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Intertwining Operators and Residues
I. Weighted Characters

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INTRODUCTION

Let G be a reductive algebraic group over a local field F of characteristic 0. Let $\Pi(G(F))$ denote the set of equivalence classes of irreducible representations of $G(F)$. The irreducible characters

$$J_G(\pi, f) = \operatorname{tr} \pi(f), \qquad \pi \in \Pi(G(F)), f \in \mathscr{H}(G(F)),$$

are linear functionals on $\mathscr{H}(G(F))$, the Hecke algebra of $G(F)$. They are invariant, in the sense that

$$J_G(\pi, h * f) = J_G(\pi, f * h), \qquad f, h \in \mathscr{H}(G(F)).$$

Characters are of course central to the harmonic analysis of $G(F)$. They also occur on the spectral side of the trace formula, in the case of the compact quotient. In the general trace formula, the analogous terms come from weighted characters. A weighted character is a certain linear form on the algebra

$$\{\pi(f) : f \in \mathscr{H}(G(F))\}$$

which is not in general the trace. Our purpose here is to study the weighted characters as functions of π.

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There is not a major distinction between the theory for real and p-adic groups, so for the introduction we shall assume that F is isomorphic to $\mathbb{R}$. Let us first describe some simple properties of ordinary characters. The set $\Pi(G(F))$ is equipped with a natural action

$$\pi_\lambda(x) = \pi(x)\, e^{\lambda(H_G(x))}, \qquad \pi \in \Pi(G(F)),\ \lambda \in \mathfrak{a}^*_{G,\mathbb{C}},$$

under the complex vector space $\mathfrak{a}^*_{G,\mathbb{C}}$ attached to the rational characters of G. Then the function

$$J_G(\pi_\lambda, f), \qquad \lambda \in \mathfrak{a}^*_{G,\mathbb{C}},$$

is analytic in λ. More generally, suppose that $M_1(F)$ is a Levi component of a parabolic subgroup of $G(F)$, and consider an induced representation

$$\sigma^G_\Lambda = \mathscr{I}_{P_1}(\sigma^G_\Lambda), \qquad P_1 \in \mathscr{P}(M_1),\ \sigma \in \Pi(M_1(F)),\ \Lambda \in \mathfrak{a}^*_{M_1,\mathbb{C}},$$

of $G(F)$. Then

$$J_G(\sigma^G_\Lambda, f), \qquad \Lambda \in \mathfrak{a}^*_{M_1,\mathbb{C}},$$

is an entire function which, in fact, belongs to the Paley–Wiener space. In other words, the Fourier transform

$$J_G(\sigma, X_1, f) = \int_{i\mathfrak{a}^*_{M_1}} J_G(\sigma^G_\Lambda, f)\, e^{-\Lambda(X_1)}\, d\Lambda, \qquad X_1 \in \mathfrak{a}_{M_1},$$

is compactly supported on $\mathscr{X}$. Another basic property is that for general π, the functional $J_G(\pi, f)$ can be expressed in terms of its values at tempered representations. Let $\Sigma(G(F))$ be the set of representations of the form σ^G_Λ as above, but with σ tempered. By analytic continuation, we can certainly express $J_G(\sigma^G_\Lambda, f)$ in terms of the values at tempered representations of $G(F)$. But it is well known that any irreducible character has a unique expansion

$$J_G(\pi, f) = \sum_{\rho \in \Sigma(G(F))} \Delta(\pi, \rho)\, J_G(\rho, f) \tag{1}$$

as a finite integral combination of standard characters.

Weighted characters are linear functionals or "distributions" on $\mathscr{H}(G(F))$ which are indexed by Levi components $M(F)$ of parabolic subgroups, and representations

$$\pi_\lambda, \qquad \pi \in \Pi(M(F)),\ \lambda \in \mathfrak{a}^*_{M,\mathbb{C}}.$$

They reduce to ordinary characters when $M = G$. The weighted character is defined by a formula

$$J_M(\pi_\lambda, f) = \operatorname{tr}(\mathscr{R}_M(\pi_\lambda, P)\, \mathscr{I}_P(\pi_\lambda, f)), \qquad P \in \mathscr{P}(M),$$

where $\mathscr{R}_M(\pi_\lambda, P)$ is a certain operator on the space of the induced representation $\mathscr{I}_P(\pi_\lambda)$. In the case of rank 1, $\mathscr{R}_M(\pi_\lambda, P)$ equals a logarithmic derivative

$$R_{\bar{P}|P}(\pi_\lambda)^{-1} \frac{d}{d\lambda} R_{\bar{P}|P}(\pi_\lambda)$$

of normalized intertwining operators, but in higher rank it is given by a more general limit process. At any rate, to define $J_M(\pi_\lambda, f)$ we must first introduce suitably normalized intertwining operators

$$R_{P'|P}(\pi_\lambda) \colon \mathscr{I}_P(\pi_\lambda) \to \mathscr{I}_{P'}(\pi_\lambda), \qquad P, P' \in \mathscr{P}(M). \tag{2}$$

This we do in Sections 2 and 3 and the Appendix. We shall show that the normalizing factors suggested by Langlands in [15(b), Appendix II] do indeed endow the intertwining operators with the desired properties. We shall also show that the matrix coefficients of $R_{P'|P}(\pi_\lambda)$ are *rational* functions of λ. We introduce the distributions $J_M(\pi_\lambda, f)$ in Section 6. The rationality of $R_{P'|P}(\pi_\lambda)$ implies that the matrix coefficients of $\mathscr{R}_M(\pi_\lambda, P)$ are rational functions of λ. It will follow that $J_M(\pi_\lambda, f)$ is a meromorphic function of $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$, with finitely many poles, each lying along a hyperplane

$$\lambda(\alpha^\vee) = c, \qquad \alpha \in \Sigma(G, A_M),\ c \in \mathbb{C},$$

defined by a root α of (G, A_M). A similar assertion applies if π_λ is replaced by an induced representation

$$\sigma^M_\Lambda, \qquad \sigma \in \Pi(M_1(F)),\ \Lambda \in \mathfrak{a}^*_{M_1,\mathbb{C}},$$

where $M_1(F)$ is a Levi subgroup of $M(F)$.

The generalization of (1) entails a comparison of the normalizing factors for π_λ and ρ_λ. If π and ρ occur in (1), their normalizing factors need not be equal. In Lemma 5.2 we shall show that the ratio

$$\tilde{r}_{P'|P}(\pi_\lambda, \rho_\lambda) = r_{P'|P}(\pi_\lambda)^{-1}\, r_{P'|P}(\rho_\lambda) \tag{3}$$

of these normalizing factors behaves in some ways like the operator (2). In particular, (3) is a rational function of λ. In Proposition 6.1 we will establish an expansion

$$J_M(\pi_\lambda, f) = \sum_{\rho \in \Sigma(M(F))} \sum_{L \in \mathscr{L}(M)} r_M^L(\pi_\lambda, \rho_\lambda) J_L(\rho_\lambda^L, f), \tag{1*}$$

where for each Levi subgroup $L(F)$ containing $M(F)$, $r_M^L(\pi_\lambda, \rho_\lambda)$ is a rational function which is defined by a limiting process from the functions (3).

We shall study the residues of $J_M(\pi_\lambda, f)$ in Sections 8 and 9. Suppose that Ω is a sequence of singular hyperplanes, which intersect at an affine space

$$\Lambda_\Omega + \mathfrak{a}_{L,\mathbb{C}}^*, \qquad \Lambda_\Omega \in \mathfrak{a}_{M,\mathbb{C}}^*, L \in \mathscr{L}(M).$$

Let Res_Ω denote the associated iterated residue. Lemma 8.1 asserts that if $L = G$,

$$\mathrm{Res}_\Omega J_M(\pi_\lambda, f), \qquad f \in \mathscr{H}(G(F)),$$

is an invariant distribution. A natural problem is to compute this distribution, or at least to express it in terms of other natural objects. We shall study this question in some detail in a future paper. Another problem in the case of general L is to find a descent formula, which relates the residue to the distributions

$$J_L(\rho, f), \qquad \rho \in \Sigma(L(F)).$$

A partial answer to this will be provided by Proposition 9.1.

One reason for studying residues is to be able to deform contours of integration. In Section 10 we shall describe a formal scheme for doing this, which is similar to that of the Paley–Wiener theorem. We will then conclude the paper with an application. It is important to understand the integral

$$\phi_M(f, \pi, X) = \int_{i\mathfrak{a}_M^*} J_M(\pi_\lambda, f)\, e^{-\lambda(X)}\, d\lambda, \qquad X \in \mathfrak{a}_M,$$

as a function of a tempered representation $\pi \in \Pi(M(F))$. In Theorem 12.1 we shall establish that $\phi_M(f, \pi, X)$ is an entire function in the natural parameters which characterize π. This is a key requirement for putting the trace formula into invariant form. An equivalent result was established as [1(a), Theorem 12.1]. However, this earlier theorem was proved by looking at orbital integrals instead of residues, and it was contingent on some hypotheses from local harmonic analysis which have yet to be completely verified.

Since the results of this paper are to be applied to the trace formula, we shall work in greater generality. We shall include the case that F is a number field, equipped with a finite set S of valuations. The weighted characters will then be functionals on $\mathscr{H}(G(F_S))$. We should also include the twisted trace formula, so we will work with disconnected groups. In the paper, we will take G to be a component of a nonconnected algebraic group over F.

Much of the material for the normalization of intertwining operators was contained in an old preprint "On the Invariant Distributions Associated to Weighted Orbital Integrals." The main step is to relate the normalizing factors to the Plancherel density. Our argument relies on an unpublished lemma of Langlands, which we have reproduced in the Appendix. I thank Langlands for communicating this result to me.

1. Intertwining Operators

Suppose that G is a connected component of an algebraic group over a field F. We shall write G^+ for the group generated by G, and G^0 for the connected component of 1 in G^+. We assume that G is reductive. Then G^+ and G^0 are reductive algebraic groups over F. We also make the assumption that $G(F)$ is not empty. Then $G(F)$ is a Zariski dense subset of G if F is infinite. As we noted in [1(e)] many of the notions which are used in the harmonic analysis of connected groups are also valid for G. Let us briefly recall some of them.

A *parabolic subset* of G is a set $P=\tilde{P}\cap G$, where $\tilde{P}$ is the normalizer in G^+ of a parabolic subgroup of G^0 which is defined over F. We shall write N_P for the unipotent radical of P. A *Levi component* of P is a nonempty set $M=\tilde{M}\cap P$, where $\tilde{M}$ is the normalizer in G^+ of a Levi component of P^0 which is defined over F. Clearly $P=MN_P$. We call any such M a *Levi subset* of G. Suppose that M is fixed. Let $\mathscr{F}(M)$ denote the parabolic subsets of G which contain M. Similarly, let $\mathscr{L}(M)$ be the collection of Levi subsets of G which contain M. Any $P\in\mathscr{F}(M)$ has a unique Levi component M_P in $\mathscr{L}(M)$ so we can write $P=M_PN_P$. As usual, we let $\mathscr{P}(M)$ denote the set of $P\in\mathscr{F}(M)$ such that $M_P=M$. Suppose that L is an element in $\mathscr{L}(M)$. Then M is a Levi subset of L. We write $\mathscr{F}^L(M)$, $\mathscr{L}^L(M)$, and $\mathscr{P}^L(M)$ for the sets above, but with G replaced by L. For any pair of elements $Q\in\mathscr{P}(L)$ and $R\in\mathscr{P}^L(M)$, there is a unique element $P\in\mathscr{P}(M)$ such that $P\subset Q$ and $P\cap L=R$. When we want to stress its dependence on Q and R, we will denote P by $Q(R)$.

For a given Levi subset M, we let A_M denote the split component of the centralizer of M in M^0. We also write

$$\mathfrak{a}_M=\mathrm{Hom}(X(M)_F,\mathbb{R}),$$

where $X(M)_F$ is the group of characters of M^+ which are defined over F. Now suppose that $P \in \mathscr{F}(M)$. We shall frequently write $A_P = A_{M_P}$ and $\mathfrak{a}_P = \mathfrak{a}_{M_P}$. The roots of (P, A_P) are defined by taking the adjoint action of A_P on the Lie algebra of N_P. We shall regard them either as characters on A_P or, more commonly, as elements in the dual space $\mathfrak{a}_P^*$ of $\mathfrak{a}_P$. The usual properties in the connected case carry over to the present setting. In particular, we can define the simple roots Δ_P of (P, A_P) and the associated "co-roots"

$$\Delta_P^\vee = \{\alpha^\vee : \alpha \in \Delta_P\}$$

in $\mathfrak{a}_P$. The roots of (P, A_P) divide $\mathfrak{a}_P$ into chambers. As usual, we shall write $\mathfrak{a}_P^+$ for the chamber on which the roots Δ_P are positive.

From now on we take F to be either a local or global field which is of characteristic 0. We also fix a finite set S of inequivalent valuations on F. Then

$$F_S = \prod_{v \in S} F_v$$

is a locally compact ring. We can regard G, G^0, and G^+ as schemes over F. Since F embeds diagonally in F_S, we can take the corresponding sets $G(F_S)$, $G^0(F_S)$, and $G^+(F_S)$ of F_S-valued points. Each is a locally compact space. Consider the homomorphism

$$H_G : G^+(F_S) \to \mathfrak{a}_G,$$

which is defined by

$$e^{\langle H_G(x), \chi \rangle} = |\chi(x)| = \prod_{v \in S} |\chi(x_v)|_v,$$

for any $x = \prod_{v \in S} x_v$ in $G^+(F_S)$ and χ in $X(G)_F$. Let us write

$$\mathfrak{a}_{G,S} = H_G(G^+(F_S))$$

for its image. If the set, $S \cap S_\infty$, of Archimedean valuations in S is not empty, then $\mathfrak{a}_{G,S} = \mathfrak{a}_G$. On the other hand, if $S \cap S_\infty$ is empty, $\mathfrak{a}_{G,S}$ could be messy. This is only a superficial difficulty, due to our definition of H_G. To avoid it, we make the assumption that if $S \cap S_\infty$ is empty then all the valuations in S divide a fixed rational prime p. In general, we set

$$\mathfrak{a}_{G,S}^\vee = \mathrm{Hom}(\mathfrak{a}_{G,S}, 2\pi\mathbb{Z}).$$

Then

$$\mathfrak{a}_{G,S}^* = \mathfrak{a}_G^* / \mathfrak{a}_{G,S}^\vee$$

is the additive character group of $\mathfrak{a}_{G,S}$. It is a compact quotient of $\mathfrak{a}_G^*$ if $S \cap S_\infty$ is empty, and is equal to $\mathfrak{a}_G^*$ otherwise.

In this paper, all integrals on groups and homogeneous spaces will be taken with respect to the invariant measures. We will usually not specify how to normalize the measures, beyond assuming that in a given context they satisfy any obvious compatibility conditions. However, there will be two exceptions. One concerns the measure on the groups $N_P(\mathbb{R})$, in the special case that $F = \mathbb{R}$. We shall discuss this in Section 3. The other, which is of no great significance, pertains to the spaces $\mathfrak{a}_M$. We fix Euclidean metrics on these spaces in a compatible way—that is, so that they are all obtained from a fixed, Weyl invariant metric on a maximal such space. Our Haar measure on each $\mathfrak{a}_M$ will then be the associated Euclidean measure. We take the corresponding dual measure on $i\mathfrak{a}_M^*$. Now the objects H_M, $\mathfrak{a}_{M,S}$, and $\mathfrak{a}_{M,S}^\vee$ can of course be defined as above. On the quotient space $i\mathfrak{a}_{M,S}^* = i\mathfrak{a}_M^*/i\mathfrak{a}_{M,S}^\vee$ we take the associated quotient measure. In case $S \cap S_\infty$ is empty, we can assume that this quotient measure is dual to the discrete measure on $\mathfrak{a}_{M,S}$.

For each $v \in S$ let K_v be a fixed maximal compact subgroup of $G^0(F_v)$. Then $K = \prod_{v \in S} K_v$ is a maximal compact subgroup of $G^0(F_S)$. If v is non-Archimedean, we assume in addition that K_v is hyperspecial. We shall only be interested in Levi subsets M which are in good relative position with respect to K. More precisely, we require that each K_v be *admissible* relative to M^0 in the sense of [1(a), Sect. 1]. From now on, M will always be understood to represent some Levi subset which has this property. The pair $(M, K_M = K \cap M^0(F_S))$ then satisfies the same conditions as (G, K).

Suppose that (τ, V_τ) is a representation of $M^+(F_S)$ which is admissible relative to K_M. For any $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$, the representation

$$\tau_\lambda(m) = \tau(m)\, e^{\lambda(H_M(m))}, \qquad m \in M^+(F_S),$$

is also admissible. For each $P \in \mathscr{P}(M)$ let $\mathscr{I}_P(\tau_\lambda)$ denote the associated induced representation. In this paper we shall usually regard it as a representation of the convolution algebra of smooth, compactly supported, K-finite functions f on $G^+(F_S)$. As such, it acts on the space $\mathscr{V}_P(\tau)$ of K-finite functions $\phi\colon K \to V_\tau$ such that

$$\phi(nmk) = \tau(m)\phi(k), \qquad n \in N_P(F_S) \cap K,\ m \in K_M,\ k \in K.$$

The operator $\mathscr{I}_P(\tau_\lambda, f)$ is defined by

$$(\mathscr{I}_P(\tau_\lambda, f)\phi)(k) = \int_{G^+(F_S)} f(y)\tau(M_P(K_P(ky)))\phi(ky)\, e^{(\lambda + \rho_P)(H_P(ky))}\, dy,$$

where we write

$$x = N_P(x)\,M_P(x)\,K_P(x), \qquad N_P(x) \in N_P(F_S),\ M_P(x) \in M^+(F_S),\ K_P(x) \in K,$$

and

$$H_P(x) = H_M(M_P(x)),$$

for any $x \in G^+(F_S)$. As always, ρ_P denotes the vector in $\mathfrak{a}_P^+$ associated with the square root of the modular function of $P^0(F_S)$. We shall be concerned with the intertwining operators

$$J_{P'|P}(\tau_\lambda)\colon \mathscr{V}_P(\tau) \to \mathscr{V}_{P'}(\tau), \qquad P, P' \in \mathscr{P}(M),$$

for these induced representations. Recall that $J_{P'|P}(\tau_\lambda)$ is defined by

$$(J_{P'|P}(\tau_\lambda)\phi)(k) = \int \tau(M_P(n))\phi(K_P(n)k)\, e^{(\lambda+\rho_P)(H_P(n))}\, dn. \tag{1.1}$$

The integral is over $N_{P'}(F_S) \cap N_P(F_S)\backslash N_{P'}(F_S)$, and converges absolutely for the real part of λ in a certain chamber.

Let us list some of the elementary properties of the intertwining operators. These are either well known or follow directly from the definition (1.1):

(J_1) $J_{P'|P}(\tau_\lambda)\,\mathscr{I}_P(\tau_\lambda, f) = \mathscr{I}_{P'}(\tau_\lambda, f)\,J_{P'|P}(\tau_\lambda)$. (This of course is the basic intertwining property.)

(J_2) $J_{P''|P}(\tau_\lambda) = J_{P''|P'}(\tau_\lambda)\,J_{P'|P}(\tau_\lambda)$, for P, P', and P'' in $\mathscr{P}(M)$, with $d(P'', P) = d(P'', P') + d(P', P)$. (We write $d(P'', P)$ for the number of singular hyperplanes in $\mathfrak{a}_M$ which separate the chambers of P'' and P.)

(J_3) Suppose that $Q \in \mathscr{P}(L)$ and $R, R' \in \mathscr{P}^L(M)$, for $L \in \mathscr{L}(M)$. Then

$$(J_{P'|P}(\tau_\lambda)\phi)_k = J_{R'|R}(\tau_\lambda)\,\phi_k, \qquad \phi \in \mathscr{V}_P(\tau),\ k \in K,$$

where $P = Q(R)$, $P' = Q(R')$, and ϕ_k is the function

$$k_1 \to \phi(k_1 k), \qquad k_1 \in K_L,$$

in $\mathscr{V}_R(\tau)$.

(J_4) If τ is unitary, then

$$J_{P'|P}(\tau_\lambda)^* = J_{P|P'}(\tau_{-\bar{\lambda}}),$$

where ( )* denotes the adjoint with respect to the Hermitian form

$$(\phi, \phi') = \int_K (\phi(k), \phi'(k))\, dk$$

on $\mathscr{V}_P(\tau)$.

(J_5) For any $w \in K$,

$$l(w)\, J_{P'|P}(\tau_\lambda)\, l(w)^{-1} = J_{wP'w^{-1}|wPw^{-1}}((w\tau)_{w\lambda}),$$

where the meaning of $w\tau$ and $w\lambda$ is clear, and $l(w)$ is the map from $\mathscr{V}_P(\tau)$ to $\mathscr{V}_{wPw^{-1}}(w\tau)$ defined by

$$(l(w)\phi)(k) = \phi(w^{-1}k).$$

I do not know whether one can prove analytic continuation for general τ. However, we can obtain everything we need from the case that τ is irreducible. Then it is well known [11(a), 13(b), 16(a)] that $J_{P'|P}(\tau_\lambda)$ can be analytically continued as a meromorphic function to all $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$ (see also Theorem 2.1 below). Our eventual goal is to study a certain rational map constructed from the intertwining operators. However, it is necessary to use properties that hold only when the operators have been suitably normalized. In Sections 2–5 we shall discuss the normalization of the operators and some related questions. Before we turn to this, we should agree on how we will attach irreducible representations to the set M.

Let $\Pi(M^+(F_S))$ denote the set of equivalence classes of irreducible (admissible) representations of $M^+(F_S)$. There is an action of the finite group

$$\Xi_{M,S} = \mathrm{Hom}(M^+(F_S)/M^0(F_S), \mathbb{C}^*) = \prod_{v \in S} \mathrm{Hom}(M^+(F_v)/M^0(F_v), \mathbb{C}^*)$$

on $\Pi(M^+(F_S))$, which is given by

$$\pi_\xi(m) = \pi(m)\, \xi(\bar{m}), \qquad \pi \in \Pi(M^+(F_S)),\ \xi \in \Xi_{M,S},\ m \in M^+(F_S).$$

Here, $\bar{m}$ stands for the projection of m onto $M^+(F_S)/M^0(F_S)$. More generally, if τ is any representation of $M^+(F_S)$, and $\zeta = (\xi, \lambda)$ is an element in $\Xi_{M,S} \times \mathfrak{a}^*_{M,\mathbb{C}}$, we shall write

$$\tau_\zeta(m) = \tau(m)\, \xi(\bar{m})\, e^{\lambda(H_M(m))}.$$

We define $\Pi(M(F_S))$ to be the subset of $\Pi(M^+(F_S))$ consisting of those π whose restriction π^0 to $M^0(F_S)$ remains irreducible. Note that π^0 is then invariant under the finite group $M^+(F_S)/M^0(F_S)$. Conversely, any irreducible representation of $M^0(F_S)$ which is invariant by this group equals π^0 for some π in $\Pi(M(F_S))$. There is a character theoretic interpretation of $\Pi(M(F_S))$. If π is any representation of $M^+(F_S)$ of finite length, let $\mathrm{tr}(\pi)$ denote the restriction of the character to $M(F_S)$. (It is sufficient here to regard $\mathrm{tr}(\pi)$ as a distribution, although it is actually known to be a function, at least for p-adic groups [7].) Then $\Pi(M(F_S))$ consists of the representations π in $\Pi(M^+(F_S))$ for which $\mathrm{tr}(\pi)$ does not vanish. It is easy to see that if $\{\pi\}$ is a set of representatives of $\Xi_{M,S}$-orbits in $\Pi(M(F_S))$, then the functions $\{\mathrm{tr}(\pi)\}$ are linearly independent. Note also that the action of $\Xi_{M,S}$ preserves $\Pi(M(F_S))$. Indeed, $\Pi(M(F_S))$ is just the subset of $\Pi(M^+(F_S))$ on which $\Xi_{M,S}$ acts freely. Moreover, the map

$$\{\pi\} \to \pi^0$$

is a bijection from the set of fixed point free orbits of $\Xi_{M,S}$ in $\Pi(M^+(F_S))$ onto the set of elements in $\Pi(M^0(F_S))$ which are invariant under $M^+(F_S)/M^0(F_S)$.

We shall write $\Pi_{\mathrm{temp}}(M(F_S))$ for the subset of representations π in $\Pi(M(F_S))$ such that π^0 is tempered.

2. Normalization

In this section we take π to a representation in $\Pi(M(F_S))$. We shall first state the properties we require of the normalization as a theorem. In the remainder of the section we show that the proof of the theorem reduces in a canonical way to a basic special case, that of F local, $G=G^0$, $\dim(A_M/A_G)=1$, and π square integrable modulo A_M. We shall discuss the special case later, for real and p-adic groups separately, in Sections 3 and 4.

THEOREM 2.1. *There exist meromophic, scalar valued functions*

$$r_{P'|P}(\pi_\lambda), \qquad P, P' \in \mathscr{P}(M),\ \pi \in \Pi(M(F_S)),$$

such that the normalized operators

$$R_{P'|P}(\pi_\lambda) = r_{P'|P}(\pi_\lambda)^{-1} J_{P'|P}(\pi_\lambda)$$

have analytic continuation as meromorphic functions of $\lambda \in \mathfrak{a}^*_{M,\mathbf{C}}$, and such that the following properties hold:

(R$_1$) $R_{P'|P}(\pi_\lambda)\,\mathscr{I}_P(\pi_\lambda, f) = \mathscr{I}_{P'}(\pi_\lambda, f)\, R_{P'|P}(\pi_\lambda)$.

(R$_2$) $R_{P''|P}(\pi_\lambda) = R_{P''|P'}(\pi_\lambda)\, R_{P'|P}(\pi_\lambda)$, *for any P, P', and P'' in $\mathscr{P}(M)$.*

(R$_3$) $(R_{P'|P}(\pi_\lambda)\phi)_k = R_{R'|R}(\pi_\lambda)\phi_k$, $\phi \in \mathscr{V}_P(\pi)$, $k \in K$, *for* $P = Q(R)$, $P' = Q(R')$, *with* $R, R' \in \mathscr{P}^L(M)$ *and* $Q \in \mathscr{P}(L)$ *(and with apologies for overuse of the symbol R).*

(R$_4$) *If π is unitary, then*

$$R_{P'|P}(\pi_\lambda)^* = R_{P|P'}(\pi_{-\bar{\lambda}}).$$

(R$_5$) $l(w)\, R_{P'|P}(\pi_\lambda)\, l(w)^{-1} = R_{wP'w^{-1}|wPw^{-1}}((w\pi)_{w\lambda})$, *for any $w \in K$.*

(R_6) *If F is Archimedean, $R_{P'|P}(\pi_\lambda)$ is a rational function of $\{\lambda(\alpha^\vee)\colon \alpha \in \Delta_P\}$; if F is a local field with residue field of order q, $R_{P'|P}(\pi_\lambda)$ is a rational function of $\{q^{-\lambda(\alpha^\vee)}\colon \alpha \in \Delta_P\}$.*

(R_7) *If π is tempered, $r_{P'|P}(\pi_\lambda)$ has neither zeros nor poles with the real part of λ in the positive chamber attached to P.*

(R_8) *Suppose that F is local, non-Archimedean, that G and π are unramified, and that K is hyperspecial. Then if $\phi \in \mathscr{V}_P(\pi)$ is fixed by K, the function $R_{P'|P}(\pi_\lambda)\phi$ is independent of λ.*

The first five properties (R_1)–(R_5) are obviously extensions of (J_1)–(J_5). Note that once the normalizing factors have been defined, the analytic continuation and (R_1) follow trivially from the corresponding properties for $J_{P'|P}(\pi_\lambda)$. Other properties, such as (R_2), are nontrivial extensions and hold only for the normalized operators. We shall reduce the proof of the theorem to the special case mentioned above. We therefore assume in this section that the functions $r_{P'|P}(\pi_\lambda)$ have been defined, and that the theorem is valid, when $S=\{v\}$, $F=F_v$, $G=G^0$, $\dim(A_M/A_G)=1$, and π is square integrable modulo A_M.

We shall first relax the condition on the rank. Assume that S, F, G, and π satisfy the constraints above, but that $\dim(A_M/A_G)$ is arbitrary. Given $P \in \mathscr{P}(M)$, let Σ_P^r be the set of reduced roots of (P, A_M). For each $\beta \in \Sigma_P^r$, define M_β to be the group in $\mathscr{L}(M)$ such that

$$\mathfrak{a}_{M_\beta} = \{H \in \mathfrak{a}_M \colon \beta(H) = 0\}.$$

Then $\dim(A_M/A_{M_\beta})=1$. Let P_β be the unique group in $\mathscr{P}^{M_\beta}(M)$ whose simple root is β. We define the normalizing factors

$$r_{P'|P}(\pi_\lambda) = \prod_{\beta \in \Sigma_P^r \cap \Sigma_{\overline{P}'}^r} r_{\overline{P}_\beta | P_\beta}(\pi_\lambda), \tag{2.1}$$

in terms of those of rank 1. The property (R_3) follows immediately. In proving (R_2), we may assume that $d(P'', P')=1$. If $d(P'', P) > d(P', P)$, (R_2) holds since it holds for $J_{P''|P}(\pi_\lambda)$ and $r_{P''|P}(\pi_\lambda)$ separately. On the other hand, if $d(P'', P) < d(P', P)$, we have

$$R_{P'|P}(\pi_\lambda) = R_{P'|P''}(\pi_\lambda)\, R_{P''|P}(\pi_\lambda)$$

for the same reason. Reducing to the case of rank 1 by (R_3) we obtain

$$R_{P'|P''}(\pi_\lambda) = R_{P''|P'}(\pi_\lambda)^{-1},$$

so (R_2) follows. The analytic continuation and the remaining properties can all be reduced to the case of rank 1 by (R_2) and (R_3).

Next assume that π is tempered. It is known that π is an irreducible constituent of an induced representation $\mathscr{I}_R^M(\sigma)$, where M_R is an admissible Levi subgroup of M and $\sigma \in \Pi(M_R(F))$ is square integrable modulo A_R. Then $\mathscr{I}_P(\pi_\lambda)$ is canonically isomorphic to a subrepresentation of $\mathscr{I}_{P(R)}(\sigma_\lambda)$. A glance at the defining integral formula reveals that $J_{P'|P}(\pi_\lambda)$ is identified with the restriction of $J_{P'(R)|P(R)}(\sigma_\lambda)$ to the corresponding invariant subspace. If we define

$$r_{P'|P}(\pi_\lambda) = r_{P'(R)|P(R)}(\sigma_\lambda), \tag{2.2}$$

the required properties will all follow from the square integrable case.

Now take π to be an arbitrary representation in $\Pi(M(F))$. (We continue to assume that $S = \{v\}$, $F = F_v$, and $G = G^0$.) The Langlands classification [15(a), 5] holds for p-adic as well as real groups. Therefore π is the Langlands quotient of a representation $\mathscr{I}_R^M(\sigma_\mu)$, where M_R is an admissible Levi subgroup of M, σ is a representation in $\Pi_{\mathrm{temp}}(M_R(F))$, and μ is a point in the chamber of $\mathfrak{a}_R^*/\mathfrak{a}_M^*$ attached to R. That is, π is equivalent to the action of $\mathscr{I}_R(\sigma_\mu)$ on the quotient of $\mathscr{V}_R(\sigma)$ by the kernel of $J_{\bar{R}|R}(\sigma_\mu)$. By (R_7) the function $r_{\bar{R}|R}(\sigma_\Lambda)$ has no pole or zero at $\Lambda = \mu$. Consequently the kernel of $J_{\bar{R}|R}(\sigma_\mu)$ equals the kernel of $R_{\bar{R}|R}(\sigma_\mu)$. Set $\Lambda = \mu + \lambda$, and define

$$r_{P'|P}(\pi_\lambda) = r_{P'(R)|P(R)}(\sigma_\Lambda). \tag{2.3}$$

It follows from (R_3), applied to the tempered representation σ, that the induced representation $\mathscr{I}_P(\pi_\lambda)$ is equivalent to the action of $\mathscr{I}_{P(R)}(\sigma_\Lambda)$ on the quotient $\mathscr{V}_{P(R)}(\sigma)/\ker R_{P(\bar{R})|P(R)}(\sigma_\Lambda)$. Under this equivalence the intertwining operator $R_{P'|P}(\pi_\lambda)$ becomes $R_{P'(R)|P(R)}(\sigma_\Lambda)$. All the required properties of $R_{P'|P}(\pi_\lambda)$, with the exception of (R_4), then follow from the corresponding properties for σ_Λ.

Assume in addition that π is unitary. It has been observed by Knapp and Zuckerman [14] that the unitarizability of the Langlands quotient implies that there is an element w in K_M such that $wRw^{-1} = \bar{R}$, $w\sigma \cong \sigma$, and $w\mu = -\mu$, and such that the inner product on V_π can be obtained from $R_{\bar{R}|R}(\sigma_\mu)$ and w. More precisely, π is unitarily equivalent to the action of $\mathscr{I}_R^M(\sigma_\mu)$ on $\mathscr{V}_R^M(\sigma)/\ker R_{\bar{R}|R}(\sigma_\mu)$ under an inner product

$$\langle \Phi, \Psi \rangle = (\delta l(w)\, R_{\bar{R}|R}(\sigma_\mu)\, \Phi, \Psi), \qquad \Phi, \Psi \in \mathscr{V}_R^M(\sigma),$$

Here $l(w)$ is as defined in Section 1 and δ is an intertwining operator from $w\sigma$ to σ, acting by multiplication on $\mathscr{V}_R^M(w\sigma)$. (Actually Knapp and Zuckerman considered only real groups, but their observation applies equally well to p-adic groups.) It follows that the induced representation $\mathscr{I}_P(\pi)$ is unitarily equivalent to the action of $\mathscr{I}_{P(R)}(\sigma_\mu)$ on $\mathscr{V}_{P(R)}(\sigma)/\ker R_{P(\bar{R})|P(R)}(\sigma_\mu)$ under the new inner product:

$$\langle \Phi, \Psi \rangle = (\delta l(w)\, R_{P(\bar{R})|P(R)}(\sigma_\mu)\, \Phi, \Psi), \qquad \Phi, \Psi \in \mathscr{V}_{P(R)}(\sigma).$$

To establish the adjoint condition (R_4), we choose vectors $\Phi \in \mathscr{V}_{P(R)}(\sigma)$ and $\Phi' \in \mathscr{V}_{P'(R)}(\sigma)$. Then

$$\begin{aligned}&\langle R_{P'(R)|P(R)}(\sigma_\mu)\, \Phi, \Phi' \rangle \\ &\quad = (\delta l(w)\, R_{P'(\bar{R})|P'(R)}(\sigma_\mu)\, R_{P'(R)|P(R)}(\sigma_\mu)\, \Phi, \Phi') \\ &\quad = (\delta l(w)\, R_{P'(\bar{R})|P(\bar{R})}(\sigma_\mu)\, R_{P(\bar{R})|P(R)}(\sigma_\mu)\, \Phi, \Phi').\end{aligned}$$

By (R_3) and the definition of δ, this equals

$$(R_{P'(R)|P(R)}(\sigma_{-\mu})\, \delta l(w)\, R_{P(\bar{R})|P(R)}(\sigma_\mu)\, \Phi, \Phi').$$

Applying (R_4) to the tempered representation σ, we see that this in turn equals

$$\begin{aligned}&(\delta l(w)\, R_{P(\bar{R})|P(R)}(\sigma_\mu)\Phi, R_{P(R)|P'(R)}(\sigma_\mu)\Phi') \\ &\quad = \langle \Phi, R_{P(R)|P'(R)}(\sigma_\mu)\Phi' \rangle.\end{aligned}$$

Translating to a formula for π, we obtain

$$R_{P'|P}(\pi)^* = R_{P|P'}(\pi).$$

Property (R_4) follows for imaginary λ by a change in the definition of π, and then for general λ by analytic continuation.

Finally, let us relax the conditions on S, F, and G. Taking these objects to be arbitrary, we write

$$\pi = \bigotimes_{v \in S} \pi_v, \qquad \pi_v \in \Pi(G(F_v)).$$

We shall require that

$$r_{P'|P}(\pi_\lambda) = \prod_{v \in S} r_{P'|P}(\pi_{v,\lambda}). \tag{2.4}$$

Then

$$R_{P'|P}(\pi_\lambda) = \bigotimes_{v \in S} R_{P'|P}(\pi_{v,\lambda}), \tag{2.4'}$$

and the theorem reduces to the case that S consists of one element v. In this case, write $M_v = M_{F_v}$ and $P_v = P_{F_v}$ for M and P, respectively, regarded as varieties over F_v. Then $P \to P_v$ embeds $\mathscr{P}(M)$ into $\mathscr{P}(M_v)$. Similarly, $P_v \to P_v^0$ embeds $\mathscr{P}(M_v)$ into $\mathscr{P}(M_v^0)$. We shall insist that

$$r_{P'|P}(\pi_\lambda) = r_{P'_v|P_v}(\pi_\lambda) = r_{(P'_v)^0|(P_v)^0}(\pi_\lambda). \tag{2.5}$$

Then

$$R_{P'|P}(\pi_\lambda) = R_{P'_v|P_v}(\pi_\lambda) = R_{(P'_v)^0|(P_v)^0}(\pi_\lambda). \tag{2.5'}$$

The definitions and properties in the theorem reduce to those for local fields and connected algebraic groups, the case we dealt with above.

3. Real Groups

In this section we assume that $G = G^0$, that S contains one Archimedean valuation v, and that $F = F_v$. Since we can always restrict scalars, we shall in fact take $F_v = \mathbb{R}$. Knapp and Stein [13(a), (b)] have given a general procedure for normalizing the intertwining operators so that some of the properties of Section 2 hold. In [15(b), Appendix II] Langlands proposed normalizing the intertwining operators in terms of L-functions. Langlands' suggestions were for any local field, but at the moment they can be carried out only for the reals, since the L-functions for p-adic groups have not been defined in general. We shall show that for a natural choice of measures on the spaces $N_{P'}(\mathbb{R}) \cap N_P(\mathbb{R})$, the normalization proposed by Langlands satisfies all the conditions of Theorem 2.1.

As we have defined them, the intertwining operators depend intrinsically on K. Having fixed K, however, we shall describe how to choose canonical measures on the spaces $N_{P'}(\mathbb{R}) \cap N_P(\mathbb{R})$. Denote real Lie algebras by the appropriate Gothic letters. Then $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebras of $G(\mathbb{R})$ and K, respectively. Let θ be the Cartan involution and let B be a $G(\mathbb{R})$-invariant bilinear form on $\mathfrak{g}$ such that the quadratic form

$$X \to -B(X, \theta X), \qquad X \in \mathfrak{g}, \tag{3.1}$$

is positive definite. Choose any maximal torus T of M which is θ-stable and defined over $\mathbb{R}$. The restriction of B to the Lie algebra $\mathfrak{t}$ of $T(\mathbb{R})$ is non-degenerate. It may be used to define a bilinear form, which we still denote by B, on the dual space of $\mathfrak{t}_{\mathbb{C}}$. This form is positive definite on the real span of the roots of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Set

$$\alpha_{P'|P} = \prod_{\alpha} \left(\tfrac{1}{2}B(\alpha, \alpha)\right)^{1/2}, \qquad P, P' \in \mathscr{P}(M),$$

where the product is taken over all roots α of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ whose restrictions to $\mathfrak{a}_M$ are roots of both (P', A_M) and $(\bar{P}, A_M)$. This number is independent of the maximal torus T. Our measure dn on $N_{P'}(\mathbb{R}) \cap N_P(\mathbb{R})$ is then defined by

$$\int_{N_{P'}(\mathbb{R}) \cap N_P(\mathbb{R})} \phi(n)\, dn = \alpha_{P'|P} \int_{\mathfrak{n}_{P'} \cap \mathfrak{n}_P} \phi(\exp X)\, dX, \quad \phi \in C_c^\infty(N_{P'}(\mathbb{R}) \cap N_P(\mathbb{R})),$$

where dX is the Euclidean measure defined by the restriction of the form (3.1) to $\mathfrak{n}_{P'} \cap \mathfrak{n}_P$. Note that if $B(\cdot,\cdot)$ is replaced by $t^2 B(\cdot,\cdot)$, $t>0$, the number of $\alpha_{P'|P}$ will be replaced by $t^{-\dim(N_{P'} \cap N_P)} \alpha_{P'|P}$, while dX will be replaced by $t^{\dim(N_{P'} \cap N_P)}\, dX$. Since $B(\cdot,\cdot)$ is uniquely determined up to scalar multiples on each of the simple factors of $\mathfrak{g}$, the measures dn are independent of B. We define $J_{P'|P}(\pi_\lambda)$ with the associated invariant measure on $N_{P'}(\mathbb{R}) \cap N_P(\mathbb{R}) \backslash N_{P'}(\mathbb{R})$.

We recall how the L-functions of representations of $M(\mathbb{R})$ are defined. To any $\pi \in \Pi(M(\mathbb{R}))$, there corresponds a map

$$\phi \colon W_{\mathbb{R}} \to {}^L M,$$

from the Weil group of $\mathbb{R}$ to the L-group of M, which is uniquely determined up to conjugation by ${}^L M^0$ [15(a)]. Let ρ be a (finite-dimensional, analytic) representation of ${}^L M$. Then $\rho \cdot \phi$ is a representation of $W_{\mathbb{R}}$ which has a decomposition $\bigoplus_\tau \tau$ into irreducible representations of $W_{\mathbb{R}}$. By definition,

$$L(s, \pi, \rho) = L(s, \rho \cdot \phi) = \prod_\tau L(s, \tau).$$

If τ is one-dimensional, it is the pullback to $W_{\mathbb{R}}$ of a quasi-character

$$x \to x^{-N} |x|^{s_1}, \qquad N \in \{0, 1\},\ s_1 \in \mathbb{C},$$

of $\mathbb{R}^*$, in which case

$$L(s, \tau) = L_{\mathbb{R}}(s + s_1) \overset{\text{defn}}{=} \pi^{-(s+s_1)/2}\, \Gamma((s+s_1)/2).$$

Otherwise, τ is the two-dimensional representation induced from a quasi-character

$$z \to z^{-N} (z\bar{z})^{s_1}, \qquad N \in \{0, 1, 2, \ldots\},\ s_1 \in \mathbb{C},$$

or

$$z \to \bar{z}^{-N} (z\bar{z})^{s_1}, \qquad N \in \{0, 1, 2, \ldots\},\ s_1 \in \mathbb{C},$$

of $\mathbb{C}^*$, in which case

$$L(s, \tau) = \Gamma_{\mathbb{C}}(s + s_1) \overset{\text{defn}}{=} 2(2\pi)^{-(s+s_1)}\, \Gamma(s + s_1)$$

(see [18]).

Let $\rho_{P'|P}$ be the adjoint representation of ${}^L M$ on the complex vector space ${}^L\mathfrak{n}_{P'} \cap {}^L\mathfrak{n}_P \backslash {}^L\mathfrak{n}_{P'}$. We shall take $\rho = \tilde{\rho}_{P'|P}$, the contragradient of $\rho_{P'|P}$. In the present context, the normalizing factors of Langlands can be taken to be

$$r_{P'|P}(\pi_\lambda) = L(0, \pi_\lambda, \tilde{\rho}_{P'|P})\, L(1, \pi_\lambda, \tilde{\rho}_{P'|P})^{-1}, \qquad \lambda \in \mathfrak{a}^*_{M,\mathbb{C}}. \tag{3.2}$$

We must show that they satisfy the conditions of Theorem 2.1. It is clear from the definition that the factors satisfy the formulas (2.1)–(2.3). Therefore, the reduction of the last section applies, and we may assume that $\dim(A_M/A_G)=1$ and that π is square integrable modulo A_M. We shall establish Theorem 2.1 under these assumptions.

Some of the conditions of Theorem 2.1 are immediate. As we mentioned earlier, the analytic continuation is known and (R_1) is equivalent to (J_1). Condition (R_3) is trivial since $\dim(A_M/A_G)=1$, while (R_8) does not pertain to real groups. From the definition (3.2) we may deduce formulas

$$r_{P'|P}(\pi_\lambda)=r_{wP'w^{-1}|wPw^{-1}}((w\pi)_{w\lambda}), \qquad w\in K,$$

and, if π is tempered,

$$\overline{r_{P'|P}(\pi_\lambda)}=r_{P|P'}(\pi_{-\bar{\lambda}}).$$

Combined with (J_4) and (J_5) they yield properties (R_4) and (R_5) of the theorem.

The remaining points of the theorem are (R_2), (R_6), and (R_7). To verify these, we must look more carefully at the map ϕ associated to π. The Weil group $W_{\mathbb{R}}$ contains a normal subgroup $\mathbb{C}^*$ of index 2; we can fix an element σ in the nontrivial coset such that $\sigma^2=-1$ and $\sigma z\sigma^{-1}=\bar{z}$. The L-group ${}^LM={}^LM^0\rtimes W_{\mathbb{R}}$ comes equipped with a distinguished maximal torus ${}^LT={}^LT^0\rtimes W_{\mathbb{R}}$. Fix $P\in\mathscr{P}(M)$. Then there are embeddings

$${}^LM\subset{}^LP\subset{}^LG.$$

Following [15(a)], choose ϕ so that its image normalizes ${}^LT^0$. Then for each $z\in\mathbb{C}^*$, $\phi(z)$ is a point in ${}^LT^0$. It is determined by a formula

$$\lambda^\vee(\phi(z))=z^{\langle\mu,\lambda^\vee\rangle}\bar{z}^{\langle\nu,\lambda^\vee\rangle}, \qquad \lambda^\vee\in L^\vee,$$

for elements $\mu,\nu\in L\otimes\mathbb{C}$ with $\mu-\nu\in L$. (Recall that $L^\vee$ is the lattice of rational characters of ${}^LT^0$ and $L=\mathrm{Hom}(L^\vee,\mathbb{Z})$ is the dual lattice.) The expression on the right is just a formal way of writing the complex number

$$z^{\langle\mu-\nu,\lambda^\vee\rangle}(z\bar{z})^{\langle\nu,\lambda^\vee\rangle}=\bar{z}^{\langle\nu-\mu,\lambda^\vee\rangle}(z\bar{z})^{\langle\mu,\lambda^\vee\rangle}.$$

The point

$$h=\phi(\sigma)=a\rtimes\sigma, \qquad a\in{}^LM^0,$$

normalizes ${}^LT^0$. We shall write $\bar{\sigma}$ for its adjoint action on ${}^LT^0$, L, and $L^\vee$. Then $\nu=\bar{\sigma}\mu$. We note that there is a canonical injection of the space

$$\mathfrak{a}^*_{M,\mathbb{C}}=X(M)\otimes\mathbb{C}$$

into $L \otimes \mathbb{C}$. If π is replaced by π_λ, $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$, ϕ will be replaced by a map ϕ_λ, in which (μ, ν, h) becomes $(\mu+\lambda, \nu+\lambda, h)$.

Let $T \subset M$ be a maximal torus over $\mathbb{R}$ whose real split component is A_M. Fix an isomorphism of $T(\mathbb{C})$ with $\mathrm{Hom}(L, \mathbb{C}^*)$. Then L and $L^\vee$ are identified with $X^*(T)$ and $X_*(T)$ respectively and $\bar\sigma$ is the same as the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ action induced from T (see [15(a), p. 50]). Let $\Sigma_P(G, T)$ be the set of roots of (G, T) which restrict to roots of (P, A_M). Then the eigenspaces of $\tilde\rho_{\bar P|P}(\phi_\lambda(\mathbb{C}^*))$ are the root spaces of $\{-\alpha^\vee : \alpha \in \Sigma_P(G, T)\}$. Consequently, the irreducible constituents τ_λ of $\tilde\rho_{\bar P|P} \cdot \phi_\lambda$ correspond to orbits of $\bar\sigma$ in $\Sigma_P(G, T)$. Consider a two-dimensional constituent, corresponding to a pair $\{\alpha, \bar\sigma\alpha\}$ of complex roots. Then τ_λ is induced from the quasi-character

$$z \to z^{\langle \mu+\lambda, \alpha^\vee \rangle} \bar z^{\langle \nu+\lambda, \alpha^\vee \rangle} = \bar z^{\langle \bar\sigma\mu - \mu, \alpha^\vee \rangle} (z\bar z)^{\langle \mu+\lambda, \alpha^\vee \rangle}$$

of $\mathbb{C}^*$. Replacing $\alpha^\vee$ by $\bar\sigma\alpha^\vee$ if necessary, we can assume that $\langle \bar\sigma\mu - \mu, \alpha^\vee \rangle$ is a nonpositive integer. Consequently

$$L(0, \tau_\lambda) L(1, \tau_\lambda)^{-1} = \Gamma_{\mathbb{C}}(\langle \mu+\lambda, \alpha^\vee \rangle)\, \Gamma_{\mathbb{C}}(\langle \mu+\lambda, \alpha^\vee \rangle + 1)^{-1}. \quad (3.3)$$

The one-dimensional constituents correspond to the real roots $\{\alpha_0\}$ in $\Sigma_P(G, T)$. There is at most one of these. If α_0 exists, let $X_{\alpha_0^\vee}$ be a root vector for $\alpha_0^\vee$, and set

$$\mathrm{Ad}(\phi(\sigma))\, X_{\alpha_0^\vee} = (-1)^{N_0} X_{\alpha_0^\vee}, \qquad N_0 = 0, 1.$$

Since

$$\phi_\lambda(z)\, X_{\alpha_0^\vee} = (z\bar z)^{\langle \mu+\lambda, \alpha_0^\vee \rangle} X_{\alpha_0^\vee}, \qquad z \in \mathbb{C}^*,$$

the one-dimensional constituent τ_λ comes from the quasi-character

$$x \to \left(\frac{x}{|x|}\right)^{-N_0} |x|^{\langle \mu+\lambda, \alpha_0^\vee \rangle} = x^{-N_0} |x|^{\langle \mu+\lambda, \alpha_0^\vee \rangle + N_0}$$

of $\mathbb{R}^*$. Consequently,

$$L(0, \tau_\lambda) L(1, \tau_\lambda)^{-1} = \Gamma_{\mathbb{R}}(\langle \mu+\lambda, \alpha_0^\vee \rangle + N_0)\, \Gamma_{\mathbb{R}}(\langle \mu+\lambda, \alpha_0^\vee \rangle + N_0 + 1)^{-1}. \quad (3.4)$$

Condition (R_7) of Theorem 2.1 is easily observed from (3.3) and (3.4). For if π is tempered, and α is as in (3.3), the real part of the number $\langle \mu, \alpha^\vee \rangle$ is nonnegative. If λ belongs to the chamber attached to P, the number $\langle \lambda, \alpha^\vee \rangle$ is real and positive. If α_0 is as in (3.4), the real part of the number $\langle \mu+\lambda, \alpha_0^\vee \rangle$ is positive. Condition (R_7) follows from the fact that the gamma function has neither zeros nor poles in the right half plane.

To establish (R_2), we must show that

$$R_{P|\bar{P}}(\pi_\lambda)\, R_{\bar{P}|P}(\pi_\lambda) = 1.$$

Incorporating λ into the representation π and then appealing to analytic continuation, we may assume that $\lambda = 0$ and π is tempered. Then (R_4) applies, and it is enough to show that

$$J_{\bar{P}|P}(\pi)^* J_{\bar{P}|P}(\pi) = |r_{\bar{P}|P}(\pi)|^2. \tag{3.5}$$

PROPOSITION 3.1. *Assuming that π is tempered, we have*

$$(J_{\bar{P}|P}(\pi)^* J_{\bar{P}|P}(\pi))^{-1} = (2\pi)^{-\dim N_P} \cdot \prod_{\alpha \in \Sigma_P(G,T)} |\mu(\alpha^\vee)| \cdot \left| \tanh\left(\frac{\pi\mu(\alpha_0^\vee)}{2i}\right)\right|^{\varepsilon_\pi}, \tag{3.6}$$

where

$$\varepsilon_\pi = \begin{cases} (-1)^{N_0}, & \textit{if } \alpha_0 \textit{ exists} \\ 0, & \textit{otherwise.} \end{cases}$$

We shall save the proof of this proposition for the Appendix. It rests on Harish-Chandra's explicit formula for the Plancherel density, and a lemma of Langlands which interprets ε_π as a sign occurring in Harish-Chandra's parametrization. The right hand side (3.6) is actually somewhat simpler than Harish-Chandra's formula. It is missing certain constants, whose absence we owe to our choice of measures on $N_P(\mathbb{R})$ and $N_{\bar{P}}(\mathbb{R})$.

Given Proposition 3.1 we have only to look at the absolute values of (3.3) and (3.4). If s is any imaginary number,

$$|\Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s+1)^{-1}|^{-2} = |2\pi s^{-1}|^{-2} = (2\pi)^{-2} s\bar{s},$$

while

$$\begin{aligned} |\Gamma_{\mathbb{R}}(s)\,\Gamma_{\mathbb{R}}(s+1)^{-1}|^{-2} &= \frac{1}{\pi}\left|\Gamma\left(\frac{s}{2}\right)\right|^{-2} \left|\Gamma\left(\frac{s+1}{2}\right)\right|^{2} \\ &= \frac{1}{\pi}\cdot\frac{1}{\pi}\left|\frac{s}{2i}\sinh\left(\frac{\pi s}{2i}\right)\right| \cdot \pi \left|\cosh\left(\frac{\pi s}{2i}\right)\right|^{-1} \\ &= (2\pi)^{-1}|s|\left|\tanh\left(\frac{\pi s}{2i}\right)\right|, \end{aligned}$$

and

$$|\Gamma_{\mathbb{R}}(s+1)\,\Gamma_{\mathbb{R}}(s+2)^{-1}|^{-2} = \left|\Gamma_{\mathbb{R}}(s+1)\,\frac{s}{2\pi}\,\Gamma_{\mathbb{R}}(s)^{-1}\right|^{-2}$$
$$= (2\pi)^{-1}\,|s|\left|\coth\left(\frac{\pi s}{2i}\right)\right|.$$

It follows from (3.3) and (3.4) that $|r_{\bar{P}|P}(\pi)|^{-2}$ equals the right hand of (3.6). This proves formula (3.5) and therefore property (R_2).

It remains to establish the rationality of

$$R_{\bar{P}|P}(\pi_\lambda) = r_{\bar{P}|P}(\pi_\lambda)^{-1}\,J_{\bar{P}|P}(\pi_\lambda).$$

Let β be the simple root of (P, A_M). Since $r_{\bar{P}|P}(\pi_\lambda)$ is a product of functions of the form (3.3) and (3.4), both $r_{\bar{P}|P}(\pi_\lambda)$ and $r_{\bar{P}|P}(\pi_\lambda)^{-1}$ can be expressed as constant multiples of products of the form

$$\prod_{i=1}^{N} \Gamma(t_i\lambda(\beta^\vee)+\zeta_i)\Gamma(t_i\lambda(\beta^\vee)+\eta_i)^{-1}, \tag{3.7}$$

with each t_i a positive real number, and $\zeta_i, \eta_i \in \mathbb{C}$. Let Γ denote a finite set of irreducible representations of K (in addition to the gamma function!). Write $J_{\bar{P}|P}(\pi_\lambda)_\Gamma$ and $R_{\bar{P}|P}(\pi_\lambda)_\Gamma$ for the restrictions of the given operators to $\mathscr{V}_P(\pi)_\Gamma$, the subspace of $\mathscr{V}_P(\pi)$ that transforms under K according to Γ. The operator $J_{\bar{P}|P}(\pi_\lambda)_\Gamma$ can be expressed simply in terms of Harish-Chandra's c-function [11(c), Lemma 11.1]. It follows from a result of Wallach [20, Theorem 7.2] that the matrix coefficients of $J_{\bar{P}|P}(\pi_\lambda)_\Gamma$ are linear combinations of functions of the form (3.7). The same is therefore true of the matrix coefficients of $R_{\bar{P}|P}(\pi_\lambda)_\Gamma$. On the other hand, by results of L. Cohn [10, Theorem 5], the inverse of the determinant of $J_{\bar{P}|P}(\pi_\lambda)_\Gamma$ is a function of the form (3.7). Therefore, the matrix coefficients of $J_{\bar{P}|P}(\pi_\lambda)_\Gamma^{-1}$ are also linear combinations of functions of the form (3.7). The same is therefore true of the matrix coefficients of $R_{\bar{P}|P}(\pi_\lambda)_\Gamma^{-1}$. Now there is an elementary estimate of the gamma function that we can apply to (3.7). Given $t>0$, and $\zeta, \eta \in \mathbb{C}$, and also a real number b, we can choose constants c and n, and a polynomial $l(z)$ such that

$$\left| l(z)\,\frac{\Gamma(tz+\zeta)}{\Gamma(tz+\eta)}\right| \leqslant c(1+|z|)^n,$$

for all $z \in \mathbb{C}$ with $\operatorname{Re} z > b$ (see, for example, [1(d), p. 33]). It follows that we may choose $l(\ )$, c, and n such that

$$|l(\lambda(\beta^\vee))|\,(\|R_{\bar{P}|P}(\pi_\lambda)_\Gamma\| + \|R_{\bar{P}|P}(\pi_\lambda)_\Gamma^{-1}\|) \leqslant c(1+|\lambda(\beta^\vee)|)^n, \tag{3.8}$$

for all $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$ with $\operatorname{Re}\lambda(\beta^\vee) \geqslant b$.

On the other hand, the functional equation (R_2) tells us that

$$\|R_{\bar{P}|P}(\pi_\lambda)_\Gamma\| = \|R_{P|\bar{P}}(\pi_\lambda)_\Gamma^{-1}\|.$$

Apply (3.8) (with the roles of P and $\bar{P}$ reversed) to the norm on the right. Since $-\beta$ is the simple root of $(\bar{P}, A_M)$, we see that

$$|l(\lambda(\beta^\vee))| \cdot \|R_{\bar{P}|P}(\pi_\lambda)_\Gamma\| \leqslant c(1+|\lambda(\beta^\vee)|)^n$$

whenever $\lambda(\beta^\vee) \leqslant b$. Combining this with (3.8), we get

$$|l(\lambda(\beta^\vee))| \cdot \|R_{\bar{P}|P}(\pi_\lambda)_\Gamma\| \leqslant c(1+|\lambda(\beta^\vee)|)^n$$

for all $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$. Thus $R_{\bar{P}|P}(\pi_\lambda)_\Gamma$, a priori a meromorphic function of the complex variable $\lambda(\beta^\vee)$, extends to a meromorphic function on the Riemann sphere. It is therefore a rational function of $\lambda(\beta^\vee)$. This establishes the final property (R_6). Therefore Theorem 2.1 holds for real groups with the normalizing factors (3.2).

Remarks. 1. Suppose that $\dim(A_M/A_G)=1$ as above, and the $G(\mathbb{R})$ does not have a compact Cartan subgroup. Then there is no real root α_0. Since each function (3.3) is rational, $r_{\bar{P}|P}(\pi_\lambda)$ is rational. Consequently $J_{\bar{P}|P}(\pi_\lambda)$ is itself in this case a rational function of λ.

2. If T is any endomorphism of $\mathcal{V}_P(\pi)$,

$$R_{P'|P}(\pi_\lambda)^{-1}\, TR_{P'|P}(\pi_\lambda) = J_{P'|P}(\pi_\lambda)^{-1}\, TJ_{P'|P}(\pi_\lambda).$$

The rationality of $R_{P'|P}(\pi_\lambda)$ is therefore a generalization of a result [11(c), Lemma 19.2] of Harish-Chandra.

3. Shahidi has used Whitaker functionals to investigate the normalizing factors (3.2). Some of the results of this section can be extracted from his paper [16(c)]. Shahidi's methods give additional information about the normalized operators that will be useful in applications of the trace formula.

4. p-ADIC GROUPS

Suppose that $G=G^0$, $S=\{v\}$, $F=F_v$, and that F_v is non-Archimedean with residue field of order q. In Lecture 15 of [9], Langlands verifies the existence of normalizing factors $r_{P'|P}(\pi_\lambda)$ such that Theorem 2.1 holds. The factors are required to satisfy (2.1)–(2.3), so it is enough to define them when $\dim(A_M/A_G)=1$ and π is square integrable modulo A_G. For a given

$P \in \mathscr{P}(M)$, with simple root β, Langlands observes that one can define a rational function $V_P(\pi, z)$ of one variable so that

$$r_{\bar{P}|P}(\pi_\lambda) = V_P(\pi, q^{-\lambda(\beta^\vee)})$$

satisfies the conditions of Theorem 2.1. The main requirement of $r_{\bar{P}|P}(\pi_\lambda)$ is that

$$(\overline{r_{\bar{P}|P}(\pi_{-\bar{\lambda}})}\, r_{\bar{P}|P}(\pi_\lambda))^{-1} = \mu_M(\pi_\lambda), \tag{4.1}$$

where

$$\mu_M(\pi_\lambda) = (J_{\bar{P}|P}(\pi_{-\bar{\lambda}})^* J_{\bar{P}|P}(\pi_\lambda))^{-1} = (J_{P|\bar{P}}(\pi_\lambda)\, J_{\bar{P}|P}(\pi_\lambda))^{-1}$$

is Harish-Chandra's μ-function, and π is taken to be unitary as well as square integrable modulo A_M.

A future concern (although not for this paper) will be to show that related groups can be assigned the same normalizing factors. We remark that the general construction above will suffice for this, provided one can show that the μ-functions can be matched. In [2] we shall carry this out for the example of inner twistings of GL_n.

It would of course be useful to define the normalizing factors in terms of L-functions, as we did for real groups. For p-adic fields and $G = GL_n$, the L-functions have been defined. Suppose that P is a standard maximal parabolic subgroup of GL_n, that $\pi = \pi_1 \times \pi_2$ is an irreducible tempered representation of $M_P(F) \cong GL_{n_1}(F) \times GL_{n_2}(F)$, and that

$$\pi_\lambda(m_1 \times m_2) = \pi(m_1 \times m_2)\, |\det m_1|^s\, |\det m_2|^{-s},$$

for $s \in \mathbb{C}$. Shahidi [16(b)] has shown that for a certain normalization of the measures on $N_P(F)$ and $N_{\bar{P}}(F)$, depending on a fixed additive character ψ of F, the factors

$$r_{\bar{P}|P}(\pi_\lambda) = L(s, \pi_1 \times \tilde{\pi}_2)(L(1+s, \pi_1 \times \tilde{\pi}_2) \mid \varepsilon(s, \pi_1 \times \tilde{\pi}_2, \psi))^{-1}$$

satisfy (4.1). Here $L(\cdot)$ and $\varepsilon(\cdot)$ are the functions defined by Jacquet, Piatetski-Shapiro, and Shalika [12]. Therefore, for GL_n the intertwining operators can be normalized by L-functions.

5. Standard Representations

For the rest of this paper G and F will be as in Section 1, with no additional restrictions. We assume that the normalizing factors $\{r_{P'|P}(\pi_\lambda)\}$ have been fixed, and satisfy the supplementary conditions (2.1)–(2.5) as well as the properties (R_1)–(R_8) of Theorem 2.1. This section will be an addendum to our discussion of normalization. We shall compare the normalizing factors for representations which are related by block equivalence. We begin by discussing how this equivalence relation, which was introduced by Vogan [19(b)], applies to the present context.

We can regard G as a scheme defined over the ring F_S. An *admissible Levi subset over* F_S will be a product $\mathcal{M} = \prod_{v \in S} M_v$, where each M_v is a Levi subset of G which is defined over F_v and for which K_v is admissible. Given such an $\mathcal{M}$, we write $A_{\mathcal{M}} = \prod_{v \in S} A_{M_v}$ and $\mathfrak{a}_{\mathcal{M}} = \bigoplus_{v \in S} \mathfrak{a}_{M_v}$. By *a root of* $(G, A_{\mathcal{M}})$ we shall understand a root of (G, A_{M_v}), for some $v \in S$. If $\sigma = \bigotimes_{v \in S} \sigma_v$ is an admissible representation of the group $\mathcal{M}^+(F_S) = \prod_v M_v^+(F_v)$ and $\Lambda = \bigoplus_v \Lambda_v$ belongs to $\mathfrak{a}^*_{\mathcal{M},\mathbb{C}}$, then

$$\sigma_\Lambda(m) = \bigotimes_v \sigma_{v,\Lambda_v}(m_v) = \bigotimes_v \sigma_v(m_v)\, e^{\Lambda_v(H_{M_v}(m_v))}, \qquad m = \prod_v m_v,$$

is also an admissible representation of $\mathcal{M}^+(F_S)$. We shall write σ_Λ^G for the equivalence class of the associated induced representation of $G^+(F_S)$.

Let $\Sigma(G(F_S))$ denote the set of (equivalence classes of) representations of $G^+(F_S)$ which equal σ_Λ^G for some $\mathcal{M}$, with σ a representation in

$$\Pi_{\mathrm{temp}}(\mathcal{M}(F_S)) = \left\{ \bigotimes_v \sigma_v : \sigma_v \in \Pi_{\mathrm{temp}}(M_v(F_v)) \right\},$$

and Λ a point in $\mathfrak{a}^*_{\mathcal{M}}$ which is regular (in the sense that $\Lambda(\beta) \neq 0$ for every root β of $(G, A_{\mathcal{M}})$). The elements in $\Sigma(G(F_S))$ are called *standard representations.* Suppose that $\rho \in \Sigma(G(F_S))$. Our definition is such that ρ^0 belongs to $\Sigma(G^0(F_S))$. It is known that ρ^0 has a unique irreducible quotient. Consequently, ρ also has a unique irreducible quotient. It is a representation in $\Pi(G(F_S))$, which we denote by $\bar{\rho}$. Moreover, $\rho \to \bar{\rho}$ is a bijection from $\Sigma(G(F_S))$ onto $\Pi(G(F_S))$.

The next proposition is a slight extension of a basic result (see [19(a), Prop. 6.6.7] and the introduction to [19(b)]). We include a proof, which is based on familiar ideas.

PROPOSITION 5.1. *Let* $\{\Pi(G(F_S))\}$ *and* $\{\Sigma(G(F_S))\}$ *denote the set of* $\Xi_{G,S}$*-orbits in* $\Pi(G(F_S))$ *and* $\Sigma(G(F_S))$*, respectively. Then there are uniquely determined complex numbers*

$$\{\Gamma(\rho, \pi), \Delta(\pi, \rho) : \pi \in \Pi(G(F_S)),\ \rho \in \Sigma(G(F_S))\},$$

with

$$\Gamma(\rho_\xi, \pi_\eta) = \xi(G)\Gamma(\rho, \pi)\eta(G)^{-1}, \tag{5.1}$$

and

$$\Delta(\pi_\eta, \rho_\xi) = \eta(G)\Delta(\pi, \rho)\xi(G)^{-1}, \qquad \xi, \eta \in \Xi_{G,S}, \tag{5.2}$$

such that

$$\operatorname{tr}(\rho) = \sum_{\pi \in \{\Pi(G(F_S))\}} \Gamma(\rho, \pi) \operatorname{tr}(\pi), \qquad \rho \in \Sigma(G(F_S)), \tag{5.3}$$

and

$$\operatorname{tr}(\pi) = \sum_{\rho \in \{\Sigma(G(F_S))\}} \Delta(\pi, \rho) \operatorname{tr}(\rho), \qquad \pi \in \Pi(G(F_S)). \tag{5.4}$$

Proof. Recall that if $\{\pi\}$ is a set of representatives of $\Xi_{G,S}$-orbits in $\Pi(G(F_S))$, the functions $\{\operatorname{tr}(\pi)\}$ are linearly independent. The uniqueness assertion follows easily from this.

To prove the existence of $\{\Gamma(\rho, \pi)\}$ and $\{\Delta(\pi, \rho)\}$ we can clearly assume that $S = \{v\}$ and $F = F_v$. Let ρ be a standard representation in $\Sigma(G(F_S))$. It has a decomposition

$$\rho = \bigoplus_{\pi} m(\rho, \pi)\pi, \qquad m(\pi, \rho) = 0, 1, 2, \ldots$$

(within the appropriate Grothendieck group), into irreducible representations of $G^+(F_S)$. Consider this decomposition as a character identity on $G(F_S)$. If π does not belong to $\Pi(G(F_S))$, its character vanishes on $G(F_S)$ and may be ignored. Consequently,

$$\operatorname{tr}(\rho) = \sum_{\pi \in \Pi(G(F_S))} m(\rho, \pi) \operatorname{tr}(\pi). \tag{5.5}$$

We define

$$\Gamma(\rho, \pi) = \sum_{\xi \in \Xi_{G,S}} m(\rho, \pi_\xi)\xi(G). \tag{5.6}$$

The formula (5.3) then follows from (5.5).

The numbers $\{\Delta(\pi, \rho)\}$ are constructed by inverting (5.5). Each representation in $\Sigma(G(F_S))$ or $\Pi(G(F_S))$ has an infinitesimal character

$$\chi: \mathscr{Z}(G) \to \mathbb{C}.$$

(If F is Archimedean, $\mathscr{Z}(G)$ is just the center of the universal enveloping algebra, while for p-adic F we take $\mathscr{Z}(G)$ to be the Bernstein center. See [3, 4].) The constituents of ρ will have the same infinitesimal character, so $\Gamma(\rho, \pi)$ vanishes if π and ρ have different infinitesimal characters.

Moreover, it is known that there are only finitely many representations in $\Pi(G(F_S))$ with a given infinitesimal character. (For real groups this is a basic result of Harish-Chandra. For p-adic groups it follows from [17, Theorem 3.9.1].) Therefore, to invert (5.5) we need consider only the finite set of π and ρ with a given infinitesimal character.

Fix a minimal parabolic subset P_0 of G over F. Then P_0^0 is a minimal parabolic subgroup of G^0 over F. The positive chamber $(\mathfrak{a}_{P_0}^*)^+$ in $\mathfrak{a}_{P_0}^*$ associated to P_0 is contained in the chamber $(\mathfrak{a}_0^*)^+ = (\mathfrak{a}_{P_0^0}^*)^+$ associated to P_0^0. As is usual, we shall write

$$\Lambda' \leqslant \Lambda, \qquad \Lambda', \Lambda \in \mathfrak{a}_0^*,$$

if $\Lambda - \Lambda'$ is a nonnegative, real linear combination of simple roots of $(P_0^0, A_{P_0^0})$. Suppose that ρ is a representation in $\Sigma(G(F_S))$. Then there are unique elements M, $P \in \mathscr{P}(M)$, $\sigma \in \Pi_{\text{temp}}(M(F_S))$ and $\Lambda \in (\mathfrak{a}_P^*)^+$, with $P \supset P_0$, such that $\rho = \mathscr{I}_P(\sigma_\Lambda)$. Set $\Lambda = \Lambda_{\bar{\rho}}$. It is a point in the closure of $(\mathfrak{a}_0^*)^+$, which is uniquely determined by the representation $\bar{\rho} \in \Pi(G(F_S))$. Now, consider the expression (5.5). The representation $\bar{\rho}$ occurs as a constituent of ρ only as the Langlands quotient. Therefore, $m(\rho, \bar{\rho}) = 1$. We claim that if π occurs on the right hand side of (5.5) with positive multiplicity, then $\Lambda_\pi \leqslant \Lambda_{\bar{\rho}}$ with equality holding only when $\pi = \bar{\rho}$. Indeed, if $G = G^0$ the assertion is well known (see [5, IV.4.13 and XI.2.13]). But if G is arbitrary, the restriction of ρ to $G^0(F_S)$ is standard. The claim therefore follows from the connected case. This establishes that the matrix

$$(m(\rho', \bar{\rho})), \qquad \rho, \rho' \in \Sigma(G(F_S)),$$

is unipotent. Its inverse is again a unipotent matrix, so that

$$\operatorname{tr}(\pi) = \sum_{\rho \in \Sigma(G(F_S))} n(\pi, \rho) \operatorname{tr}(\rho), \qquad \pi \in \Pi(G(F_S)),$$

for integers $n(\pi, \rho)$. If we define

$$\Delta(\pi, \rho) = \sum_{\eta \in \Xi_{G,S}} n(\pi, \rho_\eta) \eta(G), \tag{5.7}$$

we obtain the formula (5.4). The required formulas (5.1) and (5.2) follow immediately from the definitions. Our proof is complete. ∎

Following Vogan, we define *block equivalence* to be the equivalence relation on $\Pi(G(F_S))$ generated by

$$\{\pi \sim \bar{\rho} : \Gamma(\rho, \pi) \neq 0\}.$$

Block equivalent representations have the same infinitesimal character. It is also clear that if $\Delta(\pi, \rho) \neq 0$, then $\bar{\rho}$ and π are block equivalent.

We now return to our general Levi subset M. Suppose that $\rho \in \Sigma(M(F_S))$. Define

$$r_{P'|P}(\rho_\lambda) = r_{P'|P}(\bar{\rho}_\lambda)$$

and

$$R_{P'|P}(\rho_\lambda) = r_{P'|P}(\rho_\lambda)^{-1} J_{P'|P}(\rho_\lambda),$$

for $P, P' \in \mathscr{P}(M)$ and $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$. With the exception of (R_4), all the properties of Theorem 2.1 hold for these operators. This follows by analytic continuation from the case that ρ is tempered. If π is any representation in $\Pi(M(F_S))$, we set

$$\tilde{r}_{P'|P}(\pi_\lambda, \rho_\lambda) = r_{P'|P}(\pi_\lambda)^{-1} r_{P'|P}(\rho_\lambda). \tag{5.8}$$

PROPOSITION 5.2. *Fix $\pi \in \Pi(M(F_S))$ and $\rho \in \Sigma(M(F_S))$ with π and $\bar{\rho}$ block equivalent. Then*

$$\tilde{r}_{P''|P}(\pi_\lambda, \rho_\lambda) = \tilde{r}_{P''|P'}(\pi_\lambda, \rho_\lambda)\, \tilde{r}_{P'|P}(\pi_\lambda, \rho_\lambda) \tag{5.9}$$

for any P, P', and P''. Moreover, $\tilde{r}_{P'|P}(\pi_\lambda, \rho_\lambda)$ is a rational function of $\{\lambda(\alpha^\vee): \alpha \in \Delta_P\}$ if F is Archimedean and a rational function of $\{q^{-\lambda(\alpha^\vee)}: \alpha \in \Delta_P\}$ if F is a local field of residual order q.

Proof. Suppose that π and $\bar{\rho}$ are block equivalent to a third representation $\tau = \bar{\sigma}$, with $\sigma \in \Sigma(M(F_S))$. Then

$$\tilde{r}_{P'|P}(\pi_\lambda, \rho_\lambda) = \tilde{r}_{P'|P}(\pi_\lambda, \sigma_\lambda)\, \tilde{r}_{P'|P}(\sigma_\lambda, \rho_\lambda).$$

It is therefore enough to prove the proposition when $\Gamma(\rho, \pi) \neq 0$. By (2.5) the function $\tilde{r}_{P'|P}(\pi_\lambda, \rho_\lambda)$ is left unchanged if π is replaced by π_ξ, with $\xi \in \Xi_{G,S}$. It follows from the definition (5.6) that we may take π to be a constituent of ρ. This makes the induced space $\mathscr{V}_P(\pi)$ into a subquotient of $\mathscr{V}_P(\rho)$. Let $J_{P'|P}(\rho_\lambda)_\pi$ and $R_{P'|P}(\rho_\lambda)_\pi$ be the operators on $\mathscr{V}_P(\pi)$ obtained as subquotients of $J_{P'|P}(\rho_\lambda)$ and $R_{P'|P}(\rho_\lambda)$. The original integral formula for intertwining operators tells us that

$$J_{P'|P}(\rho_\lambda)_\pi = J_{P'|P}(\pi_\lambda).$$

Consequently

$$\begin{aligned} &R_{P'|P}(\pi_\lambda)\, R_{P'|P}(\rho_\lambda)_\pi^{-1} \\ &\quad = r_{P'|P}(\pi_\lambda)^{-1} J_{P'|P}(\pi_\lambda)\, J_{P'|P}(\rho_\lambda)_\pi^{-1} r_{P'|P}(\rho_\lambda) \\ &\quad = \tilde{r}_{P'|P}(\pi_\lambda, \rho_\lambda). \end{aligned}$$

The two assertions of the proposition follow from the properties (R_2) and (R_6), applied to both $R_{P'|P}(\pi_\lambda)$ and $R_{P'|P}(\rho_\lambda)$. ∎

COROLLARY 5.3. *Let π and π' be representations in $\Pi(M(F_S))$ which are block equivalent. Then the μ-functions $\mu_M(\pi_\lambda)$ and $\mu_M(\pi'_\lambda)$ are equal.*

Proof. By definition

$$\mu_M(\pi_\lambda) = (J_{P|\bar{P}}(\pi_\lambda)\, J_{\bar{P}|P}(\pi_\lambda))^{-1},$$

and we obtain

$$\mu_M(\pi_\lambda) = (r_{P|\bar{P}}(\pi_\lambda)\, r_{\bar{P}|P}(\pi_\lambda))^{-1}$$

from (R_2). Since a similar formula holds for π'_λ, the corollary follows from (5.9). ∎

6. THE DISTRIBUTIONS $J_M(\pi_\lambda)$

We come now to our primary objects of study. They are linear functionals on the Hecke space of $G(F_S)$. Recall that the Hecke space, $\mathscr{H}(G(F_S))$, consists of the smooth, compactly supported functions on $G(F_S)$ whose left and right translates by K span a finite dimensional space. The linear functionals, which we shall call distributions on $\mathscr{H}(G(F_S))$ (a harmless abuse of language), are obtained from a certain rational function constructed from the normalized intertwining operators. They were originally introduced in [1(a), Sect. 8], and were later shown to describe the terms in the trace formula arising from Eisenstein series [1(c), 9].

Fix a representation $\pi \in \Pi(M(F_S))$. The distributions are defined in terms of the (G, M) family

$$\mathscr{R}_P(\nu, \pi_\lambda, P_0) = R_{P|P_0}(\pi_\lambda)^{-1}\, R_{P|P_0}(\pi_{\lambda+\nu}), \qquad \nu \in i\mathfrak{a}_M^*,\ P \in \mathscr{P}(M), \tag{6.1}$$

introduced in [1(a), Sect. 7]. (For the definitions and properties of (G, M) families, we refer the reader to [1(a), Sect. 6] and, for the case that $G \neq G^0$, the remarks in [1(e), Sect. 1].) The functions (6.1) are meromorphic in ν and λ, and depend on a fixed $P_0 \in \mathscr{P}(M)$. If we take $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$ to be in general position, the function (6.1) will have no poles for $\nu \in i\mathfrak{a}_M^*$. The distributions are then defined by

$$J_M(\pi_\lambda, f) = \operatorname{tr}(\mathscr{R}_M(\pi_\lambda, P_0)\, \mathscr{I}_{P_0}(\pi_\lambda, f)), \qquad f \in \mathscr{H}(G(F_S)),$$

where

$$\mathscr{R}_M(\pi_\lambda, P_0) = \lim_{\nu \to 0} \sum_{P \in \mathscr{P}(M)} \mathscr{R}_P(\nu, \pi_\lambda, P_0)\, \theta_P(\nu)^{-1},$$

in the notation of [1(a), Sect. 6]. They are independent of P_0.

The distributions $J_M(\pi_\lambda, f)$ are meromorphic in λ. More generally, suppose that $\mathcal{M} = \prod_{v \in S} M_v$ is an admissible Levi subset of M over F_S, and that σ is a representation in

$$\Pi(\mathcal{M}(F_S)) = \left\{ \bigotimes_v \sigma_v : \sigma_v \in \Pi(M_v(F_v)) \right\}.$$

If $\Lambda = \oplus_v \Lambda_v$ is a generic point in $\mathfrak{a}^*_{\mathcal{M},\mathbb{C}}$, the induced representation σ^M_Λ belongs to $\Pi(M(F_S))$. The associated distribution $J_M(\sigma^M_\Lambda, f)$, which we will often denote simply by $J_M(\sigma_\Lambda, f)$, then extends as a meromorphic function of Λ to $\mathfrak{a}^*_{\mathcal{M},\mathbb{C}}$. We can be rather precise about its poles. Let $\Sigma_M(G, A_{\mathcal{M}})$ denote the set of roots of $(G, A_{\mathcal{M}})$ which do not vanish on $\mathfrak{a}_M$. Any $\beta \in \Sigma_M(G, A_{\mathcal{M}})$ belongs to $\Sigma_M(G, A_{M_v})$ for a unique $v \in S$. Set $q_\beta(\Lambda)$ equal to $\Lambda_v(\beta^\vee)$ if F_v is Archimedean, and equal to $q_v^{-\Lambda_v(\beta^\vee)}$ if F_v is non-Archimedean of residual order q_v. The properties (R_2), (R_3), (R_6), and (2.4′) of Section 2 tell us that the matrix coefficients of the operators

$$R_{P|P_0}(\sigma^M_\Lambda), \qquad P, P_0 \in \mathcal{P}(M),$$

are all rational functions of the variables

$$\{q_\beta(\Lambda) : \beta \in \Sigma_M(G, A_{\mathcal{M}})\}, \tag{6.2}$$

whose poles lie along hyperplanes of the form

$$\Lambda(\beta^\vee) = c, \qquad \beta \in \Sigma_M(G, A_{\mathcal{M}}),\ c \in \mathbb{C}. \tag{6.3}$$

The same is therefore true of the matrix coefficients of $\mathcal{R}_M(\sigma^M_\Lambda, P_0)$. It follows that $J_M(\sigma_\Lambda, f)$ is a meromorphic function of Λ whose poles lie along hyperplanes of the form (6.3).

It is important to relate the distributions $J_M(\pi_\lambda)$ to similar objects defined for standard representations. Suppose that $\rho \in \Sigma(M(F_S))$. Then

$$\rho = \sigma^M_{\Lambda_0}, \qquad \sigma \in \Pi_{\text{temp}}(\mathcal{M}(F_S)),\ \Lambda_0 \in \mathfrak{a}^*_{\mathcal{M}},$$

for some $\mathcal{M}$ as above. We define

$$J_M(\rho_\lambda, f) = \lim_{\Lambda \to \Lambda_0 + \lambda} J_M(\sigma_\Lambda, f), \qquad \lambda \in \mathfrak{a}^*_{M,\mathbb{C}},$$

with Λ ranging over points in $\mathfrak{a}^*_{\mathcal{M},\mathbb{C}}$ for which σ^M_Λ belongs to $\Pi(M(F_S))$. Then

$$J_M(\rho_\lambda, f) = \operatorname{tr}(\mathcal{R}_M(\rho_\lambda, P_0)\, \mathcal{I}_{P_0}(\rho_\lambda, f)),$$

where $\mathscr{R}_M(\rho_\lambda, P_0)$ comes from the (G, M) family

$$\mathscr{R}_P(\nu, \rho_\lambda, P_0) = R_{P|P_0}(\rho_\lambda)^{-1} R_{P|P_0}(\rho_{\lambda+\nu}), \qquad \nu \in i\mathfrak{a}_M^*, \ P \in \mathscr{P}(M).$$

It is clear from the remarks above that $J_M(\rho_\lambda, f)$ is a well-defined, meromorphic function of λ. Now, suppose we are also given $\pi \in \Pi(M(F_S))$. The distributions $J_M(\pi_\lambda)$ and $J_M(\rho_\lambda)$ will be related by a (G, M) family

$$r_P(\nu, \pi_\lambda, \rho_\lambda, P_0) = \Delta(\pi, \rho)\, \tilde{r}_{P|P_0}(\pi_\lambda, \rho_\lambda)^{-1}\, \tilde{r}_{P|P_0}(\pi_{\lambda+\nu}, \rho_{\lambda+\nu}),$$
$$P \in \mathscr{P}(M), \ \nu \in i\mathfrak{a}_M^*, \qquad (6.4)$$

of scalar valued functions. These functions all vanish unless π and $\bar{\rho}$ are block equivalent. The compatibility condition required of a (G, M) family follows from (2.3) and (5.9). Suppose that $L \in \mathscr{L}(M)$. There is certainly the (L, M) family

$$r_R^L(\nu, \pi_\lambda, \rho_\lambda, R_0), \qquad R, R_0 \in \mathscr{P}^L(M),$$

obtained by replacing (G, M) by (L, M) in the definition (6.4). On the other hand, for any $Q \in \mathscr{P}(L)$

$$r_R^Q(\nu, \pi_\lambda, \rho_\lambda, P_0) = r_{Q(R)}(\nu, \pi_\lambda, \rho_\lambda, P_0), \qquad R \in \mathscr{P}^L(M),$$

is also an (L, M) family. These two (L, M) families are not the same. However, it follows easily from (5.9) that the associated numbers

$$r_M^L(\pi_\lambda, \rho_\lambda, R_0) = \lim_{\nu \to 0} \sum_R r_R^L(\nu, \pi_\lambda, \rho_\lambda, R_0)\, \theta_R(\nu)^{-1}$$

and

$$r_M^Q(\pi_\lambda, \rho_\lambda, P_0) = \lim_{\nu \to 0} \sum_R r_R^Q(\nu, \pi_\lambda, \rho_\lambda, P_0)\, \theta_R(\nu)^{-1}$$

are equal. We denote their common value by $r_M^L(\pi_\lambda, \rho_\lambda)$. It is independent of R_0, P_0, and Q.

The next proposition is a generalization of (5.4).

PROPOSITION 6.1. *We have*

$$J_M(\pi_\lambda, f) = \sum_{L \in \mathscr{L}(M)} \sum_{\rho \in \{\Sigma(M(F_S))\}} r_M^L(\pi_\lambda, \rho_\lambda)\, J_L(\rho_\lambda^L, f),$$

for any $\pi \in \Pi(M(F_S))$ *and* $f \in \mathscr{H}(G(F_S))$.

Proof. By definition $J_M(\pi_\lambda, f)$ equals

$$\lim_{\nu \to 0} \sum_{P \in \mathscr{P}(M)} \operatorname{tr}(R_{P|P_0}(\pi_\lambda)^{-1} R_{P|P_0}(\pi_{\lambda+\nu})\, \mathscr{I}_{P_0}(\pi_\lambda, f))\, \theta_P(\nu)^{-1}.$$

We write this as

$$\lim_{\nu \to 0} \sum_{P \in \mathscr{P}(M)} (r_{P|P_0}(\pi_\lambda)^{-1} r_{P|P_0}(\pi_{\lambda+\nu}))^{-1} T_P(\nu, \pi_\lambda),$$

where

$$T_P(\nu, \tau_\lambda) = \mathrm{tr}(J_{P|P_0}(\tau_\lambda)^{-1} J_{P|P_0}(\tau_{\lambda+\nu}) \mathscr{I}_{P_0}(\tau_\lambda, f))$$

for any admissible representation τ of $M^+(F_S)$. It is a straightforward consequence of the integral formula (1.1) that as a function of τ, $T_P(\nu, \tau_\lambda)$ depends only on $\mathrm{tr}(\tau)$. In fact

$$\mathrm{tr}(\tau) \to T_P(\nu, \tau_\lambda)$$

extends to a linear functional on the vector space spanned by the functions $\{\mathrm{tr}(\tau)\}$. It follows from Proposition 5.1 that

$$\begin{aligned} T_P(\nu, \pi_\lambda) &= \sum_{\rho \in \{\Sigma(M(F_S))\}} \Delta(\pi, \rho)\, T_P(\nu, \rho_\lambda) \\ &= \sum_\rho \Delta(\pi, \rho)\, r_{P|P_0}(\rho_\lambda)^{-1} r_{P|P_0}(\rho_{\lambda+\nu})\, \mathrm{tr}(R_{P|P_0}(\rho_\lambda)^{-1} \\ &\quad \times R_{P|P_0}(\rho_{\lambda+\nu}) \mathscr{I}_{P_0}(\rho_\lambda, f)). \end{aligned}$$

Therefore $J_M(\pi_\lambda, f)$ equals the sum over $\rho \bullet \{\Sigma(M(F_S))\}$ of

$$\lim_{\nu \to 0} \sum_{P \in \mathscr{P}(M)} r_P(\nu, \pi_\lambda, \rho_\lambda, P_0)\, \mathrm{tr}(\mathscr{R}_P(\nu, \rho_\lambda, P_0) \mathscr{I}_{P_0}(\rho_\lambda, f))\, \theta_P(\nu)^{-1}.$$

This last expression is built out of a product of two (G, M) families. Applying [1(a), Corollary 6.5], we obtain

$$J_M(\pi_\lambda, f) = \sum_{L \in \mathscr{L}(M)} \sum_{\rho \in \{\Sigma(M(F_S))\}} r_M^L(\pi_\lambda, \rho_\lambda)\, \mathrm{tr}(\mathscr{R}_L(\rho_\lambda, P_0) \mathscr{I}_{P_0}(\rho_\lambda, f)).$$

A simple argument, similar to the derivation of [1(a), (7.8)], establishes that

$$\mathrm{tr}(\mathscr{R}_L(\rho_\lambda, P_0) \mathscr{I}_{P_0}(\rho_\lambda, f)) = J_L(\rho_\lambda^L, f).$$

The lemma follows. ∎

In the paper [1(a)], we actually defined the distributions for Schwartz functions and tempered representations. We then proved a formula

$$J_M(\pi_\lambda, f^y) = \sum_{Q \in \mathscr{F}(M)} J_M^{M_Q}(\pi_\lambda, f_{Q,y}) \tag{6.5}$$

for their behaviour under conjugation [1(a), Lemma (8.3)]. Here $J_M^{M_Q}$ stands for the distribution on $M_Q(F_S)$, while

$$f^y(x) = f(yxy^{-1}),$$

and

$$f_{Q,y}(m) = \delta_Q(m)^{1/2} \int_K \int_{N_Q(F_S)} f(k^{-1}mnk)\, v'_Q(ky)\, dn\, dk, \qquad m \in M_Q(F_S),$$

in the notation of [1(e), (2.3)]. This formula must be modified for the present situation, since conjugation at the Archimedean place does not preserve the Hecke space. Take $f \in \mathscr{H}(G(F_S))$. If $y \in G^0(F_S)$, the functions

$$(L_y f)(x) = f(y^{-1}x),$$

and

$$(R_y f)(x) = f(xy^{-1})$$

do not in general belong to $\mathscr{H}(G(F_S))$. However, if h is in the Hecke algebra of the group

$$G^0(F_S)^1 = \{ y \in G^0(F_S) \colon H_G(y) = 0 \},$$

the functions

$$L_h f = \int_{G^0(F_S)^1} h(y)(L_y f)\, dy = h * f$$

and

$$R_h f = \int_{G^0(F_S)^1} h(y)(R_y f)\, dy = f * h$$

do belong to $\mathscr{H}(G(F_S))$. More generally, for any $Q \in \mathscr{F}(M)$ the functions

$$L_{Q,h} f = \int_{G^0(F_S)^1} h(y)(L_y f)_{Q,y^{-1}}\, dy$$

and

$$R_{Q,h} f = \int_{G^0(F_S)^1} h(y)(R_y f)_{Q,y^{-1}}\, dy$$

belong to $\mathscr{H}(M_Q(F_S))$. Observe that $L_{G,h} f = L_h f$ and $R_{G,h} f = R_h f$.

LEMMA 6.2. *Fix $f \in \mathscr{H}(G(F_S))$ and $h \in \mathscr{H}(G^0(F_S)^1)$. Then*

$$J_M(\pi_\lambda, L_h f) = \sum_{Q \in \mathscr{F}(M)} J_M^{M_Q}(\pi_\lambda, R_{Q,h} f)$$

and

$$J_M(\pi_\lambda, R_h f) = \sum_{Q \in \mathscr{F}(M)} J_M^{M_Q}(\pi_\lambda, L_{Q,h} f)$$

for any $\pi \in \Pi(M(F_S))$.

Proof. Suppose first that $\pi \in \Pi_{\text{temp}}(M(F_S))$. Then the formula (6.5) may be applied. (Actually, (6.5) was proved in [1(a)] only for $G = G^0$, but the argument is identical for general G.) Since

$$L_y f = (R_y f)^{y^{-1}},$$

we have

$$J_M(\pi_\lambda, L_y f) = \sum_{Q \in \mathscr{F}(M)} J_M^{M_Q}(\pi_\lambda, (R_y f)_{Q, y^{-1}}).$$

Multiply both sides by $h(y)$ and integrate over $y \in G^0(F_S)^1$. We obtain

$$J_M(\pi_\lambda, L_h f) = \sum_{Q \in \mathscr{F}(M)} J_M^{M_Q}(\pi_\lambda, R_{Q,h} f),$$

which is the first of the required formulas. Observe that if

$$\pi = \sigma_\Lambda^M, \qquad \sigma \in \Pi_{\text{temp}}(\mathscr{M}(F_S)), \quad \Lambda \in i\mathfrak{a}_{\mathscr{M}}^*,$$

for some $\mathscr{M}$, then each side of the formula can be analytically continued to any $\Lambda \in \mathfrak{a}_{\mathscr{M},\mathbb{C}}^*$. The formula therefore holds if π is replaced by any standard representation in $\Sigma(M(F_S))$.

Now, suppose that $\pi \in \Pi(M(F_S))$ is arbitrary. Combining Lemma 6.1 with what we have just proved, we see that

$$\begin{aligned} J_M(\pi_\lambda, L_h f) &= \sum_{L \in \mathscr{L}(M)} \sum_{\rho \in \{\Sigma(M(F_S))\}} r_M^L(\pi_\lambda, \rho_\lambda) \sum_{Q \in \mathscr{F}(L)} J_L^{M_Q}(\rho_\lambda^L, R_{Q,h} f) \\ &= \sum_{Q \in \mathscr{F}(M)} \sum_{L \in \mathscr{L}^{M_Q}(M)} \sum_{\rho \in \{\Sigma(M(F_S))\}} r_M^L(\pi_\lambda, \rho_\lambda) J_L^{M_Q}(\rho_\lambda^L, R_{Q,h} f) \\ &= \sum_{Q \in \mathscr{F}(M)} J_M^{M_Q}(\pi_\lambda, R_{Q,h} f). \end{aligned}$$

Thus, the first of the required formulas holds in general. The second required formula is established the same way. ∎

It is natural to call a distribution I on $\mathscr{H}(G(F_S))$ *invariant* if

$$I(L_h f - R_h f) = 0$$

for each $f \in \mathscr{H}(G(F_S))$ and $h \in \mathscr{H}(G^0(F_S)^1)$. The last lemma asserts that

$$J_M(\pi_\lambda, L_h f - R_h f) = \sum_{Q \neq G} J_M^{M_Q}(\pi_\lambda, R_{Q,h} f) = -\sum_{Q \neq G} J_M^{M_Q}(\pi_\lambda, L_{Q,h} f),$$

and so gives the obstruction to $J_M(\pi_\lambda)$ being invariant.

7. The Distributions $J_M(\pi, X)$

It is not actually the distributions $J_M(\pi_\lambda)$ which occur in the trace formula, but rather their integrals over π. Recall that the set

$$\mathfrak{a}_{M,S} = \{H_M(m) \colon m \in M(F_S)\}$$

is a subgroup of $\mathfrak{a}_M$. It equals $\mathfrak{a}_M$ if S contains an Archimedean place, and is a lattice in $\mathfrak{a}_M$ otherwise. The additive character group

$$\mathfrak{a}^*_{M,S} = \mathfrak{a}^*_M / \mathfrak{a}^\vee_{M,S}$$

equals $\mathfrak{a}^*_M$ in the first instance, and is a compact quotient of $\mathfrak{a}^*_M$ in the second. Now, suppose that $\pi \in \Pi(M(F_S))$ is such that $J_M(\pi_\lambda, f)$ is regular for $\lambda \in i\mathfrak{a}^*_M$. This holds, for example, if π is unitary (by property (R_4) of Theorem 2.1). Then if $X \in \mathfrak{a}_{M,S}$, we define

$$J_M(\pi, X, f) = \int_{i\mathfrak{a}^*_{M,S}} J_M(\pi_\lambda, f)\, e^{-\lambda(X)}\, d\lambda, \qquad f \in \mathscr{H}(G(F_S)).$$

For a general representation $\pi \in \Pi(M(F_S))$, we define

$$J_M(\pi, X, f) = \sum_{P \in \mathscr{P}(M)} \omega_P J_M(\pi_{\varepsilon_P}, X, f)\, e^{-\varepsilon_P(X)},$$

where each ε_P is a small point in the chamber $(\mathfrak{a}^*_P)^+$, and

$$\omega_P = \mathrm{vol}(\mathfrak{a}^+_P \cap B)\, \mathrm{vol}(B)^{-1},$$

with B a ball in $\mathfrak{a}_M$ centered at the origin. By changing the contour of integration, we see that these two definitions are compatible.

The distributions $J_M(\pi, X)$ have some simple transformation properties. If $\zeta = (\xi, \lambda)$ is any element in

$$\Xi_{M,S} \times (\mathfrak{a}^*_G + i\mathfrak{a}^*_{M,S}),$$

it follows from (2.5′) that

$$J_M(\pi_\zeta, X, f) = J_M(\pi, X, f)\, \xi(M)\, e^{\lambda(X)}, \tag{7.1}$$

where the component M is understood to be diagonally embedded in $\Xi_{M,S}$. It is clear that Lemma 6.2 can be used to describe the behaviour under convolution. We have only to multiply the two formulas of the lemma by $e^{-\lambda(X)}$, and then integrate over λ. The first formula, for example, becomes

$$J_M(\pi, X, L_h f) = \sum_{Q \in \mathscr{F}(M)} J_M^{M_Q}(\pi, X, R_{Q,h} f), \qquad h \in \mathscr{H}(G^0(F_S)^1). \tag{7.2}$$

We shall sometimes need to define $J_M(\pi, X, f)$ when f is not quite in the Hecke space. Suppose that Z is a point in $\mathfrak{a}_{G,S}$. Let f^Z denote the restriction of a given function $f \in \mathscr{H}(G(F_S))$ to

$$G(F_S)^Z = \{x \in G(F_S) \colon H_G(x) = Z\}.$$

The Haar measures on $G(F_S)$ and $\mathfrak{a}_{G,S}$ determine measures on the spaces $G(F_S)^Z$. For any π,

$$\mathscr{I}_{P_0}(\pi, f^Z) = \int_{G(F_S)^Z} f^Z(x)\, \mathscr{I}_{P_0}(\pi, x)\, dx, \qquad P_0 \in \mathscr{P}(M),$$

is an operator on $\mathscr{V}_{P_0}(\pi)$. Define

$$J_M(\pi_\lambda, f^Z) = \operatorname{tr}(\mathscr{R}_M(\pi_\lambda, P_0)\, \mathscr{I}_{P_0}(\pi_\lambda, f^Z)).$$

It is clear that

$$J_M(\pi_{\lambda+\lambda_G}, f^Z) = J_M(\pi_\lambda, f^Z)\, e^{\lambda_G(Z)},$$

for any point λ_G in $\mathfrak{a}^*_{G,\mathbb{C}}$. Now take

$$Z = h_G(X), \qquad X \in \mathfrak{a}_{M,S},$$

where h_G denotes the projection onto $\mathfrak{a}_G$. By the Fourier inversion formula on $\mathfrak{a}_{G,S}$, we have

$$J_M(\pi, X, f) = \sum_{P \in \mathscr{P}(M)} \omega_P \int_{\varepsilon_P + i\mathfrak{a}^*_{M,S}/i\mathfrak{a}^*_{G,S}} J_M(\pi_\lambda, f^Z)\, e^{-\lambda(X)}\, d\lambda, \tag{7.3}$$

for $\{\varepsilon_P\}$ as above. In particular, $J_M(\pi, X, f)$ depends only on f^Z. It can be defined for any function f which has the same restriction to $G(F_S)^Z$ as some function in $\mathscr{H}(G(F_S))$.

As with $J_M(\pi_\lambda)$, the distribution $J_M(\pi, X)$ has an expansion in terms of standard representations. If ρ belongs to $\Sigma(M(F_S))$, we can define $J_M(\rho, X)$ in terms of $J_M(\rho_\lambda)$ by mimicing the discussion above. This new distribution then satisfies the obvious analogues of (7.1)–(7.3).

PROPOSITION 7.1. *Fix* $\pi \in \Pi((M(F_S))$, $X \in \mathfrak{a}_{M,S}$, *and* $f \in \mathscr{H}(G(F_S)))$. *Then*

$$J_M(\pi, X, f) = \sum_{P \in \mathscr{P}(M)} \omega_P \sum_{L \in \mathscr{L}(M)} r^L_{M,\varepsilon_P}(\pi, X, J_L(f)),$$

where for any point $\mu \in \mathfrak{a}^*_M$ *in general position,*

$$r^L_{M,\mu}(\pi, X, J_L(f))$$

equals

$$\int_{\mu + i\mathfrak{a}^*_{M,S}/i\mathfrak{a}^*_{L,S}} \sum_{\rho \in \{\Sigma(M(F_S))\}} r^L_M(\pi_\lambda, \rho_\lambda)\, J_L(\rho^L_\lambda, h_L(X), f)\, e^{-\lambda(X)}\, d\lambda.$$

Proof. Observe that

$$J_M(\pi, X, f)$$

$$= \sum_{P \in \mathscr{P}(M)} \omega_P \int_{\varepsilon_P + i\mathfrak{a}^*_{M,S}/i\mathfrak{a}^*_{G,S}} J_M(\pi_\lambda, f^{h_G(X)})\, e^{-\lambda(X)}\, d\lambda$$

$$= \sum_P \omega_P \int \sum_{L \in \mathscr{L}(M)} \sum_{\rho \in \{\Sigma(M(F_S))\}} r^L_M(\pi_\lambda, \rho_\lambda)\, J_L(\rho^L_\lambda, f^{h_G(X)})\, e^{-\lambda(X)}\, d\lambda,$$

by (7.3) and Lemma 6.1. We are assuming that ε_P is in general position. Consequently, the function

$$r^L_M(\pi_\lambda, \rho_\lambda)\, J_L(\rho^L_\lambda, f^{h_G(X)})\, e^{-\lambda(X)} \tag{7.4}$$

has no singularities which meet $\varepsilon_P + \mathfrak{a}^*_{M,S}/i\mathfrak{a}^*_{G,S}$. By a standard estimate, it is integrable over this space. We may therefore take the integral above inside the sum over L and ρ. We then decompose the resulting integral into a double integral over

$$(\varepsilon_P + i\mathfrak{a}^*_{M,S}/i\mathfrak{a}^*_{L,S}) \times (i\mathfrak{a}^*_{L,S}/i\mathfrak{a}^*_{G,S}).$$

It becomes

$$\int_{\varepsilon_P + i\mathfrak{a}^*_{M,S}/i\mathfrak{a}^*_{L,S}} r^L_M(\pi_\lambda, \rho_\lambda)\, J_L(\rho^L_\lambda, h_L(X), f)\, e^{-\lambda(X)}\, d\lambda.$$

Proposition 7.1 follows. ∎

We should keep in mind that $J_M(\pi, X, f)$ is a function not only of f but also of (π, X). Interpreted one way it is a family of distributions, and the other way it is a transform. As in [1(a)], we shall use a completely different notation when we want to emphasize this second point of view. We write ϕ_M for the map which transforms $f \in \mathscr{H}(G(F_S))$ to the function

$$\phi_M(f): \pi \to \phi_M(f, \pi) = J_M(\pi, f), \qquad \pi \in \Pi_{\text{temp}}(M(F_S)),$$

on $\Pi_{\text{temp}}(M(F_S))$. This is a linear combination of matrix entries of $\mathscr{J}_{P_0}(\pi, f)$, which for

$$\pi = \sigma_\Lambda^M, \qquad \sigma \in \Pi_{\text{temp}}(\mathscr{M}(F_S)), \ \Lambda \in i\mathfrak{a}_{\mathscr{M}}^*,$$

as in Section 6, has coefficients which are rational functions of the variables (6.2). In particular, it cannot be extended to all nontempered π. We would like to show, however, that the map

$$(\pi, X) \to \phi_M(f, \pi, X) = \int_{i\mathfrak{a}_{M,S}^*} \phi_M(f, \pi_\lambda)\, e^{-\lambda(X)}\, d\lambda = J_M(\pi, X, f)$$

can be so extended. We would also like to compare its values at arbitrary (π, X) with $J_M(\pi, X, f)$. Both of these questions are related to the residues of the function

$$\phi_M(f, \sigma_\Lambda^M) = J_M(\sigma_\Lambda^M, f).$$

We shall devote the next two sections to a study of these residues.

8. Residues

Suppose that $\mathscr{M} = \prod_{v \in S} M_v$ is an admissible Levi subset of M over F_S. For each v, write $S_v = \{v\}$ and define $\mathfrak{a}_{M_v, S_v}$ to be the image of the map

$$H_{M_v}: G(F_v) \to \mathfrak{a}_{M_v}.$$

Set

$$\mathfrak{a}_{\mathscr{M},S} = \bigoplus_{v \in S} \mathfrak{a}_{M_v, S_v},$$

$$\mathfrak{a}_{\mathscr{M},S}^\vee = \operatorname{Hom}(\mathfrak{a}_{\mathscr{M},S}, 2\pi\mathbb{Z}) = \bigoplus_{v \in S} \operatorname{Hom}(\mathfrak{a}_{M_v, S_v}, 2\pi\mathbb{Z}),$$

and

$$\mathfrak{a}_{\mathscr{M},S}^* = \mathfrak{a}_{\mathscr{M}}^* / \mathfrak{a}_{\mathscr{M},S}^\vee.$$

For the next two sections we shall keep $\mathscr{M}$ fixed. We shall also fix a representation $\sigma \in \Pi(\mathcal{M}(F_S))$ and a function a_Λ which is defined and analytic on a neighbourhood of some point Λ_0 in

$$\mathfrak{a}^*_{\mathcal{M},\mathbb{C}}/i\mathfrak{a}^\vee_{\mathcal{M},S} = \mathfrak{a}^*_{\mathcal{M}} + i\mathfrak{a}^*_{\mathcal{M},S}.$$

We propose to investigate the residues of the functions

$$a_\Lambda J_M(\sigma_\Lambda, f), \qquad f \in \mathcal{H}(G(F_S)). \tag{8.1}$$

It is clear from the discussion of Section 6 that the singularities of each of these functions lie along a set of hyperplanes of the form (6.3) which is finite modulo $i\mathfrak{a}^\vee_{\mathcal{M},S}$.

Consider a sequence

$$\mathcal{M} = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_r = \mathcal{L} \tag{8.2}$$

of embedded Levi subsets of G over F_S. We assume that for each i, $1 \leqslant i \leqslant r$,

$$\mathfrak{a}_{\mathcal{M}_i} = \{H \in \mathfrak{a}_{\mathcal{M}_{i-1}} : \beta_i(H) = 0\},$$

for some root β_i of $(G, A_{\mathcal{M}_{i-1}})$. The roots $\{\beta_i\}$ are uniquely determined up to scalar multiples. For each i let E_i be a fixed nonzero real multiple of β_i. Then the set

$$\mathcal{E}_\Omega = (E_1, \dots, E_r)$$

determines the sequence (8.2). In addition, fix a linear functional $\Lambda_\Omega \in \mathfrak{a}^*_{\mathcal{M},\mathbb{C}}$ which vanishes on $\mathfrak{a}_{\mathcal{L}}$. We shall call the pair

$$\Omega = (\mathcal{E}_\Omega, \Lambda_\Omega)$$

a *residue datum* for $\mathcal{L}$.

Take Λ_0 to be a fixed point in general position in the affine subspace $\Lambda_\Omega + \mathfrak{a}^*_{\mathcal{L},\mathbb{C}}$ of $\mathfrak{a}^*_{\mathcal{M},\mathbb{C}}$, and set

$$\Lambda(z) = \Lambda_0 + z_1 E_1 + \cdots + z_r E_r$$

for

$$z = (z_1, \dots, z_r)$$

in $\mathbb{C}^r$. Let $\Gamma_1, \dots, \Gamma_r$ be small positively oriented circles about the origin in the complex plane such that for each i, the radius of Γ_i is much smaller than that of Γ_{i+1}. Consider a meromorphic function $\psi(\Lambda)$ on a neighbourhood of Λ_0 in $\mathfrak{a}^*_{\mathscr{M},\mathbb{C}}$ whose singularities lie along hyperplanes of the form (6.3). Then

$$(2\pi i)^{-r}\int_{\Gamma_r}\cdots\int_{\Gamma_1}\psi(\Lambda(z))\,dz_1\cdots dz_r$$

is a meromorphic function of Λ_0. We denote it by

$$\operatorname*{Res}_{\Omega,\Lambda\to\Lambda_0}\psi(\Lambda)=\operatorname*{Res}_{\Omega}\psi(\Lambda_0).$$

We shall study it with $\psi(\Lambda)$ equal to the function (8.1) above.

Define a sequence

$$M=M_0\subset M_1\subset\cdots\subset M_r=L$$

of elements in $\mathscr{L}(M)$ inductively by

$$\mathfrak{a}_{M_i}=\{H\in\mathfrak{a}_{M_{i-1}}: E_i(H)=0\},\qquad 1\leqslant i\leqslant r.$$

LEMMA 8.1. *Suppose that $L=G$. Then*

$$\operatorname*{Res}_{\Omega,\Lambda\to\Lambda_0}(a_\Lambda J_M(\sigma_\Lambda,f)),\qquad f\in\mathscr{H}(G(F_S)),$$

is an invariant distribution.

Proof. We shall use the formula

$$J_M(\sigma_\Lambda, L_h f-R_h f)=\sum_{\{Q\in\mathscr{F}(M):\,Q\neq G\}}J_M^{M_Q}(\sigma_\Lambda, R_{Q,h}f),\qquad h\in\mathscr{H}(G^0(F_S)^1),$$

of Lemma 6.2. Fix $Q\in\mathscr{F}(M)$ with $Q\neq G$. Let i be the smallest integer such that M_i is not contained in M_Q. The partial residue

$$\int_{\Gamma_{i-1}}\cdots\int_{\Gamma_1}a_{\Lambda(z)}J_M^{M_Q}(\sigma_{\Lambda(z)}, R_{Q,h}f)\,dz_1\cdots dz_{i-1}$$

is a meromorphic function of $(z_i,\ldots,z_r)$. Its poles lie along affine hyperplanes obtained from roots of $(G, A_{\mathscr{M}_{i-1}})$ which vanish on $\mathfrak{a}_{M_Q}$. The hyperplane $z_i=0$ is defined by any root which is a multiple of E_i. Our choice of i means that E_i does not vanish on $\mathfrak{a}_{M_Q}$. Consequently $z_i=0$ is not a singular hyperplane of the function. It follows that

$$\operatorname*{Res}_{\Omega,\Lambda\to\Lambda_0}(a_\Lambda J_M^{M_Q}(\sigma_\Lambda, R_{Q,h}f))=0.$$

Therefore

$$\operatorname*{Res}_{\Omega, \Lambda \to \Lambda_0} (a_\Lambda J_M(\sigma_\Lambda, L_h f - R_h f)) = 0,$$

and the lemma follows. ∎

We return to the case that L is arbitrary. Our goal is to provide a simple formula for

$$\operatorname*{Res}_{\Omega, \Lambda \to \Lambda_0} (a_\Lambda J_M(\sigma_\Lambda, f)).$$

We shall postpone this until the next section. In the meantime, we shall make some comments of a general nature.

The distribution

$$\gamma(g) = \operatorname*{Res}_{\Omega, \Lambda \to \Lambda_0} (a_\Lambda J_M^L(\sigma_\Lambda, g)), \qquad g \in \mathscr{H}(L(F_S)),$$

can be extended to functions in $\mathscr{H}(L^+(F_S))$, the Hecke algebra of $L^+(F_S)$. A simple extension of Lemma 8.1 (and also Lemma 6.2) affirms that it is invariant, in the sense that

$$\gamma(g_1 * g_2) = \gamma(g_2 * g_1).$$

Let us fix a parabolic subset R in $\mathscr{P}^L(M)$. Then for each $g \in \mathscr{H}(L^+(F_S))$, the number $\gamma(g)$ can be obtained from the Taylor series of $\mathscr{I}_R(\sigma^M_{\Lambda(z)}, g)$ about $z = 0$. In fact there is a positive integer N, independent of g and also of the function a_Λ above, such that $\gamma(g)$ depends only on the Taylor coefficients of total degree no greater than N. We shall let τ denote the representation of $\mathscr{H}(L^+(F_S))$ obtained by taking the Taylor series of $\mathscr{I}_R(\sigma^M_{\Lambda(z)}, g)$ modulo terms of degree greater than N. It acts on the space of power series in z, taken modulo terms of degree greater than N, with values in $\mathscr{V}_R(\sigma^M_{\Lambda_0})$. We can of course also regard $\mathscr{I}_R(\sigma^M_{\Lambda(z)})$ as a representation of the group $L^+(F_S)$, so that τ is the representation of $\mathscr{H}(L^+(F_S))$ associated to an admissible representation (τ, V_τ) of $L^+(F_S)$. By construction,

$$\tau(g) \to \langle \gamma, \tau(g) \rangle \overset{\text{defn}}{=} \gamma(g)$$

is a well-defined invariant form on the algebra

$$\mathscr{A}_\tau = \{\tau(g) \colon g \in \mathscr{H}(L^+(F_S))\}$$

of operators on V_τ.

Let Q be an element in $\mathscr{P}(L)$, and form the induced representation

$$\mathscr{I}_Q(\tau_\lambda), \qquad \lambda \in \mathfrak{a}^*_{L,\mathbb{C}}.$$

Let us say that an operator $A(\tau)$ on $\mathscr{V}_Q(\tau)$ is *admissible* if it is represented by a K-finite kernel $A(\tau; k_1, k_2)$ with values in the algebra $\mathscr{A}_\tau$ of operators on V_τ. Define a linear form

$$T_\gamma(A(\tau)) = \int_K \langle \gamma, A(\tau; k, k) \rangle \, dk$$

on the space of admissible operators. Since γ is invariant, we have

$$T_\gamma(A(\tau) B(\tau)) = T_\gamma(B(\tau) A(\tau)) \tag{8.3}$$

for every pair $A(\tau)$ and $B(\tau)$ of admissible operators. For any function $f \in \mathscr{H}(G(F_S))$ the operator $\mathscr{I}_Q(\tau, f)$ is admissible. Its kernel is

$$\int_{M_Q(F_S)} \int_{N_Q(F_S)} f(k_1^{-1} m n k_2) \, \tau(m) \, e^{(\lambda + \rho_Q)(H_Q(m))} \, dn \, dm.$$

For obvious reasons we can refer to

$$f \to T_\gamma(\mathscr{I}_Q(\tau, f))$$

as the distribution on $G(F_S)$ *induced from* γ.

The linear form $T_\gamma(A(\tau))$ is of course closely related to our study of residues. Set $P = Q(R)$ and write $\pi = \sigma^M_{\Lambda_0}$. By induction in stages we can identify $\mathscr{V}_P(\pi)$, the Hilbert space on which $\mathscr{I}_P(\sigma^M_\Lambda)$ acts, with $\mathscr{V}_Q(\mathscr{V}_R(\pi))$. Then the operator

$$\mathscr{R}^Q_M(\sigma^M_\Lambda, P) = \lim_{\nu \to 0} \sum_{\{P_1 \in \mathscr{P}(M):\, P_1 \subset Q\}} R_{P_1|P}(\sigma^M_\Lambda)^{-1} \, R_{P_1|P}(\sigma^M_{\Lambda+\nu}) \, \theta_{P_1 \cap L}(\nu)^{-1}$$

acts on $\mathscr{V}_P(\pi)$ through the fibre. It transforms the values of a given function by the operator $\mathscr{R}^L_M(\sigma^M_\Lambda, R)$ on $\mathscr{V}_R(\pi)$. Now, suppose that $A(\sigma^M_\Lambda)$ is a holomorphic function with values in the space of operators on $\mathscr{V}_P(\pi)$. By taking the Taylor series of $A(\sigma^M_{\Lambda(z)})$, modulo terms of degree greater than N, we obtain an operator $A(\tau)$ on $\mathscr{V}_P(\tau)$. It is clear that $A(\sigma^M_\Lambda) \to A(\tau)$ is an algebra homomorphism, and that each $\mathscr{I}_P(\sigma^M_\Lambda, f)$ maps to $\mathscr{I}_Q(\tau, f)$. We shall call $A(\sigma^M_\Lambda)$ *admissible* if the corresponding operator $A(\tau)$ is admissible. In this case we have

$$\operatorname*{Res}_{\Omega, \Lambda \to \Lambda_0} a_\Lambda \operatorname{tr}(A(\sigma^M_\Lambda) \, \mathscr{R}^Q_M(\sigma^M_\Lambda, P)) = T_\gamma(A(\tau)). \tag{8.4}$$

This last formula provides the connection with residues.

Let Q' be another element in $\mathscr{P}(L)$, and set $P' = Q'(R)$. It is clear that the definition of admissible operator can be extended to linear transformations from $\mathscr{V}_Q(\tau)$ to $\mathscr{V}_{Q'}(\tau)$. Formula (8.3) and the correspondence

$A(\sigma_\Lambda^M) \to A(\tau)$ also have obvious extensions. It is easily deduced from (1.1) that the intertwining operator $J_{P'|P}(\sigma_{\Lambda+\lambda}^M)$ maps to $J_{Q'|Q}(\tau_\lambda)$. In particular, $J_{Q'|Q}(\tau_\lambda)$ can be analytically continued to a meromorphic function of λ on $\mathfrak{a}_{L,\mathbb{C}}^*$. Now, $J_{Q'|Q}(\tau_\lambda)$ is not admissible as it stands. However, let Γ be a finite subset of $\Pi(K)$. We shall show that the restriction of $J_{Q'|Q}(\tau_\lambda)$ to $\mathscr{V}_Q(\tau)_\Gamma$ is an admissible operator. (As before, $(\ )_\Gamma$ denotes the subspace that transforms under K according to representations in Γ.)

Let Γ_L be the set of irreducible representations of K_L which occur as constituents of restrictions to K_L of representations in Γ. Define

$$\theta_{\Gamma_L}(k_1) = \sum_{\mu \in \Gamma_L} \operatorname{tr} \mu(k_1), \qquad k_1 \in K_L,$$

and set

$$E_\Gamma v = \int_{K_L} \theta_{\Gamma_L}(k_1^{-1})\, \tau(k_1)\, v\, dk_1,$$

for any $v \in V_\tau$. Then E_τ is the projection of V_τ onto the finite-dimensional subspace $(V_\tau)_{\Gamma_L}$. If ϕ is any vector in $\mathscr{V}_Q(\tau)_\Gamma$, the value of $J_{Q'|Q}(\tau_\lambda)\,\phi$ at $k \in K$ equals

$$\int_{N_{Q'}(F_S) \cap N_Q(F_S) \backslash N_{Q'}(F_S)} E_\Gamma \tau(M_Q(n))\, E_\Gamma \phi(K_Q(n)k)\, e^{(\lambda + \rho_Q)(H_Q(n))}\, dn. \tag{8.5}$$

This follows from (1.1) and the fact that $J_{Q'|Q}(\tau_\lambda)$ maps $\mathscr{V}_Q(\tau)_\Gamma$ to $\mathscr{V}_{Q'}(\tau)_\Gamma$. We claim that for each $m \in L^+(F_S)$, the operator

$$E_\Gamma \tau(m)\, E_\Gamma$$

belongs to the algebra $\mathscr{A}_\tau$. To see this, choose a sequence $\{g_i\}$ of functions in $C_c^\infty(L^+(F_S))$ which approach the Dirac measure at m. Then the matrix coefficients of the operators $\tau(g_i)$ approach those of $\tau(m)$. But the functions

$$g_{i,\Gamma}(m') = \int_{K_L} \int_{K_L} \theta_{\Gamma_L}(k_1)\, g_i(k_1 m' k_2)\, \theta_{\Gamma_L}(k_2)\, dk_1\, dk_2, \qquad m' \in L^+(F_S),$$

all belong to $\mathscr{H}(L^+(F_S))$, and

$$\tau(g_{i,\Gamma}) = E_\Gamma \tau(g_i)\, E_\Gamma.$$

In particular, $\tau(g_{i,\Gamma})$ converges to $E_\Gamma \tau(m)\, E_\Gamma$. This shows that $E_\Gamma \tau(m)\, E_\Gamma$ belongs to the closure of the subspace

$$\{E_\Gamma \tau(g)\, E_\Gamma \colon g \in \mathscr{H}(L^+(F_S))\}$$

of $\mathscr{A}_\tau$. Since the subspace is actually finite dimensional, the claim follows. Now, left translation on any space of K-finite functions on K is an integral operator with K-finite kernel. It follows from (8.5) that the restriction of $J_{Q'|Q}(\tau_\lambda)$ to $\mathscr{V}_Q(\tau)_\Gamma$ is an admissible operator.

The following lemma is a consequence of this discussion. We shall use it in the next section.

LEMMA 8.2. *Suppose that we are given a finite sum*

$$A(\sigma_\Lambda^M) = \sum_{i=1}^n a_{i,\Lambda} A_i(\sigma_\Lambda^M),$$

where for each i, $a_{i,\Lambda}$ is a holomorphic function on a neighbourhood of Λ_0 and

$$A_i(\sigma_\Lambda^M) \colon \mathscr{V}_P(\pi) \to \mathscr{V}_{P'}(\pi)$$

is admissible. Then

$$\operatorname*{Res}_{\Omega, \Lambda \to \Lambda_0} \operatorname{tr}(R_{P'|P}(\sigma_\Lambda^M)^{-1} A(\sigma_\Lambda^M)\, \mathscr{R}_M^Q(\sigma_\Lambda^M, P))$$

equals

$$\operatorname*{Res}_{\Omega, \Lambda \to \Lambda_0} \operatorname{tr}(A(\sigma_\Lambda^M)\, R_{P'|P}(\sigma_\Lambda^M)^{-1}\, \mathscr{R}_M^{Q'}(\sigma_\Lambda^M, P')).$$

Proof. Since both expressions are linear in $A(\sigma_\Lambda^M)$, we can assume that $n = 1$. Write

$$R_{P'|P}(\sigma_\Lambda^M)^{-1} = r_{P'|P}(\sigma_\Lambda^M)\, J_{P'|P}(\sigma_\Lambda^M)^{-1}.$$

Each of these three functions is holomorphic in a neighbourhood of Λ_0. This follows from the general position of Λ_0 and the fact that $P \cap L = P' \cap L = R$. Define γ as above, with

$$a_\Lambda = r_{P'|P}(\sigma_\Lambda^M)\, a_{1,\Lambda}.$$

Then by (8.4), our two expressions equal

$$T_\gamma(J_{Q'|Q}(\tau)^{-1} A_1(\tau))$$

and

$$T_\gamma(A_1(\tau)\, J_{Q'|Q}(\tau)^{-1}),$$

respectively. We can certainly replace $J_{Q'|Q}(\tau)^{-1}$ by its restriction to a subspace $\mathscr{V}_{Q'}(\tau)_\Gamma$. The operator is then admissible, so the lemma follows from an obvious variant of (8.3). ∎

9. Proof of Proposition 9.1

We shall now establish the formula for

$$\operatorname*{Res}_{\Omega, \Lambda \to \Lambda_0} (a_\Lambda J_M(\sigma_\Lambda, f)).$$

It will be given in terms of a certain operator $\Gamma_\Omega(\sigma_\Lambda^M, P_0)$, which we must first describe.

We continue with the notation of the last section. The embedded subspaces

$$\mathfrak{a}_L = \mathfrak{a}_{M_r} \subset \mathfrak{a}_{M_{r-1}} \subset \cdots \subset \mathfrak{a}_{M_0} = \mathfrak{a}_M$$

are of successive codimensions 0 or 1. If $\mathfrak{a}_{M_i}$ is of codimension 1 in $\mathfrak{a}_{M_{i-1}}$, let ε_i be the *unit* vector in $\mathfrak{a}_{M_{i-1}}^*$ in the direction of the restriction of E_i to $\mathfrak{a}_{M_{i-1}}$. (Recall that we have fixed Euclidean norms on $\mathfrak{a}_M$ and $\mathfrak{a}_M^*$.) If $\mathfrak{a}_{M_i} = \mathfrak{a}_{M_{i-1}}$, take ε_i to be the zero vector. Then the nonzero vectors in $\{\varepsilon_1, ..., \varepsilon_r\}$ form an orthonormal basis of $(\mathfrak{a}_M^L)^*$. Let R_0 be the unique parabolic in $\mathscr{P}^L(M)$ for which the Levi components M_j are all standard, and on whose chamber $\mathfrak{a}_{R_0}^+$ the functions ε_j are all nonnegative. Similarly, for $1 \leqslant i \leqslant r$, let $R_i \in \mathscr{P}^L(M)$ be the parabolic for which each M_i is standard and such that the functions $\{\varepsilon_1, ..., -\varepsilon_i, ..., \varepsilon_r\}$ are all nonnegative on $\mathfrak{a}_{R_i}^+$. Fix $Q_0 \in \mathscr{P}(L)$, and define parabolics

$$P_i = Q_0(R_i), \qquad 0 \leqslant i \leqslant r,$$

in $\mathscr{P}(M)$. Set

$$R'_{P_0|P_i}(\sigma_\Lambda^M, \varepsilon_i) = \lim_{t \to 0} \frac{d}{dt} R_{P_0|P_i}(\sigma_{\Lambda + t\varepsilon_i}^M), \qquad 1 \leqslant i \leqslant r.$$

Taking an r-fold product of logarithmic derivatives, we define an operator

$$\Gamma_\Omega(\sigma_\Lambda^M, P_0) = R'_{P_0|P_r}(\sigma_\Lambda^M, \varepsilon_r)\, R_{P_0|P_r}(\sigma_\Lambda^M)^{-1} \cdots R'_{P_0|P_1}(\sigma_\Lambda^M, \varepsilon_1)\, R_{P_0|P_1}(\sigma_\Lambda^M)^{-1}$$

on $\mathscr{V}_{P_0}(\pi)$. Observe that if any of the vectors $\{\varepsilon_1, ..., \varepsilon_r\}$ is 0, the operator $\Gamma_\Omega(\sigma_\Lambda^M, P_0)$ vanishes.

Proposition 9.1. *The distribution*

$$\operatorname*{Res}_{\Omega, \Lambda \to \Lambda_0} (a_\Lambda J_M(\sigma_\Lambda, f))$$

equals

$$\operatorname*{Res}_{\Omega, \Lambda \to \Lambda_0} (a_\Lambda \operatorname{tr}(\mathscr{I}_{P_0}(\sigma_\Lambda^M, f)\, \mathscr{R}_L(\sigma_\Lambda^M, P_0)\, \Gamma_\Omega(\sigma_\Lambda^M, P_0))).$$

Proof. The proof will be by induction on the length of the residue datum Ω. Assume that the proposition holds for any datum of length r. Let Ω' be a datum of length $(r+1)$ associated to a sequence

$$\mathscr{M}=\mathscr{M}_0\subset\mathscr{M}_1\subset\cdots\subset\mathscr{M}_r\subset\mathscr{M}_{r+1}=\mathscr{L}'.$$

It is clear that Ω' is obtained from a datum Ω of length r, for which we follow the notation above. The only additional information in Ω' is E_{r+1}, a multiple of some root β_{r+1} of $(G, A_{\mathscr{L}})$, and a point $\Lambda_{\Omega'}$ in $(\Lambda_\Omega+\mathbb{C}E_{r+1})$. Let Λ_0' be a point in general position in $\Lambda_{\Omega'}+\mathfrak{a}^*_{\mathscr{L}',\mathbb{C}}$, and set

$$\Lambda_0=\Lambda_0'+z_{r+1}E_{r+1},$$

with z_{r+1} a variable point in $\mathbb{C}$. The operator $\operatorname{Res}_{\Omega',\Lambda\to\Lambda_0'}$ can be calculated by first applying $\operatorname{Res}_{\Omega,\Lambda\to\Lambda_0}$ and then integrating z_{r+1} over a small circle about the origin. It follows from our induction hypothesis that

$$\operatorname*{Res}_{\Omega',\Lambda\to\Lambda_0'}(a_\Lambda J_M(\sigma_\Lambda, f)) \tag{9.1}$$

equals the residue about $z_{r+1}=0$ of

$$\operatorname*{Res}_{\Omega,\Lambda\to\Lambda_0}(a_\Lambda \operatorname{tr}(\mathscr{I}_{P_0}(\sigma_\Lambda^M, f)\,\mathscr{R}_L(\sigma_\Lambda^M, P_0)\,\Gamma_\Omega(\sigma_\Lambda^M, P_0))). \tag{9.2}$$

We recall here that $P_0=Q_0(R_0)$, where $Q_0\in\mathscr{P}(L)$ is arbitrary, and $R_0\in\mathscr{P}^L(M)$ is chosen to be compatible with the directions $\{\varepsilon_1, ..., \varepsilon_r\}$.

The operator $\mathscr{R}_L(\sigma_\Lambda^M, P_0)$ is obtained from the (G, L) family

$$R_{Q(R_0)\mid P_0}(\sigma_\Lambda^M)^{-1}\,R_{Q(R_0)\mid P_0}(\sigma_{\Lambda+\nu}^M),\qquad \nu\in\mathfrak{a}^*_{L,\mathbb{C}},\ Q\in\mathscr{P}(L).$$

Applying [1(a), (6.5)] to this family, we see that (9.2) equals

$$\operatorname*{Res}_{\Omega,\Lambda\to\Lambda_0}\frac{a_\Lambda}{r!}\sum_{\{P=Q(R_0):\,Q\in\mathscr{P}(L)\}}\operatorname{tr}(\mathscr{I}_{P_0}(\sigma_\Lambda^M, f)\,R_{P\mid P_0}(\sigma_\Lambda^M)^{-1}$$
$$\times R^{(r)}_{P\mid P_0}(\sigma_\Lambda^M, \nu)\,\Gamma_\Omega(\sigma_\Lambda^M, P_0))\,\theta_Q(\nu)^{-1},$$

where

$$R^{(r)}_{P\mid P_0}(\sigma_\Lambda^M, \nu)=\lim_{t\to 0}\left(\frac{d}{dt}\right)^r R_{P\mid P_0}(\sigma_{\Lambda+t\nu}^M).$$

This expression does not depend on the point $\nu\in\mathfrak{a}^*_{L,\mathbb{C}}$. The only constituent of the expression which could possibly contribute a pole along the hyperplane $\Lambda_0(\beta_{r+1}^\vee)=0$ is the function

$$R_{P\mid P_0}(\sigma_\Lambda^M)^{-1}\,R^{(r)}_{P\mid P_0}(\sigma_\Lambda^M, \nu)=R_{Q(R_0)\mid Q_0(R_0)}(\sigma_\Lambda^M)^{-1}\,R^{(r)}_{Q(R_0)\mid Q_0(R_0)}(\sigma_\Lambda^M, \nu).$$

But each singular hyperplane of this function is defined by a co-root whose restriction to $\mathfrak{a}_L^*$ separates the chambers $(\mathfrak{a}_Q^*)^+$ and $(\mathfrak{a}_{Q_0}^*)^+$. Consequently, the function gives no contribution to the residue (9.1) unless the restriction of E_{r+1} to $\mathfrak{a}_L$ defines a hyperplane of this sort. In particular, (9.1) vanishes if the restriction of E_{r+1} to $\mathfrak{a}_L$ is zero. Combining this with our induction assumption, we obtain the required assertion that (9.1) vanishes in case any of the vectors $\{\varepsilon_1, ..., \varepsilon_{r+1}\}$ is zero.

We can assume, then, that $\varepsilon_1, ..., \varepsilon_{r+1}$ are all nonzero. We must fix an arbitrary parabolic $Q_0' \in \mathscr{P}(L')$. Taken together with the unit vectors $\{\varepsilon_1, ..., \varepsilon_{r+1}\}$ it determines unique parabolics

$$P_i' = Q_0'(R_i'), \qquad 0 \leqslant i \leqslant r+1,$$

in $\mathscr{P}(M)$ by the conventions above. The parabolic $Q_0 \in \mathscr{P}(L)$, which has been arbitrary, we now take to be the unique parabolic which is contained in Q_0' and for which the function $(-\varepsilon_{r+1})$ is positive on $\mathfrak{a}_{Q_0}^+$. Then

$$P_0 = Q_0(R_0) = P_{r+1}'.$$

Given $P = Q(R_0)$, with $Q \in \mathscr{P}(L)$, we note that the hyperplane in $\mathfrak{a}_L$ defined by β_{r+1} separates the chambers $\mathfrak{a}_Q^+$ and $\mathfrak{a}_{Q_0}^+$ if and only if $d(P, P_0) > d(P, P_0')$. Writing

$$\mathscr{I}_{P_0}(\sigma_\Lambda^M, f)\, R_{P|P_0}(\sigma_\Lambda^M)^{-1} = R_{P_0'|P_0}(\sigma_\Lambda^M)^{-1}\, \mathscr{I}_{P_0'}(\sigma_\Lambda^M, f)\, R_{P|P_0'}(\sigma_\Lambda^M)^{-1}$$

for each such P, we see that (9.1) can be obtained by summing the product of $(r!\, \theta_Q(v))^{-1}$ and

$$\begin{aligned} \operatorname*{Res}_{\Omega, \Lambda \to \Lambda_0} \big(a_\Lambda \operatorname{tr}(R_{P_0'|P_0}(\sigma_\Lambda^M)^{-1}\, \mathscr{I}_{P_0'}(\sigma_\Lambda^M, f)\, R_{P|P_0'}(\sigma_\Lambda^M)^{-1} \\ \times R_{P|P_0}^{(r)}(\sigma_\Lambda^M, v)\, \Gamma_\Omega(\sigma_\Lambda^M, P_0))\big) \end{aligned} \tag{9.3}$$

over $P = Q(R_0)$ in the set

$$\{P = Q(R_0) \colon Q \in \mathscr{P}(L), d(P, P_0) > d(P, P_0')\}, \tag{9.4}$$

and then taking the residue about $z_{r+1} = 0$.

The operator $\Gamma_\Omega(\sigma_\Lambda^M, P_0)$ acts on the vector space $\mathscr{V}_{P_0}(\pi) = \mathscr{V}_{Q_0}(\mathscr{V}_{R_0}(\pi))$ through the fibre. It transforms a given function from K to $\mathscr{V}_{R_0}(\pi)$ by the operator $\Gamma_\Omega(\sigma_\Lambda^M, R_0)$. The other operators in (9.3) are products of scalar valued functions of Λ with admissible operators. We can therefore apply Proposition 9.1 inductively, with G replaced by L, to the fibres of these operators. It follows that the expression (9.3) is left unchanged if

$\Gamma_\Omega(\sigma_\Lambda^M, P_0)$ is replaced by $\mathscr{R}_M^{Q_0}(\sigma_\Lambda^M, P_0)$. Consequently, we may apply Lemma 8.2 to commute the operators $R_{P_0'|P_0}(\sigma_\Lambda^M)^{-1}$ and

$$A(\sigma_\Lambda^M) = \mathscr{I}_{P_0'}(\sigma_\Lambda^M, f)\, R_{P|P_0'}(\sigma_\Lambda^M)^{-1}\, R_{P|P_0}^{(r)}(\sigma_\Lambda^M, \nu)$$

in (9.3). As a result, (9.3) equals

$$\operatorname*{Res}_{\Omega, \Lambda \to \Lambda_0} a_\Lambda \operatorname{tr}(\mathscr{I}_{P_0'}(\sigma_\Lambda^M, f)\, R_{P|P_0'}(\sigma_\Lambda^M)^{-1}\, R_{P|P_0}^{(r)}(\sigma_\Lambda^M, \nu)$$
$$\times R_{P_0'|P_0}(\sigma_\Lambda^M)^{-1}\, \Gamma_\Omega(\sigma_\Lambda^M, P_0')).$$

Now, we employ Leibnitz' rule to write

$$R_{P|P_0'}(\sigma_\Lambda^M)^{-1}\, R_{P|P_0}^{(r)}(\sigma_\Lambda^M, \nu)\, R_{P_0'|P_0}(\sigma_\Lambda^M)^{-1}$$

as

$$\sum_{j=0}^{r} \frac{r!}{(r-j)!\,(j)!} R_{P|P_0'}(\sigma_\Lambda^M)^{-1}\, R_{P|P_0'}^{(r-j)}(\sigma_\Lambda^M, \nu)\, R_{P_0'|P_0}^{(j)}(\sigma_\Lambda^M, \nu) R_{P_0'|P_0}(\sigma_\wedge^M)^{-1}.$$

It is only the operators $R_{P_0'|P_0}(\cdot)$ which can contribute a pole along $\Lambda_0(\beta_{r+1}^\vee) = 0$. If $j = 0$, these operators cancel. The corresponding term can therefore be left out of the resulting formula for (9.1). Recombining the residues in z_{r+1} and Ω, we express (9.1) finally as

$$\operatorname*{Res}_{\Omega', \Lambda \to \Lambda_0'} a_\Lambda \operatorname{tr}(\mathscr{I}_{P_0'}(\sigma_\Lambda^M, f)\, R(\Lambda)\, \Gamma_\Omega(\sigma_\Lambda^M, P_0')),$$

where $R(\Lambda)$ is the operator

$$\sum_P \theta_Q(\nu)^{-1} \sum_{j=1}^{r} \frac{1}{(r-j)!\,(j)!} R_{P|P_0'}(\sigma_\Lambda^M)^{-1}\, R_{P|P_0'}^{(r-j)}(\sigma_\Lambda^M, \nu)\, R_{P_0'|P_0}^{(j)}(\sigma_\Lambda^M, \nu)$$
$$\times R_{P_0'|P_0}(\sigma_\Lambda^M)^{-1}. \tag{9.5}$$

Here $P = Q(R_0)$ is summed over the set (9.4).

The point ν intervenes only in the expression (9.5). Since the final residue is independent of ν, we may choose the point any way we wish. Set $\nu = \nu' + s\varepsilon_{r+1}$, with ν' a point in general position in $\mathfrak{a}_{L',\mathbb{C}}^*$, and s a small complex number which approaches 0. Note that the function

$$\theta_Q(\nu)^{-1} = \operatorname{vol}(\mathfrak{a}_L^G/\mathbb{Z}(\Delta_Q^\vee)) \Big(\prod_{\alpha \in \Delta_Q} (\nu' + s\varepsilon_{r+1})(\alpha^\vee) \Big)^{-1}$$

is the only term which can contribute a singularity in s to (9.5). It has at most a pole of order 1 at $s = 0$, and this occurs precisely when some root

$\alpha \in \Delta_Q$ vanishes on $\mathfrak{a}_{L'}$. That is, when the parabolic $P = Q(R_0)$ equals $Q'(R_0')$ for some $Q' \in \mathscr{P}(L')$. On the other hand,

$$\lim_{s \to 0} R^{(r-j)}_{P|P_0'}(\sigma^M_\Lambda, \nu) = R^{(r-j)}_{P|P_0'}(\sigma^M_\Lambda, \nu'),$$

while

$$R^{(j)}_{P_0'|P_0}(\sigma^M_\Lambda, \nu) = R^{(j)}_{P_0'|P_0}(\sigma^M_\Lambda, s\varepsilon_{r+1}) = s^j R^{(j)}_{P_0'|P_0}(\sigma^M_\Lambda, \varepsilon_{r+1}),$$

since $P_0' \cap L' = P_0 \cap L'$. Therefore, the only summands in (9.5) which do not approach 0 are those with $j = 1$ and $P = Q(R_0) = Q'(R_0')$, $Q' \in \mathscr{P}(L')$. Observe that if P is of this form, and α_1 is the unique root in Δ_Q which vanishes on $\mathfrak{a}_{L'}$, then

$$\begin{aligned}
&\lim_{s \to 0} \theta_Q(\nu)^{-1} s \\
&\quad = \mathrm{vol}(\mathfrak{a}^G_L/\mathbb{Z}(\Delta^\vee_Q))\, \|\alpha_1^\vee\|^{-1} \Big(\prod_{\alpha \in \Delta_Q \setminus \{\alpha_1\}} \nu'(\alpha^\vee) \Big)^{-1} \\
&\quad = \mathrm{vol}(\mathfrak{a}^G_{L'}/\mathbb{Z}(\Delta^\vee_{Q'})) \Big(\prod_{\alpha' \in \Delta_{Q'}} \nu'((\alpha')^\vee) \Big)^{-1} \\
&\quad = \theta_{Q'}(\nu')^{-1}.
\end{aligned}$$

Consequently, the value of (9.5) at $s = 0$ is

$$\begin{aligned}
&\frac{1}{(r-1)!} \sum_{\{P = Q'(R_0'):\, Q' \in \mathscr{P}(L')\}} \theta_{Q'}(\nu')^{-1} R_{P|P_0'}(\sigma^M_\Lambda)^{-1} R^{(r-1)}_{P|P_0'}(\sigma^M_\Lambda, \nu') \\
&\qquad \times R'_{P_0'|P_0}(\sigma^M_\Lambda, \varepsilon_{r+1})\, R_{P_0'|P_0}(\sigma^M_\Lambda)^{-1}.
\end{aligned}$$

Appealing again to [1(a), (6.5)], we see that this equals

$$\mathscr{R}_{L'}(\sigma^M_\Lambda, P_0')\, R'_{P_0'|P_0}(\sigma^M_\Lambda, \varepsilon_{r+1})\, R_{P_0'|P_0}(\sigma^M_\Lambda)^{-1}.$$

Consequently, (9.1) equals

$$\begin{aligned}
&\operatorname*{Res}_{\Omega', \Lambda \to \Lambda_0'} a_\Lambda \operatorname{tr}(\mathscr{I}_{P_0'}(\sigma^M_\Lambda, f)\, \mathscr{R}_{L'}(\sigma^M_\Lambda, P_0')\, R'_{P_0'|P_0}(\sigma^M_\Lambda, \varepsilon_{r+1}) \\
&\qquad \times R_{P_0'|P_0}(\sigma^M_\Lambda)^{-1}\, \Gamma_\Omega(\sigma^M_\Lambda, P_0')). \qquad (9.6)
\end{aligned}$$

We are now essentially done. The parabolics P_i, $0 \leqslant i \leqslant r$, are all contained in Q_0. Similarly, the parabolics P_i', $0 \leqslant i \leqslant r$, are all contained in another fixed element of $\mathscr{P}(L)$. Since $P_i \cap L = P_i' \cap L$, we have

$$R'_{P_0|P_i}(\sigma^M_\Lambda, \varepsilon_i)\, R_{P_0|P_i}(\sigma^M_\Lambda)^{-1} = R'_{P_0'|P_i'}(\sigma^M_\Lambda, \varepsilon_i)\, R_{P_0'|P_i'}(\sigma^M_\Lambda)^{-1}.$$

We have already noted that $P_0 = P'_{r+1}$. It follows that

$$R'_{P'_0|P_0}(\sigma^M_\Lambda, \varepsilon_{r+1})\, R_{P'_0|P_0}(\sigma^M_\Lambda)^{-1}\, \Gamma_\Omega(\sigma^M_\Lambda, P'_0) = \Gamma_{\Omega'}(\sigma^M_\Lambda, P'_0).$$

Substituting into (9.6) gives the required formula

$$\operatorname*{Res}_{\Omega', \Lambda \to \Lambda'_0} a_\Lambda \operatorname{tr}(\mathscr{I}_{P'_0}(\sigma^M_\Lambda, f)\, \mathscr{R}_{L'}(\sigma^M_\Lambda, P'_0)\, \Gamma_{\Omega'}(\sigma^M_\Lambda, P'_0))$$

for (9.1). This completes the induction step, and the proof of the proposition. ∎

In this paper we shall use the proposition only as a vanishing assertion. Let $h_{\mathscr{L}}$ and h_M denote the natural projections of $\mathfrak{a}_{\mathscr{M}}$ onto $\mathfrak{a}_{\mathscr{L}}$ and $\mathfrak{a}_M$, respectively.

Corollary 9.2. *The distribution*

$$\operatorname*{Res}_{\Omega, \Lambda \to \Lambda_0} (a_\Lambda J_M(\sigma_\Lambda, f))$$

vanishes unless

$$\ker(h_{\mathscr{L}}) \cap \ker(h_M) = \{0\}.$$

Proof. The projection $h_{\mathscr{L}}$ is associated to a canonical splitting

$$\mathfrak{a}_{\mathscr{M}} = \mathfrak{a}^{\mathscr{L}}_{\mathscr{M}} \oplus \mathfrak{a}_{\mathscr{L}}.$$

A similar assertion holds for h_M. Consider the associated dual projections $\mathfrak{a}^*_{\mathscr{M}} \to \mathfrak{a}^*_{\mathscr{L}}$ and $\mathfrak{a}^*_{\mathscr{M}} \to \mathfrak{a}^*_M$. The kernel of the first one is spanned by $\{E_1, ..., E_r\}$. But if these vectors have images in $\mathfrak{a}^*_M$ which are linearly dependent, the operator $\Gamma_\Omega(\sigma^M_\Lambda, P_0)$ is defined to be 0. The corollary therefore follows from the proposition. ∎

10. Changes of Contour

The reason for studying residues is to be able to deform contour integrals. In this paragraph we shall set up a scheme for keeping track of the residues that arise from changes of contour. It is similar to the procedure used in the proof of the Paley–Wiener theorem [1(d), Sect. II.2], and was originally motivated by Langlands' theory of Eisenstein series [15(b), Sect. 7].

Suppose that $\mathscr{M}$ and $\sigma \in \Pi(\mathscr{M}(F_S))$ are as in the last two paragraphs, and that μ is a fixed point in general position in $\mathfrak{a}^*_{\mathscr{M}}$. Suppose also that for each Levi subset $\mathscr{L}$ over F_S which contains $\mathscr{M}$, we have fixed a point $\nu_{\mathscr{L}}$

in general position in $\mathfrak{a}_{\mathscr{L}}^*$. Let Γ be a finite subset of $\Pi(K)$, the set of equivalence classes of irreducible representations of K. We write $\mathscr{H}(G(F_S))_\Gamma$ for the space of functions in $\mathscr{H}(G(F_S))$ which transform on each side under K according to representations in Γ. The residue scheme will be determined in a canonical way from the point μ, the collection

$$\mathscr{N} = \{\nu_{\mathscr{L}} : \mathscr{L} \supset \mathscr{M}\},$$

and the set

$$\Psi = \{\psi(\Lambda) = e^{-\Lambda(\mathscr{X})} J_M(\sigma_\Lambda, f) : \mathscr{X} \in \mathfrak{a}_{\mathscr{M},S}, f \in \mathscr{H}(G(F_S))_\Gamma\}$$

of functions on $\mathfrak{a}_{\mathscr{M},\mathbb{C}}^*$. Note that the singularities of all the functions in Ψ form a set of hyperplanes of the form (6.3) which is finite modulo $i\mathfrak{a}_{\mathscr{M},S}^\vee$. Our assumption on the general position of $\nu_{\mathscr{L}}$ implies that if $\psi(\Lambda)$ belongs to Ψ and Ω is a residue datum for $\mathscr{L}$, then the function

$$\operatorname*{Res}_{\Omega, \Lambda \to (\Lambda_\Omega + \lambda)} \psi(\Lambda) = \operatorname*{Res}_{\Omega} \psi(\Lambda_\Omega + \lambda), \qquad \lambda \in \mathfrak{a}_{\mathscr{L},\mathbb{C}}^*,$$

is regular on $\nu_{\mathscr{L}} + i\mathfrak{a}_{\mathscr{L}}^*$.

PROPOSITION 10.1 *For each $\mathscr{L}$ there is a finite set*

$$R_{\mathscr{L}} = R_{\mathscr{L}}(\mu, \mathscr{N})$$

of residue data for $\mathscr{L}$ such that

$$\int_{\mu + i\mathfrak{a}_{\mathscr{M},S}^*} \psi(\Lambda)\, d\Lambda = \sum_{\mathscr{L}} \int_{\nu_{\mathscr{L}} + i\mathfrak{a}_{\mathscr{L},S}^*} \Big(\sum_{\Omega \in R_{\mathscr{L}}} \operatorname*{Res}_{\Omega} \psi(\Lambda_\Omega + \lambda) \Big)\, d\lambda \qquad (10.1)$$

for any function ψ in Ψ.

Proof. The construction is similar to that of [1(d), pp. 45–51], so our discussion will be rather brief. We shall define the sets $R_{\mathscr{L}}$ by induction on $\dim(\mathfrak{a}_{\mathscr{M}}^*/\mathfrak{a}_{\mathscr{L}}^*)$. In the process, we shall associate to each $\Omega \in R_{\mathscr{L}}$ a point μ_Ω in $\mathfrak{a}_{\mathscr{L}}^*$.

If $\mathscr{L} = \mathscr{M}$, take $R_{\mathscr{L}}$ to consist only of the trivial residue datum Ω_0, with $\mathscr{E}_{\Omega_0}$ empty and $\Lambda_{\Omega_0} = 0$. Set $\mu_{\Omega_0} = \mu$. Now assume inductively that we have defined the sets $R_{\mathscr{L}}$ and also points $\{\mu_\Omega \in \mathfrak{a}_{\mathscr{L}}^* : \Omega \in R_{\mathscr{L}}\}$, for each $\mathscr{L}$ with $\dim(\mathfrak{a}_{\mathscr{M}}^*/\mathfrak{a}_{\mathscr{L}}^*) = r$. Fix a Levi subset $\mathscr{L}'$ over F_S with $\dim(\mathfrak{a}_{\mathscr{M}}^*/\mathfrak{a}_{\mathscr{L}'}^*) = r + 1$. Then $R_{\mathscr{L}'}$ will be defined as a union over all $\mathscr{L} \supset \mathscr{L}'$ with $\dim(\mathfrak{a}_{\mathscr{M}}^*/\mathfrak{a}_{\mathscr{L}}^*) = r$, and over all $\Omega \in R_{\mathscr{L}}$, of certain sets. Consider such an $\mathscr{L}$ and a residue datum

$$\Omega = (\mathscr{E}_\Omega, \Lambda_\Omega) = ((E_1, \dots, E_r), \Lambda_\Omega),$$

in $R_{\mathscr{P}}$. By our general position assumption, $\nu_{\mathscr{P}} - \mu_{\Omega}$ does not belong to $\mathfrak{a}^*_{\mathscr{P}'}$. Let E_{r+1} be the unit vector in $\mathfrak{a}^*_{\mathscr{P}}$ which is orthogonal to $\mathfrak{a}^*_{\mathscr{P}'}$ and whose inner product with the vector $\nu_{\mathscr{P}} - \mu_{\Omega}$ is positive. (The inner product on $\mathfrak{a}^*_{\mathscr{P}}$ is constructed in the same way as that on $\mathfrak{a}^*_M$.) We shall describe the subset of $R_{\mathscr{P}'}$ associated to $\mathscr{L}$ and Ω. It is parameterized by the orbits under $i\mathfrak{a}^\vee_{\mathscr{M},S}$ of those singular hyperplanes of the function

$$\lambda \to \operatorname*{Res}_{\Omega} \psi(\Lambda_{\Omega} + \lambda), \qquad \lambda \in \mathfrak{a}^*_{\mathscr{P},\mathbb{C}}, \psi \in \Psi, \tag{10.2}$$

which are of the form

$$\zeta' E_{r+1} + \mathfrak{a}^*_{\mathscr{P}',\mathbb{C}}, \qquad \zeta' \in \mathbb{C},$$

and which intersect the set

$$\{t\mu_{\Omega} + (1-t)\,\nu_{\mathscr{P}} + \lambda : 0 < t < 1,\ \lambda \in i\mathfrak{a}^*_{\mathscr{P}}\}. \tag{10.3}$$

The residue datum $\Omega' = (\mathscr{E}_{\Omega'}, \Lambda_{\Omega'})$ attached to such a singular hyperplane is defined by

$$\mathscr{E}_{\Omega'} = (E_1, \dots, E_r, E_{r+1})$$

and

$$\Lambda_{\Omega'} = \Lambda_{\Omega} + \zeta' E_{r+1}.$$

We then take $\mu_{\Omega'}$ to be the unique point in $\mathfrak{a}^*_{\mathscr{P}'}$ such that $\Lambda_{\Omega'} + \mu_{\Omega'}$ belongs to the set (10.3).

The inductive definition is set up to account for changes of contours of integration. Standard estimates (such as the inequality (12.7) below) allow us to control the growth of a function (10.2) on the set (10.3), at least away from the singular hyperplanes. We can therefore deform the integral of (10.2) over $(\mu_{\Omega} + i\mathfrak{a}^*_{\mathscr{P},S})$ to an integral over $(\nu_{\mathscr{P}} + i\mathfrak{a}^*_{\mathscr{P},S})$. In the process, we pick up residues at the singular hyperplanes. The general position of $\nu_{\mathscr{P}}$ means that the singularities can be handled separately. It follows from our definition that the sum over $(\mathscr{L}, \Omega)$, with $\Omega \in R_{\mathscr{P}}$ and $\dim(\mathfrak{a}^*_{\mathscr{M}}/\mathfrak{a}^*_{\mathscr{P}}) = r$, of the expression

$$\int_{\mu_{\Omega} + i\mathfrak{a}^*_{\mathscr{P},S}} (\operatorname*{Res}_{\Omega} \psi(\Lambda_{\Omega} + \lambda))\, d\lambda - \int_{\nu_{\mathscr{P}} + i\mathfrak{a}^*_{\mathscr{P},S}} (\operatorname*{Res}_{\Omega} \psi(\Lambda_{\Omega} + \lambda))\, d\lambda,$$

equals the sum over $(\mathscr{L}', \Omega')$, with $\Omega' \in R_{\mathscr{P}'}$ and $\dim(\mathfrak{a}^*_{\mathscr{M}}/\mathfrak{a}^*_{\mathscr{P}'}) = r+1$, of

$$\int_{\mu_{\Omega'} + i\mathfrak{a}^*_{\mathscr{P}',S}} (\operatorname*{Res}_{\Omega'} \psi(\Lambda_{\Omega'} + \lambda))\, d\lambda.$$

The required identity (10.1) is then obtained by applying this last formula repeatedly, as r increases from 0 to $\dim(\mathfrak{a}_L/\mathfrak{a}_G)$. ∎

Remark. It is clear that the construction applies to any family of functions on $\mathfrak{a}^*_{\mathcal{M},\mathbb{C}}/i\mathfrak{a}^\vee_{\mathcal{M},S}$ whose singularities and growth properties are similar to those of Ψ.

11. The Spaces $\mathcal{H}_{\mathrm{ac}}(G(F_S))$ and $\mathcal{I}_{\mathrm{ac}}(G(F_S))$

As an application of our discussion on residues, we will study the function

$$\phi_M(f, \pi, X), \qquad \pi \in \Pi_{\mathrm{temp}}(M(F_S)),\ X \in \mathfrak{a}_{M,S}.$$

In particular, we shall show that as a function of the parameters on $\Pi_{\mathrm{temp}}(M(F_S))$, it can be analytically continued to an entire function. We will come to this in Section 12. In the present paragraph we shall simply describe some spaces of functions, in order to illustrate the properties of ϕ_M. These spaces will also be useful for another paper on the invariant trace formula.

We shall consider $\mathcal{H}(G(F_S))$ as a topological vector space. Fix a positive function

$$\|x\| = \prod_{v \in S} \|x_v\|_v, \qquad x \in G(F_S),$$

on $G(F_S)$ as in [1(b), Sect. 2]. We assume in particular that $\|\cdot\|$ satisfies [1(b), conditions (i)–(iii), p. 1253]. Suppose that N is a positive number and that Γ is a finite subset of $\Pi(K)$. We define $\mathcal{H}_N(G(F_S))_\Gamma$ to be the space of smooth functions on $G(F_S)$ which are supported on the set

$$G(F_S, N) = \{x \in G(F_S) \colon \log\|x\| \leqslant N\}$$

and which transform on each side under K according to representations in Γ. The topology on $\mathcal{H}_N(G(F_S))_\Gamma$ is that given by the semi-norms

$$\|f\|_D = \sup_{x \in G(F_S)} |Df(x)|, \qquad f \in \mathcal{H}_N(G(F_S))_\Gamma,$$

where D is a differential operator on $G(F_{S \cap S_\infty})$. (We are writing S_∞ for the set of Archimedean valuations of F.) We then define $\mathcal{H}(G(F_S))$ as the topological direct limit

$$\mathcal{H}(G(F_S)) = \varinjlim_{\Gamma} \mathcal{H}(G(F_S))_\Gamma,$$

where

$$\mathscr{H}(G(F_S))_\Gamma = \varinjlim_N \mathscr{H}_N(G(F_S))_\Gamma.$$

Suppose that $f \in \mathscr{H}(G(F_S))$. We have the invariant Fourier transform

$$f_G: \pi \to f_G(\pi) = \operatorname{tr} \pi(f), \qquad \pi \in \Pi_{\text{temp}}(G(F_S)).$$

However, it is convenient for us to take a slightly different point of view. Define

$$f_G(\pi, Z) = \operatorname{tr} \pi(f^Z), \qquad \pi \in \Pi_{\text{temp}}(G(F_S)),\ Z \in \mathfrak{a}_{G,S}.$$

Then

$$f_G(\pi, Z) = \int_{i\mathfrak{a}^*_{G,S}} f_G(\pi_\lambda)\, e^{-\lambda(Z)}\, d\lambda.$$

Thus, f_G can be interpreted in two ways, either as a function on $\Pi_{\text{temp}}(G(F_S))$ or, via the Fourier transform on $\mathfrak{a}_{G,S}$, as a function on $\Pi_{\text{temp}}(G(F_S)) \times \mathfrak{a}_{G,S}$. Note that the situation is analogous to that of the function $\phi_M(f)$. Indeed f_G is just the special case that $M = G$. We will generally lean towards the second interpretation. Then $f \to f_G$ will be regarded as a map from $\mathscr{H}(G(F_S))$ to a space of functions on $\Pi_{\text{temp}}(G(F_S)) \times \mathfrak{a}_{G,S}$. When $G = G^0$, the work of Clozel–Delorme [8(a), (b)] and Bernstein–Deligne–Kazhdan [4] provides a characterization of the image.

In order to describe the image, it is convenient to fix Euclidean inner products and Haar measures on the various spaces associated to Levi subsets $\mathscr{M} = \prod_{v \in S} M_v$. We do this for each $v \in S$ separately, by following the conventions of Section 1 (with F replaced by F_v). We obtain Euclidean norms on the spaces $\mathfrak{a}_{\mathscr{M}}$ and $\mathfrak{a}^*_{\mathscr{M}}$, and Haar measures on the groups $\mathfrak{a}_{\mathscr{M}}$, $\mathfrak{a}_{\mathscr{M},S}$, $i\mathfrak{a}^*_{\mathscr{M}}$, and $i\mathfrak{a}^*_{\mathscr{M},S}$. For any positive number N, let $C^\infty_N(\mathfrak{a}_{\mathscr{M},S})$ denote the topological vector space of smooth functions on $\mathfrak{a}_{\mathscr{M},S}$ which are supported on

$$\{\mathscr{X} \in \mathfrak{a}_{\mathscr{M},S}: \|\mathscr{X}\| \leqslant N\}.$$

Suppose that Γ is a finite subset of $\Pi(K)$, and that N is a positive number. We define $\mathscr{I}_N(G(F_S))_\Gamma$ to be the space of functions

$$\phi: \Pi_{\text{temp}}(G(F_S)) \times \mathfrak{a}_{G,S} \to \mathbb{C}$$

which satisfy the following three conditions.

1. If $\zeta = (\xi, \lambda)$ is any element in $\Xi_{G,S} \times i\mathfrak{a}_{G,S}$, then

$$\phi(\pi_\zeta, Z) = \phi(\pi, Z)\xi(G)\, e^{\lambda(Z)}.$$

2. Suppose that the restriction of π to K does not contain any representation in Γ. Then $\phi(\pi, Z)=0$.

3. Suppose that $\mathcal{M}$ is an admissible Levi subset of G over F_S, and that $\sigma \in \Pi_{\text{temp}}(\mathcal{M}(F_S))$. Then the integral

$$\phi(\sigma, \mathcal{X})=\int_{i\mathfrak{a}^*_{\mathcal{M},S}/i\mathfrak{a}^*_{G,S}} \phi(\sigma^G_\Lambda, h_G(\mathcal{X}))\, e^{-\Lambda(\mathcal{X})}\, d\Lambda, \qquad \mathcal{X} \in \mathfrak{a}_{\mathcal{M},S},$$

converges to a function of $\mathcal{X}$ which belongs to $C^\infty_N(\mathfrak{a}_{\mathcal{M},S})$.

We give $\mathcal{I}_N(G(F_S))_\Gamma$ the topology provided by the semi-norms

$$\phi \to \|\phi(\sigma, \cdot)\|_{\mathcal{M},\sigma},$$

with $\mathcal{M}$ and σ as above, and $\|\cdot\|_{\mathcal{M},\sigma}$ a continuous semi-norm on $C^\infty_N(\mathfrak{a}_{\mathcal{M},S})$. We then define $\mathcal{I}(G(F_S))$ as the topological direct limit

$$\mathcal{I}(G(F_S))=\varinjlim_{\Gamma} \mathcal{I}(G(F_S))_\Gamma,$$

where

$$\mathcal{I}(G(F_S))_\Gamma=\varinjlim_{N} \mathcal{I}_N(G(F_S))_\Gamma.$$

Note that the first condition implies that the integral

$$\int_{\mathfrak{a}_{G,S}} \phi(\pi, Z)\, dZ$$

is actually a Fourier transform on $\mathfrak{a}_{G,S}$. The other two conditions are taken from [8(a)] and [4]. For example, Condition 2 asserts that the function $\pi \to \phi(\pi, Z)$ is supported on finitely many components, in the sense of [4]. Condition 3 requires that for every $\mathcal{M}$ and σ, the function

$$\int_{\mathfrak{a}_{G,S}} \phi(\sigma^G_\Lambda, Z)\, dZ, \qquad \Lambda \in i\mathfrak{a}^*_{\mathcal{M},S},$$

belongs to the Paley-Wiener space on $i\mathfrak{a}^*_{\mathcal{M},S}$. In particular, if S consists of one discrete valuation, the function is a finite Fourier series on the torus $i\mathfrak{a}^*_{\mathcal{M},S}$.

The function $\phi_M(f)$ does not in general belong to $\mathcal{I}(M(F_S))$. To accommodate it, we must extend our definitions slightly. Suppose again that Γ is a finite subset of $\Pi(K)$. Define $\mathcal{H}_{\text{ac}}(G(F_S))_\Gamma$ to be the space of functions f on $G(F_S)$ such that for any $b \in C^\infty_c(\mathfrak{a}_{G,S})$, the function

$$f^b(x)=f(x)\, b(H_G(x)), \qquad x \in G(F_S),$$

belongs to $\mathscr{H}(G(F_S))_\Gamma$. (Here ac stands for "almost compact" support.) We give $\mathscr{H}_{ac}(G(F_S))_\Gamma$ the topology defined by the semi-norms

$$f \to \|f^b\|, \qquad f \in \mathscr{H}_{ac}(G(F_S))_\Gamma,$$

where b is any function as above, and $\|\cdot\|$ is a continuous semi-norm on $\mathscr{H}(G(F_S))_\Gamma$. Similarly, define $\mathscr{I}_{ac}(G(F_S))_\Gamma$ to be the space of functions ϕ on $\Pi_{\text{temp}}(G(F_S)) \times \mathfrak{a}_{G,S}$ such that for any b as above, the function

$$\phi^b(\pi, Z) = \phi(\pi, Z)\, b(Z), \qquad \pi \in \Pi_{\text{temp}}(G(F_S)), Z \in \mathfrak{a}_{G,S},$$

belongs to $\mathscr{I}(G(F_S))_\Gamma$. We topologize $\mathscr{I}_{ac}(G(F_S))_\Gamma$ the same way, by the semi-norms

$$\phi \to \|\phi^b\|, \qquad \phi \in \mathscr{I}_{ac}(G(F_S))_\Gamma,$$

with $\|\cdot\|$ a continuous semi-norm on $\mathscr{I}(G(F_S))_\Gamma$. We then define $\mathscr{H}_{ac}(G(F_S))$ and $\mathscr{I}_{ac}(G(F_S))$ as topological direct limits

$$\mathscr{H}_{ac}(G(F_S)) = \varinjlim_{\Gamma} \mathscr{H}_{ac}(G(F_S))_\Gamma$$

and

$$\mathscr{I}_{ac}(G(F_S)) = \varinjlim_{\Gamma} \mathscr{I}_{ac}(G(F_S))_\Gamma.$$

While we are at it, we shall define a useful space of functions that lies between $\mathscr{H}(G(F_S))$ and $\mathscr{H}_{ac}(G(F_S))$. We shall say that a function $f \in \mathscr{H}_{ac}(G(F_S))$ is *moderate* if there are positive constants c and d such that f is supported on the set

$$\{x \in G(F_S): \log\|x\| \leqslant c(\|H_G(x)\| + 1)\},$$

and such that

$$\sup_{x \in G(F_S)} (|\Delta f(x)| \exp\{-d\|H_G(x)\|\}) < \infty,$$

for any left invariant differential operator Δ on $G(F_{S_\infty \cap S})$. We shall also say that a function $\phi \in \mathscr{I}_{ac}(G(F_S))$ is *moderate* if for every Levi subset $\mathscr{M}$ over F_S, and every $\sigma \in \Pi_{\text{temp}}(\mathscr{M}(F_S))$, the function $\phi(\sigma, \cdot)$ has similar support and growth properties. Namely, there are positive constants c_1 and d_1 such that $\phi(\sigma, \cdot)$ is supported on the set

$$\{\mathscr{X} \in \mathfrak{a}_{\mathscr{M},S}: \|\mathscr{X}\| \leqslant c_1(\|h_G(\mathscr{X})\| + 1\},$$

and such that

$$\sup_{\mathscr{X} \in \mathfrak{a}_{\mathscr{M},S}} (|\Delta_1 \phi(\sigma, \mathscr{X})| \exp\{-d_1 \|h_G(\mathscr{X})\|\}) < \infty,$$

for any invariant differential operator Δ_1 on $\mathfrak{a}_{\mathscr{M}, S \cap S_\infty}$.

The notion of a moderate function will be a crucial ingredient in a certain convergence estimate required for the comparison of trace formulas. We shall see this in another paper, where we shall also show that $f \to f_G$ maps the moderate functions in $\mathscr{H}_{\text{ac}}(G(F_S))$ *onto* those in $\mathscr{I}_{\text{ac}}(G(F_S))$.

12. The Map ϕ_M

We defined the function

$$\phi_M(f)\colon (\pi, X) \to \phi_M(f, \pi, X), \qquad (\pi, X) \in \Pi_{\text{temp}}(M(F_S)) \times \mathfrak{a}_{M,S},$$

in Section 7. If Z is the projection of X onto $\mathfrak{a}_{G,S}$, the value $\phi_M(f, \pi, X)$ depends only on f^Z. Consequently, $\phi_M(f)$ is defined for any $f \in \mathscr{H}_{\text{ac}}(G(F_S))$. In this section we shall establish that ϕ_M maps $\mathscr{H}_{\text{ac}}(G(F_S))$ to $\mathscr{I}_{\text{ac}}(M(F_S))$. It is one of the main results of the paper.

It is convenient to study a slightly more general map. Suppose $\mu \in \mathfrak{a}_M^*$. Let $\phi_{M,\mu}(f)$ be the function whose value at a point (π, X) in

$$\Pi_{\text{temp}}(M(F_S)) \times \mathfrak{a}_{M,S}$$

equals

$$\phi_{M,\mu}(f, \pi, X) = J_M(\pi_\mu, X, f)\, e^{-\mu(X)}.$$

This too is defined if f is any function in $\mathscr{H}_{\text{ac}}(G(F_S))$. If $\zeta = (\xi, \lambda)$ belongs to $\Xi_{M,S} \times i\mathfrak{a}_M^*$, we have

$$\phi_{M,\mu}(f, \pi_\zeta, X) = \phi_{M,\mu}(f, \pi, X)\, \xi(M)\, e^{\lambda(X)}, \tag{12.1}$$

from (7.1). If h belongs to $\mathscr{H}(G^0(F_S)^1)$, (7.2) tells us that

$$\phi_{M,\mu}(L_h f) = \sum_{Q \in \mathscr{F}(M)} \phi_{M,\mu}^{M_Q}(R_{Q,h} f). \tag{12.2}$$

Suppose for a moment that f belongs to $\mathscr{H}(G(F_S))$. Suppose also that $b \in C_c^\infty(\mathfrak{a}_{M,S})$, that $\mathscr{M} = \prod_{v \in S} M_v$ is an admissible Levi subset of M over F_S, and that $\sigma \in \Pi_{\text{temp}}(\mathscr{M}(F_S))$. We shall need to study the function

$$\phi_{M,\mu}^b(f, \sigma, \mathscr{X}), \qquad \mathscr{X} \in \mathfrak{a}_{\mathscr{M},S}.$$

From the definitions of the last paragraph, we have

$$\phi^b_{M,\mu}(f, \sigma, \mathscr{X})$$
$$= b(h_M(\mathscr{X})) \int_{i\mathfrak{a}^*_{\mathscr{M},S}/i\mathfrak{a}^*_{M,S}} \phi_{M,\mu}(f, \sigma^M_\Lambda, h_M(\mathscr{X}))\, e^{-\Lambda(\mathscr{X})}\, d\Lambda$$
$$= b(h_M(\mathscr{X})) \sum_{P \in \mathscr{P}(M)} \omega_P \int_{\varepsilon_P + \mu + i\mathfrak{a}^*_{\mathscr{M},S}} J_M(\sigma_\Lambda, f)\, e^{-\Lambda(\mathscr{X})}\, d\Lambda.$$

Assume for simplicity that $J_M(\sigma_\Lambda, f)$ is analytic for Λ in $\mu + i\mathfrak{a}^*_{\mathscr{M},S}$. Then

$$\phi^b_{M,\mu}(f, \sigma, \mathscr{X}) = b(h_M(\mathscr{X})) \int_{\mu + i\mathfrak{a}^*_{\mathscr{M},S}} J_M(\sigma_\Lambda, f)\, e^{-\Lambda(\mathscr{X})}\, d\Lambda.$$

Our main concern will be to show that this function is compactly supported in $\mathscr{X}$. As in the proof of the classical Paley–Wiener theorem, this entails changing contours of integration. We will use Proposition 10.1 to account for the resulting residues.

THEOREM 12.1. *For each* $\mu \in \mathfrak{a}^*_M$, $\phi_{M,\mu}$ maps $\mathscr{H}_{\mathrm{ac}}(G(F_S))$ *continuously to* $\mathscr{I}_{\mathrm{ac}}(M(F_S))$.

Proof. Fix a function $b \bullet C_c^\infty(\mathfrak{a}_{M,S})$. Then

$$\phi^b_{M,\mu}(f, \pi, X) = b(X)\, \phi_{M,\mu}(f, \pi, X)$$
$$= b(X) \sum_{P \in \mathscr{P}(M)} \omega_P \int_{\varepsilon_P + \mu + i\mathfrak{a}^*_{M,S}/i\mathfrak{a}^*_{G,S}} J_M(\pi_\lambda, f^{h_G(X)})\, e^{-\lambda(X)}\, d\lambda,$$

for any $f \in \mathscr{H}_{\mathrm{ac}}(G(F_S))$. Thus, if b' is any function in $C_c^\infty(\mathfrak{a}_{G,S})$ which equals 1 on the image in $\mathfrak{a}_{G,S}$ of the support of b, we have

$$\phi^b_{M,\mu}(f) = \phi^b_{M,\mu}(f^{b'}).$$

We may therefore assume that f belongs to $\mathscr{H}(G(F_S))$. More precisely, we need only establish that

$$f \to \phi^b_{M,\mu}(f)$$

is a continuous map from $\mathscr{H}(G(F_S))$ to $\mathscr{H}(M(F_S))$. Choose a positive number N and a finite set $\Gamma \subseteq \Pi(K)$. Let Γ_M be the set of irreducible representations of K_M which are constituents of the restrictions of Γ to K_M. The theorem will follow if we can show for some $N_M > 0$, that $\phi^b_{M,\mu}$ maps $\mathscr{H}_N(G(F_S))_\Gamma$ continuously to $\mathscr{I}_{N_M}(M(F_S))_{\Gamma_M}$.

In order to prove that a function

$$\phi^b_{M,\mu}(f), \qquad f \in \mathscr{H}_N(G(F_S))_\Gamma,$$

belongs to a space $\mathscr{I}_{N_M}(M(F_S))_{\Gamma_M}$, we must establish three conditions. The first condition is just (12.1), while the second follows immediately from Frobenius reciprocity and the definition of Γ_M. The third condition, of course, is the main point. Fix $\mathscr{M}$ and σ as above. We shall show that the function

$$\mathscr{X} \to \phi^b_{M,\mu}(f, \sigma, \mathscr{X}), \qquad \mathscr{X} \in \mathfrak{a}_{\mathscr{M},S}, \tag{12.3}$$

belongs to the space $C^\infty_{N_M}(\mathfrak{a}_{\mathscr{M},S})$, with N_M depending only on N and Γ, and that it varies continuously with $f \in \mathscr{H}_N(G(F_S))_\Gamma$. This will establish the third condition, and complete the proof of the theorem.

We shall combine Proposition 10.1 with Corollary 9.2. However, we first observe that it is sufficient to prove the assertion with μ replaced by any of the points $\varepsilon_P + \mu$ in the formula for $\phi^b_{M,\mu}(f, \sigma, \mathscr{X})$ above. We may therefore assume that each function $J_M(\sigma_\Lambda, f)$ is analytic for $\Lambda \bullet (\mu + i\mathfrak{a}^*_{\mathscr{M},S})$. Consequently,

$$\phi^b_{M,\mu}(f, \sigma, \mathscr{X}) = b(h_M(\mathscr{X})) \int_{\mu + i\mathfrak{a}^*_{\mathscr{M},S}} J_M(\sigma_\Lambda, f)\, e^{-\Lambda(\mathscr{X})}\, d\Lambda.$$

Next, we assign a chamber $c_{\mathscr{L}}$ in $\mathfrak{a}_{\mathscr{L}}$ to each $\mathscr{L} \supset \mathscr{M}$. There are only finitely many such assignments, and $\mathfrak{a}_{\mathscr{M},S}$ is the corresponding finite union of the sets

$$\{\mathscr{X} \in \mathfrak{a}_{\mathscr{M},S} : h_{\mathscr{L}}(\mathscr{X}) \in \bar{c}_{\mathscr{L}},\ \mathscr{L} \supset \mathscr{M}\}. \tag{12.4}$$

We may therefore assume that $\mathscr{X}$ actually belongs to a given set (12.4). For each $\mathscr{L}$, let $c^*_{\mathscr{L}}$ be the associated chamber in $\mathfrak{a}^*_{\mathscr{L}}$, and let $\nu_{\mathscr{L}}$ be a highly regular point in general position in $c^*_{\mathscr{L}}$. Applying Proposition 10.1, with $\mathscr{N} = \{\nu_{\mathscr{L}}\}$, we see that $\phi^b_{M,\mu}(f, \sigma, \mathscr{X})$ equals

$$b(h_M(\mathscr{X})) \sum_{\mathscr{L} \supset \mathscr{M}} \sum_{\Omega \in R_{\mathscr{L}}} \int_{\nu_{\mathscr{L}} + i\mathfrak{a}^*_{\mathscr{L},S}} \operatorname*{Res}_{\Omega, \Lambda \to \Lambda_\Omega + \lambda} (e^{-\Lambda(\mathscr{X})} J_M(\sigma_\Lambda, f))\, d\lambda.$$

Corollary 9.2 (with $a_\Lambda = e^{-\Lambda(\mathscr{X})}$) then provides an important condition on $\mathscr{L}$ in order that the integral not vanish. We are thus able to write $\phi^b_{M,\mu}(f, \sigma, \mathscr{X})$ as the sum over those $\mathscr{L} \supset \mathscr{M}$ with

$$\ker(h_{\mathscr{L}}) \cap \ker(h_M) = \{0\}, \tag{12.5}$$

and over $\Omega \in R_{\mathscr{S}}$, of the product of $b(h_M(\mathscr{X}))$ with

$$\int_{\nu_{\mathscr{S}} + i\mathfrak{a}^*_{\mathscr{S},S}} \operatorname*{Res}_{\Omega, \Lambda \to \Lambda_\Omega + \lambda} (e^{-\Lambda(\mathscr{X})} \operatorname{tr}(\mathscr{R}_M(\sigma^M_\Lambda, P_0) \mathscr{I}_{P_0}(\sigma^M_\Lambda, f))) \, d\lambda. \quad (12.6)$$

As always, P_0 is any fixed element in $\mathscr{P}(M)$.

Our next step is to deform the contour of integration in (12.6). Our assumption on $\nu_{\mathscr{S}}$ ensures that none of the singularities of the integrand meets the tube over the translated chamber $(\nu_{\mathscr{S}} + c^*_{\mathscr{S}})$. Now a standard argument shows that there is a constant A, depending at most on Γ, such that

$$\|\mathscr{I}_{P_0}(\sigma^M_\Lambda, f)\|_\sigma \leqslant c_n(f) \, e^{AN\|\mathrm{Re}(\Lambda)\|}(1 + \|\mathrm{Im}(\Lambda_\infty)\|)^{-n}, \quad (12.7)$$

for any $f \in \mathscr{H}_N(G(F_S))_\Gamma$, $n \in \mathbb{R}$ and $\Lambda \in \mathfrak{a}^*_{\mathscr{M},\mathbb{C}}$. (See, for example, the first steps in the proof of [1(d), Lemma III.3.1].) Here $\|\cdot\|_\sigma$ is any norm on the finite-dimensional space $\mathscr{V}_{P_0}(\sigma)_\Gamma$, $c_n(\cdot)$ is a continuous semi-norm on $\mathscr{H}_N(G(F_S))_\Gamma$, and Λ_∞ is the projection of Λ onto

$$\bigoplus_{v \in S \cap S_\infty} \mathfrak{a}^*_{M_v,\mathbb{C}}.$$

Since $\mathscr{R}_M(\sigma^M_\Lambda, P_0)$ is a rational expression in the variables (6.3), the function

$$\operatorname{tr}(\mathscr{R}_M(\sigma^M_\Lambda, P_0) \mathscr{I}_{P_0}(\sigma^M_\Lambda, f))$$

satisfies a similar estimate for Λ in the tube over $(\nu_{\mathscr{S}} + c^*_{\mathscr{S}})$. We can therefore deform the contour of integration in (12.6) to $t\nu_{\mathscr{S}} + i\mathfrak{a}^*_{\mathscr{S},S}$, where t is a real number which approaches infinity. If $\mathscr{X}$ belongs to the support of (12.6), we obtain an inequality

$$|\nu_{\mathscr{S}}(\mathscr{X})| \leqslant AN\|\nu_{\mathscr{S}}\|,$$

with A depending at most on Γ. But we are already assuming that $h_{\mathscr{S}}(\mathscr{X})$ belongs to the closure of $c_{\mathscr{S}}$. Since $\nu_{\mathscr{S}}$ is strictly positive on the complement of the origin in this set, we can estimate $\|h_{\mathscr{S}}(\mathscr{X})\|$ in terms of $|\nu_{\mathscr{S}}(\mathscr{X})|$. Consequently, there is an A_1, depending only on Γ, such that

$$\|h_{\mathscr{S}}(\mathscr{X})\| \leqslant A_1 N, \quad (12.8)$$

if $\mathscr{X}$ belongs to the support of (12.6).

The condition (12.5) implies that

$$\|\mathscr{X}\| \leqslant \|h_{\mathscr{S}}(\mathscr{X})\| + \|h_M(\mathscr{X})\|. \quad (12.9)$$

Imposing the additional requirement that $b(h_M(\mathscr{X})) \neq 0$, we then combine (12.8) and (12.9). We obtain

$$\|\mathscr{X}\| \leqslant A_1 N + A_2,$$

for a fixed constant A_2. It follows from this that the original function (12.3) is supported on a ball whose radius depends only on N and Γ.

The proof of the theorem is essentially complete. The only additional point is to establish the continuous dependence of (12.3) on f. In this regard, it is simplest to represent the value of a continuous semi-norm on the function (12.3) in terms of the Fourier transform. The required inequality then follows easily from the estimate (12.7). Thus, the properties of (12.3) are as promised, and the theorem is proved. ∎

COROLLARY 12.2. *For each* $\mu \in \mathfrak{a}^*_{\mathscr{M}}$, $\phi_{M,\mu}$ *maps moderate functions in* $\mathscr{H}_{\mathrm{ac}}(G(F_S))$ *to moderate functions in* $\mathscr{I}_{\mathrm{ac}}(M(F_S))$.

Proof. Suppose that f is a moderate function in $\mathscr{H}_{\mathrm{ac}}(G(F_S))$. In order to show that $\phi_{M,\mu}(f)$ is a moderate function in $\mathscr{I}_{\mathrm{ac}}(M(F_S))$, we must verify two conditions. For the support condition, we must look back at the proof of the theorem. Note that the integral (12.6) depends only on the function

$$f^Z, \qquad Z = h_G(\mathscr{X}).$$

By assumption, f is supported on a set

$$\{x \in G(F_S) \colon \log\|x\| \leqslant c(\|H_G(x)\| + 1)\},$$

so we may identify f^Z with the restriction to $G(F_S)^Z$ of a function in $\mathscr{H}_N(G(F_S))$, where

$$N = c(\|Z\| + 2) = c(\|h_G(\mathscr{X})\| + 2).$$

The inequality (12.8) can therefore be written

$$\|h_{\mathscr{S}}(\mathscr{X})\| \leqslant A_1 c(\|h_G(\mathscr{X})\| + 2).$$

Combined with (12.9), this becomes

$$\|\mathscr{X}\| \leqslant A_1 c(\|h_G(\mathscr{X})\| + 2) + \|h_M(\mathscr{X})\|.$$

Since

$$\|h_G(\mathscr{X})\| \leqslant \|h_M(\mathscr{X})\|,$$

the function

$$\mathscr{X} \to \phi_{M,\mu}(f, \sigma, \mathscr{X}), \qquad \mathscr{X} \in \mathfrak{a}_{\mathscr{M},S},$$

is supported on a set

$$\{\mathscr{X} \in \mathfrak{a}_{\mathscr{M},S} : \|\mathscr{X}\| \leqslant c_1(\|h_M(\mathscr{X})\| + 1)\}.$$

This is the required support condition. The growth condition on $\phi_{M,\mu}(f, \sigma, \mathscr{X})$ is a routine matter. It follows easily from the given growth condition on f and the appropriate variant of the estimate (12.7). Therefore, $\phi_{M,\mu}(f)$ is a moderate function. ∎

COROLLARY 12.3 *The linear transformation ϕ_M maps $\mathscr{H}_{\mathrm{ac}}(G(F_S))$ continuously to $\mathscr{I}_{\mathrm{ac}}(M(F_S))$. The image of a moderate function in $\mathscr{H}_{\mathrm{ac}}(G(F_S))$ is a moderate function in $\mathscr{I}_{\mathrm{ac}}(M(F_S))$.*

Proof. This of course is just the special case of the theorem in which $\mu = 0$. ∎

APPENDIX

We shall prove Proposition 3.1. There are two steps. The first is a straightforward examination of the constants that appear in Harish-Chandra's explicit formula for the μ-function. The second is an interpretation of the sign ε_π in terms of a certain abelian character value that appears in the work of Harish-Chandra. This is an unpublished lemma of Langlands.

We adopt the notation of Section 3. In particular, $G = G^0$, M is maximal Levi subgroup of G, $P = MN_P$ is a group in $\mathscr{P}(M)$, and T is a maximal torus of M over $\mathbb{R}$ with real split component A_M. In addition, we have the cuspidal map ϕ from $W_{\mathbb{R}}$ to ${}^LM$ with

$$\lambda^\vee(\phi(z)) = z^{\langle \mu, \lambda^\vee \rangle} \bar{z}^{\langle \nu, \lambda^\vee \rangle}, \qquad z \in \mathbb{C}^*,\ \lambda^\vee \in L^\vee,$$

and

$$h = \phi(\sigma) = a \rtimes \sigma.$$

We shall write $\Sigma(M, T)$ and $W_{\mathbb{R}}(M, T)$ respectively for the set of roots and the real Weyl group of (M, T). Let ρ_M equal one-half the sum of the positive roots in $\Sigma(M, T)$ with respect to some fixed order. We should first recall how Langlands attaches a packet $\{\pi\}$ of cuspidal representations of $M(\mathbb{R})$ to ϕ. Choose a point $\lambda_0 \in L \otimes \mathbb{C}$ such that

$$\lambda^\vee(a) = e^{2\pi i \langle \lambda_0, \lambda^\vee \rangle}$$

for every element $\lambda^\vee \in L^\vee$ such that $\langle \alpha, \lambda^\vee \rangle = 0$ for all the roots α in $\Sigma(M, T)$. We have fixed an isomorphism of $T(\mathbb{C})$ with $L^\vee \otimes \mathbb{C}^*$, and we use this to identify the complex Lie algebra $\mathfrak{t}_{\mathbb{C}}$ with $L^\vee \otimes \mathbb{C}$. If

$$t = \exp H, \qquad H \in L^\vee \otimes \mathbb{C},$$

is any point in $T(\mathbb{C})$, define

$$\chi(t) = e^{\langle \lambda_0, H - \bar{\sigma} H \rangle} e^{(1/2)\langle \mu - \rho_M, H + \bar{\sigma} H \rangle}. \tag{A.1}$$

Then the packet consists of those cuspidal representations π of $M(\mathbb{R})$ whose character values at regular points $t \in T(\mathbb{R})$ are of the form

$$\pm \sum_{s \in W(M,T)} \varepsilon(s) \chi(s^{-1} t)\, t^{(s\rho_P - \rho_P)} \Big(\prod_{\beta \in \Sigma(M,T)} (1 - t^{-\beta}) \Big)^{-1}.$$

Because he uses different measures on the groups $N_P(\mathbb{R})$, Harish-Chandra's μ-function actually equals

$$\alpha^2_{\bar{P}|P} \gamma^2_{\bar{P}|P} (J_{\bar{P}|P}(\pi)^* J_{\bar{P}|P}(\pi))^{-1} \tag{A.2}$$

in our notation, where $\gamma_{\bar{P}|P}$ is the constant defined in [11(c). Sect. 2]. We must examine Harish-Chandra's explicit formula in [11(c)] for this expression.

There are two cases to consider. Assume first of all that $\dim N_P(\mathbb{R})$ is even. Then $T(\mathbb{R})$ is fundamental in $G(\mathbb{R})$. The expression (A.2) equals

$$\gamma_{\bar{P}|P} \frac{|W_{\mathbb{R}}(G, T)|}{|W_{\mathbb{R}}(M, T)|} \cdot \frac{c_M}{c_G} \cdot \Big| \prod_{\alpha \in \Sigma_P(G,T)} B(\alpha, \mu) \Big|$$

in the notation of [11(c), Theorem 24.1]. We have used the fact that the Weyl group of A_M is isomorphic to $W_{\mathbb{R}}(G, T)/W_{\mathbb{R}}(M, T)$. By [11(a), Lemma 37.3],

$$\begin{aligned} \frac{c_M}{c_G} = {} & (2\pi)^{r_M - r_G} \cdot 2^{(1/2)(\nu_M - \nu_G)} (v(T(\mathbb{R}) \cap K)/v(M(\mathbb{R}) \cap K)) \\ & \times (v(T(\mathbb{R}) \cap K)/v(K))^{-1} \frac{|W_{\mathbb{R}}(M, T)|}{|W_{\mathbb{R}}(G, T)|}, \end{aligned}$$

where

$$\begin{aligned} r_M - r_G &= \tfrac{1}{2}\{\dim(M/T) - \dim(G/T)\} = -\dim N_P, \\ \nu_M - \nu_G &= \tfrac{1}{2}\{\dim(M(\mathbb{R})/K \cap M(\mathbb{R})) - \operatorname{rank}(M(\mathbb{R})/K \cap M(\mathbb{R})) \\ &\qquad - \dim(G(\mathbb{R})/K) + \operatorname{rank}(G(\mathbb{R})/K)\} \\ &= -\tfrac{1}{2} \dim N_P, \end{aligned}$$

and

$$(v(T(\mathbb{R})\cap K)/v(M(\mathbb{R})\cap K))(v(T(\mathbb{R})\cap K)/v(K))^{-1}=v(M(\mathbb{R})\cap K\backslash K),$$

the volume of $M(\mathbb{R})\cap K\backslash K$ with respect to the invariant measure dk defined by the Euclidean structure (3.1) on $\mathfrak{k}\cap\mathfrak{m}\backslash\mathfrak{k}$. Now from [11(b), p. 45] we know that

$$v(M(\mathbb{R})\cap K\backslash K)=c\gamma_{\bar{P}|P},$$

where c is the positive constant such that

$$\int_{K\cap M(\mathbb{R})\backslash K}\phi(k)\,dk=c\int_{\bar{N}_P(\mathbb{R})}e^{2\rho_P(H_P(\bar{n}))}\,\phi(K_P(\bar{n}))\,d\bar{n},$$

for every $\phi\in C_c^\infty(K\cap M(\mathbb{R})\backslash K)$. To evaluate c, observe that if $\{X_i\colon 1\leqslant i\leqslant \dim N_P$ is an orthogonal basis of $\bar{\mathfrak{n}}_P$, $\{2^{-1/2}(X_i+\theta X_i)\colon 1\leqslant i\leqslant\dim N_P\}$ is an orthonormal basis of $\mathfrak{k}\cap\mathfrak{m}\backslash\mathfrak{k}$. Taking ϕ to be supported in a small neighbourhood of 1 we find that $c=2^{(1/2)\dim N_P}$. It follows that the expression (A.2) equals

$$\begin{aligned}&(2\pi)^{-\dim N_P}\gamma_{\bar{P}|P}^2\left|\prod_{\alpha\in\Sigma_P(G,T)}B(\alpha,\mu)\right|\\&\quad=\gamma_{\bar{P}|P}^2\cdot\alpha_{\bar{P}|P}^2(2\pi)^{-\dim N_P}\left|\prod_{\alpha\in\Sigma_P(G,T)}\mu(\alpha^\vee)\right|,\end{aligned}$$

where $\alpha^\vee$ is the co-root of α.

Next suppose that $\dim N_P$ is odd. Then there is a positive real root α_0 of (G,T). We choose a basis (H',X',Y') of the derived algebra of the centralizer of $\mathfrak{t}\cap\mathfrak{k}$ in $\mathfrak{g}$ as in [11(c), Sect. 30]. Define $\gamma=\exp\pi(X'-Y')$ as in [11(c), Sect. 30] and let B be the maximal torus obtained from T by Cayley transform. Then $B(\mathbb{R})/A_G(\mathbb{R})$ is compact. Let $\Sigma_c(G,T)$ denote the complement of $\{\alpha_0\}$ in $\Sigma_P(G,T)$, which is just the set of complex roots in $\Sigma_P(G,T)$. Then the expression (A.2) equals [11(c), p. 190]

$$\begin{aligned}&\frac{c_M}{c_G}\cdot\frac{[G(\mathbb{R}):G(\mathbb{R})^0\,T(\mathbb{R})]\,|W_{\mathbb{R}}(G(\mathbb{R})^0,B(\mathbb{R})^0)|}{|W_{\mathbb{R}}(M,T)|}\cdot B(\alpha_0,\alpha_0)^{1/2}\\&\quad\cdot\gamma_{\bar{P}|P}\cdot\mu_0(\chi)\cdot\left|\prod_{\alpha\in\Sigma_c(G,T)}B(\alpha,\mu)\right|,\end{aligned}$$

where

$$\begin{aligned}\mu_0(\chi)=&\left(\frac{\pi\mu(\alpha_0^\vee)}{i}\right)\sinh\left(\frac{\pi\mu(\alpha_0^\vee)}{i}\right)\\&\times\left\{\cosh\left(\frac{\pi\mu(\alpha_0^\vee)}{i}\right)-\frac{1}{2}(-1)^{\rho_P(\alpha_0^\vee)}(\chi(\gamma)+\chi(\gamma^{-1}))\right\}^{-1},\end{aligned}$$

and $W_{\mathbb{R}}(G(\mathbb{R})^0, B(\mathbb{R})^0)$ is the subgroup of $W_{\mathbb{R}}(G, B)$ induced from elements in the connected component $G(\mathbb{R})^0$ of $G(\mathbb{R})$.

By [11(a), Lemma 37.3],

$$\begin{aligned}\frac{c_M}{c_G} = (2\pi)^{r_M - r_G} \cdot 2^{(1/2)(v_M - v_G)} \\ \times (v(T(\mathbb{R}) \cap K)/v(M(\mathbb{R}) \cap K))(v(B(\mathbb{R}) \cap K)/v(K))^{-1} \\ \times \frac{|W_{\mathbb{R}}(M, T)|}{|W_{\mathbb{R}}(G, B)|},\end{aligned}$$

where again

$$r_M - r_G = \tfrac{1}{2}\{\dim(M/T) - \dim(G/B)\} = -\dim N_P,$$

and

$$\begin{aligned}v_M - v_G &= \tfrac{1}{2}\{\dim(M(\mathbb{R})/K \cap M(\mathbb{R})) - \operatorname{rank}(M(\mathbb{R})/K \cap M(\mathbb{R})) \\ &\quad - \dim(G(\mathbb{R})/K) + \operatorname{rank}(G(\mathbb{R})/K)\} \\ &= -\tfrac{1}{2}\dim N_P.\end{aligned}$$

Also,

$$\begin{aligned}&(v(T(\mathbb{R}) \cap K)/v(M(\mathbb{R}) \cap K))(v(B(\mathbb{R}) \cap K)/v(K))^{-1} \\ &\quad = v(T(\mathbb{R}) \cap K)/v(B(\mathbb{R}) \cap K) \cdot v(K \cap M(\mathbb{R}) \backslash K).\end{aligned}$$

Repeating an argument above, we obtain

$$v(K \cap M(\mathbb{R}) \backslash K) = 2^{(1/2)\dim NP} \cdot \gamma_{\bar{P}|P}.$$

We can write

$$\begin{aligned}&v(T(\mathbb{R}) \cap K)/v(B(\mathbb{R}) \cap K) \\ &\quad = v(T(\mathbb{R}) \cap K^0) \cdot [K^0(K \cap T(\mathbb{R})) : K^0] \\ &\qquad \cdot v(B(\mathbb{R}) \cap K^0)^{-1} \cdot [K^0(K \cap B(\mathbb{R})) : K^0]^{-1} \\ &\quad = v(B(\mathbb{R})^0/T \cap B(\mathbb{R})^0)^{-1} [G(\mathbb{R})^0 T(\mathbb{R}) : G(\mathbb{R})^0] \\ &\qquad \cdot [G(\mathbb{R})^0 B(\mathbb{R}) : G(\mathbb{R})^0]^{-1}\end{aligned}$$

since $B(\mathbb{R}) \cap K^0$ is connected. It follows that

$$\frac{c_M}{c_G} \frac{[G(\mathbb{R}) : G(\mathbb{R})^0\, T(\mathbb{R})]\, |W_{\mathbb{R}}(G(\mathbb{R})^0, B(\mathbb{R})^0)|}{|W_{\mathbb{R}}(M, T)|}$$
$$= (2\pi)^{-\dim N_P} \gamma_{\bar{P}|P} \cdot \frac{|W_{\mathbb{R}}(G(\mathbb{R})^0, B(\mathbb{R})^0)|}{|W_{\mathbb{R}}(G, B)|}$$
$$\times \frac{[G(\mathbb{R}) : G(\mathbb{R})^0]}{[G(\mathbb{R})^0\, B(\mathbb{R}) : G(\mathbb{R})^0]} \cdot v(B(\mathbb{R})^0/T(\mathbb{R}) \cap B(\mathbb{R})^0)^{-1}$$
$$= (2\pi)^{-\dim N_P} \gamma_{\bar{P}|P} v(B(\mathbb{R})^0/T(\mathbb{R}) \cap B(\mathbb{R})^0)^{-1}.$$

It is a consequence of the discussion of [11(c), Sect. 30] that $\{\exp \theta(X' - Y')\colon 0 \leqslant \theta \leqslant \pi\}$ is a set of representatives of $B(\mathbb{R})^0/T(\mathbb{R}) \cap B(\mathbb{R})^0$, and that

$$B(X' - Y', X' - Y')^{1/2} = 2B(\alpha_0, \alpha_0)^{-1/2}.$$

Therefore,

$$v(B(\mathbb{R})^0/T(\mathbb{R}) \cap B(\mathbb{R})^0) = 2\pi B(\alpha_0, \alpha_0)^{-1/2}.$$

It follows that the expression (A.2) equals

$$(2\pi)^{-\dim N_P} \gamma^2_{\bar{P}|P} \cdot B(\alpha_0, \alpha_0) \cdot \frac{\mu_0(\chi)}{2\pi} \cdot \left| \prod_{\alpha \in \Sigma_c(G, T)} B(\alpha, \mu) \right|$$
$$= \gamma^2_{\bar{P}|P} \alpha^2_{\bar{P}|P} (2\pi)^{-\dim N_P} \cdot \frac{\mu_0(\chi)}{\pi} \left| \prod_{\alpha \in \Sigma_c(G, T)} \mu(\alpha^\vee) \right|.$$

Since $\gamma^2 = 1$, the number

$$\varepsilon'_\pi = -\tfrac{1}{2}(-1)^{\rho_P(\alpha_0^\vee)}(\chi(\gamma) + \chi(\gamma^{-1})) \tag{A.3}$$

equals ± 1. If it equals 1,

$$\mu_0(\chi) = \frac{\pi\mu(\alpha_0^\vee)}{i}\, 2\sinh\left(\frac{\pi\mu(\alpha_0^\vee)}{2i}\right) \cosh\left(\frac{\pi\mu(\alpha_0^\vee)}{2i}\right) \frac{1}{2} \cosh\left(\frac{\pi\mu(\alpha_0^\vee)}{2i}\right)^{-2}$$
$$= \frac{\pi\mu(\alpha_0^\vee)}{i} \tanh\left(\frac{\pi\mu(\alpha_0^\vee)}{2i}\right).$$

Similarly if $\varepsilon'_\pi = -1$,

$$\mu_0(\chi) = \frac{\pi\mu(\alpha_0^\vee)}{i} \coth\left(\frac{\pi\mu(\alpha_0^\vee)}{2i}\right).$$

Set $\varepsilon'_\pi = 0$ if there is no real root α_0. Then collecting the facts above, we see that

$$(J_{\bar{P}|P}(\pi)^* J_{\bar{P}|P}(\pi))^{-1}$$

equals

$$(2\pi)^{-\dim N_P} \prod_{\alpha \in \Sigma_P(G,T)} |\mu(\alpha^\vee)| \left|\tanh\left(\frac{\pi\mu(\alpha_0^\vee)}{2i}\right)\right|^{\varepsilon'_\pi}.$$

Proposition 3.1 will then follow from

LEMMA A.1 (Langlands). $\varepsilon'_\pi = \varepsilon_\pi$.

Proof. Associated with the embedding into $\mathfrak{g}$ of the Lie algebra spanned by (H', X', Y') we have a homomorphism of SL_2 into G. The co-root $\alpha_0^\vee$ can be defined as the composition

$$z \to \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \to G.$$

It follows that $\gamma = \alpha_0^\vee(-1)$. Since

$$\lambda(\alpha_0^\vee(-1)) = (-1)^{\langle \lambda, \alpha_0^\vee \rangle} = e^{\langle \lambda, \pi i \alpha_0^\vee \rangle}, \qquad \lambda \in L,$$

we have

$$\tfrac{1}{2}(\chi(\gamma) + \chi(\gamma^{-1})) = \chi(\alpha_0^\vee(-1)) = e^{2\pi i \langle \lambda_0, \alpha_0^\vee \rangle}.$$

Let $a = a't$ where $a' \in ({}^LM^0)_{\mathrm{der}}$, the derived subgroup of ${}^LM$, and $t \in {}^LT^0$. Since α_0 is real, $\alpha_0^\vee$ extends to a character on ${}^LM$ which must vanish on $({}^LM^0)_{\mathrm{der}}$. Then

$$\mathrm{Ad}(\phi(\sigma))\, X_{\alpha_0^\vee} = e^{2\pi i \langle \lambda_0, \alpha_0^\vee \rangle}\, \mathrm{Ad}(a' \rtimes \sigma)\, X_{\alpha_0^\vee}.$$

We have only to show that

$$\mathrm{Ad}(a' \rtimes \sigma)\, X_{\alpha_0^\vee} = -(-1)^{\rho_P(\alpha_0^\vee)} X_{\alpha_0^\vee}.$$

This is a statement about a real reductive group G, a Levi component M of a maximal parabolic subgroup, a Cartan subgroup T of M with T/A_M compact, a real root α_0 of (G, T), and any element a' in $({}^LM^0)_{\mathrm{der}}$ normalizing ${}^LT^0$ such that $a' \rtimes \sigma$ acts as -1 on the roots of $({}^LM^0, {}^LT^0)$. We leave the reader to check that it holds if the derived group of $G(\mathbb{R})$ is locally isomorphic with $SL(2, \mathbb{R})$ or $SU(2, 1)$. We prove the statement in general by induction on the dimension of G. Let $\beta^\vee$ be the largest root of one of the simple factors of $({}^LM^0)_{\mathrm{der}}$. Let ${}^LG_1^0$ be the connected subgroup whose

Lie algebra is generated by the Lie algebra ${}^L\mathfrak{t}$ of ${}^LT$ and $\{X_{\alpha^\vee}: \beta^\vee(\alpha)=0\}$. Define ${}^LG_1 = {}^LG_1^0 \rtimes W_{\mathbb{R}}$, a subgroup of ${}^LG$, and set ${}^LM_1 = {}^LM \cap {}^LG_1$. Let ${}^LJ^0$ be the connected subgroup of ${}^LM^0$ whose Lie algebra is generated by ${}^L\mathfrak{t}$ and $\{X_{\beta^\vee}, X_{-\beta^\vee}\}$. Let a_1 be an element of ${}^LM_1^0$ which normalizes ${}^LT^0$ and takes positive roots to negative roots. Let a_2 be an element in ${}^LJ^0$ that normalizes ${}^LT^0$ and takes $\beta^\vee$ to $-\beta^\vee$. Then a_1 and a_2 commute, and $({}^LG_1, {}^LM_1, {}^LT, \alpha_0^\vee, a_1 \rtimes \sigma)$ satisfies the same assumptions as $({}^LG, {}^LM, {}^LT, \alpha_0^\vee, a' \rtimes \sigma)$. Moreover, we may assume $a' = a_1 a_2$ [15(a), p. 47]. After applying our induction hypothesis, we have only to prove that $\mathrm{Ad}(a_2) X_{\alpha_0^\vee}$ equals $X_{\alpha_0^\vee}$ times (-1) raised to the exponent

$$\frac{1}{2} \sum_{\{\gamma>0:\, \gamma(\beta^\vee)\neq 0\}} \gamma(\alpha_0^\vee). \tag{A.4}$$

Suppose that $\gamma>0$, $\gamma(\beta^\vee)\neq 0$, $\gamma(\alpha_0^\vee)\neq 0$, and $\gamma^\vee$ is not in the plane spanned by $\alpha_0^\vee$ and $\beta^\vee$. If $\gamma^\vee = \mathrm{Ad}(a_2)\gamma^\vee$ we would have $\gamma = \mathrm{Ad}(a_2)\gamma$ and $\gamma(\beta^\vee)=0$. If $\gamma^\vee = \mathrm{Ad}(\phi(\sigma))\gamma^\vee$ we would have $\gamma^\vee = \alpha_0^\vee$. Finally, if $\gamma^\vee = \mathrm{Ad}(a_2\phi(\sigma))\gamma^\vee$, $\gamma^\vee$ would be in the plane of $\alpha_0^\vee$ and $\beta^\vee$. These three possibilities are all impossible. It follows that $\gamma^\vee$, $\mathrm{Ad}(a_2)\gamma^\vee$, $\mathrm{Ad}(\phi(\sigma))\gamma^\vee$, and $\mathrm{Ad}(a_2\phi(\sigma))\gamma^\vee$ are all distinct and positive. Since

$$\gamma(\alpha_0^\vee) = (\mathrm{Ad}(a_2)\gamma)(\alpha_0^\vee) = (\mathrm{Ad}(\phi(\sigma))\gamma)(\alpha_0^\vee) = (\mathrm{Ad}(a_2\phi(\sigma))\gamma)(\alpha_0^\vee),$$

the summands for these four roots may be dropped from (A.4). We can also drop those γ with $\gamma(\alpha_0^\vee)=0$.

Thus, the sum in (A.4) can be taken over those $\gamma>0$, $\gamma(\beta^\vee)\neq 0$, which lie in the plane spanned by α_0 and β. This becomes a sum over positive roots of a Lie algebra of rank two. On the other hand, $X_{\alpha_0^\vee}$ is the zero weight vector in an irreducible $({}^LJ^0)_{\mathrm{der}}$ module of odd dimension, say $2n+1$. Since a_2 represents the nontrivial Weyl group element in $({}^LJ^0)_{\mathrm{der}}$,

$$\mathrm{Ad}(a_2) X_{\alpha_0^\vee} = (-1)^n X_{\alpha_0^\vee}.$$

We can calculate the integer (A.4), case by case, from each of the diagrams on [6, p. 276] which have a pair (α_0, β) of orthogonal roots. We calculate n from the corresponding dual diagram. It follows easily that the difference of these two integers is even. ∎

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