Orbital L-functions for GL(3)

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Conference for 80^{th} birthday of W. Casselman.

$G = G(n) = GL(n+1)$ over $\mathbb{Q}$

I. FOREWORD

Two kinds of L-functions for G

Spectral: (Standard) Automorphic L-fn's

$L(s,\pi)$, π cusp. aut rep. (Tate $GL(1)$ – 1950, Godement–Jacquet $GL(n+1)$ – 1972)

Geometric: Orbital L-fn's $L(s,R)$,

R an order in field $E/\mathbb{Q}$ of degree $(n+1)$, Z. Yun (2013)

Play parallel roles on 2 sides of trace formula.

Problem: Unlike $L(s,\pi)$, local factors of $L_p(s,R)$ of $L(s,R)$ are not explicit (except for $GL(2)$).

③

They are closely related to p-adic orb. integrals

$$O(\gamma, f_p) = \int_{G_\gamma(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)} f_p(x_p^{-1}\gamma x_p)\, dx_p,$$

$f_p = \mathbb{1}_p$, char. f^n of $G(\mathcal{O}_p)$.

__Our goal__: For $G = GL(3)$, describe __explicit__ formulas for $O(\gamma, f_p)$ + related local factors $L_p(s, R)$

__Our conclusion__: The orbital integrals $Orb(\gamma, f_p)$ have unexpected hidden structure, for $G = GL(3)$, + perhaps $G = GL(n+1)$, and possibly for any (quasisplit) group G.

④

II. TRACE FORMULA (approximation)

Test f^{n}: $f \in C_c^\infty(Z_+ \backslash G(\mathbb{A}))$, $Z_+ = \left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} : r > 0 \right\} \subset G(\mathbb{R}) \subset G(\mathbb{A})$.

$$I_{geom}(f) \text{ (geom exp}^{n}) = I_{spec}(f) \text{ (spectral exp}^{n})$$

$$I_{geom}(f) = I_{ell,reg}(f) + \text{suppl. geom terms}$$

$$I_{spec}(f) = I_{cusp}(f) + \text{supp. spectral terms}$$

$$I_{ell,reg}(f) = \sum_{\gamma \in \Gamma_{ell,reg}(G)} \mathrm{vol}(\gamma)\, O(\gamma, f)$$

$\gamma \in \Gamma_{ell,reg}(G)$: ell. reg. conj classes in $G(\mathbb{Q})$

$\mathrm{vol}(\gamma)$: $\mathrm{vol}\, Z_+ G_\gamma(\mathbb{Q}) \backslash G_\gamma(\mathbb{A})$

$O(\gamma, f)$: $\int_{G_\gamma(\mathbb{A}) \backslash G(\mathbb{A})} f(x^{-1}\gamma x)\, dx$

$$I_{cusp}(f) = \sum_{\pi \in \Pi_{cusp}(G)} \mathrm{mult}(\pi) \cdot \mathrm{tr}(\pi(f))$$

$\pi \in \Pi_{cusp}(G)$: π irred, cuspidal $\subset L^2_{disc}(Z_+ G(\mathbb{Q}) \backslash G(\mathbb{A}))$

$\mathrm{mult}(\pi)$: multiplicity in $L^2_{disc}(\cdot)$

$\mathrm{tr}(\pi(f))$: irred characters

We write

$I_{ell,reg}(f) \sim I_{cusp}(f)$ — "pretend primary geometric & spectral terms are equal"

(5)

III. BEYOND ENDOSCOPY (Langlands: dream / strategy)

Establish Principle of Functoriality by combining trace formula with general automorphic L-functions $L(s, \pi, r)$,

$$r : \hat{G} = GL(n+1, \mathbb{C}) \longrightarrow GL(N, \mathbb{C}), \text{ finite dim rep.}$$

Langlands' proposed refinement of trace formula

Given finite dim. rep r, replace spectral side by

$$I^r_{cusp}(f) = \sum_{\pi} m_\pi(r) \cdot \mathrm{mult}(\pi) \cdot \mathrm{tr}(\pi(f)),$$

where

$$m_\pi(r) = -\operatorname*{res}_{s=1} \frac{d}{ds} \left(\log L(s, \pi, r) \right).$$

Would include (subtle) information about π as a "functorial image".

Fundamental question: Is there a geom. exp^{n} $I^r_{geom}(f)$

$$I^r_{geom}(f) = I^r_{spec}(f) \; ? \quad \sim r\text{-trace formula}$$

Many hard things would have to be solved first.

IV. EARLY PROGRESS

Frenkel, Langlands, Ngo (2010): "Replace $\gamma \in \Gamma_{ell, reg}(G)$ by its char. polynomial.

$$P_\gamma(\lambda) = \det(\lambda I - \gamma) = \lambda^{n+1} - a_1\lambda^n + \dots + (-1)^{n+1} a_{n+1} = P_a(\lambda),$$

$$a = (a_1, \dots, a_n, a_{n+1}) = (b, a_{n+1}) \in \mathbb{Q}^n \times \mathbb{Q}^*.$$

Thus

$$\gamma \in \Gamma_{ell, reg}(G) \longleftrightarrow a \in \mathbb{Q}^n \times \mathbb{Q}^* \ni P_a(\lambda) \text{ irred} / \mathbb{Q}.$$

Simplification: Langlands (2004), Ali Altug (2005).

Set $f = f^\infty f_\infty$, $f_\infty \in C_c^\infty(Z_+ \backslash G(\mathbb{R}))$, $f^\infty = \mathbb{1}^\infty = \prod_p \mathbb{1}_p$,

$\uparrow$ unit f^n on $G(\mathbb{A}^\infty)$

+ consider only those $\gamma \ni$

$$O(\gamma, f) = O(\gamma, f^\infty) = \prod_p O(\gamma, \mathbb{1}_p) \neq 0.$$

Then

$$\gamma \longleftrightarrow a = (b, \varepsilon), \quad b \in \mathbb{Z}^n, \quad \varepsilon = \det(\gamma) = \pm 1,$$

so terms in $I_{ell, reg}(f)$ then corresp. to irred monic poly^s with integral coeff^s + $a_{n+1} = \varepsilon = \pm 1$.

(7) This is a fundamental change of outlook

Philosophically: 2 sides of trace (formula are) indexed by 2 classifications of Galois extensions $K/\mathbb{Q}$ (like abelian class field theory)

$$\left\{\begin{matrix}\text{splitting fields of irred} \\ \text{polynomials}\end{matrix}\right\} \text{- geometric} \longleftrightarrow \left\{\begin{matrix}\text{irred rep}^{\text{s}} \text{ of their Galois} \\ \text{groups}\end{matrix}\right\} \text{- spectral}$$

Suppose $\gamma \longrightarrow a = (b, \varepsilon)$, + $E = E_a = \mathbb{Q}[\lambda]/(p_a(\lambda))$. In

$$I_{ell,reg}(f) = \sum_{\gamma} \mathrm{vol}(\gamma) \cdot O(\gamma, f^{\infty}) \cdot O(\gamma, f_{\infty}),$$

we can then write

$$\textcircled{H}^{\varepsilon}(b, f_\infty) = O(\gamma, f_\infty) \underbrace{|D(\gamma)|^{\frac{1}{2}}}_{\text{Weyl discrim}} = O(\gamma, f_\infty) |\underbrace{D_E}_{\text{disc } E/\mathbb{Q}}|^{\frac{1}{2}} \left(\prod_p p^{\delta_p}\right),$$

and

$$\underbrace{\mathrm{vol}(\gamma)}_{\text{regulator of } E/\mathbb{Q}} = |D_E|^{\frac{1}{2}} \cdot \lim_{s \to 1} \left(\zeta_E(s)/\zeta_{\mathbb{Q}}(s)\right) = |D_E|^{\frac{1}{2}} (\zeta_E/\zeta_{\mathbb{Q}})(1).$$

class number formula for $E/\mathbb{Q}$

We get

$$I_{ell,reg}(f) = \sum_{\varepsilon = \{\pm 1\}} \sum_{b \in \mathbb{Z}^m}{}' \underbrace{(\zeta_E/\zeta_{\mathbb{Q}})(1) \cdot \prod_p \left(O(\gamma, f_p)\, p^{-\delta_p}\right)}_{\text{global coeff}^{\text{s}}} \cdot \underbrace{\textcircled{H}^{\varepsilon}(b, f_\infty)}_{\text{local test fn on } \mathbb{R}^n},$$

where $\sum'$ means sum only over those b $\ni$

$p_a(\lambda) = p_{b,\varepsilon}(\lambda)$ is irred.

V. ON POISSON SUMMATION

Question (FLN): Can we modify this so we can apply Poisson summation formula to the lattice

$\{b \in \mathbb{Z}^n\} \subset \{u \in \mathbb{R}^n\}$?

Answer for $GL(2)$: (Altug). Yes!

(Uses remarkable techniques, using explicit orbital L-$f^{\underline{ns}}$ for $GL(2)$ of Zagier (1976))

Obstructions to Poisson summation: (Solved by Altug for $GL(2)$, open in general)

(i) Function: $\Theta^{\varepsilon}(b, f_\infty)$: Does not extend to smooth $f^{\underline{n}}$ of $u \in \mathbb{R}^n$ – singular hyperplanes

Sol$^{\underline{n}}$: Multiply it by small power ~~of~~ $|D(\gamma)|^{\alpha}$, $\alpha > 0$

(ii) Coeff^{s}: $\varepsilon(b) = (\zeta_E / \zeta_{\mathbb{Q}})(1) \cdot \prod_p \left(\mathrm{Orb}(\gamma, \mathbb{1}_p)\, p^{-\delta_p}\right)$

$\gamma \longleftrightarrow a = (b, \varepsilon)$. Could use approx. $f^{\underline{n}}$ eq$^{\underline{tn}}$ for Dirichlet L-$f^{\underline{n}}$:

$$L(s, E) = \zeta_E(s) / \zeta_{\mathbb{Q}}(s)$$

to express value at $s=1$ as rest$^{\underline{n}}$ to $\{b\}$ of a smooth $f^{\underline{n}}$ ~~of~~ $u \in \mathbb{R}^n$. ~~B~~

But what about p-adic orb. integrals?

<u>Remarkable fact</u>: $c^{\varepsilon}(b) = c(a)$ equals the value at $s=1$ of the orbital L-function

$$\underbrace{L(s,R)}_{\text{orbital L-fn}} \stackrel{\text{def}}{=} \underbrace{\zeta_R(s)}_{\text{Yun zeta fn}}/\zeta_{\mathbb{Q}}(s).$$

Since it has analytic cont. + fn eq^{tn}, it also has an approx. f^{nal} eq^{tn}: we get $c^{\varepsilon}(b)$ as the rest^{n} to b of a smooth, tempered f^{n} of $u \in \mathbb{R}$.

[Recall: $\gamma \longleftrightarrow$ ~~[illegible]~~ $a = (b, \varepsilon)$

$E = E_a = \mathbb{Q}[\lambda]/(p_a(\lambda))$ – ext^{n} with $\deg(E/\mathbb{Q}) = (n+1)$

$R = R_a = \mathbb{Z}[\lambda]/(p_a(\lambda))$ – an order in $\mathcal{O}_E$]

Altug required further analysis of resulting exp^{n} for $GL(2)$, but he (eventually) obtains Poisson summation over b. ~~[illegible]~~

He then showed that the Fourier transform term with $\xi = 0$ in $\mathbb{Z}$ gives the contribution of the nontempered 1-dim. rep. of $G(\mathbb{A})$ to $I_{ell,reg}(f)$, with strong estimate for (the difference) ~~[illegible]~~

<u>Original Answer in FLN</u>: Poisson summation for any G(!), and proof that term with $\xi = 0$ in $\mathbb{Z}^n$ gives contrib. of 1-dim. rep^{s} to $I_{ell,reg}(f)$. But the abstract techniques give only weak control over the remainder term.

(10) VI. ON GL(3): $G = G(2) = GL(3)$, over nonarch. local field F of char 0, res. char. q; $f = 1\!\!1_F = CF(G(\mathcal{O}_F))$.

Local orbital integrals

Theorem 1: Suppose $\gamma \in \Gamma_{ell, reg}(G)$ is unram, so that

$$|D(\gamma)|^{\frac{1}{2}} = q^{\delta}, \qquad \delta = 3m, \quad m \in \mathbb{N}.$$

Then $O(\gamma, f)$ equals

contrib. of reg. Shalika germ

$$\boxed{1 + \underbrace{(1+q^{-1}+q^{-2})}_{L(1,E)^{-1}} \Big[(q^{3m} + 2q^{3m-1} + 3q^{3m-2} + \cdots + (3m-1)q^2) \underbrace{- 3(q+1)(q^{2m-2} + 2q^{2m-4} + 3q^{2m-6} + \cdots + (m-1)q^2)}_{\text{contrib. of subreg. Shalika germ}} \Big]}$$

Remarks: (i) Solution of a difference eq^{tn} for no $= X^{\gamma}$ function in Kottwitz (Duke Math J, 48, 1981, p. 660)

(ii) Very similar formula if $\gamma \in \Gamma_{ell, reg}(G)$ is ramified. (Ibid, for no $= X^{\gamma}$ on p. 661.)

(iii) If $\gamma \in \Gamma_{reg}(G)$ is not elliptic, the problems reduce simpler formulas for proper Levi sub gp's $M \subset G$

Example: $m = 3$, $\delta = 9$. It then follows easily from th^m that

$$\boxed{\begin{aligned} O(\gamma, f) &= 1 + (1+q^{-1}+q^{-2})(q^9 + 2q^8 + 3q^7 + 4q^6 + 2q^5 + 3q^4 + 1q^3 + 2q^2) \\ &= q^9 + 3(q^8 + 2q^7 + 3q^6 + 3q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1). \end{aligned}}$$

This pattern is clear. It is exactly the same for any m.

<u>Local orbital L-functions</u>: Use Thm 1 and the general analogue of the example above to write

$$O(\gamma, f) = 1 + L_q(1,E)^{-1}\left(q^{3m} + c_1 q^{3m-1} + c_2 q^{3m-2} + \cdots + c_{m-2}\, q^{3} + c_{m-1}\, q^{2}\right)$$

for <u>explicit</u> pos. integers $c_1, \dots, c_{m-2}$. Then inflate this expression to a fn of s by 4 operations:

(i) (Translation) <u>Multiply the expn</u> by $q^{\delta(1-s)}$.

(ii) (Dirichlet L-fn) <u>Replace</u> $L_q(1,E)^{-1}$ by $L_q(s,E)^{-1}$.

(iii) (Scaling) <u>Inflate each monomial</u> q^k to $q^{k(2s-1)}$

(iv) (Desingularization). Replace each coefficient $c_h \in \mathbb{N}$ by $\sum_{i=0}^{c_h - 1} q^{(1-s)i}$.

Write $\hat{L}(s,R)$ for the resulting function of s, where $R = R_a$ (local order), $E = E_a$ (local field extn)

<u>Theorem 2</u>: (i) $\hat{L}(1,R) = \hat{L}(0,R) = O(\gamma, f)$

(ii) (Functional Eqtn) $\hat{L}(s,R) = \hat{L}(1-s,R)$

(i) is trivial, by construction and (ii)

(ii) uses elementary but tricky combinatorics. To simplify notation one writes $x = q^{1-s}$, $y = q^{s}$, so that $q^{2s-1} = y x^{-1}$. Must then (show) resulting exp^n is symmetric under $x \leftrightarrow y$.

Example: $m=3, \delta = 9$. One sees that $\hat{L}(s,R)$ equals

$$y^9 + y^8(x^2+x+1) + y^7(x^4+x^3+2x^2+x+1)$$
$$+ y^6(x^6+x^5+2x^4+2x^3+2x^2+x) + y^5(x^6+2x^5+3x^4+2x^3+x^2)$$
$$+ y^4(x^7+2x^6+3x^5+2x^4+x^3) + y^3(x^7+2x^6+2x^5+x^4)$$
$$+ y^2(x^8+2x^7+2x^6+x^5) + y^1(x^8+x^7+x^6) + y^0(x^8+x^7) + 1\cdot x^9,$$

and then verifies symmetry under $x \leftrightarrow y$.

- Same constructions, ~~+~~ theorem + proof if $\gamma \in \Gamma_{ell,reg}(G)$ is ramified
- If $\gamma \in \Gamma_{reg}(G)$ is not elliptic, the results reduce to proper Levi subgp^s $M \subset G$

For each local case for $GL(3)$, define

$$L(s,R) = L(s,E)\,\hat{L}(s,R)\,q^{-\delta s} \quad \text{— local orbital } L\text{-f}^n$$

(with E a product of fields if γ is not elliptic.

<u>Global orbital L-functions</u>. Now suppose (for $GL(3)$) that F is global, $\gamma \in \Gamma_{ell,reg}(G)$, $E = E_a$, $R = R_a$, $\gamma \longleftrightarrow a$.

Define

$$L(s, R) = L_\infty(s, R) \cdot \prod_p L_p(s, R)$$ — global orbital L-function

<u>Corollary 3</u>: $L(s, R) = L(1-s, R)$

Follows from Theorem 2(ii) + f^{n} equation for $L(s, E)$.

(14) VII. ON THE FUTURE

- Expect to use Theorem 2 to prove Poisson summation (à la Altug) for $G = GL(3)$.
- Problem: Prove that $\zeta_R(s) = \zeta_Q(s) L(s, R)$ equals Yun's local zeta function $J_R(s)$ for $GL(3)$.

  (Done for $GL(2)$ by M. Espinosa Lara)
- Problem: Extend Altug's global results to $GL(3)/F$, for any number field $F/\mathbb{Q}$

  (Done for $GL(2)/F$ by Espinosa-Lara, Emory, Kundu, Tian)
- The local formulas of Theorem 1 and the general version of the examples (there) were not hard to prove, but they turned out to be simpler than I expected. To me, they suggest possibility of manageable formulas for $GL(n+1)$

  (See Rogawski, Contemp. Math. 53, 1986, for making guesses, and the two papers of Waldsurger on germs for $GL(n+1)$ to try to prove them.)

15 VIII. CONJECTURE/SPECULATION: G arb. quasisplit gp

- Perhaps we can hope for manageable local formulas extending Theorem 1 & the example there for any G, despite Hales ("Why p-adic harmonic analysis is not elementary")

- It is likely there is a rich, hidden structure on $I_{ell,reg}(f)$, given by duality between local Shalika germs and the (global) parametrization of nontempered reps in the automorphic discrete spectrum

geometric		spectral
{stable unip class in G, base of S.H. fibration}	$\longleftrightarrow$	{unip classes in $\hat{G}$, Dynkin classes + diagrams, Bala Carter classes}

duality of unip classes,
Poisson summation,
$G \leftrightarrow \hat{G}$ etc.