Can. J. Math., Vol. XXXVII, No. 6, 1985, pp. 1237-1274

# A MEASURE ON THE UNIPOTENT VARIETY

JAMES ARTHUR

**Introduction.** Suppose that $G$ is a reductive algebraic group defined over $\mathbf{Q}$. There occurs in the trace formula a remarkable distribution on $G(\mathbf{A})^1$ which is supported on the unipotent set. It is defined quite concretely in terms of a certain integral over $G(\mathbf{Q})\backslash G(\mathbf{A})^1$. Despite its explicit description, however, this distribution is not easily expressed locally, in terms of integrals on the groups $G(\mathbf{Q}_v)$. For many applications of the trace formula, it will be essential to do this. In the present paper we shall solve the problem up to some undetermined constants.

The distribution, which we shall denote by $J_{\text{unip}}$, was defined in **[1]** and **[3]** as one of a family $\{J_{\mathfrak{o}}\}$ of distributions. It is the value at $T = T_0$ of a certain polynomial $J^T_{\text{unip}}$. We shall recall the precise definition in Section 1. Let us just say here that for $f \in C_c^\infty(G(\mathbf{A})^1)$, $J^T_{\text{unip}}(f)$ is given as an integral over $G(\mathbf{Q})\backslash G(\mathbf{A})^1$ which converges only for $T$ in a certain chamber which depends on the support of $f$. This is a source of some difficulty. For example, since $J_{\text{unip}}(f)$ is defined by continuation in $T$ outside the domain of absolute convergence of the integral $J^T_{\text{unip}}(f)$, it is not possible, a priori, to identify $J_{\text{unip}}$ with a measure. This will be a consequence (Corollary 8.3) of our final formula for $J_{\text{unip}}$.

We shall work indirectly. From **[3]** we understand the behaviour of $J_{\text{unip}}$ under conjugation. If $y$ is any point in $G(\mathbf{A})^1$, we have

$$(1)\qquad J_{\text{unip}}(f^y) = \sum_{Q\in\mathcal{F}} |W_0^{M_Q}|\,|W_0^G|^{-1} J^{M_Q}_{\text{unip}}(f_{Q,y}).$$

(See Section 1 for a description of the undefined symbols.) On the other hand, for any finite set $S$ of valuations of $\mathbf{Q}$, there are some distributions on $G(\mathbf{Q}_S)^1$ which have the same behaviour under conjugation and which can be expressed locally in terms of integrals on the groups $G(\mathbf{Q}_v)$. They are the weighted orbital integrals

$$J_M(\gamma, f),\quad f \in C_c^\infty(G(\mathbf{Q}_S)^1),\ \gamma \in M(\mathbf{Q}_S) \cap G(\mathbf{Q}_S)^1,$$

in which $M$ is a Levi component of a parabolic subgroup of $G$. The definition of these objects, which is somewhat tricky for general $\gamma$, will be given in the paper **[6]**, along with the conjugation formula

---

Received November 19, 1984. This research was supported by an E. W. R. Steacie Memorial Fellowship, NSERC Operating Grant A3483, and also in part by the Institute for Advanced Study.

$$(2) \quad J_M(\gamma, f^y) = \sum_{Q \in \mathcal{F}(M)} J_M^{M_Q}(\gamma, f_{Q,y}).$$

Our strategy is to find a linear combination of weighted orbital integrals which differs from $J_{\mathrm{unip}}(f)$ by an invariant distribution.

Assume that $S$ contains the Archimedean valuation. We can identify $C_c^\infty(G(\mathbf{Q}_S)^1)$ with the subspace of functions in $C_c^\infty(G(\mathbf{A})^1)$ which equal the characteristic function of a maximal compact subgroup away from $S$. Suppose for the moment that $G$ has $\mathbf{Q}$-rank 1 and that $M$ is the minimal Levi subgroup. Consider the distribution

$$(3) \quad J_{\mathrm{unip}}(f) - \frac{1}{2}\,\mathrm{vol}(M(\mathbf{Q})\backslash M(\mathbf{A})^1) J_M(1, f), \quad f \in C_c^\infty(G(\mathbf{Q}_S)^1).$$

Since

$$\begin{aligned} |W_0^M|\,|W_0^G|^{-1} J_{\mathrm{unip}}^M(g) &= \frac{1}{2}\,\mathrm{vol}(M(\mathbf{Q})\backslash M(\mathbf{A})^1) g(1) \\ &= \frac{1}{2}\,\mathrm{vol}(M(\mathbf{Q})\backslash M(\mathbf{A})^1) J_M^M(1, g) \end{aligned}$$

for any $g \in C_c^\infty(M(\mathbf{Q}_S)^1)$, the distribution (3) is invariant. It also annihilates any function which vanishes on the unipotent set in $G(\mathbf{Q}_S)^1$. From this one can show that (3) equals a linear combination

$$\sum_u a^G(S, u) J_G(u, f)$$

of invariant orbital integrals over the unipotent conjugacy classes in $G(\mathbf{Q}_S)^1$. Thus, $J_{\mathrm{unip}}(f)$ may be written as a linear combination of (weighted) orbital integrals. If $G$ is of general rank, a similar argument can be made. The combinatorics are reminiscent of those involved in putting the trace formula into invariant form, and are actually a special case of Proposition 4.1 of [**3**]. We present them in Section 8.

We summarize our results as follows. There are unique constants $\{a^M(S, u)\}$, defined for every $M$ which contains a fixed minimal Levi subgroup $M_0$ and every unipotent conjugacy class $u$ in $M(\mathbf{Q}_S)$, such that

$$J_{\mathrm{unip}}(f) = \sum_M \sum_u |W_0^M|\,|W_0^G|^{-1} a^M(S, u) J_M(u, f),$$

for any $f \in C_c(G(\mathbf{Q}_S)^1)$. Moreover, $a^M(S, u)$ vanishes unless $u$ is the image of a unipotent class in $M(\mathbf{Q})$. Finally,

$$a^M(S, 1) = \mathrm{vol}(M(\mathbf{Q})\backslash M(\mathbf{A})^1)$$

for any $M$ and $S$.

It is the latter two assertions which cause most of the trouble. The problem is essentially the one mentioned above. That is, $J_{\mathrm{unip}}(f)$ is defined in terms of the polynomial $J^T_{\mathrm{unip}}(f)$, which in turn is given by a concrete expression only for certain $T$ depending on $f$. We confront the difficulty in Theorem 3.1. We find a second expression, which approximates $J^{\mathrm{T}}_{\mathrm{unip}}(f)$ for large $T$, but which is defined for $T$ in a domain which is independent of $f$. This second expression readily decomposes into a sum of terms, indexed by the unipotent classes $U$ of $G$ which are defined over $\mathbf{Q}$. Theorem 4.2 then asserts that the term corresponding to a given $U$ is asymptotic to a unique polynomial $J^T_U(f)$ in $T$ and hence that

$$J^T_{\mathrm{unip}}(F) = \sum_U J^T_U(f).$$

Theorem 4.2 is the heart of the paper. Taken with its Corollaries 4.3 and 4.4, it provides the means for us to finally prove the required properties of the constants $\{a^M(S, u)\}$ in Section 8.

Most of the burden of the proof of Theorem 4.2 falls into a technical result, Lemma 4.1. In Section 5 we reduce Lemma 4.1 to a kind of lattice point problem (Lemma 5.1). Lemma 5.1 is then proved in Section 6, by combining various elementary estimates with the Poisson summation formula.

Finally, we mention that $J_{\mathrm{unip}}$, and more generally the distributions $\{J_o\}$, are dual to the distributions $\{J_\chi\}$ in the trace formula which come from Eisenstein series. The problem of finding a local formula for $J_\chi(f)$ was solved in **[4]** and **[5]**. It is worth noting that some of the results of this paper have direct analogues for the distributions $\{J_\chi\}$. For example, Theorem 3.1 corresponds to Theorem 3.2 of **[2]**, while Theorem 4.2 corresponds to Proposition 5.1 of **[4]**. We comment further on this in Section 2.

**1. The distribution $J^T_{\mathrm{unip}}$.** Let $G$ be a reductive algebraic group defined over $\mathbf{Q}$. We fix a subgroup $M_0$ of $G$ which is defined over $\mathbf{Q}$ and which is a Levi component of a minimal parabolic subgroup of $G$ defined over $\mathbf{Q}$. Let $\mathscr{F}$ be the set of parabolic subgroups of $G$ which contain $M_0$ and are defined over $\mathbf{Q}$, and let $\mathscr{L}$ be the set of subgroups of $G$ which contain $M_0$ and are Levi components of groups in $\mathscr{F}$. Suppose that $P \in \mathscr{F}$. Then we write

$$P = M_P N_P$$

where $N_P$ is the unipotent radical of $P$ and $M_P$ belongs to $\mathscr{L}$. Let $A_P = A_{M_P}$ be the split component of the center of $M_P$. If $X(M_P)_{\mathbf{Q}}$ is the group of characters of $M_P$ which are defined over $\mathbf{Q}$,

$$\mathfrak{a}_{M_P} = \mathfrak{a}_P = \mathrm{Hom}(X(M_P)_{\mathbf{Q}}, \mathbf{R})$$

is a real vector space whose dimension equals that of $A_P$. It can be regarded in a natural way as both a quotient and a subspace of $\mathfrak{a}_{M_0}$. We shall usually write $A_0 = A_{M_0}$ and $\mathfrak{a}_0 = \mathfrak{a}_{M_0}$.

We shall also fix a maximal compact subgroup $K$ of the adèlized group $G(\mathbf{A})$ which is admissible relative to $M_0$ in the sense of Section 1 of [**3**]. Then for any $P$ we have the usual function $H_P$ from $G(\mathbf{A})$ to $\mathfrak{a}_P$. (See [**1**], p. 917.) As in previous papers we write $G(\mathbf{A})^1$ for the kernel of $H_G$ in $G(\mathbf{A})$. Then $G(\mathbf{Q})$ is a discrete subgroup of $G(\mathbf{A})^1$ such that $G(\mathbf{Q})\backslash G(\mathbf{A})^1$ has finite invariant volume.

Unless otherwise specified, any integral on a group or homogenous space will be with respect to the invariant measure. Such measures are of course determined only up to scalar multiples, which we prefer not to normalize. We assume only that in a given context they satisfy any obvious compatibility conditions.

The distribution we will study is one of those introduced in [**1**]. It depends on a minimal parabolic subgroup

$$P_0 = M_0 N_0$$

defined over $\mathbf{Q}$. It also depends on a point $T$ in $\mathfrak{a}_0^+$, the positive chamber in $\mathfrak{a}_0$ associated to $P_0$, which initially is suitably regular in the sense that its distance from the walls of $\mathfrak{a}_0^+$ is large. Let $\mathscr{U}_G$ denote the Zariski closure in $G$ of the unipotent set in $G(\mathbf{Q})$. It is a closed algebraic subvariety of $G$ which is defined over $\mathbf{Q}$. The set

$$\mathfrak{o} = \mathscr{U}_G(\mathbf{Q})$$

of rational points in $\mathscr{U}_G$ consists, of course, of the unipotent elements in $G(\mathbf{Q})$. It is one of the equivalence classes in $G(\mathbf{Q})$ defined on p. 921 of [**1**]. Let

$$J_{\text{unip}}^T(f) = J_{\mathfrak{o}}^T(f), \quad f \in C_c^\infty(G(\mathbf{A})^1),$$

be the corresponding distribution. We shall briefly recall its definition.

The minimal parabolic subgroup $P_0$ will be fixed from Sections 2 to 6. During this time all parabolic subgroups $P$ will be understood to contain $P_0$. For any such $P$, following Section 1 of [**1**], we let $\Delta_P$ denote the simple roots of $(P, A_P)$ and we let $\hat{\Delta}_P$ denote the basis of $\mathfrak{a}_P^*/\mathfrak{a}_G^*$ which is dual to the simple "co-roots" $\{\alpha^\vee : \alpha \in \Delta_P\}$. If $f \in C_c^\infty(G(\mathbf{A})^1)$ and $T \in \mathfrak{a}_0^+$ is suitably regular, $J_{\text{unip}}^T(f)$ is the integral over $x$ in $G(\mathbf{Q})\backslash G(\mathbf{A})^1$ of the function

$$\sum_P (-1)^{\dim(A_P/A_G)} \sum_{\delta \in P(\mathbf{Q})\backslash G(\mathbf{Q})} K_{P,\text{unip}}(\delta x, \delta x)\hat{\tau}_P(H_P(\delta x) - T),$$

where $\hat{\tau}_P$ is the characteristic function of

$$\{H \in \mathfrak{a}_0 : \varpi(H) > 0, \varpi \in \hat{\Delta}_P\},$$

and

$$K_{P,\text{unip}}(y, y) = \sum_{\gamma \in \mathscr{U}_{M_P}(\mathbf{Q})} \int_{N_P(\mathbf{A})} f(y^{-1}\gamma n y)\,dn.$$

As a function of $T$, $J^T_{\mathrm{unip}}(f)$ is a polynomial of total degree at most

$$d_0 = \dim \mathfrak{a}_0.$$

(See Section 2 of [3].) Its definition can therefore be extended to all $T \in \mathfrak{a}_0$. There is a point $T_0$ in $\mathfrak{a}_0$, determined uniquely by $K$ and $M_0$, such that the distribution

$$J_{\mathrm{unip}}(f) = J^{T_0}_{\mathrm{unip}}(f)$$

is independent of $P_0$. (See Section 1 and Section 2 of [3].) In Sections 7 and 8 we will confine our attention to this distribution, and we will forget about $P_0$. Then parabolic subgroups will be taken from the set $\mathscr{F}$, as for example in the formula

$$(1.1)\quad J_{\mathrm{unip}}(f^y) = \sum_{Q \in \mathscr{F}} |W_0^{M_Q}|\,|W_0^G|^{-1} J^{M_Q}_{\mathrm{unip}}(f_{Q,y}),$$

which describes the failure of $J_{\mathrm{unip}}$ to be invariant under conjugation by elements $y$ in $G(\mathbf{A})^1$. (Theorem 3.2 of [3].) Here $W_0^G$ stands for the Weyl group of $(G, A_0)$, and

$$f \to f_{Q,y}$$

is the transform from $C_c^\infty(G(\mathbf{A})^1)$ to $C_c^\infty(M_Q(\mathbf{A})^1)$ defined by formula 3.3 of [3].

**2. A remark on the truncation operator.** The distribution $J^T_{\mathrm{unip}}$ arises from the trace formula. It is the most troublesome of those terms in the formula which are associated to conjugacy classes. The terms on the other side of the trace formula are associated to "cuspidal automorphic data" $\chi$, and have been evaluated explicitly in [5]. The two kinds of terms are in a sense dual to each other, and it is worthwhile to look for analogies between them.

We obtained $J^T_{\mathrm{unip}}(f)$ by integrating an alternating sum over standard parabolic subgroups. A similar alternating sum occurs in the definition of the distributions $J^T_\chi(f)$ ([2], p. 107). However, to actually calculate $J^T_\chi(f)$ we had to introduce a second formula (see [2], Lemma 2.4 and Theorem 3.2), based on a truncation operator. We showed that $J^T_\chi(f)$ could be obtained by taking the leading term $K_\chi(x, x)$ in the alternating sum, truncating in each variable separately, and then integrating. We shall do a similar thing for $J^T_{\mathrm{unip}}(f)$ in the next section. The situations are not entirely analogous, for in this case the second formula will be asymptotic rather than exact. Moreover, it will be used in a somewhat different way.

The leading term in the alternating sum for $J_{\mathrm{unip}}(f)$ is

$$K_{\mathrm{unip}}(x, x) = K_{G,\mathrm{unip}}(x, x) = \sum_{\gamma \in \mathscr{U}_G(\mathbf{Q})} f(x^{-1}\gamma x).$$

It is the restriction to the diagonal of a function of two variables, but unlike $K_\chi(x, x)$, it is not left $G(\mathbf{Q})$-invariant in each variable separately. It is therefore more appropriate to truncate over the diagonal rather than in each variable separately. We shall show that the integral of the resulting function can be rewritten in a more elementary way.

If $P_1 \subset P_2$ are (standard) parabolic subgroups, we write $A_{P_1}^\infty$ for $A_{P_1}(\mathbf{R})^0$, the identity component of $A_{P_1}(\mathbf{R})$, and

$$A_{P_1,P_2}^\infty = A_{P_1} \cap M_{P_2}(\mathbf{A})^1.$$

Then $H_{P_1}$ maps $A_{P_1,P_2}^\infty$ isomorphically onto $\mathfrak{a}_{P_1}^{P_2}$, the orthogonal complement of $\mathfrak{a}_{P_2}$ in $\mathfrak{a}_{P_1}$. If $T_1$ and $T$ are points in $\mathfrak{a}_0$, set $A_{P_1,P_2}^\infty(T_1, T)$ equal to the set

$$\{a \in A_{P_1,P_2}^\infty : \alpha(H_{P_1}(a) - T_1) > 0, \ \alpha \in \Delta_{P_1}^{P_2};$$
$$\varpi(H_{P_1}(a) - T) < 0, \ \varpi \in \hat{\Delta}_{P_1}^{P_2}\},$$

where as in [1],

$$\Delta_{P_1}^{P_2} = \Delta_{P_1 \cap M_{P_2}} \quad \text{and} \quad \hat{\Delta}_{P_1}^{P_2} = \hat{\Delta}_{P_1 \cap M_{P_2}}.$$

From now on we shall fix $T_1$ so that $-T_1$ is suitably regular. (In [1] we denoted this point by $T_0$. In this paper we have agreed that $T_0$ should stand for the point defined by Lemma 1.1 of [3].) Suppose that $T$ is suitably regular. We define

$$F(x, T) = F^G(x, T)$$

as on p. 941 of [1]. It is the characteristic function of the compact subset of $G(\mathbf{Q})\backslash G(\mathbf{A})^1$ obtained by projecting

$$N_0(\mathbf{A}) \cdot M_0(\mathbf{A})^1 \cdot A_{P_0,G}^\infty(T_1, T) \cdot K$$

onto $G(\mathbf{Q})\backslash G(\mathbf{A})^1$.

The truncation operator $\Lambda^T$ is defined on p. 89 of [2]. It maps certain functions on $G(\mathbf{Q})\backslash G(\mathbf{A})^1$ to functions on $G(\mathbf{Q})\backslash G(\mathbf{A})^1$ which are rapidly decreasing.

LEMMA 2.1. *$\Lambda^T 1_G = F(\cdot, T)$, where $1_G$ is the function which is identically $1$ on $G(\mathbf{Q})\backslash G(\mathbf{A})^1$.*

*Proof.* The formula is a consequence of Lemma 6.4 of [1] and Lemma 1.5 of [2]. For the first lemma states that

$$\sum_P \sum_{\delta \in P(\mathbf{Q})\backslash G(\mathbf{Q})} F^P(\delta x, T)\tau_P(H_P(\delta x) - T) = 1,$$

where $\tau_P$ is the characteristic function of

$$\{H \in \mathfrak{a}_0 : \alpha(H) > 0, \ \alpha \in \Delta_P\}$$

and

$$F^P(nmk, T) = F^{M_P}(m, T), \quad n \in N_P(\mathbf{A}), m \in M_P(\mathbf{A}), k \in K.$$

The second lemma tells us that for all $x$,

$$\sum_P \sum_{\delta \in P(\mathbf{Q})\backslash G(\mathbf{Q})} (\Lambda^{T,P} 1_G)(\delta x) \cdot \tau_P(H_P(\delta x) - T) = 1,$$

where $\Lambda^{T,P}$ is the partial truncation operator defined on p. 97 of [2]. Our result is immediately obtained by induction.

Our second formula for $J^T_{\text{unip}}(f)$ will be in terms of

$$(2.1) \quad \int_{G(\mathbf{Q})\backslash G(\mathbf{A})^1} \Lambda^T_d K_{\text{unip}}(x, x) dx,$$

where $\Lambda^T_d$ refers to the truncation operator acting on the diagonal.

LEMMA 2.2. *The integral* (2.1) *is equal to*

$$(2.2) \quad \int_{G(\mathbf{Q})\backslash G(\mathbf{A})^1} F(x, T)\Bigg( \sum_{\gamma \in \mathcal{U}_G(\mathbf{Q})} f(x^{-1}\gamma x) \Bigg) dx.$$

*Proof.* Set

$$k(x, f) = K_{\text{unip}}(x, x) = \sum_{\gamma \in \mathcal{U}_G(\mathbf{Q})} f(x^{-1}\gamma x).$$

Any left invariant differential operator on $G(\mathbf{R})^1$ transforms $k(x, f)$ to a finite sum of functions of the form

$$k(x, f_i), \quad f_i \in C_c^\infty(G(\mathbf{A})^1).$$

This follows from the chain rule and the definition of $K_{\text{unip}}(x, x)$. Therefore by Lemma 1.4 of [2] and Lemma 4.3 of [1], $\Lambda^T k(\cdot, f)$ is rapidly decreasing. In particular, it is integrable over $G(\mathbf{Q})\backslash G(\mathbf{A})^1$. Writing $(\cdot, \cdot)$ for the inner product on $L^2(G(\mathbf{Q})\backslash G(\mathbf{A})^1)$, we see that (2.1) equals

$$\begin{aligned} &(\Lambda^T k(\cdot, f), 1_G) \\ &= ( (\Lambda^T)^2 k(\cdot, f), 1_G) \\ &= (\Lambda^T k(\cdot, f), \Lambda^T 1_G) \end{aligned}$$

by two results of [2] (Corollary 1.2 and Lemma 1.3). But $\Lambda^T 1_G$ is of course also rapidly decreasing, so we can repeat the argument. We obtain

$$\begin{aligned} &(\Lambda^T k(\cdot, f), \Lambda^T 1_G) \\ &= (k(\cdot, f), (\Lambda^T)^2 1_G) \\ &= (k(\cdot, f), \Lambda^T 1_G) \\ &= (k(\cdot, f), F(\cdot, T) ), \end{aligned}$$

by Lemma 2.1. This last inner product is just (2.2).

The variety $\mathcal{U}_G$ is a finite union of (geometric) unipotent conjugacy classes of $G$. The Galois group, $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$, operates on these conjugacy classes. We shall write $(\mathcal{U}_G)$ for the set of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$-orbits. Then any $U \in (\mathcal{U}_G)$ is a locally closed subset of $G$, which is defined over $\mathbf{Q}$, and which consists of a finite union of unipotent conjugacy classes of $G$. It is clear that if $R$ is any ring which contains $\mathbf{Q}$, $\mathcal{U}_G(R)$ is the disjoint union over $U \in (\mathcal{U}_G)$ of the set $U(R)$. In particular, we can write

$$K_{\mathrm{unip}}(x, x) = \sum_{U \in (\mathcal{U}_G)} K_U(x, x),$$

where

$$K_U(x, x) = \sum_{\gamma \in U(\mathbf{Q})} f(x^{-1}\gamma x).$$

In Section 4 we shall use this as a starting point to construct a decomposition of $J^T_{\mathrm{unip}}(f)$. Observe that if $K_U(x, x)$ does not vanish, the set $U(\mathbf{Q})$ is not empty, and $U$ consists of a single unipotent conjugacy class of $G$.

The following lemma is proved exactly as the last one.

LEMMA 2.3. *For any* $U \in (\mathcal{U}_G)$,

$$\int_{G(\mathbf{Q})\backslash G(\mathbf{A})^1} \Lambda^T_d K_U(x, x)dx$$

*equals*

$$\int_{G(\mathbf{Q})\backslash G(\mathbf{A})^1} F(x, T)\Big(\sum_{\gamma \in U(\mathbf{Q})} f(x^{-1}\gamma x)\Big)dx.$$

**3. An alternate characterization.** Fix a Euclidean norm $\|\cdot\|$ on $\mathfrak{a}_0$, and set

$$d(T) = \min_{\alpha \in \Delta_{P_0}} \{\alpha(T)\}.$$

In this section we shall let $T$ vary over suitably regular points such that

$$d(T) \geqq \epsilon_0\|T\|,$$

for some fixed positive number $\epsilon_0$.

THEOREM 3.1. *There is a continuous semi-norm* $\|\cdot\|$ *on* $C^\infty_{c(G(\mathbf{A})^1)}$ *such that*

$$\left| J^T_{\mathrm{unip}}(f) - \int_{G(\mathbf{Q})\backslash G(\mathbf{A})^1} \Lambda^T_d K_{\mathrm{unip}}(x, x)dx \right| \leqq \|f\|e^{-(d(T)/2)},$$

*for all* $f \in C^\infty_c(G(\mathbf{A})^1)$.

*Proof.* The theorem is a refinement of Theorem 7.1 of [**1**]. We will need to go over the proof of this result, taking $\mathfrak{o} = \mathcal{U}_G(\mathbf{Q})$, and keeping track of the dependence on $f$ and $T$.

In the proof of Theorem 7.1 of [**1**] it was established that $J^T_{\mathrm{unip}}(f)$ could be written as a sum over standard parabolic subgroups $P_1 \subset P_2$ of terms involving the function

$$F^{P_1}(nmk) = F^{M_1}(m), \quad n \in N_{P_1}(\mathbf{A}),\ m \in m_{P_1}(\mathbf{A}),\ k \in K,$$

and the characteristic function $\sigma^{P_2}_{P_1}$ on $\mathfrak{a}_0$ defined on p. 938 of [**1**]. The term corresponding to $P_1 = P_2$ equals zero if $P_1 \neq G$, and equals

$$\int_{G(\mathbf{Q})\backslash G(\mathbf{A})^1} F(x, T)\Big(\sum_{\gamma \in \mathcal{U}_G(\mathbf{Q})} f(x^{-1}\gamma x)\Big)dx$$

if $P_1 = G$. Consequently (see p. 945 of [**1**] ),

$$\left|J^T_{\mathrm{unip}}(f) - \int_{G(\mathbf{Q})\backslash G(\mathbf{A})^1} F(x, T)\Big(\sum_{\gamma \in \mathcal{U}_G(\mathbf{Q})} f(x^{-1}\gamma x)\Big)dx\right|$$

is bounded by the sum over $\{P_1, P_2 : P_1 \subsetneq P_2\}$ of

$$(3.1)\quad \int_{P_1(\mathbf{Q})\backslash G(\mathbf{A})^1} F^{P_1}(x, T)\sigma^{P_2}_{P_1}(H_{P_1}(x) - T) \times \sum_{\gamma \in \mathcal{U}_1(\mathbf{Q})} \sum_{\zeta \in \mathfrak{n}^{P_2}_{P_1}(\mathbf{Q})'} |\Phi_\gamma(x, \zeta)|\,dx,$$

where we have set

$$\mathcal{U}_1(\mathbf{Q}) = \mathcal{U}_{M_{P_1}}(\mathbf{Q}) = \mathcal{U}_G(\mathbf{Q}) \cap M_{P_1}(\mathbf{Q}),$$

and

$$\Phi_m(y, Y) = \int_{\mathfrak{n}_{P_1}(\mathbf{A})} f(y^{-1}m \exp(X)y)\psi(\langle X, Y\rangle)dX,\ Y \in \mathfrak{n}_{P_1}(\mathbf{A}),$$

for $y \in G(\mathbf{A})^1$ and $m \in M_{P_1}(\mathbf{A})^1$. (Here $\mathfrak{n}_{P_1}$ is the Lie algebra of $N_{P_1}$ and the symbols $\mathfrak{n}^{P_2}_{P_1}(\cdot)'$, and $\langle \cdot, \cdot \rangle$ are as defined on p. 945 of [**1**].) Note that $\Phi_m(y, \cdot)$ is a Schwartz-Bruhat function on $\mathfrak{n}_{P_1}(\mathbf{A})$ which varies smoothly with $m$ and $y$. Moreover, if $y$ remains within a compact set, $\Phi_m(y, \cdot)$ will vanish identically for all $m$ outside a compact subset of $M_{P_1}(\mathbf{A})^1$.

As in [**1**] we will use the decomposition

$$P_1(\mathbf{Q})\backslash G(\mathbf{A})^1 = N_{P_1}(\mathbf{Q})\backslash N_{P_1}(\mathbf{A}) \times M_{P_1}(\mathbf{Q})\backslash M_{P_1}(\mathbf{A})^1 \times \mathbf{A}^\infty_{P_1,G} \times K$$

to rewrite the integral in (3.1). It will also be convenient to decompose the resulting integral over $M_{P_1}(\mathbf{Q})\backslash M_{P_1}(\mathbf{A})^1$. Any element in this coset space on which the function $F^{P_1}(\cdot, T)$ does not vanish has a representative in

$$(N_0(\mathbf{A}) \cdot A^{\infty}_{P_0,P_1}(T_1, T) \cdot M_0(\mathbf{A})^1 \cdot K) \cap M_{P_1}(\mathbf{A})^1.$$

Therefore (3.1) is bounded by the integral over $k \in K$,

$$a \in A^{\infty}_{P_0,P_1}(T_1, T), \quad a' \in A^{\infty}_{P_1,G}$$

and $m$, $n$ and $n''$ in fixed compact fundamental domains in $M_0(\mathbf{A})^1$, $N_0(\mathbf{A}) \cap M_{P_2}(\mathbf{A})^1$ and $N_{P_2}(\mathbf{A})$ respectively, of

$$\delta_{P_0}(a'a)^{-1}\sigma^{P_2}_{P_1}(H_{P_1}(a') - T) \cdot \sum_{\gamma \in \mathscr{U}_1(\mathbf{Q})} \sum_{\zeta \in \mathfrak{N}^{P_2}_{P_1}(\mathbf{Q})'} |\Phi_\gamma(n''na'amk, \zeta)|.$$

(Here, $\delta_{P_0}$ is the modular function of $P_0(\mathbf{A})$.) In view of the definition of $\Phi$, we can write

$$\begin{aligned} |\Phi_\gamma(n''na'amk, \zeta)| &= |\Phi_\gamma(na'amk, \zeta)| \\ &= |\Phi_\gamma(a'a \cdot (a'a)^{-1}n(a'a)mk, \zeta)| \\ &= \delta_{P_1}(a'a)\, |\Phi_{a^{-1}\gamma a}((a'a)^{-1}n(a'a)mk, \mathrm{Ad}(a'a)\zeta)|. \end{aligned}$$

If we assume that $\sigma^{P_2}_{P_1}(H_{P_1}(a') - T)$ is not zero, the projection of $H_{P_0}(a'a) - T$ onto $\mathfrak{a}^{P_2}_{P_0}$ will belong to a translate of the positive chamber. (See Corollary 6.2 of [1], and if necessary the discussion below on the decomposition of the vector $H_{P_0}(a'a)$.) This means that $\{ (a'a)^{-1}n(a'a) \}$ will remain in a fixed compact set. Since

$$\delta_{P_0}(a'a) = \delta^{P_1}_{P_0}(a) \cdot \delta_{P_1}(a'a),$$

we see that (3.1) is bounded by the integral over $a' \in A^{\infty}_{P_1,G}$ and $a \in A^{\infty}_{P_0,P_1}(T_1, T)$ of

$$(3.2) \quad \delta^{P_1}_{P_0}(a)^{-1} \sum_{\gamma \in \mathscr{U}_1(\mathbf{Q})} \sigma^{P_2}_{P_1}(H_{P_1}(a') - T)$$

$$\times \sum_{\zeta \in \mathfrak{N}^{P_2}_{P_1}(\mathbf{Q})'} \int_\Gamma |\Phi_{a^{-1}\gamma a}(y, \mathrm{Ad}(a'a)\zeta)|\, dy,$$

where $dy$ stands for a Radon measure on a fixed compact subset $\Gamma$ of $G(\mathbf{A})^1$.

In order to estimate the sum over $\gamma$ in (3.2) we will need a familiar lemma. As explained in Section 5, it can be regarded as a special case of a future result (Lemma 4.1), whose proof will be given in Sections 5 and 6.

LEMMA 3.2. *Suppose that $\phi$ is a bounded, nonnegative function on $G(\mathbf{A})^1$ of compact support. Then*

$$\delta_{P_0}(a)^{-1} \sum_{\gamma \in G(\mathbf{Q})} \phi(a^{-1}\gamma a)$$

*is bounded independently of a in the set*

$$A_{P_0}^{\infty}(T_1) = \{a \in A_{P_0}^{\infty}: \alpha(H_{P_0}(a) - T_1) > 0, \alpha \in \Delta_{P_0}\}.$$

Assuming the lemma, we proceed to estimate (3.2). Let

$$\mathfrak{n}_{P_1}^{P_2} = \oplus_\lambda \mathfrak{n}_\lambda$$

be a decomposition of $\mathfrak{n}_{P_1}^{P_2}$ into eigenspaces under the action of $A_0$. Each $\lambda$ stands for a linear function on $\mathfrak{a}_0$ which vanishes on the subspace $\mathfrak{a}_{P_2}$. Choose a basis of $\mathfrak{n}_{P_1}^{P_2}(\mathbf{Q})$ with respect to which $\langle , \rangle$ is the standard positive definite inner product, and such that each basis element lies in some $\mathfrak{n}_\lambda(\mathbf{Q})$. The basis gives us a Euclidean norm on $\mathfrak{n}_{P_1}^{P_2}(\mathbf{R})$ and allows us to speak of $\mathfrak{n}_{P_1}^{P_2}(\mathbf{Z})$ and $\mathfrak{n}_\lambda(\mathbf{Z})$. Fix a large integer $n$. It follows from Lemma 3.2 (applied to the group $M_{P_1}$) and the properties of the function $\Phi_m(y, Y)$ that (3.2) is bounded by an expression

$$(3.3) \quad \|f\|_1 \sigma_{P_1}^{P_2}(H_{P_1}(a') - T) \sum_{\zeta \in \mathfrak{N}_{P_1}^{P_2}(N(f)^{-1}\mathbf{Z})'} \|\mathrm{Ad}(a'a)\zeta\|^{-n}.$$

Here $\|\cdot\|_1$ is a continuous semi-norm

$$(3.4) \quad \|f\|_1 = c_f \sum_{i=1}^{k} \sup_{x \in G(\mathbf{A})^1} |(X_i f)(x)|,$$

where $c_f$ is a number which depends only on the support of $f$ and each $X_i$ is a left invariant differential operator on $G(\mathbf{R})^1$. The number

$$N(f) = \prod_p p^{n_p(f)}$$

is a positive integer determined by the support of $\Phi_m(y, \cdot)$ at the finite completions of $\mathbf{Q}$. Defining it in terms of the Fourier transform of $\Phi_m(y, \cdot)$, we take $n_p(f)$ to be the smallest nonnegative integer such that the function

$$X \to f(y^{-1}me(X)y), \quad X \in \mathfrak{n}_{P_1}(\mathbf{A}),$$

is invariant under

$$\{X_p \in \mathfrak{n}_{P_1}^{P_2}(\mathbf{Q}_p): \|X_p\|_p \leqq p^{-n_p(f)}\}$$

for all $y \in \Gamma$ and $m \in M_{P_1}(\mathbf{A})^1$. (The norm on $\mathfrak{n}_{P_1}^{P_2}(\mathbf{Q}_p)$ is the natural one associated to our basis of $\mathfrak{n}_{P_1}^{P_2}(\mathbf{Q})$.)

Now each $\lambda$ above is a unique nonnegative integral combination of the simple roots $\Delta_{P_0}^{P_2}$. Suppose that $S$ is a subset of elements $\lambda$ with the property that for any $\alpha$ in the complement of $\Delta_{P_0}^{P_1}$ in $\Delta_{P_0}^{P_2}$, there is a $\lambda$ in $S$

whose $\alpha$ co-ordinate is positive. Let $\mathfrak{n}_S(\mathbf{Q})'$ be the set of those elements in $\mathfrak{n}_{P_1}^{P_2}(\mathbf{Q})$ whose projections onto $\mathfrak{n}_\lambda$ are nonzero if $\lambda$ belongs to $S$, and are zero otherwise. Then the sum over $\mathfrak{n}_{P_1}^{P_2}(N(f)^{-1}\mathbf{Z})'$ in (3.3) can be replaced by the double sum over all such $S$ and over $\zeta$ in $\mathfrak{n}_S(N(f)^{-1}\mathbf{Z})'$. Clearly

$$\sum_{\zeta\in\mathfrak{n}_S(N(f)^{-1}\mathbf{Z})'} \|\mathrm{Ad}(a'a)\zeta\|^{-n}$$

$$\leqq \prod_{\lambda\in S} \sum_{\xi\in\mathfrak{n}_\lambda(N(f)^{-1}\mathbf{Z})'} \|\mathrm{Ad}(a'a)\xi\|^{-n_S}$$

$$= \prod_{\lambda\in S}\Big(\sum_{\xi\in\mathfrak{n}_\lambda(N(f)^{-1}\mathbf{Z})'} \|\xi\|^{-n_S}\Big)(e^{-n_S\lambda(H_{P_0}(a'a))}),$$

where $\mathfrak{n}_\lambda(N(f)^{-1}\mathbf{Z})'$ is the set of nonzero elements in $\mathfrak{n}_\lambda(N(f)^{-1}\mathbf{Z})$ and $n_S$ is the quotient of $n$ by the number of roots in $S$. For large enough $n$ this last expression is bounded by a constant multiple of

$$N(f)^n\cdot\prod_{\lambda\in S} e^{-n_S\lambda(H_{P_0}(a'a))}.$$

It follows that (3.2) is bounded by

$$\|f\|_1\cdot N(f)^n\cdot\sigma_{P_1}^{P_2}(H_{P_1}(a')-T)\prod_{\alpha\in\Delta_{P_0}^{P_2}} e^{-k_\alpha\alpha(H_{P_0}(a'a))} \tag{3.5}$$

where $\|\cdot\|_1$ is of the form (3.4), $n$ is a positive integer, and each $k_\alpha$ is a nonnegative integer which is positive if $\alpha$ belongs to the complement of $\Delta_{P_0}^{P_1}$ in $\Delta_{P_0}^{P_2}$.

We can decompose the vector

$$H_{P_0}(a'a) = H_{P_1}(a') + H_{P_0}(a)$$

as

$$\Big(\sum_{\beta\in\Delta_{P_1}^{P_2}} t_\beta\tilde{\omega}_\beta^\vee + H^*\Big) - \Big(\sum_{\delta\in\Delta_{P_0}^{P_1}} r_\delta\delta^\vee\Big) + T,$$

where $t_\beta$ and $r_\delta$ are real numbers and $H^*$ is a vector in $\mathfrak{a}_{P_2}$. (As in [**1**], $\{\tilde{\omega}_\beta^\vee\}$ stands for the basis of $\mathfrak{a}_{P_1}^{P_2}$ which is dual to $\{\beta\in\Delta_{P_1}^{P_2}\}$.) The point $a$ belongs to $A_{P_0,P_1}^\infty(T_1, T)$ so that for each $\delta\in\Delta_{P_0}^{P_1}$, the number $r_\delta$ is nonnegative and

$$\delta(H_P(a'a)) = \delta(H_P(a)) \geqq \delta(T_1).$$

We are trying to estimate the integral of (3.5), so we can certainly assume that the number

$$\sigma_{P_1}^{P_2}(H_{P_1}(a')-T) = \sigma_{P_1}^{P_2}\Big(\sum_{\beta\in\Delta_{P_1}^{P_2}} t_\beta\tilde{\omega}_\beta^\vee + H^*\Big)$$

is not zero. It follows from Corollary 6.2 of [1] that each $t_\beta$ is positive and that $H^*$ belongs to a compact subset whose volume can be bounded by some polynomial, say

$$\prod_{\beta \in \Delta_{P_1}^{P_2}} p(t_\beta),$$

in the numbers $\{t_\beta\}$. Finally, recall that each root $\alpha \in \Delta_{P_0}^{P_2} \setminus \Delta_{P_0}^{P_1}$ projects onto a unique root $\beta \in \Delta_{P_1}^{P_2}$, and that $\alpha(\delta^\vee) \leqq 0$ for each $\delta$. It follows from these facts that

$$\begin{aligned}\prod_{\alpha \in \Delta_{P_0}^{P_2}} e^{-k_\alpha \alpha(H_{P_0}(a'a))} &\leqq c(T_1) \prod_{\alpha \in \Delta_{P_0}^{P_2} \setminus \Delta_{P_0}^{P_1}} e^{-k_\alpha \alpha(H_{P_0}(a'a))} \\ &\leqq c(T_1) \prod_{\alpha \in \Delta_{P_0}^{P_2} \setminus \Delta_{P_0}^{P_1}} e^{-\alpha(H_{P_0}(a'a))} \\ &\leqq c(T_1) \prod_{\beta \in \Delta_{P_1}^{P_2}} e^{-(t_\beta + \beta(T))},\end{aligned}$$

where

$$c(T_1) = \prod_{\delta \in \Delta_{P_0}^{P_1}} e^{-k_\delta \delta(T_1)},$$

a constant that depends only on $T_1$. We conclude that the integral over $a'$ and $a$ of (3.5) is bounded by a constant multiple of the product of

$$\|f\|_1 \cdot N(f)^n \cdot \mathrm{vol}(A_{P_0,P_1}^\infty(T_0, T))$$

and

$$\prod_{\beta \in \Delta_{P_1}^{P_2}} \left(e^{-\beta(T)} \int_0^\infty p(t_\beta) e^{-t_\beta} dt_\beta\right).$$

The second expression is certainly bounded by a constant multiple of $e^{-d(T)}$. The volume of $A_{P_0,P_1}^\infty(T_1, T)$ is certainly bounded by a polynomial in $\|T\|$. Taking into account our constraints on $T$, we see that the integral of (3.5) is bounded by a constant multiple of

$$\|f\|_1 \cdot N(f)^n \cdot e^{-(d(T)/2)}.$$

Incorporating the constant into $\|\cdot\|_1$, we obtain this quantity as a bound for our original expression

$$\left|J_{\mathrm{unip}}^T(f) - \int_{G(\mathbf{Q})\backslash G(\mathbf{A})^1} F(x, T)\left(\sum_{\gamma \in \mathscr{U}_G(\mathbf{Q})} f(x^{-1}\gamma x)\right) dx\right|.$$

Recalling Lemma 2.2, we see that Theorem 3.2 follows with

$$\|f\| = \|f\|_1 \cdot N(f)^n.$$

We shall later want to truncate functions $f$ around certain conjugacy classes. This will require that we evaluate the semi-norm $\|\cdot\|$ of the theorem at certain functions obtained from $f$. Fix a valuation $v$ of $\mathbf{Q}$. If $v$ is discrete, define

$$\rho_v : \mathbf{R} \to \mathbf{R}$$

to be the characteristic function of the interval $[-1, 1]$. If $v$ is Archimedean, let $\rho_v$ be any fixed function in $C_c^\infty(\mathbf{R})$ which vanishes outside the interval $[-1, 1]$, which equals 1 on the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$ and takes values between 0 and 1 at all other points. Suppose that $q$ is a polynomial function which is defined over $\mathbf{Q}$. For any $f \in C_c^\infty(G(\mathbf{A})^1)$ and $\epsilon > 0$, consider the function

$$f_{q,v}^\epsilon(x) = f(x)\rho_v(\epsilon^{-1}|q(x)|_v), \quad x \in G(\mathbf{A})^1,$$

where $|q(x)|_v$ is the absolute value of the $v$ component of the adèle $q(x)$. It also belongs to $C_c^\infty(G(\mathbf{A})^1)$. We would like to study $\|f_{q,v}^\epsilon\|$.

We saw that we could take

$$(3.6) \quad \|f\| = \|f\|_1 \cdot N(f)^n, \quad f \in C_c^\infty(G(\mathbf{A})^1),$$

where $\|\cdot\|_1$ is the semi-norm (3.4) and $N(f)$ is the positive integer defined in the proof of the theorem.

COROLLARY 3.3. *There is a positive integer $m$, and another semi-norm $\|\cdot\|'$ of the form* (3.6) *such that*

$$\|f_{q,v}^\epsilon\| \leqq \epsilon^{-m}\|f\|',$$

*for all $f \in C_c^\infty(G(\mathbf{A})^1)$ and $\epsilon$ with $0 < \epsilon \leqq 1$.*

*Proof.* If $v$ is Archimedean, $N(f_{q,v}^\epsilon) = N(f)$. The corollary then follows immediately from the formula (3.4) and the chain rule.

Suppose that $v$ is discrete. It is clear that

$$\|f_{q,v}^\epsilon\|_1 = \|f\|_1,$$

so we need only study the integer

$$N(f_{q,v}^\epsilon) = \prod_p p^{n_p(f_{q,v}^\epsilon)}.$$

Moreover, $n_p(f_{q,v}^\epsilon) = n_p(f)$ unless $p$ is the rational prime which defines $v$. Assume then that this is the case. Take an embedding $G \subset GL_N$ into a general linear group defined over $\mathbf{Q}$. There is a positive number $t_0$ such that for any $\epsilon > 0$ the function

$$x \to \rho_v(\epsilon^{-1}|q(x)|_v), \quad x \in \mathrm{supp}(f),$$

is bi-invariant under

$$\{k_v \in GL_N(\mathbf{Q}_v) \cap G:|k_v - I|_v < t_0\epsilon\}.$$

(Here $I$ stands for the identity matrix in $GL_N(\mathbf{Q}_v)$.) Choose parabolic subgroups $P_1 \subsetneq P_2$, and recall the notation of the proof of the theorem. It is easily established from the property above that there is a positive number $t_1$ such that for any $\epsilon$, $0 < \epsilon \leqq 1$, the function

$$X \to f^{\epsilon}_{q,v}(y^{-1}m \exp(X)y)$$

is invariant under

$$\{X_p \in \mathfrak{N}^{P_2}_{P_1}(\mathbf{Q}_p):\|X_p\|_v \leqq \min[\,(t_1\epsilon), p^{-n_p(f)}]\,\}$$

for all $y \in \Gamma$, $m \in M_{P_1}(\mathbf{A})^1$ and $f \in C^\infty_c(G(\mathbf{A})^1)$. Consequently

$$\begin{aligned} p^{n_p(f^{\epsilon}_{q,v})} &\leqq \max[\,(t_1\epsilon)^{-1}, p^{n_p(f)}] \\ &\leqq t_1^{-1}\epsilon^{-1}p^{n_p(f)}. \end{aligned}$$

The corollary follows.

**4. More distributions.** We shall show that $J^T_{\text{unip}}$ can be decomposed into a sum of distributions indexed by the orbits in $(\mathcal{U}_G)$. This is in rather close analogy with what was done in Section 5 of [**4**] for the distributions $J^T_{\chi}$. First, however, we must state a lemma.

Fix an orbit $U \in (\mathcal{U}_G)$. Its Zariski closure $\bar{U}$ is a closed subvariety of $G$ which is defined over $\mathbf{Q}$. The ideal of polynomial functions on $G$ which vanish on $U$ is of the form $(q_1, \ldots, q_l)$, where $q_1, \ldots, q_l$ are polynomials on $G$ defined over $\mathbf{Q}$. We have

$$\bar{U} = \{x \in G:q_1(x) = \ldots = q_l(x) = 0\}.$$

Fix a valuation $v$ of $\mathbf{Q}$, and recall the functions $\rho_v$ defined in Section 3. For any $f \in C^\infty_c(G(\mathbf{A})^1)$ and $\epsilon > 0$, define

$$(4.1)\quad f^{\epsilon}_{U,v}(x) = f(x)\rho_v(\epsilon^{-1}|q_1(x)|_v)\ldots\rho_v(\epsilon^{-1}|q_l(x)|_v),\quad x \in G(\mathbf{A})^1.$$

Then $f^{\epsilon}_{U,v}$ also belongs to $C^\infty_c(G(\mathbf{A})^1)$. Observe that it equals $f$ on a neighborhood of the set $\bar{U}(\mathbf{A})$.

LEMMA 4.1. *There is a positive number $r$ and a continuous semi-norm $\|\cdot\|$ on $C^\infty_c(G(\mathbf{A})^1)$ such that for any $\epsilon > 0$,*

$$\int_{G(\mathbf{Q})\backslash G(\mathbf{A})^1} F(x, T) \sum_{\gamma \in G(\mathbf{Q})\setminus \bar{U}(\mathbf{Q})} |f^{\epsilon}_{U,v}(x^{-1}\gamma x)|\,dx \leqq \|f\|\epsilon^r(1 + \|T\|)^{d_0},$$

*for each $f \in C^\infty_c(G(\mathbf{A})^1)$ and each suitably regular $T$.*

The proof of this lemma is quite long and will be given in Sections 5 and 6. Assuming it in the meantime, we will establish

THEOREM 4.2. *There are distributions*

$$\{J_U^T : U \in (\mathscr{U}_G)\}$$

*which are polynomials in $T$ of total degree at most $d_0$ such that*

$$J_{\mathrm{unip}}^T(f) = \sum_U J_U^T(f), \quad f \in C_c^\infty(G(\mathbf{A})^1), \tag{4.2}$$

*and which satisfy the following property. There is a continuous semi-norm $\|\cdot\|$ on $C_c^\infty(G(\mathbf{A})^1)$ and positive numbers $\epsilon_0$ and $\epsilon$ such that*

$$\left| J_U^T(f) - \int_{G(\mathbf{Q})\backslash G(\mathbf{A})^1} \Lambda_d^T K_U(x,x)dx \right| \leqq \|f\| e^{-\epsilon d(T)} \tag{4.3}$$

*for all $U \in (\mathscr{U}_G)$, $f \in C_c^\infty(G(\mathbf{A})^1)$ and every suitably regular $T$ with $d(T) \geqq \epsilon_0 \|T\|$.*

*Remark.* Since it is a polynomial in $T$, $J_U^T(f)$ is uniquely determined by the inequality (4.3). In particular, $J_U^T$ annihilates any function which vanishes on $U(\mathbf{A})$. Moreover, $J_U^T$ is zero if $U(\mathbf{Q})$ is empty.

*Proof.* Fix an orbit $U \in (\mathscr{U}_G)$ and let $v$ be any valuation of $\mathbf{Q}$. We shall construct $J_U^T$ by examining the behaviour of

$$J_{\mathrm{unip}}^T(f_{U,v}^\epsilon), \quad f \in C_c^\infty(G(\mathbf{A})^1),$$

as $\epsilon$ approaches 0.

We would like to estimate

$$\left| J_{\mathrm{unip}}^T(f_{U,v}^\epsilon) - \int_{G(\mathbf{Q})\backslash G(\mathbf{A})^1} \Lambda_d^T K_{\bar U}(x,x)dx \right|, \tag{4.4}$$

where

$$K_{\bar U}(x,x) = \sum_{\{U' \in (\mathscr{U}_G): U' \subset \bar U\}} K_{U'}(x,x).$$

Invoking Lemma 2.3, and observing that

$$K_{\bar U}(x,x) = \sum_{\gamma \in \bar U(\mathbf{Q})} f(x^{-1}\gamma x) = \sum_{\gamma \in \bar U(\mathbf{Q})} f_{U,v}^\epsilon(x^{-1}\gamma x),$$

we bound (4.4) by the sum of two expressions,

$$\left| J_{\mathrm{unip}}^T(f_{U,v}^\epsilon) - \int_{G(\mathbf{Q})\backslash G(\mathbf{A})^1} F(x,T)\Bigl( \sum_{\gamma \in \mathscr{U}_G(\mathbf{Q})} f_{U,v}^\epsilon(x^{-1}\gamma x) \Bigr) dx \right|$$

and

$$\int_{G(\mathbf{Q})\backslash G(\mathbf{A})^1} F(x,T) \sum_{\gamma \in \mathscr{U}_G(\mathbf{Q}) \setminus \bar U(\mathbf{Q})} |f_{U,v}^\epsilon(x^{-1}\gamma x)|dx.$$

The first expression has a bound

$$\|f^{\epsilon}_{U,\nu}\|e^{-(d(T)/2)}$$

by Theorem 3.1. This in turn is bounded by

$$\epsilon^{-lm}\|f\|'e^{-(d(T)/2)},$$

with $\|\cdot\|'$ a continuous semi-norm on $C_c^\infty(G(\mathbf{A})^1)$, as we can see with $l$ applications of Corollary 3.3. In the second expression we can sum over $G(\mathbf{Q})\backslash \bar{U}(\mathbf{Q})$ instead of $\mathscr{U}_G(\mathbf{Q})\backslash \bar{U}(\mathbf{Q})$, and then use the estimate provided by Lemma 4.1. It follows that there is a positive integer $k = lm$ and a continuous semi-norm $\|\cdot\|'$ on $C_c^\infty(G(\mathbf{A})^1)$ such that (4.4) is bounded by

$$\text{(4.5)}\quad \|f\|'(\epsilon^{-k}e^{-(d(T)/2)} + \epsilon^r(1 + \|T\|)^{d_0}).$$

In the expression (4.4) we set

$$\epsilon = \epsilon(n) = \delta^n, \quad n = 1, 2, 3, \ldots,$$

for any number $\delta$ with $0 < \delta < 1$. The result is bounded by

$$\|f\|'(e^{|\log\delta|kn-(d(T)/2)} + \delta^{rn}(1 + \|T\|)^{d_0}).$$

This in turn can clearly be bounded by an expression

$$\text{(4.6)}\quad \|f\| \cdot \delta^{rn} \cdot (1 + \|T\|)^{d_0},$$

provided that

$$d(T) \geqq C|\log \delta|n,$$

with $C$ some positive constant and $\|\cdot\|$ a continuous semi-norm on $C_c^\infty(G(\mathbf{A})^1)$. Consider this last estimate for two successive values of $n$. We obtain an inequality

$$|J^T_{\mathrm{unip}}(f^{\epsilon(n)}_{U,\nu}) - J^T_{\mathrm{unip}}(f^{\epsilon(n+1)}_{U,\nu})|$$
$$\leqq 2\|f\|\delta^{rn}(1 + \|T\|)^{d_0},$$

valid for any positive integer $n$ and any $T$ with

$$d(T) > C|\log \delta|(n + 1).$$

But

$$J^T_{\mathrm{unip}}(f^{\epsilon(n)}_{U,\nu}) - J^T_{\mathrm{unip}}(f^{\epsilon(n+1)}_{U,\nu})$$

is a polynomial in $T$ of total degree at most $d_0$. By interpolating, (see **[4]**, Lemma 5.2), we can estimate its absolute value without the restriction on $T$. We obtain a constant $A$ such that it is bounded by

$$A \cdot \|f\| \cdot \delta^{rn} \cdot (|\log \delta|(n + 1))^{d_0} \cdot (1 + \|T\|)^{d_0}$$

for all $T$. Observe that

$$\sum_{j=n}^{\infty} \delta^{rn}(|\log \delta|(n + 1))^{d_0}$$

is bounded by a constant multiple of $\delta^{rn/2}$. By telescoping the last estimate, we see that the sequence

$$\{J^T_{\mathrm{unip}}(f^{\epsilon(n)}_{U,v}) : n = 1, 2, \dots\}$$

converges. Let $J^T_{\bar{U}}(f)$ denote its limit. It is a distribution in $f$, and a polynomial in $T$ of total degree at most $d_0$. Moreover, the quantity

$$(4.7)\quad |J^T_{\mathrm{unip}}(f^{\epsilon(n)}_{U,v}) - J^T_{\bar{U}}(f)|$$

is bounded by a constant multiple of

$$\|f\|\delta^{rn/2}(1 + \|T\|)^{d_0},$$

for all $f$, $\delta$, $n$ and $T$.

Finally, we combine the estimate for (4.7) with the bound (4.6) of our original expression (4.4) (with $\epsilon = \epsilon(n)$). Fix $\delta$, and for a given suitably regular $T$ take $n$ to be the largest integer such that

$$d(T) \geqq C|\log \delta| n.$$

We find that we can define $\|\cdot\|$ and also redefine $\epsilon > 0$, so that

$$(4.8)\quad \left| J^T_{\bar{U}}(f) - \int_{G(\mathbf{Q})\backslash G(\mathbf{A})^1} \Lambda^T_d K_{\bar{U}}(x, x)dx \right| \leqq \|f\| e^{-r_0 d(T)}.$$

In particular, $J^T_{\bar{U}}(f)$ is independent of $v$ as the notation suggests.

Now $\bar{U}(\mathbf{Q})$ is the disjoint union, over those orbits $U' \in (\mathscr{U}_G)$ which are contained in $\bar{U}$, of the sets $U'(\mathbf{Q})$. Define $J^T_U(f)$ by induction on dim $U$ by

$$J^T_U(f) = J^T_{\bar{U}}(f) - \sum_{\{U' \subset \bar{U} : U' \neq U\}} J^T_{U'}(f).$$

Since $(\mathscr{U}_G)$ is finite, the required inequality (4.3) follows from (4.8). The required property (4.2) follows immediately from (4.3) and Theorem 3.1.

The following corollary will be especially important to us.

COROLLARY 4.3. *The distribution*

$$J^T_{\bar{U}}(f) = \sum_{\{U' \in (\mathscr{U}_G) : U' \subset \bar{U}\}} J^T_{U'}(f)$$

*equals*

$$\lim_{\epsilon \to 0} J^T_{\mathrm{unip}}(f^{\epsilon}_{U,v}),$$

*for any valuation $v$ of $\mathbf{Q}$. In particular, the limit is independent of $v$ and annihilates any function which vanishes on $U(\mathbf{A})$.*

*Proof.* Set $n = 1$ in the estimate for (4.7) given in the proof of the theorem. We see that there is a continuous semi-norm $\|\cdot\|$ and a positive number $r_0$ such that

$$(4.9)\quad |J_U^T(f) - J_{\text{unip}}^T(f_{U,\nu}^\delta)| \leqq \|f\| \cdot \delta^{r_0} \cdot (1 + \|T\|)^{d_0}$$

for all $f$, $T$ and $\delta > 0$. The first assertion of the corollary follows. If $f$ vanishes on $U(\mathbf{A})$ it vanishes on each of the spaces $U'(\mathbf{A})$, and so each $J_{U'}^T(f)$ equals 0. This gives the second assertion.

Of particular interest is the case that $U = \{1\}$, the class of the identity element. The corresponding distribution has a simple formula.

COROLLARY 4.4. $J_{\{1\}}^T(f) = \mathrm{vol}(G(\mathbf{Q})\backslash G(\mathbf{A})^1)f(1)$.

*Proof.* If $U = \{1\}$ we have

$$\int_{G(\mathbf{Q})\backslash G(\mathbf{A})} \Lambda_d^T K_U(x, x)dx = \int_{G(\mathbf{Q})\backslash G(\mathbf{A})^1} F(x, T)dx \cdot f(1),$$

by Lemma 2.3. This approaches

$$\mathrm{vol}(G(\mathbf{Q})\backslash G(\mathbf{A})^1) \cdot f(1)$$

by the dominated convergence theorem. The corollary is then a consequence of the theorem.

**5. Reduction of lemma 4.1.** We have still to establish Lemma 4.1, as well as Lemma 3.2 from Section 3. In this section we shall reduce the proofs to that of a third lemma, whose proof will be the content of the next section. Actually, Lemma 3.2 is much easier. It is essentially a special case of Lemma 4.1, so most of our discussion will concern this second result.

Lemma 4.1 pertains to the function $f_{U,\nu}^\epsilon$ defined by (4.1). We are required to estimate

$$\int_{G(\mathbf{Q})\backslash G(\mathbf{A})^1} F(x, T) \sum_{\gamma \in G(\mathbf{Q}) \setminus \bar{U}(\mathbf{Q})} |f_{U,\nu}^\epsilon(x^{-1}\gamma x)|dx.$$

This is bounded by the integral over $k \in K$, $a \in A_{P_0,G}^\infty(T_0, T)$ and $m$ and $n$ in fixed compact fundamental domains in $M_0(\mathbf{A})^1$ and $N_0(\mathbf{A})$ respectively, of

$$\delta_{P_0}(a)^{-1} \sum_{\gamma \in G(\mathbf{Q}) \setminus \bar{U}(\mathbf{Q})} |f_{U,\nu}^\epsilon(k^{-1}m^{-1}a^{-1}n^{-1}\gamma namk)|.$$

Since $\{a^{-1}na\}$ remains in a fixed compact set, the original expression will be no greater than

$$\int_{A_{P_0,G}^\infty(T_0,T)} \delta_{P_0}(a)^{-1} \sum_\gamma \int_\Gamma |f_{U,\nu}^\epsilon(y^{-1}a^{-1}\gamma ay)|dyda,$$

where $dy$ stands for a Radon measure on a fixed compact subset $\Gamma$ of $G(\mathbf{A})^1$. This in turn is evidently bounded by the integral over

$A^{\infty}_{P_0,G}(T_1, T)$ of

$$(5.1) \quad \delta_{P_0}(a)^{-1} \sum_{\gamma \in G_\epsilon(a)} \phi(a^{-1}\gamma a),$$

where

$$G_\epsilon(a) = \{\gamma \in G(\mathbf{Q}) \setminus \bar{U}(\mathbf{Q}) : \inf_{y \in \Gamma} \sup_{1 \leqq i \leqq l} |q_i(y^{-1}a^{-1}\gamma a y)|_v \leqq \epsilon\}$$

and

$$\phi(x) = \int_\Gamma |f(y^{-1}xy)| dy.$$

The volume of $A^{\infty}_{P_0,G}(T_1, T)$ is bounded by a multiple of $(1 + \|T\|)^{d_0}$. Therefore, to complete the proof of Lemma 4.1 it would suffice to bound (5.1) independently of $a \in A^{\infty}_{P_0}(T_1)$ by $c\epsilon^r$, where $r$ is a positive constant and $c$ is a positive number depending only on $\phi$. We might just as well take $\phi$ to be an arbitrary bounded nonnegative function on $G(\mathbf{A})^1$ of compact support. Notice that (5.1) is similar to the expression to be estimated in Lemma 3.2. If we include the case that $U$ is the empty set, with

$$q_1 = \ldots = q_l = 1,$$

the required estimate for (5.1) will also provide a proof of Lemma 3.2.

Let $P$ be a (standard) parabolic subgroup. Consider an element $\gamma$ which belongs to $P(\mathbf{Q})$ but to no (standard) parabolic subgroup $P'(\mathbf{Q})$, with $P' \subsetneq P$. Then $\gamma$ can be written

$$\gamma = \eta w \pi, \quad \eta \in N_0(\mathbf{Q}), \ \pi \in P_0(\mathbf{Q}),$$

where $w$ is an element of the Weyl group of $(G, A_0)$ whose space of fixed vectors in $\mathfrak{a}_0$ contains $\mathfrak{a}_P$ but no space $\mathfrak{a}_{P'}$, with $P' \subsetneq P$. Let $\Lambda$ be a rational representation of $G$ whose highest weight $\lambda$ is a positive integral combination of all the fundamental dominant weights in $\hat{\Delta}_{P_0} \setminus \hat{\Delta}_P$. Then $\lambda - w\lambda$ is a positive integral combination of all the roots in $\Delta^P_{P_0}$. Let $\|\cdot\|$ be a height function relative to a basis of the underlying space of $\Lambda$ which contains a highest weight vector $v$ and also the vector $\Lambda(w)v$. (See [1], p. 944.) For each $a \in A^{\infty}_{P_0}(T_1)$ the component of $\Lambda(a^{-1}\gamma a)v$ in the direction of $\Lambda(w)v$ is

$$e^{(\lambda - w\lambda)(H_{P_0}(a))}\Lambda(w)v.$$

Consequently

$$\|\Lambda(a^{-1}\gamma a)v\| \geqq e^{(\lambda - w\lambda)(H_{P_0}(a))}.$$

Fix $\phi$, and assume that $\phi(a^{-1}\gamma a) \neq 0$. Then the left hand side of this inequality will be bounded independently of $a$ and $\gamma$. It follows that for each $\alpha \in \Delta^P_{P_0}$, the number $\alpha(H_{P_0}(a))$ will be bounded independently of $a$ and $\gamma$. We may therefore write $a = ba_1$, where $a_1$ belongs to $A^{\infty}_{P,G}(T_1)$ and

$b$ belongs to a fixed compact set $B$. Notice that if we put

$$\gamma = \mu\nu, \quad \mu \in M_P(\mathbf{Q}), \nu \in N_P(\mathbf{Q}),$$

then

$$a^{-1}\gamma a = b^{-1} \cdot \mu a_1^{-1}\nu a_1 \cdot b.$$

Consequently $\mu$ itself will belong to a fixed compact set.

For each $\mu \in M_P(\mathbf{Q})$, set

$$\phi_\mu(n) = \sup_{b \in B}(\delta_{P_0}(b)^{-1}\phi(b^{-1}\mu nb)), \quad n \in N_P(\mathbf{A}).$$

Since $B$ is a fixed compact set, $\phi_\mu$ is bounded, has compact support, and vanishes for all but finitely many $\mu$. The expression (5.1) is bounded by the sum over $P$ and over $\mu \in M_P(\mathbf{Q})$ of

$$(5.2) \quad \delta_P(a_1)^{-1} \sum_\nu \phi_\mu(a_1^{-1}\nu a_1).$$

Here $a_1$ stands for the projection of $a$ onto $A^\infty_{P,G}$, and the sum is to be taken over those elements $\nu$ in $N_P(\mathbf{Q})$ such that the $\mu\nu$ does not belong to $\bar{U}(\mathbf{Q})$ and such that

$$\inf_{z \in \Gamma B} \sup_i |q_i(z^{-1}\mu a_1^{-1}\nu a_1 z)|_v \leqq \epsilon.$$

Now the functions

$$x \to q_i(z^{-1}xz), \quad z \in G, 1 \leqq i \leqq l,$$

belong to the ideal of polynomials that vanish on $\bar{U}$. It follows that there are polynomial functions

$$p_{ij}(z, x), \quad 1 \leqq i, j \leqq l,$$

on $G \times G$ such that

$$q_i(z^{-1}xz) = \sum_j p_{ij}(z, x)q_j(x).$$

If $a_1$, $\mu$ and $\nu$ are as in (5.2), we have

$$\sup_i |q_i(\mu a_1^{-1}\nu a_1)|_v \leqq c_0 \inf_{z \in \Gamma B} \sup_j |q_j(z^{-1}\mu a_1^{-1}\nu a_1 z)|_v \leqq c_0\epsilon,$$

where $c_0$ is the supremum over $(z, x)$ in the compact set

$$(\Gamma B) \times (B \cdot \operatorname{supp} \phi \cdot B)$$

of

$$\sup_i \sum_j |p_{ij}(z, zxz^{-1})|_v.$$

Then the value of (5.2) will not be decreased if the sum is taken over those

$\nu \in N_P(\mathbf{Q})$ with $\mu\nu \notin \bar{U}(\mathbf{Q})$, and with

$$\sup_i |q_i(\mu a_1^{-1} \nu a_1)|_v \leqq c_0 \epsilon.$$

It will be enough to estimate (5.2) for fixed $P$ and $\mu$. Let $\mathfrak{n}$ be the Lie algebra of $N_P$. It is an affine space equipped with an action of $A_P$. Consider the Zariski closed set of points $X$ in $\mathfrak{n}$ such that $\mu \exp X$ belongs to $\bar{U}$. The ideal $\mathscr{I}$ of polynomials on $\mathfrak{n}$ which vanish on this set is defined over $\mathbf{Q}$ and is $A_P$ invariant, since the same is true of the set itself. We can therefore write

$$\mathscr{I} = (E_1, \ldots, E_k),$$

where each $E_i$ is an $A_P$-equivariant polynomial on $\mathfrak{n}$ with rational coefficients. That is,

$$E_i(\mathrm{Ad}(a^{-1})X) = \chi_i(a)^{-1} E_i(X), \quad a \in \mathbf{A}_P,$$

with $\chi_i$ a rational character on $A_P$. However, $\mathscr{I}$ is the radical of the ideal generated by the functions

$$X \rightarrow q_j(\mu \exp X), \quad 1 \leqq j \leqq l.$$

Consequently we can find a positive integer $n$ and polynomials $F_{ij}(X)$, defined over $\mathbf{Q}$, such that

$$E_i(X)^n = \sum_{j=1}^{l} F_{ij}(X) q_j(\mu \exp X),$$

for each $i$. Let $c_1$ be the supremum over $i$, and over those points $X \in \mathfrak{n}(\mathbf{A})$ such that $\exp X$ belongs to the support of $\phi_\mu$, of

$$\sum_{j=1}^{l} |F_{ij}(X)|_v.$$

Then (5.2) is bounded by

$$(5.3) \quad \delta_P(a_1)^{-1} \sum_X \phi_\mu(a_1^{-1} \cdot \exp X \cdot a_1),$$

where $X$ is summed over those elements in $\mathfrak{n}(\mathbf{Q})$ such that

$$|E_i(\mathrm{Ad}(a_1)^{-1} X)|_v \leqq (c_1 c_0 \epsilon)^{1/n}$$

for each $i$, and such that $\mu \exp X$ does not belong to $\bar{U}(\mathbf{Q})$. But $\mu \exp X$ belongs to $\bar{U}(\mathbf{Q})$ if and only if each $E_i(X)$ vanishes. It follows that (5.3) is bounded by

$$(5.4) \quad \sum_i \left( \delta_P(a_1)^{-1} \sum_X \phi_\mu(a_1^{-1} \cdot \exp X \cdot a_1) \right),$$

with the inner sum now taken over the set

$$\{X \in \mathfrak{n}(\mathbf{Q}) : 0 < |E_i(\mathrm{Ad}(a_1)^{-1}X)|_v \leqq (c_1 c_0 \epsilon)^{1/n}\}.$$

It is certainly enough to estimate the expression in the brackets of (5.4). Therefore, we let $E$ be any rational polynomial on $\mathfrak{n}$ such that

$$E(\mathrm{Ad}(a)^{-1}U) = \chi(a)^{-1}E(U), \quad a \in A_P,\ U \in \mathfrak{n},$$

for some rational character $\chi$ of $A_P$. We shall also replace the function $\phi_\mu(\exp(\cdot))$ with an arbitrary bounded, nonnegative function $\Phi$ on $\mathfrak{n}(\mathbf{A})$ of compact support. We have reduced Lemma 4.1 (as well as Lemma 3.2) to the following assertion.

LEMMA 5.1. *There are positive numbers c and r such that for any $\epsilon > 0$ and any a in the set*

$$A_P^\infty(T_1) = \{a \in A_P^\infty : \alpha(H_P(a) - T_1) > 0,\ \alpha \in \Delta_P\},$$

*the inequality*

$$(5.5)\quad \delta_P(a)^{-1} \sum_X \Phi(\mathrm{Ad}(a)^{-1}X) \leqq c\epsilon^r$$

*holds, where X is summed over the points in* $\mathfrak{n}(\mathbf{Q})$ *with*

$$0 < |E(\mathrm{Ad}(a)^{-1}X)|_v \leqq \epsilon.$$

We shall prove this result in the next section. There is nothing to show if $P = G$, so we shall assume that $P$ is a proper (standard) parabolic subgroup of $G$.

**6. Proof of lemma 5.1.** Lemma 5.1 is really a lattice point problem. Our main tool will be the usual one: the Poisson summation formula. To be able to exploit it, we note that any bounded function of compact support can be bounded by a smooth function of compact support. Consequently, it is enough to prove the lemma with $\Phi$ a nonnegative function in $C_c^\infty(\mathfrak{n}(\mathbf{A}))$. For any $\epsilon > 0$, the function

$$\Phi_\epsilon(V) = \Phi(V)\rho_v(\epsilon^{-1}|E(V)|_v), \quad V \in \mathfrak{n}(\mathbf{A}),$$

is also in $C_c^\infty(\mathfrak{n}(\mathbf{A}))$. It is obviously enough to prove the lemma with the left hand side of (5.5) replaced by the expression

$$(6.1)\quad \delta_P(a)^{-1} \sum_{\{X \in \mathfrak{n}(\mathbf{Q}) : E(X) \neq 0\}} \Phi_\epsilon(\mathrm{Ad}(a)^{-1}X).$$

We shall first show that the existence of an $X$ which gives a nonzero contribution to (6.1) already poses a restriction on $a$ and $\epsilon$. The image under $E$ of the support of $\Phi$ is a compact subset of $\mathbf{A}$. If $X$ contributes to the sum in (6.1), the adèle

$$E(\mathrm{Ad}(a)^{-1}X) = \chi(a)^{-1}E(X)$$

will belong to this set. This gives us inequalities

$$|E(\mathrm{Ad}(a)^{-1}X)|_w \leqq b_w,$$

where $b_w$ is a positive number for each valuation $w$ of $\mathbf{Q}$ which equals 1 for almost all $w$. At the place $v$ there is the additional constraint

$$|E(\mathrm{Ad}(a)^{-1}X)|_v \leqq \epsilon.$$

Now $E(X)$ is a nonzero rational number, so

$$\prod_w |E(X)|_w = 1,$$

by the product formula. Combining the inequalities as a product, we see that

$$\chi(a)^{-1} \leqq b\epsilon,$$

where

$$b = \prod_{w \neq v} b_w.$$

Let $r_0$ and $\delta$ be small positive numbers. Given a parabolic subgroup $P_1$, with $P \subset P_1 \subsetneq G$, consider the set points $a$ in $A_P^\infty(T_1)$ such that

$$\delta \cdot \chi(a)^{r_0} \leqq \beta(a)$$

for each root $\beta$ of $(P, A_P)$ which is nontrivial on $A_{P_1}$. (Such roots exist since $P_1 \neq G$.) It is clearly possible to choose $r_0$ and $\delta$ so that any point $a$ in $A_P^\infty(T_1)$ belongs to a set of this form. (In fact, one can always arrange that $P_1$ is maximal parabolic.) We may therefore fix $P_1$ for the rest of the proof, and only estimate (6.1) for those $a$ in the corresponding set. Our two constraints on $a$ and $\epsilon$ will be used together. Combined, they will tell us that there is a constant $c_0$ such that

$$\text{(6.2)} \quad \sup_\beta(\beta(a)^{-1}) \leqq c_0\epsilon^{r_0},$$

where $\beta$ ranges over the roots of $(P, A_P)$ which are nontrivial on $A_{P_1}$. Set

$$R = P \cap M_{P_1}.$$

It is a parabolic subgroup of $M_{P_1}$. Let $\mathfrak{n}_R$ be the Lie algebra of its unipotent radical. We shall assume inductively that Lemma 5.1 holds if $(G, P, \mathfrak{n})$ is replaced by $(M_{P_1}, R, \mathfrak{n}_R)$.

Let $\mathfrak{n}_1$ be the Lie algebra of $N_{P_1}$. Then

$$\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_R.$$

Let $\mathfrak{n}_R(\mathbf{Q})^+$ be the set of elements $Y$ in $\mathfrak{n}_R(\mathbf{Q})$ such that the polynomial

$$U \to E(U + Y), \quad U \in \mathfrak{n}_1,$$

does not vanish identically. Then (6.1) is bounded by

$$B(\epsilon, a)$$
$$= \delta_P(a)^{-1} \sum_{Y \in \mathfrak{n}_R(\mathbf{Q})^+} \sum_{X_1 \in \mathfrak{n}_1(\mathbf{Q})} \Phi_\epsilon(\mathrm{Ad}(a)^{-1} X_1 + \mathrm{Ad}(a)^{-1} Y).$$

Apply the Poisson summation formula to the sum over $X_1$. The result is

$$\delta_R(a)^{-1} \sum_{Y \in \mathfrak{n}_R(\mathbf{Q})^+} \sum_{X_1 \in \mathfrak{n}_1(\mathbf{Q})} \tilde{\Phi}_\epsilon(\mathrm{Ad}(a) X_1, \mathrm{Ad}(a)^{-1} Y),$$

where

$$\tilde{\Phi}_\epsilon(U, V) = \int_{\mathfrak{n}_1(\mathbf{A})} \Phi_\epsilon(U' + V)\psi(\langle U', U\rangle) dU', \qquad U \in \mathfrak{n}_1(\mathbf{A}),\ V \in \mathfrak{n}_R(\mathbf{A}),$$

a partial Fourier transform of $\Phi_\epsilon$. Here $\psi$ is an additive character on $\mathbf{A}/\mathbf{Q}$ and $\langle , \rangle$ is the standard inner product on $\mathfrak{n}_1(\mathbf{Q})$ relative to a fixed basis $\{X_1, \ldots, X_d\}$ of eigenvectors of $A_P$. We decompose this last expression for $B(\epsilon, a)$ as the sum of

$$B_1(\epsilon, a) = \delta_R(a)^{-1} \sum_{Y \in \mathfrak{n}_R(\mathbf{Q})^+} \sum_{\{X_1 \in \mathfrak{n}_1(\mathbf{Q}): X_1 \neq 0\}} \tilde{\Phi}_\epsilon(\mathrm{Ad}(a) X_1, \mathrm{Ad}(a)^{-1} Y)$$

and

$$B_2(\epsilon, a) = \delta_R(a)^{-1} \sum_{Y \in \mathfrak{n}_R(\mathbf{Q})^+} \int_{\mathfrak{n}_1(\mathbf{A})} \Phi_\epsilon(U + \mathrm{Ad}(a)^{-1} Y) dU.$$

Now $B(\epsilon, a)$ is an increasing function of $\epsilon$. If we take $\epsilon$ to be less than 1, we have

$$B(\epsilon, a) \leqq B_1(\epsilon^s, a) + B_2(\epsilon^s, a)$$

for any number $s$, with $0 < s \leqq 1$. We shall presently show how to choose $s$ so that the constraint (6.2) leads to a good estimate.

LEMMA 6.1. *There is a bounded, nonnegative function $\Phi_R$ of compact support on $\mathfrak{n}_R(\mathbf{A})$ such that for any $\epsilon$ and $a$, $B_1(\epsilon, a)$ is bounded by*

$$\Big(\delta_R(a)^{-1} \sum_{Y \in \mathfrak{N}_R(\mathbf{Q})} \Phi_R(\mathrm{Ad}(a)^{-1} Y)\Big)\Big(\epsilon^{-1} \sup_\beta(\beta(a)^{-1})\Big)^{2\dim \mathfrak{n}_1},$$

*with $\beta$ varying as in* (6.2).

*Proof.* Let $d = \dim \mathfrak{n}_1$. The basis $\{X_1, \dots, X_d\}$ of $\mathfrak{n}_1(\mathbf{Q})$, mentioned above, gives us a natural norm $\|\cdot\|_w$ on $\mathfrak{n}_1(\mathbf{Q}_w)$ for each valuation $w$ of $\mathbf{Q}$. It also gives us a Laplacian $\Delta$ on $\mathfrak{n}_1(\mathbf{R})$.

Consider the formula above for $B_1(\epsilon, a)$. The inner sum over $X_1$ can actually be taken over the nonzero points in the lattice $\mathfrak{n}_1(N^{-1}\mathbf{Z})$, where

$$N = \prod_p p^{n_p}$$

is a positive integer defined as follows. For each prime $p$, take $n_p$ to be the smallest nonnegative integer such that $\Phi_\epsilon$ is invariant under the open compact subgroup

$$\{X_p \in \mathfrak{n}_1(\mathbf{Q}_p) : \|X_p\|_p \leqq p^{-n_p}\}.$$

The integer $N$ thus obtained is independent of $\epsilon$ if the original valuation $v$ is Archimedean. If $v$ is discrete it follows from the definition of $\Phi_\epsilon$ that $N \leqq \epsilon^{-1} N_0$, where $N_0$ is independent of $\epsilon$.

From its formula, we see that $B_1(\epsilon, a)$ is bounded by the product of

$$\sum_{\{X_1 \in \mathfrak{n}_1(N^{-1}\mathbf{Z}) : X_1 \neq 0\}} \|\mathrm{Ad}(a) X_1\|_{\mathbf{R}}^{-2d}$$

with

$$\delta_R(a)^{-1} \sum_{Y \in \mathfrak{n}_R(\mathbf{Q})} \int_{\mathfrak{n}_1(\mathbf{A})} |\Delta^d \Phi_\epsilon(U + \mathrm{Ad}(a)^{-1} y)|\, dU.$$

The first term is clearly bounded by a constant multiple of

$$(N \sup_\beta (\beta(a)^{-1}))^{2d},$$

where $\beta$ ranges over the roots of $(P, A_P)$ which are nontrivial on $A_{P_1}$. It is independent of $\epsilon$ if $v$ is infinite, and is bounded by a multiple of $\epsilon^{-2d}$ if $v$ is finite. The same is true of the second term, but with the conditions on $v$ reversed. Moreover, the second term vanishes unless $\mathrm{Ad}(a)^{-1} Y$ lies in a fixed compact subset of $\mathfrak{n}_R(\mathbf{A})$. Consequently, we can find a bounded function $\Phi_R$ of compact support on $\mathfrak{n}_R(\mathbf{A})$ such that

$$B_1(\epsilon, a) \leqq \delta_R(a)^{-1} \sum_Y \Phi_R(\mathrm{Ad}(a)^{-1} Y) \cdot \epsilon^{-2d} (\sup_\beta(\beta(a)^{-1}))^{2d},$$

as required.

Combining the lemma with (6.2) we see that $B_1(\epsilon^s, a)$ is bounded by the product of

$$(\epsilon^{-s} \cdot c_0 \epsilon^{r_0})^{2\dim \mathfrak{n}_1}$$

with

$$\delta_R(a)^{-1} \sum_{Y \in \mathfrak{n}_R(\mathbf{Q})} \Phi_R(\mathrm{Ad}(a)^{-1}Y).$$

The induction hypothesis applied to $\mathfrak{n}_R$ (with $E$ replaced by the constant polynomial 1) tells us that this second expression is bounded independently of $a$. Set $s = \frac{1}{2}r_0$. We obtain positive numbers $c_1$ and $r_1$ and an estimate

$$B_1(\epsilon^s, a) \leqq c_1\epsilon^{r_1},$$

valid for all $\epsilon > 0$ and all $a \in A_P^\infty(T_1)$ which satisfy (6.2).

We must next deal with $B_2(\epsilon^s, a)$. The estimate we will derive for this quantity will not depend on the inequality (6.2). We shall expand the polynomial $E$ in terms of the basis $\{X_1, \dots, X_d\}$ of $\mathfrak{n}_1(\mathbf{Q})$. If $V \in \mathfrak{n}_R(\mathbf{A})$ and

$$x = (x_1, \dots, x_d), \quad x_i \in \mathbf{A},$$

we can write

$$E(x_1X_1 + \dots + x_dX_d + V) = \sum_\alpha E_\alpha(V)x^\alpha, \tag{6.3}$$

where

$$\alpha = (\alpha_1, \dots, \alpha_d)$$

is a multi-index,

$$x^\alpha = x_1^{\alpha_1} \dots x_d^{\alpha_d},$$

and $E_\alpha$ is a rational polynomial on $\mathfrak{n}_R$. Clearly

$$E_\alpha(\mathrm{Ad}(a)^{-1}V) = \chi_\alpha(a)^{-1}E_\alpha(V),$$

for some rational character $\chi_\alpha$ of $A_P$. If $Y$ is an element in $\mathfrak{n}_R(\mathbf{Q})^+$, one of the coefficients $E_\alpha(Y)$ will be nonzero. We shall let $\alpha_Y$ denote the highest such multi-index, relative to the lexicographic order.

Define a function $B_2'(\epsilon, a)$ by the formula

$$\delta_R(a)^{-1} \sum_\alpha \sum_Y \int_{\mathfrak{n}_1(\mathbf{A})} \Phi_\epsilon(U + \mathrm{Ad}(a)^{-1}Y)dU,$$

in which $Y$ is summed over the set

$$\{Y \in \mathfrak{n}_R(\mathbf{Q})^+ : \alpha_Y = \alpha, |E_{\alpha_Y}(\mathrm{Ad}(a)^{-1}Y)|_v \leqq \epsilon^{1/2}\}.$$

Define $B_2''(\epsilon, a)$ similarly, $Y$ running over the same set except with

$$|E_{\alpha_Y}(\mathrm{Ad}(a)^{-1}Y)|_v > \epsilon^{1/2}.$$

Then

$$B_2(\epsilon, a) = B_2'(\epsilon, a) + B_2''(\epsilon, a).$$

The first function $B_2'(\epsilon, a)$ is clearly bounded by an expression

$$\sum_{\alpha} \left( \delta_R(a)^{-1} \sum_{Y} \Phi_R' (\mathrm{Ad}(a)^{-1} Y) \right),$$

where this time $Y$ is summed over

$$\{Y \in \mathfrak{n}_R(\mathbf{Q}) : 0 < |E_\alpha(\mathrm{Ad}(a)^{-1}Y)|_v \leqq \epsilon^{1/2}\}$$

and where

$$\Phi_R'(V) = \int_{\mathfrak{n}_1(\mathbf{R})} \Phi(U + V)dU, \quad V \in \mathfrak{n}_R(\mathbf{A}),$$

a bounded function of compact support. The sum over $\alpha$ is certainly finite. Therefore the induction hypothesis applied to $\mathfrak{n}_R$ (with $E$ replaced by $E_\alpha$) tells us that there are positive constants $c'$ and $r'$ such that

$$B_2'(\epsilon, a) \leqq c'\epsilon^{r'},$$

for all $\epsilon > 0$ and all $a \in A_P^\infty(T_1)$.

To deal with $B_2''(\epsilon, a)$ we shall make use of an elementary estimate for polynomials. Suppose that $\gamma = (\gamma_1, \ldots, \gamma_d)$ is a multi-index. For each $\delta > 0$ let $\mathscr{P}_\gamma(\delta)$ denote the set of polynomials

$$\sum e_\alpha x^\alpha, \quad e_\alpha \in \mathbf{Q}_v,$$

in $\mathbf{Q}_v[x_1, \ldots, x_d]$ which satisfy the following two conditions.

(i) $e_\alpha$ vanishes unless $\alpha_i \leqq \gamma_i$ for every $i$.

(ii) If $\alpha$ is the highest multi-index such that $e_\alpha \neq 0$, then $|e_\alpha|_v > \delta$.

Let $\Gamma_v$ be a compact subset of $\mathbf{Q}_v^d$, and set

$$\Gamma_v(p, \epsilon) = \{x \in \Gamma_v : |p(x)|_v < \epsilon\},$$

for each $p \in \mathscr{P}_\gamma(\delta)$ and $\epsilon > 0$. Then there are positive constants $C$ and $t$ such that for any $\epsilon > 0$, $\delta > 0$, and $p \in \mathscr{P}_\gamma(\delta)$,

(6.4) $\quad \mathrm{vol}\, \Gamma_v(p, \epsilon) \leqq c(\delta^{-1}\epsilon)^t.$

We leave this estimate as an exercise. Alternatively, it can be justified as a special case of Lemma 7.1 of **[6]**.

We choose the compact subset $\Gamma_v$ of $\mathbf{Q}_v^d$, as well as a compact subset $\Gamma^v$ of $(\mathbf{A}^v)^d$, so that the function

$$\Phi(x_1X_1 + \ldots + x_dX_d + V), \quad x_i \in \mathbf{A}, V \in \mathfrak{n}_R(\mathbf{A}),$$

is supported on $\Gamma_v\Gamma^v \times \mathfrak{n}_R(\mathbf{A})$. (Here $\mathbf{A}^v$ denotes those adèles which are 0 at $v$.) Look back at the definition of $B_2''(\epsilon, a)$. The integrand $\Phi_\epsilon$ is defined in terms of $\Phi$ and the polynomial

$$\sum_{\alpha} E_{\alpha}(\mathrm{Ad}(a)^{-1}Y)x^{\alpha} = E(x_1X_1 + \ldots + x_dX_d + \mathrm{Ad}(a)^{-1}Y).$$

Let $E_{\alpha}(\mathrm{Ad}(a)^{-1}Y)_v$ be the component of the adèle

$$E_{\alpha}(\mathrm{Ad}(a)^{-1}Y)$$

at $v$, and set

$$p_Y(x) = \sum_{\alpha} E_{\alpha}(\mathrm{Ad}(a)^{-1}Y)_v x^{\alpha}.$$

We can certainly choose the multi-index $\gamma$ so that every $E_{\alpha}(\mathrm{Ad}(a)^{-1}Y)_v$ vanishes unless $\alpha_i \leqq \gamma_i$ for every $i$. Then an element $Y \in \mathfrak{n}_R(\mathbf{Q})^+$ occurs in the sum which defines $B_2''(\epsilon, a)$ if and only if $p_Y$ belongs to $\mathscr{P}_{\gamma}(\epsilon^{1/2})$. It is clear that $B_2''(\epsilon, a)$ is bounded by the sum over all such $Y$, and the integral over $(x_1, \ldots, x_d)$ in $\Gamma_v(p_Y, \epsilon)\Gamma^v$, of

$$\delta_R(a)^{-1}\Phi(x_1X_1 + \ldots + x_dX_d + \mathrm{Ad}(a)^{-1}Y).$$

Therefore, by the inequality (6.4), $B_2''(\epsilon, a)$ is bounded by

$$C(\epsilon^{-1/2}\epsilon)^t \cdot \mathrm{vol}(\Gamma^v) \cdot \delta_R(a)^{-1} \sum_{Y \in \mathfrak{n}_R(\mathbf{Q})} \Phi_R''(\mathrm{Ad}(a)^{-1}Y),$$

where

$$\Phi_R''(\mathrm{Ad}(a)^{-1}Y) = \sup_{U \in \mathfrak{n}_1(\mathbf{A})} \Phi(U + \mathrm{Ad}(a)^{-1}Y).$$

Applying our original induction hypothesis to $\mathfrak{n}_R$, we see that

$$B_2''(\epsilon, a) \leqq c''\epsilon^{r''},$$

where $c''$ and $r''$ are positive numbers, independent of $\epsilon$ and $a$.

We are now essentially done. Combining our estimates we obtain

$$\begin{aligned} B(\epsilon, a) &\leqq B_1(\epsilon^s, a) + B_2'(\epsilon^s, a) + B_2''(\epsilon^s, a) \\ &\leqq c_1\epsilon^{r_1} + c'(\epsilon^s)^{r'} + c''(\epsilon^s)^{r''} \\ &\leqq c\epsilon^r, \end{aligned}$$

for positive constants $c$ and $r$. Since $B(\epsilon, a)$ was a bound for (6.1), our proof of Lemma 5.1 (and hence the earlier Lemmas 4.1 and 3.2) is finally complete.

**7. Weighted orbital integrals.** We can now return to the discussion left off in Section 4. We shall take $T = T_0$. We have already mentioned that the distribution

$$J_{\mathrm{unip}} = J_{\mathrm{unip}}^{T_0}$$

is independent of $P_0$. Set

$$J_U = J_U^{T_0}, \quad U \in (\mathscr{U}_G).$$

Then (4.2) and Corollaries 4.3 and 4.4 hold for these distributions. From Corollary 4.3 we see that $J_U$ is also independent of $P_0$. Insofar as they define polynomials in $T$, Theorem 3.1 and Theorem 4.2 provide formulas for $J_{\text{unip}}$ and $J_U$. However, these are not satisfactory. We need formulas which are given in terms of locally defined objects.

Examples of such objects are the weighted orbital integrals which are defined and studied in [**6**]. Suppose that $S$ is a finite set of places of $\mathbf{Q}$. Set

$$G(\mathbf{Q}_S)^1 = G(\mathbf{Q}_S) \cap G(\mathbf{A})^1,$$

where

$$\mathbf{Q}_S = \prod_{v \in S} \mathbf{Q}_v.$$

A weighted orbital integral is a distribution

$$f \to J_M(\gamma, f), \quad f \in C_c^\infty(G(\mathbf{Q}_S)^1),$$

on $G(\mathbf{Q}_S)^1$ which is associated to a Levi subgroup $M \in \mathscr{L}$ and a conjugacy class $\gamma$ in $M(\mathbf{Q}_S) \cap G(\mathbf{Q}_S)^1$. For this paper we need only consider the case that $\gamma$ is a unipotent conjugacy class. In fact, we shall see that it is enough to take the image in $M(\mathbf{Q}_S)$ of a unipotent conjugacy class in $M(\mathbf{Q})$.

For any $u \in \mathscr{U}_G(\mathbf{Q})$ there is an associated unipotent conjugacy class

$$u_S = \prod_{v \in S} u_v$$

in $G(\mathbf{Q}_S)^1$. Call $u$ and $u'$ $(G, S)$-equivalent if the associated conjugacy classes $u_S$ and $u'_S$ are the same. Let $(\mathscr{U}_G(\mathbf{Q}))_{G,S}$ denote the set of such equivalence classes in $\mathscr{U}_G(\mathbf{Q})$. Any element $u \in \mathscr{U}_G(\mathbf{Q})$ is contained in a unique geometric conjugacy class $U_u = U_u^G$ in $(\mathscr{U}_G)$. It depends only on the $(G, S)$ equivalence class of $u$. The set $U_u(\mathbf{Q}_S)$ breaks up into finitely many $G(\mathbf{Q}_S)^1$ conjugacy classes, one of which is $u_S$. The next lemma tells us that they are all of this form.

LEMMA 7.1. *Suppose that $U$ is any orbit in $(\mathscr{U}_G)$ such that $U(\mathbf{Q})$ is not empty. Then any $G(\mathbf{Q}_S)$ orbit in $U(\mathbf{Q}_S)$ is of form $u_S$ for some $u \in (\mathscr{U}_G(\mathbf{Q}))_{G,S}$.*

*Proof.* Let $\mathfrak{g}$ be the Lie algebra of $G$. The exponential map gives an isomorphism from the nilpotent variety of $\mathfrak{g}$ onto the unipotent variety of $G$. By assumption, $\log(U)$ is a nilpotent $G$-orbit which contains a representative in $\mathfrak{g}(\mathbf{Q})$. By the Jacobson-Morosov theorem there is a Lie algebra homomorphism

$$\phi: sl(2) \to \mathfrak{g},$$

defined over $\mathbf{Q}$, such that

$$X = \phi\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

belongs to $\log(U)$. As usual, define

$$H = \phi\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and

$$\mathfrak{g}_i = \{\xi \in \mathfrak{G}: \mathrm{ad}(H)\xi = i\xi\}.$$

Then

$$\mathfrak{p} = \bigoplus_{i \geqq 0} \mathfrak{g}_i$$

is a parabolic subalgebra of $\mathfrak{g}$ which is defined over $\mathbf{Q}$. It has unipotent radical

$$\mathfrak{n} = \bigoplus_{i > 0} \mathfrak{g}_i,$$

and Levi component

$$\mathfrak{m} = \mathfrak{g}_0,$$

both defined also over $\mathbf{Q}$.

Now take $v \in S$ and suppose that $u_v$ is an element in $U(\mathbf{Q}_v)$. There is a homomorphism

$$\phi_v: sl(2) \to \mathfrak{g},$$

defined over $\mathfrak{g}_v$, such that

$$X_v = \phi_v\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

equals $\log(u_v)$. We can introduce $H_v$, $\mathfrak{g}_{i,v}$, $\mathfrak{p}_v$ and $\mathfrak{m}_v$ as above. They are all defined over $\mathbf{Q}_v$. The vectors $X_v$ and $X$ are conjugate. By Kostant's theorem [7], $\phi_v$ and $\phi$ are also conjugate; there is an element $y \in G$ (not necessarily defined over $\mathbf{Q}_v$) such that

$$\phi = \mathrm{Ad}(y) \circ \phi_v.$$

Therefore

$$\mathfrak{p} = \mathrm{Ad}(y)\mathfrak{p}_v \quad \text{and} \quad \mathfrak{m} = \mathrm{Ad}(y)\mathfrak{m}_v.$$

Since the parabolic subalgebras $\mathfrak{p}$ and $\mathfrak{p}_v$ are both defined over $\mathbf{Q}_v$, they are conjugate over $G(\mathbf{Q}_v)$. The same is true of the Levi components $\mathfrak{m}$ and

$\mathfrak{m}_v$. Replacing $u_v$ by a $G(\mathbf{Q}_v)$-conjugate of itself if necessary, we may assume that $\mathfrak{p} = \mathfrak{p}_v$, $\mathfrak{m} = \mathfrak{m}_v$, and that $y$ belongs to the subgroup of $G$ whose Lie algebra is $\mathfrak{m}$. This forces $H_v$ and $H$ to be equal. The spaces $\mathfrak{g}_i$ and $\mathfrak{g}_{i,v}$ therefore coincide.

The spaces $\mathfrak{g}_i$ are defined over $\mathbf{Q}$ and $\mathfrak{g}_i(\mathbf{Q})$ is embedded in $\mathfrak{g}_i(\mathbf{Q}_S)$. The intersection of $U$ with $\mathfrak{g}_2$ is a subset which is open and dense in the Zariski topology. (See **[9]**.) It follows that each $G(\mathbf{Q}_S)$-orbit in $U(\mathbf{Q}_S)$ meets $\mathfrak{g}_2(\mathbf{Q}_S)$ in an open subset. We have only to show that each such open subset intersects $\mathfrak{g}_2(\mathbf{Q})$. Since $\mathfrak{g}_2(\mathbf{Q})$ is a finite dimensional vector space over $\mathbf{Q}$, the result is a consequence of the strong approximation theorem.

The lemma implies that $u \to u_S$ is a bijection from the set of $u \in (\mathscr{U}_G(\mathbf{Q}))_{G,S}$, with $U_u = U$, onto the set of $G(\mathbf{Q}_S)^1$-orbits in $U(\mathbf{Q}_S)$. We shall often drop the notation $u_S$ and simply identify $u$ with a $G(\mathbf{Q}_S)$ conjugacy class. The same will be true of $(\mathscr{U}_M(\mathbf{Q}))_{M,S}$ for any Levi subgroup $M \in \mathscr{L}$. For any $u \in (\mathscr{U}_M(\mathbf{Q}))_{M,S}$, we shall write $U_u^G$ for the induced unipotent conjugacy class of $G$ associated to $U_u^M$ and $G$ ( **[8]** ). It is the unique unipotent class in $G$ which, for any $P = MN_P$, intersects $U_u^M \cdot N_P$ in a Zariski dense open set.

For any $M \in \mathscr{L}$, $u \in (\mathscr{U}_M(\mathbf{Q}))_{M,S}$ and $f \in C_c^\infty(G(\mathbf{Q}_S)^1)$, we can take the weighted orbital integral $J_M(u, f)$. It has two properties which we should point out. The first concerns its behaviour under conjugation. Let $\mathscr{F}(M)$ be the set of $P \in \mathscr{F}$ such that $M_P$ contains $M$. Then if $y \in G(\mathbf{Q}_S)^1$,

$$(7.1)\quad J_M(u, f^y) = \sum_{Q \in \mathscr{F}(M)} J_M^{M_Q}(u, f_{Q,y}).$$

(See **[6]**, Lemma 8.1.) Note the similarity with (1.1). The second property arises from the definition. Recall from **[6]** that $J_M(u, f)$ is an integral on $U_u^G(\mathbf{Q}_S)$ with respect to a measure which is absolutely continuous relative to the sum of the invariant measures on the $G(\mathbf{Q}_S)^1$-orbits in $U_u^G(\mathbf{Q}_S)$. In particular, $J_M(u)$ annihilates any function which vanishes on $U_u^G(\mathbf{Q}_S)$. Suppose that $U$ is any class in $(\mathscr{U}_G)$ and that $v \in S$. For $\epsilon > 0$ we can define the function $f_{U,v}^\epsilon$ exactly as in (4.1). It belongs to $C_c^\infty(G(\mathbf{Q}_S)^1)$. From the definitions we deduce that

$$(7.2)\quad \lim_{\epsilon \to 0} J_M(u, f_{U,v}^\epsilon) = \begin{cases} J_M(u, f) & \text{if } U_u^G \subset \bar{U}, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the limit is independent of $v$ and annihilates any function which vanishes on $U(\mathbf{Q}_S)$. Again, recall that there is a similar property for $J_{\text{unip}}$, described in Corollary 4.3. As we shall see, properties (7.1) and (7.2) allow us to compare the weighted orbital integrals with our globally defined distributions.

**8. A comparison.** We come now to our main result. Let $S$ be any finite set of places of $\mathbf{Q}$ which contains the Archimedean place. We shall embed $C_c^\infty(G(\mathbf{Q}_S)^1)$ in $C_c^\infty(G(\mathbf{A})^1)$ by taking the product of functions in $C_c^\infty(G(\mathbf{Q}_S)^1)$ with the characteristic function of $\prod_{v \notin S} K_v$. Any function in $C_c^\infty(G(\mathbf{A})^1)$ is the image of a function in $C_c^\infty(G(\mathbf{Q}_S)^1)$ for some such $S$.

Our theorem is to be proved by induction, so we shall state it for a Levi subgroup $L$ of $G$. Everything we have done for $G$, of course, has an analogue for $L$. We shall use the same notation with the added superscript $L$. Thus we have the distribution $J_{\text{unip}}^L$ on $L(\mathbf{A})^1$ and distributions $J_M^L(u)$, $M \subset L$, on $L(\mathbf{Q}_S)^1)$.

THEOREM 8.1. *For any $S$ there are uniquely determined numbers*

$$a^M(S, u), \quad M \in \mathscr{L}, u \in (\mathscr{U}_M(\mathbf{Q}))_{M,S},$$

*such that for any $L \in \mathscr{L}$ and $f \in C_c^\infty(L(\mathbf{Q}_S)^1)$,*

$$J_{\text{unip}}^L(f) = \sum_{M \in \mathscr{L}^L} |W_0^M| \, |W_0^L|^{-1} \sum_{u \in (\mathscr{U}_M(\mathbf{Q}))_{M,S}} a^M(S, u) J_M^L(u, f). \tag{8.1}$$

*Proof.* Fix $S$ and assume inductively that the numbers $a^M(S, u)$ have been defined for any $M \subsetneq L$. Of course implicit in this assumption is the validity of (8.1) with $L$ replaced by a proper Levi subgroup. Define

$$\begin{aligned} T^L(f) = {} & J_{\text{unip}}^L(f) \\ & - \sum_{\{M \in \mathscr{L}^L : M \neq L\}} |W_0^M| \, |W_0^L|^{-1} \\ & \times \sum_{u \in (\mathscr{U}_M(\mathbf{Q}))_{M,S}} a^M(S, u) J_M^L(u, f), \end{aligned}$$

for $f \in C_c^\infty(L(\mathbf{Q}_S)^1)$. Then $T^L$ is a distribution on $L(\mathbf{Q}_S)^1$ which annihilates any function which vanishes on $\mathscr{U}_L(\mathbf{Q}_S)$. We need to show that there are uniquely determined numbers $\{a^L(S, u)\}$ such that

$$T^L(f) = \sum_{u \in (\mathscr{U}_L(\mathbf{Q}))_{L,S}} a^L(S, u) J_L^L(u, f), \tag{8.2}$$

for every $f$. Recall that $J_L^L(u)$ is just the invariant orbital integral on the conjugacy class of $u$ in $L(\mathbf{Q}_S)^1$. The distributions $\{J_L^L(u)\}$ are linearly independent so that numbers $\{a^L(S, u)\}$ must be unique. Our charge, then, is to prove their existence.

We shall first show that $T^L$ is invariant. For any $y \in L(\mathbf{Q}_S)^1$, $T^L(f^y)$ equals

$$J_{\text{unip}}^L(f^y) - \sum_{M \subsetneq L} |W_0^M| \, |W_0^L|^{-1} \sum_{u \in (\mathscr{U}_M(\mathbf{Q}))_{M,S}} a^M(S, u) J_M^L(u, f^y).$$

By (1.1) and (7.1) this is the difference between

$$\sum_{Q \in \mathscr{F}^L} |W_0^{M_Q}|\,|W_0^L|^{-1}\, J_{\mathrm{unip}}^{M_Q}(f_{Q,y})$$

and

$$\sum_{M \subsetneq L}\ \sum_{Q \in \mathscr{F}^L(M)} |W_0^M|\,|W_0^L| \sum_u a^M(S, u) J_M^{M_Q}(u, f_{Q,y}).$$

Therefore,

$$T^L(f^y) - T^L(f)$$

equals the sum over $\{Q \in \mathscr{F}^L : Q \neq L\}$ of the product of

$$|W_0^{M_Q}|\,|W_0^L|^{-1}$$

with the expression

$$J_{\mathrm{unip}}^{M_Q}(f_{Q,y}) - \sum_{M \in \mathscr{L}^{M_Q}} |W_0^M|\,|W_0^{M_Q}|^{-1} \sum_u a^M(S, u) J_M^{M_Q}(u, f_{Q,y}).$$

The assertion that this last expression vanishes is just the equation (8.1), with $L$ replaced by $M_Q$. It follows from our induction assumption. Therefore $T^L$ is an invariant distribution.

For any integer $d$, let $\mathscr{U}_{L,d}$ be the union of those orbits $U$ in $(\mathscr{U}_L)$ with $\dim U \leqq d$. It is a Zariski closed subset of $\mathscr{U}_L$ which is defined over $\mathbf{Q}$. The set

$$\mathscr{U}_{L,d}(\mathbf{Q}_S) = \prod_{v \in S} \mathscr{U}_{L,d}(\mathbf{Q}_v)$$

of $\mathbf{Q}_S$ valued points is a closed subspace of $L(\mathbf{Q}_S)$ consisting of a finite union of $L(\mathbf{Q}_S)$ conjugacy classes. Let

$$\mathscr{U}_{L,d}(\mathbf{Q}_S)' \subset \mathscr{U}_{L,d}(\mathbf{Q}_S)$$

denote the union over those orbits $U \in (\mathscr{U}_L)$ such that $\dim U \leqq d$, and such that $U(\mathbf{Q})$ is not empty, of the spaces $U(\mathbf{Q}_S)$. It is the union of those $L(\mathbf{Q}_S)$ conjugacy classes which are parametrized by elements $u \in (\mathscr{U}_L(\mathbf{Q}))_{L,S}$ with $\dim U_u^L \leqq d$. The numbers required for (8.2) are provided by the next lemma.

LEMMA 8.2. *There exist numbers*

$$a^L(S, u), \quad u \in (\mathscr{U}_L(\mathbf{Q}))_{L,S},$$

*such that for any $d$ the distribution*

$$T_d^L(f) = T^L(f) - \sum_{\{u \in (\mathscr{U}_L(\mathbf{Q}))_{L,S} : \dim U_u^L > d\}} a^L(S, u) J_L^L(u, f)$$

*annihilates any function $f \in C_c^\infty(L(\mathbf{Q}_S)^1)$ which vanishes on $\mathscr{U}_{L,d}(\mathbf{Q}_S)$.*

*Proof.* Suppose first that $d \geqq \dim \mathscr{U}_L$. Then $\mathscr{U}_{L,d}(\mathbf{Q}_S)'$ is the union of all the spaces $U(\mathbf{Q}_S)$ such that $U(\mathbf{Q})$ is not empty. Moreover, $T_d^L(f)$ is just equal to $T^L(f)$. It is the difference between the distribution

$$J_{\mathrm{unip}}^L(f) = \sum_{U \in (\mathscr{U}_L)} J_U^L(f)$$

and a sum of integrals over spaces $U(\mathbf{Q}_S)$, with $U(\mathbf{Q})$ not empty. Since $J_U^L$ is zero when $U(\mathbf{Q})$ is empty, $T_d^L$ annihilates any function which vanishes on $\mathscr{U}_{L,d}(\mathbf{Q}_S)'$. Thus, the lemma holds if $d \geqq \dim \mathscr{U}_L$.

Suppose that $d$ is arbitrary. Assume inductively that the numbers $a^L(S, u)$ have been defined for any $u$ with $\dim U_u^L > d$, and that $T_d^L$ annihilates any function which vanishes on $\mathscr{U}_{L,d}(\mathbf{Q}_S)'$. Let $\mathscr{U}_{L,d}^0$ be the union over those orbits $U$ in $(\mathscr{U}_L)$ with $\dim U = d$, and let $C^d$ be the complement of $\mathscr{U}_{L,d}^0(\mathbf{Q}_S)$ in $\mathscr{U}_{L,d}(\mathbf{Q}_S)$. Then $C^d$ equals the union over $v \in S$ and over those $U \in (\mathscr{U}_L)$ with $\dim U < d$, of the sets

$$C_{U,v}^d = U(\mathbf{Q}_v) \prod_{\{w \in S : w \neq v\}} \mathscr{U}_{L,d}(\mathbf{Q}_w).$$

It is a closed subset of $L(\mathbf{Q}_S)^1$. We shall consider the restriction of $T_d^L$ to $L(\mathbf{Q}_S)^1 \setminus C^d$, the complement of $C^d$ in $L(\mathbf{Q}_S)^1$.

The space

$$\mathscr{U}_{L,d}(\mathbf{Q}_S)' \setminus C^d = \mathscr{U}_{L,d}(\mathbf{Q}_S)' \cap \mathscr{U}_{L,d}^0(\mathbf{Q}_S)$$

is a finite disjoint union of $L(\mathbf{Q}_S)$ conjugacy classes which are closed in $L(\mathbf{Q}_S)^1 \setminus C^d$. These conjugacy classes are parametrized by the elements $u \in (\mathscr{U}_L(\mathbf{Q}))_{L,S}$ with $\dim U_u^L = d$. For each such $u$, let $L_u$ be the centralizer in $L$ of a fixed representative of $u$ in $L(\mathbf{Q})$. There is a surjective, $L(\mathbf{Q}_S)^1$-equivariant map

$$C_c^\infty(L(\mathbf{Q}_S)^1 \setminus C^d \to \bigoplus_u C_c^\infty(L(\mathbf{Q}_S)/L_u(\mathbf{Q}_S)).$$

Its kernel consists of the functions in $C_c^\infty(L(\mathbf{Q}_S)^1 \setminus C^d)$ which vanish on $\mathscr{U}_{L,d}(\mathbf{Q}_S)'$. Any invariant distribution on $L(\mathbf{Q}_S)^1 \setminus C^d$ which annihilates the kernel is the pull-back of an $L(\mathbf{Q}_S)^1$-invariant distribution on the second space. It follows that we can choose a number $a^L(S, u)$ for each $u \in (\mathscr{U}_L(\mathbf{Q}))_{L,S}$ with $\dim U_u^L = d$, such that

$$T_d^L(f) = \sum_u a^L(S, u) J_L^L(u, f)$$

for any $f \in C_c^\infty(L(\mathbf{Q}_S)^1 \setminus C^d)$.

Now if $f$ is an arbitrary function in $C_c^\infty(L(\mathbf{Q}_S)^1)$, we can set

$$T_{d-1}^L(f) = T_d^L(f) - \sum_{\{u : \dim U_u^L = d\}} a^L(S, u) J_L^L(u, f).$$

Then $T^L_{d-1}$ is an invariant distribution which is supported on $C^d$ and annihilates any function that vanishes on $\mathcal{U}_{L,d}(\mathbf{Q}_S)'$. Suppose that $f$ is a function which is assumed only to vanish on $\mathcal{U}_{L,d-1}(\mathbf{Q}_S)'$. We want to show that

$$T^L_{d-1}(f) = 0.$$

Consider the collection of sets $C^d_{U,v}$, as defined above, such that $f$ does not vanish on a neighborhood of the closure $\bar{C}^d_{U,v}$. If there are no such sets, $f$ belongs to $C^\infty_c(L(\mathbf{Q}_S)^1 \setminus C^d)$ and $T^L_{d-1}(f) = 0$. Suppose then that there are exactly $(k + 1)$ such sets, with $k \geqq 0$. Choose one set $C^d_{U,v}$ from among these. If $\epsilon > 0$, the function $f^\epsilon_{U,v}$ vanishes whenever $f$ does. Moreover, $f^\epsilon_{U,v}$ is equal to $f$ in a neighborhood of $\bar{C}^d_{U,v}$. Consequently, the function $(f - f^\epsilon_{U,v})$ will vanish in a neighborhood of the closure of all but at most $k$ sets. We may therefore assume inductively that

$$T^L_{d-1}(f - f^\epsilon_{U,v}) = 0.$$

On the other hand, since $T^L_{d-1}(f)$ is the difference between $J^L_{\mathrm{unip}}(f)$ and

$$\sum_{M \subsetneq L} |W^M_0|\,|W^L_0|^{-1} \sum_{u \in (\mathcal{U}_M(\mathbf{Q}))_{M,S}} a^M(S, u) J^L_M(u, f)$$

$$+ \sum_{\{u \in (\mathcal{U}_L(\mathbf{Q}))_{L,S} : \dim U^L_u \geqq d\}} a^L(S, u) J^L_L(u, f),$$

we will be able to write down the limit

$$\lim_{\epsilon \to 0} T^L_{d-1}(f^\epsilon_{U,v})$$

by (7.2) and Corollary 4.3. It equals

$$J^L_{\bar{U}}(f) - \sum_{M \subsetneq L} |W^M_0|\,|W^L_0|^{-1}$$

$$\times \sum_{\{u \in (\mathcal{U}_M(\mathbf{Q}))_{M,S} : U^L_u \subsetneq \bar{U}\}} a^M(S, u) J^L_M(u, f),$$

since $\dim U < d$. Since $f$ vanishes on $\mathcal{U}_{L,d-1}(\mathbf{Q}_S)'$, we see easily from the formula (4.3) that $J^L_{\bar{U}}(f) = 0$. The other terms in this last expression are clearly also equal to 0. Therefore the limit vanishes, and

$$T^L_{d-1}(f) = 0$$

as required.

We have shown that the assertion of the lemma holds if $d$ is replaced by $d - 1$. This completes the induction step and gives the proof of the lemma.

Our theorem now follows. For the lemma tells us that $T^L_d$ vanishes if

$d < 0$. This establishes (8.2), which was what remained to be proved of the theorem.

For convenience we write down separately the specialization of (8.1) to $L = G$. Since the distributions $J_M(u)$ are all measures, we have

COROLLARY 8.3. *For any* $f \in C_c^\infty(G(\mathbf{Q}_S)^1)$,

$$J_{\mathrm{unip}}(f) = \sum_{M \in \mathscr{L}} |W_0^M| \, |W_0^G|^{-1} \sum_{u \in (\mathscr{U}_M(\mathbf{Q}))_{M,S}} a^M(S, u) J_M(u, f).$$

*In particular, the restriction of* $J_{\mathrm{unip}}$ *to* $G(\mathbf{Q}_S)^1$ *is a measure.*

COROLLARY 8.4. *For any* $U \in (\mathscr{U}_G)$ *and* $f \in C_c^\infty(G(\mathbf{Q}_S)^1)$, *we have*

$$J_U(f) = \sum_{M \in \mathscr{L}} |W_0^M| \, |W_0^G|^{-1} \times \sum_{\{u \in (\mathscr{U}_M(\mathbf{Q}))_{M,S} : U_u^G = U\}} a^M(S, u) J_M(u, f).$$

*Proof.* Take any $v \in S$. Replace $f$ in (8.1) by $f^\epsilon_{U,v}$ and let $\epsilon$ approach 0. By (7.2) and Corollary 4.3 we obtain

$$\sum_{\{U' \in (\mathscr{U}_G) : U' \subset \bar{U}\}} J_{U'}(f) = \sum_{M \in \mathscr{L}} |W_0^M| \, |W_0^G|^{-1} \sum_{\{u \in (\mathscr{U}_M(\mathbf{Q}))_{M,S} : U_u^G \subset \bar{U}\}} a^M(S, u) J_M(u, f).$$

The corollary follows by increasing induction on the dimension of $U$.

For many applications of the trace formula it is probably not necessary to be able to evaluate the numbers $a^M(S, u)$. Still, we can't help wondering whether reasonable formulas exist. The methods of this paper lead naturally to only one such formula. It is an immediate consequence of the theorem and Corollary 4.4.

COROLLARY 8.5. $a^G(S, 1) = \mathrm{vol}(G(\mathbf{Q}) \backslash G(\mathbf{A})^1)$.

Finally, we should say that Theorem 8.1 and its corollaries remain true if $\mathbf{Q}$ is replaced by an algebraic number field $F$. Of course results for $\mathbf{Q}$ can always be applied to groups defined over $F$ by restricting scalars. This gives an immediate analogue of Theorem 8.1. However, restriction of scalars requires that $S$ be the set of all valuations of $F$ which lie over a finite set of valuations of $\mathbf{Q}$. On the other hand, every argument of this paper can be applied equally as well to $F$ as $\mathbf{Q}$. We have chosen to work over $\mathbf{Q}$ only to avoid introducing extra notation in our discussion of the paper [**1**]. At any rate, Theorem 8.1 and its corollaries hold for $F$ without the above restriction on $S$.

## References


1. J. Arthur, *A trace formula for reductive groups I: terms associated to classes in* $G(\mathbf{Q})$, Duke Math. J. *45* (1978), 911-952.
2. ——— *A trace formula for reductive groups II: applications of a truncation operator*, Comp. Math. *40* (1980), 87-121.
3. ——— *The trace formula in invariant form*, Ann. of Math. *114* (1981), 1-74.
4. ——— *On a family of distributions obtained from Eisenstein series I: Application of the Paley-Wiener theorem*, Amer. J. Math. *104* (1982), 1243-1288.
5. ——— *On a family of distributions obtained from Eisenstein series II: Explicit formulas*, Amer. J. Math. *104* (1982), 1289-1336.
6. ——— *The local behaviour of weighted orbital integrals*, in preparation.
7. B. Kostant, *The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group*, Amer. J. Math. *81* (1959), 973-1032.
8. G. Lusztig and N. Spaltenstein, *Induced unipotent classes*, J. London Math. Soc. *19* (1979), 41-52.
9. T. A. Springer and R. Steinberg, *Conjugacy classes*, in *Seminar on algebraic groups and related finite groups*, Lecture Notes in Math. *131* (Springer-Verlag, 1970), 167-266.


*University of Toronto,*
*Toronto, Ontario*