Proceedings of Symposia in Pure Mathematics
Volume 48 (1988)

Characters, Harmonic Analysis, and an L^2-Lefschetz Formula

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Suppose that U is a locally compact group. It is a fundamental problem to classify the irreducible unitary representations of U. A second basic problem is to decompose the Hilbert space of square integrable functions on U, or on some homogeneous quotient of U, into irreducible U-invariant subspaces. The underlying domain is often attached to a natural Riemannian manifold, and the required decomposition becomes the spectral decomposition of the Laplace-Beltrami operator. Weyl solved both problems in the case of a compact Lie group. His method, which was simple and elegant, was based on the theory of characters.

In this lecture, we shall briefly review Weyl's theory for compact groups. We shall then discuss two newer areas that could claim Weyl's work as a progenitor: the harmonic analysis on noncompact groups, and the analytic theory of automorphic forms. The three areas together form a progression that is natural in several senses; in particular, the underlying algebraic structures of each could be characterized as that of an algebraic group over the field $\mathbf{C}$, $\mathbf{R}$, or $\mathbf{Q}$. However, the latter two areas are vast. Beyond a few general comments, we can attempt nothing like a survey. We shall instead concentrate on a topic that has a particular connection to Weyl's character formula and its later generalizations. We shall describe a Lefschetz formula for the Hecke operators on L^2-cohomology. The formula deals with objects which are highly singular, but turns out to be quite simple nonetheless. It will probably play a role some day in relating the arithmetic objects discussed in Langlands' lecture [**8d**] to the analytic theory of automorphic forms.

1. Suppose that U is a compact simply connected Lie group. Any representation[1] $\tau \in \hat{U}$ is finite dimensional, and the character

$$\Theta_\tau(x) = \operatorname{tr} \tau(x), \qquad x \in U,$$

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 11F72, 22E46; Secondary 52T15.

[1]If G is a locally compact group, $\hat{G}$ denotes the set of equivalence classes of irreducible unitary representations of G.

is an invariant function on the conjugacy classes of U. Let T be a maximal torus in U. Weyl introduced the finite group[2]

$$W(U,T) = \mathrm{Norm}_U(T)/\mathrm{Cent}_U(T)$$

to characterize the conjugacy classes in U. Each conjugacy class of U intersects T in a unique $W(U,T)$-orbit. The character Θ_τ, and hence the representation τ itself, is uniquely determined by the $W(U,T)$-invariant function $\Theta_\tau(t)$, $t \in T$.

Weyl [**12a**] constructed the characters $\{\Theta_\tau\}$ from three simple facts. The first was the observation that the restriction of τ to T is a direct sum of irreducible characters of T. Accordingly, $\Theta_\tau(t)$ is a finite sum of characters of T. The second fact is that the characters $\{\Theta_\tau\}$ form an orthonormal basis of the space of square integrable class functions on U. This is an immediate consequence of the Schur orthogonality relations, established for compact groups by Peter and Weyl in [**9**]. The third fact is the Weyl integration formula

$$\int_U f(x)\,dx = |W(U,T)|^{-1} \int_T \int_U |\Delta_T^U(t)|^2 f(x^{-1}tx)\,dx\,dt, \qquad f \in C(U),$$

where dx and dt are the normalized Haar measures on U and T, and

$$\Delta_T^U(t) = \prod_{\alpha>0} (\alpha(t)^{1/2} - \alpha(t)^{-1/2}), \qquad t \in T.$$

The product here is taken over the positive roots of (U,T) relative to some ordering, and when multiplied out, the square roots all become well-defined functions of T.

The function Δ_T^U is skew-symmetric. That is,

$$\Delta_T^U(st) = \varepsilon(s)\Delta_T^U(t), \qquad s \in W(U,T),$$

where $\varepsilon(s)$ is the sign of s, regarded as a permutation of the roots. Therefore, the functions

$$\Delta_T^U(t)\Theta_\tau(t), \qquad \tau \in \hat{U},$$

are also skew-symmetric. It follows easily from the second and third facts above that these functions form an orthonormal basis, relative to the measure $|W(U,T)|^{-1}dt$, of the space of square-integrable, skew-symmetric functions on T. Now the group $W(U,T)$ acts by duality on $\hat{T}$. Let $\{\chi\}$ be a set of representatives of those orbits which are regular (in the sense that χ is fixed by only the identity in $W(U,T)$). Then the functions

$$\sum_{s\in W(U,T)} \varepsilon(s)(s\chi)(t), \qquad t \in T,$$

also form an orthonormal basis of the space of skew-symmetric functions. Choose the representatives χ so that their differentials all lie in the chamber defined by the positive roots. It is then a straightforward consequence of the first fact above that the two orthonormal bases are the same.

[2]By $\mathrm{Norm}_X(Y)$ and $\mathrm{Cent}_X(Y)$ we of course mean the normalizer and centralizer of Y in X.

This gives the classification of $\hat{U}$. There is a bijection $\tau \leftrightarrow \chi$, with the property that

$$\Theta_\tau(t) = \Delta_T^U(t)^{-1} \sum_{s \in W(U,T)} \varepsilon(s)(s\chi)(t), \tag{1.1}$$

for any point t in

$$T_{\mathrm{reg}} = \{t \in T \colon \Delta_T^U(t) \neq 0\}.$$

This identity is the famous Weyl character formula. It uniquely determines the character Θ_τ and the correspondence $\tau \leftrightarrow \chi$. By taking the limit as t approaches 1, Weyl obtained the simple formula

$$\deg(\tau) = \prod_{\alpha > 0} \frac{(d\chi, \alpha)}{(d\chi_1, \alpha)} \tag{1.2}$$

for the degree of τ. Here, $\chi_1 \in \hat{T}$ corresponds to the trivial representation of U, $d\chi$ is the differential of χ, and $(\cdot,\cdot)$ is a $W(U,T)$-invariant bilinear form on the dual of the Lie algebra of T. The Peter-Weyl theorem asserts that τ occurs in $L^2(U)$ with multiplicity equal to the degree of τ. Therefore, Weyl's classification of $\hat{U}$ also provides a decomposition of $L^2(U)$ into irreducible representations. (See pp. 377–385 of **[12b]** for a clear elucidation of a special case, and also the survey **[2b]**, in addition to the original papers **[12a]**).

EXAMPLE. Suppose that $U = \mathrm{SU}(n, \mathbf{R})$, the group of complex unitary matrices of determinant 1. One can take

$$T = \left\{ t = \begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix} : \det t = 1 \right\}.$$

The Weyl group $W(U,T)$ then becomes the symmetric group S_n. The positive regular characters χ can be identified with the set

$$\{(\lambda_1, \ldots, \lambda_n) \in \mathbf{Z}^n \colon \lambda_1 > \lambda_2 > \cdots > \lambda_n\},$$

taken modulo the diagonal action of $\mathbf{Z}$, and we have

$$\varepsilon(s)(s\chi)(t) = \mathrm{sgn}(s) e^{i\theta_1 \lambda_{s(1)}} \cdots e^{i\theta_n \lambda_{s(n)}}, \qquad s \in S_n,$$

for the summands in the Weyl character formula. The degree of the corresponding representation τ is simply equal to

$$\prod_{1 \leq i < j \leq n} \left(\frac{\lambda_i - \lambda_j}{j - i} \right).$$

There is a unique bi-invariant Riemannian metric on U whose restriction to T is given by the form $(\cdot,\cdot)$. One checks that the Laplacian acts on the subspace of $L^2(U)$ corresponding to τ by the scalar

$$(d\chi, d\chi) - (d\chi_1, d\chi_1).$$

This number represents an eigenvalue of the Laplacian, which occurs with multiplicity

$$\prod_{\alpha>0}\left(\frac{(d\chi,\alpha)}{(d\chi_1,\alpha)}\right)^2.$$

Weyl's theory thus gives something which is rather rare: an explicit description of the spectrum of the Laplacian on a Riemannian manifold. This is typical of the examples that Lie theory contributes to an increasingly large number of mathematical areas. The examples invariably have an internal structure which is both rich and computable.

2. The compact group U has a complexification. Conversely, a complex semisimple group has a compact real form, which is unique up to conjugation. Weyl exploited this connection with his "unitary trick", in order to study the finite-dimensional representations of a complex group. Now, a complex semisimple group is actually algebraic. Therefore, the choice of the group U in §1 is tantamount to a choice of a semisimple, simply connected algebraic group G over $\mathbf{C}$. In this paragraph, we shall assume that G is in fact defined over $\mathbf{R}$. In other words, we suppose that we are given an arbitrary real form $G(\mathbf{R})$ of $G(\mathbf{C})$. In this context there is again a Riemannian manifold. It is the globally symmetric space

$$X_G = G(\mathbf{R})/K_{\mathbf{R}},$$

in which $K_{\mathbf{R}}$ is a maximal compact subgroup of $G(\mathbf{R})$.

Any representation $\tau \in \hat{U}$ extends to a finite-dimensional representation of $G(\mathbf{C})$. However, the restriction of this representation of $G(\mathbf{R})$, which we shall also denote by τ, need not be unitary. The representations in the unitary dual $\widehat{G(\mathbf{R})}$ are generally infinite-dimensional. The problem of classifying $\widehat{G(\mathbf{R})}$ is much more difficult than in the compact case, and is still not completely solved. (See the lecture [**11**] of Vogan). However, Harish-Chandra found enough of the representations in $\widehat{G(\mathbf{R})}$ to be able to describe explicitly the decomposition of $L^2(G(\mathbf{R}))$ into irreducible representations.

One of Harish-Chandra's achievements was to establish a theory of characters for infinite-dimensional representations. For a given $\pi \in \widehat{G(\mathbf{R})}$, it was first shown that the operators

$$\pi(f) = \int_{G(\mathbf{R})} f(x)\pi(x)\,dx, \qquad f \in C_c^\infty(G(\mathbf{R})),$$

were of trace class, and that the functional $f \to \mathrm{tr}(\pi(f))$ was a distribution on $G(\mathbf{R})$. Harish-Chandra was able to prove that the distribution was actually a function [**5a**]. That is, there exists a locally integrable function Θ_π on $G(\mathbf{R})$ such that

$$\mathrm{tr}(\pi(f)) = \int_{G(\mathbf{R})} f(x)\Theta_\pi(x)\,dx,$$

for any $f \in C_c^\infty(G(\mathbf{R}))$. The function Θ_π is invariant on the conjugacy classes of $G(\mathbf{R})$, and is called the *character* of π. It uniquely determines the equivalence class of π.

Harish-Chandra realized that at the heart of the problem of decomposing $L^2(G(\mathbf{R}))$ lay a class $\widehat{G(\mathbf{R})}_{\text{disc}}$ of representations in $\widehat{G(\mathbf{R})}$ which behaved very much like the irreducible representations of a compact group. They are called the *discrete series*, because they are precisely the representations which occur discretely in the decomposition of $L^2(G(\mathbf{R}))$. Harish-Chandra [**5b**] classified the discrete series. He showed that discrete series exist if and only if $G(\mathbf{R})$ has a Cartan subgroup $A_0(\mathbf{R})$ which is compact. Assume that such a group exists. As in §1, we can define the real Weyl group

$$W_{\mathbf{R}}(G, A_0) = \mathrm{Norm}_{G(\mathbf{R})}(A_0)/\mathrm{Cent}_{G(\mathbf{R})}(A_0),$$

and the function

$$\Delta^G_{A_0}(t) = \prod_{\alpha>0} (\alpha(t)^{1/2} - \alpha(t)^{-1/2}), \qquad t \in A_0(\mathbf{R}).$$

Then there is a bijection $\pi \leftrightarrow \{\chi\}$ between $\widehat{G(\mathbf{R})}_{\text{disc}}$ and the $W_{\mathbf{R}}(G, A_0)$-orbits of regular elements in $\widehat{A_0(\mathbf{R})}$ such that

$$\Theta_\pi(t) = \pm\Delta^G_{A_0}(t)^{-1} \sum_{s\in W_{\mathbf{R}}(G,A_0)} \varepsilon(s)(s\chi)(t), \qquad t \in A_0(\mathbf{R})_{\text{reg}}.$$

This looks very much like the classification of representations of U. However, there are two important differences.

The real Weyl group is in general a proper subgroup of the complex Weyl group

$$W(G, A_0) = \mathrm{Norm}_G(A_0)/\mathrm{Cent}_G(A_0).$$

Replacing the compact real form U with a conjugate, if necessary, we can assume that $A_0(\mathbf{R})$ coincides with the torus T. It is then easy to show that the Weyl groups $W(G, A_0)$ and $W(U, T)$ are the same. Consequently, every regular $W(U, T)$-orbit in $\hat{T} = \widehat{A_0(\mathbf{R})}$ contains several $W_{\mathbf{R}}(G, A_0)$-orbits. It follows that $\widehat{G(\mathbf{R})}_{\text{disc}}$ is a natural disjoint union of finite subsets $\widehat{G(\mathbf{R})}_\tau$, each of order

$$w(G) = |W(G, A_0)/W_{\mathbf{R}}(G, A_0)|,$$

which are parametrized by the representations $\tau \in \hat{U}$. The packets $\widehat{G(\mathbf{R})}_\tau$ are characterized by the property that on $T_{\text{reg}} = A_0(\mathbf{R})_{\text{reg}}$, the function

$$(-1)^{(1/2)\dim(X_G)} \sum_{\pi\in\widehat{G(\mathbf{R})}_\tau} \Theta_\pi \tag{2.1}$$

equals the character Θ_τ.

The second essential difference from the compact case is that $G(\mathbf{R})$ generally has several $G(\mathbf{R})$-conjugacy classes $\{A(\mathbf{R})\}$ of Cartan subgroups. The function Θ_π must be evaluated on every Cartan subgroup if it is to be specified on all of $G(\mathbf{R})$. Harish-Chandra gave such formulas for the characters of discrete series. As an example, we shall quote the general character formula for the sum (2.1). Let $A(\mathbf{R})$ be an arbitrary Cartan subgroup of $G(\mathbf{R})$. Then the set R of real roots of $(G(\mathbf{R}), A(\mathbf{R}))$ is a root system, and it has a corresponding system $R^\vee$ of co-roots. We can choose an isomorphism

$$\mathrm{Ad}(y)\colon A_0(\mathbf{C}) \xrightarrow{\sim} A(\mathbf{C}), \qquad y \in G(\mathbf{C}),$$

over the complex numbers. This allows us to define a character

$$\chi'(\gamma) = \chi(\mathrm{Ad}(y)^{-1}\gamma), \qquad \gamma \in A(\mathbf{C}),$$

on $A(\mathbf{C})$, for every χ in $\hat{T} = \widehat{A_0(\mathbf{R})}$. One then has the formula

$$(-1)^{(1/2)\dim(X_G)} \sum_{\pi \in \widehat{G(\mathbf{R})}_\tau} \Theta_\pi(\gamma) = \Delta_A^G(\gamma)^{-1} \sum_{s \in W(G, A_0)} \varepsilon(s) \cdot \bar{c}(R^\vee_{(s\chi)'}, R_\gamma) \cdot (s\chi)'(\gamma),$$

valid for any point $\gamma \in A(\mathbf{R})_{\mathrm{reg}}$. Here, $\bar{c}(R^\vee_{(s\chi)'}, R_\gamma)$ is an integer which depends only on the systems of positive roots in $R^\vee$ and R defined by $(s\chi)'$ and γ respectively. It can be computed from a simple inductive procedure, based on the rank of R. (See [**6**].)

Thus, on the noncompact Cartan subgroups, the characters of discrete series are slightly different than the characters of finite-dimensional representations of $G(\mathbf{R})$. They are in fact closer to finite-dimensional characters for Levi subgroups of G. Let M be the centralizer in G of the $\mathbf{R}$-split component of A. Then M is a reductive subgroup of G, and the $\mathbf{R}$-split component A_M of the center of M is the same as the original $\mathbf{R}$-split component of A. We can write

$$\Delta_A^G(\gamma) = \Delta_M^G(\gamma)\Delta_A^M(\gamma)(-1)^{|R_\gamma \cap (-R^+)|}, \qquad \gamma \in A(\mathbf{R}),$$

where, if $\mathfrak{g}$ and $\mathfrak{m}$ denote the Lie algebras of G and M,

$$\Delta_M^G(\gamma) = |\det(1 - \mathrm{Ad}(\gamma))_{\mathfrak{g}/\mathfrak{m}}|^{1/2}.$$

It can be shown that the function

$$\Phi_M(\gamma, \tau) = \Delta_M^G(\gamma)^{-1}(-1)^{(1/2)\dim(X_G)} \sum_{\pi \in \widehat{G(\mathbf{R})}_\tau} \Theta_\pi(\gamma), \qquad \gamma \in A(\mathbf{R})_{\mathrm{reg}}, \tag{2.2}$$

extends to a continuous function on $A(\mathbf{R})$. This function is not smooth at the singular hyperplanes defined by real roots. However, on each connected component of

$$\{\gamma \in A(\mathbf{R})\colon \alpha(\gamma) \neq 1,\ \alpha \in R\},$$

$\Phi_M(\gamma, \tau)$ is an integral linear combination of finite-dimensional characters of $M(\mathbf{R})$.

EXAMPLE. Suppose that $G = \mathrm{SU}(p, q)$. The conjugacy classes of Cartan subgroups are represented by groups

$$\{A_r(\mathbf{R})\colon 0 \leq r \leq \min(p, q)\},$$

in which the split component A_{M_r} of A_r has dimension r. The corresponding Levi subgroup M_r is the product of $\mathrm{SU}(p - r, q - r)$ with an abelian group. The real root system R consists simply of r copies of $\{\pm\alpha\}$, the root system of type A_1. The constant

$$\bar{c}(R_{(s\chi)'}, R_\gamma)$$

can be determined from the case of SL(2). It equals 2^r in chambers where the function

$$(\chi, \gamma) \to (s\chi)'(\gamma)$$

is bounded, and it vanishes otherwise.

The harmonic analysis of $L^2(G(\mathbf{R}))$ is based on the classification of the discrete series. Harish-Chandra [**5c**] showed that the subrepresentation of $L^2(G(\mathbf{R}))$ which decomposed continuously could be understood in terms of the discrete series on Levi subgroups of G. More precisely, $L^2(G(\mathbf{R}))$ is a direct integral of irreducible representations obtained by inducing discrete series from parabolic subgroups. As in §1, the harmonic analysis is closely related to the spectral decomposition of a Laplacian. If (σ, V_σ) is a finite-dimensional unitary representation on $K_\mathbf{R}$, one can form the homogeneous vector bundle

$$G(\mathbf{R}) \times_{K_\mathbf{R}} V_\sigma$$

over the Riemannian symmetric space X_G. The Laplacian on X_G then gives an operator on the space of square-integrable sections. A knowledge of the spectral decomposition of this space of sections, for arbitrary σ, is equivalent to the knowledge of the decomposition of $L^2(G(\mathbf{R}))$. Harish-Chandra's theory therefore provides another example of an explicit description of the spectrum of a Laplacian, this time for a noncompact Riemannian manifold.

3. Now suppose that Γ is a discrete subgroup of $G(\mathbf{R})$. We assume that Γ is a congruence subgroup of an arithmetic group. The associated problem in harmonic analysis is to decompose the right regular representation of R of $G(\mathbf{R})$ on $L^2(\Gamma\backslash G(\mathbf{R}))$. The theory has some similarities with that of $L^2(G(\mathbf{R}))$. Write

$$R = R_{\text{disc}} \oplus R_{\text{cont}},$$

where R_{disc} is a direct sum of irreducible representations, and R_{cont} decomposes continuously. Then the decomposition of R_{cont} can be described in terms of the decomposition of the analogues of R_{disc} for Levi subgroups of G. This description is part of the theory of Eisenstein series, initiated by Selberg [**10**], and established for general groups by Langlands [**8c**].

The remaining problem, then, is to decompose R_{disc}. This is much harder than the corresponding problem for $L^2(G(\mathbf{R}))$. In fact, one does not really expect ever to obtain a complete description of the decomposition of R_{disc}. Rather, one wants to establish relations or "reciprocity laws" between the decompositions of the representations R_{disc} for different groups. Such relations are summarized in Langlands' functoriality conjecture, and are very deep. (See [**8a**, **2a**, **1a**].) The best hope for studying functoriality seems to be through the trace formula. The trace formula is a nonabelian analogue of the Poisson summation formula, which relates the study of R_{disc} to the harmonic analysis on $G(\mathbf{R})$. It provides a reasonably explicit expression for the trace of the operator

$$R_{\text{disc}}(f) = \int_{G(\mathbf{R})} f(x) R_{\text{disc}}(x)\, dx,$$

for a suitable function f on $G(\mathbf{R})$. (Actually, $R_{\text{disc}}(f)$ is not known to be of trace class in general, so one must group the terms in a certain way to insure convergence.) The trace formula was introduced by Selberg [**10**] in the case of compact quotient and for some groups of rank one. For general rank, we refer the reader to the surveys [**1b**] and [**1d**]. We shall not try to summarize the trace formula here, or to discuss the limited applications to functoriality that it has so far yielded. We shall instead discuss an application of the trace formula to L^2-cohomology. This is appropriate in the present symposium, for in the end there is a surprising connection with the character formulas that began with Weyl.

Assume that Γ has no elements of finite order. Then

$$X_\Gamma = \Gamma \backslash G(\mathbf{R})/K_{\mathbf{R}} = \Gamma \backslash X_G$$

is a locally symmetric Riemannian manifold. Choose an irreducible representation τ of $G(\mathbf{C})$ on a finite-dimensional Hilbert space V_τ. Restricting τ to the subgroup Γ of $G(\mathbf{C})$, we define a locally constant sheaf

$$\mathcal{F}_\tau = V_\tau \times_\Gamma X_G$$

on X_Γ. Let $A^q_{(2)}(X_\Gamma, \mathcal{F}_\tau)$ be the space of smooth q-forms ω on X_Γ with values in $\mathcal{F}_\tau$, such that ω and $d\omega$ are both square integrable. Then $A^*_{(2)}(X_\Gamma, \mathcal{F}_\tau)$ is a differential graded algebra. Its cohomology

$$H^*_{(2)}(X_\Gamma, \mathcal{F}_\tau) = \bigoplus_q H^q_{(2)}(X_\Gamma, \mathcal{F}_\tau)$$

is the L^2-cohomology of X_Γ (with coefficients in $\mathcal{F}_\tau$). Assume that $G(\mathbf{R})$ has a compact Cartan subgroup $A_0(\mathbf{R})$. Then Borel and Casselman [**3**] have shown that the groups $H^q_{(2)}$ are finite-dimensional. We would like to compute the L^2-Euler characteristic

$$\sum_q (-1)^q \dim(H^q_{(2)}(X_\Gamma, \mathcal{F}_\tau)).$$

More generally, we shall consider the L^2-Lefschetz numbers of Hecke operators.

Hecke operators are best discussed in terms of adèles. As partial motivation for the introduction of adèle groups, consider the question of how one obtains congruence subgroups Γ. The main step is to choose a $\mathbf{Q}$-structure for $G(\mathbf{R})$. In other words, assume that the algebraic group G is defined over $\mathbf{Q}$. Assume also that $G(\mathbf{R})$ has no compact simple factors. Let

$$\mathbf{A} = \mathbf{R} \times \mathbf{A}_0 = \mathbf{R} \times \mathbf{Q}_2 \times \mathbf{Q}_3 \times \cdots$$

be the ring of adèles of $\mathbf{Q}$. Then $G(\mathbf{A})$ is a locally compact group, which contains $G(\mathbf{Q})$ as a discrete subgroup. Suppose that K_0 is an open compact subgroup of the group $G(\mathbf{A}_0)$ of points over the finite adèles. Since G is simply connected, the strong approximation theorem asserts that

$$G(\mathbf{A}) = G(\mathbf{Q}) \cdot G(\mathbf{R})K_0.$$

It follows immediately that there are homeomorphisms

$$\Gamma\backslash G(\mathbf{R}) \xrightarrow{\sim} G(\mathbf{Q})\backslash G(\mathbf{A})/K_0 \tag{3.1}$$

and

$$X_\Gamma = \Gamma\backslash G(\mathbf{R})/K_\mathbf{R} \xrightarrow{\sim} G(\mathbf{Q})\backslash G(\mathbf{A})/K_\mathbf{R}K_0, \tag{3.2}$$

where

$$\Gamma = G(\mathbf{Q})K_0 \cap G(\mathbf{R}),$$

a congruence subgroup. Every congruence subgroup is obtained in this way. In other words, given the $\mathbf{Q}$-structure on $G(\mathbf{R})$, a choice of Γ amounts to a choice of an open compact subgroup K_0 of $G(\mathbf{A}_0)$. Suppose that h belongs to the Hecke algebra $\mathcal{H}_{K_0}$ of compactly supported functions on $G(\mathbf{A}_0)$ which are bi-invariant under K_0. If ϕ is any form in $A^q_{(2)}(X_\Gamma, \mathcal{F}_\tau)$, we define

$$h\phi = \int_{G(\mathbf{A}_0)} h(g)(g^*\phi)\,dg,$$

where

$$(g^*\phi)(x) = \phi(xg),$$

for any point x in

$$X_\Gamma \cong G(\mathbf{Q})\backslash G(\mathbf{A})/K_\mathbf{R}K_0.$$

Then $h\phi$ is also a form in $A^q_{(2)}(X_\Gamma, \mathcal{F}_\tau)$. We obtain an operator

$$H^q_{(2)}(h, \mathcal{F}_\tau) \colon H^q_{(2)}(X_\Gamma, \mathcal{F}_\tau) \to H^q_{(2)}(X_\Gamma, \mathcal{F}_\tau).$$

The problem is to compute the Lefschetz number

$$\mathcal{L}_\tau(h) = \sum_q (-1)^q \operatorname{tr}(H^q_{(2)}(h, \mathcal{F}_\tau)).$$

Of course, if we set h equal to 1_{K_0}, the characteristic function of K_0 divided by the volume of K_0, we obtain the L^2-Euler characteristic.

The homeomorphism (3.1) is compatible with right translation by $G(\mathbf{R})$. It follows that the representations R and R_{disc} of $G(\mathbf{R})$ extend to representations of $G(\mathbf{R}) \times \mathcal{H}_{K_0}$. It turns out that one can find a function $f_\tau \in C_c^\infty(G(\mathbf{R}))$ such that

$$\mathcal{L}_\tau(h) = \operatorname{tr}(R_{\mathrm{disc}}(f_\tau \times h)).$$

One can then evaluate the right-hand side by the trace formula. The result is a rather simple formula for $\mathcal{L}_\tau(h)$. Foregoing the details, which appear in [**1c**], we shall be content to state the final answer.

We must first describe the main ingredients of the formula. Suppose that M is a Levi component of a parabolic subgroup of G which is defined over $\mathbf{Q}$. If M

contains a Cartan subgroup $A(\mathbf{R})$ which is compact modulo $A_M(\mathbf{R})^0$, we have the function

$$\Phi_M(\gamma, \tau), \qquad \gamma \in A(\mathbf{R})_{\mathrm{reg}},$$

defined by (2.2) in terms of characters of discrete series. Let us extend this to an $M(\mathbf{R})$-invariant function on $M(\mathbf{R})$ which vanishes unless γ is conjugate to an element in $A(\mathbf{R})$. If M does not contain such a Cartan subgroup, we simply set $\Phi_M(\gamma, \tau) = 0$. This function represents the contribution from τ. Now, suppose that γ belongs to $M(\mathbf{Q})$. Let M_γ denote the centralizer of γ in M. We assume that $\Phi_M(\gamma, \tau)$ does not vanish, so in particular, γ is semisimple. The contribution from $h \in \mathcal{H}_{K_0}$ will be of the form

$$h_M(\gamma) = \int_{K_{0,\max}} \int_{N_P(\mathbf{A}_0)} \int_{M_\gamma(\mathbf{A}_0)\backslash M(\mathbf{A}_0)} h(k^{-1}m^{-1}\gamma mnk)\, dm\, dn\, dk,$$

where $P = MN_P$ is a parabolic subgroup over $\mathbf{Q}$ with Levi component M, and $K_{0,\max}$ is a suitable maximal compact subgroup of $G(\mathbf{A}_0)$. This is essentially an invariant orbital integral of h, and is easily seen to be independent of P. The third ingredient in the formula will be a constant $\chi(M_\gamma)$, which is closely related to the (classical) Euler characteristic of the symmetric space of M_γ. Let $\overline{M}_\gamma$ be any reductive group over $\mathbf{Q}$ which is an inner twist of M_γ and such that $\overline{M}_\gamma(\mathbf{R})/A_M(\mathbf{R})^0$ is compact. Then

$$\chi(M_\gamma) = (-1)^{(1/2)\dim(X_{M_\gamma})} \mathrm{vol}(\overline{M}_\gamma(\mathbf{Q})\backslash\overline{M}_\gamma(\mathbf{A}_0)) w(M_\gamma),$$

provided that G has no factors of type E_8. This relies on a theorem of Kottwitz [**7**], which requires the Hasse principle. Otherwise, $\chi(M_\gamma)$ must be given by a slightly more complicated formula.

We can now state the L^2-Lefschetz formula. It is

$$(3.3)\quad \mathcal{L}_\tau(h) = \sum_M (-1)^{\dim(A_M)} |W(G, A_M)|^{-1} \sum_{\gamma \in (M(\mathbf{Q}))} \chi(M_\gamma)\Phi_M(\gamma, \tau) h_M(\gamma),$$

where h is any element in the Hecke algebra $\mathcal{H}_{K_0}$ and

$$W(G, A_M) = \mathrm{Norm}_G(A_M)/\mathrm{Cent}_G(A_M).$$

The outer sum is over the conjugacy classes of Levi subgroups M in G, while the inner sum is over the conjugacy classes of elements γ in $M(\mathbf{Q})$. Since $\Phi_M(\gamma, \tau)$ vanishes unless γ is $\mathbf{R}$-elliptic in M, we can restrict the inner sum to such elements. It can actually be taken over a finite set, which depends only on the support of h. We therefore have a finite closed formula for the Lefschetz number $\mathcal{L}_\tau(h)$.

4. We shall conclude with a few general, and perhaps obvious, remarks. Suppose that the symmetric space X_Γ has complex Hermitian structure. Then one has the Baily-Borel compactification $\overline{X}_\Gamma$ of X_Γ, which is a complex projective algebraic variety with singularities. The Goresky-Macpherson intersection homology is a theory for singular spaces which satisfies Poincaré duality. Zucker's conjecture asserts that the L^2-cohomology of X_Γ is isomorphic to the intersection homology of $\overline{X}_\Gamma$. It can be regarded as an analogue of the de Rham theorem for the noncompact space X_Γ. The L^2-cohomology is of course the analytic ingredient, and the intersection homology represents the geometric ingredient.

Zucker's conjecture opens the possibility of interpreting (3.3) as a fixed point formula. The geometric interpretation of the Hecke operators is easy to describe. For

$$\Gamma = G(\mathbf{Q})K_0 \cap G(\mathbf{R}),$$

as above, there is a bijection

$$\Gamma\backslash G(\mathbf{Q})/\Gamma \xrightarrow{\sim} K_0\backslash G(\mathbf{A}_0)/K_0$$

between double coset spaces. Suppose that $h \in \mathcal{H}_{K_0}$ is given by the characteristic function of a coset

$$\Gamma g \Gamma, \qquad g \in G(\mathbf{Q}).$$

Let Γ' be any subgroup of finite index in $\Gamma \cap g\Gamma g^{-1}$. Then the map

$$\Gamma' x \to (\Gamma x, \Gamma g^{-1} x) \tag{4.1}$$

is an embedding of $X_{\Gamma'}$ into $(X_\Gamma \times X_\Gamma)$. The resulting correspondence gives the Hecke operator on cohomology. Goresky and Macpherson [**4**] have proved a general Lefschetz fixed point theorem for intersection homology. One could apply it to the correspondence (4.1) and try to duplicate the formula (3.3). However, the correspondence (4.1) does not intersect the diagonal in a nice way. Moreover, the singularities of $\overline{X}_\Gamma$ are quite bad. It therefore seems remarkable that the formula (3.3) is as simple as it is. It is another instance of a general theory which works out nicely in the examples arising from Lie theory.

What one would really like is an interpretation of (3.3) as a fixed point formula in characteristic p, as was done in [**8b**] for $G = \mathrm{GL}(2)$. Our condition that G be simply connected was purely for simplicity. If we allow G to be an arbitrary reductive group over $\mathbf{Q}$, some of the spaces

$$X_{K_0} = G(\mathbf{Q})\backslash G(\mathbf{A})/K_{\mathbf{R}}K_0$$

will be associated to Shimura varieties, as in [**8d**]. They will admit natural definitions over number fields, which are compatible with the action of the Hecke operators. A similar assertion should also apply to the compactifications $\overline{X}_{K_0}$. One could then take the reduction modulo a good prime, and consider the action of the Frobenius. The intersection homology has an l-adic analogue. What is wanted is a formula for the Lefschetz number of the composition of a power of the Frobenius with an arbitrary Hecke correspondence at the invertible primes. The resulting formula could then be compared to (3.3), for suitably chosen h. Of course, one would need to know the structure of the points $\bmod p$, discussed in [**8d**], and more elaborate information on the points at infinity. Along the way, one would have to be able to explain the geometric significance of the discrete series characters $\Phi_M(\gamma, \tau)$. Such things are far from known, at least to me. I mention them only to emphasize that the formula (3.3) is just a piece of a larger puzzle.

The ultimate goal is of course to prove reciprocity laws between the arithmetic information conveyed by l-adic representations of Galois groups, and the analytic information wrapped up in the Hecke operators on L^2-cohomology. A comparison of the two Lefschetz formulas would lead to generalizations of the results in [8b] for GL(2). However, as the analytic representative, the formula (3.3) is still somewhat deficient. To get an idea of what more is needed, consider the case that the highest weight of τ is regular. Then one can show that the L^2-cohomology of X_{K_0} is concentrated in the middle dimension. Moreover, the cohomology has a Hodge decomposition

$$H^*_{(2)}(X_{K_0}, \mathcal{F}_\tau) = \bigoplus_{p+q=(1/2)\dim(X_G)} H^{p,q}_{(2)}(X_{K_0}, \mathcal{F}_\tau).$$

Since the Hecke operators commute with the Hodge group $S(\mathbf{R}) \cong \mathbf{C}^*$, there is also a decomposition

$$H^*_{(2)}(h, \mathcal{F}_\tau) = \bigoplus_{p,q} H^{p,q}_{(2)}(h, \mathcal{F}_\tau),$$

for each $h \in \mathcal{H}_{K_0}$. What is required for the comparison is a formula for each number

$$\mathrm{tr}(H^{p,q}_{(2)}(h, \mathcal{F}_\tau)), \tag{4.2}$$

rather than just the sum of traces provided by (3.3). However, this will have to wait until the trace formula has been stabilized. Then one would be able to write (4.2) as a linear combination of Lefschetz numbers attached to endoscopic groups of G. These could then be evaluated by (3.3).

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