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THE LOCAL BEHAVIOUR OF WEIGHTED ORBITAL INTEGRALS

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Introduction. Let G be a reductive algebraic group over a local field F of characteristic 0. The invariant orbital integrals

$$J_G(\gamma, f) = |D(\gamma)|^{1/2} \int_{G_\gamma(F)\backslash G(F)} f(x^{-1}\gamma x)\, dx, \qquad \gamma \in G(F),\ f \in C_c^\infty(G(F)),$$

are obtained by integrating f with respect to the invariant measure on the conjugacy class of γ. They are of considerable importance for the harmonic analysis of $G(F)$. Invariant orbital integrals are also of interest because they occur on the geometric side of the trace formula, in the case of compact quotient. For the general trace formula, the analogous terms are weighted orbital integrals [3]. They are obtained by integrating f over the conjugacy class of γ, but with respect to a measure which is not in general invariant. Weighted orbital integrals may also play a role in the harmonic analysis of $G(F)$, but this is not presently understood. Our purpose here is to study the weighted orbital integrals as functions of γ. In particular, we shall show that they retain some of the basic properties of ordinary orbital integrals.

Recall a few of the main features of the invariant orbital integrals. If F is an Archimedean field, they satisfy the differential equations

$$J_G(\gamma, zf) = \partial(h_T(z)) J_G(\gamma, f), \qquad \gamma \in T_{\mathrm{reg}}(F), \tag{1}$$

where $T_{\mathrm{reg}}(F)$ is the set of regular points in a maximal torus of $G(F)$, z is an element in the center of the universal enveloping algebra, and $\partial(h_T(z))$ is the corresponding invariant differential operator on $T(F)$. If F is a p-adic field, there are no differential equations. Instead, one has the Shalika germ expansion

$$J_G(\gamma, f) = \sum_{u \in (\mathscr{U}_G(F))} \Gamma(\gamma, u) J_G(u, f) \tag{2}$$

about 1, or more generally about any semisimple point in $G(F)$. The coefficients $\{\Gamma(\gamma, u)\}$ are functions of regular points γ near 1 and are indexed by the unipotent conjugacy classes u in $G(F)$. If F is either Archimedean or p-adic, the

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values of invariant orbital integrals at singular points can in principle be expressed in terms of their values at nearby regular points. The simplest example of this phenomenon occurs when γ belongs to the Levi component $M(F)$ of a parabolic subgroup of $G(F)$. One can form the induced class γ^G in $G(F)$, which may be singular in $G(F)$ even if γ is regular in $M(F)$. Then

$$J_G(\gamma^G, f) = \lim_{a \to 1} J_G(a\gamma, f), \tag{3}$$

where a takes values in $A_M(F)$, the split component of the center of $M(F)$. (Actually, γ^G can be a union of several conjugacy classes in $G(F)$, in which case the left-hand side of (3) is defined as a sum of several orbital integrals.)

Weighted orbital integrals are distributions on $G(F)$ which are indexed by Levi components $M(F)$, and elements $\gamma \in M(F)$. They reduce to invariant orbital integrals when $M = G$. If the centralizer $G_\gamma(F)$ is contained in $M(F)$, the weighted orbital integral is given by the formula

$$J_M(\gamma, f) = |D(\gamma)|^{1/2} \int_{G_\gamma(F)\backslash G(F)} f(x^{-1}\gamma x) v_M(x)\, dx,$$

where $v_M(x)$ is the volume of a certain convex hull. However, for general elements $\gamma \in M(F)$, the definition is more delicate and will be a consequence of Theorem 5.2 and Corollary 6.2. We will end up defining $J_M(\gamma, f)$ as a limit

$$J_M(\gamma, f) = \lim_{a \to 1} \sum_{L \in \mathscr{L}(M)} r_M^L(\gamma, a) J_L(a\gamma, f), \tag{3*}$$

where a takes small regular values in $A_M(F)$ and for each Levi component $L(F) \supset M(F)$, $r_M^L(\gamma, a)$ is a certain real-valued function. At the same time, we shall show that the distribution $J_M(\gamma, f)$ is given by an absolutely continuous measure on γ^G. The analogy between (3*) and (3) is clear. Notice that for any such a, $G_{a\gamma}(F)$ is contained in $L(F)$ so the distributions on the right side of (3*) are given by the integral formula above.

In part 2 we shall treat the case of p-adic F. We will derive a germ expansion

$$J_M(\gamma, f) \sim \sum_{L \in \mathscr{L}(M)} \sum_{u \in (\mathscr{U}_L(F))} g_M^L(\gamma, u) J_L(u, f) \tag{2*}$$

about 1, or more generally about any semisimple point in $M(F)$ (Proposition 9.1). In (2*), γ ranges over $G(F)$-regular points in $M(F)$ which are close to 1. The equivalence of the two sides of the formula means that as functions of γ, they differ by an orbital integral on $M(F)$. In particular, the coefficients $g_M^L(\gamma, u)$ are really equivalence classes of germs of functions of γ. We shall also show (Lemma 9.2) that in certain cases the germs about an arbitrary semisimple point σ in $M(F)$ can be expressed in terms of the germs about 1 in $G_\sigma(F)$. In §10 we shall investigate a homogeneity property and expand the germ

$$g_M^G(\gamma^t, v^t), \qquad t \in F^*, v \in (\mathscr{U}_G(F)),$$

in terms of

$$g_M^L(\gamma, u), \qquad u \in (\mathscr{U}_L(F)).$$

We shall deal with Archimedean F in part 3. In Proposition 11.1, we will derive a differential equation

$$(1^*) \qquad J_M(\gamma, zf) = \sum_{L \in \mathscr{L}(M)} \partial_M^L(\gamma, z_L) J_L(\gamma, f), \qquad \gamma \in T_{\text{reg}}(F),$$

when $T(F)$ is contained in $M(F)$. Here $\partial_M^L(\gamma, z_L)$ is a differential operator on $T_{\text{reg}}(F)$ which depends only on the image z_L of z in the center of the universal enveloping algebra for $L(F)$. We include γ in the notation to emphasize that if $L \neq M$, $\partial_M^L(\gamma, z_L)$ has variable coefficients. In the case $L = M$, $\partial_M^M(\gamma, z_M)$ is equal to the invariant differential operator $\partial(h_T(z))$. Proposition 11.1 is proved by a simple invariance argument, but the differential operators $\partial_M^G(\gamma, z)$ can also be constructed from the radial decomposition of z (Lemma 12.1). This formulation gives qualitative information which is useful for comparing weighted orbital integrals on different groups. We will conclude part 3 by looking at the behaviour of

$$J_M(\gamma, f), \qquad \gamma \in T_{\text{reg}}(F),$$

as γ approaches the singular set.

The coefficients $r_M^L(\gamma, a)$ in (3*) are generalizations of functions used by Flicker [10]. We shall indicate briefly how they are constructed. The essential difficulty arises when $\gamma = u$ is unipotent in $M(F)$, so let us assume this to be the case. The problem is that

$$a \to J_M(au, f), \qquad a \in A_M(F),$$

blows up at $a = 1$. Suppose first that the Levi component $M(F)$ is maximal. Then it turns out that the function

$$J_M(au, f) - 2\|\beta^\vee\| \rho(\beta, u) \log|a^\beta - a^{-\beta}| J_G(au, f)$$

has a limit at $a = 1$. Here β is either of the two reduced roots of $(G(F), A_M(F))$, and $\rho(\beta, u)$ is a uniquely determined positive number which we shall introduce in §3. As a function,

$$\rho(\beta, u), \qquad u \in \mathscr{U}_M(F),$$

is lower semicontinuous on the F-rational unipotent variety of M, and $\rho(\beta, u)$ depends only on the geometric conjugacy class of u. Now, suppose that the Levi component $M(F)$ is arbitrary. For each reduced root β of (G, A_M), we can define the number $\rho(\beta, u)$ as above. If $L(F) \supset M(F)$, there is a real vector space $\mathfrak{a}_M^L$ whose chambers correspond to parabolic subgroups R of L with Levi component M. Then $r_M^L(u, a)$ equals the volume in $\mathfrak{a}_M^L$ of the convex hull of the points

$$X_R(u, a) = \sum_{\beta} \rho(\beta, u) \log|a^{\beta} - a^{-\beta}| \beta^{\vee},$$

where for each R, β is summed over the reduced roots of (R, A_M).

The main result in part 1 is the existence of the limit in (3*). The proof is in two stages, one algebraic (§4) and one analytic (§§5–7). The algebraic part, which is due to Langlands, is a key step. It is based on the geometry of the Grothendieck–Springer resolution and establishes the continuity of a certain function on the product of $A_M(F)$ with a subspace of the unipotent variety. The analytic part, although fairly long, is based on familiar notions. Beginning with the formula of R. Rao for a unipotent orbital integral, we make various changes of variable and eventually reduce the question to an elementary problem (Lemma 6.1) in real analysis. At the end of part 1, having finally completed the definition (3*), we shall derive a descent property (Theorem 8.1) for weighted orbital integrals. This is used in the proof of the main result of [3] and will also be required for Lemma 9.2.

Since the results of this paper are to be applied to the trace formula, it is best to work in a little greater generality. We shall allow F to be a number field, equipped with a finite set S of valuations. Then the weighted orbital integrals will be distributions on $G(F_S)$. We also want to include the twisted trace formula, so we will work with disconnected groups. In the paper we will take G to be a component of a nonconnected reductive algebraic group over F.

Contents

Part 1: The general definition

§1. Assumptions on G. We would like our discussion to include twisted weighted orbital integrals considered in [9]. This is accomplished by working with nonconnected groups. Suppose that $\tilde{G}$ is an algebraic group, not necessarily connected, which is defined over a field F. We shall not focus on $\tilde{G}$ itself, but rather on a fixed connected component G of $\tilde{G}$. Given G, let us write G^+ for the subgroup of $\tilde{G}$ generated by G and G^0 for the connected component of 1 in G^+. We shall assume that G^+ is reductive. We also make the assumption that $G(F)$ is not empty. Then $G(F)$ is a Zariski dense subset of G if F is infinite.

Many of the usual notions for connected groups extend to G. For example, we can form the polynomial

$$\det((t+1)-\mathrm{Ad}(x)) = \sum_k D_k(x)t^k, \qquad x \in G.$$

The smallest integer r for which $D_r(x)$ does not vanish identically is called the *rank* of G. Choose an element $\gamma \in G$ which is *G-regular*, in that it belongs to the set

$$G_{\mathrm{reg}} = \{x \in G: D_r(x) \neq 0\},$$

and let T_0 be the connected component of the centralizer of γ in G^0. Then T_0 is a torus in G^0 ([7, Lemma 1]). We shall call the variety

$$T = T_0\gamma$$

a *maximal torus* in G. (Of course, T itself is not an algebraic torus. It is an affine variety on which the torus T_0 acts simply transitively.) Given T, set

$$T_{\mathrm{reg}} = T \cap G_{\mathrm{reg}}.$$

Then the map

$$T_{\mathrm{reg}} \times T_0 \backslash G^0 \to G,$$

given by

$$(\gamma, x) \to x^{-1}\gamma x,$$

is an open immersion.

If T is a maximal torus in G, let H_T^0 be the centralizer of T_0 in G^0. We claim that H_T^0 is a maximal torus in G^0. To see this, fix an element $\gamma \in T_{\mathrm{reg}}$. According to a result of Steinberg (Theorem 7.5 of [26]), the element γ normalizes a Borel subgroup B^0 of G^0 and a maximal torus H^0 of B^0. In particular, σ normalizes the chamber in H^0 associated to B^0. We can therefore find a point in this chamber which commutes with γ. In other words, T_0 contains a point which is G^0-regular. Consequently, $H_T^0 = H^0$ and H_T^0 is a maximal torus in G^0, as claimed. We shall write

$$H_T = H_T^0 T.$$

Define a parabolic subgroup of G^+ over F to be the normalizer in G^+ of a parabolic subgroup of G^0 which is defined over F. We define a *parabolic subset* of G to be a nonempty set of the form $P = P^+ \cap G$, where P^+ is a parabolic subgroup of G^+ over F. A *Levi component* of P will be a set $M = M^+ \cap P$, where M^+ is the normalizer in G^+ of some Levi component of P^0 which is defined over F. We shall call such an M a *Levi subset* of G. Both P and M are subvarieties of G which are defined over F. It is clear that

$$P^0 = P^+ \cap G^0$$

and

$$M^0 = M^+ \cap G^0.$$

Let N_P denote the unipotent radical of P^0. Then $P = MN_P$. If $P^+ \cap G^0$ is a minimal parabolic subgroup of G^0 over F, then P^+ meets every connected component of G^+.

We shall use the symbol M, without comment, to denote a Levi subset of G. Let $\mathscr{F}(M)$ be the collection of parabolic subsets of G which contain M, and let $\mathscr{L}(M)$ be the collection of Levi subsets of G which contain M. Any $P \in \mathscr{F}(M)$ has a unique Levi component M_P in $\mathscr{L}(M)$. As usual, we write $\mathscr{P}(M)$ for the set of $P \in \mathscr{F}(M)$ with $M_P = M$. If L belongs to $\mathscr{L}(M)$, then M is a Levi subset of L. We shall write $\mathscr{F}^L(M)$, $\mathscr{L}^L(M)$, and $\mathscr{P}^L(M)$ for the sets above, but with G replaced by L. (In general, if our notation calls for a superscript L, we shall often suppress the superscript if $L = G$.)

Let A_M denote the split component of the centralizer of M in M^0. It is a split torus over F. Let $X(M)_F$ be the group of characters of M^+ which are defined over F, and set

$$\mathfrak{a}_M = \mathrm{Hom}(X(M)_F, \mathbb{R}).$$

Then $\mathfrak{a}_M$ is a real vector space whose dimension equals that of A_M. Observe that $A_M \subset A_{M^0}$ and $\mathfrak{a}_M \subset \mathfrak{a}_{M^0}$. It is convenient to fix a Euclidean metric $\|\cdot\|$ on the space $\mathfrak{a}_M$, which we assume is the restriction of a Weyl invariant metric on a maximal such space. This provides us with a Euclidean measure on $\mathfrak{a}_M$ and also on any subspace of $\mathfrak{a}_M$.

Now, suppose that $P \in \mathscr{P}(M)$. We shall frequently write $A_P = A_{M_P}$ and $\mathfrak{a}_P = \mathfrak{a}_{M_P}$. The roots of (P, A_P) are defined with respect to the adjoint action of A_P on the Lie algebra of N_P. We shall regard them either as characters on A_P or as elements in the dual space $\mathfrak{a}_P^*$ of $\mathfrak{a}_P$. As in the connected case, we can define the simple roots Δ_P of (P, A_P) and the chamber $\mathfrak{a}_P^+$ in $\mathfrak{a}_P$ associated to P. Let Q be a second set in $\mathscr{F}(M)$, such that $P \subset Q$. Then there are canonical embeddings $\mathfrak{a}_Q \subset \mathfrak{a}_P$, $\mathfrak{a}_Q^* \subset \mathfrak{a}_P^*$, and canonical complementary subspaces $\mathfrak{a}_P^Q \subset \mathfrak{a}_P$ and $(\mathfrak{a}_P^Q)^* \subset \mathfrak{a}_P^*$. Let Δ_P^Q denote the set of roots in Δ_P which vanish on $\mathfrak{a}_Q$. It can be identified with the set of simple roots of the parabolic subset $P \cap M_Q$ of M_Q. In §1 of the paper [5] we introduced "co-roots"

$$\{\beta^\vee\colon \beta \in \Delta_{P^0}\}.$$

For each $\alpha \in \Delta_P$ define

$$\alpha^\vee = \sum_\beta \beta^\vee,$$

where β ranges over the roots in Δ_{P^0} whose restriction to $\mathfrak{a}_P$ equals α. Then

$$\Delta_P^\vee = \{\alpha^\vee\colon \alpha \in \Delta_P\}$$

is a basis of $\mathfrak{a}_P^G$. For nonminimal parabolics, the co-roots are not really natural objects, and the definition is somewhat arbitrary. However, one can see easily that the function

$$\theta_P(\lambda) = \operatorname{vol}\left(\mathfrak{a}_P^G/\mathbb{Z}(\Delta_P^\vee)\right)^{-1} \prod_{\alpha \in \Delta_P} \lambda(\alpha^\vee), \qquad \lambda \in \mathfrak{a}_{P,\mathbf{C}}^*,$$

does not depend on how the co-roots are chosen. In any case, we take $\hat{\Delta}_P = \{\varpi_\alpha\colon \alpha \in \Delta_P\}$ and $\{\varpi_\alpha^\vee\colon \alpha \in \Delta_P\}$ to be the bases of $(\mathfrak{a}_P^G)^*$ and $\mathfrak{a}_P^G$ which are dual to $\Delta_P^\vee$ and Δ_P, respectively.

We shall need the notion of a (G, M)-family. For connected groups (the case here that $G = G^0$), this was introduced in §6 of [6]. However, the definitions and results of §6 of [6] rely only on the formal properties of the chambers $\{\mathfrak{a}_P^+\colon P \in \mathscr{P}(M)\}$. These properties hold for arbitrary G, so we shall quote freely from §6 of [6] without being troubled that G is now more general. Thus, a (G, M)-family is a set of functions $\{c_P(\lambda)\colon P \in \mathscr{P}(M)\}$ of $\lambda \in i\mathfrak{a}_M^*$ with the property that if P and P' are adjacent, and λ belongs to the hyperplane spanned by the common wall of the two associated chambers in $i\mathfrak{a}_M^*$, then $c_P(\lambda) = c_{P'}(\lambda)$. Associated to any (G, M)-family $\{c_P(\lambda)\}$ is an important smooth function

$$c_M(\lambda) = \sum_{P \in \mathscr{P}(M)} c_P(\lambda)\theta_P(\lambda)^{-1}$$

on $i\mathfrak{a}_M^*$ (Lemma 6.2 of [6]). In addition, for any $Q \in \mathscr{F}(M)$, there is a smooth function $c_Q'(\lambda)$ on $i\mathfrak{a}_Q^*$ defined by formula (6.3) of [6]. As in [6], we shall let c_M and c_Q' denote the values of these functions at $\lambda = 0$.

From now on, we take F to be either a local or a global field of characteristic 0. We fix a finite set S of valuations on F. Put

$$F_S = \prod_{v \in S} F_v,$$

where F_v denotes the completion of F at v. Then F_S is a locally compact ring which is equipped with the absolute value

$$|x| = \prod_{v \in S} |x_v|_v, \qquad x \in F_S.$$

We can regard G, G^0, and G^+ as schemes over F. Since F embeds diagonally in F_S, we can take the corresponding sets $G(F_S)$, $G^0(F_S)$, and $G^+(F_S)$ of F_S-valued points. Each is a locally compact space. Both $G(F_S)$ and $G^0(F_S)$ can be expressed as products over $v \in S$ of sets of F_v-valued points. Both are contained in $G^+(F_S)$. We define a homomorphism

$$H_G\colon G^+(F_S) \to \mathfrak{a}_G$$

by

$$e^{\langle H_G(x), \chi\rangle} = |\chi(x)|, \qquad x \in G^+(F_S),\ \chi \in X(G)_F.$$

Similarly, for any M we have a homomorphism

$$H_M\colon M^+(F_S) \to \mathfrak{a}_M,$$

which restricts to a function on $M(F_S)$ or $M^0(F_S)$.

For each $v \in S$, let K_v be a maximal compact subgroup of $G^0(F_v)$ which is admissible relative to M in the sense of §1 of [6]. Then $K = \prod_{v \in S} K_v$ is a maximal compact subgroup of $G^0(F_S)$. If P is any element in $\mathscr{F}(M)$,

$$G^+(F_S) = P^+(F_S)K = N_P(F_S)M_P^+(F_S)K.$$

This follows from the connected case and the fact that P^+ meets every component of G^+. For any point

$$x = n_P m_P k_P, \qquad n_P \in N_P(F_S),\ m_P \in M_P^+(F_S),\ k \in K,$$

in $G^+(F_S)$, define

$$H_P(x) = H_{M_P}(m_P).$$

Set

$$v_P(\lambda, x) = e^{-\lambda(H_P(x))}, \qquad \lambda \in \mathfrak{a}^*_{P,\mathbf{C}}.$$

Then, for any $x \in G^+(F_S)$,

$$\{v_P(\lambda, x): P \in \mathscr{P}(M)\}$$

is a (G, M)-family of functions of $\lambda \in i\mathfrak{a}_M^*$. (See page 40 of [6].) It is the main ingredient in the definition of a weighted orbital integral.

§2. Weighted orbital integrals. Suppose that $\gamma = \prod_{v \in S} \gamma_v$ is an element in $G(F_S)$. For each v, we write G_{γ_v} for the identity component of the centralizer of γ_v in G^0. It is a connected algebraic group, defined over F_v, which is reductive if γ_v is semisimple. We regard

$$G_\gamma = \prod_{v \in S} G_{\gamma_v}$$

as a scheme over F_S. It is clear that

$$G_\gamma(F_S) = \prod_{v \in S} G_{\gamma_v}(F_v).$$

In the special case that γ belongs to $G(F)$, as an element embedded diagonally in $G(F_S)$, G_γ is just the identity component of the centralizer of γ in G^0, and in particular is a group defined over F. In general, if M is given and $P \in \mathscr{P}(M)$, we can obviously define group schemes M_γ and P_γ in the same way.

There is a Jordan decomposition for elements in $G(F_S)$. Any $\gamma \in G(F_S)$ can be decomposed uniquely as $\gamma = \sigma u$, where σ is a semisimple element in $G(F_S)$ and u is a unipotent element in $G_\sigma(F_S)$. If γ belongs to $G(F)$, then σ and u will belong to $G(F)$ and $G_\sigma(F)$, respectively. Define

$$D(\gamma) = D^G(\gamma) = \prod_{v \in S} \det(1 - \mathrm{Ad}(\sigma_v))_{\mathfrak{g}/\mathfrak{g}_{\sigma_v}},$$

where $\sigma = \prod_{v \in S} \sigma_v$ and $\mathfrak{g}$ and $\mathfrak{g}_{\sigma_v}$ are the Lie algebras of G and G_{σ_v}, respectively. Observe that $D(\gamma) = D_r(\gamma)$ if and only if γ is G-regular. In general, $D(\gamma)$ is an element in F_S which depends only on the $G^0(F_S)$ orbit of the semisimple constituent of γ. The absolute value

$$|D(\gamma)| = \prod_{v \in S} |D(\gamma_v)|_v, \qquad \gamma \in G(F_S),$$

is an upper semicontinuous function of γ on $G(F_S)$.

We require a compactness lemma.

LEMMA 2.1. *Given the semisimple element σ in $G(F_S)$, we can find an invariant neighborhood Δ_σ of 1 in $G_\sigma(F_S)$ with the following property: for every compact subset Δ of $G(F_S)$ there is a compact subset Σ of $G_\sigma(F_S) \setminus G^0(F_S)$ such that*

$$y^{-1}\sigma\Delta_\sigma y \cap \Delta = \varnothing, \qquad y \in G_\sigma(F_S) \setminus G^0(F_S),$$

unless y belongs to Σ.

Proof. Clearly it is enough to prove the lemma for each of the groups $G(F_v)$, $v \in S$. We may therefore assume that S contains just one v, and that $F = F_v = F_S$. We can also certainly restrict our attention to a fixed maximal torus T_0 of G_σ, defined over F. The lemma is then equivalent to the following assertion: Given T_0 and a small neighborhood ω_T of σ in

$$T(F) = \sigma T_0(F),$$

we can choose a compact Σ for every compact Δ such that if

$$y^{-1}\omega_T y \cap \Delta \neq \varnothing, \qquad y \in G^0(F),$$

then the projection of y onto $G_\sigma(F)\backslash G^0(F)$ belongs to Σ. If $G = G^0$, this is a result of Harish-Chandra. (See [12, Theorem 1] for Archimedean F and [14, Lemma 19] for discrete F.) In the case of base change for real groups, the assertion has been proved by Shelstad [23, Theorem 4.2.1].

Harish-Chandra's proof of Lemma 19 of [14] can actually be applied to the general case. Let F' be a finite extension of F over which the torus H_T^0, defined in §1, splits. The map

$$G_\sigma(F)\backslash G^0(F) \to G_\sigma(F')\backslash G^0(F')$$

is a continuous injection, and its image is closed. (See pages 52–53 of [14].) It is therefore enough to prove the assertion with F replaced by F'. But

$$G^0(F') = B^0(F')K',$$

where K' is a compact subgroup of $G^0(F')$ and

$$B^0 = NH_T^0$$

is a Borel subgroup of G^0 defined over F'. We therefore need only consider elements y in $B^0(F')$.

We can write H_T^0 as a product $T_0 \times S_0$, where S_0 is a split torus over F' which is normalized by $\mathrm{ad}(\sigma)$ and for which the endomorphism

$$s \to s^{-1}\sigma s\sigma^{-1}, \qquad s \in S_0,$$

has finite kernel. We thus have a surjective map

$$N_\sigma(F')\backslash N(F') \times S_0(F') \to B_\sigma^0(F')\backslash B^0(F'),$$

with finite fibres. Suppose that

$$y = ns, \qquad n \in N_\sigma(F')\backslash N(F'),\ s \in S_0(F'),$$

and that

$$y^{-1}\omega_T y \cap \Delta' \neq \varnothing,$$

for a given compact subset Δ' of $G(F)$. We have only to show that y lies in a compact subset of

$$N_\sigma(F')\backslash N(F') \times S_0(F'),$$

which depends only on Δ'.

Observe that

$$y^{-1}\omega_T y \subset N(F')T(F') \cdot s^{-1}\sigma s\sigma^{-1}.$$

Consequently, s must lie in a fixed compact subset Σ'_S of $S_0(F')$. This implies that

$$n^{-1}\omega_T n$$

meets the compact set

$$\{s\Delta' s^{-1}: s \in \Sigma'_S\}.$$

It follows that there is a compact subset Δ'_N of $N(F')$, depending only on Δ', such that

$$t^{-1}n^{-1}tN_\sigma(F')n \cap \Delta'_N \neq \varnothing$$

for some $t \in \omega_T$. But we can then argue as on pages 53–54 of [14]. The conclusion is that n lies in a fixed compact subset Σ'_N of $N_\sigma(F')\backslash N(F')$. (This may also be deduced from an integration formula similar to Lemma 2.2 of [1].) As required, we have established that y lies in $\Sigma'_N \Sigma'_S$, a compact set which depends only on Δ'. □

The space $C_c^\infty(G(F_S))$ of smooth, compactly supported functions on $G(F_S)$ is defined in the usual fashion. Our objects of study are distributions on $G(F_S)$ which are indexed by elements $\gamma \in M(F_S)$. We can define them initially, however, only for elements $\gamma \in M(F_S)$ for which M_γ equals G_γ. (If $\gamma = \sigma u$ is the Jordan decomposition, this condition is equivalent to the equality of M_σ and G_σ.) Assume that γ has this property. We set

$$(2.1)\quad J_M(\gamma, f) = |D(\gamma)|^{1/2} \int_{M^0(F_S)\backslash G^0(F_S)} \int_{\mathcal{O}_\gamma(M^0(F_S))} f(x^{-1}\mu x) v_M(x)\, d\mu\, dx,$$

where

$$\mathcal{O}_\gamma(M^0(F_S)) = \{m^{-1}\gamma m: m \in M^0(F_S)\}$$

and f is any function in $C_c^\infty(G(F_S))$. Implicit in the definition is a choice of a $G^0(F_S)$-invariant measure on $M^0(F_S)\setminus G^0(F_S)$ and an $M^0(F_S)$-invariant measure on the orbit $\mathcal{O}_\gamma(M^0(F_S))$. The convergence of the integrals is assured by the last lemma and the existence [20] of invariant measures on $\mathcal{O}_\gamma(M^0(F_S))$. It is clear that $J_M(\gamma)$ depends only on the orbit $\mathcal{O}_\gamma(M^0(F_S))$. The choice of the element γ from the orbit allows us to combine our two measures into a $G^0(F_S)$-invariant measure on $G_\gamma(F_S)\setminus G^0(F_S)$. Since

$$v_M(mx) = v_M(x), \qquad m \in M^0(F_S),$$

we obtain an alternate formula

$$J_M(\gamma, f) = |D(\gamma)|^{1/2} \int_{G_\gamma(F_S)\setminus G^0(F_S)} f(x^{-1}\gamma x) v_M(x)\, dx. \tag{2.1*}$$

Our definition is actually a mild generalization of the ones given in [6, §8] and [9]. In the earlier formulations, γ was assumed to be a G-regular semisimple element in $M(F_S)$. However, the integral formula given in §8 of [6] is of the same form as (2.1*), and the two can be subjected to similar manipulations.

We should perhaps stress that although $J_M(\gamma)$ is a distribution on $G(F_S)$, it depends directly on the field F. The dependence is through the Levi subset M, or, more precisely, the split component A_M. There is no need to include this in the notation, however, as long as we remember to regard M as an object over F.

Weighted orbital integrals are of course generalizations of the invariant orbital integrals $J_G(\gamma)$. In this paper we intend to show that weighted orbital integrals retain many of the properties of invariant orbital integrals. However, we might first recall that there is one basic difference. The distribution $J_M(\gamma)$ is not in general invariant. Suppose that y is any point in $G^0(F_S)$. Then for γ and f as above, and

$$f^y(x) = f(yxy^{-1}),$$

we have

$$J_M(\gamma, f^y) = \sum_{Q \in \mathcal{F}(M)} J_M^{M_Q}(\gamma, f_{Q,y}). \tag{2.2}$$

Here $J_M^{M_Q}(\gamma)$ stands for the weighted orbital integral on $M_Q(F_S)$ and $f_{Q,y}$ is the function in $C_c^\infty(M_Q(F_S))$ defined by (3.3) of [6]. That is,

(2.3)

$$f_{Q,y}(m) = \delta_Q(m)^{1/2} \int_K \int_{N_Q(F_S)} f(k^{-1}mnk) v_Q'(ky)\, dn\, dk, \qquad m \in M_Q(F_S),$$

where δ_Q is the modular function of the group $Q(F_S)$. Formula (2.2) was proved in Lemma 8.2 of [6] for γ regular. For arbitrary γ (with $M_\gamma = G_\gamma$, of course) it is proved exactly the same way. In view of (2.2), the true analogues of the invariant orbital integrals are really the invariant distributions introduced in §10 of [6]. We shall return to these in another paper.

The distribution $J_M(\gamma)$ does depend on the maximal compact subgroup K as well as on M. However, suppose that y is an element in $G(F)$. Then we have

$$J_{y^{-1}My}(y^{-1}\gamma y, f^y) = J_M(\gamma, f), \tag{2.4}$$

where the distribution on the left is taken with respect to the maximal compact subgroup $y^{-1}Ky$. This follows immediately from the fact that $v_{y^{-1}My}(y^{-1}xy)$, taken with respect to $y^{-1}Ky$, equals $v_M(x)$.

It will be useful to have notation to describe whether a function of γ equals an orbital integral near a given point. Suppose that σ is a semisimple element in $M(F_S)$ and that ϕ_1 and ϕ_2 are functions defined on an open subset Σ of $\sigma M_\sigma(F_S)$. We assume that the closure of Σ contains an $M_\sigma(F_S)$-invariant neighbourhood of σ in $\sigma M_\sigma(F_S)$. We shall say that ϕ_1 is (M,σ)-*equivalent* to ϕ_2, and we shall write

$$\phi_1(\gamma) \overset{(M,\sigma)}{\sim} \phi_2(\gamma)$$

if the difference is an (invariant) orbital integral on $M(F_S)$ near σ; that is, if there is a function $h \in C_c^\infty(M(F_S))$ and a neighbourhood U of σ in $M(F_S)$ such that

$$\phi_1(\gamma) - \phi_2(\gamma) = J_M^M(\gamma, h), \qquad \gamma \in \Sigma \cap U.$$

Implicit in this definition is the assumption that the function

$$\phi_1(\gamma) - \phi_2(\gamma), \qquad \gamma \in \Sigma \cap U,$$

depends on a choice of an $M^0(F_S)$-invariant measure on $\mathcal{O}_\gamma(M^0(F_S))$.

LEMMA 2.2. *Suppose that $M_\sigma = G_\sigma$. Then for any $f \in C_c^\infty(G(F_S))$,*

$$J_M(\gamma, f) \overset{(M,\sigma)}{\sim} 0, \qquad \gamma \in \sigma M_\sigma(F_S).$$

Proof. The function $J_M(\gamma, f)$ is defined for all elements γ in $\sigma M_\sigma(F_S)$ which are sufficiently close to σ. Indeed, for any such γ we have $G_\gamma \subset G_\sigma$, so that $G_\gamma = M_\gamma$. We apply the formula (2.1*). Decomposing the integral into a double integral over

$$M_\gamma(F_S)\backslash M^0(F_S) \times M^0(F_S)\backslash G^0(F_S),$$

we obtain

$$J_M(\gamma, f) = |D(\gamma)|^{1/2} \int_{M^0(F_S)\backslash G^0(F_S)} \int_{M_\gamma(F_S)\backslash M^0(F_S)} f(y^{-1}x^{-1}\gamma xy) v_M(y)\, dx\, dy.$$

By Lemma 2.1, we can restrict y to a compact set which is independent of γ. The lemma then follows from the fact that

$$|D(\gamma)|^{1/2} |D^M(\gamma)|^{-1/2}, \qquad \gamma \in M(F_S),$$

is an $M^0(F_S)$-invariant function which is smooth for γ near σ. □

Let $A_{M,\mathrm{reg}}$ be the set of elements $a \in A_M$ such that G_a is contained in M^0. It is an open subvariety of A_M which is defined over F. Suppose that γ is *any* point in $M(F_S)$. Then for any $a \in A_{M,\mathrm{reg}}(F_S)$ which is close to 1, $a\gamma$ will be a point in $M(F_S)$ with the property that $M_{a\gamma} = G_{a\gamma}$. The distribution $J_M(a\gamma)$ is therefore defined. We propose to investigate its behaviour as a approaches 1.

§3. **Polynomials on unipotent orbits.** Our goal is to define distributions $J_M(\gamma)$ when γ is an arbitrary orbit in $M(F_S)$. We shall later use a descent argument to reduce the problem to the case of unipotent classes. In the next two sections we shall study some functions that arise from this special case. In these sections we assume that $G^0 = G$. We shall also assume that S contains one element and F is local, so that $F = F_S$.

Since $G^0 = G$, M is a Levi subgroup of G. Let $\mathcal{U}_M$ denote the Zariski closure in M of the set of unipotent elements in $M(F)$. It is an algebraic variety which is defined over F. Notice that $\mathcal{U}_M$ is a union of unipotent conjugacy classes in M. We shall write $(\mathcal{U}_M)$ for the set of those conjugacy classes in $\mathcal{U}_M$ (over $\bar{F}$) which have a rational representative. (This notation is slightly different from that of [3].) For any $U \in (\mathcal{U}_M)$ it is clear that the set $U(F) = M(F) \cap U$ of rational points is Zariski dense in U.

Suppose that P_1 belongs to $\mathcal{P}(M)$. We shall write $N_1 = N_{P_1}$, and we shall write $\Sigma^r_{P_1}$ for the set of reduced roots of (P_1, A_M). If $a \in A_M$ and $u \in \mathcal{U}_M$, then

$$n \to (au)^{-1} n^{-1} (au) n, \qquad n \in N_1,$$

is a polynomial mapping from N_1 to itself. It is invertible if a belongs to $A_{M,\mathrm{reg}}$, or equivalently, if the function

$$\prod_{\beta \in \Sigma^r_{P_1}} (a^\beta - a^{-\beta}) \tag{3.1}$$

does not vanish. Consequently, for any such a and any unipotent element

$$\pi = uv, \qquad u \in \mathcal{U}_M,\ v \in N_1,$$

in P_1, we can define $n \in N_1$ uniquely by

$$a\pi = n^{-1}aun. \tag{3.2}$$

The function $(a, \pi) \to n$ is the product of a negative power of (3.1) with a morphism from $A_M \times \mathcal{U}_M N_1$ to N_1 which is defined over F. We shall exploit the connection with finite-dimensional representations to study $v_P(\lambda, n)$ as a function of a and π.

There is a natural embedding of the character lattice $X(A_M)$ into $\mathfrak{a}_M^*$. We write $Wt(\mathfrak{a}_M)$ for the set of elements in $\mathfrak{a}_M^*$ which are extremal weights of irreducible finite-dimensional F-rational representations of G. Then $Wt(\mathfrak{a}_M)$ is a subgroup of finite index in $X(A_M)$ and in particular a lattice in $\mathfrak{a}_M^*$. For each $\omega \in Wt(\mathfrak{a}_M)$ fix $(\Lambda_\omega, V_\omega, \phi_\omega, \|\cdot\|)$, with Λ_ω an irreducible representation of G on a vector space V_ω, defined over F, ϕ_ω an extremal vector in V_ω with weight ω, and $\|\cdot\|$ a norm (height) function on $V_\omega(F)$ which is stabilized by K and for which ϕ_ω is a unit vector. Suppose that for a given $P \in \mathcal{P}(M)$, ω is P-dominant. That is, ω is nonnegative on the intersection of $\mathfrak{a}_M^G$ with the chamber $\mathfrak{a}_P^+$ of P. Then for any point

$$x = pk, \qquad p \in P(F),\ k \in K,$$

in $G(F)$ we have

$$v_P(\omega, x) = e^{-\omega(H_P(x))} = \left\|\Lambda_\omega(p^{-1})\phi_\omega\right\|.$$

It follows that

$$v_P(\omega, x) = \left\|\Lambda_\omega(x^{-1})\phi_\omega\right\|. \tag{3.3}$$

Now $\Lambda_\omega(n^{-1})\phi_\omega$ is a polynomial in n with values in V_ω. It follows that $v_P(\omega, n)$, as a function of (a, π), is the product of the absolute value of a negative power of (3.1) with the norm of a polynomial in (a, π) with values in V_ω. (By *polynomial* we mean of course a morphism between algebraic varieties.)

Suppose that β is a reduced root of (G, A_M). Let M_β be the group in $\mathcal{L}(M)$ such that

$$\mathfrak{a}_{M_\beta} = \{H \in \mathfrak{a}_M : \beta(H) = 0\}.$$

Then $\dim(A_M/A_{M_\beta}) = 1$. Let P_β be the unique group in $\mathcal{P}^{M_\beta}(M)$ whose simple root is β. We assume that the groups P, $P_1 \in \mathcal{P}(M)$ are chosen so that $P \cap M_\beta = P_\beta$ and $P_1 \cap M_\beta = \overline{P}_\beta$. Suppose that $\pi_\beta = u\nu_\beta$, where ν_β is restricted to lie in $\overline{N}_\beta = N_1 \cap M_\beta$. Then if

$$a\pi_\beta = n_\beta^{-1} a u n_\beta,$$

as in (3.2), n_β will also lie in $\overline{N}_\beta$. As a function of a, n_β is invariant under A_{M_β}.

Choose $\omega \in Wt(\mathfrak{a}_M)$ such that $\omega(\beta^\vee)$ is positive. Then

$$v_{P_\beta}(\omega, n_\beta) = v_P(\omega, n_\beta) = \left\| \Lambda_\omega\left(n_{\bar{\beta}}^{-1}\right)\phi_\omega \right\|.$$

Any rational function on (A_M/A_{M_β}) has a Laurent expansion in $(a^\beta - a^{-\beta})$ about $a = 1$, so we can write

$$\Lambda_\omega\left(n_{\bar{\beta}}^{-1}\right)\phi_\omega = \sum_{k \in \mathbf{Z}} c_k(\pi_\beta)(a^\beta - a^{-\beta})^{-k},$$

for a near 1. The coefficients $c_k(\pi_\beta)$ are polynomials from $\mathscr{U}_M \bar{N}_\beta$ to V_ω which are defined over F. They vanish for almost all positive k.

Consider the restriction of the functions $c_k(\pi_\beta)$ to a set $U\bar{N}_\beta$, where U is a class in $(\mathscr{U}_M)$. For each $u \in U(F)$, let $\rho(\beta, u)$ denote the product of $\omega(\beta^\vee)^{-1}$ with the largest k such that c_k does not vanish identically on $U\bar{N}_\beta$. Notice that $c_0(\pi_\beta)$ does not vanish identically, since its component in the direction of ϕ_ω is 1. Consequently, $\rho(\beta, u)$ is nonnegative. Define a function

$$r_\beta(\lambda, u, a) = |a^\beta - a^{-\beta}|^{\rho(\beta, u)\lambda(\beta^\vee)}, \qquad \lambda \in \mathfrak{a}^*_{M,\mathbf{C}}. \tag{3.4}$$

It depends only on the conjugacy class of u in M. The number $\rho(\beta, u)$ is evidently characterized by the property that the limit

$$\lim_{a \to 1} r_\beta(\lambda, u, a) v_{P_\beta}(\lambda, n_\beta)$$

exists and does not vanish identically in $\pi_\beta \in U(F)\bar{N}_\beta(F)$. Therefore, $\rho(\beta, u)$ and $r_\beta(\lambda, u, a)$ are both independent of ω. We shall later need to know that

$$r_\beta(\lambda, u, a) = r_{-\beta}(-\lambda, u, a). \tag{3.5}$$

This fact is easily deduced from the existence of an involution on G which acts as (-1) on A_M. It also follows from the results of Langlands in the next section. We note, finally, that by (3.5) and our definition,

$$\rho(\beta, u)\omega(\beta^\vee)$$

is an integer for any $\omega \in Wt(\mathfrak{a}_M)$.

Now, as before, assume that P_1 is an arbitrary group in $\mathscr{P}(M)$. The functions

$$\prod_{\beta \in \Sigma_P^r \cap \Sigma_{\bar{P}_1}^r} r_\beta(\lambda, u, a), \qquad P \in \mathscr{P}(M),$$

form a (G, M)-family of the special sort considered in §7 of [2]. Assume that a, $\pi = uv$, and n are related as in (3.2), and set

$$w_P(\lambda, a, \pi) = \left(\prod_{\beta \in \Sigma_P^r \cap \Sigma_{\bar{P}_1}^r} r_\beta(\lambda, u, a) \right) v_P(\lambda, n) \tag{3.6}$$

for $P \in \mathscr{P}(M)$. These functions also form a (G, M)-family, since they are defined as products from two (G, M)-families. Note that for any k in

$$K_M = K \cap M(F),$$

we have

$$w_P(\lambda, a, k^{-1}\pi k) = w_P(\lambda, a, \pi). \tag{3.7}$$

Suppose that ω is any element in $Wt(\mathfrak{a}_M)$ which is P-dominant. By (3.3) we have

$$w_P(\omega, a, \pi) = \| W_\omega(a, \pi) \|,$$

where

$$W_\omega(a, \pi) = \Bigg(\prod_{\beta \in \Sigma_P^r \cap \Sigma_{\bar{P}_1}^r} (a^\beta - a^{-\beta})^{\rho(\beta, u)\omega(\beta^\vee)} \Bigg) \Lambda_\omega(n^{-1})\phi_\omega. \tag{3.8}$$

For any class $U \in (\mathscr{U}_M)$ we shall let U^+ denote the Zariski closure of U. (We had best not use the usual notation for closure, since $\bar{P}_1 = M\bar{N}_1$ denotes the parabolic opposite to P_1.) The function

$$W_\omega(a, \pi), \qquad (a, \pi) \in A_{M,\mathrm{reg}} \times UN_1,$$

is the product of a negative power of (3.1) with a polynomial from $A_M \times U^+N_1$ which is defined over F. In general, $w_P(\lambda, a, \pi)$ is the exponential of the value of λ at some vector in $\mathfrak{a}_M$. It follows that if $\{\omega_1, \dots, \omega_n\}$ is any subset of $\mathfrak{a}_M^*$ consisting of P-dominant elements in $Wt(\mathfrak{a}_M)$, and

$$\lambda = \Sigma_{i=1}^n \lambda_i \omega_i, \qquad \lambda_i \in \mathbb{C},$$

then

$$w_P(\lambda, a, \pi) = \prod_{i=1}^n \| W_{\omega_i}(a, \pi) \|^{\lambda_i}. \tag{3.9}$$

Fix the class $U \in (\mathscr{U}_M)$. The next lemma is essentially a consequence of our definitions.

LEMMA 3.1. *Suppose that $P \in \mathscr{P}(M)$ is adjacent to P_1. Then for all π in an open dense subset of $U(F)N_1(F)$, the limit*

$$\lim_{a \to 1} w_P(\lambda, a, \pi)$$

exists and is nonzero.

Proof. Since P is adjacent to P_1, there is a unique root β in $\Delta_P \cap \Delta_{\bar{P}_1}$. Define M_β and P_β as above. Then $P \cap M_\beta = P_\beta$ and $P_1 \cap M_\beta = \bar{P}_\beta$. Write

$$\pi = \pi_\beta \nu', \qquad \pi_\beta \in UN_{\bar{P}_\beta}, \ \nu' \in N_1 \cap N_P.$$

If n is defined by (3.2), and

$$n = n_\beta n', \qquad n_\beta \in N_{\bar{P}_\beta},\ n' \in N_1 \cap N_P,$$

then

$$a\pi_\beta = n_\beta^{-1} a u n_\beta,$$

as above. Since β is also the unique root in $\Sigma_P^r \cap \Sigma_{\bar{P}_1}^r$, we have

$$\begin{aligned} w_P(\lambda, a, \pi) &= r_\beta(\lambda, u, a) v_P(\lambda, n) \\ &= r_\beta(\lambda, u, a) v_{P_\beta}(\lambda, n_\beta). \end{aligned}$$

We have seen that for generic π_β this function has a nonzero limit at $a = 1$. □

§4. **A technique of Langlands.** In this section we shall extend Lemma 3.1 to any pair of groups P and P_1 in $\mathcal{P}(M)$. The results here are due to Langlands.* They hinge on an application of Zariski's main theorem to the Grothendieck–Springer resolution (or rather, to its analogue for arbitrary parabolics). We continue with the assumptions and notation of §3. The main result is

LEMMA 4.1. *Suppose that P is any group in $P(M)$. Then for all π in an open dense subset of $U(F)N_1(F)$, the limit*

$$\lim_{a\to 1} w_P(\lambda, a, \pi)$$

exists and is nonzero.

Proof. The lemma will be proved by induction on dist(P, P_1), the number of singular hyperplanes which separate the chambers of P and P_1. It follows from the definition that $w_{P_1}(\lambda, a, \pi) = 1$. Assume then that P and $P_1' = MN_1'$ are adjacent, with

$$\mathrm{dist}(P, P_1) = \mathrm{dist}(P_1', P_1) + 1,$$

and that the lemma is valid with P replaced by P_1'. To prove the lemma, we must show that for π in an open dense subset of $U(F)N_1(F)$, the function

$$w_P(\lambda, a, \pi) w_{P_1'}(\lambda, a, \pi)^{-1} \tag{4.1}$$

has a nonzero limit at $a = 1$.

Assume that a, $\pi = uv$, and n are related as in (3.2), and write

$$n = m'n'k', \qquad m' \in M(F),\ n' \in N_1'(F),\ k' \in K.$$

Set $u' = (m')^{-1}um'$ and define the element $\pi' = u'v'$ in $U(F)N_1'(F)$ so that a,

*I would like to thank Langlands for communicating his unpublished results to me.

π', and n' are related by the analogue of (3.2). Then

$$a\pi = n^{-1}aun = (k')^{-1}(n')^{-1}au'n'k' = (k')^{-1}a\pi'k'. \tag{4.2}$$

Let β be the unique root in $\Delta_P \cap \Delta_{\overline{P}_1'}$. The function (4.1) equals

$$\begin{aligned} &v_P(\lambda, n)v_{P_1'}(\lambda, n)^{-1}r_\beta(\lambda, u, a) \\ &\quad = \big(v_P(\lambda, m')v_P(\lambda, n')\big)\big(v_{P_1'}(\lambda, m')v_{P_1'}(\lambda, n')\big)^{-1}r_\beta(\lambda, u', a) \\ &\quad = r_\beta(\lambda, u', a)v_P(\lambda, n') \\ &\quad = w_P(\lambda, a, \pi'), \end{aligned}$$

since $v_{P_1'}(\lambda, n')$ equals 1, $v_P(\lambda, m')$ equals $v_{P_1'}(\lambda, m')$, and $r_\beta(\lambda, u, a)$ depends only on the conjugacy class of u. Now π' is uniquely defined as an element in $(U(F)N_1'(F))^{K_M}$, the space of orbits of $K_M = K \cap M(F)$ in $U(F)N_1'(F)$. It depends only on the image of π in $(U(F)N_1(F))^{K_M}$. Therefore, $(a, \pi) \to \pi'$ determines a continuous mapping

$$A_{M,\mathrm{reg}}(F) \times \big(U(F)N_1(F)\big)^{K_M} \to \big(U(F)N_1'(F)\big)^{K_M}. \tag{4.3}$$

If we can show that π' extends to a reasonable function of a in a neighborhood of 1 in $A_M(F)$, our lemma will follow from Lemma 3.1.

Let U^G be the unipotent conjugacy class which is obtained by inducing U to G in the sense of Lusztig–Spaltenstein [19]. It is defined as the unique class in $(\mathcal{U}_G)$ such that the set

$$U^{P_1} = U^G \cap UN_1$$

is dense in UN_1. It is independent of P_1. (See [19].) Write $(U^{P_1}(F))^{K_M}$ for the space of K_M orbits in $U^{P_1}(F)$.

LEMMA 4.2. *The mapping* (4.3) *has a continuous extension to an open subset of* $A_M(F) \times (U(F)N_1(F))^{K_M}$ *which contains* $\{1\} \times (U^{P_1}(F))^{K_M}$. *It maps the latter set homeomorphically onto* $(U^{P_1'}(F))^{K_M}$.

Proof. The idea of the proof is to define the map $(a, \pi) \to \pi'$ algebraically by means of a birational correspondence. Let $\mathcal{S}$ be the set of pairs (g, P_1h) in $G \times P_1 \backslash G$ such that g belongs to the parabolic subgroup $P_1^h = h^{-1}P_1h$. It is an algebraic variety which is defined over F. For each $w \in W_0$, the restricted Weyl group of G on a maximal split torus which contains A_M, there is an affine coordinate system $P_1 \times \overline{N}_1 \to \mathcal{S}$ given by

$$(p, \bar{n}) \to (p^{\bar{n}w}, P_1\bar{n}w), \qquad p \in P_1,\ \bar{n} \in \overline{N}_1.$$

The Bruhat decomposition insures that these coordinate patches cover $\mathcal{S}$.

If (g, P_1h) is any point in $\mathscr{S}$, $g^{h^{-1}} = hgh^{-1}$ is a point in P_1 which is determined up to P_1-conjugacy. Its projection onto M determines a conjugacy class in M. Let $\mathscr{S}(U)$ be the subvariety of points (g, P_1h) in $\mathscr{S}$ such that the associated conjugacy class in M belongs to $\overline{U}A_M$. By taking the semisimple constituent of this class, we get a map from $\mathscr{S}(U)$ onto A_M. Observe that the intersection of $\mathscr{S}(U)$ with any of the coordinate patches above equals $(U^+A_MN_1) \times \overline{N}_1$. Clearly $U^+A_MN_1$ is an irreducible F-closed subset of P_1, so $\mathscr{S}(U)$ is indeed a closed subvariety of $\mathscr{S}$ which is defined over F. Notice also that the projection onto the first factor gives a morphism $\mathscr{S} \to G$. We therefore obtain a morphism $\mathscr{S}(U) \to G \times A_M$ which is defined over F.

Let $\mathscr{S}'(U)$ be the variety defined as above, but with P_1 replaced by P_1', and define $\mathscr{C}(U)$ as the fibre product

$$\begin{tikzcd}[column sep=small] & \mathscr{C}(U) \arrow[dl] \arrow[dr] & \\ \mathscr{S}(U) \arrow[dr] & & \mathscr{S}'(U) \arrow[dl] \\ & G \times A_M & \end{tikzcd}.$$

Let $\mathscr{S}(U)_{\text{reg}}$, $\mathscr{S}'(U)_{\text{reg}}$, and $\mathscr{C}(U)_{\text{reg}}$ be the Zariski open subsets of $\mathscr{S}(U)$, $\mathscr{S}'(U)$, and $\mathscr{C}(U)$, respectively, which map to $G \times A_{M,\text{reg}}$. Notice that the inverse image in $\mathscr{S}(U)_{\text{reg}}$ of a point (g, a) in $G \times A_{M,\text{reg}}$ is

$$\left\{\left((au)^h, P_1h\right): u \in U^+,\ h \in P_1 \backslash G,\ (au)^h = g\right\}.$$

This set is empty if g is not conjugate to an element in aU^+. However, if g is conjugate to an element in aU^+, the set contains exactly one point as follows from the fact that the centralizer of au is contained in M. Similar remarks apply to the inverse image of (g, a) in $\mathscr{S}'(U)_{\text{reg}}$. Consequently, the maps $\mathscr{C}(U)_{\text{reg}} \to \mathscr{S}(U)_{\text{reg}}$ and $\mathscr{C}(U)_{\text{reg}} \to \mathscr{S}'(U)_{\text{reg}}$ are isomorphisms. Composing the second of these with the inverse of the first, we obtain an isomorphism from $\mathscr{S}(U)_{\text{reg}}$ to $\mathscr{S}'(U)_{\text{reg}}$ which is defined over F. It is the algebraic realization of (4.3).

Let $\mathscr{C}_1(U)$ be the closure of $\mathscr{C}(U)_{\text{reg}}$. It is a closed (irreducible) subvariety of $\mathscr{S}(U) \times \mathscr{S}'(U)$, and its projection onto each factor is a birational map. Let $\mathscr{S}_1(U)$ be the set of points in $\mathscr{S}(U)$ at which the resulting birational transformation from $\mathscr{S}(U)$ to $\mathscr{S}'(U)$ is defined. It is an open subset of $\mathscr{S}(U)$ which is defined over F and contains $\mathscr{S}(U)_{\text{reg}}$. The birational transformation induces an isomorphism from $\mathscr{S}_1(U)$ onto a Zariski open subset $\mathscr{S}_1'(U)$ of $\mathscr{S}'(U)$ which is defined over F. We shall show that $\mathscr{S}_1(U)$ contains the set

$$\{(p, P_1): p \in U^{P_1}\}. \tag{4.4}$$

We shall use Zariski's main theorem ([16], p. 280). Since $P_1' \backslash G$ is a projective variety, the morphism $\mathscr{S}'(U) \to G \times A_M$ is projective ([16], p. 103). Therefore, the morphism $\mathscr{C}(U) \to \mathscr{S}(U)$, which is just obtained by a change of base, is also projective. In particular, the image of $\mathscr{C}_1(U)$ in $\mathscr{S}(U)$ is closed in $\mathscr{S}(U)$ and hence equal to $\mathscr{S}(U)$. Zariski's main theorem asserts that the set of points in $\mathscr{S}'(U)$ corresponding to any normal point in $\mathscr{S}(U)$ is connected. Take any point (p, P_1) in the set (4.4). Such a point is smooth in $\mathscr{S}(U)$ and hence normal. Since U^{P_1} and $U^{P_1'}$ are both contained in U^G, we can write $p = (p')^h$ for elements $p' \in U^{P_1'}$ and $h \in G$. Choose any point in $\mathscr{S}'(U)$ which corresponds to (p, P_1). It equals $(q^x, (P_1')^x)$ for elements $q \in U^+N_1'$ and $x \in G$ such that $p = q^x$. Consequently, $q = (p')^{hx^{-1}}$. By Proposition 3.2(b) of [24], q belongs to UN_1' rather than just to its closure. Part (e) of the same proposition then asserts that hx^{-1} belongs to z_iP_1', with $\{z_1, \dots, z_r\}$ a fixed set of representatives of the connected components of the centralizer of p' in G. It follows that our arbitrary point equals $((p')^h, (P_1')^{z_i^{-1}h})$. In other words, the set of points corresponding to (p, P_1) is finite. By Zariski's main theorem, the set consists of exactly one point. Therefore, the correspondence is defined at (p, P_1).

We have shown that the isomorphism

$$\mathscr{S}(U)_{\text{reg}} \to \mathscr{S}'(U)_{\text{reg}}$$

has a continuous extension to an open subset of $\mathscr{S}(U)$ which contains (4.4), and, moreover, that (4.4) is mapped isomorphically onto

$$\{(p', P_1'): p' \in U^{P_1'}\}.$$

A similar assertion holds for the associated map between F-valued points. Now suppose that $a \in A_{M,\text{reg}}(F)$ and $\pi \in U(F)N_1(F)$. Then $(a\pi, P_1)$ is an F-valued point in $\mathscr{S}(U)_{\text{reg}}$. Its image in $\mathscr{S}'(U)_{\text{reg}}$ is

$$((a\pi')^{k'}, (P_1')^{k'}),$$

where $\pi' \in (U(F)N_1'(F))^{K_M}$ and $k' \in K_M \backslash K$ are defined by (4.2). To recover π', project onto the second factor and recall that

$$(P_1' \backslash G)(F) = P_1'(F) \backslash G(F) \cong K_M \backslash K.$$

This gives k' and hence π', by conjugation of the first factor. Lemma 4.2 follows. □

We can now complete the proof of Lemma 4.1. In the expression (4.1), take π in $U^{P_1}(F)$, an open dense subset of $U(F)N_1(F)$. We have already seen that (4.1) equals $w_P(\lambda, a, \pi')$. Lemma 4.1 therefore follows from Lemma 3.1, Lemma 4.2, and the fact that a composition of continuous functions is continuous. □

Lemma 4.1 has a formulation in terms of the functions $W_\omega(a, \pi)$ defined in §3.

COROLLARY 4.3. *For each* $\omega \in Wt(\mathfrak{a}_M)$ *the function*

$$W_\omega(a, \pi), \qquad (a, \pi) \in A_{M,\mathrm{reg}} \times UN_1,$$

is the restriction to $A_{M,\mathrm{reg}} \times UN_1$ *of a polynomial from* $A_M \times U^+N_1$ *to* V_ω *which is defined over* F *and does not vanish at* $a = 1$.

Proof. We already know that $W_\omega(a, \pi)$ is the product of a polynomial with a negative power of the function (3.1). Choose $P \in \mathscr{P}(M)$ so that ω is P-dominant. Then

$$\|W_\omega(a, \pi)\| = w_P(\omega, a, \pi).$$

This has a nonzero limit at $a = 1$ for all π in an open dense subset of $U(F)N_1(F)$. But any such set is Zariski dense in U^+N_1. The corollary follows. □

§5. Statement of Theorem 5.2. We now return to the general setting in which F is either local or global. Let γ be an arbitrary element in $M(F_S)$, with Jordan decomposition σu. Then $\sigma = \prod_{v \in S}\sigma_v$, with each σ_v a semisimple element in $M(F_v)$, and $u = \prod_{v \in S}u_v$, with each u_v a unipotent element $M_{\sigma_v}(F_v)$. Our definition of weighted orbital integral applies only when G_γ equals M_γ, which of course is not generally the case. We must take an element $a = \prod_{v \in S}a_v$ in $A_{M,\mathrm{reg}}(F_S)$ which is close to 1. Then $a\gamma$ is an element in $M(F_S)$ with $G_{a\gamma} = M_{a\gamma}$, so that $J_M(a\gamma)$ is defined.

For each v, a_v belongs to $A_{M,\mathrm{reg}}(F_v)$, a set which is contained in $A_{M_{\sigma_v},\mathrm{reg}}(F_v)$. We therefore have the function $r_\beta(\lambda, u_v, a_v)$ introduced in (3.4) (but now with (G_{σ_v}, F_v) in place of (G, F)). For each $P \in \mathscr{P}(M)$, define

$$r_P(\lambda, \gamma, a) = \prod_{v \in S}\prod_\beta r_\beta(\tfrac{1}{2}\lambda, u_v, a_v), \qquad \lambda \in \mathfrak{a}^*_{M,\mathbf{C}}, \tag{5.1}$$

where β ranges over the reduced roots of $(P_{\sigma_v}, A_{M_{\sigma_v}})$. The restriction of any such β to $\mathfrak{a}_M \subset \mathfrak{a}_{M_{\sigma_v}}$ belongs to Σ^r_P, so

$$\{r_P(\lambda, \gamma, a) : P \in \mathscr{P}(M)\}$$

is a (G, M)-family of the special sort considered in §7 of [2]. It depends only on the $M^0(F_S)$-orbit of γ in $M(F_S)$.

Suppose that $L \in \mathscr{L}(M)$. There is certainly an (L, M)-family

$$\{r^L_R(\lambda, \gamma, a) : R \in \mathscr{P}^L(M)\}$$

obtained by letting (L, M) play the role of (G, M) in the definition above. On the other hand, for any $Q \in \mathscr{P}(L)$ we can define an (L, M)-family $\{r^Q_R(\lambda, \gamma, a)\}$ by

$$r^Q_R(\lambda, \gamma, a) = r_{Q(R)}(\lambda, \gamma, a), \qquad R \in \mathscr{P}^L(M),$$

where $Q(R)$ is the unique parabolic subset in $\mathscr{P}(M)$ which is contained in Q and whose intersection with L is R. These two (L, M)-families are not the same, but they are closely related.

LEMMA 5.1. *For any* $Q \in \mathscr{P}(L)$,

$$r_M^Q(\gamma, a) = r_M^L(\gamma, a).$$

Proof. The number on the left equals

$$\lim_{\lambda \to 0} \left(\sum_{R \in \mathscr{P}^L(M)} r_R^Q(\lambda, \gamma, a) \theta_R(\lambda)^{-1} \right)$$

by definition. Note that $r_R^Q(\lambda, \gamma, a)$ equals the product of $r_R^L(\lambda, \gamma, a)$ with

$$\prod_{v \in S} \prod_{\alpha} r_\alpha\left(\tfrac{1}{2}\lambda, u_v, a_v\right),$$

in which α ranges over the roots of $(Q(R)_{\sigma_v}, A_{M_{\sigma_v}})$ which do not vanish on $\mathfrak{a}_{L_{\sigma_v}}$. This second factor depends on Q but is independent of R. Its value at $\lambda = 0$ is 1. Therefore, $r_M^Q(\gamma, a)$ equals

$$\lim_{\lambda \to 0} \left(\sum_{R \in \mathscr{P}^L(M)} r_R^L(\lambda, \gamma, a) \theta_R(\lambda)^{-1} \right),$$

which is just $r_M^L(\gamma, a)$. □

We can now state Theorem 5.2. It provides the definition for $J_M(\gamma)$ and is one of the main results of the paper.

THEOREM 5.2. *For each* $f \in C_c^\infty(G(F_S))$, *the limit*

$$\lim_{a \to 1} \sum_{L \in \mathscr{L}(M)} r_M^L(\gamma, a) J_L(a\gamma, f)$$

exists.

The proof of the theorem will be given in the next two sections. It reduces to a question concerning integrals over finite regions of Euclidean space of logarithms of polynomials. A key ingredient is Corollary 4.3. Let us see how we will be led to this result.

First, we should recall a familiar construction from the theory of unipotent classes. (See [25].) Given a valuation v in S, let $\mathfrak{m}_v$ be the Lie algebra of $M_v = M_{\sigma_v}$. By the Jacobson–Morosov theorem there is a Lie algebra homomorphism

$$\Psi_v : s\ell(2) \to \mathfrak{m}_v,$$

defined over F_v, such that

$$u_v = \exp\left(\Psi_v\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right).$$

Set

$$H = \Psi_v\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$\mathfrak{m}_{v,i} = \{\xi \in \mathfrak{m}_v : \mathrm{ad}(H)\xi = i\xi\}, \qquad i \in \mathbb{Z}.$$

Then $\bigoplus_{i \geqslant 0} \mathfrak{m}_{v,i}$ is a parabolic subalgebra of $\mathfrak{m}_v$. Define

$$Z_v = \{p_v^{-1} u_v p_v\},$$

where p_v ranges over the normalizer in $M_v(F_v)$ of $\bigoplus_{i \geqslant 0} \mathfrak{m}_{v,i}(F_v)$. It is known that if

$$\mathfrak{u}_v = \bigoplus_{i \geqslant 2} \mathfrak{m}_{v,i},$$

then Z_v is an open subset of $\exp(\mathfrak{u}_v(F_v))$. (See, for example, Lemmas 1 and 4 of [20].) Suppose that K_{M_v} is a good maximal compact subgroup of $M_v(F_v)$. Then

$$(Z_v)^{K_{M_v}} = \left\{k_v^{-1}\zeta_v k_v : k_v \in K_{M_v}, \zeta_v \in Z_v\right\}$$

is the conjugacy class of u_v in $M_v(F_v)$. R. Rao [20] has given an explicit description of the $M_v(F_v)$-invariant integral on this conjugacy class. It is of the form

$$\int_{K_{M_v}}\int_{\log(Z_v)} |J_v(X_v)|_v^{1/2} \phi_v\left(k_v^{-1}\exp(X_v)k_v\right) dX_v\, dk_v, \quad \phi_v \in C_c(M_v(F_v)),$$

where dX_v is a Haar measure on $\mathfrak{u}_v(F_v)$ and J_v is a polynomial on $\mathfrak{u}_v(F_v)$ which is defined over F_v.

We shall write $Z_S = \prod_{v \in S} Z_v$ and $K_{M_\sigma} = \prod_{v \in S} K_{M_v}$. If

$$\zeta = \prod_{v \in S} \exp(X_v), \qquad X_v \in \mathfrak{u}_v(F_v),$$

set

$$d\zeta = \prod_{v \in S}\left(|J_v(X_v)|_v^{1/2}\, dX_v\right).$$

Then the invariant measure on the $M_\sigma(F_S)$-conjugacy class of u is given by

$$\int_{K_\sigma}\int_{Z_S}\phi(k^{-1}\zeta k)\,d\zeta\,dk, \qquad \phi\in C_c(M_\sigma(F_S)).$$

For each $v\in S$, let K_{σ_v} be a maximal compact subgroup of $G_{\sigma_v}(F_v)$ which is admissible relative to M_{σ_v}, in the sense of §1 of [6]. Set $K_\sigma=\prod_{v\in S}K_{\sigma_v}$. We can assume that the group K_{M_σ} above equals $K_\sigma\cap M_\sigma(F_S)$. For each $P\in\mathscr{P}(M)$ and $v\in S$, the group $P_v=P_{\sigma_v}$ belongs to $\mathscr{P}(M_v)$, and

$$G_{\sigma_v}(F_v)=P_v(F_v)K_{\sigma_v}.$$

If $x=\prod_{v\in S}x_v$ is any point in $G_\gamma(F_S)$, set

$$(5.2)\qquad v_P(\lambda,\sigma,x)=\prod_{v\in S}v_{P_v}(\lambda,x_v),\qquad P\in\mathscr{P}(M),\ \lambda\in\mathfrak{a}^*_{M,\mathbf{C}}.$$

If

$$K_\sigma=K\cap G_\sigma(F_S),$$

then

$$v_P(\lambda,\sigma,x)=v_P(\lambda,x).$$

In any case, (5.2) is a (G,M)-family and we have the function

$$v_L(\sigma,x)=\lim_{\lambda\to 0}\sum_{Q\in\mathscr{P}(L)}v_Q(\lambda,\sigma,x)\theta_Q(\lambda)^{-1},\qquad \lambda\in i\mathfrak{a}^*_L,$$

associated to any $L\in\mathscr{L}(M)$. Now, fix $R=\prod_{v\in S}R_v$, where for each v, R_v is a parabolic subgroup of G_{σ_v} with Levi component M_v. Let $n=\prod_{v\in S}n_v$ and $\zeta=\prod_{v\in S}\zeta_v$ be elements in $N_R(F_S)$ and Z_S, respectively. As in (3.2), we define an element $\pi=\prod_{v\in S}\pi_v$ in

$$\Pi_S=Z_SN_R(F_S)$$

by

$$a_v\pi_v=n_v^{-1}a_v\zeta_vn_v,\qquad v\in S,$$

so that

$$(5.3)\qquad a\pi=n^{-1}a\zeta n,\qquad v\in S.$$

In the proof of the theorem we shall be confronted with the expression

$$(5.4)\qquad \sum_{L\in\mathscr{L}(M)}r_M^L(\gamma,a)v_L(\sigma,n).$$

We must be able to rewrite it in terms of π.

By Lemma 5.1 and the formula given by Corollary 6.5 of [6], (5.4) equals the number

$$\lim_{\lambda\to 0}\Big(\sum_{P\in\mathscr{P}(M)} r_P(\lambda,\gamma,a)v_P(\lambda,\sigma,n)\theta_P(\lambda)^{-1}\Big) \tag{5.5}$$

obtained from a product of (G, M)-families. Now

$$r_P(\lambda,\gamma,a)v_P(\lambda,\sigma,n) = \prod_{v\in S}\Big(\prod_{\beta\in\Sigma^r_{P_v}} r_\beta\big(\tfrac{1}{2}\lambda,\zeta_v,a_v\big)\Big)v_{P_v}(\lambda,n_v),$$

since $r_\beta(\frac{1}{2}\lambda, u_v, a_v)$ depends only on the M_v conjugacy class of u_v. It follows from (3.4) and (3.5) that

$$\begin{aligned}&\prod_{\beta\in\Sigma^r_{P_v}} r_\beta\big(\tfrac{1}{2}\lambda,\zeta_v,a_v\big)\\ &\quad=\Big(\prod_{\beta\in\Sigma^r_{P_v}\cap\Sigma^r_{\bar R_v}} r_\beta(\lambda,\zeta_v,a_v)\Big)\Big(\prod_{\alpha\in\Sigma^r_{R_v}} r_\alpha\big(\tfrac{1}{2}\lambda,u_v,a_v\big)\Big).\end{aligned}$$

Then, by the definition (3.6),

$$r_P(\lambda,\gamma,a)v_P(\lambda,\sigma,n)$$

is the product of

$$w_P(\lambda,a,\sigma\pi) = \prod_{v\in S} w_{P_v}(\lambda,a_v,\pi_v) \tag{5.6}$$

with

$$\prod_{v\in S}\prod_{\alpha\in\Sigma^r_{R_v}} r_\alpha\big(\tfrac{1}{2}\lambda,u_v,a_v\big).$$

This last number is independent of P and equals 1 at $\lambda = 0$. It therefore gives no contribution to (5.5). We have established

LEMMA 5.3. *For n and π related as above,*

$$\prod_{L\in\mathscr{L}(M)} r^L_M(\gamma,a)v_L(\sigma,n)$$

equals the number

$$w_M(a,\sigma\pi) = \lim_{\lambda\to 0}\Big(\sum_{P\in\mathscr{P}(M)} w_P(\lambda,a,\sigma\pi)\theta_P(\lambda)^{-1}\Big). \qquad\square$$

Define the set $Wt(\mathfrak{a}_M)$ of extremal weights exactly as in §3. For any v there is a natural map $\omega \to \omega_v$ from $Wt(\mathfrak{a}_M)$ into $Wt(\mathfrak{a}_{M_{\sigma_v}})$. Given any $P \in \mathscr{P}(M)$, suppose that $\{\omega_1, \dots, \omega_n\}$ is a basis of $\mathfrak{a}_M^*$ of P-dominant elements in $Wt(\mathfrak{a}_M)$ and that

$$\lambda = \Sigma_{i=1}^n \lambda_i \omega_i, \qquad \lambda_i \in \mathbf{C}.$$

Then by (3.9),

$$w_P(\lambda, a, \sigma\pi) = \prod_{v \in S} \prod_{i=1}^{n} \left\| W_{\omega_{i,v}}(a_v, \pi_v) \right\|^{\lambda_i}. \tag{5.7}$$

LEMMA 5.4. *For any $a \in A_{M,\mathrm{reg}}(F_S)$ and $\pi \in \Pi_S$, we can write $w_M(a, \sigma\pi)$ as a finite sum*

$$\sum_{\Omega} c_\Omega \left(\prod_{(w,v) \in \Omega} \log \left\| W_{\omega_v}(a_v, \pi_v) \right\| \right),$$

where each Ω is a finite disjoint union of pairs

$$(\omega, v), \qquad \omega \in Wt(\mathfrak{a}_M),\ v \in S.$$

Proof. The lemma follows from (5.7) and a general property (formula (6.5) of [6]) applied to the (G, M)-family

$$\{ w_P(\lambda, a, \sigma\pi) \colon P \in \mathscr{P}(M) \}. \qquad \square$$

In the next section we can apply Corollary 4.3 to each function $W_{\omega_v}(a_v, \pi_v)$. The variable π_v lies in $Z_v N_{R_v}(F_v)$, which is an open subset of $\exp(\mathfrak{u}_v(F_v) + \mathfrak{n}_{R_v}(F_v))$. ($\mathfrak{n}_{R_v}$ is the Lie algebra of N_{R_v}.) Let U_v be the conjugacy class in $(\mathscr{U}_{M_v})$ which contains Z_v. Since $\exp(\mathfrak{u}_v + \mathfrak{n}_{R_v})$ is Zariski closed in $U_v^+ N_{R_v}$, the restriction to $\exp(\mathfrak{u}_v + \mathfrak{n}_{R_v})$ of a polynomial on U_v^+ remains a polynomial. On the other hand,

$$\left\{ k_v^{-1} \exp(X_v) k_v \colon k_v \in K_{\sigma_v} \cap M_v,\ X_v \in \mathfrak{u}_v + \mathfrak{n}_{R_v} \right\}$$

is Zariski dense in $U_v^+ N_{R_v}$. It follows from (3.7), (3.8), and Corollary 4.3 that the function

$$(a_v, X_v) \to W_{\omega_v}(a_v, \exp(X_v)),$$

can be extended to a polynomial in $a_v \in A_M$ and $X_v \in (\mathfrak{u}_v + \mathfrak{n}_{R_v})$ which is defined over F_v and which does not vanish identically at $a_v = 1$.

§6. Reduction of the proof. We shall reduce the proof of Theorem 5.2 to an elementary result (Lemma 6.1), whose verification we will postpone until §7. At the end of this section we shall give the general definition of $J_M(\gamma, f)$ and then state two corollaries of Theorem 5.2.

We are required to show that the expression

$$\sum_{L\in\mathscr{L}(M)} r_M^L(\gamma, a)J_L(a\gamma, f), \qquad a\in A_{M,\mathrm{reg}}(F_S), \tag{6.1}$$

has a limit at $a = 1$. Since $G_{a\gamma}(F_S) = M_\gamma(F_S)$, we can write (6.1) as

$$|D(a\gamma)|^{1/2}\int_{M_\gamma(F_S)\backslash G^0(F_S)} f(x^{-1}a\gamma x)\Big(\sum_{L\in\mathscr{L}(M)} r_M^L(\gamma, a)v_L(x)\Big)\,dx.$$

The Jordan decomposition of $a\gamma$ is $a\sigma\cdot u$. It is clear that

$$M_\gamma(F_S)\subset M_\sigma(F_S)\subset G_\sigma(F_S)\subset G^0(F_S).$$

We decompose the integral into a triple integral over (m, x, y) in

$$\big(M_\gamma(F_S)\backslash M_\sigma(F_S)\big)\times\big(M_\sigma(F_S)\backslash G_\sigma(F_S)\big)\times\big(G_\sigma(F_S)\backslash G^0(F_S)\big).$$

The expression becomes

$$|D(a\sigma)|^{1/2}\int\!\!\int\!\!\int f(y^{-1}\sigma x^{-1}am^{-1}umxy)\Big(\sum_{L\in\mathscr{L}(M)} r_M^L(\gamma, a)v_L(xy)\Big)\,dm\,dx\,dy.$$

The integral in m gives the invariant integral over the conjugacy class of u in $M_\sigma(F_S)$ which, as we remarked in §5, can be expressed as a double integral over (ζ, k) in $Z_S\times K_{M_\sigma}$. Since

$$v_L(xy) = v_L(kxy), \qquad k\in K_{M_\sigma},$$

the integral over k may be incorporated into the integral over x. Therefore (6.1) equals

$$|D(a\sigma)|^{1/2}\int\!\!\int\Big(\int_{Z_S} f(y^{-1}\sigma x^{-1}a\zeta xy)\Big(\sum_{L\in\mathscr{L}(M)} r_M^L(\gamma, a)v_L(xy)\Big)\,d\zeta\Big)\,dx\,dy.$$

For any $Q\in\mathscr{F}(M)$ and $x\in G_\sigma(F_S)$, let $K_{Q_\sigma}(x)$ be the point in K_σ such that $xK_{Q_\sigma}(x)^{-1}$ belongs to $Q_\sigma(F_S)$. It is uniquely determined modulo left translation by $K_\sigma\cap M_{Q_\sigma}(F_S)$. Clearly

$$v_P(\lambda, xy) = v_P(\lambda, \sigma, x)v_P\big(\lambda, K_{P_\sigma}(x)y\big), \qquad P\in\mathscr{P}(L),$$

a product of two (G, M)-families. By Lemma 6.3 of [6],

$$v_L(xy) = \sum_{Q \in \mathscr{F}(L)} v_L^Q(\sigma, x) v_Q'\big(K_{Q_\sigma}(x) y\big),$$

where v_Q' is defined by the formula [6, (6.3)]. Consequently, (6.1) equals the integral over y in $G_\sigma(F_S) \backslash G^0(F_S)$ and the sum over $Q \in \mathscr{F}(M)$ of the product of $|D(a\sigma)|^{1/2}$ with

$$(6.2) \qquad \int_{M_\sigma(F_S) \backslash G_\sigma(F_S)} \int_{Z_S} f(y^{-1}\sigma x^{-1} a \zeta x y)$$
$$\times \Bigg(\sum_{L \in \mathscr{L}^{M_Q}(M)} r_M^L(\gamma, a) v_L^Q(\sigma, x) v_Q'\big(K_{Q_\sigma}(x) y\big)\Bigg)\, d\zeta\, dx.$$

Suppose for the moment that $Q \in \mathscr{F}(M)$ is fixed. If $v \in S$, the Levi component of $Q_v = Q_{\sigma_v}$ contains M_v. Let R_v be a parabolic subgroup of the former group with Levi component the latter group, and set $R = \prod_{v \in S} R_v$. Since

$$M_\sigma(F_S) \backslash G_\sigma(F_S) = N_R(F_S) \times N_{Q_\sigma}(F_S) \times K_\sigma,$$

we can decompose the double integral over x and ζ in (6.2) into a multiple integral over (ζ, n, n_Q, k) in

$$Z_S \times N_R(F_S) \times N_{Q_\sigma}(F_S) \times K_\sigma.$$

The integrand becomes

$$f\big(y^{-1}\sigma k^{-1} n_Q^{-1} n^{-1} a \zeta n n_Q k y\big) v_Q'(ky) \Bigg(\sum_{L \in \mathscr{L}^{M_Q}(M)} r_M^L(\gamma, a) v_L^Q(\sigma, n)\Bigg).$$

The sum in the brackets immediately suggests Lemma 5.3. Indeed, if we define

$$\Pi_S^Q = Z_S N_R(F_S)$$

and set

$$n_Q^{-1} n^{-1} a \zeta n n_Q = a \pi \nu_Q, \qquad \pi \in \Pi_S^Q,\ \nu_Q \in N_{Q_\sigma}(F_S),$$

then ζ, n, and π are related by the equation (5.3) (or, rather, by its analogue with G replaced by M_Q). Lemma 5.3, applied to M_Q instead of G, tells us that the sum in the brackets $w_M^Q(a, \sigma\pi)$. Therefore, we shall rewrite the triple integral over (ζ, n, n_Q) as a double integral over (π, ν_Q) in $\Pi_S^Q \times N_{Q_\sigma}(F_S)$. (The measure $d\pi$ on Π_S^Q is of course the product of our measure $d\zeta$ on Z_S with the Haar measure on $N_R(F_S)$.) This final change of variable introduces a Jacobian

$$\big|D^{G_\sigma}(a)\big|^{-1/2} \delta_R(a)^{1/2} \delta_{Q_\sigma}(a)^{1/2}.$$

The upshot of all the discussion is that (6.1) equals the product of

$$(6.3) \qquad \left|D^G(a\sigma)D^{G_\sigma}(a)^{-1}\right|^{1/2}$$

with the integral over $y \in G_\sigma(F_S)\backslash G^0(F_S)$ and the sum over $Q \in \mathscr{F}(M)$ of

$$(6.4) \qquad \delta_R(a)^{1/2}\int_{\Pi_S^Q}\Phi_{Q,y}^\sigma(a\pi)w_M^Q(a,\sigma\pi)\,d\pi,$$

where

$$\Phi_{Q,y}^\sigma(m) = \delta_{Q_\sigma}(m)^{1/2}\int_{K_\sigma}\int_{N_{Q_\sigma}(F_S)} f(y^{-1}\sigma k^{-1}mnky)v_Q'(ky)\,dn\,dk$$

for any $m \in M_{Q_\sigma}(F_S)$. It is clear that $\Phi_{Q,y}^\sigma$ is a function in $C_c^\infty(M_{Q_\sigma}(F_S))$ which depends smoothly on y. We must show that the entire expression has a limit at $a = 1$.

The limit at $a = 1$ of (6.3) exists and equals $|D^G(\sigma)|^{1/2}$. To deal with what remains we apply Lemma 2.1. The points

$$k^{-1}a\pi nk, \qquad k \in K_\sigma,\ \pi \in \Pi_S^Q,\ n \in N_{Q_\sigma}(F_S),$$

will all belong to a given invariant neighborhood Δ_σ of 1 in $G_\sigma(F_S)$ as long as the element $a \in A_{M,\mathrm{reg}}(F_S)$ is sufficiently close to 1. This we can always assume. We restrict y to a fixed set of representatives of $G_\sigma(F_S)$ in $G^0(F_S)$. Then Lemma 2.1 tells us that (6.4) vanishes unless y belongs to a fixed compact set. We are left with showing that (6.4) approaches a limit uniformly for y in compact subsets.

The function $w_M^Q(a,\sigma\pi)$ which occurs in (6.4) can be written as a finite sum

$$\sum_\Omega c_\Omega\Bigg(\prod_{(\omega,v)\in\Omega}\log\left\|W_{\omega_v}^Q(a_v,\pi_v)\right\|\Bigg),$$

where each Ω is a finite disjoint union of pairs $(\omega, v) \in Wt(\mathfrak{a}_M) \times S$ and π equals $\prod_{v\in S}\pi_v$. (This assertion is Lemma 5.4, with G replaced by M_Q. The notation $W_{\omega_v}^Q(a_v,\pi_v)$ refers to the analogue for M_Q of $W_{\omega_v}(a_v,\pi_v)$.) The integral in (6.4) is over

$$\Pi_S^Q = Z_SN_R(F_S) = \prod_{v\in S}\big(Z_vN_{R_v}(F_v)\big),$$

an open subset of

$$\prod_{v\in S}\exp\big(\mathfrak{u}_v(F_v) + \mathfrak{n}_{R_v}(F_v)\big),$$

where $\mathfrak{n}_{R_v}$ is the Lie algebra of N_{R_v}. For each $v \in S$, let $\{X_{v,1}, \ldots, X_{v,d_v}\}$ be a basis of the F_v-vector space $\mathfrak{u}_v(F_v) + \mathfrak{n}_{R_v}(F_v)$. Then for $\omega \in Wt(\mathfrak{a}_M)$,

$$(x_{v,1}, \ldots, x_{v,d_v}) \to W_{\omega_v}^Q\big(a_v, \exp(x_{v,1}X_{v,1} + \cdots + x_{v,d_v}X_{v,d_v})\big)$$

is a polynomial in d_v variables with coefficients in $V_{\omega_v}(F_v)$.

Fix a finite set Ω as above. For any $(\omega, v) \in \Omega$, let $\mathscr{P}(F_v^{d_v}, V_{\omega_v})$ be the set of polynomials in d_v variables with coefficients in $V_{\omega_v}(F_v)$. We give $\mathscr{P}(F_v^{d_v}, V_{\omega_v})$ the direct-limit topology inherited from the finite-dimensional subspaces of polynomials of bounded degrees. Let $\mathscr{P}^+(F_v^{d_v}, V_{\omega_v})$ denote the set of nonzero elements in $\mathscr{P}(F_v^{d_v}, V_{\omega_v})$, and define

$$\mathscr{P}^+(\Omega) = \bigoplus_{(\omega, v) \in \Omega} \mathscr{P}^+\big(F_v^{d_v}, V_{\omega_v}\big).$$

We shall set $F_S^d = \prod_{v \in S}(F_v^{d_v})$, where $d = \{d_v : v \in S\}$. Let $C_c(F_S^d)$ be the space of continuous (complex-valued) functions on F_S^d of compact support. It is the topological direct limit of subspaces (equipped with the usual supremum norm) consisting of functions supported on a given compact set. We state the next lemma in terms of the dual space $C_c(F_S^d)^*$, which we equip with the weak-* topology.

LEMMA 6.1. *Let $\mathcal{O}$ be an open subset of F_S^d. Then if*

$$p = \bigoplus_{(\omega, v) \in \Omega} p_{\omega, v}, \qquad p_{\omega, v} \in \mathscr{P}^+\big(F_v^{d_v}, V_{\omega_v}\big),$$

is an element in $\mathscr{P}^+(\Omega)$ and

$$\lambda_p(x) = \left| \prod_{(\omega, v) \in \Omega} \big(\log\| p_{\omega, v}(x_v) \|\big) \right|,$$

with $x = \prod_{v \in S} x_v$ in $\mathcal{O}$, then each integral

$$\lambda_p(\phi) = \int_{\mathcal{O}} \phi(x) \lambda_p(x)\, dx, \qquad \phi \in C_c\big(F_S^d\big),$$

is absolutely convergent. Moreover, $p \to \lambda_p$ is a continuous function from $\mathscr{P}^+(\Omega)$ to $C_c(F_S^d)^*$.

We shall prove this lemma in the next section. Assuming it for now, let us finish the proof of Theorem 5.2.

Let $\mathcal{N}$ be a small neighborhood of 1 in $A_M(F_S)$. If

$$x = \prod_{v \in S} x_v = \prod_{v \in S} \prod_{i=1}^{d_v} x_{v,i},$$

set

$$e(x) = \prod_{v \in S} e(x_v) = \prod_{v \in S} \exp\left(x_{v,1}X_{v,1} + \cdots + x_{v,d_v}X_{v,d_v}\right).$$

The expression (6.4) equals

$$\int_{\mathcal{O}} \delta_R(a)^{1/2} j(x) \Phi^{\sigma}_{Q,y}(ae(x)) w^Q_M(a, \sigma e(x))\, dx,$$

where $\mathcal{O}$ is an open subset of F_S^d and

$$j(x) = \prod_{v \in S} \left|J_v\left(x_{v,1}X_{v,1} + \cdots + x_{v,d_v}X_{v,d_v}\right)\right|_v^{1/2},$$

a continuous function on F_S^d. Now for Ω as in the lemma,

$$\bigoplus_{(\omega, v) \in \Omega} W^Q_{\omega_v}(a_v, e(x_v)), \qquad a = \prod_{v \in S} a_v \in \mathcal{N},$$

is a continuous function from $\mathcal{N}$ to $\mathcal{P}^+(\Omega)$. This follows from Corollary 4.3. (See the remarks following Lemma 5.4.) Lemma 6.1, combined with the formula above for $w^Q_M(a, \sigma\pi)$, then tells us that the function which maps $a \in \mathcal{N}$ to the linear form

$$\int_{\mathcal{O}} \phi(x) w^Q_M(a, \sigma e(x))\, dx, \qquad \phi \in C_c(F_S^d),$$

in $C_c(F_S^d)^*$ is continuous. On the other hand, the map

$$(a, y) \to \delta_R(a)^{1/2} j(\cdot) \Phi^{\sigma}_{Q,y}(ae(\cdot))$$

is a continuous function from $\mathcal{N} \times G^0(F_S)$ to $C_c(F_S^d)$. Therefore, since the natural pairing on $C_c(F_S^d) \times C_c(F_S^d)^*$ is continuous, the expression (6.4) is a continuous function on $\mathcal{N} \times G^0(F_S)$. Having agreed that the integral in y can be taken over a compact subset of $G^0(F_S)$, we see that the original expression (6.1) can be extended to a continuous function of $a \in \mathcal{N}$. This is just the assertion of the theorem. □

Having proved Theorem 5.2 (modulo Lemma 6.1), we can define

$$J_M(\gamma, f) = \lim_{a \to 1} \sum_{L \in \mathcal{L}(M)} r^L_M(\gamma, a) J_L(a\gamma, f). \tag{6.5}$$

In the special case that $G_\gamma = M_\gamma$, we have

$$r^L_M(\gamma, a) = \begin{cases} 1, & \text{if } L = M, \\ 0, & \text{if } L \neq M, \end{cases}$$

and our definition coincides with the original one given in §2. It is clear that $J_M(\gamma)$ depends only on the $M^0(F_S)$-orbit of γ. In our notation, we shall sometimes identify γ with its $M^0(F_S)$-orbit. More generally, if Γ is a finite union of $M^0(F_S)$-orbits $\{\gamma_i\}$, we shall write

$$J_M(\Gamma, f) = \sum_i J_M(\gamma_i, f).$$

It is convenient to introduce the notion of an induced space of orbits. Given the element $\gamma \in M(F_S)$, define γ^G to be the union of those $G^0(F_S)$-orbits $\{\gamma_i\}$ in $G(F_S)$ which for any $P \in \mathcal{P}(M)$ intersect $\gamma N_P(F_S)$ in an open set. This is a simple generalization of the definition in [19]. There are only finitely many such $G^0(F_S)$-orbits, and they all belong to a single geometric orbit. If γ is G-regular, γ^G consists of one orbit, that of γ itself. If γ is unipotent, each γ_i is contained in the induced geometric conjugacy class of γ. The induced space γ^G can also be characterized analytically by a formula

$$J_G(\gamma^G, f) = \lim_{a \to 1} J_G(a\gamma, f), \qquad f \in C_c^\infty(G(F_S)), \tag{6.6}$$

where a ranges over elements in $A_{M,\mathrm{reg}}(F_S)$ which are close to 1. To establish this formula, note that the right-hand side equals

$$\lim_{a\to 1} |D^G(a\gamma)|^{1/2} \int_{M^0(F_S)\backslash G^0(F_S)} \int_{M_\gamma(F_S)\backslash M^0(F_S)} f(x^{-1}m^{-1}a\gamma m x)\, dm\, dx.$$

If $P \in \mathcal{P}(M)$, this in turn equals

$$\lim_{a\to 1} |D^M(\gamma)|^{1/2} \delta_P(a\gamma)^{1/2} \int_K \int_{N_P(F_S)} \int_{M_\gamma(F_S)\backslash M^0(F_S)} f(k^{-1}am^{-1}\gamma m n k)\, dm\, dn\, dk$$

$$= |D^M(\gamma)|^{1/2} \delta_P(\gamma)^{1/2} \int_K \int_{N_P(F_S)} \int_{M_\gamma(F_S)\backslash M^0(F_S)} f(k^{-1}m^{-1}\gamma m n k)\, dm\, dn\, dk,$$

by a standard change-of-variables formula. This last expression is obviously the integral of f over the invariant measure on γ^G. It therefore equals the left-hand side of (6.6).

COROLLARY 6.2. *The distribution $J_M(\gamma, f)$ is given by the integral of f relative to a measure on γ^G which is absolutely continuous with respect to the invariant measure class.*

Proof. The changes of variables introduced in the proof of the theorem allow us to write the invariant integral over γ^G as

$$|D(\sigma)|^{1/2} \int_{G_\sigma(F_S)\backslash G^0(F_S)} \int_{K_\sigma} \int_{N_Q(F_S)} \int_{\Pi_S^Q} f(y^{-1}\sigma k^{-1}\pi n k y)\, d\pi\, dn\, dk\, dy$$

for any $Q \in \mathscr{F}(M)$. On the other hand, $J_M(\gamma, f)$ is the value of (6.1) at $a = 1$, which by the discussion above equals

$$|D(\sigma)|^{1/2} \int \sum_Q \int \int \int f(y^{-1}\sigma k^{-1}\pi n k y) v_M^Q(1, \pi) v_Q'(ky)\, d\pi\, dn\, dk\, dy.$$

This is a multiple integral whose absolute convergence is assured by Lemmas 2.1 and 6.1. It is therefore absolutely continuous with respect to the invariant measure. □

COROLLARY 6.3. *Suppose that $L_1 \in \mathscr{L}(M)$. Then the limit at $a = 1$ of the expression*

$$\sum_{L \in \mathscr{L}(L_1)} r_{L_1}^L(\gamma, a) J_L(a\gamma, f), \qquad a \in A_{M,\mathrm{reg}}(F_S),$$

exists and equals $J_{L_1}(\gamma^{L_1}, f)$.

Proof. The proof of the existence of the limit is similar to that of Theorem 5.2. We shall forgo the details. To evaluate the limit, set

$$a = ba_1, \qquad b \in A_{M,\mathrm{reg}}(F_S),\ a_1 \in A_{L_1,\mathrm{reg}}(F_S),$$

and let b approach 1. Since

$$\lim_{b \to 1} r_{L_1}^L(\gamma, ba_1) = r_{L_1}^L(\gamma, a_1), \qquad L \in \mathscr{L}(L_1),$$

we obtain

$$\lim_{a_1 \to 1} \sum_{L \in \mathscr{L}(L_1)} r_{L_1}^L(\gamma, a_1) \lim_{b \to 1} J_L(ba_1\gamma, f).$$

It is clear that $L_{a_1\gamma}$ equals $G_{a_1\gamma}$. Applying Lemma 2.2, we see that $J_L(ba_1\gamma, f)$ equals the invariant orbital integral of a function on $L(F_S)$ for b near 1. It then follows easily from (6.6) that

$$\lim_{b \to 1} J_L(ba_1\gamma, f) = J_L\big((a_1\gamma)^L, f\big) = J_L\big(a_1\gamma^{L_1}, f\big).$$

Therefore, the given limit equals

$$\lim_{a_1 \to 1} \sum_{L \in \mathscr{L}(L_1)} r_{L_1}^L(\gamma, a_1) J_L\big(a_1\gamma^{L_1}, f\big) = J_{L_1}(\gamma^{L_1}, f),$$

as required. □

§7. Proof of Lemma 6.1. The purpose of this section is to establish Lemma 6.1 and thereby complete the proof of Theorem 5.2. One way to prove the lemma would be to use the resolution of singularities. By blowing up affine space, one could replace each set

$$\{x_v\colon p_{\omega,v}(x_v)=0\}$$

by a divisor with normal crossings.[1] However, we shall instead treat the lemma as an exercise in elementary analysis.

The elements in $C_c(F_S^d)^*$ are Radon measures. The operation which restricts such a measure to an open subset of F_S^d defines a continuous projection on $C_c(F_S^d)^*$. We conclude that it suffices to prove the lemma with $\mathcal{O}=F_S^d$. Next, we note that functions of the form

$$\phi(x)=\prod_{v\in S}\phi_v(x_v),\qquad \phi_v\in C_c\big(F_v^{d_v}\big),\ x_v\in F_v^{d_v},$$

are dense in $C_c(F_S^d)$. Since the Haar measure on F_S^d is the product of Haar measures on the spaces $F_v^{d_v}$, it will be enough to prove the lemma in the case that S contains one valuation v. Therefore, we shall assume in this section that F is a local field and $F_S=F_v=F$. We take $d=d_v$ to be any positive integer. Finally, we see that by setting

$$V=\bigoplus_{(\omega,v)\in\Omega}V_{\omega_v},$$

we may assume that all our polynomial take values in the same space.

Let

$$\alpha=(\alpha_1,\ldots,\alpha_d)$$

be a multi-index of nonnegative integers. Let $\mathscr{P}_\alpha(F^d,V)$ be the set of polynomials in $\mathscr{P}(F^d,V)$ whose coefficients vanish for every multi-index α' with $\alpha_i'>\alpha_i$ for some i. It is convenient to write $\|p\|$ for the supremum of the norms of the coefficients of any polynomial p in $\mathscr{P}_\alpha(F^d,V)$. If $\delta>0$, let $\mathscr{P}_\alpha^\delta(F^d,V)$ denote the set of $p\in\mathscr{P}_\alpha(F^d,V)$ with $\|p\|>\delta$. It is clear that the space $\mathscr{P}^+(F^d,V)$ is the union over α and δ of the subsets $\mathscr{P}_\alpha^\delta(F^d,V)$. Therefore, it is good enough to fix α and δ and also a positive integer n, and to prove the analogue of Lemma 6.1 for n-tuples

$$\bigoplus_{i=1}^n p_i,\qquad p_i\in\mathscr{P}_\alpha^\delta(F^d,V).$$

We must first prove a supplementary lemma.

[1] I thank the referee for this observation.

LEMMA 7.1. *Let Γ be a compact subset of F^d. Then there are positive constants C and t such that for any $\varepsilon > 0$, $\delta > 0$, and $p \in \mathscr{P}_\alpha^\delta(F^d, V)$, the volume of the set*

$$\Gamma(p, \varepsilon) = \{x \in \Gamma: \|p(x)\| < \varepsilon\}$$

is bounded by $C(\delta^{-1}\varepsilon)^t$.

Proof. Suppose that we can prove the lemma with $V = F$. Then the result will hold for arbitrary V, as we see immediately by fixing a basis of V. The constants C and t will depend at most on the dimension of V. Notice also that if we can prove the lemma for $\delta = 1$, it will follow for any δ. Therefore, we shall assume that $V = F$ and $\delta = 1$.

We shall first prove the lemma for $d = 1$. For the moment, then, α is just a positive integer. Assume inductively that the lemma is true (for any V and δ) with α replaced by any smaller positive integer. Suppose that $0 < \varepsilon \leqslant 1$ and that $p \in \mathscr{P}_\alpha^1(F, F)$. We can write

$$p(x) = (x - r)p'(x),$$

where r belongs to an extension field E of F, with $\deg(E/F) \leqslant \alpha$, and p' belongs to $\mathscr{P}_{\alpha-1}(F, E)$. The norm function on E is of course the extension of the valuation on F. By our induction assumption, the lemma applies to p' with fixed constants C' and t'. Since

$$\|p_1 p_2\| \leqslant (\deg p_1 + 1)\|p_1\|\,\|p_2\|,$$

for any polynomials p_1 and p_2 in $\mathscr{P}(F, E)$, we have

$$\|p'\| > \min\left\{\tfrac{1}{2}, \frac{1}{2|r|}\right\}.$$

To estimate the volume of $\Gamma(p, \varepsilon)$, we examine the equation

$$|p(x)| = |x - r|\,|p'(x)|$$

is two separate cases. First suppose that $|r| \geqslant |\Gamma| + 1$, (where $|\Gamma| = \sup_{x \in \Gamma}|x|$). Then the set $\Gamma(p, \varepsilon)$ is contained in $\Gamma(p', \varepsilon/(|r| - |\Gamma|))$. Since $\|p'\| > 1/2|r|$, the induction assumption gives

$$\operatorname{vol}\Gamma(p, \varepsilon) \leqslant C'\left(2|r|\left(\frac{\varepsilon}{|r| - |\Gamma|}\right)\right)^{t'} \leqslant C'(2(|\Gamma| + 1))^{t'}\varepsilon^{t'}.$$

The other case is that $|r| < |\Gamma| + 1$. Then if x belongs to $\Gamma(p, \varepsilon)$, either x belongs to $\Gamma(p', \varepsilon^{1/2})$ or $|x - r| < \varepsilon^{1/2}$. Since $\|p'\| > \frac{1}{2}(|\Gamma| + 1)^{-1}$ in this case, the induction assumption gives

$$\operatorname{vol} \Gamma(p, \varepsilon) \leqslant 2\varepsilon^{1/2} + C'\big(2(|\Gamma| + 1)\varepsilon^{1/2}\big)^{t'}.$$

It follows that for any ε and p,

$$\operatorname{vol} \Gamma(p, \varepsilon) \leqslant C\varepsilon^{t},$$

with $C = 2 + C'(2|\Gamma| + 2)^{t'}$ and $t = \min\{\frac{1}{2}, t'/2\}$. The lemma is therefore true if $d = 1$.

Suppose now that d is arbitrary. Assume inductively that the lemma is true if d is replaced by $d - 1$. Choose a compact subset Γ_1 of F such that Γ is contained in the Cartesian product Γ_1^d. Suppose that $p \in \mathscr{P}_\alpha^1(F^d, F)$ and $0 < \varepsilon \leqslant 1$. If $x = (x_1, \dots, x_d)$ and $\alpha = (\alpha_1, \dots, \alpha_d)$, set $\hat{x} = (x_2, \dots, x_d)$ and $\hat{\alpha} = (\alpha_2, \dots, \alpha_d)$. Then

$$p(x) = \sum_{j=1}^{\alpha_1} p_j(\hat{x})(x_1)^j,$$

with $p_j \in \mathscr{P}_{\hat{\alpha}}(F^{d-1}, F)$. Since $\|p\| > 1$, there is a j with $\|p_j\| > 1$. Suppose that x belongs to $\Gamma(p, \varepsilon)$. We consider separately the cases that $|p_j(\hat{x})| \geqslant \varepsilon^{1/2}$ and $|p_j(\hat{x})| < \varepsilon^{1/2}$. In the first instance we apply what we have already proved (the case that $d = 1$) to p, regarded as a polynomial in x_1. In the second case we apply the induction assumption. We obtain

$$\begin{aligned}\operatorname{vol} \Gamma(p, \varepsilon) &\leqslant C_1(\varepsilon^{-1/2}\varepsilon)^{t_1} \cdot \operatorname{vol}\big(\Gamma_1^{d-1}\big) + \operatorname{vol}(\Gamma_1) \cdot C_{d-1}(\varepsilon^{1/2})^{t_{d-1}} \\ &\leqslant C\varepsilon^{t}\end{aligned}$$

for positive constants C and t. This establishes Lemma 7.1. □

We can now prove Lemma 6.1. Fix a continuous function ϕ on F^d which is supported on the compact set Γ. Fix also a multi-index α and a positive number δ. Suppose that

$$p = \bigoplus_{i=1}^{n} p_i$$

is an n-tuple, with each $p_i \in \mathscr{P}_\alpha^\delta(F^d, F)$. We shall estimate the integral

$$\int_{\Gamma(p, \varepsilon)} \lambda_p(x)\, dx,$$

where

$$\lambda_p(x) = \left| \prod_{i=1}^{n} \big(\log|p_i(x)|\big) \right|,$$

and

$$\Gamma(p, \varepsilon) = \bigcup_{i=1}^{n} \Gamma(p_i, \varepsilon).$$

For each point x in $\Gamma(p, \varepsilon)$ there is an i such that $|p_i(x)| \leqslant |p_j(x)|$ for all $j \neq i$. Then x belongs to $\Gamma(p_i, \varepsilon)$, and if ε is sufficiently small, as we may assume,

$$\lambda_p(x) \leqslant |\log|p_i(x)||^n.$$

It follows that

$$\int_{\Gamma(p, \varepsilon)} \lambda_p(x)\, dx \leqslant \sum_{i=1}^{n} \int_{\Gamma(p_i, \varepsilon)} |\log|p_i(x)||^n\, dx.$$

If

$$\Gamma(i, k, \varepsilon) = \Gamma\left(p_i, 2^{-k}\varepsilon\right) - \Gamma\left(p_i, 2^{-(k+1)}\varepsilon\right),$$

we see that

$$\begin{aligned}\int_{\Gamma(p_i, \varepsilon)} |\log|p_i(x)||^n\, dx &\leqslant \sum_{k=0}^{\infty} \int_{\Gamma(i, k, \varepsilon)} |\log|p_i(x)||^n\, dx \\ &\leqslant \sum_{k=0}^{\infty} \mathrm{vol}(\Gamma(i, k, \varepsilon)) \cdot \left|\log(2^{-(k+1)}\varepsilon)\right|^n \\ &\leqslant \sum_{k=0}^{\infty} C(\delta^{-1}2^{-k}\varepsilon)^t((k+1)\log 2 + |\log \varepsilon|)^n\end{aligned}$$

for positive constants C and t given by Lemma 7.1. It follows that there are positive constants C_1 and t_1 such that

$$\int_{\Gamma(p, \varepsilon)} \lambda_p(x)\, dx \leqslant C_1(\delta^{-1}\varepsilon)^{t_1} \tag{7.1}$$

for any p and ε. In particular, since $\lambda_p(x)$ is bounded on the complement of $\Gamma(p, \varepsilon)$ in Γ, the integral

$$\int_{F^d} \phi(x)\lambda_p(x)\, dx$$

is absolutely convergent. This is the first assertion of Lemma 6.1.

Fix

$$p^0 = \left(p_1^0, \ldots, p_n^0\right), \qquad p_i^0 \in \mathscr{P}_\alpha^\delta(F^d, F).$$

If p is any n-tuple as above, and $\varepsilon > 0$,

$$\left| \int_{F^d} \phi(x)\lambda_p(x)\,dx - \int_{F^d} \phi(x)\lambda_{p^0}(x)\,dx \right|$$

is bounded by the sum of

$$\|\phi\|_\infty \int_{\Gamma(p^0,\varepsilon)} \lambda_p(x)\,dx, \tag{7.2}$$

$$\|\phi\|_\infty \int_{\Gamma(p^0,\varepsilon)} \lambda_{p^0}(x)\,dx, \tag{7.3}$$

and

$$\|\phi\|_\infty \int_{\Gamma-\Gamma(p^0,\varepsilon)} |\lambda_p(x) - \lambda_{p^0}(x)|\,dx. \tag{7.4}$$

Now p is close to p^0 precisely when all the coefficients of the polynomials

$$p_i - p_i^0, \qquad 1 \leqslant i \leqslant n,$$

are small. In particular, the set $\Gamma(p^0,\varepsilon)$ will be contained in $\Gamma(p, 2\varepsilon)$ for all p sufficiently close to p^0. Therefore, both (7.2) and (7.3) may be estimated by the inequality (7.1). The integrand in (7.4) is bounded on Γ uniformly for all p sufficiently close to p^0. Therefore, by dominated convergence, (7.4) approaches 0 as p approaches p^0. It follows that

$$\lim_{p\to p^0} \int_{F^d} \phi(x)\lambda_p(x)\,dx = \int_{F^d} \phi(x)\lambda_{p^0}(x)\,dx.$$

This is the second and last assertion of Lemma 6.1. □

§8. A formula of descent. We return to our study of weighted orbital integrals. As before, γ will be a fixed element in $M(F_S)$. We are going to establish a descent property for $J_M(\gamma)$ which will be useful later. However, we shall first check that the behaviour of $J_M(\gamma)$ under conjugation is the same as that described in §2.

LEMMA 8.1. *If $f \in C_c^\infty(G(F_S))$ and $y \in G^0(F_S)$,*

$$J_M(\gamma, f^y) = \sum_{Q\in\mathscr{F}(M)} J_M^{M_Q}(\gamma, f_{Q,y}).$$

Proof. By definition

$$J_M(\gamma, f^y) = \lim_{a\to 1} \sum_{L\in\mathscr{L}(M)} r_M^L(\gamma, a) J_L(a\gamma, f^y).$$

Of course, a is assumed to be a small element in $A_{M,\mathrm{reg}}(F_S)$. Therefore, $G_{a\gamma}(F_S) = L_{a\gamma}(F_S)$ for each $L \in \mathscr{L}(M)$, so the formula (2.2) applies to $J_L(a\gamma, f^y)$. We obtain

$$J_M(\gamma, f) = \lim_{a \to 1} \sum_{L \in \mathscr{L}(M)} r_M^L(\gamma, a) \sum_{Q \in \mathscr{F}(L)} J_L^{M_Q}(a\gamma, f_{Q,y})$$

$$= \sum_{Q \in \mathscr{F}(M)} \lim_{a \to 1} \sum_{L \in \mathscr{L}^{M_Q}(M)} r_M^L(\gamma, a) J_L^{M_Q}(a\gamma, f_{Q,y})$$

$$= \sum_{Q \in \mathscr{F}(M)} J_M^{M_Q}(\gamma, f_{Q,y}),$$

as required. □

We also note in passing that the formula (2.4) holds for our arbitrary element γ. This follows from the formula

$$r_{y^{-1}My}^{y^{-1}Ly}(y^{-1}\gamma y, y^{-1}ay) = r_M^L(\gamma, a),$$

which is a consequence of the definitions.

Suppose that σ is a *semisimple* element in $M(F_S)$ and that

$$\gamma = \sigma\mu, \qquad \mu \in M_\sigma(F_S).$$

We assume that the following three conditions are satisfied:

(i) σ belongs to $G(F)$.

(ii) The space $\mathfrak{a}_{M_\sigma}$ equals $\mathfrak{a}_M$.

(iii) $G_\gamma(F_S)$ is contained in $G_\sigma(F_S)$.

According to the first condition, G_σ is a (connected) reductive group defined over F and M_σ is a Levi subgroup of G_σ. The space $\mathfrak{a}_{M_\sigma}$, defined as in §1 but with G replaced by M_σ, always contains $\mathfrak{a}_M$; the second condition asserts the equality of the two. This is equivalent to the assertion that σ is an F-elliptic element in $M(F)$. The third condition is equivalent to the equality of $G_\gamma(F_S)$ with $(G_\sigma)_\mu(F_S)$. In other words, $G_\gamma(F_S)$ may be regarded as an object associated to the group G_σ. Let us write $\mathscr{F}^\sigma(M_\sigma)$ and $\mathscr{P}^\sigma(M_\sigma)$ for the sets of parabolic subgroups of G_σ whose Levi components contain and respectively equal M_σ.

We are going to express $J_M(\gamma, f)$ in terms of weighted orbital integrals on the groups

$$\{M_R(F_S) \colon R \in \mathscr{F}^\sigma(M_\sigma)\}.$$

This will of course entail dealing with (G_σ, M_σ)-families. We shall always assume that the Euclidean norm on the space $\mathfrak{a}_{M_\sigma}$ is the same as the one on $\mathfrak{a}_M$. In particular, the Euclidean measure on $\mathfrak{a}_{M_\sigma}$ will be that of $\mathfrak{a}_M$.

We can write $J_M(\gamma, f)$ as the limit at $a = 1$ of

$$|D(a\gamma)|^{1/2} \int_{G_\sigma(F_S)\backslash G^0(F_S)} \int_{M_\gamma(F_S)\backslash G_\sigma(F_S)} f(y^{-1}\sigma x^{-1} a\mu xy) \times \Big(\sum_{L\in\mathcal{L}(M)} r_M^L(\gamma, a) v_L(xy) \Big)\, dx\, dy, \tag{8.1}$$

since $G_{a\gamma}(F_S)$ equals $M_\gamma(F_S)$. It is necessary to transform $r_M^L(\gamma, a)$ and $v_L(xy)$ into functions associated with G_γ.

Lemma 8.2. *Suppose that $L \in \mathcal{L}(M)$. Then*

$$r_M^L(\gamma, a) = \begin{cases} r_{M_\sigma}^{L_\sigma}(\mu, a), & \text{if } \mathfrak{a}_L = \mathfrak{a}_{L_\sigma}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. In order to deal with the (G, M)-family

$$\{ r_P(\lambda, \gamma, a) \colon P \in \mathcal{P}(M) \},$$

we must write down the Jordan decomposition of γ. Let $\mu = \eta \cdot u$ be the Jordan decomposition of μ. Then $\gamma = \sigma\eta \cdot u$ is the Jordan decomposition of γ. Suppose that P belongs to $\mathcal{P}(M)$. Then P_σ is a group in $\mathcal{P}^\sigma(M_\sigma)$, and condition (iii) above implies that $P_{\sigma\eta}(F_S) = (P_\sigma)_\eta(F_S)$. It then follows from the definition (5.1) that

$$r_P(\lambda, \gamma, a) = r_{P_\sigma}(\lambda, \mu, a).$$

As a consequence of this, we may write

$$\begin{aligned} r_M(\gamma, a) &= \lim_{\lambda\to 0} \sum_{P\in\mathcal{P}(M)} r_{P_\sigma}(\lambda, \mu, a)\theta_P(\lambda)^{-1} \\ &= \lim_{\lambda\to 0} \sum_{R\in\mathcal{P}^\sigma(M_\sigma)} r_R(\lambda, \mu, a) \Big(\sum_{\{P\in\mathcal{P}(M)\colon P_\sigma = R\}} \theta_P(\lambda)^{-1} \Big). \end{aligned}$$

Suppose first of all that $\mathfrak{a}_G = \mathfrak{a}_{G_\sigma}$. Then $\mathfrak{a}_M^G = \mathfrak{a}_{M_\sigma}^{G_\sigma}$. If R belongs to $\mathcal{P}^\sigma(M_\sigma)$, the set $\Delta_R^\vee = \{\alpha^\vee \colon \alpha \in \Delta_R\}$ is a basis of $\mathfrak{a}_M^G$. For each root $\alpha \in \Delta_R$, there is a unique reduced root α_1 of (G, A_M) such that $\alpha_1^\vee$ is a positive multiple of $\alpha^\vee$. The set $F^\vee = \{\alpha_1^\vee \colon \alpha \in \Delta_R\}$ is also a basis of $\mathfrak{a}_M^G$. Applying Lemma 7.2 of [2], we obtain

$$\begin{aligned} \sum_{\{P\in\mathcal{P}(M)\colon P_\sigma = R\}} \theta_P(\lambda)^{-1} &= \operatorname{vol}\big(\mathfrak{a}_M^G/\mathbb{Z}(F^\vee)\big) \prod_{\alpha\in\Delta_R} \lambda(\alpha_1^\vee)^{-1} \\ &= \operatorname{vol}\big(\mathfrak{a}_{M_\sigma}^{G_\sigma}/\mathbb{Z}(\Delta_R^\vee)\big) \prod_{\alpha\in\Delta_R} \lambda(\alpha^\vee)^{-1} \\ &= \theta_R(\lambda)^{-1}. \end{aligned}$$

Therefore, in this case we have

$$r_M(\gamma, a) = \lim_{\lambda \to 0} \Big(\sum_{R \in \mathscr{P}^\sigma(M_\sigma)} r_R(\lambda, \mu, a)\theta_R(\lambda)^{-1} \Big)$$

$$= r_{M_\sigma}(\mu, a).$$

Next, suppose that $\mathfrak{a}_G$ is a proper subspace of $\mathfrak{a}_{G_\sigma}$. Let ζ be a vector in the orthogonal complement of $\mathfrak{a}_G$ in $\mathfrak{a}_{G_\sigma}$. Take

$$\lambda = t(\lambda_0 + \zeta), \qquad t \in \mathbb{R}, \lambda_0 \in \mathfrak{a}_{M_\sigma}^{G_\sigma},$$

and calculate the limit above by letting t approach 0. Since

$$r_R(t(\lambda_0 + \zeta), \mu, a) = r_R(t\lambda_0, \mu, a), \qquad R \in \mathscr{P}^\sigma(M_\sigma),$$

we see that $r_M(\gamma, a)$ equals

$$\frac{1}{d!} \sum_{R \in \mathscr{P}^\sigma(M_\sigma)} \lim_{t \to 0} \Big(\Big(\frac{d}{dt} \Big)^d r_R(t\lambda_0, \mu, a) \Big) \Big(\sum_{\{P \in \mathscr{P}(M):\, P_\sigma = R\}} \theta_P(\lambda_0 + \zeta)^{-1} \Big),$$

where $d = \dim \mathfrak{a}_M^G$. This expression is independent of ζ (and also λ_0). By taking ζ to be very large, we see that $r_M(\gamma, a)$ equals 0.

We have so far proved the lemma if $L = G$. However, $r_M^L(\gamma, a)$ is defined by replacing the underlying group G with L. The lemma is therefore true in general. □

Take $L \in \mathscr{L}(M)$. In view of the last lemma, we need only consider the case that $\mathfrak{a}_L = \mathfrak{a}_{L_\sigma}$. Suppose that $x \in G_\sigma(F_S)$ and $y \in G^0(F_S)$. We saw in §6 that

$$v_L(xy) = \sum_{Q \in \mathscr{F}(L)} v_L^Q(\sigma, x) v_Q'(K_{Q_\sigma}(x)y).$$

Lemma 8.3. *Suppose that as above, x belongs to $G_\sigma(F_S)$ and that $\mathfrak{a}_L = \mathfrak{a}_{L_\sigma}$. Then*

$$v_L^G(\sigma, x) = \begin{cases} v_{L_\sigma}^{G_\sigma}(x), & \text{if } \mathfrak{a}_G = \mathfrak{a}_{G_\sigma}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We have

$$v_L^G(\sigma, x) = \lim_{\lambda \to 0} \sum_{P \in \mathscr{P}(L)} v_P(\lambda, \sigma, x)\theta_P(\lambda)^{-1}$$

$$= \lim_{\lambda \to 0} \sum_{R \in \mathscr{P}^\sigma(L_\sigma)} v_R(\lambda, x) \Big(\sum_{\{P \in \mathscr{P}(L):\, P_\sigma = R\}} \theta_P(\lambda)^{-1} \Big).$$

Arguing exactly as in the proof of the last lemma, we conclude that $v_L^G(\sigma, x)$ equals $v_{L_\sigma}^{G_\sigma}(x)$ if $\mathfrak{a}_G = \mathfrak{a}_{G_\sigma}$, and is 0 otherwise. □

More generally, suppose that $Q \in \mathscr{F}(L)$, and that

$$x = nmk, \qquad n \in N_{Q_\sigma}(F_S),\ m \in M_{Q_\sigma}(F_S),\ k \in K_\sigma.$$

It follows from the lemma, with (G, x) replaced by (M_Q, m), that $v_L^Q(\sigma, x)$ equals $v_{L_\sigma}^{Q_\sigma}(x)$ if $\mathfrak{a}_Q = \mathfrak{a}_{Q_\sigma}$, and is 0 otherwise. Therefore, the formula above for $v_L(xy)$ gives us

COROLLARY 8.4. *If R is any group in $\mathscr{F}^\sigma(M_\sigma)$, set*

$$v_R'(z) = \sum_{\{Q \in \mathscr{F}(M):\ Q_\sigma = R,\ \mathfrak{a}_Q = \mathfrak{a}_R\}} v_Q'(z), \qquad z \in G^0(F_S). \tag{8.2}$$

Then

$$v_L(xy) = \sum_{R \in \mathscr{F}^\sigma(M_\sigma)} v_{L_\sigma}^R(x) v_R'(K_R(x)y).$$ □

Observe that $L \to L_\sigma$ is a bijection from the set of elements $L \in \mathscr{L}(M)$, with $\mathfrak{a}_L = \mathfrak{a}_{L_\sigma}$, onto the set of Levi subgroups of G_σ which contain M_σ. Combining Lemma 8.2 with Corollary 8.4, we see that

$$\sum_{L \in \mathscr{L}(M)} r_M^L(\gamma, a) v_L(xy)$$

equals

$$\sum_{R \in \mathscr{F}^\sigma(M_\sigma)} \sum_{S \in \mathscr{L}^{M_R}(M_\sigma)} r_{M_\sigma}^S(\mu, a) v_S^R(x) v_R'(K_R(x)y).$$

The descent formula can now be proved with some changes of variable similar to those of §6. Substitute the formula just proved into the original expression (8.1). In the resulting expression, take the sum over R outside the integral over x, and then replace the integral over x with a triple integral over (m_R, n_R, k) in

$$M_\gamma(F_S) \backslash M_{R_\sigma}(F_S) \times N_R(F_S) \times K_\sigma.$$

Finally, write

$$n_R^{-1} m_R^{-1} a\mu m_R n_R = m_R^{-1} a\mu m_R n, \qquad n \in N_R(F_S),$$

and change the integral over n_R to one over n. Putting in the appropriate

Jacobian, we find that (8.1) equals

$$\left|D^G(a\gamma)^{G_\sigma}(a\gamma)^{-1}\right|^{1/2}$$
$$\times \int_{G_\sigma(F_S)\backslash G^0(F_S)} \sum_{R\in\mathscr{F}^\sigma(M_\sigma)} \sum_{S\in\mathscr{L}^{M_R}(M_\sigma)} r^S_{M_\sigma}(\mu, a) J_S^{M_R}(a\mu, \Phi_{R,y})\, dy,$$

where

$$(8.3)\qquad \Phi_{R,y}(m) = \delta_R(m)^{1/2}\int_{K_\sigma}\int_{N_R(F_S)} f(y^{-1}\sigma k^{-1}mnky)v'_R(ky)\, dn\, dk$$

for $m \in M_R(F_S)$. It is clear that $\Phi_{R,y}$ is a function in $C_c^\infty(M_R(F_S))$ which depends smoothly on y. (For the reader who might remember a similar function defined in §6, the relation is

$$\Phi_{R,y} = \sum_{\{Q\in\mathscr{F}(M):\, Q_\sigma = R,\, \mathfrak{a}_Q=\mathfrak{a}_R\}} \Phi^\sigma_{Q,y}.)$$

It is now easy to read off the limit at $a = 1$ of (8.1). The result is

THEOREM 8.5. *Suppose that M and $\gamma = \sigma\mu$ are given as above and satisfy the conditions* (i), (ii), *and* (iii). *Then $J_M(\gamma, f)$ equals*

$$\left|D^G(\sigma\mu)D^{G_\sigma}(\mu)^{-1}\right|^{1/2}\int_{G_\sigma(F_S)\backslash G^0(F_S)}\left(\sum_{R\in\mathscr{F}^\sigma(M_\sigma)} J_{M_\sigma}^{M_R}(\mu, \Phi_{R,y})\right) dy. \qquad \square$$

We will need a slight extension of the result for the paper [3]. Suppose that T is an arbitrary point in $\mathfrak{a}_M$. For each $z \in G^0(F_S)$, the functions

$$v_P(\lambda, z, T) = v_P(\lambda, z)e^{\lambda(T)}, \qquad P \in \mathscr{P}(M),$$

form a (G, M)-family. Since $e^{\lambda(T)}$ is independent of P and equals 1 at $\lambda = 0$,

$$v_L(z, T) = v_L(z), \qquad L \in \mathscr{L}(M).$$

Suppose that $x \in G_\sigma(F_S)$ and $y \in G^0(F_S)$. Then, as in §6, we have a decomposition

$$v_P(\lambda, xy, T) = v_P(\lambda, \sigma, x)v_P(\lambda, K_{P_\sigma}(x)y, T), \qquad P \in \mathscr{P}(L),$$

as a product of (G, L)-families. Applying Lemma 6.3 of [5], we obtain

$$v_L(xy) = \sum_{Q\in\mathscr{F}(L)} v_L^Q(\sigma, x)v'_Q(K_{Q_\sigma}(x)y).$$

We can substitute this formula into the expression (8.1). The proof of the theorem then leads to

COROLLARY 8.6. *Under the assumptions of Theorem* 8.5, $J_M(\gamma, f)$ *equals*

$$|D^G(\sigma\mu)D^{G_\sigma}(\mu)^{-1}|^{1/2}\int_{G_\sigma(F_S)\backslash G^0(F_S)}\Big(\sum_{R\in\mathscr{F}^\sigma(M_\sigma)} J_{M_\sigma}^{M_R}(\mu, \Phi_{R,y,T})\Big)\,dy,$$

where

$$\Phi_{R,y,T}(m) = \delta_R(m)^{1/2}\int_{K_\sigma}\int_{N_R(F_S)} f(y^{-1}\sigma k^{-1}mnky)\,v_R'(ky, T)\,dn\,dk$$

for $m \in M_R(F_S)$ *and for*

$$v_R'(z, T) = \sum_{\{Q\in\mathscr{F}(M):\ Q_\sigma = R,\ \mathfrak{a}_Q = \mathfrak{a}_R\}} v_Q'(z,T), \qquad z \in G^0(F_S). \qquad \square$$

An important special case, obviously, is when μ is unipotent. Then $\gamma = \sigma\mu$ is the Jordan decomposition of γ, and $D^G(\sigma\mu)D^{G_\sigma}(\mu)^{-1}$ equals $D^G(\sigma)$. We obtain

COROLLARY 8.7. *Suppose that* $\mu = u$ *is unipotent. Then* $J_M(\gamma, f)$ *equals*

$$|D^G(\sigma)|^{1/2}\int_{G_\sigma(F_S)\backslash G^0(F_S)}\Big(\sum_{R\in\mathscr{F}^\sigma(M_\sigma)} J_{M_\sigma}^{M_R}(u, \Phi_{R,y,T})\Big)\,dy. \qquad \square$$

Part 2: p-adic groups

§9. A germ expansion. We now consider properties of weighted orbital integrals which depend on a given local field. We shall treat the p-adic case first. The theory of Shalika germs [22] is an important component of the study of invariant orbital integrals. It turns out there there is a parallel theory for weighted orbital integrals. Its existence depends on our having defined the weighted orbital integrals at singular points in $M(F_S)$.

For the next two sections, we shall assume that S consists of one non-Archimedean valuation v, and that $F = F_v = F_S$. We shall also fix a semisimple element σ in $M(F)$. The germs at σ associated to $J_M(\gamma, f)$ are (M, σ)-equivalence classes of functions of γ and are defined on the G-regular elements in $\sigma M_\sigma(F)$ which are close to σ. They are designed to measure the obstruction to $J_M(\gamma, f)$ being an (invariant) orbital integral of a function on $M(F)$.

Given $L \in \mathscr{L}(M)$, let

$$\big(\sigma\mathscr{U}_{L_\sigma}(F)\big)$$

denote the finite set of orbits in $\sigma\mathscr{U}_{L_\sigma}(F)$ under conjugation by the group

$$L(F,\sigma)=\mathrm{Cent}\big(\sigma, L^0(F)\big).$$

PROPOSITION 9.1. *There are uniquely determined (M,σ)-equivalence classes of functions*

$$\gamma\to g_M^G(\gamma,\delta),\qquad \gamma\in\sigma M_\sigma(F)\cap G_{\mathrm{reg}},$$

parametrized by the classes $\delta\in(\sigma\mathscr{U}_{G_\sigma}(F))$ such that for any $f\in C_c^\infty(G(F))$,

$$J_M(\gamma,f)\overset{(M,\sigma)}{\sim}\sum_{L\in\mathscr{L}(M)}\ \sum_{\delta\in(\sigma\mathscr{U}_{L_\sigma}(F))} g_M^L(\gamma,\delta)J_L(\delta,f).$$

Proof. The defining equation is understood to hold for all G and, in particular, if G is replaced by any set $L\in\mathscr{L}(M)$. The uniqueness is then equivalent to the linear independence of the distributions

$$f\to J_G(\delta,f),\qquad \delta\in\big(\sigma\mathscr{U}_{G_\sigma}(F)\big).$$

This is a well-known fact which follows, for example, from the partial order on the orbit set $(\sigma\mathscr{U}_{G_\sigma}(F))$. We can therefore concentrate on establishing the existence of the germs.

Suppose first that $M=G$. Then $J_M(\gamma,f)$ is just the invariant orbital integral, and

$$J_M(\gamma,f)\overset{(M,\sigma)}{\sim}0$$

by definition. We can therefore define

$$g_M^M(\gamma,\delta)\overset{(M,\sigma)}{\sim}0,\qquad \delta\in\big(\sigma\mathscr{U}_{M_\sigma}(F)\big).$$

Now take M to be arbitrary. We must define the germs $g_M^L(\gamma,\delta)$ so that for any f, the function

$$K_M(\gamma,f)=J_M(\gamma,f)-\sum_{L\in\mathscr{L}(M)}\ \sum_{\delta\in(\sigma\mathscr{U}_{L_\sigma}(F))} g_M^L(\gamma,\delta)J_L(\delta,f)$$

is (M,σ)-equivalent to 0. For each $L\in\mathscr{L}(M)$ with $L\neq G$, we assume inductively that the germs $g_M^L(\gamma, \delta)$ have been defined, and that

$$K_M^L(\gamma, h) \overset{(M,\sigma)}{\sim} 0, \qquad h \in C_c^\infty(L(F)),$$

For every $f \in C_c^\infty(G(F))$, the function

$$K'_M(\gamma, f) = J_M(\gamma, f) - \sum_{\{L \in \mathscr{L}(M):\ L \neq G\}} \sum_{\delta \in (\sigma \mathscr{U}_{L_\sigma}(F))} g_M^L(\gamma, \delta) J_L(\delta, f)$$

is then defined. Let

$$f_\delta, \qquad \delta \in \left(\sigma \mathscr{U}_{G_\sigma}(F)\right),$$

be functions in $C_c^\infty(G(F))$ such that

$$J_G(\delta_1, f_\delta) = \begin{cases} 1, & \delta_1 = \delta, \\ 0, & \delta_1 \neq \delta, \end{cases}$$

for $\delta, \delta_1 \in (\sigma \mathscr{U}_{G_\sigma}(F))$. We then define the remaining germs by

$$g_M^G(\gamma, \delta) = K'_M(\gamma, f_\delta), \qquad \delta \in \left(\sigma \mathscr{U}_{G_\sigma}(F)\right).$$

Observe that for any δ,

$$\begin{aligned} K_M(\gamma, f_\delta) &= K'_M(\gamma, f_\delta) - \sum_{\delta_1 \in (\sigma \mathscr{U}_{G_\sigma}(F))} g_M^G(\gamma, \delta_1) J_G(\delta_1, f_\delta) \\ &= K'_M(\gamma, f_\delta) - g_M^G(\gamma, \delta) \\ &= 0. \end{aligned}$$

It remains for us to show for any f that $K_M(\gamma, f)$ is (M, σ)-equivalent to 0. Define

$$f' = f - \sum_{\delta \in (\sigma \mathscr{U}_{G_\sigma}(F))} J_G(\delta, f) f_\delta.$$

Then

$$\begin{aligned} K_M(\gamma, f) &= K_M(\gamma, f') + \sum_\delta J_G(\delta, f) K_M(\gamma, f_\delta) \\ &= K_M(\gamma, f'). \end{aligned}$$

On the other hand,

$$J_G(\delta, f') = 0$$

for every $\delta \in (\sigma\mathscr{U}_{G_\sigma}(F))$. We can therefore make use of a standard argument to represent f' as a finite sum

$$\Sigma(g^y - g), \qquad g \in C_c^\infty(G(F)),\ y \in G^0(F),$$

on some $G^0(F)$-invariant neighborhood of σ in $G(F)$. (See the discussion accompanying Proposition 4 of [17].) Therefore, if $\gamma \in \sigma M_\sigma(F)$ is close to σ,

$$K_M(\gamma, f) = K_M(\gamma, f') = \sum K_M(\gamma, g^y - g).$$

By Lemma 8.1 we can write

$$K_M(\gamma, g^y - g)$$

as the difference between

$$\sum_{\{Q \in \mathscr{F}(M):\ Q \neq G\}} J_M^{M_Q}(\gamma, g_{Q,y})$$

and

$$\sum_{L \in \mathscr{L}(M)} \sum_{\{Q \in \mathscr{F}(L):\ Q \neq G\}} \sum_{\delta \in (\sigma\mathscr{U}_{L_\sigma}(F))} g_M^L(\gamma, \delta) J_L^{M_Q}(\delta, g_{Q,y}).$$

It follows that

$$K_M(\gamma, g^y - g) = \sum_{\{Q \in \mathscr{F}(M):\ Q \neq G\}} K_M^{M_Q}(\gamma, g_{Q,y}).$$

By our induction hypothesis, this function is (M, σ)-equivalent to 0. Thus,

$$K_M(\gamma, f) \overset{(M,\sigma)}{\sim} 0,$$

and the proposition follows. □

Remark. If $G_\sigma = M_\sigma$, it follows inductively from Lemma 2.2 that the germs all vanish.

The most important case is when $\sigma = 1$. Then $G = G^0$, and the germs $g_M^G(\gamma, \delta)$ are parametrized by unipotent conjugacy classes in $G(F)$. It turns out that the germs for general σ have descent properties which reduce their study to the unipotent case. We shall prove one lemma in this direction.

Observe that each class $\delta \in (\sigma\mathscr{U}_{L_\sigma}(F))$ is a finite union of classes σu, with u a unipotent conjugacy class in $\mathscr{U}_{L_\sigma}(F)$. The set of all such u, which we shall denote simply by

$$\{u:\ \sigma u \sim \delta\},$$

has a transitive action under the finite group

$$\iota^L(\sigma) = L_\sigma(F)\backslash L^0(F,\sigma),$$

where $L^0(F, \sigma)$ denotes the centralizer of σ in $L^0(F)$.

LEMMA 9.2. *Suppose that σ is a semisimple element in $M(F)$ such that $\mathfrak{a}_{M_\sigma} = \mathfrak{a}_M$, and that $L \in \mathscr{L}(M)$. Then if*

$$\gamma = \sigma\mu, \qquad \mu \in M_\sigma(F),$$

is G-regular, we have

$$g_M^L(\gamma, \delta) = \varepsilon^L(\sigma) \sum_{\{u:\ \sigma u \sim \delta\}} g_{M_\sigma}^{L_\sigma}(\mu, u), \qquad \delta \in \left(\sigma\mathscr{U}_{L_\sigma}(F)\right),$$

where $\varepsilon^L(\sigma)$ equals 1 if $\mathfrak{a}_L = \mathfrak{a}_{L_\sigma}$ and is 0 otherwise.

Proof. We can of course assume that γ is close to σ. Then $\gamma = \sigma\mu$ satisfies the conditions of §8. By Theorem 8.5, $J_M(\gamma, f)$ equals the product of

$$\left|D^G(\sigma\mu)D^{G_\sigma}(\mu)^{-1}\right|^{1/2}$$

and

$$\int_{G_\sigma(F)\backslash G^0(F)} \left(\sum_{R \in \mathscr{F}^\sigma(M_\sigma)} J_{M_\sigma}^{M_R}(\mu, \Phi_{R,y}) \right) dy.$$

Since μ is close to 1,

$$\left|D^G(\sigma\mu)D^{G_\sigma}(\mu)^{-1}\right|^{1/2} = \left|\det(1 - \mathrm{Ad}(\sigma\mu))_{\mathfrak{g}/\mathfrak{g}_\sigma}\right|^{1/2} = \left|D^G(\sigma)\right|^{1/2}.$$

Moreover, for any $\Phi \in C_c^\infty(M_\sigma(F))$,

$$J_{M_\sigma}^{M_R}(\mu, \Phi) \overset{(M_\sigma, 1)}{\sim} \sum_{L_\sigma \in \mathscr{L}^{M_R}(M_\sigma)} \ \sum_{u \in (\mathscr{U}_{L_\sigma}(F))} g_{M_\sigma}^{L_\sigma}(\mu, u) J_{L_\sigma}^{M_R}(u, \Phi),$$

by Proposition 9.1. Substitute this into the expression above and interchange the sums over R and L_σ. To any $L_\sigma \in \mathscr{L}(M_\sigma)$ there corresponds a unique element $L \in \mathscr{L}(M)$. Conversely, an element $L \in \mathscr{L}(M)$ arises this way precisely when $\varepsilon^L(\sigma) = 1$. It follows that as a function of μ, $J_M(\sigma\mu, f)$ is $(M_\sigma, 1)$-equivalent to the sum over $L \in \mathscr{L}(M)$ and $u \in (\mathscr{U}_{L_\sigma}(F))$ of the product of

$$\varepsilon^L(\sigma) g_{M_\sigma}^{L_\sigma}(\mu, u)$$

with

$$|D^G(\sigma)|^{1/2} \int_{G_\sigma(F)\backslash G^0(F)} \sum_{R \in \mathscr{F}^\sigma(L_\sigma)} J_{L_\sigma}^{M_R}(u, \Phi_{R,y})\, dy.$$

Applying Corollary 8.7, we obtain

$$J_M(\sigma\mu, f) \overset{(M_\sigma,1)}{\sim} \sum_{L \in \mathscr{L}(M)} \sum_{u \in (\mathscr{U}_{L_\sigma}(F))} \varepsilon^L(\sigma) g_{M_\sigma}^{L_\sigma}(\mu, u) J_L(\sigma u, f).$$

We want to show that $g_M^L(\gamma, \delta)$ is (M, σ)-equivalent to the germ

$$\tilde{g}_M^L(\gamma, \delta) = \varepsilon^L(\sigma) \sum_{\{u:\ \sigma u \sim \delta\}} g_{M_\sigma}^{L_\sigma}(\mu, u).$$

It follows easily from Lemma 2.1 that (M, σ)-equivalence of functions of γ is the same as $(M_\sigma, 1)$-equivalence of the corresponding functions of μ. Therefore, what we have already established may be written in the form

$$J_M(\gamma, f) \overset{(M,\sigma)}{\sim} \sum_{L \in \mathscr{L}(M)} \sum_{\delta \in (\sigma\mathscr{U}_{L_\sigma}(F))} \tilde{g}_M^L(\gamma, \delta) J_L(\delta, f).$$

Assume inductively that

$$g_M^L(\gamma, \delta) = \tilde{g}_M^L(\gamma, \delta)$$

for any $L \neq G$. Then

$$\sum_{\delta \in (\sigma\mathscr{U}_{G_\sigma}(F))} \left(g_M^G(\gamma, \delta) - \tilde{g}_M^G(\gamma, \delta)\right) J_G(\delta, f) = 0.$$

It follows that

$$g_M^G(\gamma, \delta) = \tilde{g}_M^G(\gamma, \delta), \qquad \delta \in \left(\sigma\mathscr{U}_{G_\sigma}(F)\right),$$

as required. □

§10. Homogeneity of germs. We would expect the germs introduced in the last section to behave like Shalika germs. In this section we shall establish a formula which is the analogue of the homogeneity property [11, Theorem 14(1)] and [21, Theorem 1.2(4)] for Shalika germs.

We continue to assume that $F = F_v = F_S$ is a non-Archimedean local field. We shall consider only the special case of §9, that $\sigma = 1$. In particular, we assume in this section that $G = G^0$. Given a unipotent element $u \in \mathscr{U}_M(F)$, define

$$d^M(u) = \tfrac{1}{2}(\dim M_u - \operatorname{rank} M).$$

This number is a nonnegative half integer, which depends only on the geometric conjugacy class of u. Recall that the induced space u^G is a finite union of unipotent $G(F)$-conjugacy classes. For any $w \in G(F)$, set

$$[u^G\colon w] = \begin{cases} 1, & \text{if } w \in u^G, \\ 0, & \text{otherwise.} \end{cases}$$

If $[u^G\colon w] = 1$, it is clear that

$$d^G(w) = d^M(u).$$

The exponential map provides a homeomorphism from an Ad($G(F)$)-invariant neighborhood of 0 in $\mathfrak{g}(F)$ into an ad($G(F)$)-invariant neighborhood of 1 in $G(F)$. If $t \in F$, and

$$x = \exp X, \qquad X \in \mathfrak{g}(F),$$

set

$$x^t = \exp tX.$$

We shall regard this map as a germ, in the sense that is maps a sufficiently small invariant neighborhood of 1 in $G(F)$ to another such neighborhood. It is clear that

$$gx^tg^{-1} = \left(gxg^{-1}\right)^t, \qquad g \in G(F).$$

In particular,

$$u \to u^t, \qquad u \in \mathscr{U}_G(F),$$

induces a permutation on the set of unipotent conjugacy classes in $G(F)$. If t belongs to F^*, it follows easily from the Jacobson–Morosov theorem that u^t and u are conjugate over the algebraic closure of F. It t belongs to $(F^*)^2$, u^t and u are actually $G(F)$-conjugate. (See the discussion in §1 of [21].) There is a compatible way to choose invariant measures on the $G(F)$-orbits of u and u^t. For applying the Jacobson–Morosov theorem as in §5, we see that all the $G(F)$-orbits within the geometric conjugacy class of u can be associated to a single vector space $\mathfrak{u}_v(F)$. The measure

$$d\zeta = |J_v(X)|^{1/2}\, dX, \qquad X \in \mathfrak{u}_v(F),$$

introduced in §5 then provides invariant measures on all the associated $G(F)$-

orbits. However, the map

$$x \to x^t, \qquad x \in \mathcal{O}_u(G(F)),$$

does not preserve the compatible measures. If S is any bounded measurable subset of $\mathcal{O}_u(G(F))$, and

$$S^t = \{x^t \colon x \in S\},$$

a mild generalization of Lemma 7 of [21] establishes that

$$\text{(10.1)} \qquad \operatorname{vol}(S^t) = |t|^{(1/2)\dim(G/G_u)} \operatorname{vol}(S).$$

To state the homogeneity property, we must introduce another (G, M)-family. Take u to be an element in $\mathcal{U}_M(F)$. For each reduced root β of (G, A_M), we defined the number $\rho(\beta, u)$ in §5. It depends only on the geometric conjugacy class of u. Given $P \in \mathcal{P}(M)$, define

$$c_P(\nu, u, t) = \prod_\beta |t|^{-(1/2)\rho(\beta, u)\nu(\beta^\vee)}, \qquad \nu \in i\mathfrak{a}_M^*,$$

where the product is taken over the reduced roots β of (P, A_M). Then

$$c_P(\nu, u, t), \qquad P \in \mathcal{P}(M),\ \nu \in i\mathfrak{a}_M^*,$$

is a (G, M)-family.

LEMMA 10.1. *Suppose that $a \in A_{M,\mathrm{reg}}(F)$ is close to* 1. *Then*

$$r_M^L(u, a) = \sum_{M_1 \in \mathcal{L}^L(M)} c_M^{M_1}(u, t) r_{M_1}^L(u, a^t)$$

for any $t \in F^*$ and $L \in \mathcal{L}(M)$.

Proof. For each $P \in \mathcal{P}(M)$, we have

$$r_P(\nu, u, a) = \prod_\beta |a^\beta - a^{-\beta}|^{(1/2)\rho(\beta, u)\nu(\beta^\vee)}, \qquad \nu \in i\mathfrak{a}_M^*,$$

where the product is again taken over the reduced roots β of (P, A_M). Suppose that

$$a = \exp H,$$

where H is a point in the Lie algebra of $A_M(F)$ which is close to 0. Then

$$\begin{aligned}|a^{\beta} - a^{-\beta}| &= \left|\left(1 + \beta(H) + \frac{\beta(H)^2}{2!} + \cdots\right) - \left(1 - \beta(H) + \frac{\beta(H)^2}{2!} - \cdots\right)\right| \\ &= |2\beta(H)|,\end{aligned}$$

since H is small and $|\cdot|$ is a non-Archimedean valuation. Similarly,

$$|(a^t)^{\beta} - (a^t)^{-\beta}| = |2t\beta(H)|,$$

so that

$$|a^{\beta} - a^{-\beta}| = |t^{-1}|\,|(a^t)^{\beta} - (a^t)^{-\beta}|.$$

We obtain

$$r_P(\nu, u, a) = c_P(\nu, u, t) r_P(\nu, u, a^t),$$

a product of (G, M)-families. The lemma then follows from Corollary 6.5 of [6]. □

PROPOSITION 10.2. *Suppose that $t \in F^*$ and $w \in (\mathcal{U}_G(F))$. Then*

$$g_M^G(\gamma^t, w^t) \overset{(M,1)}{\sim} |t|^{d^G(w)} \sum_{L \in \mathcal{L}(M)} \sum_{u \in (\mathcal{U}_L(F))} g_M^L(\gamma, u) c_L(u, t)[u^G : w].$$

Proof. Suppose that $f \in C_c^{\infty}(G(F))$ is given. Define

$$f^t(x^t) = f(x).$$

Then f^t is the restriction of a function in $C_c^{\infty}(G(F))$ to an invariant neighborhood of 1 in $G(F)$. Suppose that γ is a point in $M(F)$ which is G-regular and close to 1. We have

$$\begin{aligned}J_M(\gamma, f) &= |D(\gamma)|^{1/2} \int_{G_{\gamma}(F)\backslash G(F)} f(x^{-1}\gamma x) v_M(x)\, dx \\ &= |D(\gamma)|^{1/2} \int_{G_{\gamma}(F)\backslash G(F)} f^t(x^{-1}\gamma^t x) v_M(x)\, dx.\end{aligned}$$

The centralizer $T = G_\gamma$ is a maximal torus in G and is equal to G_{γ^t}. We choose invariant measures on the orbits of γ and γ^t induced by a fixed invariant measure on $T(F)\backslash G(F)$. Consequently,

$$J_M(\gamma, f) = \left(|D(\gamma)|\,|D(\gamma^t)|^{-1}\right)^{1/2} J_M(\gamma^t, f^t).$$

Let $\{X_i\}$ be a basis of $\mathfrak{g}/\mathfrak{g}_\gamma$ such that

$$\mathrm{Ad}(\gamma)X_i = \xi_i(\gamma)X_i, \qquad \gamma \in T(F),$$

where ξ_i is a character of T. Since γ is close to 1,

$$|1 - \xi_i(\gamma^t)| = |t|\,|1 - \xi_i(\gamma)|.$$

It follows that

$$\begin{aligned} |D(\gamma^t)|^{1/2} &= \prod_i |1 - \xi_i(\gamma^t)|^{1/2} \\ &= \prod_i \left(|t|^{1/2}|1 - \xi_i(\gamma)|^{1/2}\right) \\ &= |t|^{(1/2)(\dim G - \mathrm{rank}\, G)}|D(\gamma)|^{1/2} \end{aligned}$$

if γ is close to 1. Therefore,

$$J_M(\gamma^t, f^t) = |t|^{(1/2)(\dim G - \mathrm{rank}\, G)} J_M(\gamma, f).$$

We take the germ expansion of each side of this equation. The left-hand side is $(M, 1)$-equivalent to

$$\sum_{L_1 \in \mathscr{L}(M)} \sum_{w \in (\mathscr{U}_{L_1}(F))} g_M^{L_1}(\gamma^t, w^t) J_{L_1}(w^t, f^t), \tag{10.2}$$

since $w \to w^t$ is a bijection on $(\mathscr{U}_{L_1}(F))$. The right-hand side is $(M, 1)$-equivalent to

$$\sum_{L \in \mathscr{L}(M)} \sum_{u \in (\mathscr{U}_L(F))} |t|^{(1/2)(\dim G - \mathrm{rank}\, G)} g_M^L(\gamma, u) J_L(u, f). \tag{10.3}$$

We must relate the distributions $J_L(u, f)$ with $J_{L_1}(w^t, f^t)$.

By the definition (6.5) and Lemma 10.1,

$$J_L(u, f) = \lim_{a \to 1} \sum_{\{L_2, L_1:\ L_2 \supset L_1 \supset L\}} c_L^{L_1}(u, t) r_{L_1}^{L_2}(u, a^t) J_{L_2}(au, f),$$

where each a is a small regular point in $A_L(F)$. Now, by repeating an argument above, and noting that $G_a = L$, we see that

$$|D(a^t u^t)|^{1/2} = |t|^{(1/2)\dim(G/L)} |D(au)|^{1/2}.$$

Consequently,

$$J_{L_2}(au, f) = |D(au)|^{1/2} \int_{G_{au}(F)\backslash G(F)} f(x^{-1} aux) v_{L_2}(x)\, dx$$

$$= |t|^{-(1/2)\dim(G/L)} |D(a^t u^t)|^{1/2} \int_{G_{au}(F)\backslash G(F)} f^t(x^{-1} a^t u^t x) v_{L_2}(x)\, dx.$$

It is easy to see that

$$G_{au} = L_u = L_{u^t} = G_{a^t u^t}.$$

However, the compatible invariant measures on the orbits of au and $a^t u^t$ do not induce the same measure on

$$G_{au}(F) \backslash G(F) = G_{a^t u^t}(F) \backslash G(F).$$

By (10.1) (applied to L instead of G), the invariant measures induced on the coset spaces differ by the constant

$$|t|^{-(1/2)\dim(L/L_u)}.$$

It follows that

$$J_{L_2}(au, f) = |t|^{-(1/2)\dim(G/L_u)} J_{L_2}(a^t u^t, f^t).$$

We have thus far shown that $J_L(u, f)$ equals

$$|t|^{-(1/2)\dim(G/L_u)} \lim_{a \to 1} \sum_{L_2 \supset L_1 \supset L} c_L^{L_1}(u, t) r_{L_1}^{L_2}(u, a^t) J_{L_2}(a^t u^t, f^t).$$

But $r_{L_1}^{L_2}(u, a^t)$ depends only on the geometric conjugacy class of u, so that

$$r_{L_1}^{L_2}(u, a^t) = r_{L_1}^{L_2}(u^t, a^t).$$

Moreover, for any L_1,

$$\begin{aligned}
&\lim_{a\to 1} \sum_{L_2 \in \mathscr{L}(L_1)} r_{L_1}^{L_2}(u^t, a^t) J_{L_2}(a^t u^t, f^t)\\
&= \lim_{a\to 1} \sum_{L_2 \in \mathscr{L}(L_1)} r_{L_1}^{L_2}(u^t, a) J_{L_2}(a u^t, f^t)\\
&= J_{L_1}\big((u^t)^{L_1}, f^t\big)\\
&= \sum_{w \in (\mathscr{U}_{L_1}(F))} J_{L_1}(w^t, f^t)[u^{L_1}: w],
\end{aligned}$$

by Corollary 6.3. It follows that $J_L(u, f)$ equals

$$|t|^{-(1/2)\dim(G/L_u)} \sum_{L_1 \in \mathscr{L}(L)} \sum_{w \in (\mathscr{U}_{L_1}(F))} c_L^{L_1}(u, t) J_{L_1}(w^t, f^t)[u^{L_1}: w].$$

We have established that (10.3) equals the sum over $L_1 \in \mathscr{L}(M)$ and $w \in (\mathscr{U}_{L_1}(F))$ of

$$\sum_{L \in \mathscr{L}^{L_1}(M)} \sum_{u \in (\mathscr{U}_L(F))} |t|^{(1/2)(\dim L_u - \operatorname{rank} L)} g_M^L(\gamma, u) c_L^{L_1}(u, t) J_{L_1}(w^t, f^t)[u^{L_1}: w].$$

Now, if $[u^{L_1}: w] \neq 0$, we have

$$|t|^{(1/2)(\dim L_u - \operatorname{rank} L)} = |t|^{d^L(u)} = |t|^{d^{L_1}(w)}.$$

It follows that the sum over $L_1 \in \mathscr{L}(M)$ and $w \in (\mathscr{U}_{L_1}(F))$ of the product of

(10.4)

$$g_M^{L_1}(\gamma^t, w^t) - \sum_{L \in \mathscr{L}^{L_1}(M)} \sum_{u \in (\mathscr{U}_L(F))} |t|^{d^{L_1}(w)} g_M^L(\gamma, u) c_L^{L_1}(u, t)[u^{L_1}: w],$$

with

$$J_{L_1}(w^t, f^t),$$

is $(M, 1)$-equivalent to 0. We are required to show that (10.4) is $(M, 1)$-equivalent to 0. We can assume inductively that this is so if $L_1 \neq G$. For a given $w_1 \in (\mathscr{U}_G(F))$, choose f so that for each $w \in (\mathscr{U}_G(F))$,

$$J_G(w^t, f^t) = \begin{cases} 1, & w = w_1 \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 10.2 follows. □

Remark. Suppose that t equals a positive rational integer ℓ. Then x^t is just the ℓth power of x. In this special case, Proposition 10.2 will be used for the comparison of germs that arise in base change.

Part 3: Real groups

§11. Differential equations. We come finally to the case of real groups. Harish-Chandra's theory for invariant orbital integrals is based on the differential equations associated to the center of the universal enveloping algebra. We shall examine analogous differential equations satisfied by weighted orbital integrals.

For the rest of the paper, we assume that S consists of one Archimedean valuation v, and that $F = F_v = F_S$. We shall regard $G^0(F)$ as a real Lie group. Let

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{g}(F) \otimes_{\mathbf{R}} \mathbf{C}$$

be its complexified Lie algebra, and let $\mathscr{G}$ be the universal enveloping algebra. We shall write

$$\mathscr{Z} = \mathscr{Z}_G$$

for the center of $\mathscr{G}$. Recall that there is an injective map

$$z \to z_M, \qquad z \in \mathscr{Z}_G,$$

from $\mathscr{Z}_G$ into $\mathscr{Z}_M$.

For the rest of the paper we shall fix a maximal torus T of G which is defined over F. We assume that T_0 contains A_M. We shall also assume that T_0 is stable under the Cartan involution of $G^0(F)$ defined by K.

PROPOSITION 11.1. *There are uniquely determined differential operators*

$$\partial_M^G(\gamma, z), \qquad \gamma \in T_{\mathrm{reg}}(F),$$

on $T_{\mathrm{reg}}(F)$, parametrized by elements $z \in \mathscr{Z}$, such that for any $f \in C_c^\infty(G(F))$,

$$J_M(\gamma, f) = \sum_{L \in \mathscr{L}(M)} \partial_M^L(\gamma, z_L) J_L(\gamma, f).$$

Proof. The defining equation is understood to hold for all G, in particular if G is replaced by any set $L \in (M)$. Now, consider

$$f \to J_M(\gamma, f) \tag{11.1}$$

as a linear map from $C_c^\infty(G(F))$ to the space of germs of smooth functions near a given point in $T_{\mathrm{reg}}(F)$. According to the definition (2.1*),

$$J_M(\gamma, f) = |D(\gamma)|^{1/2} \int_{T_0(F)\backslash G^0(F)} f(x^{-1}\gamma x) v_M(x)\, dx.$$

Since

$$(11.2) \qquad (\gamma, x) \to x^{-1}\gamma x, \qquad \gamma \in T_{\mathrm{reg}}(F),\ x \in T_0(F)\backslash G^0(F),$$

is an open immersion into $G(F)$, the map (11.1) is surjective. Consequently, the differential operators

$$\partial_M^G(\gamma, z)$$

are uniquely determined by the required formula. We can therefore concentrate on their existence.

Fix the element $z \in \mathscr{Z}$. For each $L \in \mathscr{L}(M)$, with $L \neq G$, we assume inductively that the differential operators ∂_M^L have been defined and that

$$J_M^L(\gamma, z_L h) = \sum_{L_1 \in \mathscr{L}^L(M)} \partial_M^{L_1}(\gamma, z_{L_1}) J_{L_1}^L(\gamma, h)$$

for any $h \in C_c^\infty(L(F))$. The distribution

$$K'_M(\gamma, z, f) = J_M(\gamma, zf) - \sum_{\{L \in \mathscr{L}(M):\ L \neq G\}} \partial_M^L(\gamma, z_L) J_L(\gamma, f),$$

$f \in C_c^\infty(G(F))$, is then defined. Notice that this distribution is supported on the $G^0(F)$-orbit of γ. We claim that it is also invariant. To see this, take an element $y \in G^0(F)$ and consider the expression

$$(11.3) \qquad K'_M(\gamma, z, f^y - f).$$

By Lemma 8.1, we can write (11.3) as the difference between

$$\sum_{\{Q \in \mathscr{F}(M):\ Q \neq G\}} J_M^{M_Q}(\gamma, (zf)_{Q,y})$$

and

$$\sum_{L \in \mathscr{L}(M)} \sum_{\{Q \in \mathscr{F}(L):\ Q \neq G\}} \partial_M^L(\gamma, z_L) J_L^{M_Q}(\gamma, f_{Q,y}).$$

According to (2.3),

$$(zf)_{Q,y} = \delta_Q(m)^{1/2} \int_K \int_{N_Q(F)} (zf)(k^{-1}mnk) v'_Q(ky)\, dn\, dk$$

for any $m \in M_Q(F)$. The differential operator z certainly commutes with right translation by elements $k \in K$. It then follows easily from the definition of z_{M_Q}

that

$$(zf)_{Q,y} = z_{M_Q} f_{Q,y}.$$

Thus, (11.3) equals

$$\sum_{\{Q \in \mathscr{F}(M):\, Q \neq G\}} \left(J_M^{M_Q}\big(\gamma, z_{M_Q} f_{Q,y}\big) - \sum_{L \in \mathscr{F}^{M_Q}(M)} \partial_M^L(\gamma, z_L) J_L^{M_Q}\big(\gamma, f_{Q,y}\big) \right).$$

According to our induction assumption, this vanishes. Consequently, $K'_M(\gamma, z, \cdot)$ is an invariant distribution, as claimed.

We can use the immersion (11.2) to pull back $K'_M(\gamma, z, \cdot)$. The resulting distribution on

$$T_{\mathrm{reg}}(F) \times \big(T_0(F) \backslash G^0(F)\big)$$

is supported at $\{\gamma\}$ in the first factor and is $G^0(F)$-invariant in the second factor. We leave the reader to check that any $G^0(F)$-invariant distribution on $T_0(F) \backslash G^0(F)$ is a multiple of the invariant measure. It follows easily that there is a differential operator $\partial_M^G(\gamma, z)$ on $T_{\mathrm{reg}}(F)$ such that

$$K'_M(\gamma, z, f) = \partial_M^G(\gamma, z) J_G(\gamma, f).$$

The required formula then follows immediately from the definition of $K'_M(\gamma, z, f)$. □

§12. Comparison with the radial decomposition. The differential operators $\partial_M^G(\gamma, z)$ were obtained in a nonconstructive way from a simple invariance argument. They also have a more complicated but constructive description in terms of Harish-Chandra's radial decomposition of z. We shall review this description, which in the case of $G = G^0$ was introduced in §5 of [1].

Let $\mathfrak{t}_{\mathbf{C}}$ be the complexified Lie algebra of $T_0(F)$. We can identify the symmetric algebra $S(\mathfrak{t}_{\mathbf{C}})$ with its image $\mathscr{T}$ in $\mathscr{G}$. Let $\mathfrak{q}_{\mathbf{C}}$ be the direct sum of the nonzero eigenspaces of $T(F)$ in $\mathfrak{g}_{\mathbf{C}}$, and let $\mathscr{Q}$ be the image of $S(\mathfrak{q}_{\mathbf{C}})$ in $\mathscr{G}$. Then for each element $\gamma \in T_{\mathrm{reg}}(F)$, there is a linear isomorphism

$$\Gamma_\gamma : \mathscr{Q} \otimes \mathscr{T} \to \mathscr{G}$$

which is uniquely defined by

$$\Gamma_\gamma(X_1 \cdots X_k \otimes u) = \big(L_{\mathrm{Ad}(\gamma^{-1})X_1} - R_{X_1}\big) \cdots \big(L_{\mathrm{Ad}(\gamma^{-1})X_k} - R_{X_k}\big) u$$

for $X_1, \ldots, X_k$ in $\mathfrak{q}_{\mathbf{C}}$ and $u \in \mathscr{T}$. Here $L_X g = Xg$ and $R_X g = gX$ for $X \in \mathfrak{g}_{\mathbf{C}}$ and $g \in \mathscr{G}$. This is a routine extension to nonconnected groups of a result [12, Lemma 22] of Harish-Chandra. Let $\mathscr{Q}'$ be the subspace of codimension 1 in $\mathscr{Q}$ consisting of elements with zero constant term. Then for any $z \in \mathscr{Z}$, there is a unique element $\beta(\gamma, z)$ in $\mathscr{T}$ such that $z - \beta(\gamma, z)$ belongs to $\Gamma_\gamma(\mathscr{Q}' \otimes \mathscr{T})$. In other words, there are elements

$$\{X_i \colon 1 \leqslant i \leqslant r\}$$

in $\mathscr{Q}'$ which are adjoint invariant under $T(F)$, linearly independent elements

$$\{u_i \colon 1 \leqslant i \leqslant r\}$$

in $\mathscr{T}$, and analytic functions

$$\{a_i \colon 1 \leqslant i \leqslant r\}$$

on $T_{\text{reg}}(F)$, such that

$$z = \beta(\gamma, z) + \sum_{i=1}^{k} a_i(\gamma)\Gamma_\gamma(X_i \otimes u_i). \tag{12.1}$$

As on pages 229–231 of [1], the decomposition (12.1) provides a formula for $J_M(\gamma, zf)$. We describe the result. Associated to the elements $\beta(\gamma, z)$ and u_i in $S(\mathfrak{t}_{\mathbf{C}})$ we of course have the differential operators $\partial(\beta(\gamma, z))$ and $\partial(u_i)$ on $T_{\text{reg}}(F)$. Define new differential operators on $T_{\text{reg}}(F)$ by

(12.2)

$$\partial(\alpha(\gamma, z)) = |D(\gamma)|^{1/2}\partial(\beta(\gamma, z)) \circ |D(\gamma)|^{-1/2}, \qquad \alpha(\gamma, z) \in S(\mathfrak{t}_{\mathbf{C}}),$$

and

$$\partial_i(\gamma, z) = |D(\gamma)|^{1/2} a_i(\gamma)\partial(u_i) \circ |D(\gamma)|^{-1/2}, \qquad 1 \leqslant i \leqslant r. \tag{12.3}$$

Let

$$X \to D_X$$

be the anti-isomorphism of $\mathscr{G}$ into the algebra of right $G^0(F)$-invariant differential operators on $G(F)$. Then $J_M(\gamma, zf)$ equals the sum of

$$\partial(\alpha(\gamma, z))J_M(\gamma, f)$$

and

$$\sum_{i=1}^{r} \partial_i(\gamma, z)\left(|D(\gamma)|^{1/2} \int_{T_0(F)\backslash G^0(F)} f(x^{-1}\gamma x)\big(D_{X_i} v_M(x)\big)\, dx\right). \tag{12.4}$$

We shall calculate $D_{X_i} v_M(x)$. Set

$$d = \dim(A_M/A_G).$$

Then by formula (6.5) of [1],

$$v_M(x) = \frac{1}{d!} \sum_{P \in \mathscr{P}(M)} \nu(H_P(x))^d \theta_P(\nu)^{-1}.$$

The expression on the right is independent of the point $\nu \in i\mathfrak{a}_M^*$. Consider an element $P = MN_P$ in $\mathscr{P}(M)$. If $\mathfrak{p}^1$ denotes the Lie algebra of

$$\{x \in P(F) \colon H_P(x) = 0\},$$

the Lie algebra of $P^0(F)$ is the direct sum of $\mathfrak{p}^1$ and $\mathfrak{a}_M$. The group $A_M(F)$ is contained in $T_0(F)$ and acts on $T(F)$ by translation. In particular, X_i is invariant under $A_M(F)$. As on page 223 of [1], we see that there is a unique element $\mu_P(X_i)$ in the symmetric algebra $S(\mathfrak{a}_{M,\mathbf{C}})$ such that

$$X_i - \mu_P(X_i)$$

belongs to $\mathfrak{p}^1\mathscr{G}$. For each nonnegative integer m, let $\mu_{P,m}(X_i)$ denote the homogeneous component of $\mu_P(X_i)$ of degree m. We will write

$$\langle \mu_{P,m}(X_i), \nu \rangle, \qquad \nu \in i\mathfrak{a}_M^*,$$

for the corresponding homogeneous polynomial of degree m on $i\mathfrak{a}_M^*$. It follows from our definitions that

$$\begin{aligned} \frac{1}{d!} D_{X_i} \nu(H_P(x))^d &= \frac{1}{d!} D_{\mu_P(X_i)} \nu(H_P(x))^d \\ &= \sum_{m=0}^{d} \frac{1}{(m-d)!} \nu(H_P(x))^{d-m} \langle \mu_{P,m}(X_i), \nu \rangle. \end{aligned}$$

Thus,

$$D_{X_i} v_M(x)$$

equals

$$C_i(x) = \sum_{P \in \mathscr{P}(M)} \sum_{m=0}^{d} \frac{1}{(d-m)!} \nu(H_P(x))^{d-m} \langle \mu_{P,m}(X_i), \nu \rangle \theta_P(\nu)^{-1}.$$

The expression on the right is of course independent of ν. It is clear that C_i is a smooth function on $T_0(F)\setminus G^0(F)$ whose value at 1 equals the number

$$c_i = \sum_{P\in\mathcal{P}(M)} \langle \mu_{P,d}(X_i), \nu\rangle \theta_P(\nu)^{-1}. \tag{12.5}$$

LEMMA 12.1. *If $M = G$,*

$$\partial_M^G(\gamma, z) = \partial(\alpha(\gamma, z)).$$

If $M \neq G$,

$$\partial_M^G(\gamma, z) = \sum_{i=1}^r c_i \partial_i(\gamma, z).$$

Proof. Suppose first that $M = G$. Then

$$v_M(x) = v_G(x) = 1$$

and

$$D_{X_i} v_M(x) = 0, \qquad 1 \leqslant i \leqslant r.$$

The first assertion follows.

Assume now that $M \neq G$. We have seen that the expression (12.4) equals

$$J_M(\gamma, zf) - \partial(\alpha(\gamma, z))J_M(\gamma, f).$$

Since

$$\partial(\alpha(\gamma, z)) = \partial_G^G(\gamma, z) = \partial_M^M(\gamma, z_M),$$

this in turn equals

$$\sum_{\{L\in\mathcal{L}(M):\ L\neq M\}} \partial_M^L(\gamma, z_L) J_L(\gamma, f) \tag{12.6}$$

by Proposition 11.1. We shall extract the second assertion from the equality of (12.4) and (12.6).

Fix a small open set U in $T_{\mathrm{reg}}(F)$. Then

$$(\gamma, x) \to x^{-1}\gamma x, \qquad \gamma \in U,\ x \in T_0(F)\setminus G^0(F),$$

is an open injective map from

$$U \times \big(T_0(F)\setminus G^0(F)\big)$$

into $G(F)$. Let ϕ be an arbitrary smooth function on U, and let $\{\psi_n\}$ be a sequence in $C_c^\infty(T_0(F)\setminus G^0(F))$ which approaches the Dirac measure at 1. For each n, we choose a function $f_n \in C_c^\infty(G(F))$ such that

$$f_n(x^{-1}\gamma x) = |D(\gamma)|^{-1/2}\phi(\gamma)\psi_n(x), \qquad x \in U,\, x \in T_0(F)\setminus G^0(F).$$

When $f = f_n$, the expression (12.4) equals

$$\sum_{i=1}^{r} \left(\partial_i(\gamma, z)\phi(\gamma)\right)\int_{T_0(F)\setminus G^0(F)} \psi_n(x) C_i(x)\, dx.$$

This approaches

$$\sum_{i=1}^{r} c_i \partial_i(\gamma, z)\phi(\gamma)$$

as n approaches infinity. On the other hand, the value of (12.6) at $f = f_n$ equals

$$\sum_{L \supsetneq M} \left(\partial_M^L(\gamma, z_L)\phi(\gamma)\right)\int_{T_0(F)\setminus G^0(F)} \psi_n(x) v_L(x)\, dx.$$

Since

$$v_L(1) = \begin{cases} 1, & L = G, \\ 0, & L \neq G, \end{cases}$$

this approaches

$$\partial_M^G(\gamma, z)\phi(\gamma).$$

The function ϕ was arbitrary, so the second assertion of the lemma follows. □

COROLLARY 12.2. *For each z there is a positive integer p such that*

$$D(\gamma)^p \partial_M^G(\gamma, z), \qquad \gamma \in T_{\mathrm{reg}}(F),$$

extends to an analytic differential operator on $T(F)$.

Proof. The corollary follows from the lemma, our various definitions, and a result of Harish-Chandra [12, Lemma 23], which implies that for large p, the differential operators

$$D(\gamma)^p \partial_i(\gamma, z)$$

all extend to $T(F)$. □

The lemma provides an algorithm for computing the differential operators through manipulations in the universal enveloping algebra. If z is the Casimir operator, this leads to a simple formula for $\partial_M^G(\gamma, z)$. (See [4] for the case of real rank 1.) However, the other elements in $\mathscr{Z}$ are not so amenable, and for these the algorithm is not practical. Nevertheless, Lemma 12.1 does give useful qualitative information about the differential operators that can be applied in general.

Let us consider the triplet $(\mathfrak{g}_{\mathbf{C}}, T(F), A_M(F))$ as an independent entity. Then we are presented with the following objects: a complex reductive Lie algebra $\mathfrak{g}_{\mathbf{C}}$, a manifold $T(F)$ equipped with a simple transitive action of a real torus $T_0(F)$, a compatible map

$$\mathrm{Ad}\colon T(F) \to GL(\mathfrak{g}_{\mathbf{C}}), \tag{12.7}$$

a real split torus $A_M(F)$, and an embedding of $A_M(F)$ into $T_0(F)$. The algebra $\mathscr{Z}$ is of course determined by $\mathfrak{g}_{\mathbf{C}}$. It is also apparent that the subset $T_{\mathrm{reg}}(F)$ of $T(F)$ is uniquely defined by $\mathfrak{g}_{\mathbf{C}}$ and $T(F)$ (and the map (12.7)). In fact, it follows from the definitions that the entire decomposition (12.1) is determined by $\mathfrak{g}_{\mathbf{C}}$ and $T(F)$. Since the function $D(\gamma)$ is also determined by $\mathfrak{g}_{\mathbf{C}}$ and $T(F)$, the same is true of the differential operators $\partial(\alpha(\gamma, z))$ and $\partial_i(\gamma, z)$. Finally, it follows from (12.5) that the constants c_i are determined by our triplet. Therefore, the formulas from the lemma imply

COROLLARY 12.3. *The differential operators $\partial_M^G(\gamma, z)$ are uniquely determined by the triplet $(\mathfrak{g}_{\mathbf{C}}, T(F), A_M(F))$.* □

This corollary tells us that the differential operators can be matched for groups which differ by an inner twisting. In fact, somewhat more is true. The algebraic closure $\bar{F}$ of F is isomorphic to $\mathbf{C}$. The associated space $T(\bar{F})$ is a complex manifold, and (12.7) extends to a holomorphic map

$$\mathrm{Ad}\colon T(\bar{F}) \to GL(\mathfrak{g}_{\mathbf{C}}).$$

In the definitions (12.2) and (12.3), we can replace $|D(\gamma)|^{1/2}$ by any branch of $D(\gamma)^{1/2}$. It then follows from the other definitions that the entire construction extends holomorphically from $T_{\mathrm{reg}}(F)$ to $T_{\mathrm{reg}}(\bar{F})$. Therefore, by Lemma 12.1, the analytic differential operators $\partial_M^G(\gamma, z)$ extend to holomorphic differential operators on $T_{\mathrm{reg}}(\bar{F})$. The construction also behaves in the obvious way under conjugation by $M^0(\bar{F})$. In particular,

$$\partial_M^G(m^{-1}\gamma m, z) = \partial_M^G(\gamma, z), \qquad m \in M^0(\bar{F}).$$

It follows from these remarks and Corollary 12.2 that the differential operators $\partial_M^G(\gamma, z)$ can be computed in general from the special case that G^0 splits over F and H_T^0 is an F-split torus.

Let $\mathfrak{h}_{T,\mathbf{C}}$ be the complexified Lie algebra of the Cartan subgroup $H_T^0(F)$ of $G^0(F)$. Then the Harish-Chandra map

$$z \to h(z), \qquad z \in \mathscr{Z},$$

sends $\mathscr{Z}$ isomorphically onto the Weyl invariant elements in $S(\mathfrak{h}_{T,\mathbf{C}})$. We can write

$$\mathfrak{h}_{T,\mathbf{C}} = \mathfrak{t}_{\mathbf{C}} \oplus \mathfrak{s}_{\mathbf{C}},$$

where $\mathfrak{s}_{\mathbf{C}}$ is spanned by the $\mathrm{Ad}(T(F))$-eigenvectors in $\mathfrak{h}_{T,\mathbf{C}}$ with nonzero eigenvalue. Then

$$S(\mathfrak{h}_{T,\mathbf{C}}) = S(\mathfrak{t}_{\mathbf{C}}) \otimes S(\mathfrak{s}_{\mathbf{C}}),$$

so that

$$S(\mathfrak{h}_{T,\mathbf{C}}) = S(\mathfrak{t}_{\mathbf{C}}) \oplus S(\mathfrak{h}_{T,\mathbf{C}})\mathfrak{s}_{\mathbf{C}}.$$

Let $h_T(z)$ denote the projection of $h(z)$ onto $S(\mathfrak{t}_{\mathbf{C}})$, relative to this last decomposition. Then

$$z \to h_T(z)$$

is a homomorphism of $\mathscr{Z}$ into $S(\mathfrak{t}_{\mathbf{C}})$. We can expect the image of h_T to be the space of invariants in $S(\mathfrak{t}_{\mathbf{C}})$ under the normalizer of $\mathfrak{t}_{\mathbf{C}}$ in the Weyl group. (See [8, Theorem 1.2].) In any case, it is clear that $S(\mathfrak{t}_{\mathbf{C}})$ is a finite module over $h_T(\mathscr{Z})$.

LEMMA 12.4. $\partial_G^G(\gamma, z) = \partial(h_T(z)), \qquad z \in \mathscr{Z}.$

Proof. By the first assertion of Lemma 12.1, we need only show that $h_T(z)$ equals $\alpha(\gamma, z)$. If $G = G^0$, so that $h_T(z) = h(z)$, this is a standard result of Harish-Chandra [12, Theorem 2]. Harish-Chandra's proof entails differentiating the characters of finite-dimensional representations. Rather than deal with twisted characters, we shall use a different argument.

According to the remarks above, we may assume that H_T^0 is an F-split torus. Then G^0 has a Borel subgroup

$$B^0 = N_B H_T^0$$

which is normalized by T and is defined over F. We can write

$$J_G(\gamma, f) = |D(\gamma)|^{1/2} \int_K \int_{N_B(F)} \int_{T_0(F)\backslash H_T^0(F)} f(k^{-1}n^{-1}h^{-1}\gamma hnk)\, dh\, dn\, dk.$$

Since γ is a regular element in $T(F)$, we can change variables in the integral over $N_B(F)$. The usual formula gives

$$J_G(\gamma, f) = \delta_{B^0}(\gamma)^{1/2} \int_K \int_{N_B(F)} \int_{T_0(F)\backslash H_T^0(F)} f(k^{-1}h^{-1}\gamma hnk)\, dh\, dn\, dk.$$

It then follows from the definition of the Harish-Chandra map that

$$J_G(\gamma, zf) = \partial(h_T(z))J_G(\gamma, f), \qquad z \in \mathscr{Z}.$$

Therefore,

$$\partial(\alpha(\gamma, z))J_G(\gamma, f) = \partial(h_T(z))J_G(\gamma, f).$$

Since any smooth function of $\gamma \in T_{\mathrm{reg}}(F)$ is locally of the form $J_G(\gamma, f)$, we obtain

$$\partial(\alpha(\gamma, z)) = \partial(h_T(z)),$$

as required. □

§13. Behaviour near singular points. For real groups, the most important singularities of weighted orbital integrals are at the semiregular points. As with invariant orbital integrals, a knowledge of the behavior around semiregular points, in combination with the differential equations, suffices for most applications. We have investigated this behaviour in [1]. For the sake of completeness, we shall recapitulate the main result.

Recall that an element $\sigma \in T(F)$ is *semiregular* if the derived group of $G_\sigma(F)$ is three-dimensional (as a real Lie group). Suppose that this is so, and that the derived group is noncompact. Then $F \cong \mathbb{R}$. Let β be one of the two roots of (G_σ, T_0) and let $\beta^\vee \in \mathfrak{t}(F)$ be the corresponding co-root. (Here $\mathfrak{t}$ is the Lie algebra of T_0.) Notice that β extends to a morphism

$$t \to t^\beta, \qquad t \in T,$$

from T to GL_1 over F, such that $\sigma^\beta = 1$ and

$$(tt_0) = t^\beta t_0^\beta, \qquad t \in T,\ t_0 \in T_0.$$

Let M_1 be the Levi subset in $\mathscr{L}(M)$ such that

$$A_{M_1} = \{a \in A_M \colon a^\beta = 1\}.$$

We shall consider the function

(13.1)

$$J_M^\beta(\gamma, f) = J_M(\gamma, f) + \|\beta_M^\vee\| \log|\gamma^\beta - \gamma^{-\beta}|\, J_{M_1}(\gamma, f), \qquad \gamma \in T_{\mathrm{reg}}(F),$$

where $\|\beta_M^\vee\|$ denotes the norm of the projection of $\beta^\vee$ onto $\mathfrak{a}_M$. The reader can check that

$$J_M^\beta(\sigma a, f) = \sum_{L \in \mathscr{L}(M)} r_M^L(\sigma, a) J_L(a\sigma, f)$$

for $a \in A_M(F)$. In particular, the function on the left is continuous at $a = 1$. We are interested in the behaviour of its derivatives.

Let

$$C\colon T_0 \to T_{01}$$

be the Cayley transform in G_σ. Then C is an inner automorphism on G_σ which maps T_0 to a torus T_{01} in G_σ which is F-anisotropic modulo the center of G_σ. Its differential, which we also write as C, maps $\beta^\vee$ to a co-root $\beta_1^\vee$ for T_{01}. Set

$$\gamma_r = \sigma \exp(r\beta^\vee), \qquad r \in \mathbb{R},$$

and

$$\delta_s = \sigma \exp(s\beta_1^\vee), \qquad s \in \mathbb{R}.$$

Suppose that

$$\partial(u), \qquad u \in S(\mathfrak{t}_{\mathbb{C}}),$$

is an invariant differential operator on $T(F)$. Then if w_β denotes the simple reflection about β, we define an invariant differential operator

$$\partial(u_1) = \partial(Cw_\beta u - Cu)$$

on

$$T_1(F) = \sigma T_{01}(F).$$

Lemma 13.1. *Let $n_\beta(A_M)$ denote the angle between $\beta^\vee$ and the subspace $\mathfrak{a}_M$ of $\mathfrak{t}(F)$. Then the limit*

$$\lim_{r \to 0+} \big(\partial(u) J_M^\beta(\gamma_r, f) - \partial(u) J_M^\beta(\gamma_{-r}, f)\big) \tag{13.2}$$

equals

$$n_\beta(A_M) \lim_{s \to 0} \partial(u_1) J_{M_1}(\delta_s, f).$$

This lemma is essentially Theorem 6.1 of [1]. In [1] we were working only with connected groups, but the extension of the proof to arbitrary G is formal. There is a minor difference between our definition of $J_M^\beta(\gamma_r, f)$ and the analogous distribution introduced on page 227 of [1]. In place of the function

$$\log|(\gamma_r)^\beta - (\gamma_r)^{-\beta}| = \log|e^{2r} - e^{-2r}| = \log|e^r - e^{-r}| + \log(e^r + e^{-r})$$

in (13.1), we had

$$\log|e^r - e^{-r}|$$

in [1]. However, since the function

$$\log(e^r + e^{-r}) J_{M_1}(\gamma_r, f)$$

is smooth at $r = 0$, this discrepancy does not affect the limit (13.2). In [1], we in fact proved that the two one-sided limits in (13.2) each exist.

We shall conclude with a general estimate for the derivatives of $J_M(\gamma, f)$ near an arbitrary singularity. The technique is due to Harish-Chandra [15, Lemma 48] and exploits the differential equations. We shall apply the technique to weighted orbital integrals in much the same way as in the proof of Lemma 8.1 of [1].

Let Δ be a compact subset of $T(F)$, and set

$$\Delta_{\mathrm{reg}} = \Delta \cap G_{\mathrm{reg}}.$$

LEMMA 13.2. *For every element $u \in S(\mathfrak{t}_{\mathbf{C}})$ there is a positive integer q and a continuous seminorm c on $C_c^\infty(G(F))$ such that*

$$|\partial(u) J_M(\gamma, f)| \leqslant c(f) |D^G(\gamma)|^{-q} \tag{13.3}$$

for any $\gamma \in \Delta_{\mathrm{reg}}$ and $f \in C_c^\infty(G(F))$.

Proof. Suppose first that $u = 1$. By Corollary 7.4 of [1], we can choose a positive integer p and a continuous seminorm c on $C_c^\infty(G(F))$ so that

$$|J_M(\gamma, f)| \leqslant c(f) \left|(\log |D^G(\gamma)|)\right|^p$$

for γ near the singular set of $T(F)$. In particular, (13.3) holds with $q = 1$.

Next, suppose that

$$u = h_T(z)$$

for some $z \in \mathscr{Z}$. Then by Proposition 11.1 and Lemma 12.4,

$$\partial(u) J_M(\gamma, f) = J_M(\gamma, zf) - \sum_{\{L \in \mathscr{L}(M):\ L \neq M\}} \partial_M^L(\gamma, z_L) J_L(\gamma, f).$$

Having already dealt with the case that $u = 1$, we need only estimate

$$\partial_M^L(\gamma, z_L) J_L(\gamma, f)$$

for $L \supsetneqq M$. Applying Corollary 12.2 to L, we can choose p so that

$$\partial_M^L(\gamma, z) = (D^G(\gamma))^{-p} \sum_j \xi_j(\gamma) \partial(u_j)$$

for elements $u_j \in S(\mathfrak{t}_{\mathbf{C}})$ and analytic functions ξ_j on $T(F)$. Assume inductively that the lemma holds if M is replaced by L. Then for each j there is a positive integer q_j and a continuous seminorm c_j such that

$$|\partial(u_j)J_L(\gamma, f)| \leqslant c_j(f)|D^G(\gamma)|^{-q_j}, \qquad \gamma \in \Delta_{\mathrm{reg}}.$$

The required estimate follows.

Suppose, finally, that u is arbitrary. This is the point at which we use Harish-Chandra's method. We shall simply state the basic formula from the proof of Lemma 48 of [15] in a form that applies in the present context. (See also the discussion on page 13 of lecture 3 of [18].) The fact that we are dealing here with a nonconnected group is of no consequence, for $S(\mathfrak{t}_{\mathbf{C}})$ is still a finite module over

$$h_T(\mathscr{Z}) = \{h_T(z) : z \in \mathscr{Z}\}.$$

Introduce a distance function for neighboring points in $T(F)$ by exponentiating a norm on $\mathfrak{t}(F)$, and let

$$\tau(\gamma), \qquad \gamma \in T(F),$$

denote the distance from γ to the singular set

$$T(F) - T_{\mathrm{reg}}(F).$$

We need only establish (13.3) for points $\gamma \in \Delta_{\mathrm{reg}}$ with $\tau(\gamma)$ small. Set

$$\varepsilon = \tfrac{1}{6}\tau(\gamma).$$

Then Harish-Chandra's argument provides functions $E_{1,\varepsilon}, \ldots, E_{k,\varepsilon}$ and β_ε on $T_0(F)$ and elements $u_1, \ldots, u_k$ in $h_T(\mathscr{Z})$ such that

$$\partial(u)J_M(\gamma, f)$$

equals the difference between

$$\text{(13.4)} \qquad \sum_{j=1}^{k} \int_{T_0(F)} \partial(u_i)J_M(\gamma t, f)E_{j,\varepsilon}(t)\, dt$$

and

$$\text{(13.5)} \qquad \int_{T_0(F)} J_M(\gamma t, f)\beta_\varepsilon(t)\, dt.$$

The functions $E_{j,\varepsilon}$ and β_ε are independent of f but depend on γ through the number ε. They are all supported on the ball of radius 3ε. Moreover, each $E_{j,\varepsilon}$ is bounded, while β_ε satisfies an estimate

$$|\beta_\varepsilon(t)| \leqslant c\varepsilon^{-p} = c(\tau(\gamma)/6)^{-p}.$$

We can certainly produce bounds of the form

$$c_1\tau(\gamma)^{\ell} \leqslant |D^G(\gamma)| \leqslant c_2\tau(\gamma),$$

since γ belongs to the compact set Δ. In particular, we can write the previous estimate in the form

$$|\beta_\varepsilon(t)| \leqslant c'|D^G(\gamma)|^{-p}.$$

Applying the special case established above to the elements $u_0 = 1, u_1, \ldots, u_k$, we obtain

$$|\partial(u_i)J_M(\gamma t, f)| \leqslant c_i(f)|D^G(\gamma t)|^{-q_i}.$$

Suppose that for a given t, one of the integrands in (13.4) or (13.5) does not vanish. Then

$$\tau(\gamma t) \geqslant 3\varepsilon = \tfrac{1}{2}\tau(\gamma),$$

and we obtain an estimate

$$|\partial(u_i)J_M(\gamma t, f)| \leqslant c_i'(f)|D^G(\gamma)|^{-q_i'}.$$

The required estimate (13.3) then follows. □

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