Proc. Indian Acad. Sci. (Math. Sci.), Vol. 97, Nos 1–3, December 1987, pp. 3–19.

The characters of supercuspidal representations as weighted orbital integrals

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Abstract. Weighted orbital integrals are the terms which occur on the geometric side of the trace formula. We shall investigate these distributions on a p-adic group. We shall evaluate the weighted orbital integral of a supercuspidal matrix coefficient as a multiple of the corresponding character.

Keywords. Supercuspidal representation; weighted orbital integrals.

1. Introduction

Let G be a reductive algebraic group over a non-Archimedean local field F of characteristic 0. Suppose that π is a (smooth) supercuspidal representation of $G(F)$ on a complex vector space V. Let $f(x)$ be a finite sum of matrix coefficients

$$\xi(\pi(x)^{-1}v), \quad x\in G(F), \quad v\in V, \quad \xi\in V^*.$$

Then f is a locally constant function on $G(F)$ which is compactly supported modulo the split component A_G of the centre of G. If Θ_π is the character of π, set

$$\Theta_\pi(f) = \int_{A_G(F)\backslash G(F)} f(x)\Theta_\pi(x)\mathrm{d}x.$$

We are going to study the weighted orbital integrals of f.

Suppose that M is a Levi component of some parabolic subgroup of G which is defined over F. The set $\mathscr{P}(M)$ of all parabolic subgroups over F with Levi component M is parametrized by the chambers in a real vector space $\mathbf{a}_M$. The weight factor for orbital integrals is a certain function $v_M(x)$ on $M(F)\backslash G(F)$ which arises in the theory of automorphic forms; it is defined as the volume of the convex hull in $\mathbf{a}_M/\mathbf{a}_G$ of a set of points indexed by $\mathscr{P}(M)$. Suppose that γ is a G-regular element in $M(F)$ which is M-elliptic over F. This means that the centralizer of γ in $M(F)$ is compact modulo $A_M(F)$. The object of this paper is to prove the following result.

Theorem: *The weighted orbital integral*

$$\int_{A_M(F)\backslash G(F)} f(x^{-1}\gamma x)v_M(x)\mathrm{d}x$$

* Supported in part by NSERC Operating Grant A3483.

equals

$$(-1)^{\dim(A_M/A_G)}\Theta_\pi(f)\Theta_\pi(\gamma).$$

Observe that the second expression depends on a choice of invariant measure on $A_G(F)\backslash G(F)$; the first expression depends on choices of invariant measures on $\mathbf{a}_M/\mathbf{a}_G$ and $A_M(F)\backslash G(F)$. There is a compatibility requirement between the implicit measure on $A_M(F)/A_G(F)$ and the measure on $\mathbf{a}_M/\mathbf{a}_G$.

The theorem is a p-adic version of a similar result for real groups ([1], Theorem 9.1). It tells us that the character values of π on a non-compact torus can be recovered as the weighted orbital integrals of a matrix coefficient of π. There is reason to believe that the result is part of a larger theory. Kazhdan has suggested the possibility of proving a local trace formula for G. The idea would be to try to compute the trace of the left-right convolution operator of a pair of functions, acting on the discrete spectrum of $L^2(G(F))$. Our theorem could be regarded as a special case, in which one of the two functions is the matrix coefficient f. A different special case of this (as yet undiscovered) trace formula is provided by work of Waldspurger [6]. We hope to return to the question on another occasion.

For $G = SL(2)$, the theorem was first established by Kazhdan (unpublished). I am indebted to him for enlightening conversations.

2. Positive orthogonal sets

Let us recall the precise definition of $v_M(x)$. It depends on a special maximal compact subgroup K of $G(F)$ which is in good position relative to M. (This means that the vertex of K in the building of G lies in the apartment of a maximal split torus of M.) For any parabolic subgroup $P \in \mathscr{P}(M)$, with Levi decomposition $P = MN_P$, and any point $x \in G(F)$, we can write

$$x = n_P(x)m_P(x)k_P(x), \tag{1}$$

with $n_P(x) \in N_P(F)$, $m_P(x) \in M(F)$ and $k_P(x) \in K$. Set

$$H_P(x) = H_M(m_P(x)),$$

where H_M is the usual map from $M(F)$ to the real vector space

$$\mathbf{a}_M = \mathrm{Hom}(X(M)_F, \mathbb{R}),$$

given by

$$\exp(\langle H_M(m), \chi\rangle) = |\chi(m)|, \quad m \in M(F), \quad \chi \in X(M)_F.$$

There is a canonical map from $\mathbf{a}_M$ onto $\mathbf{a}_G$, whose kernel we denote by $\mathbf{a}_M^G$. Since $X(M)_F$ embeds into the character group $X(A_M)$ of A_M, there is also a compatible embedding of $\mathbf{a}_G$ into $\mathbf{a}_M$, and therefore a canonical decomposition

$$\mathbf{a}_M = \mathbf{a}_M^G \oplus \mathbf{a}_G.$$

The function $v_M(x)$ equals the volume of the convex hull of the projection of

$$\{-H_P(x): P\in\mathscr{P}(M)\}$$

onto $\mathbf{a}_M/\mathbf{a}_G \cong \mathbf{a}_M^G$.

It is convenient to choose a suitable Euclidean metric $\|\cdot\|$ on $\mathbf{a}_M$, and to use this to normalize the Haar measures on $\mathbf{a}_M$, $\mathbf{a}_G$ and $\mathbf{a}_M/\mathbf{a}_G$. These measures then determine Haar measures on $A_M(F)$, $A_G(F)$ and $A_M(F)/A_G(F)$. Indeed,

$$\kappa_M = A_M(F)\cap K$$

is the maximal (open) compact subgroup of $A_M(F)$, and H_M maps $A_M(F)/\kappa_M$ injectively onto a lattice in $\mathbf{a}_M$. We take the Haar measure on $A_M(F)$ such that

$$\mathrm{vol}(\kappa_M) = \mathrm{vol}(\mathbf{a}_M/H_M(A_M(F))).$$

The cases of $A_G(F)$ and $A_M(F)/A_G(F)$ are similar, and we have

$$\mathrm{vol}(\kappa_M/\kappa_G) = \mathrm{vol}(\mathbf{a}_M/H_M(A_M(F)) + \mathbf{a}_G). \tag{2}$$

The points $\{-H_P(x)\}$ form a positive orthogonal set. In general, we say that a set

$$\mathscr{Y} = \{Y_P: P\in\mathscr{P}(M)\}$$

of points in $\mathbf{a}_M$ is a positive orthogonal set for M if it has the following property. For any pair P and P' of adjacent groups in $\mathscr{P}(M)$, whose chambers in $\mathbf{a}_M$ share the wall determined by a simple root α in $\Delta_P\cap(-\Delta_{P'})$ of (P, A_M), we have

$$Y_P - Y_{P'} = r\alpha^\vee,$$

for a non-negative number r. As usual, Δ_P is the set of simple roots of (P, A_M), and $\alpha^\vee\in\mathbf{a}_M$ is the "co-root" associated to α. Suppose that $\mathscr{Y}$ has this property. Then the volume in $\mathbf{a}_M/\mathbf{a}_G$ of the convex hull of $\{Y_P\}$ can be expressed analytically as

$$\lim_{\lambda\to 0}\sum_{P\in\mathscr{P}(M)} \exp[\lambda(Y_P)]\theta_P(\lambda)^{-1}, \tag{3}$$

where

$$\theta_P(\lambda) = \mathrm{vol}(\mathbf{a}_M^G/\mathbb{Z}(\Delta_P^\vee))^{-1}\prod_{\alpha\in\Delta_P}\lambda(\alpha^\vee).$$

(See [3], p. 36.) As in [4], we shall write $d(\mathscr{Y})$ for the smallest of the numbers

$$\{\alpha(Y_P): \alpha\in\Delta_P,\ P\in\mathscr{P}(M)\}.$$

Fix such a $\mathscr{Y}$, and let

$$Q = M_Q N_Q,\quad M_Q\supset M,$$

be an element in the set $\mathscr{F}(M)$ of parabolic subgroups of G over F which contain M.

Then

$$\{Y_{P\cap M_Q} = Y_P : P \in \mathscr{P}(M),\ P \subset Q\} \tag{4}$$

is a positive orthogonal set for M, but relative to M_Q instead of G. As above, $\mathbf{a}_M$ is the direct sum of the spaces $\mathbf{a}_M^{M_Q} = \mathbf{a}_M^Q$ and $\mathbf{a}_{M_Q} = \mathbf{a}_Q$. We shall write $S_M^Q(\mathscr{Y})$ for the convex hull of (4) in $\mathbf{a}_M$, taken modulo $\mathbf{a}_Q$, and we shall let $\sigma_M^Q(\cdot, \mathscr{Y})$ stand for the characteristic function of $S_M^Q(\mathscr{Y})$. The vectors (4) all project onto the same point Y_Q in $\mathbf{a}_Q$. Moreover, if we fix the Levi component $L = M_Q$ instead of Q, the set

$$\mathscr{Y}_L = \{Y_Q : Q \in \mathscr{P}(L)\}$$

is a positive orthogonal set for L. For simplicity, we shall usually denote $S_L^G(\mathscr{Y}_L)$ and $\sigma_L^G(\cdot, \mathscr{Y}_L)$ by $S_L(\mathscr{Y})$ and $\sigma_L(\cdot, \mathscr{Y})$ respectively.

The following geometric property is a restatement of Lemmas 3.1 and 3.2 of [4].

Lemma 1: There is a positive constant δ_M with the following property. If $\mathscr{Y}$ is any positive orthogonal set for M and $L \supset M$ is as above, and if

$$H_M = H_M^L \oplus H_L, \quad H_M^L \in \mathbf{a}_M^L, \quad H_L \in \mathbf{a}_L,$$

is a point in $\mathbf{a}_M$ such that

$$\|H_M^L\| \leqslant \delta_M d(\mathscr{Y}),$$

then H_M belongs to $S_M(\mathscr{Y})$ if and only if H_L belongs to $S_L(\mathscr{Y}_L)$. □

Another example of a positive orthogonal set is provided by the Weyl orbit of a point. Let $M_0 \subset M$ be a fixed Levi component of some minimal parabolic subgroup over F, and let W_0 be the Weyl group of (G, A_{M_0}). Our metric on $\mathbf{a}_M$ is understood to be the restriction of a Euclidean metric $\|\cdot\|$ on $\mathbf{a}_{M_0}$ which is invariant under W_0. Choose an element $P'_0 \in \mathscr{P}(M_0)$, and let $T_{P'_0}$ be a point in $\mathbf{a}_{M_0}$ which lies in the chamber associated to P'_0. The Weyl group W_0 acts simply transitively on $\mathscr{P}(M_0)$, and

$$\mathscr{T} = \{T_{P_0} = sT_{P'_0} : P_0 = sP'_0,\ s \in W_0\} \tag{5}$$

is a positive orthogonal set for M_0. By the discussion above (with (M, L) replaced by (M_0, M)), we obtain a positive orthogonal set

$$\mathscr{T}_M = \{T_P : P \in \mathscr{P}(M)\}$$

for M, and it is not hard to show that

$$d(\mathscr{T}_M) \geqslant d(\mathscr{T}),$$

provided that M is not equal to G. We are of course free to vary the original point $T_{P'_0}$. In future we shall want to choose $T_{P'_0}$ so that the number $d(\mathscr{T})$ is large, and of an order of magnitude comparable to the norm

$$\|\mathscr{T}\| = \|T_{P'_0}\|.$$

We shall actually work with a combination of the two examples. For a given

$x \in G(F)$, set

$$\mathscr{Y}(x, \mathscr{T}) = \{Y_P(x, \mathscr{T}) = T_P - H_{\bar{P}}(x) \colon P \in \mathscr{P}(M)\}, \tag{6}$$

where $\bar{P}$ denotes the parabolic subgroup opposite to P. Because it is a difference of positive orthogonal sets, rather than a sum, $\mathscr{Y}(x, \mathscr{T})$ need not be a positive orthogonal set. However, if $d(\mathscr{T})$ is large with respect to x, the positivity of $\mathscr{T}$ dominates, and $\mathscr{Y}(x, \mathscr{T})$ becomes a positive orthogonal set. We shall assume this in what follows.

3. The main geometric lemma

We shall now begin the proof of the theorem. Suppose that $\mathscr{T}$ is defined by (5). Let $u(x, \mathscr{T})$ denote the characteristic function in $A_G(F)\backslash G(F)$ of the set of points

$$x = k_1 h k_2, \quad k_1, k_2 \in K, \quad h \in A_G(F)\backslash A_{M_0}(F),$$

such that the projection onto $\mathbf{a}_{M_0}^G$ of $H_{M_0}(h)$ lies in the convex hull $S_{M_0}(\mathscr{T})$. Since K corresponds to a special vertex,

$$G(F) = K A_{M_0}(F) K.$$

We can consequently force $u(x, \mathscr{T})$ to be identically equal to 1 on any given compact subset of $A_G(F)\backslash G(F)$ simply by choosing $\mathscr{T}$ so that $d(\mathscr{T})$ is sufficiently large.

Our starting point for the study of the matrix coefficient f is a simple consequence of results of Harish-Chandra.

Lemma 2: Suppose that f and γ are as in the theorem. Then

$$\Theta_\pi(f)\Theta_\pi(\gamma) = \int_{A_G(F)\backslash G(F)} f(x^{-1}\gamma x) u(x, \mathscr{T}) \mathrm{d}x,$$

whenever $d(\mathscr{T})$ is sufficiently large.

Proof: If g is any function in $C_c^\infty(G(F))$, Theorem 9 of [5] tells us that

$$\Theta_\pi(f)\Theta_\pi(g) = \int_{A_G(F)\backslash G(F)} \left(\int_{G(F)} f(x^{-1}yx) g(y) \mathrm{d}y \right) \mathrm{d}x. \tag{7}$$

Assume that

$$g(y) = \mathrm{vol}(K)^{-1} \int_K g_0(kyk^{-1}) \mathrm{d}k,$$

where g_0 is supported on a small neighbourhood Ω of γ. The right hand side of (7) can then be written

$$\mathrm{vol}(K)^{-1} \int_{A_G(F)\backslash G(F)} \left(\int_{G(F)} \int_K f(x^{-1}k^{-1}ykx) g_0(y) \mathrm{d}k \mathrm{d}y \right) \mathrm{d}x.$$

It is a straightforward consequence of [5, Lemma 13] that the integrand in x is supported on a compact subset of $A_G(F)\backslash G(F)$ which depends only on Ω. Now let g_0 approach the Dirac measure at γ. The left hand side of (7) approaches

$$\Theta_\pi(f)\Theta_\pi(\gamma),$$

while the right hand side converges to

$$\mathrm{vol}(K)^{-1}\int_{A_G(F)\backslash G(F)}\left(\int_K f(x^{-1}k^{-1}\gamma kx)\mathrm{d}k\right)\mathrm{d}x.$$

This last integrand in x is compactly supported. We can therefore multiply it with $u(x,\mathcal{T})$ without changing its value, as long as $d(\mathcal{T})$ is sufficiently large. The expression becomes

$$\mathrm{vol}(K)^{-1}\int_{A_G(F)\backslash G(F)}\left(\int_K f(x^{-1}k^{-1}\gamma kx)\mathrm{d}k\right)u(x,\mathcal{T})\mathrm{d}x$$

$$=\mathrm{vol}(K)^{-1}\int_{A_G(F)\backslash G(F)}\int_K f(x^{-1}k^{-1}\gamma kx)u(x,\mathcal{T})\mathrm{d}k\mathrm{d}x$$

$$=\int_{A_G(F)\backslash G(F)} f(x^{-1}\gamma x)u(x,\mathcal{T})\mathrm{d}x,$$

since $u(x,\mathcal{T})$ is bi-invariant under K. This establishes the required formula. □

In view of the lemma, we may write

$$\Theta_\pi(f)\Theta_\pi(\gamma)=\int_{A_G(F)\backslash G(F)} f(x^{-1}\gamma x)u(x,\mathcal{T})\mathrm{d}x$$

$$=\int_{A_M(F)\backslash G(F)} f(x^{-1}\gamma x)\left(\int_{A_M(F)/A_G(F)} u(ax,\mathcal{T})\mathrm{d}a\right)\mathrm{d}x.$$

By assumption, the centralizer of γ in $G(F)$ is compact modulo $A_M(F)$. Therefore, the last integral over x may be taken over a compact set of representatives of $A_M(F)\backslash G(F)$ in $G(F)$.

Our task then is to evaluate the integral

$$\int_{A_M(F)/A_G(F)} u(ax,\mathcal{T})\mathrm{d}a.$$

The main step is to express the integral in terms of the set $\mathcal{Y}(x,\mathcal{T})$ given by (6).

Lemma 3: For any compact subset Γ of $G(F)$ and any $\delta>0$, there is a positive constant $c(\Gamma,\delta)$ with the following property. If x belongs to Γ, a belongs to $A_M(F)$, and $\mathcal{T}$ is such that

$$d(\mathcal{T})\geqslant\delta\,\|\mathcal{T}\|\geqslant c(\Gamma,\delta), \tag{8}$$

then $u(ax,\mathcal{T})$ equals 1 if and only if $H_M(a)$ belongs to $S_M(\mathcal{Y}(x,\mathcal{T}))$.

Proof: If Q is a group in $\mathscr{F}(M)$, we write τ_Q for the characteristic function of

$$\{H \in \mathbf{a}_{M_0} : \alpha(H) > 0,\ \alpha \in \Delta_Q\}.$$

It is known that

$$\sum_{Q \in \mathscr{F}(M)} \sigma_M^Q(H, \mathscr{T})\tau_Q(H - T_Q) = 1, \tag{9}$$

for $\mathscr{T}$ as in (5), and any $H \in \mathbf{a}_M$. This is a general property of positive orthogonal sets which is easily deduced, for example, from Langlands' combinatorial lemma ([1], Lemma 2.3),([2], Lemma 6.3). We shall actually apply the result with $H = H_M(a)$, and $\mathscr{T}$ replaced by the set

$$\varepsilon\mathscr{T} = \{\varepsilon T_{P_0} : P_0 \in \mathscr{P}(M_0)\},$$

for a certain $\varepsilon > 0$. Having been given δ, we choose ε so that $2\varepsilon\delta^{-1}$ is smaller than the numbers δ_M and δ_{M_0} provided by Lemma 1.

Fix the elements $a \in A_M(F)$ and $x \in \Gamma$. The left hand side of (9) is a sum of characteristic functions, so there is a unique group $Q \in \mathscr{F}(M)$ such that

$$\sigma_M^Q(H_M(a), \varepsilon\mathscr{T})\tau_Q(H_M(a) - \varepsilon T_Q) = 1.$$

Once Q is determined, we can write

$$\begin{aligned} ax &= am_{\bar{Q}}(x)n_{\bar{Q}}(x)k_{\bar{Q}}(x) \\ &= ad(am_{\bar{Q}}(x))n_{\bar{Q}}(x)\cdot am_{\bar{Q}}(x)k_{\bar{Q}}(x). \end{aligned}$$

Consider a root α of (Q, A_Q). Since $H_M(a)$ is the sum of a vector in $\mathbf{a}_Q^+$, the positive chamber of Q, with a convex linear combination of points

$$\{\varepsilon T_P : P \in \mathscr{P}(M),\ P \subset Q\},$$

we have

$$\alpha(H_M(a)) \geqslant \varepsilon \inf_{\{P : P \subset Q\}} \alpha(T_P) \geqslant \varepsilon d(\mathscr{T}).$$

Having fixed ε, we choose $c(\Gamma, \delta)$ so that $\varepsilon c(\Gamma, \delta)$ is large. Then $\varepsilon d(\mathscr{T})$ will be large whenever $\mathscr{T}$ satisfies (8), and $ad(a)$ will act by contraction on $n_{\bar{Q}}(x)$. In particular, we can force the point

$$ad(am_{\bar{Q}}(x))n_{\bar{Q}}(x)$$

to be close to 1, uniformly for x in Γ. We may therefore assume that the point lies in the open compact subgroup K. Consequently, ax belongs to the double coset

$$Kam_{\bar{Q}}(x)K.$$

The next step is to write

$$am_{\bar{Q}}(x) = k_1hk_2, \quad h \in A_{M_0}(F), k_1, k_2 \in K \cap M_Q(F). \tag{10}$$

Then ax belongs to KhK. Observe also that

$$H_Q(h) = H_Q(a) + H_Q(m_{\bar{Q}}(x)),$$

so that

$$H_Q(h) = H_Q(a) + H_{\bar{Q}}(x). \tag{11}$$

We write

$$H_M(a) = H_M^Q(a) + H_Q(a), \quad H_M^Q(a) \in \mathbf{a}_M^Q,$$

for the decomposition of $H_M(a)$ relative to the direct sum $\mathbf{a}_M = \mathbf{a}_M^Q \oplus \mathbf{a}_Q$. Similarly

$$H_{M_0}(h) = H_{M_0}^Q(h) + H_Q(h), \quad H_{M_0}^Q(h) \in \mathbf{a}_{M_0}^Q.$$

Then there is a constant $c(\Gamma)$ such that

$$\| H_{M_0}^Q(h) \| \leqslant \| H_M^Q(a) \| + c(\Gamma),$$

for any $x \in \Gamma$ and $a \in A_M(F)$, and for h defined by (10). This follows easily from the standard properties of height functions on $G(F)$. Now, we are assuming that

$$\sigma_M^Q(H_M^Q(a), \varepsilon\mathscr{T}) = 1,$$

so that $H_M^Q(a)$ belongs to the convex set $S_M^Q(\varepsilon\mathscr{T})$. It follows that $\| H_M^Q(a) \|$ is bounded by the norm of the projection of any of the vectors

$$\{\varepsilon T_P : P \in \mathscr{P}(M), P \subset Q\}$$

onto $\mathbf{a}_M^Q$. Therefore,

$$\| H_M^Q(a) \| \leqslant \varepsilon \| T_P \| \leqslant \varepsilon \| \mathscr{T} \|. \tag{12}$$

Choose $c(\Gamma, \delta)$ to be so large that $\varepsilon\delta^{-1}c(\Gamma, \delta)$ is greater than the constant $c(\Gamma)$ above. Then

$$\| H_{M_0}^Q(h) \| \leqslant 2\varepsilon\delta^{-1} d(\mathscr{T}) \leqslant \delta_{M_0} d(\mathscr{T})$$

whenever $\mathscr{T}$ satisfies (8). Recall that the function

$$u(ax, \mathscr{T}) = u(h, \mathscr{T})$$

equals 1 if and only if $H_{M_0}(h)$ belongs to $S_{M_0}(\mathscr{T})$. It follows from Lemma 1 that $u(ax, \mathscr{T})$ equals 1 if and only if $H_Q(h)$ belongs to $S_{M_Q}(\mathscr{T})$.

We are also assuming that

$$\tau_Q(H_Q(a) - \varepsilon T_Q) = \tau_Q(H_M(a) - \varepsilon T_Q) = 1.$$

In particular, $H_Q(a)$ lies in the positive chamber $\mathbf{a}_Q^+$. More precisely,

$$\alpha(H_Q(a)) \geqslant \varepsilon\alpha(T_Q) \geqslant \varepsilon d(\mathscr{T}),$$

for any root $\alpha \in \Delta_Q$. We can make this number as large as we wish, for $\mathscr{T}$ satisfying (8), simply by taking $c(\Gamma, \delta)$ large enough. Now $H_Q(a)$ is related to $H_Q(h)$ by equation (11). Since $H_{\bar{Q}}(x)$ remains bounded, we can assume that $H_Q(h)$ also lies in $\mathbf{a}_Q^+$. But according to ([1] Lemma 3.2), the intersection of $\mathbf{a}_Q^+$ with $S_{M_Q}(\mathcal{T})$ is the set

$$\{H \in \mathbf{a}_Q^+ : \varpi(H - T_Q) < 0, \varpi \in \hat{\Delta}_Q\},$$

where $\hat{\Delta}_Q$ is the dual basis of $\Delta_Q^\vee$. Thus, $u(ax, \mathcal{T})$ equals 1 if and only if each of the numbers

$$\varpi(H_Q(h) - T_Q) = \varpi(H_Q(a) - Y_Q(x, \mathcal{T})), \quad \varpi \in \hat{\Delta}_Q,$$

is negative. We have now only to retrace our steps. Since $H_Q(a)$ lies in $\mathbf{a}_Q^+$, the last condition is equivalent to the assertion that $H_Q(a)$ lies in $S_{M_Q}(\mathcal{Y}(x, \mathcal{T}))$. Moreover, $d(\mathcal{T})$ is large relative to x, so we can assume that

$$d(\mathcal{Y}(x, \mathcal{T})) \geq \tfrac{1}{2} d(\mathcal{T}).$$

It follows from (12) that

$$\begin{aligned} \| H_M^Q(a) \| &\leq \varepsilon \| \mathcal{T} \| \\ &\leq \varepsilon \delta^{-1} d(\mathcal{T}) \\ &\leq 2\varepsilon\delta^{-1} d(\mathcal{Y}(x, \mathcal{T})) \\ &\leq \delta_M d(\mathcal{Y}(x, \mathcal{T})), \end{aligned}$$

whenever $\mathcal{T}$ satisfies (8). Applying Lemma 1 again, we conclude that $H_Q(a)$ belongs to $S_{M_Q}(\mathcal{Y}(x, \mathcal{T}))$ if and only if $H_M(a)$ belongs to $S_M(\mathcal{Y}(x, \mathcal{T}))$. This is equivalent to the original condition that $u(ax, \mathcal{T})$ equals 1, so the proof of the lemma is complete. □

As an identity of characteristic functions, the lemma asserts that

$$u(ax, \mathcal{T}) = \sigma_M(H_M(a), \mathcal{Y}(x, \mathcal{T})), \quad \alpha \in A_M(F)/A_G(F),$$

for x and $\mathcal{T}$ as stated. It follows that $\Theta_\pi(f)\Theta_\pi(\gamma)$ equals

$$\int_{A_M(F)\backslash G(F)} f(x^{-1}\gamma x) \left(\int_{A_M(F)/A_G(F)} \sigma_M(H_M(a), \mathcal{Y}(x, \mathcal{T})) da \right) dx.$$

However, the integral

$$\int_{A_M(F)/A_G(F)} \sigma_M(H_M(a), \mathcal{Y}(x, \mathcal{T})) da$$

is not equal to the volume of $S_M(\mathcal{Y}(x, \mathcal{T}))$. For

$$\{H_M(a) : a \in A_M(F)/A_G(F)\}$$

is a lattice in $\mathbf{a}_M/\mathbf{a}_G$; the integral is multiple of the number of lattice points in $S_M(\mathcal{Y}(x, \mathcal{T}))$. We must find a way to relate this to the volume.

It will actually be convenient to replace $A_M(F)$ by a subgroup. Suppose that A'_M

is any subgroup of finite index in $A_M(F)$, which contains $A_G(F)$. Combining Lemmas 2 and 3 as above, we obtain

$$\Theta_\pi(f)\Theta_\pi(\gamma)=\int_{A'_M\backslash G(F)} f(x^{-1}\gamma x)\left(\int_{A'_M/A_G(F)}\sigma_M(H_M(a),\mathscr{Y}(x,\mathscr{T}))\mathrm{d}a\right)\mathrm{d}x, \quad (13)$$

a formula which holds whenever $\mathscr{T}$ satisfies the conditions (8).

4. Counting lattice points

For each reduced root β of (G, A_{M_0}), we have the co-root $\beta^\vee$. Any such $\beta^\vee$ defines an element in the lattice

$$X_*(A_{M_0})=\mathrm{Hom}(X(A_{M_0}),\mathbb{Z})$$

in $\mathbf{a}_{M_0}$. Suppose that $P\in\mathscr{P}(M)$ and that α is a root in Δ_P. For any given $P_0\in\mathscr{P}(M_0)$, with $P_0\subset P$, there is a unique root $\beta\in\Delta_{P_0}$ whose restriction to A_M equals α; the "co-root" $\alpha^\vee\in\Delta_P^\vee$ is, by definition, the projection of $\beta^\vee$ onto $\mathbf{a}_M$. The lattice $\mathbb{Z}(\Delta_P^\vee)$ in $\mathbf{a}_M^G$, generated by $\Delta_P^\vee$, is the projection of $\mathbb{Z}(\Delta_{P_0}^\vee)$ onto $\mathbf{a}_M^G$. Since $\mathbb{Z}(\Delta_{P_0}^\vee)$ is independent of P_0, $\mathbb{Z}(\Delta_P^\vee)$ is independent of P. The lattice $\mathbb{Z}(\Delta_P^\vee)$ need not be contained in

$$X_*(A_M)=\mathrm{Hom}(X(A_M),\mathbb{Z}).$$

However, it is easily seen to be a subgroup of

$$\mathrm{Hom}(X(M)_F,\mathbb{Z}),$$

which is in turn a finite extension of $X_*(A_M)$. Consequently, there is an integer k such that $k\mathbb{Z}(\Delta_P^\vee)$ is a subgroup of $X_*(A_M)$.

Recall that

$$\exp(\langle H_M(m),\chi\rangle)=|\chi(m)|,$$

for any $\chi\in X(M)_F$ and $m\in M(F)$. It follows easily that $H_M(A_M(F))$ equals the lattice

$$\log(q_F)X_*(A_M)$$

in $\mathbf{a}_M$, where q_F is the degree of the residue field of F. Define

$$\Lambda_{M,k}=k\log(q_F)\mathbb{Z}(\Delta_P^\vee)=\log(q_F^k)\mathbb{Z}(\Delta_P^\vee)$$

for any $P\in\mathscr{P}(M)$ and any positive integer k. For any such P, the vectors

$$\mu_{\alpha,k}=k\log(q_F)\alpha^\vee,\quad \alpha\in\Delta_P,$$

form a $\mathbb{Z}$-basis of $\Lambda_{M,k}$. We fix k so that $\Lambda_{M,k}$ is contained in $H_M(A_M(F))$. Set

$$A_{M,k}=\{a\in A_M(F): H_M(a)\in\Lambda_{M,k}\}.$$

Then

$$A'_{M,k}=A_{M,k}A_G(F)$$

is a subgroup of finite index in $A_M(F)$; it is this group which we will employ in the formula (13). The first step will be to calculate the integral

$$\int_{A'_{M,k}/A_G(F)} \sigma_M(H_M(a), \mathscr{Y}(x, \mathscr{T}))da. \tag{14}$$

The kernel of H_M in $A'_{M,k}$ equals the group

$$\kappa_M = A_M(F) \cap K.$$

It follows easily that the quotient of $A'_{M,k}/A_G(F)$ by κ_M/κ_G is isomorphic under H_M to $\Lambda_{M,k}$. We can therefore write (14) as the product of the volume of κ_M/κ_G with the number of points in the intersection of $\Lambda_{M,k}$ with $S_M(\mathscr{Y}(x, \mathscr{T}))$. Consequently, (14) may be rewritten as

$$\operatorname{vol}(\kappa_M/\kappa_G) \lim_{\lambda \to 0} \left\{ \sum_{\xi} e^{\lambda(\xi)} \right\}, \tag{15}$$

the sum being taken over ξ in $\Lambda_{M,k} \cap S_M(\mathscr{Y}(x, \mathscr{T}))$. We shall calculate this by the method in ([1], § 3).

Take λ to be a point in $\mathbf{a}^*_{M,\mathbb{C}}$ whose real part $\lambda_{\mathbb{R}} \in \mathbf{a}^*_M$ is regular. If $P \in \mathscr{P}(M)$, we shall write

$$\Delta_P^\lambda = \{\alpha \in \Delta_P : \lambda_{\mathbb{R}}(\alpha^\vee) < 0\}.$$

Let ϕ_P^λ denote the characteristic function of the set of $H \in \mathbf{a}_M$ such that $\varpi_\alpha(H) > 0$ for each $\alpha \in \Delta_P^\lambda$, and $\varpi_\alpha(H) \leqslant 0$ for any α in the complement of Δ_P^λ in Δ_P. (Recall that

$$\hat{\Delta}_P = \{\varpi_\alpha : \alpha \in \Delta_P\}$$

is the basis of $(\mathbf{a}_M^G)^*$ which is dual to $\{\alpha^\vee : \alpha \in \Delta_P\}$.) It follows easily from Langlands' combinatorial lemma that

$$\sum_{P \in \mathscr{P}(M)} (-1)^{|\Delta_P^\lambda|} \phi_P^\lambda(H - Y_P(x, \mathscr{T})), \quad H \in \mathbf{a}_M,$$

equals the characteristic function of $S_M(\mathscr{Y}(x, \mathscr{T}))$. (See Lemma 3.2 of [1] for the special case that H lies in the complement of a finite set of hyperplanes. The general case follows in the same way from [2], Lemma 6.3.) Therefore, the expression in the brackets in (15) equals

$$\sum_{\xi \in \Lambda_{M,k}} (-1)^{|\Delta_P^\lambda|} \phi_P^\lambda(\xi - Y_P(x, \mathscr{T})) \exp(\lambda(\xi)). \tag{16}$$

We shall write Y_P^λ for the extreme point in

$$\{\xi \in \Lambda_{M,k} : \phi_P^\lambda(\xi - Y_P(x, \mathscr{T})) = 1\}. \tag{17}$$

That is,

$$Y_P^\lambda = Y_P(x, \mathscr{T}) + \sum_{\alpha \in \Delta_P^\lambda} t_\alpha \mu_{\alpha,k} - \sum_{\alpha \in \Delta_P - \Delta_P^\lambda} (1 - t_\alpha)\mu_{\alpha,k},$$

for positive numbers t_α, with $0 < t_\alpha \leqslant 1$. The set (17) can then be written as

$$\left\{ Y_P^\lambda + \sum_{\alpha \in \Delta_P^\lambda} n_\alpha \mu_{\alpha,k} - \sum_{\alpha \in \Delta_P - \Delta_P^\lambda} n_\alpha \mu_{\alpha,k} \right\},$$

where each n_α ranges over all positive integers. Expression (16) becomes a multiple geometric series, which equals

$$(-1)^{|\Delta_P^\lambda|} \exp[\lambda(Y_P^\lambda)] \prod_{\alpha \in \Delta_P^\lambda} (1 - \exp[\lambda(\mu_{\alpha,k})])^{-1} \prod_{\alpha \in \Delta_P - \Delta_P^\lambda} (1 - \exp[-\lambda(\mu_{\alpha,k})])^{-1}.$$

If $\lambda_{\mathbf{R}}$ belongs to the negative chamber $-(\mathbf{a}_P^*)^+$ of P in $\mathbf{a}_M^*$, we shall denote Y_P^λ simply by

$$Y_P^+ = Y_P(x, \mathscr{T})^+ = (T_P - H_{\bar{P}}(x))^+.$$

Then for general λ,

$$Y_P^+ = Y_P^\lambda + \sum_{\alpha \in \Delta_P - \Delta_P^\lambda} \mu_{\alpha,k}.$$

Expression (16) may therefore be written as

$$\exp[\lambda(Y_P^+)] \prod_{\alpha \in \Delta_P} (\exp[\lambda(\mu_{\alpha,k})] - 1)^{-1}.$$

We have shown that (14) equals

$$\mathrm{vol}(\kappa_M/\kappa_G) \lim_{\lambda \to 0} \sum_{P \in \mathscr{P}(M)} (\exp[\lambda(Y_P^+)] \prod_{\alpha \in \Delta_P} (\exp[\lambda(\mu_{\alpha,k})] - 1)^{-1}).$$

Let us rewrite this last formula for (14) as

$$\mathrm{vol}(\kappa_M/\kappa_G) \lim_{\lambda \to 0} \sum_{P \in \mathscr{P}(M)} c_P(\lambda, x, \mathscr{T}) d_P(\lambda) \theta_P(\lambda)^{-1},$$

where

$$c_P(\lambda, x, \mathscr{T}) = \exp[\lambda(Y_P^+)] = \exp[\lambda((T_P - H_{\bar{P}}(x))^+)], \tag{18}$$

and

$$d_P(\lambda) = \theta_P(\lambda) \prod_{\alpha \in \Delta_P} (\exp[\lambda(\mu_{\alpha,k})] - 1)^{-1}.$$

We leave the reader to check that

$$\{Y_P^+ : P \in \mathscr{P}(M)\}$$

is a positive orthogonal set for M. This implies that $\{c_P(\lambda, x, \mathscr{T})\}$ is a (G, M) family, in the language of ([3], § 6). Moreover, $\{d_P(\lambda)\}$ is also a (G, M) family. Applying ([3], Lemma 6.3) to the product of (G, M) families in the expression above, we see that (14) equals

$$\mathrm{vol}(\kappa_M/\kappa_G) \sum_{Q \in \mathscr{F}(M)} c_M^Q(x, \mathscr{T}) d'_Q.$$

This follows the notation of ([3], § 6). In particular,

$$c_M^Q(x, \mathscr{T}) = \lim_{\lambda \to 0} \sum_{\{P \in \mathscr{P}(M): P \subset Q\}} \exp[\lambda((T_P - H_{\bar{P}}(x))^+)]\theta_P(\lambda)^{-1}.$$

Next, we substitute the formula we have just established for (14) into the identity (13). We see that $\Theta_\pi(f)\Theta_\pi(\gamma)$ equals

$$\mathrm{vol}(\kappa_M/\kappa_G) \sum_{Q \in \mathscr{F}(M)} d'_Q \int_{A'_{M,k}\backslash G(F)} f(x^{-1}\gamma x) c_M^Q(x, \mathscr{T}) \mathrm{d}x.$$

For any group $Q \in \mathscr{F}(M)$ we have

$$c_M^Q(x, \mathscr{T}) = c_M^Q(m_{\bar{Q}}(x), \mathscr{T}).$$

It follows easily from this fact that

$$\int_{A'_{M,k}\backslash G(F)} f(x^{-1}\gamma x) c_M^Q(x, \mathscr{T}) \mathrm{d}x$$

is a multiple of

$$\int_K \int_{N_{\bar{Q}}(F)} \int_{A'_{M,k}\backslash M_{\bar{Q}}(F)} f(k^{-1}m^{-1}\gamma mnk) c_M^Q(m, \mathscr{T}) \mathrm{d}m \, \mathrm{d}n \, \mathrm{d}k.$$

Since f is a supercusp form, this expression vanishes for any $Q \neq G$. Consequently,

$$\Theta_\pi(f)\Theta_\pi(\gamma) = \mathrm{vol}(\kappa_M/\kappa_G) d'_G \int_{A'_{M,k}\backslash G(F)} f(x^{-1}\gamma x) c_M(x, \mathscr{T}) \mathrm{d}x.$$

Now, by definition,

$$d'_G = d'_G(0) = \lim_{\lambda \to 0} d_P(\lambda),$$

for any $P \in \mathscr{P}(M)$. Therefore

$$\begin{aligned} d'_G &= \mathrm{vol}(\mathbf{a}_M^G/\mathbb{Z}(\Delta_P^\vee))^{-1} \lim_{\lambda \to 0} \prod_{\alpha \in \Delta_P} (\lambda(\alpha^\vee)(\exp[\lambda(\mu_{\alpha,k})] - 1)^{-1}) \\ &= \mathrm{vol}(\mathbf{a}_M^G/\mathbb{Z}(\Delta_P^\vee))^{-1} \prod_{\alpha \in \Delta_P} (\lambda(\alpha^\vee)\lambda(\mu_{\alpha,k})^{-1}) \\ &= \mathrm{vol}(\mathbf{a}_M^G/\Lambda_{M,k})^{-1}. \end{aligned}$$

On the other hand, it follows from (2) that

$$\begin{aligned} \mathrm{vol}(\kappa_M/\kappa_G) &= \mathrm{vol}(\mathbf{a}_M/H_M(A_M(F)) + \mathbf{a}_G) \\ &= \mathrm{vol}(\mathbf{a}_M/\Lambda_{M,k} + \mathbf{a}_G)|A_M(F)/A'_{M,k}|^{-1} \\ &= \mathrm{vol}(\mathbf{a}_M^G/\Lambda_{M,k})|A_M(F)/A'_{M,k}|^{-1}, \end{aligned}$$

since the map

$$A_M(F)/A'_{M,k} \to (H_M(A_M(F)) + \mathbf{a}_G)/(\Lambda_{M,k} + \mathbf{a}_G)$$

is an isomorphism. Our formula becomes

$$\Theta_\pi(f)\Theta_\pi(\gamma) = |A_M(F)/A'_{M,k}|^{-1} \int_{A'_{M,k}\backslash G(F)} f(x^{-1}\gamma x)c_M(x,\mathscr{T})\mathrm{d}x. \tag{19}$$

It is valid whenever $\mathscr{T}$ satisfies the conditions (8)

5. Completion of the proof

The formula (19) is close to that of the theorem. The only problem is that it depends on $(T_P - H_{\bar{P}}(x))^+$, rather than the vector $T_P - H_{\bar{P}}(x)$. To overcome this, we shall average $\mathscr{T}$ over a certain compact domain.

Observe that $\Lambda_{M,k}$ is the projection onto $\mathbf{a}_M^G$ of the lattice

$$\Lambda_{M_0,k} = k\log(q_F)\mathbb{Z}(\Delta_{P_0}^\vee), \quad P_0 \in \mathscr{P}(M_0),$$

in $\mathbf{a}_{M_0}^G$. Choose an element P'_0 in $\mathscr{P}(M_0)$, and let $\mathscr{D}$ denote the compact fundamental domain

$$\left\{u = \sum_{\beta\in\Delta_{P'_0}} u_\beta \mu_{\beta,k} \cdot 0 \leqslant u_\beta \leqslant 1\right\}$$

for $\Lambda_{M_0,k}$ in $\mathbf{a}_{M_0}^G$. (Recall that $\{\mu_{\beta,k}\}$ is a basis of $\Lambda_{M_0,k}$ consisting of positive multiples of the co-roots $\Delta_{P'_0}^\vee$). Suppose that $P\in\mathscr{P}(M)$. Then there is an element $s\in W_0$ such that $P_0 = sP'_0$ contains P. For each $\alpha\in\Delta_P$, let $\beta(\alpha)$ be the unique root in $\Delta_{P'_0}$ such that the restriction of $s\beta(\alpha)$ onto $\mathbf{a}_M$ equals α. Then $\mu_{\alpha,k}$ is the projection of $s(\mu_{\beta(\alpha),k})$ onto $\mathbf{a}_M$. Given a vector $u\in\mathscr{D}$ as above, set

$$u_P = \sum_{\alpha\in\Delta_P} u_{\beta(\alpha)}\mu_{\alpha,k}.$$

This notation of course holds if M_0 is used instead of M, and the set

$$\mathscr{T}_u = \{T_{P_0} - u_{P_0} : P_0 \in \mathscr{P}(M_0)\}$$

satisfies similar conditions to $\mathscr{T}$. We may therefore replace $c_M(x,\mathscr{T})$ by $c_M(x,\mathscr{T}_u)$ on the right hand side of (19).

Observe that

$$c_P(\lambda, x, \mathscr{T}_u) = \exp[\lambda((T_P - u_P - H_{\bar{P}}(x))^+)], \quad P\in\mathscr{P}(M).$$

Define

$$c_P(\lambda, x, \mathscr{T}, u) = \exp[\lambda((T_P - u_P - H_{\bar{P}}(x))^+ + u_P)], \quad P\in\mathscr{P}(M),$$

so that

$$c_P(\lambda, x, \mathscr{T}_u) = c_P(\lambda, x, \mathscr{T}, u)\exp[-\lambda(u_P)].$$

This is a product of two (G, M) families. We can therefore apply Lemma 6.3 of [3] to decompose $c_M(x, \mathcal{T}_u)$ into a sum over $Q \in \mathcal{F}(M)$. The second (G, M) family is independent of x. By arguing as in § 4, we see that the contribution of any $Q \neq G$ to the integral

$$\int_{A'_{M,k}\backslash G(F)} f(x^{-1}\gamma x) c_M(x, \mathcal{T}_u) \mathrm{d}x$$

vanishes. We may therefore replace $c_M(x, \mathcal{T}_u)$ by $c_M(x, \mathcal{T}, u)$, the term corresponding to $Q = G$. Since this is valid for any $u \in \mathcal{D}$, we may integrate over $\mathcal{D}$ if we choose. It follows that (19) remains valid if the function $c_M(x, \mathcal{T})$ is replaced by

$$\int_{\mathcal{D}} c_M(x, \mathcal{T}, u) \mathrm{d}u.$$

Now,

$$\int_{\mathcal{D}} c_M(x, \mathcal{T}, u) \mathrm{d}u$$

$$= \int_{\mathcal{D}} \lim_{\lambda \to 0} \sum_{P \in \mathcal{P}(M)} (c_P(\lambda, x, \mathcal{T}, u)\, \theta_P(\lambda)^{-1}) \mathrm{d}u$$

$$= \lim_{\lambda \to 0} \sum_{P \in \mathcal{P}(M)} \left(\left(\int_{\mathcal{D}} c_P(\lambda, x, \mathcal{T}, u) \mathrm{d}u \right) \theta_P(\lambda)^{-1} \right).$$

Thus, we have only to compute

$$\int_{\mathcal{D}} E((Y_P - u_P)^+ + u_P) \mathrm{d}u, \tag{20}$$

where

$$E((Y_P - u_P)^+ + u_P) = \exp[\lambda((Y_P - u_P)^+ + u_P)],$$

with

$$Y_P = Y_P(x, \mathcal{T}) = T_P - H_{\bar{P}}(x).$$

This integral can be written as a multiple integral, over the cube

$$\left\{ \prod_{\alpha \in \Delta_P} r_\alpha : 0 \leqslant r_\alpha \leqslant 1 \right\},$$

of the function

$$E\left(\left(Y_P - \sum_\alpha r_\alpha \mu_{\alpha,k} \right)^+ + \sum_\alpha r_\alpha \mu_{\alpha,k} \right).$$

Recall that Y_P^+ is the unique point in $\Lambda_{M,k} + \mathbf{a}_G$ of the form

$$Y_P + \sum_{\alpha \in \Delta_P} t_\alpha \mu_{\alpha,k},$$

where $0 < t_\alpha \leqslant 1$. Taking the integrals in r_α separately over the intervals $[0, 1 - t_\alpha]$ and $[1 - t_\alpha, 1]$, we can change variables; we obtain the integral over $\{r_\alpha\}$ of

$$E\left(Y_P + \sum_\alpha r_\alpha \mu_{\alpha,k}\right).$$

It follows that (20) equals

$$\int_{\mathscr{D}} E(Y_P + u_P)\mathrm{d}u.$$

We have shown that

$$\int_{\mathscr{D}} c_M(x, \mathscr{T}, u)\mathrm{d}u = \int_{\mathscr{D}} \bar{v}_M(x, \mathscr{T}, u)\mathrm{d}u,$$

where

$$\begin{aligned}\bar{v}_P(\lambda, x, \mathscr{T}, u) &= \exp[T_P + u_P - H_{\bar{P}}(x)]\\ &= \exp[-\lambda(H_{\bar{P}}(x))]\exp[\lambda(T_P + u_P)].\end{aligned}$$

This is again a product of (G, M) families. We apply Lemma 6.3 of [3] once more, and decompose $\bar{v}_M(x, \mathscr{T}, u)$ into a sum over $Q \in \mathscr{F}(M)$. Since the second (G, M) family is independent of x, the contribution of any $Q \neq G$ to the integral

$$\int_{A'_{M,k}\backslash G(F)} f(x^{-1}\gamma x) \int_{\mathscr{D}} \bar{v}_M(x, \mathscr{T}, u)\mathrm{d}u\mathrm{d}x = \int_{\mathscr{D}} \int_{A'_{M,k}\backslash G(F)} f(x^{-1}\gamma x)\bar{v}_M(x, \mathscr{T}, u)\mathrm{d}x\mathrm{d}u$$

vanishes. The term corresponding to $Q = G$ is just $\bar{v}_M(x)$, where

$$\bar{v}_P(\lambda, x) = \exp[-\lambda(H_{\bar{P}}(x))], \quad P \in \mathscr{P}(M).$$

This is of course independent of u, so the integral over $\mathscr{D}$ disappears. The formula (19) becomes

$$\Theta_\pi(f)\Theta_\pi(\gamma) = |A_M(F)/A'_{M,k}|^{-1} \int_{A'_{M,k}\backslash G(F)} f(x^{-1}\gamma x)\bar{v}_M(x)\mathrm{d}x.$$

The (G, M)-family $\{\bar{v}_P(\lambda, x)\}$ is slightly different from the original (G, M) family

$$v_P(\lambda, x) = \exp[-\lambda(H_P(x))], \quad P \in \mathscr{P}(M).$$

Observe, however, that

$$\begin{aligned}\bar{v}_M(x) &= \lim_{\lambda \to 0} \sum_{P \in \mathscr{P}(M)} \exp[-\lambda(H_{\bar{P}}(x))]\theta_P(\lambda)^{-1}\\ &= (-1)^{\dim(A_M/A_G)} \sum_{P \in \mathscr{P}(M)} \exp[-\lambda(H_{\bar{P}}(x))]\theta_{\bar{P}}(\lambda)^{-1}\end{aligned}$$

$$= (-1)^{\dim(A_M/A_G)} \sum_{P \in \mathscr{P}(M)} \exp[-\lambda(H_P(x))]\theta_P(\lambda)^{-1}$$

$$= (-1)^{\dim(A_M/A_G)} v_M(x),$$

since

$$\theta_{\bar{P}}(\lambda) = (-1)^{\dim(A_M/A_G)}\theta_P(\lambda).$$

In other words, $\Theta_\pi(f)\Theta_\pi(\gamma)$ equals

$$|A_M(F)/A'_{M,k}|^{-1}(-1)^{\dim(A_M/A_G)} \int_{A'_{M,k}\backslash G(F)} f(x^{-1}\gamma x)v_M(x)\mathrm{d}x.$$

Now, it is well known that the function $v_M(x)$ is left invariant under $M(F)$. In particular, the integrand is left invariant under $A_M(F)$. We may therefore change the domain of integration to $A_M(F)\backslash G(F)$, if we multiply by the index $|A_M(F)/A'_{M,k}|$. We obtain the identity of $\Theta_\pi(f)\Theta_\pi(\gamma)$ with

$$(-1)^{\dim(A_M/A_G)} \int_{A_M(F)\backslash G(F)} f(x^{-1}\gamma x)v_M(x)\mathrm{d}x.$$

This completes the proof of the theorem. □

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