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The Trace Formula for Noncompact Quotient

1. In [12] and [13] Selberg introduced a trace formula for a compact, locally symmetric space of negative curvature. There is a natural algebra of operators on any such space which commute with the Laplacian. The Selberg trace formula gives the trace of these operators. Selberg also pointed out the importance of deriving such a formula when the symmetric space is assumed only to have finite volume. Then the Laplace operator will have continuous as well as discrete spectrum; it is the trace of the restriction of the operator to the discrete spectrum that is sought. Selberg gave such a formula for the quotient of the upper half plane by $\mathrm{SL}(2, \boldsymbol{Z})$. (See also [6] and [8].) Selberg also suggested how to extend the formula to any locally symmetric space of rank 1. Spaces of rank 1 are the easiest noncompact ones to handle for they can be compactified in a natural way by adding a finite number of points. I have recently obtained a trace formula for spaces of higher rank. In this article I shall illustrate the formula by looking at a typical example.

2. Let $\tilde{X}$ be the space of n by n symmetric positive definite matrices of determinant 1. The group $G = \mathrm{SL}(n, \boldsymbol{R})$ acts transitively on $\tilde{X}$ as isometries by

$$g: p \to gp\,{}^t g, \qquad p \in \tilde{X}, \qquad g \in G.$$

Since the isotropy subgroup of the identity matrix is $K = \mathrm{SO}(n, \boldsymbol{R})$, we can identify $\tilde{X}$ with the space of cosets G/K. Suppose that Γ is a discrete subgroup of G. Then the locally symmetric space

$$X = \Gamma \backslash \tilde{X}$$

can be identified with the space $\Gamma \backslash G/K$ of double cosets. We are interested in the spectrum of the Laplacian on $L^2(X)$. Let $\mathscr{H}_K$ be the space of smooth, compactly supported functions on G which are left and right invariant under K. It is a commutative algebra under convolution,

$$(f_1 * f_2)(u) = \int_G f_1(y) f_2(y^{-1}u)\,dy, \qquad u \in G.$$

For any $f \in \mathscr{H}_K$, define the operator $R(f)$ on $L^2(\Gamma\backslash G/K)$ by

$$\big(R(f)\phi\big)(x) = \int_G f(y)\phi(xy)\,dy, \qquad \phi \in L^2(\Gamma\backslash G/K).$$

This gives a homomorphism of the algebra $\mathscr{H}_K$ into the algebra of bounded operators on $L^2(\Gamma\backslash G/K)$. The corresponding representation of $\mathscr{H}_K$ on $L^2(X)$ commutes with the Laplacian. Since the Laplacian can be approximated by operators $R(f)$, the problem of the spectral decomposition of the Laplacian on $L^2(X)$ is included in that of the spectral decomposition of $\mathscr{H}_K$ on $L^2(\Gamma\backslash G/K)$.

Suppose that $f \in \mathscr{H}_K$ and $\phi \in L^2(\Gamma\backslash G/K)$. Then

$$\begin{aligned}\big(R(f)\phi\big)(x) &= \int_G f(y)\phi(xy)\,dy = \int_G f(x^{-1}y)\phi(y)\,dy \\ &= \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)\phi(y)\,dy,\end{aligned}$$

since G is unimodular and ϕ is left Γ invariant. Thus, $R(f)$ is an integral operator with a smooth kernel

$$K(x, y) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y), \qquad x, y \in \Gamma\backslash G.$$

If $\Gamma\backslash G$ is compact, the trace of the operator will be obtained by integrating the kernel over the diagonal

$$\int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(x^{-1}\gamma x)\,dx = \operatorname{tr} R(f).$$

Selberg's formula is obtained by grouping together those elements in Γ with the same eigenvalues and taking the integral separately of each such term. The result is a sum of G-invariant integrals over semisimple conjugacy classes of G.

3. From now on, we will take Γ to be the discrete subgroup $\mathrm{SL}(n, \mathbf{Z})$ of G. Then $\Gamma\backslash G$ has finite invariant volume, but is no longer compact. The integral of $K(x, y)$ over the diagonal does not converge.

However, it is possible to modify $K(x, x)$ by some functions on $\Gamma\backslash G/K$ which are supported near infinity and which reflect the various directions in which the integral can diverge. The functions are parametrized by the standard parabolic subgroups

$$P_\pi = N_\pi M_\pi,$$

indexed by partitions

$$\pi = (n_1, \ldots, n_r), \quad n_1 + \ldots + n_r = n,$$

of n. The group M_π is the intersection of G with

$$\mathrm{GL}(n_1, \boldsymbol{R}) \times \ldots \times \mathrm{GL}(n_r, \boldsymbol{R}),$$

embedded diagonally in $\mathrm{GL}(n, \boldsymbol{R})$, while N_π is the group of matrices which differ from the identity by a matrix with entries only above the diagonal blocks of M_π. It is easy to show (using a variant of Gram–Schmidt orthogonalization, for example) that any $x \in G$ can be decomposed as

$$x = nmk,$$

with $k \in K$, $n \in N_\pi$ and the element

$$m = m_1 \ldots m_r, \quad m_i \in \mathrm{GL}(n_i, \boldsymbol{R}),$$

belonging to M_π. The decomposition is not unique, but the vector

$$H_\pi(x) = (\log|\det m_1|, \ldots, \log|\det m_r|),$$

which lies in the vector space

$$\mathfrak{a}_\pi = \left\{(u_1, \ldots, u_r) \in \boldsymbol{R}^r\colon \sum u_i = 0\right\},$$

is uniquely determined by x. Note that if π_0 is the partition $(1, \ldots, 1)$ corresponding to the minimal parabolic subgroup, there is a natural projection

$$(t_1, \ldots, t_n) = T \to T_\pi = (t_1 + \ldots + t_{n_1}, t_{n_1+1} + \ldots + t_{n_2}, \ldots)$$

of $\mathfrak{a}_{\pi_0}$ onto $\mathfrak{a}_\pi$ such that $\big(H_{\pi_0}(x)\big)_\pi = H_\pi(x)$.

The modified kernel depends on a truncation parameter

$$T = (t_1, \ldots, t_n)$$

in $\mathfrak{a}_{\pi_0}$ such $t_j - t_{j+1}$ is large for each j. For any partition π let $\hat{\tau}_\pi$ be the characteristic function of the set of vectors $(u_1, \ldots, u_r)$ in $\mathfrak{a}_\pi$ such that

$$u_1 + \ldots + u_i > u_{i+1} + \ldots + u_r$$

for each $i = 1, \ldots, r-1$. The modified kernel is

$$\sum_{\pi}(-1)^{|\pi|+1}\sum_{\delta\in P_\pi\cap\Gamma\backslash G}\int_{N_\pi}\sum_{\gamma\in M_\pi} f(x^{-1}\delta^{-1}\gamma n\delta x)\hat{\tau}_\pi\big(H_\pi(\delta x)-T_\pi\big)\,dn,$$

where $|\pi| = r$ denotes the length of π. Note that if $\pi = (n)$, so that $M_\pi = G$, the function $\hat{\tau}_\pi$ is identically one and the group N_π is trivial. The corresponding summand is $K(x, x)$ itself. The other summands, as functions of x, are defined on $\Gamma\backslash G$ and are supported only near infinity.

Let $\mathcal{O}$ be the set of equivalence classes in $\Gamma = \mathrm{SL}(n, \boldsymbol{Z})$ of matrices with the same (complex) eigenvalues. The modified kernel can be written

$$\sum_{o\in\mathcal{O}} k_o^T(x, f),$$

where

$$k_o^T(x,f) = \sum_{\pi}(-1)^{|\pi|+1}\sum_{\delta\in P_\pi\cap\Gamma\backslash G}\int_{N_\pi}\sum_{\gamma\in M_\pi\cap o} f(x^{-1}\delta^{-1}\gamma n\delta x)\hat{\tau}_\pi\big(H_\pi(\delta x)-T_\pi\big)\,dn.$$

As we would hope, the function $k_o^T(x, f)$ is integrable. (One actually has to prove that $\sum_{o}\int_{\Gamma\backslash G}|k_o^T(x,f)|\,dx$ is finite [1, Theorem 7.1].) The integral

$$\int_{\Gamma\backslash G} k_o^T(x, f)\,dx,$$

defined a priori only if $t_i - t_{i+1}$ is large for each i, turns out to be a polynomial in T [3, Proposition 2.3]. We let $J_o(f)$ denote its value at $T = 0$. The left hand side of our trace formula will be

$$\sum_{o\in\mathcal{O}} J_o(f).$$

It is a generalization of the formula for compact quotient. For if the class o intersects no proper parabolic subgroup P_π, as is always the case when the quotient is compact, there are no correction terms and $J_o(f)$ is just a G invariant integral over a semisimple conjugacy class in G. In general, though, $J_o(f)$ is more complicated. If o contains only semisimple matrices, $J_o(f)$ will still be an integral over a semisimple conjugacy class, but sometimes with respect to a measure which is not G invariant. If o contains matrices which are not semisimple, $J_o(f)$ will be a sum of integrals over several conjugacy classes.

The proof of integrability requires some knowledge of the geometry $\Gamma\backslash G$ near infinity. If C is a compact fundamental domain for $N_{\pi_0}\cap\Gamma$ in N_{π_0}, the set

$$S = C\left\{\begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} : a_j > 0,\ a_j/a_{j+1} > \frac{2}{\sqrt{3}}\right\}K$$

is an approximate fundamental domain for Γ in G [7]. This means that $\Gamma S = G$, but only finitely many Γ translates of S intersect S. In particular, there are $(n-1)$ independent co-ordinates which can approach infinity. One studies the function $k_o^T(x, f)$ as x approaches infinity in the direction of each partition $\pi = (n_1, \ldots, n_r)$, in the sense that if

$$x = n\begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix}k, \quad n \in C, \quad k \in K,$$

the co-ordinates a_{n_i}/a_{n_i+1} are each large, but all the other co-ordinates a_j/a_{j+1} remain within a compact set.

4. The other main difficulty in the noncompact case is the existence of continuous spectrum. This means that the right hand side of the formula for compact quotient has also to be seriously modified. The continuous spectrum has been completely characterized in terms of the discrete spectrum of spaces of lower dimension. It is handled by means of Eisenstein series, whose study was begun by Selberg, and completed by Langlands [9], [11]. If π is a partition of n, let M_π^1 be the subgroup of elements

$$m = m_1 \ldots m_r, \quad m_i \in \mathrm{GL}(n_i, \boldsymbol{R}),$$

in M_π such that $|\det m_i| = 1$ for each i; let A_π be the subgroup of elements m such that each m_i is a positive multiple of the identity matrix. Then M_π is the direct product of M_π^1 and A_π. If $K_\pi = M_\pi \cap K$, we can define a convolution algebra $\mathscr{H}_{K_\pi}$ of functions on M_π^1 exactly as above. Eisenstein series are associated to eigenfunctions of $\mathscr{H}_{K_\pi}$ in $L^2(\Gamma \cap M_\pi \backslash M_\pi^1/K_\pi)$. Suppose that ϕ is such an eigenfunction. Set

$$\phi_\pi(x) = \phi(m),$$

for any element

$$x = mank, \quad m \in M_\pi^1, \quad a \in A_\pi, \quad n \in N_\pi, \quad k \in K.$$

If λ belongs to $\mathfrak{a}_\pi^* \otimes \boldsymbol{C}$, the space of complex linear functions on $\mathfrak{a}_\pi$, the Eisenstein series is defined by

$$E(x, \phi, \lambda) = \sum_{\delta \in \Gamma \cap P_\pi \backslash \Gamma} \phi_\pi(\delta x) e^{(\lambda + \varrho_\pi)(H_\pi(\delta x))},$$

where ϱ_π is the linear functional which maps any vector $u = (u_1, \ldots, u_r)$ in $\mathfrak{a}_\pi$ to the dot product

$$\left(-\frac{n-1}{2}, -\frac{n-3}{2}, \ldots, \frac{n-3}{2}, \frac{n-1}{2}\right)\cdot(\underbrace{u_1, \ldots, u_1}_{n_1}, \ldots, \underbrace{u_r, \ldots, u_r}_{n_r}).$$

The Eisenstein series actually converges only for certain λ, but Langlands shows that it can be analytically continued to all λ as a meromorphic function which has no poles when λ is purely imaginary. There is a functional equation which relates $E(x, \phi, \lambda)$ to the Eisenstein series in which the co-ordinates of λ are permuted by an element w in $S_{|\pi|}$ (the symmetric group on $|\pi|$ letters). For then

$$w\pi = (n_{w(1)}, \ldots, n_{w(r)})$$

is another partition to which one can associate an eigenfunction $w\phi$ and a linear functional $w\lambda$. One can choose an orthonormal basis $\mathscr{B}_\pi$ of the subspace of $L^2(\Gamma \cap M_\pi \backslash M_\pi^1 / K_\pi)$ spanned by the eigenfunctions such that $w\mathscr{B}_\pi = \mathscr{B}_{w\pi}$ for each w, and on which the functional equations are especially simple. For any $\phi \in \mathscr{B}_\pi$ the functional equation is just

$$E(x, \phi, \lambda) = m(w, \phi, \lambda) E(x, w\phi, w\lambda),$$

with $m(w, \phi, \lambda)$ a meromorphic function of λ. When λ is purely imaginary, $m(w, \phi, \lambda)$ has absolute value 1. It can be decomposed

$$m(w, \phi, \lambda) = \prod_{\{(i,j):\, i<j,\ w(i)>w(j)\}} m_\phi(\lambda_i - \lambda_j),$$

where $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $m_\phi(z)$ is a meromorphic function of one complex variable. $m_\phi(z)$ equals the classical function

$$\frac{\pi^{\frac{z}{2}} \Gamma\left(\frac{z}{2}\right) \zeta(z)}{\pi^{\frac{z+1}{2}} \Gamma\left(\frac{z+1}{2}\right) \zeta(z+1)}$$

if $\pi = \pi_0$, but is obtained from a more general L-function for arbitrary π. (See [10].)

The importance of the function $E(\cdot, \phi, \lambda)$ is, of course, that it is an eigenfunction of $\mathscr{H}_K$. Indeed, it is not difficult to see from the definition that for any $f \in \mathscr{H}_K$,

$$R(f) E(\cdot, \phi, \lambda) = E\big(\cdot, R_\pi(\hat{f}_{\pi,\lambda})\phi, \lambda\big),$$

where R_π denotes the action of $\mathscr{H}_{K_\pi}$ on $L^2(\Gamma\cap M_\pi\backslash M_\pi^1/K_\pi)$, and

$$\hat{f}_{\pi,\lambda}(m) = \int_{N_\pi}\int_{A_\pi} f(man)e^{(\lambda+\varrho_\pi)(H_\pi(a))}da\,dn, \qquad m \in M_\pi^1.$$

For each λ, $f\to\hat{f}_{\pi,\lambda}$ is in fact a homomorphism from $\mathscr{H}_K$ to $\mathscr{H}_{K_\pi}$. We are assuming that ϕ is an eigenfunction of $\mathscr{H}_{K_\pi}$; that is,

$$R_\pi(g)\phi = h_\phi(g)\phi, \qquad g \in \mathscr{H}_{K_\pi},$$

for a complex valued homomorphism h_ϕ of $\mathscr{H}_{K_\pi}$. It follows that

$$h_{\phi,\lambda}: f\to h_\phi(\hat{f}_{\pi,\lambda})$$

is a complex valued homomorphism of $\mathscr{H}_K$, and

$$R(f)E(\cdot,\phi,\lambda) = h_{\phi,\lambda}(f)E(\cdot,\phi,\lambda).$$

If $\pi = (n)$, $E(\cdot,\phi,\lambda)$ is just ϕ, which by assumption is square integrable. However if $\pi \neq (n)$, $E(\cdot,\phi,\lambda)$ will not be square integrable, and so will not lie in the discrete spectrum, Suppose that

$$\pi' = (n_1',\ldots,n_r')$$

is another partition of n, which equals $w\pi$ for some permutation w. Then if λ is purely imaginary and $\phi \in \mathscr{B}_\pi$ there is an asymptotic formula

$$e^{\varrho_{\pi'}(H_{\pi'}(x))}E(x,\phi,\lambda) \sim \sum_{\{w:\ w\pi=\pi'\}} m(w,\phi,\lambda)\cdot(w\phi)_{\pi'}(x)e^{(w\lambda)(H_{\pi'}(x))}$$

as x approaches infinity in the direction of π'. Since the function on the right is oscillatory and not square integrable in this direction, $E(\cdot,\phi,\lambda)$ cannot be square integrable. Incidentally, from this we recognize the functions $\{m(w,\phi,\lambda)\}$ as higher dimensional analogues of the classical scattering matrix.

5. Langlands shows that as π, $\phi \in \mathscr{B}_\pi$ and $\lambda \in i\mathfrak{a}_\pi^*$ vary, the Eisenstein series exhaust the spectrum. This gives a second formula

$$\sum_\pi \frac{1}{|\pi|}\sum_{\phi\in\mathscr{B}_\pi}\int_{i\mathfrak{a}_\pi^*} h_{\phi,\lambda}(f)E(x,\phi,\lambda)\overline{E(y,\phi,\lambda)}d\lambda$$

for the kernel $K(x,y)$. (It is convenient to take $d\lambda$ to be the measure on $i\mathfrak{a}_\pi^*$ which is dual to the Lebesgue measure associated to the basis

$$(1,-1,0,\ldots,0),\ldots,(0,\ldots,0,1,-1)$$

of $\mathfrak{a}_\pi$.) The summand with $\pi = (n)$ is just the kernel of the restriction of $R(f)$ to the discrete spectrum.

We have already discussed how to truncate $K(x, y)$ so that it can be integrated over the diagonal. The main result of [2] is that the second formula may be truncated in an apparently different way, more suitable to calculation, without changing the integral. The resulting integral is therefore a polynomial in the variable T of truncation. From its value at $T = 0$ we would hope to extract the trace of $R(f)$ on the discrete spectrum together with some terms. The answer turns out to be simpler than one has a right to expect. We will do no more than quote it.

Consider a partition

$$\pi = (n_1, \ldots, n_r)$$

of n. Let λ be a fixed point in $i\mathfrak{a}_\pi^*$ and let

$$\xi = (\xi_1, \ldots, \xi_r), \qquad \xi_i \in i\boldsymbol{R},$$

be a variable point in $i\mathfrak{a}_\pi^*$. (The co-ordinates ξ_i of ξ are uniquely determined modulo diagonal vectors $(\xi_0, \ldots, \xi_0)$.) Suppose for the moment that ϕ is any vector in $\mathscr{B}_\pi$. It is a simple exercise to show that

$$\sum_{w \in S_r} \frac{m(w, \phi, \lambda)^{-1} m(w, \phi, \lambda + \xi)}{(\xi_{w(1)} - \xi_{w(2)}) \ldots (\xi_{w(r-1)} - \xi_{w(r)})}$$

is a regular function of $\xi \in i\mathfrak{a}_\pi^*$—despite the apparent singularities from the denominator. Let $\mu_\pi(\phi, \lambda)$ be its value at $\xi = 0$. It is an interesting rational expression in the functions $m(w, \phi, \lambda)$ and their derivatives, which reduces to a logarithmic derivative if $r = 2$. More generally, suppose that π_1 is a partition of n which is finer than π. Then $i\mathfrak{a}_\pi^*$ is naturally embedded in $i\mathfrak{a}_{\pi_1}^*$ and $S_r = S_{|\pi|}$ represents certain cosets in $S_{|\pi_1|}$ modulo the subgroup of permutations in S_{π_1} which leave $i\mathfrak{a}_\pi^*$ pointwise fixed. Consequently, the expression above makes sense if ϕ is taken to be a vector in $\mathscr{B}_{\pi_1}$. It too is regular in $\xi \in i\mathfrak{a}_\pi^*$, so we continue to denote its value at $\xi = 0$ by $\mu_\pi(\phi, \lambda)$. Given π_1, let $\mathscr{B}_{\pi_1}(\pi)$ be the set of vectors ϕ in $\mathscr{B}_{\pi_1}$ such that $w\phi = \phi$ for each w in the subgroup of $S_{|\pi_1|}$ which leaves $i\mathfrak{a}_\pi^*$ pointwise fixed. For this set to be nonempty, π_1 must necessarily be of the form

$$\Big(\underbrace{\frac{n_1}{d_1}, \ldots, \frac{n_1}{d_1}}_{n_1}, \ldots, \underbrace{\frac{n_r}{d_r}, \ldots, \frac{n_r}{d_r}}_{n_r}\Big),$$

where each d_i is a divisor of n_i.

The formula for the integral of the second truncated kernel (at $T = 0$) ends up being

$$\sum_{\pi} \sum_{\pi_1} \sum_{\phi \in \mathscr{B}_{\pi_1}(\pi)} \frac{1}{r!(d_1 \ldots d_r)^2} \int_{i\mathfrak{a}_\pi^*} \mu_\pi(\phi, \lambda) h_{\phi,\lambda}(f) d\lambda$$

with the numbers $r, d_1, \ldots, d_r$ related to π and π_1 as above. (This formula is a special case of the main result, Theorem 8.2, of [4].) Actually, the terms must be grouped in a certain way to ensure convergence. This is because one does not know that $R(f)$ is of trace class on the discrete spectrum. Suppose for simplicity that the complication is not present. If π_1 equals (n) so does π, and the corresponding term is just the trace of $R(f)$ on the discrete spectrum. Our final formula then expresses this trace as

$$\sum_{o \in \mathcal{O}} J_o(f) - \sum_{\pi_1, \pi, \phi} \frac{1}{r!(d_1 \ldots d_r)^2} \int_{i\mathfrak{a}_\pi^*} \mu_\pi(\phi, \lambda) h_{\phi,\lambda}(f) d\lambda,$$

where the sum is over partitions π_1 and π with $\pi_1 \neq (n)$, and vectors $\phi \in \mathscr{B}_{\pi_1}(\pi)$. We reiterate that the only terms left over from the case of compact quotient correspond to classes o which meet no proper parabolic subgroup. All the other terms are peculiar to the noncompact setting.

In general, though, we do not know that $R(f)$ has a trace on the discrete spectrum. The most that can be said at present is that $R(f)$ is of trace class on the space of cusp forms, a subspace of the discrete spectrum. Grouping the terms slightly differently will then give a formula for the trace of $R(f)$ on the cusp forms.

6. The main applications of the trace formula are actually to be found in a more general situation. We change notation slightly, writing $K_{\boldsymbol{R}}$ for $\mathrm{SO}(n, \boldsymbol{R})$ and $G(\boldsymbol{R})$ for $\mathrm{SL}(n, \boldsymbol{R})$, with G now standing for the algebraic group $\mathrm{SL}(n)$. The adèle group $G(\boldsymbol{A})$ is defined as the group of elements

$$(g_{\boldsymbol{R}}, g_2, g_3, \ldots, g_p, \ldots),$$

with $g_{\boldsymbol{R}} \in G(\boldsymbol{R})$ and $g_p \in G(\boldsymbol{Q}_p)$ for every prime number p, so that g_p actually belongs to the compact group $K_p = G(\boldsymbol{Z}_p)$ for almost all p. It is a locally compact group in which $G(\boldsymbol{Q})$ embeds diagonally as a discrete subgroup. It is not hard to show that natural embedding of $G(\boldsymbol{R})$ into $G(\boldsymbol{A})$ induces a diffeomorphism

$$G(\boldsymbol{Z})\backslash G(\boldsymbol{R})/K_{\boldsymbol{R}} \xrightarrow{\simeq} G(\boldsymbol{Q})\backslash G(\boldsymbol{A})/K,$$

where

$$K = K_{\mathbf{R}} \times K_2 \times K_3 \times \ldots \times K_p \times \ldots$$

The algebra $\mathscr{H}_{K_{\mathbf{R}}}$, whose action on $L^2(G(\mathbf{Z})\backslash G(\mathbf{R})/K_{\mathbf{R}})$ we have been looking at, is now seen to be part of a larger algebra. Let $\mathscr{H}_K$ be the space of smooth, compactly supported functions on $G(\mathbf{A})$ which are left and right K invariant. It is also a commutative algebra under convolution. It acts on the space $L^2(G(\mathbf{Q})\backslash G(\mathbf{A})/K)$ (and hence also on $L^2(G(\mathbf{Z})\backslash G(\mathbf{R})/K_{\mathbf{R}})$ and on $L^2(X)$). Thus, by introducing the adèles, we can see that the spectral decomposition of $L^2(X)$ comes with some rich extra structure that is not apparent at first glance. Everything we have discussed above extends and we obtain a trace formula for any function in $\mathscr{H}_K$. Note that an eigenvalue of $\mathscr{H}_K$ will be a formal product

$$h = h_{\mathbf{R}} \cdot h_2 \cdot h_3 \ldots h_p \ldots$$

of homomorphisms. It is the relationship of these local homomorphisms with each other that is expected to carry the interesting number theoretic information.

More generally, there is no reason to ask that functions be invariant under K. The associated convolution algebra will no longer be abelian, but that does not matter. Nor does G have to be $\mathrm{SL}(n)$. It can be any reductive algebraic group over $\mathbf{Q}$. With arguments that follow the general pattern outlined above one can establish a trace formula for any operator $R(f)$ on $L^2(G(\mathbf{Q})\backslash G(\mathbf{A}))$, with f a smooth, compactly supported function on $G(\mathbf{A})$. For more details, we refer the reader to the survey article [5].

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