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THE INVARIANT TRACE FORMULA. II. GLOBAL THEORY

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CONTENTS

INTRODUCTION

The purpose of this article is to prove an explicit invariant trace formula. In the preceding paper [1(j)], we studied two families of invariant distributions. Now we shall exhibit these distributions as terms on the two sides of the invariant trace formula. We refer the reader to the introduction of [1(j)], which contains a general discussion of the problem. In this introduction, we shall describe the formula in more detail.

Let G be a connected reductive algebraic group over a number field F, and let f be a function in the Hecke algebra on $G(\mathbb{A})$. We already have a "coarse" invariant trace formula

$$\sum_{\mathfrak{o}\in\mathcal{O}} I_{\mathfrak{o}}(f) = \sum_{\chi\in\mathcal{X}} I_{\chi}(f), \tag{1}$$

which was established in an earlier paper [1(c)]. This will be our starting point here. The terms on each side of (1) are invariant distributions, but as they stand, they are not explicit enough to be very useful. After recalling the formula (1) in §2, we shall study the two sides separately in §§3 and 4. These two sections

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are the heart of the paper. Building on earlier investigations of noninvariant distributions [1(e), 1(g)], we shall establish finer expansions for each side of (1). The resulting identity

$$\begin{aligned}\sum_M |W_0^M||W_0^G|^{-1} \sum_{\gamma\in(M(F))_{M,S}} a^M(S,\gamma)I_M(\gamma,f)\\ = \sum_{t\geq 0}\sum_M |W_0^M||W_0^G|^{-1}\int_{\Pi(M,t)} a^M(\pi)I_M(\pi,f)\,d\pi\end{aligned} \tag{2}$$

will be our explicit trace formula. The terms $I_M(\gamma,f)$ and $I_M(\pi,f)$ in (2) are essentially the invariant distributions studied in [1(j)]. The functions $a^M(S,\gamma)$ and $a^M(\pi)$ depend only on a Levi subgroup M, and are global in nature. They are strongly dependent on the discrete subgroup $M(F)$ of $M(\mathbb{A})$. We refer the reader to §§3 and 4 for more detailed description of these objects, as well as the sets $(M(F))_{M,S}$ and $\Pi(M,t)$.

The paper [1(c)] relied on certain hypotheses in local harmonic analysis. Some of these have since been resolved by the trace Paley-Wiener theorems. Others concern the density of characters in spaces of invariant distributions, and are not yet known is general. In fact, to even define the invariant distributions $I_M(\gamma,f)$ and $I_M(\pi,X,f)$, we had to introduce an induction hypothesis in [1(j)]. This hypothesis remained in force throughout [1(j)], and will be carried into this paper. We shall finally settle the matter in §5. We shall show that the invariant distributions in the trace formula are all supported on characters. Using [1(j), Theorem 6.1] we shall first establish in Lemma 5.2 that the distributions on the right-hand side of (2) have the required property. We shall then use formula (2) itself to deduce the same property of the distributions on the left (Theorem 5.1). This is a generalization of an argument introduced by Kazhdan in his Maryland lectures (see [8, 10]). Theorem 6.1 (in [1(j)]) and Theorem 5.1 (here) are actually simple versions of a technique that can be applied more generally. They provide a good introduction to the more complicated versions used for base change [2, §§II.10, II.17].

It is not known whether the right-hand side of (2) converges as a double integral over t and π. It is a difficulty which originates with the Archimedean valuations of F. On the other hand, some result of this nature will definitely be required for many of the applications of the trace formula. In §6 we shall prove a weak estimate (Corollary 6.5) for the rate of convergence of the sum over t. It will be stated in terms of multipliers for the Archimedean part $\prod_{v\in S_\infty} G(F_v)$ of $G(\mathbb{A})$. One would then hope that by varying the multipliers, one could separate the terms according to their Archimedean infinitesimal character. For base change, this is in fact what happens. One can use the estimate to eliminate the problems caused by the Archimedean primes [2, §II.15]. In general, Corollary 6.5 seems to be a natural device for isolating the contributions of a given infinitesimal character.

It is useful to have simple versions of the trace formula for functions

$$f = \prod_v f_v$$

that are suitably restricted. Since the terms in (2) are all invariant distributions, we will be able to impose conditions on f strictly in terms of its orbital integrals. If at one place v the semisimple orbital integrals of f_v are supported on the elliptic set, then all the terms with $M \neq G$ on the right-hand side of (2) vanish. If the same thing is true at two places, the terms with $M \neq G$ on the left-hand side also vanish. These two assertions comprise Theorem 7.1. They are simple consequences of the descent and splitting formulas in [1(j), §§8-9]. We shall also see that in certain cases the remaining terms take a particularly simple form (Corollaries 7.3, 7.4, 7.5).

As with the preceding paper [1(j)], we shall conclude (§8) by discussing the example of $GL(n)$. Groups related to $GL(n)$ by inner twisting or cyclic base change are the simplest examples of general rank for which one can attempt a comparison of trace formulas. However, one must first establish some properties of the trace formula of $GL(n)$ itself. By imposing less stringent conditions than those of §7, we shall establish more delicate vanishing properties. The resulting formula for $GL(n)$ is then what should be compared with the twisted trace formula over a cyclic extension.

1. Assumptions on G

Let G be a connected component of a reductive algebraic group over a number field F. We assume that $G(F) \neq \emptyset$. As in previous papers, we shall write G^+ for the group generated by G, and G^0 for the connected component of 1 in G^+. The component G/F will remain fixed throughout the paper except in §5.

We shall fix a minimal Levi subgroup M_0 of G^0 over F. This was the point of view in the paper [1(g)], and we shall freely adopt the notation at the beginning of [1(g)]. In particular, we have the maximal F-split torus $A_0 = A_{M_0}$ of G^0 and the real vector space $\mathfrak{a}_0 = \mathfrak{a}_{M_0}$. On $\mathfrak{a}_0$, we fix a Euclidean norm which is invariant under the restricted Weyl group W_0 of G^0. We also have the finite collection $\mathcal{L} = \mathcal{L}^G$ of (nonempty) Levi subsets $M \subset G$ for which M^0 contains M_0, and the finite collection $\mathcal{F} = \mathcal{F}^G$ of (nonempty) parabolic subsets $P \subset G$ such that P^0 contains M_0. These collections can of course also be defined with G^0 in place of G in which case we shall write $\mathcal{L}^0 = \mathcal{L}^{G^0}$ and $\mathcal{F}^0 = \mathcal{F}^{G^0}$. Observe that $M \to M^0$ is a map from $\mathcal{L}$ into $\mathcal{L}^0$ which is neither surjective nor injective. Finally, we have the maximal compact subgroup

$$K = \prod_v K_v = \prod_v (K_v^+ \cap G^0(\mathbb{F}_v))$$

of $G^0(\mathbb{A})$. Set
$$K_v^G = K_v^+ \cap G(F_v) \quad \text{and} \quad K^G = \prod_v K_v^G .$$

In [1(g)], we studied the geometric side of the (noninvariant) trace formula as a distribution on $C_c^\infty(G(\mathbb{A})^1)$. However, to deal with the other side of the trace formula, and to exploit the present knowledge of invariant harmonic analysis, we need to work with K-finite functions. This was the point of view of [1(i)] and [1(j)]. We shall also make use of the notation from §1 of these two papers, often without comment. In §11 of [1(i)] we defined the Hecke spaces $\mathscr{H}(G(F_S))$ and $\mathscr{H}_{\mathrm{ac}}(G(F_S))$, where S is any finite set of valuations of F with the closure property. Recall that $\mathscr{H}_{\mathrm{ac}}(G(F_S))$ consists of the Hecke functions f on $G(F_S)$ of "almost compact" support, in the sense that for any $b \in C_c^\infty(\mathfrak{a}_{G,S})$, the function
$$f^b(x) = f(x)b(H_G(x)), \qquad x \in G(F_S),$$
belongs to $\mathscr{H}(G(F_S))$. Let S_{ram} be the finite set of valuations of F at which G is ramified. (By agreement, S_{ram} contains S_∞, the set of Archimedean valuations of F.) Suppose that S contains S_{ram}. We can multiply any function on $G(F_S)$ with the characteristic function of $\prod_{v \notin S} K_v^G$, thereby identifying it with a function on $G(\mathbb{A})$. This allows us to define the adèlic Hecke spaces
$$\mathscr{H}(G(\mathbb{A})) = \varinjlim_S \mathscr{H}(G(F_S))$$
and
$$\mathscr{H}_{\mathrm{ac}}(G(\mathbb{A})) = \varinjlim_S \mathscr{H}_{\mathrm{ac}}(G(F_S)) .$$
Similarly, we can define the Hecke space
$$\mathscr{H}(G(\mathbb{A})^1) = \varinjlim_S \mathscr{H}(G(F_S)^1),$$
on $G(\mathbb{A})^1$. The terms in the trace formula are actually distributions on $\mathscr{H}(G(\mathbb{A})^1)$. However, the restriction map $f \to f^1$ sends $\mathscr{H}_{\mathrm{ac}}(G(\mathbb{A}))$ to $\mathscr{H}(G(\mathbb{A})^1)$, and we shall usually regard the terms as distributions on $\mathscr{H}(G(\mathbb{A}))$ or $\mathscr{H}_{\mathrm{ac}}(G(\mathbb{A}))$ that factor through this map.

In §11 of [1(i)] we also defined function spaces $\mathscr{I}(G(F_S))$ and $\mathscr{I}_{\mathrm{ac}}(G(F_S))$ on
$$\Pi_{\mathrm{temp}}(G(F_S)) \times \mathfrak{a}_{G,S}.$$
Let $\Pi(G(\mathbb{A}))$ (respectively $\Pi_{\mathrm{unit}}(G(\mathbb{A})), \Pi_{\mathrm{temp}}(G(\mathbb{A}))$) denote the set of equivalence classes of irreducible admissible (respectively unitary, tempered) representations of $G^+(\mathbb{A})$ whose restrictions to $G^0(\mathbb{A})$ remain irreducible. Observe that the disconnected group
$$\Xi_{\mathbb{A}} = \varinjlim_S \Xi_S = \varinjlim_S \mathrm{Hom}(G^+(F_S)/G^0(F_S), \mathbb{C}^*)$$

acts freely on each of these sets. We shall write $\{\Pi(G(\mathbb{A}))\}$, $\{\Pi_{\text{unit}}(G(\mathbb{A}))\}$, and $\{\Pi_{\text{temp}}(G(\mathbb{A}))\}$ for the sets of orbits. They correspond to the sets of representations of $G^0(\mathbb{A})$ obtained by restriction. Suppose that S contains S_{ram}. Then $\mathfrak{a}_{G,S} = \mathfrak{a}_G$. We can identify any function ϕ on $\Pi_{\text{temp}}(G(F_S)) \times \mathfrak{a}_G$ with the function on $\Pi_{\text{temp}}(G(\mathbb{A})) \times \mathfrak{a}_G$ whose value at

$$(\pi, X), \qquad \pi = \otimes_v \pi_v, \ X \in \mathfrak{a}_G,$$

equals

$$\phi\left(\bigotimes_{v\in S} \pi_v, X\right) \prod_{v \notin S} \operatorname{tr}\left(\int_{K_v^G} \pi_v(k_v)\, dk_v\right).$$

With this convention, we then define

$$\mathscr{I}(G(\mathbb{A})) = \varinjlim_{S} \mathscr{I}(G(F_S))$$

and

$$\mathscr{I}_{\text{ac}}(G(\mathbb{A})) = \varinjlim_{S} \mathscr{I}_{\text{ac}}(G(F_S)).$$

Keep in mind that any of our definitions can be transferred from G to a Levi component $M \in \mathscr{L}$. In particular, we have spaces $\mathscr{I}(M(\mathbb{A}))$ and $\mathscr{I}_{\text{ac}}(M(\mathbb{A}))$. It is easy to see that the maps $f \to f_M$ and $f \to \phi_M(f)$, described in [1(i)] and [1(j)], extend to continuous maps from $\mathscr{H}_{\text{ac}}(G(\mathbb{A}))$ to $\mathscr{I}_{\text{ac}}(M(\mathbb{A}))$.

We are going to use the local theory of [1(j)] to study the trace formula. Because the Archimedean twisted trace Paley-Wiener theorem has not yet been established in general, the result of [1(j)] apply only if G equals G^0, or if G is an inner twist of a component

$$G^* = \underbrace{(GL(n) \times \cdots \times GL(n))}_{l} \rtimes \theta^*.$$

We shall therefore assume that G is of this form. However, we shall write the paper as if it applied to a general nonconnected group. With the exception of a Galois cohomology argument in the proof of Theorem 5.1, and a part of the appendix which relies on the Archimedean trace Paley-Wiener theorem, the arguments of this paper all apply in general.

Suppose that $M \in \mathscr{L}$ and that S is a finite set of valuations of F with the closure property. In [1(j)] we defined invariant distributions

$$I_M(\gamma, f) = J_M(\gamma, f) - \sum_{L\in\mathscr{L}_0(M)} \hat{I}_M^L(\gamma, \phi_L(f)), \qquad \gamma \in M(F_S),$$

and

$$I_M(\pi, X, f) = J_M(\pi, X, f) - \sum_{L\in\mathscr{L}_0(M)} \hat{I}_M^L(\pi, X, \phi_L(f)),$$

$\pi \in \Pi(M(F_S))$, $X \in \mathfrak{a}_{M,S}$, with $f \in \mathscr{H}_{\text{ac}}(G(F_S))$. (Recall that $\mathscr{L}_0(M)$ denotes the set of Levi subsets L of G with $M \subset L \subsetneq G$.) These definitions were contingent on an induction hypothesis which we must carry into this paper. We assume that for any S, and for any elements $M \in \mathscr{L}$ and $L \in \mathscr{L}_0(M)$, the distributions

$$I_M^L(\gamma), \qquad \gamma \in M(F_S),$$

on $\mathscr{H}(L(F_S))$ are all supported on characters. (A distribution attached to G is supported on characters, we recall, if it vanishes on every function f such that $f_G = 0$.) Then the distributions $I_M(\gamma)$, and, thanks to Theorem 6.1 of [1(j)], also the distributions $I_M(\pi, X)$, are well defined. In Corollary 5.3 we shall complete the induction argument by showing that the condition holds when L is replaced by G.

The distributions $I_M(\gamma)$ and $I_M(\pi, X)$ have many parallel properties. However, there is one essential difference between the two. If $\pi \in \Pi(M(\mathbb{A}))$ and $X \in \mathfrak{a}_M$, it is easy to see that $I_M(\pi, X)$ can be defined as a distribution on $\mathscr{H}(G(\mathbb{A}))$ or even $\mathscr{H}_{\mathrm{ac}}(G(\mathbb{A}))$. This is a consequence of the original definition of $J_M(\pi, X)$ in terms of normalized intertwining operators, and in particular, the property (R_8) of [1(i), Theorem 2.1]. On the other hand, if γ belongs to $M(\mathbb{A})$, there seems to be no simple way to define $I_M(\gamma)$ as a distribution on $\mathscr{H}(G(\mathbb{A}))$. This circumstance is responsible for a certain lack of symmetry in the trace formula. The terms on the geometric side depend on a suitably large finite set S of valuations, while the terms on the spectral side do not.

If $G(\mathbb{A})$ is replaced by $G(\mathbb{A})^1$, we can obviously define the sets $\Pi(G(\mathbb{A})^1)$, $\Pi_{\mathrm{unit}}(G(\mathbb{A})^1)$ and $\Pi_{\mathrm{temp}}(G(\mathbb{A})^1)$ as above. The terms on the spectral side of the trace formula will depend on elements $M \in \mathscr{L}$ and representations $\pi \in \Pi_{\mathrm{unit}}(M(\mathbb{A})^1)$. We shall generally identify a representation $\pi \in \Pi_{\mathrm{unit}}(M(\mathbb{A})^1)$ with the corresponding orbit

$$\{\pi_\mu : \mu \in i\mathfrak{a}_M^*\}$$

of $i\mathfrak{a}_M^*$ in $\Pi_{\mathrm{unit}}(M(\mathbb{A}))$. With this convention, let us agree to write

$$J_M(\pi, f) = J_M(\pi_\mu, 0, f)$$

and

$$I_M(\pi, f) = I_M(\pi_\mu, 0, f), \qquad f \in \mathscr{H}_{\mathrm{ac}}(G(\mathbb{A})),$$

for the values of the distributions at $X = 0$. The two terms on the right are independent of μ, and are therefore well defined functions of π. They also depend only on the restriction f^1 of f to $G(\mathbb{A})^1$. This notation pertains also to the map f_G. For if π is an arbitrary representation in $\Pi(G(\mathbb{A}))$ and $X \in \mathfrak{a}_G$, we have

$$f_G(\pi, X) = J_G(\pi, X, f) = I_G(\pi, X, f).$$

Therefore, if π belongs to $\Pi_{\mathrm{unit}}(G(\mathbb{A})^1)$, it makes sense to write

$$f_G(\pi) = f_G(\pi_\mu, 0) = \mathrm{tr}\,\pi(f^1), \qquad \mu \in i\mathfrak{a}_G^*, f \in \mathscr{H}_{\mathrm{ac}}(G(\mathbb{A})).$$

2. The invariant trace formula: first version

The first version of the noninvariant trace formula is summarized in [1(b), §5] and [1(c), (2.5)]. (See also [7].) It is an identity

$$\sum_{o\in\mathscr{O}} J_o(f) = J(f) = \sum_{\chi\in\mathscr{X}} J_\chi(f), \qquad f\in C_c^\infty(G(\mathbb{A})^1), \tag{2.1}$$

in which a certain distribution J on $C_c^\infty(G(\mathbb{A})^1)$ is expanded in two different ways. The sets $\mathscr{O} = \mathscr{O}(G, F)$ and $\mathscr{X} = \mathscr{X}(G, F)$ parametrize orbit theoretic and representation theoretic data respectively, but the corresponding terms are not given as explicitly as one would like.

Suppose that $J_*(f)$ stands for one of the summands in (2.1). Then J_* is a distribution on $C_c^\infty(G(\mathbb{A})^1)$ which behaves in a predictable way,

$$J_*(f^y) = \sum_{Q\in\mathscr{F}} |W_0^Q||W_0^G|^{-1} J_*^{M_Q}(f_{Q,y}), \qquad y\in G^0(\mathbb{A}),$$

under conjugation [1(c), Theorem 3.2; 7]. Since we want to take f to be in $\mathscr{H}_{\mathrm{ac}}(G(\mathbb{A}))$, we cannot use this formula. However, as in the proof of Lemma 6.2 of [1(i)], we can easily transform it to an alternate formula

$$J_*(L_h f) = \sum_{Q\in\mathscr{F}} |W_0^{M_Q}||W_0^G|^{-1} J_*^{M_Q}(R_{Q,h} f), \tag{2.2}$$

which makes sense for functions $f \bullet \mathscr{H}_{\mathrm{ac}}(G(\mathbb{A}))$ and $h \bullet \mathscr{H}(G^0(\mathbb{A})^1)$. Let $\mathscr{L}_0$ denote the set of elements $L \bullet \mathscr{L}$ with $L\neq G$. We then define an invariant distribution

$$I_*(f) = I_*^G(f), \qquad f\in\mathscr{H}_{\mathrm{ac}}(G(\mathbb{A})),$$

inductively by setting

$$I_*(f) = J_*(f) - \sum_{M\in\mathscr{L}_0} |W_0^M||W_0^G|^{-1} \hat{I}_*^M(\phi_M(f)), \qquad f\in\mathscr{H}_{\mathrm{ac}}(G(\mathbb{A})). \tag{2.3}$$

The invariance of I_* follows from (2.2) and the analogous formula [1(i), (12.2)] for ϕ_M (see [1(c), Proposition 4.1]). Implicit in the definition is the induction assumption that for any $L\in\mathscr{L}_0$, the distribution I_*^L is defined and is supported on characters. This is what allows us to write $\hat{I}_*^L$. Observe that this induction hypothesis is our second of the paper. However, in §§3 and 4 we shall establish explicit formulas for I_o and I_χ in terms of $I_M(\gamma)$ and $I_M(\pi)$ respectively. This will reduce the second induction hypothesis to the primary one adopted in §1.

It is a simple matter to substitute (2.3) for each of the terms in (2.1). The result is an identity

$$\sum_{o\in\mathscr{O}} I_o(f) = I(f) = \sum_{\chi\in\mathscr{X}} I_\chi(f), \qquad f\in\mathscr{H}_{\mathrm{ac}}(G(\mathbb{A})), \tag{2.4}$$

in which the invariant distribution

$$(2.5)\qquad I(f)=J(f)-\sum_{M\in\mathcal{L}_0}|W_0^M||W_0^G|^{-1}\hat{I}^M(\phi_M(f)),\qquad f\in\mathcal{H}_{\mathrm{ac}}(G(\mathbb{A})),$$

is expanded in two different ways (see [1(c), Proposition 4.2]). This is the first version of the invariant trace formula. It was established in [1(c)] modulo certain hypotheses in local harmonic analysis. In later papers [1(g)] and [1(e)], we found more explicit formulas for the terms $J_{\mathfrak{o}}(f)$ and $J_\chi(f)$ in (2.1). The purpose of this paper is to convert these formulas into explicit expansions of each side of the invariant formula (2.4). In the process, we will establish the required properties of local harmonic analysis.

3. The geometric side

We shall derive a finer expansion for the left-hand side of (2.4). The result will be a sum of terms, indexed by orbits in $G(F)$, which separate naturally into local and global constituents. We shall first review the results of [1(g)], which provide a parallel expansion for the noninvariant distributions on the left-hand side of (2.1).

Recall that $\mathcal{O}=\mathcal{O}(G,F)$ is the set of equivalence classes in $G(F)$, in which two elements in $G(F)$ are considered equivalent if their semisimple Jordan components belong to the same $G^0(F)$-orbit. The formulas in [1(g)] were stated in terms of another equivalence relation on $G(F)$, which is intermediate between that of $\mathcal{O}$ and $G^O(F)$-conjugacy. It depends on a finite set S of valuations of F. The (G,S)-equivalence classes are defined to be the sets

$$G(F)\cap(\sigma U)^{G^0(F)}=\{\delta^{-1}\sigma u\delta:\,\delta\in G^0(F),\,u\in U\cap G^0(F)\}$$

in which σ is a semisimple element in $G(F)$, and U is a unipotent conjugacy class in $G_\sigma(F_S)$. Any class $\mathfrak{o}$ in $\mathcal{O}$ breaks up into a finite set $(\mathfrak{o})_{G,S}$ of (G,S)-equivalence classes. The first main result of [1(g)] is Theorem 8.1, an expansion

$$(3.1)\qquad J_{\mathfrak{o}}(f)=\sum_{M\in\mathcal{L}}|W_0^M||W_0^G|^{-1}\sum_{\gamma\in(M(F)\cap\mathfrak{o})_{M,S}}{}' a^M(S,\gamma)J_M(\gamma,f),$$

for any $\mathfrak{o}\in\mathcal{O}$ and any $f\in C_c^\infty(G(F_S)^1)$. Here S is any finite set of valuations of F which contains a certain set $S_{\mathfrak{o}}$ determined by $\mathfrak{o}$. The distributions $J_M(\gamma,f)$ are purely local, in the sense that they depend only on γ as an element in $M(F_S)$. The functions $a^M(S,\gamma)$ are what carry the global information. These were defined by formula (8.1) of [1(g)] (and also Theorem 8.1 of [1(f)]), in the case that S contains $S_{\mathfrak{o}}$.

Suppose that $M\in\mathcal{L}$. A semisimple element $\sigma\in M(F)$ is said to be F-elliptic in M if the split component of the center of M_σ equals A_M. Suppose that S is any finite set of valuations of F which contains S_∞. We shall write

$$K_S^M=\prod_{v\in S}K_v^M=\prod_{v\in S}(K_v^+\cap M(F_v)).$$

Suppose that γ is an element in $M(F)$ with semisimple Jordan component σ. Set $i^M(S,\sigma)$ equal to 1 if σ is F-elliptic in M, and if for every $v \notin S$, the set

$$\mathrm{ad}(M^0(F_v))\sigma = \{m^{-1}\sigma m : m \in M^0(F_v)\}$$

intersects the compact set K_v^M. Otherwise set $i^M(S,\sigma)$ equal to 0. Then define

$$a^M(S,\gamma) = i^M(S,\sigma)|\iota^M(\sigma)|^{-1} \sum_{\{u:\, \sigma u \sim \gamma\}} a^{M_\sigma}(S,u), \tag{3.2}$$

in the notation of [1(g), (8.1)]. This definition matches the one in [1(g)] in the special case that S contains S_o, where o is the class in $\mathcal{O}$ which contains σ.

The second main result of [1(g)] is Theorem 9.2, an expansion

$$J(f) = \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma\in(M(F))_{M,S}} a^M(S,\gamma)J_M(\gamma,f), \tag{3.3}$$

for any $f \in C_\Delta^\infty(G(F_S)^1)$. Here, Δ is a compact neighborhood in $G(\mathbb{A})^1$, and S is any finite set of valuations of F which contains a certain set S_Δ determined by Δ. This latter set is large enough so that Δ is the product of a compact neighborhood in $G(F_{S_\Delta})^1$ with the characteristic function of $\prod_{v\notin S_\Delta} K_v^G$, and by definition,

$$C_\Delta^\infty(G(F_S)^1) = C_\Delta^\infty(G(\mathbb{A})^1) \cap C_c^\infty(G(F_S)^1).$$

In [1(g)] we neglected to write down the general definition (3.2) for $a^M(S,\gamma)$. This is required for the expansion (3.3) to make sense.

Proposition 3.1. *Suppose that S is a finite set of valuations which contains S_o, and that f is a function in $\mathcal{H}_{\mathrm{ac}}(G(F_S))$. Then*

$$I_o(f) = \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma\in(M(F)\cap o)_{M,S}} a^M(S,\gamma)I_M(\gamma,f).$$

Proof. By definition, $I_o(f)$ equals the difference between $J_o(f)$ and

$$\sum_{L\in\mathcal{L}_0} |W_0^L||W_0^G|^{-1}\hat{I}_o^L(\phi_L(f)).$$

We can assume inductively that if $L \in \mathcal{L}_0$, the proposition holds for I_o^L. Since ϕ_L maps $\mathcal{H}_{\mathrm{ac}}(G(F_S))$ to $\mathcal{I}_{\mathrm{ac}}(L(F_S))$, we obtain

$$\hat{I}_o^L(\phi_L(f)) = \sum_{M\in\mathcal{L}^L} |W_0^M||W_0^L|^{-1} \sum_{\gamma\in(M(F)\cap o)_{M,S}} a^M(S,\gamma)\hat{I}_M^L(\gamma,\phi_L(f)).$$

This is valid whenever S contains the finite set S_o^L associated to L. A look at the conditions defining S_o on p. 203 of [1(g)] reveals that S_o contains S_o^L, so we can certainly take any $S \supset S_o$. Combining this formula with (3.1), we write $I_o(f)$ as

$$\sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma\in(M(F)\cap o)_{M,S}} a^M(S,\gamma)\left(J_M(\gamma,f) - \sum_{L\in\mathcal{L}_0(M)} \hat{I}_M^L(\gamma,\phi_L(f))\right).$$

The expression in brackets on the right is just equal to $I_M(\gamma, f)$, so we obtain the required formula for $I_o(f)$. □

The original induction assumption of §1 implies that for any $L \in \mathscr{L}_0$, the distributions $I_M^L(\gamma)$ are all supported on characters. The last proposition provides an expansion for I_o^L in terms of the distributions $I_M^L(\gamma)$. Therefore, I_o^L is also supported on characters. Thus, half of the second induction hypothesis adopted in §2 is subsumed in the original assumption. In §4 we shall take care of the rest of the second induction hypothesis.

To be able to exploit the last proposition effectively, we shall establish an important support property of the distributions $I_M(\gamma)$. Fix an element $M \in \mathscr{L}$, a finite set S_1 of valuations containing S_{ram}, and a compact neighborhood Δ_1 in $G(F_{S_1})$. Let $\mathscr{H}_{\Delta_1}(G(F_{S_1}))$ denote the set of functions in $\mathscr{H}(G(F_{S_1}))$ which are supported on Δ_1.

Lemma 3.2. *There is a compact subset Δ_1^M of $M(F_{S_1})^1$ such that for any finite set $S \supset S_1$, and any f in the image of $\mathscr{H}_{\Delta_1}(G(F_{S_1}))$ in $\mathscr{H}(G(F_S))$, the function*

$$\gamma \to I_M(\gamma, f), \qquad \gamma \in M(F_S)^1,$$

is supported on the set

$$\mathrm{ad}(M^0(F_S))(\Delta_1^M K_S^M) = \{m^{-1}cm : m \in M^0(F_S), c \in \Delta_1^M K_S^M\}.$$

Proof. Suppose that

$$\mathscr{M}_1 = \prod_{v \in S_1} M_v$$

is a Levi subset of M defined over F_{S_1}. Then for each $v \in S_1$, M_v is a Levi subset of M which is defined over F_v. Let $M_v(F_v)'$ be the set of elements $\gamma_v \in M_v(F_v)$ whose semisimple component σ_v satisfies the following two conditions.

(i) The connected centralizer M_{σ_v} of σ_v in M^0 is contained in M_v^0.

(ii) σ_v is an F_v-elliptic point in M_v.

Set

$$\mathscr{M}_1(F_{S_1})' = \prod_{v \in S_1} M_v(F_v)'.$$

Consider the restriction of the map

$$H_{\mathscr{M}_1} = \bigoplus_{v \in S_1} H_{M_v} : \mathscr{M}_1(F_{S_1}) \to \mathfrak{a}_{\mathscr{M}_1} = \bigoplus_{v \in S_1} \mathfrak{a}_{M_v}$$

to $\mathscr{M}_1(F_{S_1})'$. The map is certainly constant on the orbit of

$$\mathscr{M}_1^0(F_{S_1}) = \prod_{v \in S_1} M_v^0(F_v).$$

The F_v-elliptic set in $M_v(F_v)$ has a set of representatives which is compact modulo $A_{M_v}(F_v)$. It follows easily that as a map on the space of

$\mathscr{M}_1^0(F_{S_1})$-orbits in $\mathscr{M}_1(F_{S_1})'$, $H_{\mathscr{M}_1}$ is proper. To prove the lemma, we shall combine this fact with the descent and splitting properties of $I_M(\gamma, f)$. The argument is quite similar to that of [1(c), Lemma 12.2].

We may assume that

$$\Delta_1 = \prod_{v \in S_1} \Delta_v$$

and

$$f = \prod_{v \in S} f_v,$$

so that f_v belongs to $\mathscr{H}_{\Delta_v}(G(F_v))$ if v belongs to S_1, and f_v equals the characteristic function of K_v^G if v belongs to the complement of S_1 in S. Suppose that

$$\gamma = \prod_{v \in S} \gamma_v$$

is an element in $M(F_S)^1$ such that $I_M(\gamma, f) \neq 0$. For each $v \in S_1$, let σ_v be the semisimple part of γ_v, and let A_{σ_v} be the split component of the center of M_{σ_v}. Set M_v equal to the centralizer of A_{σ_v} in M. Then γ_v belongs to $M_v(F_v)'$. In other words, if

$$\mathscr{M}_1 = \prod_{v \in S_1} M_v,$$

the element

$$\gamma_1 = \prod_{v \in S_1} \gamma_v$$

belongs to $\mathscr{M}_1(F_{S_1})'$. If we were to replace γ by an $M^0(F_{S_1})$-conjugate, $\mathscr{M}_1$ would be similarly conjugated, but I_M would remain nonzero. Now there are only finitely many $M^0(F_{S_1})$-orbits of Levi subsets $\mathscr{M}_1$ over F_{S_1}. It is therefore sufficient to fix $\mathscr{M}_1$, and to consider only those elements γ such that γ_1 belongs to $\mathscr{M}_1(F_{S_1})'$.

For each valuation w in $S - S_1$, we set $M_w = M$. We then define a Levi subset

$$\mathscr{M} = \mathscr{M}_1 \times \left(\prod_{w \in S - S_1} M_w \right) = \prod_{v \in S} M_v$$

of M over F_S. Regarding γ as an element in $\mathscr{M}(F_S)$, we can form the induced class

$$\gamma^M = \prod_{v \in S} \gamma_v^M.$$

But $M_{v, \gamma_v} = M_{\gamma_v}$ for each v, so γ^M is just the $\mathscr{M}^0(F_S)$-orbit of γ. Applying Corollary 9.2 of [1(j)], we obtain

$$I_{\mathscr{M}}(\gamma, f) = \sum_{\mathscr{L} \in \mathscr{L}(\mathscr{M})} d_{\mathscr{M}}^G(M, \mathscr{L}) \prod_{v \in S} \hat{I}_{M_v}^{L_v}(\gamma_v, f_{v, L_v}) \neq 0.$$

Recalling the definition of the constants $d^G_{\mathscr{M}}(M, \mathscr{L})$ in [1(j), §9], we find that we can choose

$$\mathscr{L} = \prod_{v \in S} L_v, \qquad L_v \in \mathscr{L}(M_v),$$

so that the natural map $\mathfrak{a}^G_{\mathscr{M}} \to \mathfrak{a}^G_M \oplus \mathfrak{a}^G_{\mathscr{L}}$ is an isomorphism, and so that

$$\hat{I}^{L_v}_{M_v}(\gamma_v, f_{v,L_v}) \neq 0, \qquad v \in S. \tag{3.4}$$

Suppose first that w is a valuation in the complement of S_1 in S. Since f_w is the characteristic function of K^G_w, Lemma 2.1 of [1(j)] tells us that

$$\hat{I}^{L_w}_{M_w}(\gamma_w, f_{w,L_w}) = I^{L_w}_M(\gamma_w, f_{w,Q_w}) = J^{L_w}_M(\gamma_w, f_{w,Q_w})$$

for any $Q_w \bullet \mathscr{P}(L_w)$. The function on the right is a weighted orbital integral, and by Corollary 6.2 of [1(h)], it is the integral with respect to a measure on the induced class γ^G_w. Therefore, the class γ^G_w must intersect K^G_w. Combining the definition of the induced class γ^G_w with the standard properties of the special maximal compact group K_w, we find that the $M^0(F_w)$-orbit of γ_w intersects K^M_w. Notice in particular that $H_M(\gamma_w) = 0$.

We turn, finally, to the valuations in S_1. It remains for us to show that the $M^0(F_{S_1})$-orbit of γ_1 intersects a compact subset Δ^M_1 of $M(F_{S_1})$ which depends only on Δ_1. We are already assuming that γ_1 belongs to $\mathscr{M}_1(F_{S_1})'$, so by the discussion above, we need only show that $H_{\mathscr{M}_1}(\gamma_1)$ lies in a fixed compact subset of $\mathfrak{a}_{\mathscr{M}_1}$. Set

$$\mathscr{L}_1 = \prod_{v \in S_1} L_v.$$

It is clear that the natural map

$$\mathfrak{a}_{\mathscr{M}_1} \to \mathfrak{a}_M \oplus \mathfrak{a}_{\mathscr{L}_1}$$

is injective. But the image of $H_{\mathscr{M}_1}(\gamma_1)$ in $\mathfrak{a}_M$ equals

$$H_M(\gamma_1) = H_M(\gamma) - \sum_{w \in S - S_1} H_M(\gamma_w) = 0,$$

since γ belongs to $M(F_S)^1$. We have only to show that the image of $H_{\mathscr{M}_1}(\gamma_1)$ in $\mathfrak{a}_{\mathscr{L}_1}$, namely the vector

$$H_{\mathscr{L}_1}(\gamma_1) = \bigoplus_{v \in S_1} H_{L_v}(\gamma_v),$$

lies in a compact subset of $\mathfrak{a}_{\mathscr{L}_1}$ which depends only on Δ_1. For any $v \bullet S_1$, the distribution $\hat{I}^{L_v}_{M_v}(\gamma_v, f_{v,L_v})$ depends only on the restriction of f_v to the set

$$\{x_v \bullet G(F_v) : H_{L_v}(x_v) = H_{L_v}(\gamma_v)\}.$$

It follows from (3.4) that $H_{L_v}(\gamma_v)$ belongs to $H_{L_v}(\Delta_v)$, the image of the support of f_v. In other words, $H_{\mathscr{L}_1}(\gamma_1)$ belongs to $\bigoplus_{v \in S_1} H_{L_v}(\Delta_v)$, a compact set which depends only on Δ_1. This completes the proof of the lemma. $\square$

Suppose that f belongs to $\mathscr{H}(G(\mathbb{A}))$. We shall write $\operatorname{supp}(f)$ for the support of f. There exists a finite set S of valuations of F, which contains S_{ram}, such that f is the image of a function in $\mathscr{H}(G(F_S))$. We shall write $V(f)$ for the minimal such set. If S is any such set and γ is a point in $(M(F))_{M,S}$, we shall understand $I_M(\gamma, f)$ to mean the value of the distribution $I_M(\gamma)$ at f, regarded as a function in $\mathscr{H}(G(F_S))$. Since we are thinking of $I_M(\gamma)$ as a local object, this convention is quite reasonable. It simply means that when $\gamma \in (M(F))_{M,S}$ parametrizes such a distribution, we should treat γ as a point in $M(F_S)$ rather than $M(F)$.

Theorem 3.3. *Suppose that* $f \in \mathscr{H}(G(\mathbb{A}))$. *Then*

$$I(f) = \sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in (M(F))_{M,S}} a^M(S, \gamma) I_M(\gamma, f),$$

where S *is any finite set of valuations which is sufficiently large, in a sense that depends only on* $\operatorname{supp}(f)$ *and* $V(f)$. *The inner series can be taken over a finite subset of* $(M(F))_{M,S}$ *which also depends only on* $\operatorname{supp}(f)$ *and* $V(f)$.

Proof. By (2.4) and Proposition 3.1, we have

$$I(f) = \sum_{\mathfrak{o} \in \mathscr{O}} \sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in (M(F) \cap \mathfrak{o})_{M,S}} a^M(S, \gamma) I_M(\gamma, f),$$

where S is any finite set of valuations that contains $S_{\mathfrak{o}}$. We shall use Lemma 3.2 to show that the sum over $\mathfrak{o}$ is finite.

Choose any finite set $S_1 \supset S_{\mathrm{ram}}$, and a compact neighbourhood Δ_1 in $G(F_{S_1})$, such that f belongs to $\mathscr{H}_{\Delta_1}(G(F_{S_1}))$. Assume that S contains S_1. Suppose that a class $\mathfrak{o}$ gives a nonzero contribution to the sum above. Then there is an $M \in \mathscr{L}$, and an element $\gamma \in (M(F) \cap \mathfrak{o})_{M,S}$ such that

$$a^M(S, \gamma) I_M(\gamma, f) \neq 0.$$

The nonvanishing of $a^M(S, \gamma)$ implies that for each $v \notin S$, the image of γ in $M(F_v)$ lies in

$$\operatorname{ad}(M^0(F_v))K_v^M.$$

The image of γ in $M(F_S)$ then lies in $M(F_S)^1$, and therefore belongs to a set

$$\operatorname{ad}(M^0(F_S))(\Delta_1^M K_S^M),$$

by Lemma 3.2. It follows that the $M^0(\mathbb{A})$-orbit of γ meets the compact set $\Delta_1^M K^M$, and in particular that

$$\operatorname{ad}(G^0(\mathbb{A}))\mathfrak{o} \cap \Delta_1^M K^M \neq \emptyset.$$

By Lemma 9.1 of [1(g)], $\mathfrak{o}$ must belong to a finite subset $\mathscr{O}_1$ of $\mathscr{O}$. Since Δ_1^M depends only on Δ_1, $\mathscr{O}_1$ clearly depends only on $\operatorname{supp}(f)$ and $V(f)$. The required expansion for $I(f)$ then holds if S is any finite set which contains the union of S_1 with the sets S_o, as o ranges over $\mathcal{O}_1$. This establishes the first assertion of the theorem. The union over $o \in \mathcal{O}_1$ of the sets

$$(M(F) \cap o)_{M,S}$$

is certainly a finite subset of $(M(F))_{M,S}$, so the second assertion also follows. □

4. THE SPECTRAL SIDE

We shall derive a finer expansion for the right-hand side of (2.4). The result will be a sum of terms, indexed by irreducible representations, which separate naturally into local and global constituents. Again, there is a parallel expansion for the noninvariant distributions on the right-hand side of (2.1). It is provided by the results of [1(e)] and [7]. However, these results are not immediately in the form we want, and it is necessary to review them in some detail.

The set $\mathcal{X} = \mathcal{X}(G, F)$ consists of cuspidal automorphic data [1(b), 7]. It is the set of orbits

$$\chi = \{s_0(L_0, r_0) : s_0 \in W_0\} = \{s(L_0, r_0) : s \in W_0^G\},$$

where L_0 is a Levi subgroup in $\mathcal{L}^0 = \mathcal{L}^{G^0}$, r_0 is an irreducible cuspidal automorphic representation of $L_0(\mathbb{A})^1$, and the pair (L_0, r_0) is fixed by some element in the Weyl set W_0^G of isomorphisms of $\mathfrak{a}_0$ induced from G. (We have indexed the Levi subgroup with the subscript 0 to emphasize that it need not be of the form M^0 for some $M \in \mathcal{L}$.) The set $\mathcal{X}$ has been used to describe the convergence of the spectral side, which is more delicate than that of the geometric side. However, for applications that involve a comparison of trace formulas, it is easier to handle the convergence by keeping track of Archimedean infinitesimal characters.

Set

$$F_\infty = F_{S_\infty} = \prod_{v \in S_\infty} F_v.$$

Regarding $G^0(F_\infty)$ as a real Lie group, we can define the Abelian Lie algebra

$$\mathfrak{h} = i\mathfrak{h}_K \oplus \mathfrak{h}_0$$

as in §3 of [1(d)]. Then $\mathfrak{h}_0$ is the Lie algebra of a fixed maximal real split torus in $M_0(F_\infty)$, and $\mathfrak{h}_K$ is a fixed Cartan subalgebra of the centralizer of $\mathfrak{h}_0$ in

$$K_\infty = \prod_{v \in S_\infty} K_v.$$

The complexification $\mathfrak{h}_{\mathbb{C}}$ is a Cartan subalgebra of the complex Lie algebra of $G^0(F_\infty)$, and the real form $\mathfrak{h}$ is invariant under the complex Weyl set W^G of $G(F_\infty)$. (By definition, W^G equals $\mathrm{Ad}(\varepsilon)W$, where ε is any element in $G(F_\infty)$ which normalizes $\mathfrak{h}_{\mathbb{C}}$, and W is the complex Weyl group of $G^0(F_\infty)$

with respect to $\mathfrak{h}$.) It is convenient to fix a Euclidean norm $\|\cdot\|$ on $\mathfrak{h}$ which is invariant under W^G. We shall also write $\|\cdot\|$ for the dual Hermitian norm on $\mathfrak{h}_{\mathbb{C}}^*$. To any representation $\pi \in \Pi(M(\mathbb{A}))$ we can associate the induced representation π^G of $G^+(\mathbb{A})$. Let ν_π denote the infinitesimal character of its Archimedean constituent; it is a W-orbit in $\mathfrak{h}_{\mathbb{C}}^*$. We shall actually be more concerned with the case that π is a representation in $\Pi(M(\mathbb{A})^1)$. Then ν_π is determined a priori only as an orbit of $\mathfrak{a}_{M,\mathbb{C}}^*$ in $\mathfrak{h}_{\mathbb{C}}^*$. However, this orbit has a unique point of smallest norm in $\mathfrak{h}_{\mathbb{C}}^*$ (up to translation by W) and it is this point which we shall denote by ν_π. If t is a nonnegative real number, let $\Pi_{\mathrm{unit}}(M(\mathbb{A})^1, t)$ denote the set of representations $\pi \in \Pi_{\mathrm{unit}}(M(\mathbb{A})^1)$ such that

$$\|\mathscr{I}\mathrm{m}(\nu_\pi)\| = t,$$

where $\mathscr{I}\mathrm{m}(\nu_\pi)$ is the imaginary part of ν_π relative to the real form $\mathfrak{h}^*$ of $\mathfrak{h}_{\mathbb{C}}^*$. We adopt similar notation when M is replaced by a group $L_0 \in \mathscr{L}^0$. In particular, if

$$\chi = \{s(L_0, r_0): s \in W_0^G\}$$

is any class in $\mathscr{X}$, we set $\nu_\chi = \nu_{r_0}$.

Suppose that L_0 is a Levi subgroup in $\mathscr{L}^0$. Set

$$A_{L_0,\infty} = A_{L_0,\mathbb{Q}}(\mathbb{R})^0,$$

where $A_{L_0,\mathbb{Q}}$ is the split component of the center of the group obtained by restricting scalars from F to $\mathbb{Q}$. Let

$$L^2_{\mathrm{disc},t}(L_0(F)A_{L_0,\infty} \backslash L_0(\mathbb{A})) \tag{4.1}$$

be the subspace of $L^2(L_0(F)A_{L_0,\infty}\backslash L_0(\mathbb{A}))$ which decomposes under $L_0(\mathbb{A})$ as a direct sum of representations in $\Pi_{\mathrm{unit}}(L_0(\mathbb{A}), t)$. For any group Q_0 in

$$\mathscr{P}^0(L_0) = \mathscr{P}^{G^0}(L_0)$$

and a point $\Lambda \in \mathfrak{a}_{L_0,\mathbb{C}}^*$, let

$$\rho_{Q_0,t}(\Lambda): x \to \rho_{Q_0,t}(\Lambda, x)$$

be the induced representation of $G^0(\mathbb{A})$ obtained from (4.1). If Q_0' is another group in $\mathscr{P}^0(L_0)$, the theory of Eisenstein series provides an intertwining operator $M_{Q_0'|Q_0}(\Lambda)$ from $\rho_{Q_0,t}(\Lambda)$ to $\rho_{Q_0',t}(\Lambda)$.

Lemma 4.1. *The representation $\rho_{Q_0,t}(\Lambda)$ is admissible.*

Proof. The assertion is that the restriction of $\rho_{Q_0,t}(\Lambda)$ to K contains each irreducible representation with only finite multiplicity. Since admissibility is preserved under parabolic induction, it is enough to show that the representation of $L_0(\mathbb{A})$ on (4.1) is admissible. To this end, we may assume that

$L_0 = G^0 = G$. The assertion is then a consequence of Langlands' theory of Eisenstein series [12, Chapter 7]. For one of the main results of [12] is a decomposition

$$L^2_{\mathrm{disc},t}(G(F)A_{G,\infty} \backslash G(\mathbb{A})) = \bigoplus_{\chi} L^2_{\mathrm{disc},\chi}(G(F)A_{G,\infty} \backslash G(\mathbb{A})),$$

where χ ranges over the data in $\mathscr{X}$ such that $\|\mathscr{I}\mathrm{m}(\nu_\chi)\|$ equals t, and each corresponding summand is an admissible $G(\mathbb{A})$-module. On the other hand, the set of all χ whose associated cuspidal representations contain the restrictions of a given K-type have discrete infinitesimal characters. That is, the associated points $\{\nu_\chi\}$ form a discrete subset of $B + i\mathfrak{a}^*_M$, with B a compact ball about the origin in $\mathfrak{a}^*_M$. It follows that there are only finitely many modules $L^2_{\mathrm{disc},\chi}$ in the direct sum above which contain a given K-type. The lemma follows. □

The representation $\rho_{Q_0,t}(\Lambda)$ of $G^0(\mathbb{A})$ does not in general extend to the group generated by $G(\mathbb{A})$. However, suppose that s is an element in W_0^G with representative w in $G(F)$. We can always translate functions on $G^0(\mathbb{A})$ on the right by elements in $G(\mathbb{A})$ if at the same time we translate on the left by w^{-1}. Therefore, if y belongs to $G(\mathbb{A})$, we can define a linear map $\rho_{Q_0,t}(s,\Lambda,y)$ from the underlying Hilbert space of $\rho_{Q_0,t}(\Lambda)$ to that of $\rho_{sQ_0,t}(s\Lambda)$ such that

$$\rho_{Q_0,t}(s,\Lambda,y_1yy_2) = \rho_{sQ_0,t}(s\Lambda,y_1)\rho_{Q_0,t}(s,\Lambda,y)\rho_{Q_0,t}(\Lambda,y_1), \tag{4.2}$$

for any points y_1 and y_2 in $G^0(\mathbb{A})$. This map depends only on the image of s in $W_0^G/W_0^{L_0}$. In particular, it is well defined for any element in $W^G(\mathfrak{a}_{L_0})$, the normalizer of $\mathfrak{a}_{L_0}$ in W_0^G. Suppose that s is an element in $W^G(\mathfrak{a}_{L_0})$ which fixes Λ. If f is a function in $\mathscr{H}(G(\mathbb{A}))$, we write

$$\rho_{Q_0,t}(s,\Lambda,f^1) = \int_{G(A)^1} f(x)\rho_{Q_0,t}(s,\Lambda,x)\,dx\,.$$

Then

$$M_{Q_0|sQ_0}(\Lambda)\rho_{Q_0,t}(s,\Lambda,f^1)$$

is an operator of trace class on the underlying Hilbert space of $\rho_{Q_0,t}(\Lambda)$. According to (4.2), its trace is an invariant distribution, which by Lemma 4.1 can be written as a finite linear combination of irreducible characters

$$\mathrm{tr}\,\pi(f^1) = f_G(\pi), \qquad \pi \in \Pi_{\mathrm{unit}}(G(\mathbb{A})^1, t).$$

Observe that each such irreducible character is determined in the expression only up to the orbit of π under the group $\Xi_{\mathbb{A}}$. As in §1, we write $\{\Pi_{\mathrm{unit}}(G(\mathbb{A})^1,t)\}$ for the set of such orbits in $\Pi_{\mathrm{unit}}(G(\mathbb{A})^1,t)$.

Consider the expression

$$\sum_{L_0\in\mathscr{L}^0} |W_0^{L_0}||W_0^G|^{-1}\sum_s |\det(s-1)_{\mathfrak{a}^G_{L_0}}|^{-1}\,\mathrm{tr}(M_{Q_0|sQ_0}(0)\rho_{Q_0,t}(s,0,f^1)), \tag{4.3}$$

where Q_0 stands for any element in $\mathscr{P}^0(L_0)$ and s is summed over the Weyl set

$$W^G(\mathfrak{a}_{L_0})_{\text{reg}} = \{s \in W^G(\mathfrak{a}_{L_0}) : \det(s-1)_{\mathfrak{a}^G_{L_0}} \neq 0\}.$$

This is just the "discrete part" of the formula for

$$\sum_{\{\chi \in \mathscr{X} : \|\mathscr{I}\mathrm{m}(\nu_\chi)\| = t\}} J_\chi(f), \qquad f \in \mathscr{H}(G(\mathbb{A})),$$

provided by Theorem 8.2 of [1(e)]. (For the case $G \neq G^0$, see the final lecture of [7].) According to the remarks above, we can rewrite (4.3) as

$$\sum_{\pi \in \{\Pi_{\text{unit}}(G(\mathbb{A})^1, t)\}} a^G_{\text{disc}}(\pi) f_G(\pi), \tag{4.4}$$

a finite linear combination of characters. The complex valued function

$$a_{\text{disc}}(\pi) = a^G_{\text{disc}}(\pi), \qquad \pi \in \Pi_{\text{unit}}(G(\mathbb{A})^1, t),$$

which is defined by the equality of (4.3) and (4.4), is the primary global datum for the spectral side.

It is convenient to work with a manageable subset of $\{\Pi_{\text{unit}}(G(\mathbb{A})^1, t)\}$ which contains the support of $a^G_{\text{disc}}(\pi)$. Let $\Pi_{\text{disc}}(G, t)$ denote the subset of $\Xi_{\mathbb{A}}$-orbits in $\{\Pi_{\text{unit}}(G(\mathbb{A})^1, t)\}$ which are represented by irreducible constituents of induced representations

$$\sigma^G_\lambda, \qquad M \in \mathscr{L}, \sigma \in \Pi_{\text{unit}}(M(\mathbb{A})^1, t), \lambda \in i\mathfrak{a}^*_M / i\mathfrak{a}^*_G,$$

where σ_λ satisfies the following two conditions.

(i) $a^M_{\text{disc}}(\sigma) \neq 0$.
(ii) There is an element $s \in W^G(\mathfrak{a}_M)_{\text{reg}}$ such that $s\sigma_\lambda = \sigma_\lambda$.

Observe that the restriction to $G^0(\mathbb{A})$ of any representation in $\Pi_{\text{disc}}(G, t)$ is an irreducible constituent of an induced representation

$$\rho_{Q_0, t}(0), \qquad Q_0 \in \mathscr{F}^0.$$

From the last lemma we obtain

Lemma 4.2. *Suppose that Γ is a finite subset of $\Pi(K)$. Then there are only finitely many (orbits of) representations $\pi \in \Pi_{\text{disc}}(G, t)$ whose restrictions to K contain an element in Γ. In particular, there are only finitely many orbits $\pi \in \{\Pi(G(\mathbb{A})^1, t)\}$ which contain an element in Γ and such that $a^G_{\text{disc}}(\pi) \neq 0$.* $\square$

Before going on, we note the following lemma for future reference.

Lemma 4.3. *Suppose that ξ is a one dimensional character on $G^+(\mathbb{A})^1$ which is trivial on $G^0(F)$. Then*

$$a^G_{\mathrm{disc}}(\xi\pi) = a^G_{\mathrm{disc}}(\pi), \qquad \pi \in \Pi_{\mathrm{unit}}(G(\mathbb{A})^1, t),$$

where

$$(\xi\pi)(x) = \xi(x)\pi(x), \qquad x \in G^+(\mathbb{A}).$$

Proof. If the character ξ belongs to $\Xi_{\mathbb{A}}$, the assertion of the lemma is of course part of the definition of a^G_{disc}. In general, observe that we can use ξ to define a linear operator $\rho_{Q_0}(\xi)$ on the underlying Hilbert spaces of the representations $\rho_{Q_0,t}(0)$. It has the property that

$$\rho_{Q_0}(\xi)^{-1} M_{Q_0|sQ_0}(0)\rho_{Q_0,t}(s, 0, f^1)\rho_{Q_0}(\xi) = M_{Q_0|sQ_0}(0)\rho_{Q_0,t}(s, 0, \xi f^1),$$

where

$$(\xi f)(x) = \xi(x) f(x), \qquad x \in G(\mathbb{A})^1.$$

Therefore, (4.3) remains unchanged if f is replaced by ξf. The lemma follows. $\square$

The remaining global ingredient is a function constructed from the global normalizing factors [1(e),§6]. We shall recall briefly how it is defined. Suppose that $M \in \mathscr{L}$ and that $\pi = \bigotimes_v \pi_v$ belongs to $\Pi_{\mathrm{disc}}(M, t)$. The restriction of π to $M^0(\mathbb{A})$ is an irreducible constituent of some representation

$$\rho_{R_0,t}(0), \qquad L_0 \in \mathscr{L}^{M^0}, R_0 \in \mathscr{P}^{M^0}(L_0).$$

If $P \in \mathscr{P}(M)$, we can form the induced representation

$$\mathscr{I}_P(\pi_\lambda), \qquad \lambda \in \mathfrak{a}^*_{M,\mathbb{C}}.$$

Its restriction to $G^0(\mathbb{A})$ is a subrepresentation of $\rho_{Q_0,t}(\lambda)$, where Q_0 is the group $P^0(R_0)$ in $\mathscr{P}^0(L_0)$ which is contained in P^0 and whose intersection with M^0 is R_0. If $P' \in \mathscr{P}(M)$ and $Q_0' = (P')^0(R_0)$, the operator

$$J_{P'|P}(\pi_\lambda) = \prod_v J_{P'|P}(\pi_{v,\lambda}),$$

defined as an infinite product of unnormalized intertwining operators, is therefore equivalent to the restriction of $M_{Q_0'|Q_0}(\lambda)$ to an invariant subspace. The theory of Eisenstein series tells us that the infinite product converges for certain λ, and can be analytically continued to an operator valued function which is unitary when $\lambda \in i\mathfrak{a}^*_M$. But we also have the normalized intertwining operator

$$R_{P'|P}(\pi_\lambda) = \prod_v R_{P'|P}(\pi_{v,\lambda}) = \prod_v (r_{P'|P}(\pi_{v,\lambda})^{-1} J_{P'|P}(\pi_{v,\lambda})),$$

described in [1(i)]. The infinite product reduces to a finite product at any smooth vector. It follows that the infinite product

$$r_{P'|P}(\pi_\lambda) = \prod_v r_{P'|P}(\pi_{v,\lambda})$$

of local normalizing factors converges for certain λ and can be continued as a meromorphic function which is analytic for $\lambda \in i\mathfrak{a}_M^*$. Moreover,

$$r_{P''|P}(\pi_\lambda) = r_{P''|P'}(\pi_\lambda) r_{P'|P}(\pi_\lambda),$$

if P'' is a third element in $\mathscr{P}(M)$.

For a fixed $P' \in \mathscr{P}(M)$, we define the (G, M)-family

$$r_P(\nu, \pi_\lambda, P') = r_{P|P'}(\pi_\lambda)^{-1} r_{P|P'}(\pi_{\lambda+\nu}), \qquad P \in \mathscr{P}(M), \nu \in i\mathfrak{a}_M^*.$$

Since

$$r_{P|P'}(\pi_{v,\lambda}) = \prod_{\alpha \in \Sigma_P^r \cap \Sigma_{\bar{P}'}^r} r_\alpha(\pi_v, \lambda(\alpha^\vee))$$

for each v [1(i),§2], we have

$$r_{P|P'}(\pi_\lambda) = \prod_{\alpha \in \Sigma_P^r \cap \Sigma_{\bar{P}'}^r} r_\alpha(\pi, \lambda(\alpha^\vee)),$$

where $r_\alpha(\pi, z)$ equals an infinite product

$$\prod_v r_\alpha(\pi_v, z), \qquad z \in \mathbb{C},$$

which converges in some half-plane. Therefore, the (G, M)-family is of the special sort considered in §7 of [1(e)]. In particular, if $L \in \mathscr{L}(M)$ and $Q \in \mathscr{P}(L)$, the number

$$r_M^L(\pi_\lambda) = \lim_{\nu \to 0} \sum_{\{P \in \mathscr{P}(M):\ P \subset Q\}} r_P(\nu, \pi_\lambda, P') \theta_P^Q(\nu)^{-1}$$

can be expressed in terms of logarithmic derivatives

$$r_\alpha(\pi_\lambda, 0)^{-1} r'_\alpha(\pi_\lambda, 0), \qquad \alpha \in \Sigma^r(L, A_M),$$

and is independent of Q and P' [1(e), Proposition 7.5]. As a function of $\lambda \in i\mathfrak{a}_M^*$, it is a tempered distribution [1(e), Lemma 8.4].

For a given Levi subset $M \in \mathscr{L}$, let $\Pi(M, t)$ denote the disjoint union over $M_1 \in \mathscr{L}^M$ of the sets

$$\Pi_{M_1}(M, t) = \{\pi = \pi_{1,\lambda} : \pi_1 \in \Pi_{\text{disc}}(M_1, t), \lambda \in i\mathfrak{a}_{M_1}^*/i\mathfrak{a}_M^*\}.$$

We define a measure $d\pi$ on $\Pi(M, t)$ by setting

$$\int_{\Pi(M,t)} \phi(\pi)\, d\pi = \sum_{M_1 \in \mathscr{L}^M} |W_0^{M_1}| |W_0^M|^{-1} \sum_{\pi_1 \in \Pi_{\text{disc}}(M_1, t)} \int_{i\mathfrak{a}_{M_1}^*/i\mathfrak{a}_M^*} \phi(\pi_{1,\lambda})\, d\lambda,$$

for any suitable function ϕ on $\Pi(M, t)$. The global constituent of the spectral side of the trace formula is the function

$$a^M(\pi) = a_{\text{disc}}^{M_1}(\pi_1) r_{M_1}^M(\pi_{1,\lambda}), \tag{4.5}$$

defined for any point

$$\pi = \pi_{1,\lambda}, \qquad \pi \in \Pi_{\text{disc}}(M_1, t),\ \lambda \in i\mathfrak{a}_{M_1}^*/i\mathfrak{a}_M^*,$$

in $\Pi_{M_1}(M, t)$. In our notation we should keep in mind that π_1 is a representation in $\Pi_{\mathrm{unit}}(M_1(\mathbb{A})^1)$ (determined modulo $\Xi_{\mathbb{A}}$), so that $\{\pi_{1,\lambda}\}$ stands for the associated orbit of $i\mathfrak{a}^*_{M_1}/i\mathfrak{a}^*_M$ in $\Pi_{\mathrm{unit}}(M_1(\mathbb{A}) \cap M(\mathbb{A})^1)$. In practice, however, we shall usually identify $\pi = \pi_{1,\lambda}$ with the induced representation $\pi^M_{1,\lambda}$ in $\{\Pi_{\mathrm{unit}}(M(\mathbb{A})^1)\}$. In this sense, the invariant distribution

$$I_M(\pi, f) = I_M(\pi_\mu, 0, f), \qquad \mu \in i\mathfrak{a}^*_M, f \in \mathscr{H}(G(\mathbb{A})),$$

studied in [1(j)] is defined. It will be the local constituent of the spectral side.

Theorem 4.4. *Suppose that* $f \in \mathscr{H}(G(\mathbb{A}))$. *Then*

$$I(f) = \sum_{t \geq 0} \sum_{M \in \mathscr{L}} |W_0^M| \, |W_0^G|^{-1} \int_{\Pi(M, t)} a^M(\pi) I_M(\pi, f) \, d\pi,$$

where the integral and outer sum each converge absolutely.

Proof. Set

$$J_t(f) = \sum_{\{\chi \in \mathscr{X} : \, \|\mathscr{I}\mathrm{m}(\nu_\chi)\| = t\}} J_\chi(f).$$

We shall apply the formula for $J_\chi(f)$ provided by Theorem 8.2 of [1(e)] (and the analogue in [7] for $G \neq G^0$). Then $J_t(f)$ equals the sum over $M_1 \in \mathscr{L}$, $L_0 \in \mathscr{L}^{M_1^0}$, and $s \in W^{M_1}(\mathfrak{a}_{L_0})_{\mathrm{reg}}$, of the product of

$$|W_0^{L_0}| \, |W_0^G|^{-1} |\det(s - 1)_{\mathfrak{a}^{M_1}_{L_0}}|^{-1}$$

with

$$\int_{i\mathfrak{a}^*_{M_1}} \mathrm{tr}(\mathscr{M}_{M_1}(\Lambda, Q_0) M_{Q_0|sQ_0}(0) \rho_{Q_0,t}(s, \Lambda, f^1)) \, d\Lambda.$$

Here, Q_0 is an element in $\mathscr{P}^0(L_0)$, and the operator

$$\mathscr{M}_{M_1}(\Lambda, Q_0) = \lim_{\nu \to 0} \sum_{P_1 \in \mathscr{P}(M_1)} \mathscr{M}_{P_1}(\nu, \Lambda, Q_0) \theta_{P_1}(\nu)^{-1}$$

is obtained from the (G, M_1)-family

$$\mathscr{M}_{P_1}(\nu, \Lambda, Q_0) = M_{P_1^0(R_0)|Q_0}(\Lambda)^{-1} M_{P_1^0(R_0)|Q_0}(\Lambda + \nu),$$

for $P_1 \in \mathscr{P}(M_1)$ and $\nu \bullet i\mathfrak{a}^*_{M_1}$. As above, R_0 is a fixed parabolic subgroup of M_1^0 with Levi component L_0. We can assume that $Q_0 = P^0(R_0)$ for some fixed element P in $\mathscr{P}(M_1)$.

The trace of the operator

$$\mathscr{M}_{M_1}(\Lambda, Q_0) M_{Q_0|sQ_0}(0) \rho_{Q_0,t}(s, \Lambda, f)$$

vanishes except on an invariant subspace on which the representation $\rho_{Q_0,t}(\Lambda)$ reduces to a sum of induced representations

$$\mathscr{I}_P(\pi_{1,\Lambda}), \qquad \pi_1 \in \Pi_{\mathrm{disc}}(M_1, t).$$

(Actually, $\rho_{Q_0,t}(\Lambda)$ is only a representation of $G^0(\mathbb{A})$, so we really mean the restriction of $\mathscr{I}_P(\pi_{1,\Lambda})$ to this group.) With this interpretation, the intertwining operator $M_{P_1^0(R_0)|Q_0}(\Lambda)$ corresponds to a direct sum of operators

$$J_{P_1|P}(\pi_{1,\Lambda}) = r_{P_1|P}(\pi_{1,\Lambda})R_{P_1|P}(\pi_{1,\Lambda}), \qquad \pi_1 \in \Pi_{\mathrm{disc}}(M_1, t).$$

Therefore, $\mathscr{M}_{M_1}(\Lambda, Q_0)$ corresponds to a direct sum of operators

$$\lim_{\nu\to 0} \sum_{P_1\in\mathscr{P}(M_1)} r_{P_1}(\nu, \pi_{1,\Lambda}, P)\mathscr{R}_{P_1}(\nu, \pi_{1,\Lambda}, P)\theta_{P_1}(\nu)^{-1}.$$

This last expression is obtained from a product of (G, M)-families. By Corollary 6.5 of [1(c)] it equals

$$\sum_{M\in\mathscr{L}(M_1)} r_{M_1}^M(\pi_{1,\Lambda})\mathscr{R}_M(\pi_{1,\Lambda}, P).$$

We now apply the definition of a_{disc}^G. Given the observations above, we use the equality of (4.3) and (4.4) (with G replaced by M_1) to rewrite $J_t(f)$ as the sum over $M_1 \in \mathscr{L}$ and $M \in \mathscr{L}(M_1)$ of the product of $|W_0^{M_1}||W_0^G|^{-1}$ with

$$\sum_{\pi_1\in\Pi_{\mathrm{disc}}(M_1,t)} \int_{i\mathfrak{a}_{M_1}^*} a_{\mathrm{disc}}^{M_1}(\pi_1) r_{M_1}^M(\pi_{1,\Lambda}) \operatorname{tr}(\mathscr{R}_M(\pi_{1,\Lambda}, P)\mathscr{I}_P(\pi_{1,\Lambda}, f))\, d\Lambda.$$

Observe that $r_{M_1}^M(\pi_{1,\Lambda})$ depends only on the projection λ of Λ onto $i\mathfrak{a}_{M_1}^*/i\mathfrak{a}_M^*$. Moreover, by the definition in [1(i),§7], we have

$$\int_{i\mathfrak{a}_M^*} \operatorname{tr}(\mathscr{R}_M(\pi_{1,\Lambda+\mu}, P)\mathscr{I}_P(\pi_{1,\Lambda+\mu}, f))\, d\mu$$
$$= \int_{i\mathfrak{a}_M^*} J_M(\pi_{1,\Lambda+\mu}^M, f)\, d\mu = J_M(\pi_{1,\Lambda}^M, 0, f) = J_M(\pi_{1,\lambda}^M, f),$$

if P is any element in $\mathscr{P}(M)$. (Since λ stands for a coset of $i\mathfrak{a}_M^*$ in $i\mathfrak{a}_{M_1}^*$, it is understood that $\pi_{1,\lambda}^M$ is a representation in $\Pi_{\mathrm{unit}}(M(\mathbb{A})^1)$. This justifies the notation of the last line.) Decomposing the original integral over Λ into a double integral of (λ, μ) in

$$(i\mathfrak{a}_{M_1}^*/i\mathfrak{a}_M^*) \times (i\mathfrak{a}_M^*),$$

we obtain

$$J_t(f) = \sum_{M\in\mathscr{L}} \sum_{M_1\in\mathscr{L}^M} |W_0^{M_1}||W_0^G|^{-1}$$
$$\times \sum_{\pi_1\in\Pi_{\mathrm{disc}}(M_1,t)} \int_{i\mathfrak{a}_{M_1}^*/i\mathfrak{a}_M^*} a_{\mathrm{disc}}^{M_1}(\pi_1) r_{M_1}^M(\pi_{1,\lambda}) J_M(\pi_{1,\lambda}^M, f)\, d\lambda$$
$$= \sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{M_1\in\mathscr{L}^M} \int_{\Pi_{M_1}(M,t)} a^M(\pi) J_M(\pi, f)\, d\pi$$
$$= \sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi(M,t)} a^M(\pi) J_M(\pi, f)\, d\pi.$$

The convergence of the integral and the justification for our use of Fubini's theorem follow from the fact that $r^M_{M_1}(\pi_{1,\lambda})$ is tempered.

Set

$$I_t(f) = \sum_{\{\chi\in\mathscr{X}:\ \|\mathscr{I}\mathrm{m}(\nu_\chi)\|=t\}} I_\chi(f)\,.$$

Since the invariant χ expansion converges absolutely to $I(f)$, we have

$$I(f) = \sum_{t\geq 0} I_t(f), \tag{4.6}$$

the series converging absolutely. From the definition of $I_\chi(f)$, we obtain

$$I_t(f) = J_t(f) - \sum_{L\in\mathscr{L}_0} |W_0^L||W_0^G|^{-1}\hat{I}_t^L(\phi_L(f)).$$

Assume inductively that

$$I_t^L(g) = \sum_{M\in\mathscr{L}^L} |W_0^M||W_0^L|^{-1}\int_{\Pi(M,t)} a^M(\pi)I_M^L(\pi,g)\,d\pi$$

for any $L\in\mathscr{L}_0$ and any $g \in \mathscr{H}(L(\mathbb{A}))$. Combined with the formula above for $J_t(f)$, this tells us that $I_t(f)$ equals

$$\sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1}\int_{\Pi(M,t)} a^M(\pi)\left(J_M(\pi,f) - \sum_{L\in\mathscr{L}_0(M)}\hat{I}_M^L(\pi,\phi_L(f))\right)d\pi\,.$$

It follows that

$$I_t(f) = \sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1}\int_{\Pi(M,t)} a^M(\pi)I_M(\pi,f)\,d\pi\,. \tag{4.7}$$

The theorem follows immediately from (4.6) and (4.7). □

The definitions in this paragraph have obvious analogues if the real number t is replaced by a fixed datum $\chi\in\mathscr{X}$. In particular, if $\|\mathscr{I}\mathrm{m}(\nu_\chi)\| = t$, we have a subrepresentation $\rho_{Q_0,\chi}(\Lambda)$ of $\rho_{Q_0,t}(\Lambda)$. As in earlier papers, we shall sometimes write $\mathscr{A}^2_{Q_0,\chi}$ for the space of K-finite vectors in the underlying Hilbert space of $\rho_{Q_0,\chi}(\Lambda)$. Then for any $s \in W_0^G$ and $f\in\mathscr{H}(G(\mathbb{A}))$, $\rho_{Q_0,\chi}(s,\Lambda,f^1)$ is a map from $\mathscr{A}^2_{Q_0,\chi}$ to $\mathscr{A}^2_{sQ_0,\chi}$. The definitions also provide functions $a^{M_1}_{\mathrm{disc},\chi}$ and a^M_χ on respective subsets

$$\Pi_{\mathrm{disc}}(M_1,\chi)\subset\Pi_{\mathrm{disc}}(M_1,t),\qquad M_1\in\mathscr{L},$$

and

$$\Pi(M,\chi)\subset\Pi(M,t),\qquad M\in\mathscr{L}.$$

The proof of Theorem 4.4 yields

Corollary 4.5. *Suppose that* $f \in \mathscr{H}(G(\mathbb{A}))$ *and* $\chi \in \mathscr{X}$. *Then*

$$I_\chi(f) = \sum_{M \in \mathscr{L}} |W_0^M| |W_0^G|^{-1} \int_{\Pi(M,\chi)} a_\chi^M(\pi) I_M(\pi, f)\, d\pi. \quad \square$$

For any element $L \in \mathscr{L}_0$, the corollary provides an expansion for I_χ^L in terms of the distributions

$$I_M^L(\pi) = I_M^L(\pi_\mu, 0), \qquad \mu \in i\mathfrak{a}_M^*.$$

But our original induction assumption of §1 implies that the distributions $I_M^L(\pi_\mu, 0)$ are supported on characters. This is a consequence of Theorem 6.1 of [1(j)]. Therefore, the distributions I_χ^L are also supported on characters. We have thus shown that the entire second induction assumption, adopted in §2, is subsumed in the original one.

5. Completion of the induction argument

We shall now show that all the distributions which occur in the invariant trace formula are supported on characters. These are local objects, so we shall not start off with the number field F that has been fixed up until now. Rather, we take a local field F_1 of characteristic 0, and a connected component G_1 of a reductive group over F_1, in which $G_1(F_1) \neq \varnothing$. As usual, we shall assume either that $G_1 = G_1^0$, or that G_1 is an inner twist of a component

$$G^* = (GL(n) \times \cdots \times GL(n)) \rtimes \theta^*.$$

Theorem 5.1. *For any* G_1/F_1 *as above, and any Levi subset* M_1 *of* G_1 (*with respect to* F_1), *the distributions*

$$I_{M_1}(\gamma_1, f_1), \qquad \gamma_1 \in M_1(F_1), f \in \mathscr{H}(G_1(F_1)),$$

are supported on characters.

Proof. Fix a positive integer N_1, and assume that the theorem is valid for any G_1/F_1 with $\dim_{F_1}(G_1) < N_1$. Having made this induction assumption, we fix G_1 and F_1 such that $\dim_{F_1}(G_1) = N_1$. If $L_1 \bullet \mathscr{L}_0(M_1)$, the distributions $I_{M_1}^{L_1}(\gamma_1)$ are by hypothesis supported on characters. This matches the induction assumption of §2 of [1(j)] that allowed us to define $I_{M_1}(\gamma_1)$ in the first place.

Let f_1 be a fixed function in $\mathscr{H}(G_1(F_1))$ such that

$$f_{1,G_1} = 0.$$

We must show that the distributions all vanish on f_1. It is convenient to fix M_1 and to make a second induction assumption that

$$I_{L_1}(\delta_1, f_1) = 0, \qquad \delta_1 \in L_1(F_1), \tag{5.1}$$

for any $L_1 \in \mathscr{L}(M_1)$ with $L_1 \neq M_1$. We must then show that $I_{M_1}(\gamma_1, f_1)$ vanishes for each $\gamma_1 \in M_1(F_1)$.

If γ_1 is an arbitrary point in $M_1(F_1)$, we can write

$$\begin{aligned} I_{M_1}(\gamma_1, f_1) &= \lim_{a\to 1} \sum_{L_1\in\mathscr{L}(M_1)} r_{M_1}^{L_1}(\gamma_1, a) I_{L_1}(a\gamma_1, f_1) \\ &= \lim_{a\to 1} I_{M_1}(a\gamma_1, f_1), \end{aligned}$$

by (5.1) and [1(j), (2.2)]. Since a stands for a small regular point in $A_{M_1}(F_1)$, we may assume without loss of generality that $G_{1,\gamma_1} = M_{1,\gamma_1}$. But now we can apply [1(j), (2.3)]. This formula asserts that the function

$$\gamma \to I_{M_1}(\gamma, f_1)$$

coincides with the orbital integral of a function on $M_1(F_1)$, for all points γ whose semisimple part is close to that of γ_1. It is known that the orbital integral of a function on $M_1(F_1)$ is completely determined by its values at regular semisimple points. For p-adic F_1, this is Theorem 10 of [9(c)]. If F_1 is Archimedean, the result is due also to Harish-Chandra. The proof, which was never actually published, uses the Archimedean analogues of the techniques of [9(c)]. In any case, it follows that if $I_{M_1}(\gamma, f_1)$ vanishes whenever γ is G_1-regular, it vanishes for all γ_1. We may therefore assume that γ_1 itself is G_1-regular. We can also assume that γ_1 is an F_1-elliptic point in $M_1(F_1)$. For γ_1 would otherwise belong to a proper Levi subset M of M_1 defined over F_1, and we would be able to write

$$I_{M_1}(\gamma_1, f_1) = \sum_{L\in\mathscr{L}(M)} d_M^G(M_1, L)\hat{I}_M^L(\gamma_1, f_{1,L}),$$

by the descent property [1(j), Corollary 8.3]. Since $d_M^G(M_1, L) = 0$ unless L is properly contained in G, the expression vanishes by our first induction assumption. Thus, it remains for us to show that $I_{M_1}(\gamma_1, f_1)$ vanishes when γ_1 is a fixed point in $M_1(F_1)$ which is G_1-regular and F_1-elliptic. For this basic case we shall use the global argument introduced by Kazhdan (see [8] and [10]).

Suppose that G is a component of a reductive group over some number field F, with $G(F) \neq \emptyset$, such that $F_{v_1} \cong F_1$ and $G_{v_1} = G_1$ for a valuation v_1 of F. Then

$$\dim_F(G) = \dim_{F_1}(G_1) = N_1.$$

It follows from Corollary 9.3 of [1(j)] and our induction assumption on N_1 that for any S, the distributions

$$I_M^L(\gamma), \qquad M \in \mathscr{L}, L \in \mathscr{L}_0(M), \gamma \in M(F_S),$$

are all supported on characters. Therefore, G/F satisfies the conditions of §1, and we can apply the results of §§3 and 4.

Lemma 5.2. *Suppose that*

$$f = \prod_v f_v, \qquad f_v \in \mathscr{H}(G(F_v)),$$

is a function in $\mathscr{H}(G(\mathbb{A}))$ *such that* $f_{v_1} = f_1$. *Then* $I(f) = 0$.

Proof. Consider the spectral expansion

$$I(f) = \sum_{t \geq 0} \sum_{M \in \mathscr{L}} |W_0^M| |W_0^G|^{-1} \int_{\Pi(M,t)} a^M(\pi) I_M(\pi, f)\, d\pi$$

of Theorem 4.4. We shall show that the distributions

$$I_M(\pi, f) = I_M(\pi_\mu, 0, f), \qquad \mu \in i\mathfrak{a}_M^*, M \in \mathscr{L}, \pi \in \Pi(M, t),$$

which occur on the right, vanish. In doing this, we will make essential use of the fact that π is unitary.

It is clearly enough to establish the vanishing of the Fourier transform

$$I_M(\pi, \mathscr{X}, f) = \int I_M(\pi_\Lambda, f) e^{-\Lambda(\mathscr{X})}\, d\Lambda,$$

where, for a large finite set S of valuations, $\mathscr{X}$ belongs to the vector space of elements in $\bigoplus_{v \in S} \mathfrak{a}_{M,v}$ whose components sum to 0. The integral is over the imaginary dual vector space. According to the splitting formula [1(j), Proposition 9.4], we can write $I_M(\pi, \mathscr{X}, f)$ as a finite sum of products, over $v \in S$, of distributions on the spaces $\mathscr{H}(L(F_v))$, $L \in \mathscr{L}(M)$. But if $L \in \mathscr{L}_0(M)$, our induction hypothesis, combined with Theorem 6.1 of [1(j)], tells us that the distributions

$$\hat{I}_M^L(\pi_1, X_1, f_{1,L}), \qquad \pi_1 \in \Pi_{\text{unit}}(M(F_{v_1})), X_1 \in \mathfrak{a}_{M,v_1},$$

are well defined. They must then vanish, since $f_{1,L} = 0$. It is therefore enough to show that the distributions

$$I_M(\pi_1, X_1, f_1), \qquad \pi_1 \in \Pi_{\text{unit}}(M(F_{v_1})), X_1 \in \mathfrak{a}_{M,v_1},$$

vanish. (Recall that by an abuse of notation, we denoted these distributions by $\hat{I}_M(\pi_1, X_1, f_{1,G})$ in the splitting formula.)

The formula [1(j), (3.2)] gives an expansion for $I_M(\pi_1, X_1, f_1)$ in terms of the distributions associated to standard representations $\rho \in \Sigma(M(F_v))$. Only those ρ with $\Delta(\rho, \pi_1) \neq 0$ can occur in the expansion (see [1(i), §§5–6]). Since π_1 is unitary, this implies that ρ has a unitary central character. It is sufficient to establish that for any such ρ and any point $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$ with a small real part, the distributions

$$I_L(\rho_\lambda^L, h_L(X_1), f_1), \qquad L \in \mathscr{L}(M), X_1 \in \mathfrak{a}_{M,v_1}, \tag{5.2}$$

all vanish. Since its central character is unitary, ρ must either be tempered or be induced from a proper parabolic subset of M. If ρ is tempered,

$$I_L(\rho_\lambda^L, h_L(X_1), f_1) = \begin{cases} f_{1,G}(\rho_\lambda^G, h_G(X_1)), & \text{if } L = G, \\ 0, & \text{otherwise}, \end{cases}$$

by Lemma 3.1 of [1(j)]. But $f_{1,G} = 0$, so the distribution vanishes even if $L = G$. In the other case,

$$\rho = \rho_1^M, \qquad M_1 \subsetneq M, \ \rho_1 \in \Sigma(M_1(F_{v_1})),$$

and we can make use of the descent property [1(j), Corollary 8.5]. We obtain an expression for a Fourier transform of (5.2) in terms of the distributions

$$\hat{I}_{M_1}^{M_2}(\rho_{1,\lambda}, Y_1, f_{1,M_2}), \qquad M_2 \in \mathscr{L}_0(M_1), Y_1 \in \mathfrak{a}_{M_1, v_1}.$$

Since $M_2 \neq G$, the distributions are well defined, and therefore vanish. Thus, the distribution (5.2) vanishes in all cases. In other words, the spectral expansion reduces to 0, and $I(f)$ vanishes. □

We must decide how to choose G, F and v_1 in order to prove the theorem. Our original element γ_1 in $M_1(F_1)$ belongs to a unique "maximal torus"

$$T_1 = T_{1,0}\gamma_1$$

in M_1. By definition, $T_{1,0}$ is the connected centralizer G_{1,γ_1} of γ_1 in G_1^0. It is a torus in M_1^0 which is F_1-anisotropic modulo A_{M_1}. Let $E_1 \supset F_1$ be a finite Galois extension over which G_1 and T_1 split. Choose any number field E, with a valuation w_1, such that $E_{w_1} \cong E_1$. The Galois group, $\mathrm{Gal}(E_1/F_1)$, can be identified with the decomposition group of E at w_1, and therefore acts on E. Let F be the fixed field in E of this group, and let v_1 be a valuation of F which w_1 divides. Then $F_1 \cong F_{v_1}$ and $\mathrm{Gal}(E_1/F_1) = \mathrm{Gal}(E/F)$. We can therefore use G_1 to twist the appropriate Chevalley group and "maximal torus" over F. We obtain a component G and "maximal torus" T defined over F, with $G(F)$ and $T(F)$ not empty, such that $G_1 = G_{v_1}$ and $T_1 = T_{v_1}$. Moreover, the construction is such that $M_1 = M_{v_1}$ and $\mathfrak{a}_{M_1} = \mathfrak{a}_M$, where M is a Levi subset of G which contains T and is defined over F. It follows that

$$I_{M_1}(\gamma_1, f_1) = I_M(\gamma_1, f_1).$$

But the set $T(F)$ is dense in $T(F_{v_1})$. We can therefore approximate our G-regular point γ_1 by elements $\gamma \in T(F)$. Since $I_M(\gamma_1, f_1)$ is continuous in (regular) γ_1, we have only to show that $I_M(\gamma, f_1) = 0$ for any fixed G-regular element γ in $T(F)$. We can use the trace formula to do this.

We shall choose a suitable function

$$f = \prod_v f_v, \qquad f_v \in \mathscr{H}(G(F_v)),$$

in $\mathscr{H}(G(\mathbb{A}))$, and apply Lemma 5.2. Observe first that T is F_{v_1}-anisotropic modulo A_M. This means that T is contained in no proper Levi subset of M (relative to F_{v_1}). We can always replace F by a finite extension in which v_1 splits completely. We may therefore assume that T is also F_{v_2}-anisotropic modulo A_M, where v_2 is another valuation of F. Let $V = \{v_1, v_2, \dots, v_k\}$ be a large finite set of valuations of F which contains v_1 and v_2, and outside of which G and T are unramified. At $v = v_1$, we have already been given our function $f_{v_1} = f_1$. If v is any of the other valuations in V, let f_v be any function which is supported on a very small open neighborhood of γ in $G(F_v)$, and such that

$$\hat{I}_M^M(\gamma, f_{v,M}) = I_G(\gamma, f_v) = 1.$$

If v lies outside of V, let f_v equal the characteristic function of K_v^G. Then $f = \prod_v f_v$ certainly belongs to $\mathscr{H}(G(\mathbb{A}))$. It follows from Lemma 5.2 and Theorem 3.3 that

$$\sum_{L\in\mathscr{L}} |W_0^L||W_0^G|^{-1} \sum_{\delta\in(L(F))_{L,S}} a^L(S,\delta)I_L(\delta,f) = 0. \tag{5.3}$$

Since $V = V(f)$, the shrinking of the functions $f_{v_2}, \dots, f_{v_k}$ around γ does not increase $V(f)$. Nor does it increase the support of f. It follows that in (5.3), the set S may be chosen to be independent of f, and the sums over δ can be taken over finite sets which are also independent of f.

Suppose that $L \in \mathscr{L}$ and $\delta \in (L(F))_{L,S}$. We apply the splitting formula [1(j), Corollary 9.2] to $I_L(\delta,f)$. If $L \subset L_1 \subsetneq G$, we have

$$\hat{I}_L^{L_1}(\delta, f_{v_1,L_1}) = 0,$$

by assumption. It follows that

$$I_L(\delta,f) = I_L(\delta,f_{v_1}) \cdot \prod_{v\neq v_1} \hat{I}_L^L(\delta, f_{v,L}).$$

Now the function f_{v_2} is supported on the F_{v_2}-anisotropic set in $M(F_{v_2})$. This means that $f_{v_2,L} = 0$ unless L contains a conjugate of M. On the other hand, if L contains a conjugate

$$wMw^{-1}, \qquad w \in W_0,$$

of M, we can write

$$I_L(\delta, f_{v_1}) = I_{w^{-1}Lw}(w^{-1}\delta w, f_{v_1}),$$

by [1(j), (2.4*)]. If M is properly contained in $w^{-1}Lw$, this vanishes by (5.1). Thus, the contribution of L to (5.3) vanishes unless L is conjugate to M. Since the contributions from different conjugates of M are equal, we obtain

$$\sum_{\delta\in(M(F))_{M,S}} a^M(S,\delta)\left(I_M(\delta,f_{v_1})\prod_{v\neq v_1} I_G(\delta,f_v)\right) = 0. \tag{5.4}$$

Once again, δ can be summed over a finite set which is independent of how we shrink f.

The orbital integrals

$$I_G(\delta, f_{v_j}), \qquad 2 \le j \le k,$$

vanish unless δ is close to the $G^0(F_{v_j})$-orbit of γ. In particular, the sum in (5.4) need only be taken over elements δ which are regular semisimple. Consequently,

$$a^M(S,\delta) = |M_\delta(F)\setminus M(F,\delta)|^{-1}\operatorname{vol}(M_\delta(F)\setminus M_\delta(\mathbb{A})^1),$$

by Theorem 8.2 of [1(g)]. Moreover, the (M, S)-equivalence classes of regular semisimple elements in $M(F)$ are just $M^0(F)$-orbits. It follows that

$$\sum_{\delta} c(\delta) I_M(\delta, f_{v_1}) = 0, \tag{5.5}$$

where δ is summed over those $M^0(F)$-orbits in $M(F)$ which are $G^0(F_{v_j})$-conjugate to γ for $2 \leq j \leq k$, and which meet K_v^G for v outside of V, and where

$$c(\delta) = |M_\delta(F) \setminus M(F, \delta)|^{-1} \operatorname{vol}(M_\delta(F) \setminus M_\delta(\mathbb{A})^1) \cdot \prod_{v \in S - V} I_G(\delta, f_v).$$

We must show that every such δ is also $G^0(F_{v_1})$-conjugate to γ. As in [10, Appendix], we use an argument from Galois cohomology.

For the first time in this paper we shall explicitly invoke our limiting hypothesis on G. If G is an inner twist of the component

$$G^* = (GL(n) \times \cdots \times GL(n)) \rtimes \theta^*,$$

then any two elements in $G(F)$ which are in the same G^0-orbit are actually in the same $G^0(F)$-orbit. There is nothing further to prove in this case. We can assume therefore that $G = G^0$. Then T is a maximal torus (in the usual sense) in G. The set of $G(F_v)$-conjugacy classes in $G(F_v)$ which are contained in the G-conjugacy class of γ is known to be in bijective correspondence with a subset of

$$H^1(F_v, T) = H^1(\operatorname{Gal}(\overline{F}_v/F_v), T(\overline{F}_v)).$$

A similar assertion holds for $G(F)$-conjugacy classes. Let E/F be a finite Galois extension which is unramified outside V, and over which T splits. Then $H^1(F_v, T)$ equals $H^1(\operatorname{Gal}(E_w/F_v), T(E_w))$, and Tate-Nakayama theory provides an isomorphism between this group and

$$\{\lambda^\vee \bullet X_*(T) \colon \operatorname{Norm}_{E_w/F_v}(\lambda^\vee) = 0\}/\{\lambda^\vee - \sigma\lambda^\vee \colon \lambda^\vee \in X_*(T), \sigma \bullet \operatorname{Gal}(E_w/F_v)\}, \tag{5.6}$$

and an isomorphism between

$$H^1(\operatorname{Gal}(E/F), T(\mathbb{A}_E)/T(E))$$

and

$$\{\lambda^\vee \in X_*(T) \colon \operatorname{Norm}_{E/F}(\lambda^\vee) = 0\}/\{\lambda^\vee - \sigma\lambda^\vee \colon \lambda^\vee \in X_*(T), \sigma \bullet \operatorname{Gal}(E/F)\}. \tag{5.7}$$

Here w stands for a fixed valuation on E which lies above a given v. Moreover, there is an exact sequence

$$H^1(\operatorname{Gal}(E/F), T(E)) \to \bigoplus_v H^1(\operatorname{Gal}(E_w/F_v), T(E_w)) \to H^1(\operatorname{Gal}(E/F), T(\mathbb{A}_E)/T(E)).$$

The first map is compatible with the embedding of $G(F)$-conjugacy classes into $\prod_v G(F_v)$, and the second arrow is given by the natural map

$$\bigoplus_v \lambda_v \to \sum_v \lambda_v$$

from the direct sum of modules (5.6) into (5.7). Now, consider the conjugacy class of γ. Any δ which occurs in the sum (5.5) maps to an element $\bigoplus_v \lambda_v$ such that $\sum_v \lambda_v = 0$. If v is one of the valuations $v_2, \dots, v_k$, δ is $G(F_v)$-conjugate to γ, so that $\lambda_v = 0$. If v lies outside V, δ is $M^0(F_v)$-conjugate to an element in K_v^G. Since (G, T) is unramified at v, we again have $\lambda_v = 0$ [11(a), Proposition 7.1]. It follows that $\lambda_{v_1} = 0$. In other words, δ is $G(F_{v_1})$-conjugate to γ, as we wanted to prove.

We are now done. For if δ is an element in $M(F)$ which is $G^0(F_{v_1})$-conjugate to γ, we have $\delta = y^{-1}\gamma y$, for some element $y \bullet M^0(F_{v_1})K_{v_1}$ which normalizes M^0. It follows from [1(j), (2.4*)] that

$$I_M(\delta, f_{v_1}) = I_M(\gamma, f_1).$$

But for any δ which occurs in the sum (5.5), the constant $c(\delta)$ is strictly positive. It follows from (5.5) that

$$I_M(\gamma, f_1) = 0.$$

As we noted earlier, this implies that

$$I_{M_1}(\gamma_1, f_1) = 0,$$

for our original point $\gamma_1 \bullet M_1(F_1)$. Theorem 5.1 is proved. □

Corollary 5.3. *Suppose that G/F is as in §1. Then for any S and any $M \in \mathscr{L}$, the distributions*

$$I_M(\gamma), \qquad \gamma \bullet M(F_S),$$

are supported on characters.

Proof. The corollary follows immediately from the theorem and Corollary 9.3 of [1(j)]. □

Corollary 5.3 justifies the primary induction assumption of §1. In particular, the distributions which occur in the invariant trace formula are all supported on characters. We have at last finished the extended induction argument, begun originally in [1(j)].

6. A CONVERGENCE ESTIMATE

It is not known that the spectral expansion for $I(f)$ provided by Theorem 4.4 converges as a multiple integral over t, M and π. The main obstruction is the trace class problem. This is essentially the question of showing that the operators

$$\bigoplus_{t\geq 0} \rho_{Q,t}(\Lambda, f), \qquad Q \in \mathscr{F}^0, f \in \mathscr{H}(G(\mathbb{A})),$$

are of trace class. We shall instead prove an estimate for the rate of convergence of the χ-expansion. The estimate is an extension of some of the arguments used in the derivation of the trace formula. Although rather weak, it seems to be a natural tool for those applications which entail a comparison of trace formulas.

The estimate will be stated in terms of multipliers. Recall [1(d)] that multipliers are associated to elements in $\mathscr{E}(\mathfrak{h})^W$, the convolution algebra of compactly supported W-invariant distributions on $\mathfrak{h}$. For $\alpha \in \mathscr{E}(\mathfrak{h})^W$ and $f \in \mathscr{H}(G(\mathbb{A}))$, f_α is the new function in $\mathscr{H}(G(\mathbb{A}))$ such that

$$\pi(f_\alpha) = \hat{\alpha}(\nu_\pi)\pi(f), \qquad \pi \in \Pi(G(\mathbb{A})).$$

Similarly, for any function $\phi \in \mathscr{I}(G(\mathbb{A}))$, there is another function $\phi_\alpha \in \mathscr{I}(G(\mathbb{A}))$ such that

$$\phi_\alpha(\pi) = \hat{\alpha}(\nu_\pi)\phi(\pi), \qquad \pi \in \Pi_{\text{temp}}(G(\mathbb{A})).$$

(As in §11 of [1(i)], we shall sometimes regard ϕ as a function on $\Pi_{\text{temp}}(G(\mathbb{A}))$ instead of the product $\Pi_{\text{temp}}(G(\mathbb{A}))\times \mathfrak{a}_G$. Then two interpretations are of course related by the Fourier transform

$$\phi(\pi, X) = \int_{i\mathfrak{a}_G^*} \phi(\pi_\lambda)e^{-\lambda(X)}\, d\lambda, \qquad \lambda \in \mathfrak{a}_G,$$

on $i\mathfrak{a}_G^*$.) Suppose that α belongs to the subalgebra $C_c^\infty(\mathfrak{h})^W$. Then we have

$$\phi_\alpha(\pi, X) = \int_{\mathfrak{a}_G} \phi(\pi, Z)\alpha_G(\pi, X - Z)\, dZ, \tag{6.1}$$

where

$$\alpha_G(\pi, Z) = \int_{i\mathfrak{a}_G^*} \hat{\alpha}(\nu_\pi + \lambda)e^{-\lambda(Z)}\, d\lambda, \qquad Z \in \mathfrak{a}_G.$$

Formula (6.1) is useful because it makes sense even if ϕ belongs to the larger space $\mathscr{I}_{\text{ac}}(G(\mathbb{A}))$. For if X remains within a compact set, the function

$$Z \to \alpha_G(\pi, X - Z)$$

is supported on a fixed compact set. It follows that $\phi \to \phi_\alpha$ extends to an action of $C_c^\infty(\mathfrak{h})^W$ on $\mathscr{I}_{\text{ac}}(G(\mathbb{A}))$ such that (6.1) holds. Similarly, $f \to f_\alpha$ extends to an action of $C_c^\infty(\mathfrak{h})^W$ on $\mathscr{H}_{\text{ac}}(G(\mathbb{A}))$. Recall that if $f \in \mathscr{H}_{\text{ac}}(G(\mathbb{A}))$ and $X \in \mathfrak{a}_G$, f^X is the restriction of f to

$$G(\mathbb{A})^X = \{x \in G(\mathbb{A}) : H_G(x) = X\},$$

and

$$\pi(f^X) = \int_{G(\mathbb{A})^X} f(x)\pi(x)\, dx, \qquad \pi \in \Pi(G(\mathbb{A})).$$

Then we have

$$\pi(f_\alpha^X) = \pi((f_\alpha)^X) = \int_{\mathfrak{a}_G} \pi(f^Z)\alpha_G(\pi, X - Z)\, dZ. \tag{6.2}$$

Setting $X = 0$, we obtain the formula

$$\pi(f_\alpha^1) = \int_{\mathfrak{a}_G} \pi(f^Z)\alpha_G(\pi, -Z)\, dZ, \qquad \pi \in \Pi(G(\mathbb{A})), \tag{6.2$'$}$$

for the restriction f_α^1 of f_α to $G(\mathbb{A})^1$.

We do not want f to be an arbitrary function in $\mathscr{H}_{\mathrm{ac}}(G(\mathbb{A}))$. We must insist in some mild support and growth conditions on the functions f^Z as Z gets large. Fix a height function

$$\|x\| = \prod_v \|x_v\|_v, \qquad x \in G(\mathbb{A}),$$

on $G(\mathbb{A})$ as in §§2 and 3 of [1(d)]. We shall say that a function $f \in \mathscr{H}_{\mathrm{ac}}(G(\mathbb{A}))$ is *moderate* if there are positive constants c and d such that f is supported on

$$\{x \in G(\mathbb{A}) : \log\|x\| \le c(\|H_G(x)\| + 1)\},$$

and such that

$$\sup_{x \in G(\mathbb{A})} (|\Delta f(x)| \exp\{-d\|H_G(x)\|\}) < \infty,$$

for any left invariant differential operator Δ on $G(F_\infty)$. In a similar fashion, one can define the notion of a moderate function in $\mathscr{I}_{\mathrm{ac}}(G(\mathbb{A}))$. (We shall recall the precise definition in the appendix.)

It is not hard to show that the map $f \to f_G$ sends moderate functions in $\mathscr{H}_{\mathrm{ac}}(G(\mathbb{A}))$ to moderate functions in $\mathscr{I}_{\mathrm{ac}}(G(\mathbb{A}))$. Conversely, we have

Lemma 6.1. *Suppose that Γ is a finite subset of $\Pi(K)$ and that ϕ is a moderate function in $\mathscr{I}_{\mathrm{ac}}(G(\mathbb{A}))_\Gamma$. Then there is a moderate function $f \in \mathscr{H}_{\mathrm{ac}}(G(\mathbb{A}))_\Gamma$ such that $f_G = \phi$.*

This lemma can be regarded as a variant of the trace Paley-Wiener theorem. We shall postpone its proof until the appendix.

We shall write $C_N^\infty(\mathfrak{h})^W$, as usual, for the set of functions in $C_c^\infty(\mathfrak{h})^W$ which are supported on the ball of radius N.

Lemma 6.2. *Suppose that f is a moderate function in $\mathscr{H}_{\mathrm{ac}}(G(\mathbb{A}))$. Then there is a constant c such that for any $\alpha \in C_N^\infty(\mathfrak{h})^W$, with $N > 0$, the function f_α is supported on*

$$\{x \in G(\mathbb{A}) : \log\|x\| \le c(\|H_G(x)\| + N + 1)\}.$$

Proof. We can use the direct product decomposition $G(\mathbb{A}) = G(\mathbb{A})^1 \times A_{G,\infty}$ to identify each of the restricted functions f^X, $X \in \mathfrak{a}_G$, with a function in $\mathscr{H}(G(\mathbb{A})^1)$. The lemma then follows from Proposition 3.1 of [1(d)] and the appropriate variant of (6.2). $\square$

We are now ready to state our convergence estimate. Fix a finite subset Γ of $\Pi(K)$. If $L_0 \in \mathscr{L}^0$ and $\chi \in \mathscr{X}(G,F)$, a variant of the definition of §4 provides a set $\Pi_{\mathrm{disc}}(L_0,\chi)$ of irreducible representations of $L_0(\mathbb{A})^1$. Let $\Pi_{\mathrm{disc}}(L_0,\chi)_\Gamma$ be the subset of representations in $\Pi_{\mathrm{disc}}(L_0,\chi)$ which contain representations in the restriction of Γ to $K \cap L_0(\mathbb{A})$.

Lemma 6.3. *Suppose that ϕ is a moderate function in $\mathscr{I}_{\mathrm{ac}}(G(\mathbb{A}))_\Gamma$. Then there are constants C and k such that for any subset $\mathscr{X}_1$ of $\mathscr{X}(G,F)$ and any $\alpha \in C_N^\infty(\mathfrak{h})^W$, with $N > 0$, the expression*

$$\sum_{\chi \in \mathscr{X}_1} |\hat{I}_\chi(\phi_\alpha)|$$

is bounded by the supremum over $\chi \in \mathscr{X}_1$, $L_0 \in \mathscr{L}^0$, $\Lambda \in i\mathfrak{a}_{L_0}^*$ and $\sigma \in \Pi_{\mathrm{disc}}(L_0,\chi)_\Gamma$ of

$$Ce^{kN}|\hat{\alpha}(\nu_\sigma + \Lambda)|.$$

Proof. By Lemma 6.1 there is a moderate function f in $\mathscr{H}_{\mathrm{ac}}(G(\mathbb{A}))_\Gamma$ such that $f_G = \phi$. Then

$$\hat{I}_\chi(\phi_\alpha) = I_\chi(f_\alpha) = I_\chi(f_\alpha^1),$$

for $\chi \in \mathscr{X}$ and $\alpha \in C_N^\infty(\mathfrak{h})^W$. By Lemma 6.2, the function f_α^1 is supported on a set

$$\{x \in G(\mathbb{A})^1 : \log\|x\| \leq c(1+N)\},$$

where the constant c depends only on f. We are first going to estimate the sum $\sum_{\chi \in \mathscr{X}_1} |J_\chi(f_\alpha)|$ of noninvariant distributions. We shall appeal to two results (Proposition 2.2 and Lemma A.1) of [1(d)] which apply to the case that $G = G^0$. The results for general G, which require slightly different notation, can be extracted from [7]. We shall simply quote them.

Fix a minimal parabolic subgroup $Q_0 \in \mathscr{P}^0(M_0)$ for G^0. Proposition 2.2 of [1(d)] applies to the distribution $J_\chi^T(f_\alpha)$, where T is a point in $\mathfrak{a}_0$ such that the function

$$d_{Q_0}(T) = \min_{\alpha \in \Delta_{Q_0}} \{\alpha(T)\}$$

is suitably large. The assertion is that there is a constant C_0 such that if

$$d_{Q_0}(T) > C_0 c(1+N), \tag{6.3}$$

and if f_α is as above, then $J_\chi^T(f_\alpha)$ equals an expression

$$\sum_{\{Q \in \mathscr{F}^0 :\ Q \supset Q_0\}} \int_{i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*} \Psi_{Q,\chi}^T(\Lambda, f_\alpha)\, d\Lambda.$$

Here,

$$\Psi_{Q,\chi}^T(\Lambda, f_\alpha) = |\mathscr{P}(M_Q)|^{-1} \operatorname{tr}(\Omega_{Q|sQ,\chi}^T(s\Lambda)\rho_{Q,\chi}(s,\Lambda,f_\alpha^1)),$$

where s is any element in W_0^G, $\rho_{Q,\chi}(s, \Lambda, f_\alpha^1)$ is the linear map from $\mathscr{A}^2_{Q,\chi}$ to $\mathscr{A}^2_{sQ,\chi}$ discussed in §4, and $\Omega^T_{Q|sQ,\chi}(s\Lambda)$ is the linear map from $\mathscr{A}^2_{sQ,\chi}$ to $\mathscr{A}^2_{Q,\chi}$ such that for any pair of vectors $\phi \in \mathscr{A}^2_{Q,\chi}$ and $\phi_s \in \mathscr{A}^2_{sQ,\chi}$,

$$(\Omega^T_{Q|sQ,\chi}(s, \Lambda)\phi_s, \phi)$$

equals

$$\int_{G(F)A_{G,\infty}\backslash G^0(\mathbb{A})} \Lambda^T E_{sQ}(x, \phi_s, s\Lambda) \cdot \overline{\Lambda^T E_Q(x, \phi, \Lambda)}\, dx.$$

(E_Q stands for the Eisenstein series associated to Q, and Λ^T is the truncation operator.) Therefore,

$$\sum_{\chi \in \mathscr{X}_1} |J^T_\chi(f_\alpha)| \tag{6.4}$$

is bounded by

$$\sum_{\chi \in \mathscr{X}_1} \sum_{Q \supset Q_0} |\mathscr{P}(M_Q)|^{-1} \int_{i\mathfrak{a}^*_Q/i\mathfrak{a}^*_G} \|\Omega^T_{Q|sQ,\chi}(s\Lambda)\rho_{Q,\chi}(s, \Lambda, f^1_\alpha)\|_1\, d\Lambda,$$

where $\|\cdot\|_1$ denotes the trace class norm.

Suppose that f is bi-invariant under an open compact subgroup K_0 of $G^0(\mathbb{A}_{\mathrm{fin}})$. According to Lemma A.1 of [1(d)], there are constants C_{K_0} and d_0 such that

$$\sum_{\chi \in \mathscr{X}_1} \sum_{Q \supset Q_0} |\mathscr{P}(M_Q)|^{-1} \int_{i\mathfrak{a}^*_Q/i\mathfrak{a}^*_G} \|\rho_{Q,\chi}(\Lambda, \Delta^m)^{-1}_{K_0} \cdot \Omega^T_{Q|sQ,\chi}(s\Lambda)_{K_0}\|_1\, d\Lambda$$

is bounded by

$$C_{K_0}(1 + \|T\|)^{d_0}, \tag{6.5}$$

where Δ^m is a certain left invariant differential operator on $G^0(F_\infty)^1$ and $(\cdot)_{K_0}$ denotes the restriction of a given operator to the space of K_0-invariant vectors. In order to exploit this estimate, we note that

$$\|\Omega^T_{Q|sQ}(s\Lambda)\rho_{Q,\chi}(s, \Lambda, f^1_\alpha)\|_1$$

is no greater than

$$\|\rho_{Q,\chi}(\Lambda, \Delta^m)^{-1}_{K_0} \cdot \Omega^T_{Q|sQ,\chi}(s\Lambda)_{K_0}\|_1 \cdot \|\rho_{Q,\chi}(s, \Lambda, \Delta^m f^1_\alpha)\|.$$

It follows that (6.4) is bounded by the product of (6.5) with

$$\sup_{\chi \in \mathscr{X}_1} \sup_{Q \supset Q_0} \sup_{\Lambda \in i\mathfrak{a}^*_Q/i\mathfrak{a}^*_G} \|\rho_{Q,\chi}(s, \Lambda, \Delta^m f^1_\alpha)\|. \tag{6.6}$$

Now $J^T_\chi(f_\alpha)$ is a polynomial in T, and $J_\chi(f_\alpha)$ is defined as its value at a fixed point T_0 [1(c), §2]. We can certainly interpolate $J_\chi(f_\alpha)$ from the values of $J^T_\chi(f_\alpha)$ in which T satisfies (6.3) [1(d), Lemma 5.2]. It follows that there is a constant C'_{K_0}, depending only on K_0, such that the original sum $\sum_{\chi\in\mathscr{X}_1}|J_\chi(f_\alpha)|$ is bounded by the product of (6.6) with $C'_{K_0}(1+N)^{d_0}$.

Consider the expression (6.6). For a given Q, write

$$\rho_{Q,\chi}(s,\Lambda,\Delta^m f^1_\alpha)=\bigoplus_{\sigma\in\Pi_{\mathrm{disc}}(M_Q,\chi)_\Gamma}\rho_{Q,\chi,\sigma}(s,\Lambda,\Delta^m f^1_\alpha),$$

where $\rho_{Q,\chi,\sigma}$ denotes the representation induced from the isotypical component of σ. Then

$$\|\rho_{Q,\chi}(s,\Lambda,\Delta^m f^1_\alpha)\|\le\sup_{\{\sigma\in\Pi_{\mathrm{disc}}(M_Q,\chi)_\Gamma\}}\|\rho_{Q,\chi,\sigma}(s,\Lambda,\Delta^m f^1_\alpha)\|.$$

Since

$$\Delta^m f^1_\alpha=(\Delta^m f)^1_\alpha,$$

the formula $(6.2')$ leads to an inequality

$$\begin{aligned}&\|\rho_{Q,\chi,\sigma}(s,\Lambda,\Delta^m f^1_\alpha)\|\\&\quad\le\int_{\mathfrak{a}_G}\|\rho_{Q,\chi,\sigma}(s,\Lambda,\Delta^m f^Z)\|\cdot|\alpha_G(\sigma^G_\Lambda,-Z)|\,dZ\\&\quad\le\int_{\mathfrak{a}_G}\left(\int_{G(\mathbb{A})^Z}|(\Delta^m f)(x)|\cdot\|\rho_{Q,\chi,\sigma}(s,\Lambda,x)\|\,dx\right)\cdot|\alpha_G(\sigma^G_\Lambda,-Z)|\,dZ.\end{aligned}$$

The operator $\rho_{Q,\chi,\sigma}(s,\Lambda,x)$ is unitary, and has norm equal to 1. Observe also that the function $\alpha_G(\sigma_\Lambda,Z)$ vanishes unless $\|Z\|\le N$. It follows that

$$\|\rho_{Q,\chi,\sigma}(s,\Lambda,\Delta^m f^1_\alpha)\|\le\left(\int_{G(\mathbb{A})_N}|\Delta^m f(x)|\,dx\right)\cdot\sup_{Z\in\mathfrak{a}_G}(|\alpha_G(\sigma^G_\Lambda,Z)|),$$

where

$$G(\mathbb{A})_N=\{x\in G(\mathbb{A}):\|H_G(x)\|\le N\}.$$

Since f is moderate, the intersection of its support with $G(\mathbb{A})_N$ is contained in a set

$$\{x\in G(\mathbb{A}):\log\|x\|\le c(N+1)\},$$

whose volume depends exponentially on N. Moreover, the supremum of $|\Delta f(x)|$ on $G(\mathbb{A})_N$ is bounded by a function which also depends exponentially on N. It follows that

$$\int_{G(\mathbb{A})_N}|\Delta^m f(x)|\,dx\le C_0e^{k_0N},$$

for constants C_0 and k_0 which are independent of N. On the other hand, we can write

$$\begin{aligned}\sup_{Z\in\mathfrak{a}_G}|\alpha_G(\sigma^G_\Lambda,Z)|&\le\int_{i\mathfrak{a}^*_G}|\hat{\alpha}(\nu_\sigma+\Lambda+\mu)|\,d\mu\\&\le C_G\sup_{\mu\in i\mathfrak{a}^*_G}\left((1+\|\Lambda+\mu\|^2)^{\dim\mathfrak{a}_G}|\hat{\alpha}(\nu_\sigma+\Lambda+\mu)|\right),\end{aligned}$$

where

$$C_G = \int_{i\mathfrak{a}_G^*} (1 + \|\mu\|^2)^{-\dim \mathfrak{a}_G} \, d\mu .$$

Combining these facts, we see that the expression (6.6) is bounded by the product of $C_G C_0 e^{k_0 N}$ with the supremum over $\chi \in \mathscr{X}_1$, $Q \supset Q_0$, $\Lambda \in i\mathfrak{a}_Q^*$ and $\sigma \in \Pi_{\text{disc}}(M_Q, \chi)_\Gamma$ of

$$(1 + \|\Lambda\|^2)^{\dim \mathfrak{a}_G} |\hat{\alpha}(\nu_\sigma + \Lambda)| .$$

We can now state an estimate for

$$\sum_{\chi \in \mathscr{X}_1} |J_\chi(f_\alpha)| . \tag{6.7}$$

In order to remove the dependence on Q_0, we shall replace the supremum over Q by one over $L_0 \in \mathscr{L}^0$. Choose positive constants C_1' and k_1' such that

$$C_{K_0}'(1 + N)^{d_0} C_G \cdot C_0 e^{k_0 N} \leq C_1' e^{k_1' N} .$$

Then (6.7) is bounded by the supremum over $\chi \in \mathscr{X}_1$, $L_0 \in \mathscr{L}^0$, $\Lambda \in i\mathfrak{a}_{L_0}^*$ and $\sigma \in \Pi_{\text{disc}}(L_0, \chi)_\Gamma$ of

$$C_1' e^{k_1' N} (1 + \|\Lambda\|^2)^{\dim \mathfrak{a}_G} |\hat{\alpha}(\nu_\sigma + \Lambda)| .$$

To remove the factors $(1 + \|\Lambda\|^2)$ from the estimate, we require a simple lemma.

Lemma 6.4. *For any integer $m \geq 1$ we can choose a bi-invariant differential operator z on $G(F_\infty)$, and multipliers $\alpha_1 \in C_c^m(\mathfrak{h})^W$ and $\alpha_2 \in C_c^\infty(\mathfrak{h})^W$ such that $f = (zf)_{\alpha_1} + f_{\alpha_2}$, for any function $f \in \mathscr{H}_{\text{ac}}(G(\mathbb{A}))$.*

Proof. This follows from a standard argument, which was first applied to the trace formula by Duflo and Labesse (see for example [1(a), Lemma 4.1].) For any m, one obtains a W-invariant differential operator ζ with constant coefficients on $\mathfrak{h}$, and functions $\alpha_1 \in C_c^m(\mathfrak{h})^W$ and $\alpha_2 \in C_c^\infty(\mathfrak{h})^W$, such that $\zeta\alpha_1 + \alpha_2$ is the Dirac measure at the origin in $\mathfrak{h}$. Let z be the inverse image of ζ under the Harish-Chandra map. Then

$$f = f_{(\zeta\alpha_1 + \alpha_2)} = (zf)_{\alpha_1} + f_{\alpha_2} ,$$

as required. □

Returning to the proof of Lemma 6.3, we apply Lemma 6.4, with m large, to our moderate function f. We see that (6.7) is bounded by

$$\sum_{\chi \in \mathscr{X}_1} |J_\chi((zf)_{\alpha_1 * \alpha}| + \sum_{\chi \in \mathscr{X}_1} |J_\chi(f_{\alpha_2 * \alpha})| .$$

Since the function zf is also moderate, we can apply the estimate we have obtained to each of these sums. Notice that

$$\begin{aligned}
&\sup_{\chi, L_0, \Lambda, \sigma} ((1+\|\Lambda\|^2)^{\dim \mathfrak{a}_G} |(\alpha_i * \alpha)^\wedge(\nu_\sigma + \Lambda)|) \\
&\quad \leq \sup((1+\|\Lambda\|^2)^{\dim \mathfrak{a}_G} |\hat{\alpha}_i(\nu_\sigma+\Lambda)| \cdot |\hat{\alpha}(\nu_\sigma+\Lambda)|) \\
&\quad \leq \sup((1+\|\Lambda\|^2)^{\dim \mathfrak{a}_G} |\hat{\alpha}_i(\nu_\sigma+\Lambda)|) \cdot \sup |\hat{\alpha}(\nu_\sigma+\Lambda)| \\
&\quad \leq \sup((1+\|\nu_\sigma+\Lambda\|^2)^{\dim \mathfrak{a}_G} |\hat{\alpha}_i(\nu_\sigma+\Lambda)|) \cdot \sup |\hat{\alpha}(\nu_\sigma+\lambda)|.
\end{aligned}$$

But the real parts of the points ν_σ lie in a fixed bounded set, and the functions $\hat{\alpha}_i$ decrease rapidly on cylinders (in a sense that depends on m). Therefore

$$\sup((1+\|\nu_\sigma+\Lambda\|^2)^{\dim \mathfrak{a}_G} |\hat{\alpha}_i(\nu_\sigma+\Lambda)|) < \infty.$$

It follows that there are positive constants C_1 and k_1 such that (6.7) is bounded by

$$C_1 e^{k_1 N} \sup_{\chi \in \mathscr{X}_1} \sup_{L_0, \Lambda, \sigma} (|\hat{\alpha}(\nu_\sigma+\Lambda)|).$$

We must convert this into an estimate for

$$\sum_{\chi \in \mathscr{X}_1} |\hat{I}_\chi(\phi_\alpha)|. \tag{6.8}$$

Suppose that $M \in \mathscr{L}_0$. It follows from Corollary 12.3 of [1(i)] that the function $\phi_M(f)$ in $\mathscr{I}_{\mathrm{ac}}(M(\mathbb{A}))$ is also moderate. Since

$$\hat{I}_\chi^M(\phi_M(f_\alpha)) = \hat{I}_\chi^M(\phi_M(f)_\alpha),$$

we can apply the lemma inductively to $\phi_M(f_\alpha)$. We obtain constants C_M and k_M, depending only on f, such that

$$\sum_{\chi \in \mathscr{X}_1} |\hat{I}_\chi^M(\phi_M(f_\alpha))|$$

is bounded by

$$C_M e^{k_M N} \sup_{\chi \in \mathscr{X}_1} \sup_{L_0, \Lambda, \sigma} (|\hat{\alpha}(\nu_\sigma+\Lambda)|).$$

The required estimate for (6.8) then follows from the estimate for (6.7) and the formula

$$\hat{I}_\chi(\phi_\alpha) = J_\chi(f_\alpha) - \sum_{M \in \mathscr{L}_0} |W_0^M| |W_0^G|^{-1} \hat{I}_\chi^M(\phi_M(f_\alpha)). \quad \square$$

We shall restate the lemma in a simple form that is convenient for applications. Let $\mathfrak{h}_u^*$ denote the set of elements ν in $\mathfrak{h}_{\mathbb{C}}^*/i\mathfrak{a}_G^*$ such that $\overline{\nu} = s\nu$ for some element $s \in W$ of order 2. Here $\overline{\nu}$ stands for the conjugation of $\mathfrak{h}_{\mathbb{C}}^*$ relative to $\mathfrak{h}^*$. As is well known, the infinitesimal character ν_π of any unitary representation $\pi \in \Pi_{\mathrm{unit}}(G^0(\mathbb{A})^1)$ belongs to $\mathfrak{h}_u^*$. Observe that if r and T are nonnegative real numbers, the set

$$\mathfrak{h}_u^*(r, T) = \{\nu \in \mathfrak{h}_u^* : \|\mathscr{R}\mathrm{e}(\nu)\| \leq r, \|\mathscr{I}\mathrm{m}(\nu)\| \geq T\}$$

is invariant under W. (An element $\nu \in \mathfrak{h}_u^*$ is only a coset of $i\mathfrak{a}_G^*$ in $\mathfrak{h}_{\mathbb{C}}^*$, but $\|\nu\|$ is understood to be the minimum value of the norm on the coset.) Let $\mathfrak{h}^1$ be the orthogonal complement of $\mathfrak{a}_G$ in $\mathfrak{h}$. Then $\mathfrak{h}_u^*$ can be identified with a subset of the complex dual space of $\mathfrak{h}^1$.

Corollary 6.5. *Choose any function $f \in \mathscr{H}_{\mathrm{ac}}(G(\mathbb{A}))$. Then there are positive constants C, k and r such that*

$$\sum_{t>T} |I_t(f_\alpha)| \le Ce^{kN} \sup_{\nu\in\mathfrak{h}_u^*(r,T)} (|\hat{\alpha}(\nu)|),$$

for any $T>0$ and any $\alpha \in C_N^\infty(\mathfrak{h}^1)^W$, with $N>0$.

Proof. Lemma 6.3 is stated for multipliers in $C_N^\infty(\mathfrak{h})^W$, but it is equally valid if α belongs to $C_N^\infty(\mathfrak{h}^1)^W$. To see this, apply the lemma to the sequence

$$\alpha_n(H+Z) = \alpha(H)\beta_n(Z), \qquad H \in \mathfrak{h}^1, Z \in \mathfrak{a}_G,$$

in $C_c^\infty(\mathfrak{h})^W$, where $\beta_n \in C_c^\infty(\mathfrak{a}_G)$ approaches the Dirac measure at 1. The (upper) limits of each side of the resulting inequality give the analogous inequality for α. Notice that f_α^1 depends only on f^1, so that f can indeed be an arbitrary function in $\mathscr{H}_{\mathrm{ac}}(G(\mathbb{A}))$.

We shall apply this version of the lemma to the given α, with $\phi = f_G$, and with

$$\mathscr{X}_1 = \{\chi \in \mathscr{X} : \|\mathscr{I}\mathrm{m}(\nu_\chi) > T\|\}.$$

Then

$$\sum_{t>T} |I_t(f_\alpha)| = \sum_{\chi\in\mathscr{X}_1} |\hat{I}_\chi(\phi_\alpha)|.$$

Choose a finite subset Γ of $\Pi(K)$ such that ϕ belongs to $\mathscr{I}_{\mathrm{ac}}(G(\mathbb{A}))_\Gamma$. There is a positive number r such that if π is any representation in $\Pi_{\mathrm{unit}}(G^0(\mathbb{A}))$ whose K-spectrum meets Γ, the point ν_π belongs to

$$\{\nu \in \mathfrak{h}_u^* : \|\mathscr{R}\mathrm{e}(\nu)\| \le r\}.$$

If χ, L_0, Λ and σ are elements in $\mathscr{X}_1$, $\mathscr{L}^0$, $i\mathfrak{a}_{L_0}^*$ and $\Pi_{\mathrm{disc}}(L_0, \chi)_\Gamma$, as in the inequality of the lemma, the point $\nu_\sigma + \Lambda$ then belongs to $\mathfrak{h}_u^*(r, T)$. The corollary follows. □

Remark. Suppose that $\mathfrak{h}^2$ is any vector subspace of $\mathfrak{h}$ which contains $\mathfrak{h}^1$. Then there will be an obvious variant of Corollary 6.5 for multipliers $\alpha \in C_N^\infty(\mathfrak{h}^2)^W$. For this, f must again be taken to be a moderate function in $\mathscr{H}_{\mathrm{ac}}(G(\mathbb{A}))$.

7. Simpler forms of the trace formula

The full trace formula is the identity

$$\sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma\in(M(F))_{M,S}} a^M(S,\gamma) I_M(\gamma, f)$$

$$= \sum_{t\ge 0}\sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi(M,t)} a^M(\pi) I_M(\pi, f)\, d\pi, \qquad f \in \mathscr{H}(G(\mathbb{A})),$$

given by the two expansions for $I(f)$ in Theorems 3.3 and 4.4. In this section we shall investigate how the formula simplifies if conditions are imposed on f. The conditions will be invariant, in the sense that they depend only on the image of f in $\mathcal{I}(G(\mathbb{A}))$. Equivalently, the conditions will depend only on the (invariant) orbital integrals of f.

We shall say that a function $f \in \mathcal{H}(G(\mathbb{A}))$ is *cuspidal* at a valuation v_1 if f is a finite sum of functions $\prod_v f_v$, $f_v \in \mathcal{H}(G(F_v))$, such that

$$f_{v_1,M} = 0, \qquad M \in \mathcal{L}_0.$$

This is implied by the vanishing of the orbital integral $I_G(\gamma_1, f_{v_1})$, for any G-regular element $\gamma_1 \in G(F_{v_1})$ which is not F_{v_1}-elliptic.

Theorem 7.1. (a) *If f is cuspidal at one place v_1, we have*

$$I(f) = \sum_{t \geq 0} \sum_{\pi \in \Pi_{\mathrm{disc}}(G,t)} a^G_{\mathrm{disc}}(\pi) I_G(\pi, f).$$

(b) *If f is cuspidal at two places v_1 and v_2, we have*

$$I(f) = \sum_{\gamma \in (G(F))_{G,S}} a^G(S, \gamma) I_G(\gamma, f).$$

Proof. We can assume that $f = \prod_v f_v$, with

$$f_{v_1,M} = 0, \qquad M \in \mathcal{L}_0.$$

Part (a) will be a special case of the spectral expansion

$$I(f) = \sum_{t \geq 0} \sum_{M \in \mathcal{L}} |W_0^M| \, |W_0^G|^{-1} \int_{\Pi(M,t)} a^M(\pi) I_M(\pi, f)\, d\pi.$$

The main step is to show that if $M \in \mathcal{L}_0$, then

$$I_M(\pi, f) = 0, \qquad \pi \in \Pi_{\mathrm{unit}}(M(\mathbb{A})^1).$$

But this is very similar to the proof of Lemma 5.2. Using the splitting formula [1(j), Proposition 9.4], we reduce the problem to showing that

$$I_M(\pi_1, X_1, f_{v_1}) = 0, \qquad \pi_1 \in \Pi_{\mathrm{unit}}(M(F_{v_1})),\ X_1 \in \mathfrak{a}_{M,v_1},\ M \in \mathcal{L}_0.$$

We then apply the expansion [1(j), (3.2)] into standard representations, and the descent formula [1(j), Corollary 8.5]. Since π_1 is unitary, the required vanishing formula follows as in Lemma 5.2. In particular, the terms with $M \neq G$ in the spectral expansion all vanish. Moreover,

$$\begin{aligned}
&\int_{\Pi(G,t)} a^G(\pi) I_G(\pi, f)\, d\pi \\
&= \sum_{M_1 \in \mathcal{L}} |W_0^{M_1}| \, |W_0^G|^{-1} \sum_{\pi_1 \in \Pi_{\mathrm{disc}}(M_1,t)} \int_{i\mathfrak{a}^*_{M_1}/i\mathfrak{a}^*_G} a^{M_1}_{\mathrm{disc}}(\pi_1) r^G_{M_1}(\pi_{1,\lambda}) I_G(\pi^G_{1,\lambda}, f)\, d\lambda \\
&= \sum_{\pi \in \Pi_{\mathrm{disc}}(G,t)} a^G_{\mathrm{disc}}(\pi) I_G(\pi, f),
\end{aligned}$$

since

$$I_G(\pi_1^G, f) = \hat{I}_{M_1}(\pi_1, f_{M_1}) = 0, \qquad M_1 \neq G, \ \pi_1 \in \Pi_{\text{unit}}(M_1(\mathbb{A}^1)).$$

Part (a) follows.

Suppose that f is also cuspical at a second place v_2. Part (b) will be a special case of the geometric expansion

$$I(f) = \sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma\in(M(F))_{M,S}} a^M(S,\gamma)I_M(\gamma, f).$$

The set S is large enough that it contains v_1 and v_2, and so that f belongs to $\mathscr{H}(G(F_S))$. Write

$$f = f_1 f_2, \qquad f_i \in \mathscr{H}(G(F_{S_i})),$$

where S_1 and S_2 are disjoint sets of valuations with the closure property, which contain v_1 and v_2 respectively, and whose union is S. From the splitting formula [1(j), Proposition 9.1], we obtain

$$I_M(\gamma, f) = \sum_{L_1, L_2\in\mathscr{L}(M)} d_M^G(L_1, L_2)\hat{I}_M^{L_1}(\gamma, f_{1,L_1})\hat{I}_M^{L_2}(\gamma, f_{2,L_2}).$$

The distributions on the right vanish unless $L_1 = L_2 = G$. Moreover, $d_M^G(G, G) = 0$ unless $M = G$. It follows that if $M \neq G$, the distribution $I_M(\gamma, f)$ equals 0, and the corresponding term in the geometric expansion vanishes. This gives (b). □

Corollary 7.2. *Suppose that f is cuspidal at two places. Then*

$$\sum_{\gamma\in(G(F))_{G,S}} a^G(S,\gamma)I_G(\gamma, f) = \sum_{t\geq 0}\sum_{\pi\in\Pi_{\text{disc}}(G,t)} a_{\text{disc}}^G(\pi)I_G(\pi, f). \quad \square$$

For simplicity, we shall assume that $G = G^0$ in the rest of §7. We shall also assume that $f \in \mathscr{H}(G(\mathbb{A}))$ is such that

$$f = \prod_v f_v, \qquad f_v \in \mathscr{H}(G(F_v)).$$

With additional invariant restrictions on f we shall be able to simplify the trace formula further.

Corollary 7.3. *Suppose there is a place v_1 such that*

$$\operatorname{tr}(\pi_1(f_{v_1})) = 0, \qquad \pi_1 \in \Pi_{\text{unit}}(G(F_{v_1})),$$

whenever π_1 is a constituent of a (properly) induced representation

$$\sigma_1^G, \sigma_1 \in \Pi_{\text{unit}}(M(F_{v_1})), \qquad M \in \mathscr{L}_0.$$

Then

$$I(f) = \sum_{t\geq 0} \operatorname{tr}(R_{\text{disc},t}(f)),$$

where $R_{\text{disc},t}$ denotes the representation of $G(\mathbb{A})$ on $L^2_{\text{disc},t}(G(F)A_{G,\infty}\backslash G(\mathbb{A}))$.

Proof. If M belongs to $\mathscr{L}_0$ the condition implies that

$$\operatorname{tr}(\sigma_1^G(f_{v_1}))=0, \qquad \sigma_1\in\Pi_{\text{temp}}(M(F_{v_1})),$$

so that $f_{v_1,M}=0$. Therefore f is cuspidal at v_1. Applying part (a) of the theorem, we obtain

$$\begin{aligned} I(f) &= \sum_{t\geq 0}\sum_{\pi\in\Pi_{\text{disc}}(G,t)} a^G_{\text{disc}}(\pi)I_G(\pi,f)\\ &= \sum_{t\geq 0}\sum_{M\in\mathscr{L}} |W_0^M|\,|W_0^G|^{-1}\\ &\quad\times\sum_{s\in W(\mathfrak{a}_M)_{\text{reg}}} |\det(s-1)_{\mathfrak{a}_M^G}|^{-1}\operatorname{tr}(M_{Q|sQ}(0)\rho_{Q,t}(s,0,f)),\end{aligned}$$

in the notation of §4. Here, Q is any element in $\mathscr{P}(M)$. If $M\neq G$,

$$\operatorname{tr}(M_{Q|sQ}(0)\rho_{Q,t}(s,0,f))$$

is a linear combination of characters of unitary induced representations. It vanishes by assumption. If $M=G$,

$$\operatorname{tr}(M_{Q|sQ}(0)\rho_{Q,t}(s,0,f))=\operatorname{tr}(\rho_{G,t}(0,f))=\operatorname{tr}(R_{\text{disc},t}(f)),$$

by definition. The corollary follows. □

Corollary 7.4. *Suppose there is a place* v_1 *such that*

$$I_G(\gamma_1,f_{v_1})=0$$

for any element $\gamma_1\in G(F_{v_1})$ *which is not semisimple and* F_{v_1}*-elliptic. Suppose also that* f *is cuspidal at another place* v_2. *Then*

$$I(f)=\sum_{\gamma\in\{G(F)_{\text{ell}}\}}\operatorname{vol}(G(F,\gamma)A_{G,\infty}\backslash G(\mathbb{A},\gamma))\int_{G(\mathbb{A},\gamma)\backslash G(\mathbb{A})} f(x^{-1}\gamma x)\,dx,$$

where $\{G(F)_{\text{ell}}\}$ *denotes the set of* $G(F)$*-conjugacy classes of* F*-elliptic elements in* $G(F)$, *and* $G(F,\gamma)$ *and* $G(\mathbb{A},\gamma)$ *denote the centralizers of* γ *in* $G(F)$ *and* $G(\mathbb{A})$.

Proof. The conditions imply that f is cuspidal at v_1 and v_2. We can therefore apply the formula

$$I(f)=\sum_{\gamma\in(G(F))_{G,S}} a^G(S,\gamma)I_G(\gamma,f)$$

of the theorem. If an element $\gamma\in G(F)$ is not F-elliptic, it is not F_{v_1}-elliptic, and $I_G(\gamma,f)=0$. The corollary then follows from Theorem 8.2 of [1(g)] and the definition of $I_G(\gamma,f)$. □

The conditions of Corollaries 7.3 and 7.4 sometimes arise naturally. For example, if v_1 is discrete, Kottwitz [11(b)] has introduced a simple function

f_{v_1} which satisfies the conditions of Corollary 7.4. Kottwitz also establishes a version of this corollary in [11(b)]. He imposes stronger conditions at v_2, but derives a formula without resorting to the invariant trace formula.

For another example, take $G = GL(n)$. Suppose that f is cuspidal at v_1. Any element $\gamma_1 \in G(F_{v_1})$ which is not F_{v_1}-elliptic belongs to a $G(F_{v_1})$-conjugacy class

$$\delta_1^G, \qquad \delta_1 \in M(F_{v_1}), \; M \in \mathscr{L}_0.$$

Consequently,

$$I_G(\gamma_1, f_{v_1}) = \hat{I}_M^M(\delta_1, f_{v_1,M}) = 0.$$

Therefore, the first condition of Corollary 7.4 is satisfied. Moreover, it is known that any induced unitary representation

$$\sigma_1^G, \qquad \sigma_1 \in \Pi_{\mathrm{unit}}(M(F_{v_1})), \; M \in \mathscr{L},$$

is irreducible ([3], [15]). Since

$$\mathrm{tr}(\sigma_1^G(f_{v_1})) = f_{v_1,M}(\sigma_1) = 0, \qquad M \in \mathscr{L}_0,$$

the condition of Corollary 7.3 also holds. Combining Corollaries 7.3 and 7.4, we obtain

Corollary 7.5. *Assume that $G = GL(n)$ and that f is cuspidal at two places v_1 and v_2. Then*

$$\sum_{\gamma \in \{G(F)_{\mathrm{ell}}\}} \mathrm{vol}(G(F, \gamma)A_{G,\infty} \setminus G(\mathbb{A}, \gamma)) \int_{G(\mathbb{A},\gamma)\backslash G(\mathbb{A})} f(x^{-1}\gamma x)\, dx$$
$$= \sum_{t \geq 0} \mathrm{tr}(R_{\mathrm{disc},t}(f)). \quad \square$$

8. The example of $GL(n)$. Global vanishing properties

The simple versions of the trace formula were obtained by placing rather severe restrictions on f. In many applications, one will need to prove that certain terms vanish for less severely restricted functions. We can illustrate this with the example of $GL(n)$, begun in §10 of [1(j)].

Adopt the notation of [1(j),§10]. Then

$$\eta\colon G \to G^* = \underbrace{(GL(n) \times \cdots \times GL(n))}_{l} \rtimes \theta^*$$

is a given inner twist, and G' stands for the group $GL(n)$, embedded diagonally in $(G^*)^0$. Let us write $\mathscr{L}'$ for the set of Levi subgroups of G' which contain the group of diagonal matrices. For each $L \in \mathscr{L}'$, we have the partition

$$\mathfrak{p}(L) = (n_1, \ldots, n_r), \qquad n_1 \geq n_2 \geq \cdots \geq n_r,$$

of n such that

$$L \cong GL(n_1) \times \cdots \times GL(n_r).$$

Suppose that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are partitions of n. We shall write $\mathfrak{p}_1 \le \mathfrak{p}_2$, as in [1(c), §14], if there are groups $L_1 \subset L_2$ in $\mathcal{L}'$ such that $\mathfrak{p}_1 = \mathfrak{p}(L_1)$ and $\mathfrak{p}_2 = \mathfrak{p}(L_2)$.

We shall assume that $\eta(M_0)$ is contained in a standard Levi subgroup of $(G^*)^0$, and that the restriction of η to A_{M_0} is defined over F. Then the map

$$M \to M' = \eta(M^0) \cap G', \qquad M \in \mathcal{L},$$

is an injection of $\mathcal{L}$ into $\mathcal{L}'$. The image of this map is easy to describe. For as in [1(j), §10], we can assume that

$$G^0(E) = \underbrace{GL(\tfrac{n}{d}, D \otimes E) \times \cdots \times GL(\tfrac{n}{d}, D \otimes E)}_{l_1},$$

where E/F is a cyclic extension of degree $l_E = l l_1^{-1}$, d is a divisor of n, and D is a division algebra of degree d^2 over F. The minimal group M' in the image corresponds to the partition $\mathfrak{p}(d) = (d, \dots, d)$. The other groups in the image correspond to partitions $(n_1, \dots, n_r)$ such that d divides each n_i. For each valuation v, we shall write d_v for the order of the invariant of the division algebra at v. Then d is the least common multiple of the integers d_v.

In [1(j), §10], we described the norm mapping $\gamma \to \gamma'$ from (orbits in) $G(F)$ to (conjugacy classes in) $G'(F)$. It can be defined the same way for any element $M \in \mathcal{L}$. We also investigated certain functions on the local groups $G'(F_v)$. Let $f' = \prod_v f'_v$ be a fixed function in $\mathcal{H}(G'(\mathbb{A}))$ whose local constituents satisfy [1(j), (10.1)]. That is, the orbital integrals of f'_v vanish at the G'-regular elements which are not local norms.

Proposition 8.1. *Suppose that $L \in \mathcal{L}'$ and that $\delta \in L(F)$. Embed δ in $(L(F))_{L,S}$, where $S \supset S_{\mathrm{ram}}$ is a large finite set of valuations. Then*

$$I_L(\delta, f') = 0,$$

unless $L = M'$ and $\delta = \gamma'$, for elements $M \in \mathcal{L}$ and $\gamma \in M(F)$.

Proof. In the orbital integral, δ is to be considered as a point in $L(F_S)$. We must therefore regard $f' = \prod_{v\in S} f'_v$ as an element in $\mathcal{H}(G'(F_S))$. Assume that $I_L(\delta, f') \neq 0$. We must deduce that $L = M'$ and $\delta = \gamma'$.

The first part of the proof is taken from p. 73 of [1(c)]. Applying the splitting formula [1(j), Corollary 9.2], we obtain

$$I_L(\delta, f') = \sum_{\{L_v\}} d(\{L_v\}) \hat{I}_L^{L_v}(\delta, f'_{v,L_v}), \tag{8.1}$$

where the sum is taken over collections $\{L_v \in \mathcal{L}(L) : v \in S\}$, and $d(\{L_v\})$ is a constant which vanishes unless

$$\mathfrak{a}_L^G = \bigoplus_{v\in S} \mathfrak{a}_L^{L_v}. \tag{8.2}$$

By assumption, the left hand side of (8.1) is nonzero. Therefore, there is a collection $\{L_v\}$ for which (8.2) holds, and such that $\hat{I}_L^{L_v}(\delta, f'_{v,L_v}) \neq 0$ for each $v \in S$. This implies that

$$\mathfrak{p}(d_v) \leq \mathfrak{p}(L_v), \qquad v \in S.$$

Our first task is to show that $\mathfrak{p}(d) \leq \mathfrak{p}(L)$. Let p be any rational prime, and let p^k be the highest power of p which divides d. Since d is the least common multiple of $\{d_v\}$, there is a valuation $v \in S$ such that p^k divides d_v. But the invariants of a central simple algebra sum to 0, so there must be a valuation $w \in S$, distinct from v, such that p^k also divides d_w. It follows that $\mathfrak{p}(p^k) \leq \mathfrak{p}(L_v)$ and $\mathfrak{p}(p^k) \leq \mathfrak{p}(L_w)$. Since $\mathfrak{a}_L^{L_v} \cap \mathfrak{a}_L^{L_w} = \{0\}$, we can apply Lemma 14.1 of [1(c)]. The result is that $\mathfrak{p}(p^k) \leq \mathfrak{p}(L)$. In other words, the integer p^k divides each of the numbers $n_1, \ldots, n_r$ which make up the partition $\mathfrak{p}(L)$. The same is therefore true of the integer d, so that $\mathfrak{p}(d) \leq \mathfrak{p}(L)$. In other words, $L = M'$ for an element $M \in \mathscr{L}$.

The next step is to show that δ belongs to the set

$$\begin{aligned} M'(F_S)^M &= \prod_{v \in S} M'(F_v)^M \\ &= \prod_v \{m_v \in M'(F_v) : \xi(m_v) \in N_{E_v/F_v}(E_v^*), \xi \in X(M')_F\}. \end{aligned}$$

Assume the contrary. Then there is a character $\xi \in X(M')_F$ such that $\xi(\delta)$ is not a local norm at some place. Consequently, $\xi(\delta)$ is not a global norm. It follows from global class field theory that $\xi(\delta)$ is not a local norm at two places v_1 and v_2. We can assume that v_1 and v_2 both belong to S, and that the sets $S_1 = S - \{v_2\}$ and $S_2 = \{v_2\}$ both have the closure property. (In other words, if S contains an Archimedean valuation, so does S_1.) Define

$$f'_i = \prod_{v \in S_i} f'_v, \qquad i = 1, 2.$$

Then by the splitting formula [1(j), Proposition 9.1], we have

$$I_L(\delta, f') = \sum_{L_1, L_2 \in \mathscr{L}(L)} d_L^G(L_1, L_2) \hat{I}_L^{L_1}(\delta, f'_{1,L_1}) \hat{I}_L^{L_2}(\delta, f'_{2,L_2}).$$

It follows that there is a pair L_1, $L_2 \in \mathscr{L}(L)$ such that $d_L^G(L_1, L_2) \neq 0$, and

$$\hat{I}_L^{L_i}(\delta, f'_{i,L_i}) \neq 0, \qquad i = 1, 2.$$

Now, by Lemma 10.1 of [1(j)], we can write

$$\xi(\delta) = \xi_1(\delta)\xi_2(\delta), \qquad \xi_1 \in X(L_1)_F, \ \xi_2 \in X(L_2)_F.$$

Suppose that $\xi_1(\delta)$ is a global norm. Then it is everywhere a local norm, so that $\xi_2(\delta)$ is not a local norm at v_2. It follows without difficulty from the given property of f'_{v_2} that $\hat{I}_L^{L_2}(\delta, f'_{2,L_2})$ vanishes. This is a contradiction. On the other hand, if $\xi_1(\delta)$ is not a global norm, it is not a local norm at two places in S. At least one of these places must belong to S_1. It follows easily that $\hat{I}_L^{L_1}(\delta, f'_{1,L_1})$ vanishes. This too is a contradiction. It follows that δ belongs to the set $M'(F_S)^M$.

The final step is to apply [1(j), Proposition 10.2]. This vanishing result was stated only for local fields, but by the splitting formula it extends immediately to $G'(F_S)$. Since $I_{M'}(\delta, f')$ does not vanish, and since δ belongs to $M'(F_S)^M$, the element δ must belong to a smaller set

$$M'(F_S)_M = \prod_{v \in S} M'(F_v)_M.$$

(The set $M'(F_v)_M$ was defined in the preamble to [1(j), Proposition 10.2].) Now, any element in $M'(F_v)_M$ is the local norm of an element in $M(F_v)$ [1(j), Lemma 10.4]. Since S is large, this implies that δ is everywhere a local norm. One can then show that δ is the global norm of an element in $M(F)$ (see [2, Lemma I.1.2). In other words, $\delta = \gamma'$, for some element $\gamma \in M(F)$. This completes the proof of the proposition. □

Proposition 8.2. *Suppose that $L_1 \subset L$ are elements in $\mathscr{L}'$ and that $S \supset S_{\mathrm{ram}}$ is a large finite set of valuations. Then*

$$I_L(\pi, Y, f') = 0,$$

for any $Y \in \mathfrak{a}_L$ and any induced representation

$$\pi = \pi_1^L, \qquad \pi_1 \in \Pi(L_1(F_S)),$$

unless both L_1 and L are the images of elements in $\mathscr{L}$.

Proof. Suppose that $I_L(\pi, Y, f') \neq 0$. Using the splitting formula [1(j), Proposition 9.4], we first argue as at the beginning of the proof of the last proposition. This establishes that $L = M'$ for some element $M \in \mathscr{L}$. We then apply the local vanishing property [1(j), Proposition 10.3]. This proves that $L_1 = M_1'$ for another $M_1 \in \mathscr{L}$. □

Propositions 8.1 and 8.2 are the first steps toward comparing the trace formulas of G and G'. They assert that for functions f' on $G'(\mathbb{A})$ as above, the distributions vanish at data which do not come from G. The trace formula for G' becomes

$$\sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in (M(F))_{M,S}} a^{M'}(S, \gamma') I_{M'}(\gamma', f')$$
$$= \sum_{t \geq 0} \sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi(M', t)} a^{M'}(\pi) I_{M'}(\pi, f')\, d\pi.$$

It is considerably harder to compare the terms which remain with the corresponding terms for G. This problem will be one of the main topics of [2].

APPENDIX. THE TRACE PALEY-WIENER THEOREMS

We shall prove Lemma 6.1. The result can be extracted from the trace Paley-Wiener theorems [6(a), 6(b), 5 and 14] for real and p-adic groups. For implicit in these papers is the existence of a continuous section $\phi \to f$ from $\mathscr{I}(G(F_S))_\Gamma$ to $\mathscr{H}(G(F_S))_\Gamma$, in which the growth and support properties of f can be estimated in terms of those of ϕ. I am indebted to J. Bernstein for explaining this to me in the p-adic case.

Suppose that S is any finite set of valuations of F with the closure property. The notion of a moderate function $f \in \mathscr{H}_{\mathrm{ac}}(G(F_S))$ can be characterized in terms of the behavior of the functions

$$f^b(x) = f(x)b(H_G(x)), \qquad b \in C_c^\infty(\mathfrak{a}_{G,S}).$$

Indeed f will be moderate if and only if there are positive constants c and d such that for any $N > 0$, and any $b \in C_N^\infty(\mathfrak{a}_{G,S})$,

(i) f^b belongs to $\mathscr{H}_{c(N+1)}(G(F_S))$, the set of functions in $\mathscr{H}(G(F_S))$ supported on the ball of radius $c(N+1)$, and

(ii) $\|f^b\| \leq \delta(b)d^N$.

Here,

$$\|h\| = \sup_{x\in G(F_S)} |\Delta h(x)|, \qquad h \in \mathscr{H}(G(F_S)), \tag{A.1}$$

where Δ is an arbitrary (but fixed) left invariant differential operator on $G(F_{S_\infty \cap S})$, while

$$\delta(b) = \sum_{k=1}^{r} \sup_{X\in\mathfrak{a}_{G,S}} |D_k b(X)|, \qquad b \in C_c^\infty(\mathfrak{a}_{G,S}), \tag{A.2}$$

for invariant differential operators $D_1, \dots, D_r$ on $\mathfrak{a}_{G,S}$ which depend only on Δ. (If S consists of one discrete valuation, we take Δ and $\{D_k\}$ to be constants.) The reader can check that this definition is equivalent to the one in §6. Similarly, the notion of a moderate function $\phi \in \mathscr{I}_{\mathrm{ac}}(G(F_S))$ can be defined in terms of the behavior of the functions

$$\phi^b(\pi, X) = \phi(\pi, X)b(X), \qquad b \in C_c^\infty(\mathfrak{a}_{G,S}).$$

More precisely, ϕ is said to be *moderate* if there are positive constants c and d such that for any $N > 0$, and any $b \in C_N^\infty(\mathfrak{a}_{G,S})$,

(i) ϕ^b belongs to $\mathscr{I}_{c(N+1)}(G(F_S))$, and

(ii) $\|\phi^b\|' \leq \delta(b)d^N$.

Recall [1(i)] that $\mathscr{I}_{c(N+1)}(G(F_S))$ is the set of $\psi \in \mathscr{I}(G(F_S))$ such that for every Levi subset $\mathscr{M} = \prod_{v\in S} M_v$ of G over F_S, and every representation

$$\sigma = \bigotimes_v \sigma_v, \qquad \sigma_v \in \Pi_{\mathrm{temp}}(M_v(F_v)),$$

the function

$$\psi(\sigma, \mathscr{X}) = \int_{i\mathfrak{a}^*_{\mathscr{M},S}/i\mathfrak{a}_{G,S}} \psi(\sigma^G_\Lambda, h_G(\mathscr{X}))e^{-\Lambda(\mathscr{X})}\, d\Lambda, \qquad \mathscr{X} \in \mathfrak{a}_{\mathscr{M},S},$$

is supported on the ball of radius $c(N+1)$. In the second condition, it is understood that

$$\|\psi\|' = \sup_{\mathscr{X}\in\mathfrak{a}_{\mathscr{M},S}} |\Delta'\psi(\sigma, \mathscr{X})|, \qquad \psi \in \mathscr{I}(G(F_S)), \tag{A.3}$$

where Δ' is an arbitrary invariant differential operator on $\mathfrak{a}_{\mathscr{M},S\cap S_\infty}$ for some fixed $\mathscr{M}$ and σ, while $\delta(b)$ is a seminorm on $C_c^\infty(\mathfrak{a}_{G,S})$ of the form (A.2) which depends only on Δ'.

Lemma A.1. *Suppose that Γ is a finite subset of $\Pi(K)$. Then there is a continuous linear map*

$$h: \mathscr{I}(G(F_S))_\Gamma \to \mathscr{H}(G(F_S))_\Gamma$$

with the following four properties.

(a) $h(\phi)_G = \phi$, $\phi \in \mathscr{I}(G(F_S))_\Gamma$.

(b) $h(\phi^b) = h(\phi)^b$, $b \in C_c^\infty(\mathfrak{a}_{G,S})$.

(c) *There is a positive constant c such that for each $N > 0$, the image under h of $\mathscr{I}_N(G(F_S))_\Gamma$ is contained in $\mathscr{H}_{c(N+1)}(G(F_S))_\Gamma$.*

(d) *There is a positive constant d such that*

$$\|h(\phi)\| \le \|\phi\|' d^N, \qquad \phi \in \mathscr{I}_N(G(F_S))_\Gamma,\ N > 0,$$

where $\|\cdot\|$ is an arbitrary seminorm of the form (A.1), while $\|\cdot\|'$ is a finite sum of seminorms (A.3) which depends only on $\|\cdot\|$.

Lemma 6.1 follows easily from Lemma A.1. Take $S \supset S_{\mathrm{ram}}$ to be a large finite set of valuations of F, and let ϕ be a moderate function in $\mathscr{I}_{\mathrm{ac}}(G(F_S))_\Gamma$. Let $\{b_i\}$ be a smooth partition of unity for $\mathfrak{a}_G$ and set

$$f = \sum_i h(\phi^{b_i}).$$

Then f obviously belongs to $\mathscr{H}_{\mathrm{ac}}(G(F_S))_\Gamma$. We have

$$f_G = \sum_i h(\phi^{b_i})_G = \sum_i \phi^{b_i} = \phi.$$

Suppose that $N > 0$ and that $b \in C_N^\infty(\mathfrak{a}_G)$. Then

$$f^b = \sum_i h(\phi^{b_i b}) = h\left(\sum_i \phi^{b_i b}\right) = h(\phi^b).$$

The required support and growth properties of f^b then follow from conditions (c) and (d) of Lemma A.1. $\square$

The main point, then, is to establish Lemma A.1. It is evident that we can treat the valuations in S separately. We shall therefore assume that S consists of one valuation v. To simplify the notation, we shall also assume that F itself is a local field (rather than a number field), so that $F = F_v = F_S$.

Suppose first that F is non-Archimedean. In this case, the space $\mathfrak{a}_{G,v}$ is discrete, and the required condition (b) presents no problem. For if h satisfies all the conditions but this one, and if

$$b_X(Z) = \begin{cases} 1, & Z = X, \\ 0, & Z \neq X, \end{cases}$$

for elements X, $Z \in \mathfrak{a}_{G,v}$, the map

$$\phi \to \sum_{X \in \mathfrak{a}_{G,v}} h(\phi^{b_X})^{b_X}$$

will satisfy all the required conditions. It is therefore enough to construct a map h for which the conditions (a), (c), and (d) hold.

The Bernstein center is a direct sum

$$\mathscr{Z}(G(F)) = \bigoplus_{\chi} \mathscr{Z}(G(F))_{\chi}$$

of components indexed by supercuspidal data χ. Recall that a supercuspidal datum is a Weyl orbit

$$\chi = \{s_0(L_0, r_0) : s_0 \in W_0\} = \{s(L_0, r_0) : s \in W_0^G\},$$

where L_0 is a Levi subgroup of G^0 and r_0 is an irreducible supercuspidal representation of $L_0(F)^1$ which is fixed by some element in W_0^G. The definition, in fact, is in precise analogy with that of a cuspidal automorphic datum, given in §4. We also recall that $\mathscr{Z}(G(F))_{\chi} = \mathscr{Z}(G^0(F))_{\chi}$ is isomorphic to the algebra of finite Fourier series on the torus $\{r_{0,\Lambda} : \Lambda \in i\mathfrak{a}^*_{L_0,v}\}$ which are invariant under the stabilizer of the torus in W_0. Let $\mathscr{X}(F)_{\Gamma}$ denote the finite set of data χ such that r_0 contains a representation in the restriction of Γ to $K \cap L_0(F)$. Then $\mathscr{X}(F)_{\Gamma}$ is a finite set, and

$$\mathscr{Z}(G(F))_{\Gamma} = \bigoplus_{\chi \in \mathscr{X}(F)_{\Gamma}} \mathscr{Z}(G(F))_{\chi}$$

is a finitely generated algebra over $\mathbb{C}$. Let $z_1 = 1, z_2, \ldots, z_n$ be a fixed finite set of generators. There are actions $\phi \to z\phi$ and $f \to zf$ of $\mathscr{Z}(G(F))_{\Gamma}$ on $\mathscr{I}(G(F))_{\Gamma}$ and $\mathscr{H}(G(F))_{\Gamma}$, and the module $\mathscr{I}(G(F))_{\Gamma}$ is finitely generated over $\mathscr{Z}(G(F))_{\Gamma}$. Let $\phi_1 = 1, \phi_2, \ldots, \phi_m$ be a generating set. Then any function $\phi \in \mathscr{I}(G(F))_{\Gamma}$ can be written as a finite sum

$$\phi = \sum_{j=1}^{m} \sum_{\gamma} c_{\gamma}^{j} (z^{\gamma} \phi_j), \tag{A.4}$$

where $\{c_\gamma^j\}$ are complex numbers, and where

$$z^\gamma = z^{\gamma_1} \cdots z_n^{\gamma_n},$$

for any n-tuple $\gamma = (\gamma_1, \dots, \gamma_n)$ of nonnegative integers. Assume that the functions

$$X \to \phi_j(\pi, X), \qquad \pi \in \Pi_{\mathrm{temp}}(G(F)), \ X \in \mathfrak{a}_{G,v},$$

are supported at $X = 0$. Then by the trace Paley-Wiener theorem, there are functions $f_1, \dots, f_m$ in $\mathscr{H}(G(F)^1)_\Gamma$ such that $(f_j)_G = \phi_j$. We are going to define

$$h(\phi) = \sum_{j=1}^{m} \sum_{\gamma} c_\gamma^j (z^\gamma f_j). \tag{A.5}$$

However, the expansion (A.4) for ϕ is not unique. We must convince ourselves that it can be defined linearly in terms of ϕ in a way which is sensitive to the growth and support properties of ϕ.

We can identify each $\phi \in \mathscr{I}(G(F))_\Gamma$ with a collection of functions

$$\phi_o(\sigma_\Lambda) = \int_{\mathfrak{a}_{G,v}} \phi(\sigma_\Lambda^G, X)\, dX, \qquad \Lambda \in i\mathfrak{a}_{M,v}^*,$$

in which

$$o = (M, \sigma), \qquad M \in \mathscr{L}, \ o \in \Pi_{\mathrm{temp}}(M(F)^1),$$

ranges over a finite set of pairs which depends only on Γ. Each ϕ_o is a finite Fourier series which is symmetric under the stabilizer W_o of the orbit $\{\sigma_\Lambda\}$ in $W(\mathfrak{a}_M)$. The size of the support of ϕ is determined by the largest degree of a nonvanishing Fourier coefficient. Let $\|\phi\|'$ denote the largest absolute value of any of the Fourier coefficients. It is a continuous seminorm on $\mathscr{I}(G(F))_\Gamma$ of the form (A.3).

Let us embed $\mathscr{I}(G(F))_\Gamma$ into the space $\mathscr{J}(G(F))_\Gamma$ of collections

$$\psi = \{\psi_o(\sigma_\Lambda)\}$$

of finite Fourier series which have no symmetry condition. Then $\mathscr{J}(G(F))_\Gamma$ is also a finite $\mathscr{Z}(G(F))_\Gamma$-module. By averaging each function over W_o, we obtain a $\mathscr{Z}(G(F))_\Gamma$-linear projection $\psi \to \overline{\psi}$ from $\mathscr{J}(G(F))_\Gamma$ onto $\mathscr{I}(G(F))_\Gamma$. We can assume that our generating set for $\mathscr{I}(G(F))_\Gamma$ is of the form

$$\phi_j = \overline{\psi}_j, \qquad 1 \le j \le m,$$

where the functions $\psi_1 = 1, \psi_2, \dots, \psi_m$ generate $\mathscr{J}(G(F))_\Gamma$. Now, for each $o = (M, \sigma)$, we fix a basis of the lattice $\mathfrak{a}_{M,v}$. This allows us to identify the corresponding functions with finite sums

$$\psi_o = \sum_\beta b_{\beta,o} y^\beta, \qquad b_{\beta,o} \in \mathbb{C}, \tag{A.6}$$

in which

$$\beta = (\beta_1, \dots, \beta_d), \qquad d = d_o = \dim \mathfrak{a}_M,$$

runs over $\mathbb{Z}^d$, and

$$y^\beta = y_1^{\beta_1} \cdots y_d^{\beta_d}$$

denotes the function on $\{\sigma_\Lambda\}$ whose βth Fourier coefficient is 1 and whose other Fourier coefficients vanish. The functions y_i and y_i^{-1} of course belong to $\mathcal{J}(G(F))_\Gamma$, so we can define finite expansions

$$y_i^{\pm 1} = \sum_j \sum_\gamma (\delta_\gamma^j)^\pm (z^\gamma \psi_j), \qquad (\delta_\gamma^j)^\pm \in \mathbb{C}.$$

Substituting these expressions into the βth term of (A.6), and iterating $|\beta| = |\beta_1| + \cdots + |\beta_d|$ times, we obtain an expansion

$$\psi_o = \sum_j \sum_\gamma c_{\gamma, o}^j (z^\gamma \psi_j), \tag{A.7}$$

which is now well defined. If $\beta_{\max}$ and $\gamma_{\max}$ index the nonvanishing coefficients of greatest total degree in the expansions (A.6) and (A.7), one sees that

$$|\gamma_{\max}| \leq c(|\beta_{\max}| + 1)$$

and

$$\sup(|c_{\gamma, o}^j|) \leq \sup(|b_{\beta, o}|) \cdot d^{|\beta_{\max}|},$$

for constants c and d which depend only on Γ. Finally, observe that if ψ_o equals an element ϕ_o in $\mathcal{J}(G(F))_\Gamma$, we can project each side of (A.7) onto $\mathcal{J}(G(F))_\Gamma$. We obtain a canonical expansion

$$\phi_o = \sum_j \sum_\gamma c_{\gamma, o}^j (z^\gamma \phi_j).$$

We have shown how to define the expansion (A.4) in a way that depends linearly on ϕ. Moreover, if ϕ belongs to $\mathcal{J}_N(G(F))_\Gamma$ and $\gamma_{\max}$ indexes the nonvanishing coefficient of highest degree in (A.4), we have

$$|\gamma_{\max}| \leq c(N + 1) \tag{A.8}$$

and

$$\sup(|c_\gamma^j|) \leq \|\phi\|' d^N, \tag{A.9}$$

for fixed constants c and d. We are thus free to define $h(\phi)$ by (A.5). It remains to check conditions (c) and (d) of Lemma A.1.

Let K_0 be an open compact subgroup of $G^0(F)$ which lies in the kernel of each of the representations in Γ. Set g_0 equal to the characteristic function of K_0 divided by the volume of K_0. Then g_0 acts by convolution on $\mathcal{H}(G(F))_\Gamma$ as the identity. The algebra $\mathcal{Z}(G(F))_\Gamma$ acts on $\mathcal{H}(G^0(F))$, so we can set

$$g_i = z_i g_0, \qquad 1 \leq i \leq n.$$

These functions each belong to $\mathcal{H}(G^0(F))$, and they commute with each other under convolution. Consequently, for any $\gamma = (\gamma_1, \ldots, \gamma_n)$, the function

$g^\gamma = g^{\gamma_1} * \cdots * g^{\gamma_n}$ is well defined and belongs to $\mathscr{H}(G^0(F))$. Since $\mathscr{Z}(G(F))_\Gamma$ acts as an algebra of multipliers on $\mathscr{H}(G(F))_\Gamma$, the function (A.5) can be written

$$h(\phi) = \sum_j \sum_\gamma c_\gamma^j (g^\gamma * f_j).$$

To estimate the support of $h(\phi)$, we use the inequalities

$$\operatorname{supp}(g * h) \subset \operatorname{supp}(g) \cdot \operatorname{supp}(h), \qquad g \in \mathscr{H}(G^0(F)), h \in \mathscr{H}(G(F)),$$

and

$$\|xy\| \leq \|x\| \cdot \|y\|,$$

both of which are easily established. It follows that $h(\phi)$ is supported on a set

$$\{x \in G(F) : \log\|x\| \leq c_1(|\gamma_{\max}| + 1)\},$$

where c_1 is a constant which is independent of f. The support condition (c) of the lemma then follows from (A.8). To establish the growth condition (d), we may assume that $\|\cdot\|$ is the supremum norm on $\mathscr{H}(G(F))$. Then

$$\|g * h\| \leq \|g\|_1 \|h\|, \qquad g \in \mathscr{H}(G^0(F)), h \in \mathscr{H}(G(F)),$$

where $\|\cdot\|_1$ is the L_1-norm. Condition (d) then follows from (A.8) and (A.9). This completes the proof for non-Archimedean F.

Next, suppose that F is Archimedean. If $G \neq G^0$, we must invoke our assumption that G is an inner twist of

$$G^* = (GL(n) \times \cdots \times GL(n)) \rtimes \theta^*,$$

in order to have the trace Paley-Wiener theorem (see [2, Lemma I.7.1]). We shall say no more about this case. For one can obtain Lemma A.1 from the trace Paley-Wiener theorem by arguing as in the connected case below. We assume from now on that $G = G^0$. In this case the lemma is implicit in the work of Clozel-Delorme [6(a), 6(b)]. They construct a function $f = h(\phi)$ for every ϕ, and they give an estimate for the support of f which is stronger than our required condition (c). Our main tasks, then, are to convince ourselves that the map $\phi \to h(\phi)$ is well defined, and to check the growth conditions (d). We shall only sketch the argument.

The analogy between real and p-adic groups becomes clearer if we describe the steps of Clozel-Delorme in a slightly different order from that presented in [6(a)]. Let $\mathscr{D}_K(G(F)^1)_\Gamma$ be the space of distributions on $G(F)^1$ which are supported on K, and which transform under K according to representations in Γ. For a typical example, take $\mu \in \Gamma$, and let X be an element in $\mathscr{U}(\mathfrak{g}(F)^1)^K$, the centralizer of K in the universal enveloping algebra. Then the distribution

$$X_\mu : f \to \int_K (Xf)(k) \operatorname{tr}(\mu(k))\, dk, \qquad f \in C_c^\infty(G(F)^1),$$

belongs to $\mathscr{D}_K(G(F)^1)_\Gamma$. Suppose that D is any element in $\mathscr{D}_K(G(F)^1)_\Gamma$. Since it is a compactly supported distribution, it can be evaluated at a smooth function from $G(F)$ to some vector space. In particular, one can evaluate D on the function $\pi(x)$, for $\pi \in \Pi(G(F))$, to obtain an operator $\pi(D)$. Set

$$D_G(\pi) = \mathrm{tr}(\pi(D)), \qquad \pi \in \Pi_{\mathrm{temp}}(G(F)).$$

Then D_G is a scalar valued function on $\Pi_{\mathrm{temp}}(G(F))$. Let us write $\Delta_G(G(F)^1)_\Gamma$ for the space of complex valued functions δ on $\Pi_{\mathrm{temp}}(G(F))$ which satisfy the following two conditions.

(i) $\delta(\pi) = 0$, unless π contains a representation in Γ.

(ii) For any Levi subgroup $M \in \mathscr{L}$, and any $\sigma \in \Pi_{\mathrm{temp}}(M(F))$, the function

$$\Lambda \to \delta(\sigma_\Lambda^G), \qquad \Lambda \in \mathfrak{a}_{M,\mathbb{C}}^*,$$

is a polynomial which is invariant under $\mathfrak{a}_{G,\mathbb{C}}^*$.

It is easy to see that the map $D \to D_G$ sends $\mathscr{D}_K(G(F)^1)_\Gamma$ into $\Delta_K(G(F)^1)_\Gamma$. One of the main steps in the proof of Clozel-Delorme can be interpreted as an assertion that the map is surjective. In fact, any function $\delta \in \Delta_K(G(F)^1)_\Gamma$ is the image of a finite sum of distributions X_μ. This is obtained by combining the characterization of the action of $\mathscr{Z}(\mathfrak{g}(F)^1)^K$ on a minimal K-type ([6(a), Theorem 2] and [6(b), Theorem 2]) with the reduction argument based on Vogan's theory of minimal K-types [6(a), p. 435].

Smooth multipliers on $G(F)$ map $\mathscr{D}_K(G(F)^1)_\Gamma$ to $\mathscr{H}(G(F))_\Gamma$. More precisely, if $D \in \mathscr{D}_K(G(F)^1)_\Gamma$ and $\alpha \in C_c^\infty(\mathfrak{h})^W$, there is a unique function D_α in $\mathscr{H}(G(F))_\Gamma$ such that

$$\pi(D_\alpha) = \hat{\alpha}(\nu_\pi)\pi(D), \qquad \pi \in \Pi(G(F)) \tag{A.10}$$

(see [6(a), Lemma 6]). Observe also that if δ belongs to $\Delta_K(G(F)^1)_\Gamma$, the function

$$\delta_\alpha(\pi) = \delta(\pi)\hat{\alpha}(\nu_\pi), \qquad \pi \in \Pi_{\mathrm{temp}}(G(F)),$$

belongs to $\mathscr{I}(G(F))_\Gamma$. It is clear that $(D_\alpha)_G = D_{G,\alpha}$. The second main step of Clozel-Delorme can be interpreted as an assertion that over $C_c^\infty(\mathfrak{h})^W$, the module $\mathscr{I}(G(F))_\Gamma$ has a finite set of generators in $\Delta_K(G(F)^1)_\Gamma$. In other words, there is a finite set $\delta_1, \dots, \delta_m$ of elements in $\Delta_K(G(F)^1)_\Gamma$ with the property that any function $\phi \in \mathscr{I}(G(F))_\Gamma$ can be written

$$\phi = \delta_{1,\alpha_1} + \cdots + \delta_{m,\alpha_m}, \tag{A.11}$$

for multipliers $\alpha_1, \dots, \alpha_m$ in $C_c^\infty(\mathfrak{h})^W$. Fix elements $D_1, \dots, D_m$ in $\mathscr{D}_K(G(F)^1)_\Gamma$ such that

$$(D_j)_G = \delta_j, \qquad 1 \le j \le m.$$

We are going to define

$$h(\phi) = D_{1,\alpha_1} + \cdots + D_{m,\alpha_m}. \tag{A.12}$$

However, we shall first indicate briefly how the expansion (A.11) can be defined in terms of ϕ so that it has the appropriate properties.

As in the p-adic case, we can identify each $\phi \in \mathscr{I}(G(F))_\Gamma$ with a collection of functions

$$\phi_{\mathscr{o}}(\Lambda) = \int_{\mathfrak{a}_G} \phi(\sigma_\Lambda^G, X)\, dX, \qquad \Lambda \in i\mathfrak{a}_M^*,$$

in which

$$\mathscr{o} = (M, \sigma), \qquad M \in \mathscr{L}, \ \sigma \in \Pi_{\mathrm{temp}}(M(F)^1),$$

ranges over a finite set of pairs. For each $\phi_{\mathscr{o}}$, one constructs a Paley-Wiener function $\Phi_{\mathscr{o}}$ on $\mathfrak{h}_\mathbb{C}^*$ by following the procedure on p. 439 of [6(a)]. Clozel and Delorme then appeal to a result in [13], which asserts that

$$PW(\mathfrak{h}_\mathbb{C}^*) = \hat{u}_1 PW(\mathfrak{h}_\mathbb{C}^*)^W + \cdots + \hat{u}_d PW(\mathfrak{h}_\mathbb{C}^*)^W,$$

for elements $u_1 = 1, u_2, \ldots, u_d$ in $S(\mathfrak{h}^1)$, the symmetric algebra on $\mathfrak{h}^1$. Indeed, one need only take $\{u_i\}$ to be homogeneous elements which form a basis of the quotient field of $S(\mathfrak{h})$ over that of $S(\mathfrak{h})^W$. From the corollary of Lemma 11 of [9(a)], one can then construct continuous projections

$$PW(\mathfrak{h}_\mathbb{C}^*) \to \hat{u}_i PW(\mathfrak{h}_\mathbb{C}^*)^W, \qquad 1 \le i \le d,$$

whose sum is the identity. Apply the decomposition to $\phi_{\mathscr{o}}$ and then restrict the functions obtained to the affine subspaces $\nu_\sigma + \mathfrak{a}_{M,\mathbb{C}}^*$ of $\mathfrak{h}_\mathbb{C}^*$. This provides a well-defined expansion (A.11) for $\phi_{\mathscr{o}}$. The expansion for ϕ is then the corresponding sum over $\mathscr{o}$. In particular, we take $\{\delta_1, \ldots, \delta_m\}$ to be the union over $\mathscr{o}$ of the sets of d functions

$$\Lambda \to \hat{u}_i(\nu_\sigma + \Lambda), \qquad \Lambda \in i\mathfrak{a}_M^*.$$

It follows that the expansion (A.11) is given by a well-defined linear map

$$\phi \to (\alpha_1, \ldots, \alpha_m), \qquad \alpha_i \in C_c^\infty(\mathfrak{h})^W. \tag{A.13}$$

The map h is then determined by (A.12).

It is clear from the definitions that $h(\phi)_G$ equals ϕ. The other conditions of the lemma come from properties of the map (A.13). For one can check that the map commutes with the natural action of $i\mathfrak{a}_G^*$ on $\mathscr{I}(G(F))_\Gamma$ and $C_c^\infty(\mathfrak{h})^W$. This gives the required condition (b). If $\phi \in \mathscr{I}_N(G(F))_\Gamma$, $N > 0$, it can be shown that each α_i belongs to $C_N^\infty(\mathfrak{h})^W$. Since the support of a function (or distribution) behaves well under the action of a multiplier, condition (c) follows. To prove (d), first note that a seminorm (A.1) is continuous on the Schwartz space of $G(F)$. It follows from the corollary of Theorem 13.1 of [9(b)] that the value of any such seminorm on $h(\phi)$ is bounded by a finite sum of continuous seminorms, evaluated at classical Schwartz functions

$$\lambda \to \mathscr{I}_P(\sigma_\lambda, h(\phi)), \qquad \lambda \in i\mathfrak{a}_P^*.$$

Here $P \in \mathscr{F}(M_0)$, $\sigma \in \Pi_{\text{temp}}(M_P(F))$ and $\mathscr{I}_P(\sigma_\lambda)$ is the induced representation of $G(F)$. We are assuming that $h(\phi)$ is given by (A.12), so that

$$\mathscr{I}_P(\sigma_\lambda, h(\phi)) = \sum_{i=1}^{m} \hat{\alpha}_i(\nu_\sigma + \lambda)\mathscr{I}_P(\sigma_\lambda, D_i).$$

But for any k there is a seminorm $\|\cdot\|_k'$ on $\mathscr{I}(G(F))$ of the form (A.3) such that

$$\sup_i |\hat{\alpha}_i(\nu_\sigma + \lambda)| \le \|\phi\|_k' e^{\|\nu_\sigma\| N}(1 + \|\lambda\|)^{-k},$$

for any $\lambda \in i\mathfrak{a}_P^*$ and any $\phi \in \mathscr{I}_N(G(F))$, $N > 0$. This is a consequence of the continuity properties of the map (A.13). The final condition (d) of the lemma follows. □

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