem gives an expansion for $I_{M,\mu}(\pi, X, f)$ into a sum over L_1 and L. Consider those terms in which $L = M$. It follows from the definitions that

$$D_{M,\mu}^{M,\mathcal{N}_M}(\pi, F) = \operatorname{tr}\pi(F(0)), \quad F \in \mathcal{H}(\mathfrak{a}_M, M(F)).$$

In particular, this distribution is independent of μ and $\mathcal{N}$. Therefore, the terms with $L = M$ depend only on the point $\nu = \nu_M$. Suppose for a moment that $\mu = \nu$ and $\mathcal{N} = \mathcal{N}(\nu)$. Then

$$D_{M,\mu}^{L,\mathcal{N}_L}(\pi) = 0, \quad L \neq M.$$

Applying the theorem in this case, we see that $I_{M,\nu}(\pi, X, f)$ equals the sum of those terms in the general expansion in which $L = M$. Therefore, the difference (4.6) equals the sum of those terms with $L \supsetneq M$. □

REMARKS: 1. Look again at the special case that $\mu = \nu$ and $\mathcal{N} = \mathcal{N}(\nu)$. The expansion for $I_{M,\mu}(\pi, X, f)$ contains only those terms with $L = M$. We have

$$(D_{M,\mu}^{M,\mathcal{N}_M} \cdot r_{M,\mu}^{L_1})(\pi, X, I_{L_1}(f)) = r_{M,\mu}^{L_1}(\pi, X, I_{L_1}(f)),$$

in the notation described at the end of §2, so the expansion is just

$$I_{M,\mu}(\pi, X, f) = \sum_{L_1 \in \mathcal{L}(M)} r_{M,\mu}^{L_1}(\pi, X, I_{L_1}(f)).$$

The theorem in this case is equivalent to (1.3).

2. Suppose that π is tempered. Then $I_{M,\nu}(\pi, X, f)$ vanishes if ν is near 0. Corollary 4.2 may therefore be interpreted as an inductive procedure for computing the distributions $I_M(\pi_\mu, X, f)$ in terms of residues. We shall discuss this in more detail in §7.

§5. Completion of the induction argument

Given Theorem 4.1, it is easy for us to show that the invariant distributions defined by residues are supported on characters. Fix a residue datum

$$\Omega = (\mathcal{E}_\Omega, \Lambda_\Omega)$$

for (G, M), and a representation $\pi \in \Pi(M(F))$.

LEMMA 5.1: *The data $\mu \in \mathfrak{a}_M^*$ and $\mathcal{N} = \{\nu_L\}$ of §4 may be chosen so that $R_G(\mu, \mathcal{N})$ consists only of Ω.*

Proof: This will follow easily from the definition. Recall that

$$\mathscr{E}_\Omega = (E_1, \ldots, E_r)$$

is orthogonal basis of $(\mathfrak{a}_M^G)^*$, and that

$$M = M_0 \subset M_1 \subset \cdots \subset M_r = G$$

is a sequence of Levi subgroups such that

$$\mathfrak{a}_{M_i} = \{H \in \mathfrak{a}_{M_{i-1}} : E_i(H) = 0\}, \quad 1 \leqslant i \leqslant r.$$

Let ε be a small positive number, and define a sequence

$$0 < \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_r = \varepsilon,$$

in which each ε_i is much smaller than ε_{i+1}. We then define

$$\mu = \mathrm{Re}(\Lambda_\Omega) - (\varepsilon_1 E_1 + \cdots + \varepsilon_r E_r).$$

Let μ_{M_i} be the projection of μ onto $\mathfrak{a}_{M_i}^*$, and set

$$\nu_{M_i} = \mu_{M_i} + 2\varepsilon_{i+1} E_{i+1}, \quad 0 \leqslant i \leqslant r - 1.$$

This defines the points $\nu_L \in \mathcal{N}$ when $L = M_i$. For the other elements $L \in \mathscr{L}(M)$ we can take $\nu_L = 0$. Then

$$\int_{\mu + i\mathfrak{a}_{M,v}^*} \psi(\Lambda) d\Lambda$$

equals

$$\sum_{k=0}^{r} \left(\frac{1}{2\pi i}\right)^k \int_{\nu_{M_k} + i\mathfrak{a}_{M_k,v}^*} \int_{\Gamma_k} \cdots \int_{\Gamma_1} \psi(\Lambda_\Omega^k + \lambda + z_1 E_1 + \cdots$$
$$\cdots + z_k E_k) dz_1 \ldots dz_k d\lambda.$$

Here

$$\psi(\Lambda) = J_M(\pi_\Lambda, f) e^{-\Lambda(X)}, \quad f \in \mathscr{H}(G(F)),$$

while Λ_Ω^k is the projection of Λ_Ω onto $(\mathfrak{a}_M^{M_k})_\mathbb{C}^*$ and $\Gamma_1, \ldots, \Gamma_r$ are small circles about the origin in $\mathbb{C}$ such that the radius of each Γ_i is much smaller than that of Γ_{i+1}. We may therefore take $R_L(\mu, \mathcal{N}_L)$ to be empty unless L equals some M_k, in which case it consists of a single residue datum

$$\Omega_k = ((E_1, \ldots, E_k), \Lambda_\Omega^k).$$

In particular,

$$R_G(\mu, \mathcal{N}) = \{\Omega\},$$

as required. □

THEOREM 5.2: *Suppose that a_Λ is an analytic function in a neighbourhood of Λ_Ω in $\mathfrak{a}_{M,C}^*$. Then the distribution*

$$\operatorname*{Res}_{\Omega,\Lambda\to\Lambda_\Omega} (a_\Lambda J_M(\pi_\Lambda, f)), \quad f \in \mathscr{H}(G(F)),$$

is supported on characters.

Proof: We shall apply the formula of Theorem 4.1, with μ and $\mathcal{N}$ as in the last lemma. The term with $L = L_1 = G$ in the formula equals

$$\int_{i\mathfrak{a}_{G,v}^*} \operatorname*{Res}_{\Omega,\Lambda\to\Lambda_\Omega+\lambda} (J_M(\pi_\Lambda, f) e^{-\Lambda(X)})\, d\lambda. \tag{5.1}$$

This equals the difference between

$$I_{M,\mu}(\pi, X, f)$$

and

$$\sum_{\{L_1 \supset L \supset M : L \neq G\}} (\hat{D}_{M,\mu}^{L,\mathcal{N}_L} \cdot r_{L,\nu_L}^{L_1})(\pi, X, I_{L_1}(f)).$$

Since $L \neq G$, $\hat{D}_{M,\mu}^{L,\mathcal{N}_L}$ is well defined, by our induction assumption. Suppose that f is such that $f_G = 0$. It follows from [1(f), Theorem 6.1] and [1(g), Theorem 5.1] that $I_{M,\mu}(\pi, X, f)$ and $I_L(f)$ both vanish. Consequently, the expression (5.1) vanishes. But the point $X \in \mathfrak{a}_{M,v}$ in (5.1) is arbitrary. Taking a finite linear combination of such expressions, over different values of X, we can match Taylor series of a_Λ at Λ_Ω up to any given degree. It follows that

$$\operatorname*{Res}_{\Omega,\Lambda\to\Lambda_\Omega} (a_\Lambda J_M(\pi_\Lambda, f)) = 0.$$

The given distribution is therefore supported on characters. □

With Theorem 5.1 we have completed the induction argument begun in §1. In particular, the distributions $\hat{D}_{M,\mu}^{G,\mathcal{N}_G}$ of §4 are well defined, and Theorem 4.1 and Corollary 4.2 make sense as stated.

§6. Cuspidal functions

Suppose that f belongs to $\mathscr{H}(G(F))$. If L is a Levi subgroup of G, the function

$$f_L : \pi \to \operatorname{tr}(\mathscr{I}_Q(\pi, f)), \quad Q \in \mathscr{P}(L),\ \pi \in \Pi_{\text{temp}}(L(F)),$$

belongs to $\mathscr{I}(L(F))$. We shall say that f is *cuspidal* if f_L vanishes whenever $L \neq G$. Assume that this is the case. Then, as we shall see, there is a considerable simplification in Theorem 4.1.

Suppose that M, π, μ and $\mathcal{N}$ are as in Theorem 4.1. Consider a term in the expansion for $I_{M,\mu}(\pi, X, f)$ corresponding to $L \subset L_1$. Suppose first that $L \subsetneq L_1$. We claim that the function

$$I_{L_1,\lambda}^X(f) : (\Lambda, \varrho) \to e^{-(\Lambda+\lambda)(X)} I_{L_1}(\varrho_\lambda^{L_1}, h_{L_1}(X), f), \quad \Lambda \in \mathfrak{a}_{M,\mathbb{C}}^*,\ \varrho \in \Sigma(L(F)),$$

vanishes identically. By the descent formula [1(f), Corollary 8.5], we can express the Fourier transform

$$\int_{i\mathfrak{a}_{L,v}^*/i\mathfrak{a}_{L_1,v}^*} I_{L_1}(\varrho_\lambda^{L_1}, h_{L_1}(X), f) e^{-\lambda(X)}\, d\lambda$$

as

$$\sum_{L_2 \in \mathscr{L}(L)} d_L^G(L_1, L_2) \hat{I}_L^{L_2}(\varrho, X, f_{L_2}).$$

Since $L \subsetneq L_1$, the constant $d_L^G(L_1, L_2)$ will vanish unless $L_2 \subsetneq G$. Our claim then follows from the fact that f is cuspidal. It follows from this that

$$(\hat{D}_{M,\mu}^{L,\mathcal{N}_L} \cdot r_{L,\nu_L}^{L_1})(\pi, X, I_{L_1}(f)) = 0.$$

In other words, we can discard the terms in Theorem 4.1 with $L \subsetneq L_1$. The term corresponding to $L = L_1$ is just

$$\hat{D}_{M,\mu}^{L,\mathcal{N}_L}(\pi, X, I_{L,\nu_L}(f)) = \hat{D}_{M,\mu}^{L,\mathcal{N}_L}(\pi, I_{L,\nu_L}^X(f)),$$

where

$$I^X_{L,\nu_L}(f) : (\Lambda, \varrho) \to I_{L,\nu_L}(\varrho, h_L(X), f)e^{-\Lambda(X)}, \quad \Lambda \in \mathfrak{a}^*_{M,\mathbb{C}}, \varrho \in \Sigma(L(F)).$$

We obtain the following corollary of Theorem 4.1.

COROLLARY 6.1: *If f is cuspidal,*

$$I_{M,\mu}(\pi, X, f) = \sum_{L \in \mathscr{L}(M)} \hat{D}^{L,\mathscr{N}_L}_{M,\mu}(\pi, X, I_{L,\nu_L}(f)). \qquad \square$$

For the rest of this section, ε will be a small point in $\mathfrak{a}^*_M$ in general position. We now consider the special case that $\mathscr{N}$ equals $\mathscr{N}(\varepsilon)$. In this case, the associated residue scheme is essentially that of the real Paley–Wiener theorem. The summand

$$D^\varepsilon_{M,\mu}(\pi, X, f) = \hat{D}^{G,\mathscr{N}}_{M,\mu}(\pi, X, I_{G,\nu_G}(f))$$

corresponding to $L = G$ is the leading term in the expansion of Corollary 6.1. It equals

$$\sum_{\Omega \in R_G(\mu,\varepsilon)} \operatorname*{Res}_{\Omega, \Lambda \to \Lambda_\Omega} (e^{-\Lambda(X)} J_M(\pi_\Lambda, f^{h_G(X)})), \tag{6.1}$$

and consists entirely of residues. We shall show that if π is unitary, this is the only term in the expansion.

COROLLARY 6.2: *Assume that f is cuspidal, that π has unitary central character, and that $\varepsilon \in \mathfrak{a}^*_M$ is a small point in general position. Then*

$$I_{M,\mu}(\pi, X, f) = D^\varepsilon_{M,\mu}(\pi, X, f).$$

Proof: Consider the expansion given by Corollary 6.1. Since f is cuspidal, the argument preceding Corollary 6.1 tells us that the function

$$I_{L,\varepsilon_L}(\varrho, h_L(X), f)$$

vanishes if $\varrho \in \Sigma(L(F))$ is properly induced. Now $\hat{D}^{L,\mathscr{N}_L}_{M,\mu}(\pi)$ is supported at those $\varrho \in \Sigma(L(F))$ with unitary central character. Any such ϱ which is not properly induced must be tempered. However, if ϱ is tempered, and ε_L is sufficiently small, we have

$$I_{L,\varepsilon_L}(\varrho, X, f) = 0, \quad L \neq G,$$

by [1(f), Lemmas 3.3 and 4.5]. It follows that the terms in the expansion with $L \neq G$ must vanish. □

The last formula allows us to express the map

$$\theta_M : \tilde{\mathscr{H}}_{\mathrm{ac}}(G(F)) \to \tilde{\mathscr{I}}_{\mathrm{ac}}(M(F)),$$

introduced in [1(f), §4], in terms of residues. For each $P \in \mathscr{P}(M)$, let ν_P be a point in the associated chamber $(\mathfrak{a}_P^*)^+$ in $\mathfrak{a}_M^*$ whose distance from the walls is very large. We shall then write

$$D_M^\varepsilon(\pi, X, f) = \sum_{P \in \mathscr{P}(M)} \omega_P(X) D_{M,\nu_P}^\varepsilon(\pi, X, f), \tag{6.2}$$

where $\omega_P(X)$ is the ratio defined as in [1(f), §4]. That is,

$$\omega_P(X) = \mathrm{vol}(\mathfrak{a}_P^+ \cap B_X)\,\mathrm{vol}(B_X)^{-1}, \quad X \in \mathfrak{a}_M,\ P \in \mathscr{P}(M),$$

with B_X a small ball in $\mathfrak{a}_M$ centered at X. In particular, suppose that X is a regular point in $\mathfrak{a}_{M,v}$. Then X belongs to a unique chamber $\mathfrak{a}_P^+$, and

$$D_M^\varepsilon(\pi, X, f) = D_{M,\nu_P}^\varepsilon(\pi, X, f).$$

Combining Corollary 6.2 with [1(f), Lemma 4.7], we obtain

COROLLARY 6.3: *Assume that f is cuspidal, that $\pi \in \Pi_{\mathrm{temp}}(M(F))$ and that $X \in \mathfrak{a}_{M,v}$. Then*

$$\theta_M(f, \pi, X) = D_M^\varepsilon(\pi, X, f).$$ □

For the rest of this paragraph we assume that $F = \mathbb{R}$. We shall also assume that M contains a maximal torus T over $\mathbb{R}$ which is $\mathbb{R}$-anisotropic modulo A_M. Let $\Pi_{\mathrm{disc}}(G(\mathbb{R}))$ denote the set of representations π in $\Pi_{\mathrm{temp}}(G(\mathbb{R}))$ which are square integrable modulo $A_G(\mathbb{R})$. The vector space $i\mathfrak{a}_G^*$ acts on $\Pi_{\mathrm{disc}}(G(\mathbb{R}))$ in the usual way, and the set of orbits can be identified with the discrete series, $\Pi_{\mathrm{disc}}(G(\mathbb{R})^1)$, of

$$G(\mathbb{R})^1 = \{x \in G(\mathbb{R}) : H_G(x) = 0\}.$$

For $\pi \in \Pi_{\mathrm{disc}}(G(\mathbb{R}))$ and $\gamma \in T_{\mathrm{reg}}(\mathbb{R})$, the set of G-regular points in $T(\mathbb{R})$, we set

$$I_M^G(\gamma, \pi) = |D^G(\gamma)|^{1/2}\Theta_\pi(\gamma),$$

where Θ_π is the character of π and $D^G(\gamma)$ is the usual discriminant. This function is not constant on the $i\mathfrak{a}_G^*$-orbit of π. However, its product with the function

$$f_G(\tilde{\pi}, H_G(\gamma)) = \int_{i\mathfrak{a}_G^*} \mathrm{tr}(\tilde{\pi}_\lambda(f))e^{-\lambda(H_G(\gamma))}\,d\lambda$$

is constant on the orbit, and depends only on the image of π in $\Pi_{\mathrm{disc}}(G(\mathbb{R})^1)$. Here, $\tilde{\pi}$ denotes the contragredi of the representation π.

We shall now bring in the distributions $I_M(\gamma, f)$. Suppose that $L \in \mathscr{L}(M)$. According to the descent formula [1(f), Corollary 8.2],

$$I_L(\gamma, f) = \sum_{L' \in \mathscr{L}(M)} d_M^G(L, L')\hat{I}_M^{L'}(\gamma, f_{L'}), \quad f \in \mathscr{H}(G(\mathbb{R})).$$

If $L \neq M$, the constant $d_M^G(L, L')$ will vanish unless $L' \subsetneq G$. It follows that

$$I_L(\gamma, f) = 0, \quad \gamma \in T_{\mathrm{reg}}(\mathbb{R}),\ L \supsetneq M, \tag{6.3}$$

whenever f is cuspidal.

We are going to establish the following variant of the main result of [1(a)]. It will be used in another paper on the traces of Hecke operators.

THEOREM 6.4: *Suppose that $f \in \mathscr{H}(G(\mathbb{R}))$ is such that the function f_G is supported on $\Pi_{\mathrm{disc}}(G(\mathbb{R}))$. Then $I_M(\gamma, f)$ equals*

$$(-1)^{\dim(A_M/A_G)}\mathrm{vol}(T(\mathbb{R})/A_M(\mathbb{R})^0)^{-1} \sum_{\pi \in \Pi_{\mathrm{disc}}(G(\mathbb{R})^1)} I_M^G(\gamma, \pi)f_G(\tilde{\pi}, H_G(\gamma)),$$

for any point $\gamma \in T_{\mathrm{reg}}(\mathbb{R})$.

Combining this theorem with our results on reśidues, we will also prove

THEOREM 6.5: *Suppose that $f \in \mathscr{H}(G(\mathbb{R}))$ is such that the function f_G is supported on $\Pi_{\mathrm{disc}}(G(\mathbb{R}))$. Then*

$$\sum_{\pi \in \Pi_{\mathrm{disc}}(G(\mathbb{R})^1)} I_M^G(\gamma, \pi)f_G(\tilde{\pi}, H_G(\gamma))$$

$$= (-1)^{\dim(A_M/A_G)} \sum_{\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{R})^1)} I_M^M(\gamma, \pi)D_M^\varepsilon(\tilde{\pi}, H_M(\gamma), f),$$

for any point $\gamma \in T_{\mathrm{reg}}(\mathbb{R})$.

We shall first establish a direct connection of $I_M(\gamma, f)$ with the residues.

LEMMA 6.6: *Suppose that f and γ are given as in the two theorems. Then*

$$I_M(\gamma, f) = \mathrm{vol}(T(\mathbb{R})/A_M(\mathbb{R})^0)^{-1} \sum_{\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{R})^1)} I_M^M(\gamma, \pi) D_M^{\varepsilon}(\pi, H_M(\gamma), f).$$

Proof: Notice that our condition on f implies that the function is cuspidal. According to [1(f), (2.6)], $I_M(\gamma, f)$ satisfies a differential equation

$$I_M(\gamma, zf) = \sum_{L \in \mathscr{L}(M)} \partial_M^L(\gamma, z_L) I_L(\gamma, f), \quad \gamma \in T_{\mathrm{reg}}(\mathbb{R}),$$

for every element z in the center of the universal enveloping algebra. We know that $\partial_M^M(\gamma, z)$ equals $\partial(h_T(z))$, the invariant differential operator on $T(\mathbb{R})$ obtained from the Harish-Chandra map. (See for example Lemma 12.4 of [1(d)].) Therefore, by (6.3), the differential equations simplify to

$$I_M(\gamma, zf) = \partial(h_T(z)) I_M(\gamma, f), \quad \gamma \in T_{\mathrm{reg}}(\mathbb{R}). \tag{6.4}$$

Since the distribution $I_M(\gamma, f)$ is supported on characters, it depends only on f_G. But f_G is a finite sum of eigenfunctions of the center of the universal enveloping algebra, each having regular infinitesimal character. As is well known, this severely limits the solutions of the equations (6.4). For γ lying in a given connected component of $T_{\mathrm{reg}}(\mathbb{R})$, we can write $I_M(\gamma, f)$ as a sum

$$\sum_{\xi} c_{\xi}(H_G(\gamma)) \xi(\gamma), \tag{6.5}$$

where ξ ranges over the regular quasi-characters of $T(\mathbb{R}) \cap G(\mathbb{R})^1$, and $c_{\xi} = 0$ for almost all ξ.

According to the expansion [1(f), (4.11)], we can also write

$$I_M(\gamma, f) = \sum_{L \in \mathscr{L}(M)} {}^c\hat{I}_M^L(\gamma, \theta_L(f)).$$

We would like to show that ${}^cI_M(\gamma, f)$ vanishes if $M \neq G$. Assume inductively that this is so whenever G is replaced by L, with $M \subsetneq L \subsetneq G$. We make a second induction assumption that if $L \supsetneq M$ and $Y \in \mathfrak{a}_L$, the function

$$\theta_L(f, \tilde{\pi}, Y), \quad \pi \in \Pi_{\mathrm{temp}}(L(\mathbb{R})),$$

is supported on $\Pi_{\mathrm{disc}}(L(\mathbb{R}))$. Then $\theta_L(f)$ is the image in $\tilde{\mathscr{I}}_{\mathrm{ac}}(L(\mathbb{R}))$ of a function in $\tilde{\mathscr{H}}_{\mathrm{ac}}(L(\mathbb{R}))$ which satisfies the same conditions at f. The first induction hypothesis then implies that

$$^c\hat{I}^L_M(\gamma, \theta_L(f)) = 0,$$

if $M \subsetneq L \subsetneq G$. We can therefore write

$$I_M(\gamma, f) = \hat{I}^M_M(\gamma, \theta_M(f)) + {}^cI_M(\gamma, f).$$

The function f is cuspidal. Combining [1(f), Lemma 4.7] with the descent property [1(f), Corollary 8.5], we see that $\theta_M(f)$ is the image in $\tilde{\mathscr{I}}_{\mathrm{ac}}(M(\mathbb{R}))$ of a cuspidal function in $\tilde{\mathscr{H}}_{\mathrm{ac}}(M(\mathbb{R}))$. This function is certainly K-finite, so the orbital integral $\hat{I}^M_M(\gamma, \theta_M(f))$ can be expanded in terms of characters. From the standard orthogonality properties of characters, we obtain an expression

$$\mathrm{vol}(T(\mathbb{R})/A_M(\mathbb{R})^0)^{-1} \sum_\pi I^M_M(\gamma, \pi)\theta_M(f, \tilde{\pi}, H_M(\gamma)),$$

in which the sum is over a finite set of representations $\pi \in \Pi_{\mathrm{temp}}(M(\mathbb{R}))$ whose characters do not vanish on the elliptic set. By Corollary 6.3, this in turn equals

$$\mathrm{vol}(T(\mathbb{R})/A_M(\mathbb{R})^0)^{-1} \sum_\pi I^M_M(\gamma, \pi)D^\varepsilon_M(\tilde{\pi}, H_M(\gamma), f). \tag{6.6}$$

We have shown that the difference between (6.5) and (6.6) equals ${}^cI_M(\gamma, f)$.

Suppose that $H_M(\gamma)$ lies in the chamber $\mathfrak{a}^+_P$, $P \in \mathscr{P}(M)$. Identifying the Lie algebra of $A_M(\mathbb{R})$ with $\mathfrak{a}_M$, we replace γ by a translate

$$\gamma \exp X,$$

where X lies in

$$(\mathfrak{a}^G_P)^+ = \{Y \in \mathfrak{a}^+_P : h_G(Y) = 0\}.$$

The resulting functions of X given by (6.5) and (6.6) are both analytic. In fact, they are both $(\mathfrak{a}^G_P)^+$-finite, in the sense that their translates by $(\mathfrak{a}^G_P)^+$ span a finite dimensional space. On the other hand, [1(f), Lemma 4.4] tells us that ${}^cI_M(\gamma \exp X, f)$ is a compactly supported function of X. An analytic function and a compactly supported function can only be equal if they are both zero. Therefore, (6.5) equals (6.6), and ${}^cI_M(\gamma, f)$ vanishes. This completes the first induction argument. Since the quasi-characters in (6.5) are all regular, the sum in (6.6) can be taken over $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{R})^1)$. But (6.6) equals $\hat{I}_M^M(\gamma, \theta_M(f))$, and we have seen that $\theta_M(f)$ is cuspidal. It follows that for any $Y \in \mathfrak{a}_M$, the function

$$\theta_M(f, \tilde{\pi}, Y), \quad \pi \in \Pi_{\mathrm{temp}}(M(\mathbb{R})),$$

is supported on $\Pi_{\mathrm{disc}}(M(\mathbb{R}))$. Therefore, the second induction argument is also complete.

We have actually established the lemma in the course of the two induction arguments. To recapitulate, we note that the expansion [1(f), (4.11)] reduces to

$$I_M(\gamma, \mathrm{f}) \;=\; \hat{I}_M^M(\gamma, \theta_M(f)).$$

The orbital integral on the right then has an expansion

$$\mathrm{vol}(T(\mathbb{R})/A_M(\mathbb{R})^0)^{-1} \sum_{\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{R})^1)} I_M^M(\gamma, \pi)\theta_M(f, \tilde{\pi}, H_M(\gamma))$$

into characters of discrete series on $M(\mathbb{R})^1$. The required formula of Lemma 6.6 is then a consequence of Corollary 6.3. □

Proof of Theorem 6.4: This theorem is an invariant version of Theorem 9.1 of [1(a)]. It is established by showing that as functions of γ, both sides satisfy the same differential equations, boundary conditions, and growth conditions. This was done in full detail in [1(a)], so we shall be quite brief.

The differential equations for $I_M(\gamma, f)$ are given by the formula (6.4), established in the proof of the last lemma. There is a boundary condition for each real root β of $(G(\mathbb{R}), T(\mathbb{R}))$. It follows from (6.3) that the function $I_M^\beta(\gamma, f)$, referred to in [1(f), §2], is just equal to $I_M(\gamma, f)$. If $\partial(u)$ is any invariant differential operator on $T(\mathbb{R})$, the boundary condition becomes

$$\lim_{r\to 0} (\partial(u)I_M(\gamma_r, f) - \partial(u)I_M(\gamma_{-r}, f)) \;=\; n_\beta(A_M) \lim_{s\to 0} \partial(u_1)I_{M_1}(\delta_s, f), \tag{6.7}$$

in the notation of [1(f), (2.7)]. A similar argument shows that $I_M(\gamma, f)$ is smooth across the hypersurface defined by an imaginary root of $(G(\mathbb{R}), T(\mathbb{R}))$. The growth condition we would expect is for $I_M(\gamma, f)$ to be rapidly decreasing on $T_{\mathrm{reg}}(\mathbb{R})$. However, the uniqueness argument works equally well if we only establish that $I_M(\gamma, f)$ is bounded. We shall apply Lemma 6.6.

Suppose that $H_M(\gamma)$ lies in the chamber $\mathfrak{a}_P^+$, $P \in \mathscr{P}(M)$. Then the distribution

$$D_M^\varepsilon(\tilde{\pi}, H_M(\gamma), f) = D_{M,\nu_P}^\varepsilon(\tilde{\pi}, H_M(\gamma), f)$$

equals

$$\sum_{\Omega \in R_G(\nu_P, \varepsilon)} \operatorname*{Res}_{\Omega, \Lambda \to \Lambda_\Omega} (e^{-\Lambda(H_M(\gamma))} J_M(\tilde{\pi}_\Lambda, f^{H_G(\gamma)})).$$

But $R_G(\nu_P, \varepsilon)$ is the residue scheme of the real Paley–Wiener theorem. In particular, the points

$$\{\mathrm{Re}(\Lambda_\Omega) : \Omega \in R_G(\nu_P, \varepsilon)\}$$

all lie in the closure of the dual chamber for P. That is,

$$\mathrm{Re}\, \Lambda_\Omega(X) \geqslant 0, \quad \Omega \in R_G(\nu_P, \varepsilon),\ X \in \mathfrak{a}_P^+.$$

It follows from Lemma 6.6 that $I_M(\gamma, f)$ is bounded for $\gamma \in T_{\mathrm{reg}}(\mathbb{R})$.

Now consider the other side of the formula we are trying to prove. From the character theory of discrete series, the function

$$\begin{aligned} \tilde{I}_M(\gamma, f) &= (-1)^{\dim(A_M/A_G)} \mathrm{vol}(T(\mathbb{R})/A_M(\mathbb{R})^0)^{-1} \\ &\quad \times \sum_{\pi \in \Pi_{\mathrm{disc}}(G(\mathbb{R})^1)} I_M(\gamma, \pi) f_G(\tilde{\pi}, H_G(\gamma)), \quad \gamma \in T_{\mathrm{reg}}(\mathbb{R}), \end{aligned}$$

satisfies the same differential equations, boundary conditions and growth conditions as $I_M(\gamma, f)$. The theorem is to be proved by induction on $\dim A_M$. If $M = G$, the required formula follows directly from the orthogonality properties of characters of discrete series. (In the more difficult case of Schwartz functions, it is a standard result of Harish-Chandra.) In general, we can assume inductively that

$$I_{M_1}(\delta_s, f) = \tilde{I}_{M_1}(\delta_s, f),$$

for M_1 and δ_s as in (6.7). Consequently, the difference

$$I_M(\gamma, f) - \tilde{I}_M(\gamma, f)$$

is smooth across the hypersurface defined by a real root. Theorem 6.4 then follows from a standard uniqueness argument. (See §9 of [1(a)].) $\square$

Proof of Theorem 6.5: This follows immediately from Theorem 6.4 and Lemma 6.6. □

I do not know quite what to make of Theorem 6.5. It expresses the character values of discrete series on noncompact tori as sums of residues of intertwining operators. The formula is reminiscent of Osborne's conjecture, which has been proved by Hecht and Schmid [3, Theorem 3.6]. However, it provides somewhat different information. Suppose for simplicity that $A_G = \{1\}$, and that f is a pseudo-coefficient. That is, $f_G(\tilde{\pi}') = 1$ for a fixed representation π' in $\Pi_{\mathrm{disc}}(G(\mathbb{R}))$, and f_G vanishes at all the other points in $\Pi_{\mathrm{temp}}(G(\mathbb{R}))$. Then the left side of the formula in Theorem 6.5 equals

$$I_M^G(\gamma, \pi') = |G^G(\gamma)|^{1/2}\Theta_{\pi'}(\gamma).$$

The invariant distribution

$$D_M^\varepsilon(\tilde{\pi}, H_M(\gamma), f) = D_{M,\nu_P}^\varepsilon(\tilde{\pi}, H_M(\gamma), f), \quad H_M(\gamma) \in \mathfrak{a}_P^+,$$

on the other side is obtained by combining residues of intertwining operators according to the scheme of the real Paley–Wiener theorem. It follows that the right hand side of the formula can be regarded as a sum of pairs

$$(\pi, \Lambda), \quad \pi \in \Pi_{\mathrm{disc}}(M(\mathbb{R})), \ \Lambda \in \mathfrak{a}_M^*,$$

in which Λ belongs to the chamber ${}^+\mathfrak{a}_P^*$ in $\mathfrak{a}_M^*$ which is dual to $\mathfrak{a}_P^+$, and π' occurs as a composition factor of the representation

$$d_\Lambda \mathscr{I}_P(\pi_\Lambda).$$

In particular, Theorem 6.5 implies that π' occurs as a composition factor of an induced representation for every character exponent of π'.

§7. Conclusions

In the introduction, we claimed that the residues of $\{J_M(\pi_\Lambda, f)\}$, the distributions $\{I_M(\pi, X, f)\}$, and the asymptotic behaviour of $\{I_M(\gamma, f)\}$ could all be computed from each other. Let us summarize how this can be done.

The main point is to compute the distributions

$$I_M(\pi, X, f), \quad \pi \in \Pi(M(F)), \tag{7.1}$$

from the residues. Formula (1.3) gives the values of (7.1) in terms of distributions

$$I_M(\varrho, X, f), \quad \varrho \in \Sigma(M(F)).$$

We shall assume inductively that we can compute these latter distributions if G is replaced by a proper subgroup or if M is replaced by a strictly larger group. (The case that $M = G$ is trivial.) Now, a given standard representation $\varrho \in \Sigma(M(F))$ is of the form

$$\pi^M_{1,\Lambda_1}, \quad M_1 \subset M, \pi_1 \in \Pi_{\text{temp}}(M_1(F)), \Lambda_1 \in \mathfrak{a}^*_{M_1,\mathbb{C}}.$$

Suppose that $M_1 \subsetneq M$. Then the descent formula [1(f), Corollary 8.5] allows us to write the Fourier transform

$$\int_{i\mathfrak{a}^*_{M_1,v}/i\mathfrak{a}^*_{M,v}} I_M(\pi^M_{1,\Lambda_1+\lambda}, h_M(X_1), f)e^{-\lambda(X_1)}\,d\lambda, \quad X_1 \in \mathfrak{a}_{M_1,v},$$

as a linear combination of similar distributions on proper Levi subgroups of G. In other words, $I_M(\varrho, X, f)$ is the inverse Fourier transform of a finite sum of functions we can compute inductively. This leaves undecided only the case that $M_1 = M$. It follows that the general distributions (7.1) can be computed from distributions of the form

$$I_{M,\mu}(\pi, X, f) = e^{-\mu(X)} I_M(\pi_\mu, X, f), \quad \pi \in \Pi_{\text{temp}}(M(F)), \mu \in \mathfrak{a}^*_M.$$

Recall that $I_{M,\mu}(\pi, X, f)$ is a rather straightforward function of μ. It is locally constant on the complement of a finite set of affine hyperplanes which are defined by coroots. Moreover, the mean value property [1(f), Lemma 3.2] gives the value at any μ in terms of the values at nearby points, so we can take $\mu \in \mathfrak{a}^*_M$ to be in general position. We may as well also assume that $M \neq G$. Then by [1(f), Lemma 3.1],

$$I_{M,\varepsilon}(\pi, X, f) = I_M(\pi, X, f) = 0,$$

for ε near 0. It therefore suffices to compute the difference

$$I_{M,\mu}(\pi, X, f) - I_{M,\nu}(\pi, X, f),$$

for any points $\mu, \nu \in \mathfrak{a}^*_M$ in general position. We apply Corollary 4.2. The difference becomes a sum over $L_1 \supset L \supsetneq M$ of the distributions

$$(\hat{D}^{L,\mathcal{N}_L}_{M,\mu} \cdot r^{L_1}_{L,\nu_L})(\pi, X, I_{L_1}(f)) = \int_{\nu_L + i\mathfrak{a}^*_{L,v}/i\mathfrak{a}^*_{L_1,v}} (\hat{D}^{L,\mathcal{N}_L}_{M,\mu} \cdot r^{L_1}_{L,\lambda})(\pi, I^X_{L_1,\lambda}(f))\,d\lambda.$$

Suppose that the residues of $J_M^L(\pi_\Lambda, f)$ are known. What does this mean? According to Lemma 2.1 and Theorem 5.2,

$$F \to \operatorname{Res}_{\Omega} J_M^L(\pi_\Lambda, F(\Lambda)), \quad F \in \mathscr{H}(\mathfrak{a}_M, L(F)),$$

can be regarded as a distribution of finite support in the function

$$\Phi(\Lambda, \varrho) = \hat{F}(\Lambda, \varrho), \quad \Lambda \in \mathfrak{a}_{M,\mathbb{C}}^*, \varrho \in \Sigma(L(F)),$$

for any residue datum Ω for (L, M). We assume that we can calculate it explicitly. This presupposes a knowledge of the poles of $J_M^L(\pi_\Lambda, F(\Lambda))$, which in turn determines $R_L(\mu, \mathscr{N}_L)$ and $\hat{D}_{M,\mu}^{L,\mathscr{N}_L}(\pi)$. We will then be able to write

$$\hat{D}_{M,\mu}^{L,\mathscr{N}_L}(\pi, \Phi) = \sum_{i,j} (\Delta_{ij}(\pi)\Phi)(\Lambda_i, \varrho_j^L), \quad \Phi \in \mathscr{I}(\mathfrak{a}_M, L(F)), \tag{7.2}$$

for Levi subgroups $M_j \subset L$, standard representations $\varrho_j \in \Sigma(M_j(F))$, points $\Lambda_i \in \mathfrak{a}_{M,\mathbb{C}}^*$, and differential operators $\Delta_{ij}(\pi)$ on $\mathfrak{a}_{M,\mathbb{C}}^* \times \mathfrak{a}_{M_j,\mathbb{C}}^*$, all of which we can determine explicitly. Now the integrand

$$(\hat{D}_{M,\mu}^{L,\mathscr{N}_L} \cdot r_{L,\lambda}^{L_1})(\pi, I_{L_1,\lambda}^X(f))$$

above comes from the (L_1, L) family given by (7.2), in which

$$\Phi = \tilde{r}_{Q|Q_0,\lambda}^{-1} \cdot \tilde{r}_{Q|Q_0,\lambda+\zeta} \cdot I_{L_1,\lambda}^X(f), \quad Q \in \mathscr{P}(L), \zeta \in i\mathfrak{a}_L^*.$$

That is

$$\Phi(\Lambda, \varrho) = \tilde{r}_{Q|Q_0}(\pi_{\Lambda+\lambda}^L, \varrho_\lambda)^{-1} \tilde{r}_{Q|Q_0}(\pi_{\Lambda+\lambda+\zeta}^L, \varrho_{\lambda+\zeta}) e^{-(\Lambda+\lambda)(X)} I_{L_1}(\varrho_\lambda^{L_1}, h_{L_1}(X), f).$$

The functions $\tilde{r}_{Q|Q_0}$ come from normalizing factors, which we regard as known. Moreover, we can calculate $I_{L_1}(\varrho_\lambda^{L_1}, h_{L_1}(X), f)$ inductively, since $L_1 \supsetneq M$. Therefore, we can evaluate (7.2) for the given Φ. This allows us to calculate the integrand, and the expansion given by Corollary 4.2. Thus, Corollary 4.2 gives an inductive procedure for computing the distributions (7.1) in terms of residues.

In fact, all we need to compute are the one-dimensional residues. For we can cross the singular hyperplanes one at a time. Suppose that μ and ν lie on opposite sides of a singular hyperplane

$$\Lambda(\alpha^\vee) = c, \quad \alpha \in \Sigma(G, A_M), c \in \mathbb{C},$$

and that these two points differ by a small multiple of α. Define

$$\mathcal{N} = \{\nu_L : L \in \mathcal{L}(M)\},$$

as above, by taking ν_L to be the projection of ν onto $\mathfrak{a}_L^*$. Then the distributions

$$D_{M,\mu}^{L,\mathcal{N}_L}, \quad L \supsetneq M,$$

all vanish except when L is the Levi subset defined by

$$\mathfrak{a}_L = \{H \in \mathfrak{a}_M : \alpha(H) = 0\}.$$

For this exceptional L,

$$(D_{M,\mu}^{L,\mathcal{N}_L})(\pi, F) = \frac{1}{2\pi \mathrm{i}} \int_C J_M^L(\pi_{\mu+z\alpha}, F(\mu + z\alpha))\,\mathrm{d}z,$$

for any $F \in \mathcal{H}(\mathfrak{a}_M, L(F))$. This is just an old fashioned residue, in which C is a small positively oriented circle in the complex plane. The center of C is of course the point z_0 such that $\mu + z_0\alpha$ lies on the given singular hyperplane. Thus, the distributions (7.1) can ultimately be understood in terms of the one-dimensional residues.

Conversely, it is easy to compute residues from the distributions. Suppose that Ω is an arbitrary residue datum for (G, M). According to Lemma 5.1, we can choose μ and $\mathcal{N}$ so that $R_G(\mu, \mathcal{N})$ consists only of the residue datum Ω. Then

$$D_{M,\mu}^{G,\mathcal{N}}(\pi, X, f) = \operatorname*{Res}_{\Omega, \Lambda \to \Lambda_\Omega} (\mathrm{e}^{-\Lambda(X)} J_M(\pi_\Lambda, f^{h_G(X)})), \quad f \in \mathcal{H}(G(F)).$$

Since X is an arbitrary point in $\mathfrak{a}_{M,v}$, the expression on the right is sufficient to determine the residue

$$\operatorname*{Res}_{\Omega, \Lambda \to \Lambda_\Omega} J_M(\pi_\Lambda, F(\Lambda)),$$

for any function $F \in \mathcal{H}(\mathfrak{a}_M, G(F))$. But the expression on the left equals

$$I_{M,\mu}(\pi, X, f) - \sum_{\{L_1 \supset L : L \neq G\}} (\hat{D}_{M,\mu}^{L,\mathcal{N}_L} \cdot r_{L,\nu_L}^{L_1})(\pi, X, I_{L_1}(f)).$$

Thus, the residues can be computed inductively from the distributions $I_{M,\mu}(\pi, X)$. Observe that it suffices to know all the distributions in the special case of a maximal Levi subgroup. For these will determine the one dimensional residues, and as we have seen, these in turn determine the distributions in the case of general rank.

The distributions (7.1) can be used to construct maps

$$\theta_M, {}^c\theta_M : \tilde{\mathscr{H}}_{\mathrm{ac}}(G(F)) \to \tilde{\mathscr{I}}_{\mathrm{ac}}(M(F))$$

which determine the asymptotic behaviour of $I_M(\gamma, f)$. This is treated in [1(f), §4–5], so we shall not discuss it further. Let us consider instead the converse question. How can everything be determined from the asymptotic behaviour of $I_M(\gamma, f)$? Again, we need only assume such knowledge in the case of a maximal Levi subgroup.

Suppose that $\dim(A_M/A_G) = 1$. From the formula [1(f), (4.11)], we know that

$$I_M(\gamma, f) = {}^cI_M(\gamma, f) + \hat{I}_M^M(\gamma, \theta_M(f)).$$

If X belongs to a chamber $\mathfrak{a}_P^+$, with $P \in \mathscr{P}(M)$, we have

$$\theta_M(f, \pi, X) = I_{M,\nu_P}(\pi, X, f),$$

by [1(f), Lemma 4.7]. Assume that $\pi \in \Pi_{\mathrm{temp}}(M(F))$. Then $I_{M,\varepsilon}(\pi, X, f)$ vanishes for any small point ε in $\mathfrak{a}_M^*$, and by Corollary 4.2,

$$I_{M,\nu_P}(\pi, X, f) = D_{M,\nu_P}^{\varepsilon}(\pi, X, f).$$

This is just the distribution associated to the one dimensional residue scheme, with $\mu = \nu_P$ and $\nu = \varepsilon$. It equals a finite sum of residues

$$\frac{1}{2\pi i}\int_{C_k} J_M(\pi_{z\alpha}, f)e^{-z\alpha(X)}\,dz, \tag{7.3}$$

where α is the reduced root of (P, A_M) and k indexes the finite set of points z_k in the right half plane at which the function

$$z \to R_{\bar{P}|P}(\pi_{z\alpha})^{-1}\frac{d}{dz}R_{\bar{P}|P}(\pi_{z\alpha}), \quad z \in \mathbb{C},$$

has a pole. For each k, C_k is a small positively oriented circle about z_k. Now, consider

$$\hat{I}_M^M(\gamma, \theta_M(f))$$

as a function of $\alpha(H_M(\gamma))$. If this variable is positive, the function equals a finite sum of terms

$$p_k(\alpha(H_M(\gamma)))e^{-z_k\alpha(H_M(\gamma))}, \tag{7.4}$$

where each p_k is a polynomial. These terms are characterized by their exponents, and are uniquely determined from the asymptotic values of $\hat{I}_M^M(\gamma, \theta_M(f))$. But (7.4) is just the orbital integral of the function of π defined by the residue (7.3). Moreover, $\hat{I}_M^M(\gamma, \theta_M(f))$ equals $I_M(\gamma, f)$ for $H_M(\gamma)$ outside a compact set. It follows that all the one dimensional residues can be obtained from the asymptotic behaviour of $I_M(\gamma, f)$, in the case of maximal Levi subsets. We have observed that these in turn determine the distributions $\{I_M(\pi, X, f)\}$, the asymptotic behaviour of $\{I_M(\gamma, f)\}$, and the residues of $\{J_M(\pi_\Lambda, f)\}$, all for general M.

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