ON A FAMILY OF DISTRIBUTIONS OBTAINED FROM EISENSTEIN SERIES II: EXPLICIT FORMULAS

By JAMES ARTHUR*

Introduction. The purpose of this paper is to find explicit formulas for those terms in the trace formula which arise from Eisenstein series. The paper is a continuation of [1(g)]. (We refer the reader to the introduction of [1(g)] for a general discussion as well as a description of the notation we will use below.) We have already solved the most troublesome analytic problem. The difficulties which remain are largely combinatorial.

Our principal results are Theorems 4.1, 8.1 and 8.2. Theorem 4.1 contains an explicit formula for a polynomial

$$T \to P^T(B)$$

which was introduced in [1(g)]. (This polynomial depends not only on a test function $B \in C_c^\infty(i\mathfrak{h}^*/i\mathfrak{a}_G^*)$, but also on a fixed K finite function $f \in C_c^\infty(G(\mathbf{A})^1)$ and a fixed class $\chi \in \mathfrak{X}$.) We will prove Theorem 4.1, not without some effort, from an asymptotic formula for $P^T(B)$ from the previous paper ([1(g), Theorem 7.1]). We will then be able to calculate $J_\chi^T(f)$ by substituting into the formula

$$J_\chi^T(f) = \lim_{\epsilon \to 0} P^T(B^\epsilon)$$

of [1(g), Theorem 6.3]. This will lead directly to Theorem 5.2, which is the resulting formula for

$$J_\chi(f) = J_\chi^{T_0}(f).$$

The distributions J_χ give the terms in the trace formula which arise from Eisenstein series.

Manuscript received November 20, 1981.
*Partially supported by NSERC Grant A3483.

American Journal of Mathematics, Vol. 104, No. 6, pp. 1289–1336
0002-9327/82/1046-1289 $01.50

The formula for $J_\chi(f)$ provided by Theorem 5.2 is not ideal. For one thing, it is only valid for a K finite function f. What is worse, perhaps, it contains the test function B^ϵ and a limit as ϵ approaches zero. The second half of the paper is designed to rectify these defects. An arbitrary function in $C_c^\infty(G(\mathbf{A})^1)$ is certainly a limit of K finite functions. Moreover, the pointwise limit of B^ϵ as ϵ approaches zero will be 1. In view of these facts, we will be rescued by the dominated convergence theorem provided we can show that a certain multiple integral is absolutely convergent. This will be accomplished in Theorem 8.1. We will state the improved version of the formula for J_χ as Theorem 8.2.

Our main tools center around the idea of a (G, M) family of functions, introduced in [1(e), Section 6]. This notion will be useful in handling the combinatorial problems of this paper, and it will even be needed to state our final formula for J_χ. A (G, M) family is parametrized by the finite set of parabolic subgroups of G with Levi component M. Therefore, it is best not to fix a minimal parabolic subgroup, as we did in [1(g)]. In Section 1 we will recast some of the ideas of [1(g)] into a form that does not involve *standard* parabolic subgroups. We will open Section 2 by reviewing the main properties of (G, M) families. We will then begin an investigation of the asymptotic formula for $P^T(B)$ inherited from [1(g)].

Some (G, M) families can be represented geometrically by finite sets of points, of the kind which surfaced in an earlier paper [1(a)] (under the name "A_M-orthogonal sets"). A typical example, the set of restricted Weyl group translates of T, occurs prominently in the asymptotic formula for $P^T(B)$. In Section 3 we will examine some of the properties of (G, M) families of this sort. In Section 4 we will return to our study of the asymptotic formula for $P^T(B)$. The formula provides a concrete function of T which is asymptotic to the polynomial $P^T(B)$ as T approaches infinity in a certain way. The function is given as a finite sum of integrals of expressions, each of which comes from a product of two (G, M) families. In each case, one (G, M) family is independent of T, while the other is obtained from the set of Weyl translates of T. Each integral will be transformed in a natural way by the Fourier inversion formula. This will enable us to find polynomials of T which are asymptotic to the given integrals. The sum of these polynomials will give the explicit formula for $P^T(B)$ that comprises Theorem 4.1.

The second half of our paper is based on the hypothesis that the intertwining operators between induced representations on the local groups $G(\mathbf{Q}_v)$ can all be suitably normalized. This hypothesis, which was also made in [1(e), Section 7], is discussed in Section 6. Next, in Section 7, we will prove a combinatorial lemma for a special kind of (G, M) family. It will allow us to express certain functions as products of logarithmic derivatives. We will then use this result in Section 8 to reduce the proof of Theorem 8.1 to the case of parabolic rank one. Section 8 also contains our discussion of Theorem 8.2 and a reduction of its proof to Theorem 8.1. Finally, in Section 9, we will complete the proof of Theorem 8.1 by verifying it in the case of parabolic rank one.

We should note that our final formula for J_χ in Theorem 8.2 really is a direct generalization of the results known for rank one. It is interesting to compare it with the formulas of Selberg in [6(a)] and the terms (vi), (vii) and (viii) on p. 517 of [2]. Our final formula also generalizes the one obtained for GL_3 in [1(d)].

We should also point out an obvious omission from this paper. We have not discussed the invariant distributions

$$\{I_\chi : \chi \in \mathfrak{X}\}$$

defined in [1(e)]. Formulas for these distributions would be interesting, and can in fact be derived from Theorem 8.2. However, they would take us too far from the focus of the present paper.

I would like to thank Mrs. Frances Mitchell for her usual superb typing job.

1. The operators $M_{P_1|P}(s, \lambda)$. Let G be a reductive algebraic group defined over $\mathbf{Q}$. We shall adopt the notation and conventions of the preceding paper [1(g)] (especially Sections 1 and 7), often without further comment. There will be one important difference, however. We will not fix a minimal parabolic subgroup. Instead, we fix a subgroup M_0 of G, defined over $\mathbf{Q}$, which is a Levi component of some minimal parabolic subgroup of G (defined over $\mathbf{Q}$). This is the point of view of the paper [1(e)]. As in [1(e)], K will stand for a fixed maximal compact subgroup of $G(\mathbf{A})$ which is admissible relative to M_0. In this paper, a *Levi subgroup* (of G) will mean a subgroup of G which contains M_0 and is a Levi component of some parabolic subgroup of G. It is a reductive subgroup of G which is defined over $\mathbf{Q}$. If $M \subset L$ are Levi subgroups, we denote the set of Levi subgroups of L which contain M by $\mathcal{L}^L(M)$. Also, we let $\mathcal{F}^L(M)$ be the set of parabolic subgroups of L defined over $\mathbf{Q}$ which contain M, and let $\mathcal{P}^L(M)$ be the set of groups in $\mathcal{F}^L(M)$ for which M is a Levi component. Each of these three sets is finite. (If $L = G$, we shall usually denote the sets by $\mathcal{L}(M)$, $\mathfrak{F}(M)$ and $\mathcal{P}(M)$.) Suppose that $R \in \mathfrak{F}^L(M)$. Then $R = N_R M_R$, where N_R is the unipotent radical of R and M_R is the unique Levi component of R which contains M. If Q is a group in $\mathcal{P}(L)$, let $Q(R)$ be the unique group in $\mathfrak{F}(M)$ which is contained in Q. Then $M_{Q(R)} = M_R$.

Suppose that $M \in \mathcal{L}(M_0)$ is any Levi subgroup. For any parabolic subgroup P in $\mathcal{P}(M)$ we have the group A_P and the vector spaces $\mathfrak{a}_P$ and $\mathfrak{a}_P^*$. We shall often denote them by A_M, $\mathfrak{a}_M$ and $\mathfrak{a}_M^*$, since they depend only on M. To any $P \in \mathcal{P}(M)$ there correspond chambers

$$\mathfrak{a}_M(P) = \{H \in \mathfrak{a}_M : \alpha(H) > 0,\ \alpha \in \Delta_P\}$$

and

$$\mathfrak{a}_M^*(P) = \{\lambda \in \mathfrak{a}_M^* : \lambda(\alpha^\vee) > 0,\ \alpha \in \Delta_P\}.$$

The restriction of the map

$$H_P : G(\mathbf{A}) \to \mathfrak{a}_M$$

to $M(\mathbf{A})$ is also independent of P. We denote it by H_M. Suppose that $L \in \mathcal{L}(M)$. There is a natural surjective map from $\mathfrak{a}_M$ to $\mathfrak{a}_L$ whose kernel we will denote by $\mathfrak{a}_M^L$. The norm $\|\cdot\|$ on $\mathfrak{a}_0$, fixed in [1(g)], comes from a Euclidean scalar product so that $\mathfrak{a}_M^L$ has an orthogonal complement in $\mathfrak{a}_M$. We identify it with $\mathfrak{a}_L$. In other words, $\mathfrak{a}_M = \mathfrak{a}_L \oplus \mathfrak{a}_M^L$. We also have a decomposition $\mathfrak{a}_M^* = \mathfrak{a}_L^* \oplus (\mathfrak{a}_M^L)^*$ of the dual space. (These decompositions are independent of $\|\cdot\|$; in fact in [1(c)] they were introduced without the aide of a Euclidean inner product.)

In the present setting the functional equations connected with Eisenstein series take a slightly different form. They are easily derived from the usual ones, for which the reader can consult [4(b), Appendix II] or [5]. Suppose that M and M_1 are Levi subgroups. As usual, let $W(\mathfrak{a}_M, \mathfrak{a}_{M_1})$ be the set of isomorphisms from $\mathfrak{a}_M$ onto $\mathfrak{a}_{M_1}$ obtained by restricting elements in W_0, the Weyl group of (G, A_0), to $\mathfrak{a}_M$. Each $s \in W(\mathfrak{a}_M, \mathfrak{a}_{M_1})$ has a representative w_s in $G(\mathbf{Q})$. Given $s \in W(\mathfrak{a}_M, \mathfrak{a}_{M_1})$, $P \in \mathcal{P}(M)$ and $P_1 \in \mathcal{P}(M_1)$, define $(M_{P_1|P}(s, \lambda)\phi)(x)$ to be

$$\int_{N_{P_1}(\mathbf{A}) \cap w_s N_P(\mathbf{A}) w_s^{-1} \backslash N_{P_1}(\mathbf{A})} \phi(w_s^{-1} n x) e^{(\lambda + \rho_P)(H_P(w_s^{-1} n x))} e^{-(s\lambda + \rho_{P_1})(H_{P_1}(x))} dn,$$

for $\phi \in \mathcal{A}^2(P)$ and $x \in G(\mathbf{A})$. The integral converges only for $\mathrm{Re}(\lambda)$ in a certain chamber, but $M_{P_1|P}(s, \lambda)$ can be analytically continued to a meromorphic function of $\lambda \in \mathfrak{a}^*_{P,\mathbf{C}}$ with values in the space of linear maps from $\mathcal{A}^2(P)$ to $\mathcal{A}^2(P_1)$. If χ is a class in $\mathfrak{X}$ and $\pi \in \Pi(M(\mathbf{A}))$, $M_{P_1|P}(s, \lambda)$ maps the subspace $\mathcal{A}^2_{\chi,\pi}(P)$ to $\mathcal{A}^2_{\chi,s\pi}(P_1)$. The main functional equations are

$$E(x, \phi, \lambda) = E(x, M_{P_1|P}(s, \lambda)\phi, s\lambda) \tag{1.1}$$

and

$$M_{P_2|P}(s_1 s, \lambda) = M_{P_2|P_1}(s_1, s\lambda) M_{P_1|P}(s, \lambda), \tag{1.2}$$

for $s_1 \in W(\mathfrak{a}_{M_1}, \mathfrak{a}_{M_2})$ and $P_2 \in \mathcal{P}(M_2)$. Another important, but more elementary formula holds when M and M_1 are both contained in a Levi subgroup L, and r belongs to $W^L(\mathfrak{a}_M, \mathfrak{a}_{M_1})$, the subset of elements in $W(\mathfrak{a}_M, \mathfrak{a}_{M_1})$ which leave $\mathfrak{a}_L$ pointwise fixed. Suppose that $R \in \mathcal{P}^L(M)$, $R_1 \in \mathcal{P}^L(M_1)$ and $Q \in \mathcal{P}(L)$. Then for any $k \in K$ and $\phi \in \mathcal{A}(Q(R))$, the function

$$\phi_k : m \to \phi(mk), \qquad m \in M(\mathbf{A}),$$

belongs to $\mathcal{A}^2(R)$, and

$$(M_{Q(R_1)|Q(R)}(r, \lambda)\phi)_k = M_{R_1|R}(r, \lambda)\phi_k. \tag{1.3}$$

In particular, the left hand side depends only on the projection of λ onto $(\mathfrak{a}^L_M)^*_{\mathbf{C}}$.

We should point out two other functional equations, which follow easily from the definitions. Suppose $t \in W_0$. If $M \in \mathcal{L}(M_0)$ and $P \in \mathcal{P}(M)$ then $tM = w_t M w_t^{-1}$ is another Levi subgroup, and $tP = w_t P w_t^{-1}$ is a parabolic subgroup which belongs to $\mathcal{P}(tM)$. The restriction of t to $\mathfrak{a}_M$ defines an element in $W(\mathfrak{a}_M, \mathfrak{a}_{tM})$. We can associate a linear transformation

$$t : \mathcal{A}^2(P) \to \mathcal{A}^2(tP)$$

to this element by defining

$$(t\phi)(x) = \phi(w_t^{-1}x), \qquad x \in G(\mathbf{A}).$$

Now, by Lemma 1.1 of [1(e)] there is a vector T_0 in $\mathfrak{a}_0$ such that $H_{M_0}(w_t^{-1})$ equals $T_0 - t^{-1}T_0$. One can check that if $\lambda \in \mathfrak{a}^*_{M,\mathbf{C}}$ and $y \in G(\mathbf{A})$,

$$\lambda(H_P(x)) = \lambda(t^{-1}H_{tP}(w_t y) + T_0 - t^{-1}T_0).$$

From this it follows that

$$tM_{P_1|P}(s, \lambda) = M_{tP_1|P}(ts, \lambda)e^{-(s\lambda+\rho_{P_1})(T_0-t^{-1}T_0)} \tag{1.4}$$

and

$$M_{P_1|P}(s, \lambda)t^{-1} = M_{P_1|tP}(st^{-1}, t\lambda)e^{(\lambda+\rho_P)(T_0-t^{-1}T_0)} \tag{1.5}$$

for $s \in W(\mathfrak{a}_M, \mathfrak{a}_{M_1})$ and $P_1 \in \mathcal{P}(M_1)$.

Suppose that $P_0 \in \mathcal{P}(M_0)$ is a minimal parabolic subgroup, and T is a suitably regular point in $\mathfrak{a}_0(P_0)$. That is, the number

$$d_{P_0}(T) = \min_{\alpha\in\Delta_{P_0}} \{\alpha(T)\}$$

is sufficiently large. A major part of this paper is devoted to finding an explicit formula for the associated distribution $J^T_\chi(f)$. Here χ is a class in $\mathfrak{X}$ which will remain fixed for the rest of the paper, and f is a function in $C_c^\infty(G(\mathbf{A})^1)$ which, until further notice, is assumed to be K-finite. Our starting point will be Theorems 6.3 and 7.1 of [1(g)]. Let B be an arbitrary, but fixed, function in $C_c^\infty(i\mathfrak{h}^*/i\mathfrak{a}^*_G)^W$. For each $P \supset P_0$ and $\pi \in \Pi(M_P(\mathbf{A}))$, the function

$$B_\pi(\lambda) = B(iY_\pi + \lambda), \qquad \lambda \in i\mathfrak{a}^*_P/i\mathfrak{a}^*_G,$$

defined in Section 6 of [1(g)], belongs to $C_c^\infty(i\mathfrak{a}^*_P/i\mathfrak{a}^*_G)$. Our distribution may be evaluated from a polynomial $P^T(B)$, which can in turn be obtained as the asymptotic value (in T) of the expression

(1.6)

$$\sum_{P\supset P_0} \sum_{\pi\in\Pi(M_P(\mathbf{A})^1)} |\mathcal{P}(M_P)|^{-1} \int_{i\mathfrak{a}^*_P/i\mathfrak{a}^*_G} \operatorname{tr}(\omega^T_{\chi,\pi}(P, \lambda)\rho_{\chi,\pi}(P, \lambda, f))B_\pi(\lambda)d\lambda.$$

(We are preserving the convention, introduced in Section 7 of [1(g)], of identifying any representation π of $M_P(\mathbf{A})^1$ with a representation of $M_P(\mathbf{A})$ which is trivial on $A_P(\mathbf{R})^0$.)

The operator $\omega^T_{\chi,\pi}(P, \lambda)$ acts on the space $\mathfrak{A}^2_{\chi,\pi}(P)$. It is the value at $\lambda' = \lambda$ of

$$\sum_{P_1 \supset P_0} \sum_{t,t' \in W(\mathfrak{a}_P, \mathfrak{a}_{P_1})} M_{P_1|P}(t, \lambda)^{-1} M_{P_1|P}(t', \lambda') e^{(t'\lambda' - t\lambda)(T)} \theta_{P_1}(t'\lambda' - t\lambda)^{-1}.$$

(In the paper [1(g)], throughout which P_0 was fixed, we wrote $M(t, \lambda)$ for $M_{P_1|P}(t, \lambda)$. This was reasonable, since P and P_1, as the standard parabolic subgroups associated to $\mathfrak{a}_P$ and $\mathfrak{a}_{P_1}$, were uniquely determined by t.) We can write $t' = ts$, where s is a uniquely determined element in the group

$$W(\mathfrak{a}_P) = W(\mathfrak{a}_P, \mathfrak{a}_P).$$

Then $\omega^T_{\chi,\pi}(P, \lambda)$ becomes the value at $\lambda' = \lambda$ of the sum over $P_1 \supset P_0$, $t \in W(\mathfrak{a}_P, \mathfrak{a}_{P_1})$ and $s \in W(\mathfrak{a}_P)$ of

$$M_{P_1|P}(t, \lambda)^{-1} M_{P_1|P}(ts, \lambda') e^{(t(s\lambda' - \lambda))(T)} \theta_{P_1}(t(s\lambda' - \lambda))^{-1}.$$

By (1.4) we have

$$\begin{aligned} &M_{P_1|P}(t, \lambda)^{-1} M_{P_1|P}(ts, \lambda') \\ &\qquad = (tM_{Q|P}(\lambda) e^{(\lambda + \rho_Q)(T_0 - t^{-1}T_0)})^{-1} (tM_{Q|P}(s, \lambda') e^{(s\lambda' + \rho_Q)(T_0 - t^{-1}T_0)}) \\ &\qquad = M_{Q|P}(\lambda)^{-1} M_{Q|P}(s, \lambda') e^{(s\lambda' - \lambda)(T_0 - t^{-1}T_0)}, \end{aligned}$$

where Q equals $t^{-1}P_1$, another group in $\mathcal{P}(M)$, and

$$M_{Q|P}(\lambda) = M_{Q|P}(1, \lambda).$$

Notice also that

$$\theta_{P_1}(t(s\lambda' - \lambda)) = \theta_Q(s\lambda' - \lambda).$$

Define $Y_Q(T)$ to be the projection onto $\mathfrak{a}_M$ of the point

$$t^{-1}(T - T_0) + T_0.$$

Then

$$e^{(t(s\lambda'-\lambda))(T)}e^{(s\lambda'-\lambda)(T_0-t^{-1}T_0)} = e^{(s\lambda'-\lambda)(Y_Q(T))},$$

and $\omega^T_{\chi,\pi}(P, \lambda)$ is the value at $\lambda' = \lambda$ of

$$\sum_{s\in W(\mathfrak{a}_P)} \sum_{Q\in\mathcal{P}(M_P)} M_{Q|P}(\lambda)^{-1}M_{Q|P}(s, \lambda')e^{(s\lambda'-\lambda)(Y_Q(T))}\theta_Q(s\lambda' - \lambda)^{-1}.$$

We have obtained a convenient formula for the function

$$\mathrm{tr}(\omega^T_{\chi,\pi}(P, \lambda)\rho_{\chi,\pi}(P, \lambda, f))$$

in (1.6). It is the function obtained by setting $\lambda' = \lambda$ in the sum over $s \in W(\mathfrak{a}_P)$ of

(1.7)

$$\sum_{Q\in\mathcal{P}(M_P)} \mathrm{tr}[M_{Q|P}(\lambda)^{-1}M_{Q|P}(s, \lambda')\rho_{\chi,\pi}(P, \lambda, f)]e^{(s\lambda'-\lambda)(Y_Q(T))}\theta_Q(s\lambda' - \lambda)^{-1}.$$

The next few sections will be devoted to a study of this expression.

2. (G, M) families. Fix parabolic subgroups

$$P \supset P_0, \qquad P_0 \in \mathcal{P}(M_0),$$

and for simplicity set $M = M_P$. We shall also fix $\pi \in \Pi(M(\mathbf{A})^1)$ and $s \in W(\mathfrak{a}_M)$. As a function of $(\lambda', \lambda) \in \mathfrak{a}^*_{M,\mathbf{C}} \times \mathfrak{a}^*_{M,\mathbf{C}}$, the expression (1.7) is meromorphic. We would like to show that it is regular on $i\mathfrak{a}^*_M \times i\mathfrak{a}^*_M$, and to investigate its value at $\lambda' = \lambda$.

We will use the notion of a (G, M) family, introduced in Section 6 of [1(e)]. Recall that a (G, M) family is a set of smooth functions

$$c_Q(\Lambda), \qquad \Lambda \in i\mathfrak{a}^*_M,$$

indexed by the groups Q in $\mathcal{P}(M)$, which satisfy a certain compatibility condition. Namely, if Q and Q' are adjacent groups in $\mathcal{P}(M)$ and Λ lies in the hyperplane spanned by the common wall of the chambers of Q and Q' in $i\mathfrak{a}^*_M$, then $c_Q(\Lambda) = c_{Q'}(\Lambda)$. A basic result (Lemma 6.2 of [1(e)]) asserts that if $\{c_Q(\Lambda)\}$ is a (G, M) family, then

$$c_M(\Lambda) = \sum_{Q \in \mathcal{P}(M)} c_Q(\Lambda)\theta_Q(\Lambda)^{-1} \tag{2.1}$$

extends to a smooth function on $i\mathfrak{a}_M^*$. A second result, which is at the root of most of the calculations of this paper, concerns products of (G, M) families. Suppose that $\{d_Q(\Lambda): Q \in \mathcal{P}(M)\}$ is another (G, M) family. Then the function (2.1) associated to the (G, M) family

$$\{(cd)_Q(\Lambda) = c_Q(\Lambda)d_Q(\Lambda): Q \in \mathcal{P}(M)\}$$

is given by

$$(cd)_M(\Lambda) = \sum_{S \in \mathcal{F}(M)} c_M^S(\Lambda)d_S'(\Lambda), \tag{2.2}$$

([1(e), Lemma 6.3]). For any $S \in \mathcal{F}(M)$, $c_M^S(\Lambda)$ is the function (2.1) associated to the (M_S, M) family

$$\{c_R^S(\Lambda) = c_{S(R)}(\Lambda): R \in \mathcal{P}^{M_S}(M)\},$$

and $c_S'(\Lambda)$ is a certain smooth function on $i\mathfrak{a}_M^*$ which depends only on the projection of Λ onto $i\mathfrak{a}_{M_S}^*$. The values at $\Lambda = 0$ of the functions $c_M(\Lambda)$ and $c_S'(\Lambda)$ are of special interest. Following a convention from [1(e)], we shall often suppress $\Lambda = 0$ from the notation, writing $c_M = c_M(0)$ and $c_S' = c_S'(0)$.

We should also recall, for later use, that for any (G, M) family $\{c_Q(\Lambda): Q \in \mathcal{P}(M)\}$ and any $L \in \mathcal{L}(M)$ there is associated a natural (G, L) family. Let Λ be constrained to lie in $i\mathfrak{a}_L^*$ and choose $Q_1 \in \mathcal{P}(L)$. The compatibility condition implies that the function

$$c_Q(\Lambda), \qquad Q \in \mathcal{P}(M), \qquad Q \subset Q_1,$$

is independent of Q. We denote it by $c_{Q_1}(\Lambda)$. Then

$$\{c_{Q_1}(\Lambda): Q_1 \in \mathcal{P}(L)\}$$

is a (G, L) family. We write

$$c_L(\Lambda) = \sum_{Q_1 \in \mathcal{P}(L)} c_{Q_1}(\Lambda)\theta_{Q_1}(\Lambda)^{-1}$$

for the corresponding function (2.1).

We return to our original problem. The formula (2.1) is our clue for dealing with (1.7). It suggests that we define

$$\Lambda = s\lambda' - \lambda.$$

The only possible singularities of (1.7) on $i\mathfrak{a}_M^* \times i\mathfrak{a}_M^*$ are at the zeros of the functions

$$\theta_Q(s\lambda' - \lambda) = \theta_Q(\Lambda).$$

Therefore in proving its regularity we can study (1.7) as a function of Λ. Define

$$c_Q(\Lambda) = e^{\Lambda(Y_Q(T))}, \qquad Q \in \mathcal{P}(M),$$

and

$$d_Q(\Lambda) = \mathrm{tr}(M_{Q|P}(\lambda)^{-1} M_{Q|P}(s, \lambda')\rho_{\chi,\pi}(P, \lambda, f)), \qquad Q \in \mathcal{P}(M).$$

Then $\{c_Q(\Lambda)\}$ is a (G, M) family. (See the first example in Section 7 of [1(e)].) We would like to show the same of $\{d_Q(\Lambda)\}$. The right hand side of the formula for $d_Q(\Lambda)$ is analytic at any imaginary values of λ' and λ, so d_Q is a smooth function on $i\mathfrak{a}_M^*$. Let Q' be a group in $\mathcal{P}(M)$ which is adjacent to Q. Then

$$\begin{aligned} d_{Q'}(\Lambda) &= \mathrm{tr}(M_{Q'|P}(\lambda)^{-1} M_{Q'|P}(s, \lambda')\rho_{\chi,\pi}(P, \lambda, f)) \\ &= \mathrm{tr}(M_{Q|P}(\lambda)^{-1} \cdot M_{Q'|Q}(\lambda)^{-1} M_{Q'|Q}(s\lambda') \cdot M_{Q|P}(s, \lambda')\rho_{\chi,\pi}(P, \lambda, f)). \end{aligned}$$

If Λ belongs to the hyperplane $i\mathfrak{a}_1^*$ spanned by the common wall of the chambers of Q and Q' in $i\mathfrak{a}_M^*$, the points λ and $s\lambda'$ will have the same projections onto $i\mathfrak{a}_M^*/i\mathfrak{a}_1^*$. In this case

$$M_{Q'|Q}(\lambda)^{-1} M_{Q'|Q}(s\lambda') = 1$$

by (1.3), and

$$\begin{aligned} d_{Q'}(\Lambda) &= \mathrm{tr}(M_{Q|P}(\lambda)^{-1} M_{Q|P}(s, \lambda')\rho_{\chi,\pi}(P, \lambda, f)) \\ &= d_Q(\Lambda). \end{aligned}$$

This is the required condition. Therefore $\{d_Q(\Lambda)\}$ is a (G, M) family.

The expression (1.7) equals

$$\sum_{Q \in \mathcal{P}(M)} c_Q(\Lambda) d_Q(\Lambda) \theta_Q(\Lambda)^{-1},$$

which extends to a smooth function of $\Lambda \in i\mathfrak{a}_M^*$. It follows that (1.7) extends to a smooth function of $(\lambda', \lambda) \in i\mathfrak{a}_M^* \times i\mathfrak{a}_M^*$.

Keep in mind that s is a fixed element in the Weyl group of $\mathfrak{a}_M = \mathfrak{a}_P$. It can be represented by an element in the normalizer of M in G. Let L be the smallest Levi subgroup in $\mathcal{L}(M)$ which contains a representative of s. Then

$$\mathfrak{a}_L = \{H \in \mathfrak{a}_M : sH = H\},$$

the space of fixed vectors of s in $\mathfrak{a}_M$, and $\mathfrak{a}_M^L$, the orthogonal complement of $\mathfrak{a}_L$ in $\mathfrak{a}_M$, is an invariant subspace of s.

From now on we will take $\lambda' = \lambda + \zeta$, where ζ is restricted to lie in the subspace $i\mathfrak{a}_L^*$. Then $s\zeta = \zeta$, and

$$\Lambda = (s\lambda - \lambda) + \zeta$$

is the decomposition of Λ relative to

$$i\mathfrak{a}_M^* = i(\mathfrak{a}_M^L)^* \oplus i\mathfrak{a}_L^*.$$

If λ_L is the projection of λ onto $i\mathfrak{a}_L^*$, the map

$$(\lambda, \zeta) \to (\Lambda, \lambda_L), \qquad \lambda \in i\mathfrak{a}_M^*, \qquad \zeta \in i\mathfrak{a}_L^*,$$

is a linear automorphism of the vector space $i\mathfrak{a}_M^* \oplus i\mathfrak{a}_L^*$. In particular, the points λ and $\lambda' = \lambda + \zeta$ are uniquely determined by Λ and λ_L. Write

$$c_Q(T, \Lambda) = e^{\Lambda(Y_Q(T))}, \qquad Q \in \mathcal{P}(M), \tag{2.3}$$

and

(2.4)

$$d_Q(\lambda_L, \Lambda) = \operatorname{tr}(M_{Q|P}(\lambda)^{-1} M_{Q|P}(s, \lambda + \zeta) \rho_{\chi,\pi}(P, \lambda, f)), \qquad Q \in \mathcal{P}(M),$$

to keep track of all the variables. These of course are still (G, M) families (in the variable Λ), and for $\lambda' = \lambda + \zeta$ as above, the expression (1.7) equals

$$\sum_{Q \in \mathcal{P}(M)} c_Q(T, \Lambda) d_Q(\lambda_L, \Lambda) \theta_Q(\Lambda)^{-1}.$$

Now apply (2.2). We obtain

$$\sum_{S \in \mathcal{F}(M)} c_M^S(T, \Lambda) d_S'(\lambda_L, \Lambda).$$

To evaluate (1.7) at $\lambda' = \lambda$ we set $\zeta = 0$. This simply entails replacing Λ in this last expression by $s\lambda - \lambda$.

We have established

LEMMA 2.1. *The expression* (1.7) *extends to a smooth function of* $(\lambda', \lambda) \in (i\mathfrak{a}_M^* \times i\mathfrak{a}_M^*)$. *Its value at* $\lambda' = \lambda$ *equals*

$$\sum_{S \in \mathcal{F}(M)} c_M^S(T, s\lambda - \lambda) d_S'(\lambda_L, s\lambda - \lambda),$$

where $\{c_Q(T, \Lambda)\}$ *and* $\{d_Q(\lambda_L, \Lambda)\}$ *are the* (G, M) *families defined by* (2.3) *and* (2.4) *respectively.* □

3. Some remarks on convex hulls. The dependence of the expressions in Section 1 on T has been isolated in the functions $c_M^S(T, \Lambda)$. These functions are closely related to certain convex hulls.

Suppose in general that M is a group in $\mathcal{L}(M_0)$ and that

$$\mathcal{Y} = \{Y_Q : Q \in \mathcal{P}(M)\}$$

is a set of points in $\mathfrak{a}_M$ indexed by the groups in $\mathcal{P}(M)$. Assume that for every pair (Q', Q) of adjacent groups in $\mathcal{P}(M)$, $Y_{Q'} - Y_Q$ is a positive multiple of the co-root $\alpha^\vee$, where α is the unique root in $\Delta_{Q'} \cap (-\Delta_Q)$. Then $\mathcal{Y}$ is what we called in [1(a)] a positive A_M-orthogonal set. It is clear that the functions

$$c_Q(\Lambda) = e^{\Lambda(Y_Q)}, \qquad Q \in \mathcal{P}(M),$$

form a (G, M) family. From the discussion in Section 6 of [1(e)] it follows that the function

$$c_M(\Lambda) = \sum_{Q \in \mathcal{P}(M)} e^{\Lambda(Y_Q)} \theta_Q(\Lambda)^{-1}$$

is the Fourier transform of the characteristic function of the convex hull of $\mathcal{Y}$ (regarded as a compactly supported distribution on $\mathfrak{a}_M$). More generally, suppose that $L \in \mathcal{L}(M)$ and that S is a group in $\mathcal{P}(L)$. The set

$$\mathcal{Y}_M^S = \{Y_{S(R)} : R \in \mathcal{P}^L(M)\}$$

is another positive A_M-orthogonal set, but this time with the underlying group being L instead of G. All the elements in $\mathcal{Y}_M^S$ project onto a common point in $\mathfrak{a}_L$, which we denote by Y_S. The convex hull of $\mathcal{Y}_M^S$ is the translate by Y_S of a set of positive measure in $\mathfrak{a}_M^L$. We shall denote its characteristic function by χ_M^S. Then

$$c_M^S(\Lambda) = \int_{Y_S + \mathfrak{a}_M^L} \chi_M^S(H) e^{\Lambda(H)} dH. \tag{3.1}$$

Finally, observe that the collection

$$\mathcal{Y}_L = \{Y_S : S \in \mathcal{P}(L)\}$$

is a positive A_L-orthogonal set.

LEMMA 3.1. *Suppose that H is a point in $\mathfrak{a}_M$ which belongs to the convex hull of $\mathcal{Y}$. Then the projection of H onto $\mathfrak{a}_L$ belongs to the convex hull of $\mathcal{Y}_L$.*

Proof. A necessary and sufficient condition that H belong to the convex hull of $\mathcal{Y}$ is that the projection of H onto $\mathfrak{a}_G$ equal Y_G and that

$$\varpi(Y_P - H) \geq 0$$

for every $P \in \mathcal{P}(M)$ and $\varpi \in \hat{\Delta}_P$. (See [1(a), Lemma 3.2]. Following the custom of previous papers we will let $\hat{\Delta}_P$ denote the basis of $(\mathfrak{a}_M^G)^*$ which is dual to the basis $\{\alpha^\vee : \alpha \in \Delta_P\}$ of $\mathfrak{a}_M^G$.) We are assuming that H belongs to the convex hull of $\mathcal{Y}$. Let H_L be the projection of H onto $\mathfrak{a}_L$. We must show for any $S \in \mathcal{P}(L)$ and $\varpi \in \hat{\Delta}_S$ that $\varpi(Y_S - H)$ is nonnegative. Let R be any group in $\mathcal{P}^L(M)$. Any linear function ϖ in $\hat{\Delta}_S$ vanishes on $\mathfrak{a}_M^L$, so that

$$\varpi(Y_S - H_L) = \varpi(Y_{S(R)} - H).$$

Since $\hat{\Delta}_S$ is a subset of $\hat{\Delta}_{S(R)}$, this number is nonnegative by hypothesis. It follows that H_L belongs to the convex hull of $\mathcal{Y}_L$. $\square$

Let $d(\mathcal{Y})$ be the smallest of the numbers

$$\{\alpha(Y_P):P \in \mathcal{P}(M),\ \alpha \in \Delta_P\}.$$

Let us assume that $d(\mathcal{Y})$ is positive. This means that for each P, Y_P belongs to the chamber $\mathfrak{a}_M(P)$. The next lemma provides a partial converse to the last one.

LEMMA 3.2. *There is a positive constant C_1, depending only on G, with the following property. Let H be a point in $\mathfrak{a}_M$ whose projection, H_L, onto $\mathfrak{a}_L$ belongs to the convex hull of $\mathcal{Y}_L$. Then H belongs to the convex hull of $\mathcal{Y}$, provided that*

$$\|H - H_L\| \leq C_1 d(\mathcal{Y}).$$

Proof. We shall actually show how to choose C_1 so that H belongs to the convex hull of

$$\bigcup_{S \in \mathcal{P}(L)} \mathcal{Y}_M^S = \{Y_{S(R)}:S \in \mathcal{P}(L), R \in \mathcal{P}^L(M)\},$$

a subset of $\mathcal{Y}$. Let

$$U = H - H_L,$$

the projection of H onto $\mathfrak{a}_M^L$. By hypothesis,

$$H_L = \sum_{S \in \mathcal{P}(L)} r_S Y_S,$$

for numbers r_S, $0 \leq r_S \leq 1$, such that

$$\sum_{S \in \mathcal{P}(L)} r_S = 1.$$

Then

$$H = H_L + U = \sum_{S \in \mathcal{P}(L)} r_S(Y_S + U).$$

It is clearly enough to show that $Y_S + U$ belongs to the convex hull of $\mathcal{Y}_M^S$. Suppose this is not so. Then by [1(a), Lemma 3.2] there is a group $R \in \mathcal{P}^L(M)$ and a linear function $\varpi \in \hat{\Delta}_R$ such that the number

$$\varpi(Y_{S(R)} - (Y_S + U)) = \varpi(Y_{S(R)} - U)$$

is negative. It is well known that the chamber

$$\mathfrak{a}_M(R) = \{H \in \mathfrak{a}_M : \alpha(H) > 0, \alpha \in \Delta_R\}$$

is contained in its dual chamber

$$\{H \in \mathfrak{a}_M : \varpi(H) > 0, \varpi \in \hat{\Delta}_R\}.$$

It follows that there is an $\alpha \in \Delta_R$ such that $\alpha(Y_{S(R)} - U)$ is also negative. Since Δ_R is a subset of $\Delta_{S(R)}$, we have

$$d(\mathcal{Y}) \leq \alpha(Y_{S(R)}) < \alpha(U) \leq \|\alpha\| \, \|U\|.$$

Let C_1 be the smallest of the numbers

$$\{\|\alpha\|^{-1} : \alpha \in \Delta_Q, Q \in \mathcal{P}(M)\}.$$

Then if $\|U\| \leq C_1 d(\mathcal{Y})$, the inequality above is contradicted. In other words, $Y_S + U$ belongs to the convex hull of $\mathcal{Y}_M^S$. □

The example we have in mind, of course, is the set

$$\mathcal{Y}(T) = \{Y_Q(T) : Q \in \mathcal{P}(M)\},$$

defined in Section 1. With this example comes a fixed pair of groups $P \in \mathcal{P}(M)$ and $P_0 \in \mathcal{P}(M_0)$, with $P \supset P_0$. Suppose that $Q \in \mathcal{P}(M)$ and that $\alpha \in \Delta_Q$. Remember that $Y_Q(T)$ is the projection onto $\mathfrak{a}_M$ of $t^{-1}(T - T_0) + T_0$, where t is any element in W_0 such that $P_1 = tQ$ contains P_0. Now

$$\alpha(Y_Q(T)) = \alpha(t^{-1}(T - T_0) + T_0) = \beta(T - T_0 + tT_0),$$

where $\beta = t\alpha$ is a root in Δ_{P_1}. There is a unique root β_0 in Δ_{P_0} whose restriction to $\mathfrak{a}_{P_1}$ equals β. It is a simple exercise (which we leave to the reader) to check that if X is any point in the chamber $\mathfrak{a}_0(P_0)$, then $\beta(X) \geq \beta_0(X)$. We always take T to be in this chamber, so that

$$\beta(T) \geq \beta_0(T) \geq d_{P_0}(T).$$

It follows that there is a constant C_0 such that

$$d(\mathcal{Y}(T)) \geq d_{P_0}(T) - C_0. \tag{3.2}$$

We are assuming that T is suitably regular in $\mathfrak{a}_0(P_0)$, so that $d(\mathcal{Y}(T))$ is positive. It follows easily that $\mathcal{Y}(T)$ is a *positive*, A_M-orthogonal set. In particular, the two lemmas above apply.

4. Evaluation of $P^T(B)$. We continue with a fixed group $P_0 \in \mathcal{P}(M_0)$ and a function $B \in C_c^\infty(i\mathfrak{h}^*/i\mathfrak{a}_G^*)^W$. We are going to find a formula for the polynomial $P^T(B)$. For the moment, $P \supset P_0$, $M = M_P$, π, s and L will also remain fixed, as in Section 2. Consider, in the notation of Lemma 2.1, the integral

$$\int_{i\mathfrak{a}_M^*/i\mathfrak{a}_G^*} \Big(\sum_{S \in \mathcal{F}(M)} c_M^S(T, s\lambda - \lambda) d_S'(\lambda_L, s\lambda - \lambda) \Big) B_\pi(\lambda) d\lambda. \tag{4.1}$$

We shall show that it is asymptotic to a polynomial in T, which we will calculate explicitly. This will allow us to obtain $P^T(B)$ simply by summing over P, π and s.

Let μ_λ be the projection of λ onto $i(\mathfrak{a}_M^L)^*$. Then $\lambda = \mu_\lambda + \lambda_L$, and

$$s\lambda - \lambda = s\mu_\lambda - \mu_\lambda.$$

We shall decompose the integral (4.1) as a double integral over $i(\mathfrak{a}_M^L)^*$ and $i\mathfrak{a}_L^*/i\mathfrak{a}_G^*$. Notice that

$$F_s : \mu \to s\mu - \mu, \qquad \mu \in i(\mathfrak{a}_M^L)^*,$$

is a linear isomorphism of $i(\mathfrak{a}_M^L)^*$. We can use it to change variables in the integral over $i(\mathfrak{a}_M^L)^*$, as long as we remember to divide by its determinant. It follows that (4.1) equals the product of

$$|\det(s - 1)_{\mathfrak{a}_M^L}|^{-1}$$

with the sum over $S \in \mathfrak{F}(M)$ of

$$\int_{i(\mathfrak{a}_M^L)^*} \int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} c_M^S(T, \mu) d_S'(\lambda, \mu) B_\pi(F_s^{-1}(\mu) + \lambda) d\lambda d\mu. \tag{4.2}$$

Let S be a fixed group in $\mathfrak{F}(M)$. We write $\chi_M^S(T, \cdot)$ for the characteristic function in $\mathfrak{a}_M$ of the convex hull of the set $\mathcal{Y}_M^S(T)$. The formula (3.1) becomes

$$c_M^S(T, \mu) = \int_{Y_S(T) + \mathfrak{a}_M^{M_S}} \chi_M^S(T, H) e^{\mu(H)} dH,$$

which we can substitute into the expression (4.2). We obtain

$$\int_{Y_S(T) + \mathfrak{a}_M^{M_S}} \chi_M^S(T, H) \phi_S(H) dH, \tag{4.3}$$

where

$$\phi_S(H) = \int_{i(\mathfrak{a}_M^L)^*} \int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} e^{\mu(H)} d_S'(\lambda, \mu) B_\pi(F_s^{-1}(\mu) + \lambda) d\lambda d\mu,$$

for any $H \in \mathfrak{a}_M$. The function $d_S'(\lambda, \mu)$ is smooth in each variable, and the function

$$(\mu, \lambda) \to B_\pi(F_s^{-1}(\mu) + \lambda), \qquad \mu \in i(\mathfrak{a}_M^L)^*, \qquad \lambda \in i\mathfrak{a}_L^*/i\mathfrak{a}_G^*,$$

is smooth and compactly supported. It follows that ϕ_S, which is clearly $\mathfrak{a}_L$ invariant, is a Schwartz function on $\mathfrak{a}_M/\mathfrak{a}_L$. We are going to study the behaviour of (4.3) as T approaches infinity strongly in $\mathfrak{a}_0(P_0)$. To this end, we assume that

$$d_{P_0}(T) \geq \delta \| T \|,$$

for a fixed positive constant δ, and let $\| T \|$ approach infinity.

There are two cases to consider. Suppose first of all that S does not belong to $\mathfrak{F}(L)$. This means that $\mathfrak{a}_S$ is not contained in $\mathfrak{a}_L$. In particular, there is a root α of (G, A_M) which vanishes on $\mathfrak{a}_L$ but does not vanish on

$\mathfrak{a}_S$. In fact by choosing the sign of α properly we can arrange that the restriction of α to $\mathfrak{a}_S$ is a root of (S, A_S). It follows that α is a root of $(S(R), A_M)$ for any group R in $\mathcal{P}^{M_S}(M)$. Consequently

$$\alpha(Y_{S(R)}(T)) \geq d(\mathcal{Y}(T))$$

for each R. Suppose that H is a point in $\mathfrak{a}_M$ such that $\chi_M^S(T, H) \neq 0$. Then H belongs to the convex hull of $\{Y_{S(R)}(T): R \in \mathcal{P}^{M_S}(M)\}$. This implies that

$$\alpha(H) \geq d(\mathcal{Y}(T)).$$

Let U_H be the projection of H onto $\mathfrak{a}_M^L$. Since α vanishes on $\mathfrak{a}_L$, the orthogonal complement of $\mathfrak{a}_M^L$, $\alpha(H)$ is bounded above by a constant multiple of $\| U_H \|$. On the other hand,

$$d(\mathcal{Y}(T)) \geq d_{P_0}(T) - C_0 \geq \delta \| T \| - C_0,$$

by (3.2) and our restriction on T. Consequently there is a constant C such that

$$\| T \| \leq C(1 + \| U_H \|),$$

whenever $\chi_M^S(T, H)$ does not vanish. It follows, for any positive n, that (4.3) is bounded by

$$c_n \| T \|^{-n} \int_{Y_S(T) + \mathfrak{a}_M^{M_S}} \chi_M^S(T, H) dH,$$

where c_n is the constant

$$\sup_{U \in \mathfrak{a}_M^L} \{(C(1 + \| U \|))^n | \phi_S(U) | \}.$$

It is of course finite, ϕ_S being a Schwartz function on $\mathfrak{a}_M / \mathfrak{a}_L \cong \mathfrak{a}_M^L$. Now

$$\int_{Y_S(T) + \mathfrak{a}_M^{M_S}} \chi_M^S(T, H) dH$$

is a polynomial in T. It follows that (4.3) approaches zero as $\|T\|$ approaches infinity.

Next suppose that S belongs to $\mathfrak{F}(L)$. Then

$$\mathfrak{a}_M^{M_S} = \mathfrak{a}_M^L \oplus \mathfrak{a}_L^{M_S}.$$

Since the function ϕ_S is $\mathfrak{a}_L$ invariant, we can write (4.3) as

$$\int_{Y_S(T)+\mathfrak{a}_L^{M_S}} \left(\int_{\mathfrak{a}_M^L} \phi_S(U) \chi_M^S(T, H + U) dU \right) dH. \tag{4.4}$$

We shall apply the lemmas of the last section to the set $\mathcal{Y}_M^S(T)$ (with the role of G taken by M_S). Let $\chi_L^S(T, \cdot)$ be the characteristic function in $\mathfrak{a}_M$ of the positive A_L-orthogonal set $\mathcal{Y}_L^S(T)$. Lemma 3.1 tells us that the function

$$\tilde{\chi}_M^S(T, H + U) = \chi_L^S(T, H) - \chi_M^S(T, H + U), \qquad U \in \mathfrak{a}_M^L, \qquad H \in \mathfrak{a}_L,$$

is nonnegative. Since it is the difference of two characteristic functions it must vanish whenever the first one, $\chi_L^S(T, H)$, equals zero. Lemma 3.2 tells us that it also vanishes whenever

$$\|U\| \leq C_1 d(\mathcal{Y}_M^S(T)).$$

However,

$$d(\mathcal{Y}_M^S(T)) \geq d(\mathcal{Y}(T)) \geq d_{P_0}(T) - C_0 \geq \delta \|T\| - C_0.$$

We are letting $\|T\|$ approach infinity, so we essentially can discard C_0. In fact, we can find a constant C with the property that $\tilde{\chi}_M^S(T, H + U)$ vanishes unless $\chi_L^S(T, H) = 1$ *and* $\|U\| \geq C\|T\|$. It follows that the difference between (4.4) and

$$\int_{Y_S(T)+\mathfrak{a}_L^{M_S}} \chi_L^S(T, H) dH \int_{\mathfrak{a}_M^L} \phi_S(U) dU \tag{4.5}$$

is bounded in absolute value by

$$\int_{Y_S(T)+\mathfrak{a}_L^{M_S}} \chi_L^S(T,H)dH \int_{\{U\in\mathfrak{a}_M^L:\|U\|\geq C\|T\|\}} |\phi_S(U)|dU,$$

This, for any $n \geq 0$, is in turn bounded by

$$c_n\|T\|^{-n}\int_{Y_S(T)+\mathfrak{a}_L^{M_S}} \chi_L^S(T,H)dH,$$

where

$$c_n = \int_{\mathfrak{a}_M^L} (C^{-1}\|U\|)^n|\phi_S(U)|dU.$$

Now formula (3.1) asserts that

$$\int_{X_S(T)+\mathfrak{a}_L^{M_S}} \chi_L^S(T,H)dH = c_L^S(T,0) = c_L^S(T).$$

This is a polynomial in T. It follows that the difference between (4.4) and (4.5) approaches zero as $\|T\|$ approaches infinity.

It is easy to evaluate (4.5). The definition of ϕ_S is given as a Fourier transform of a certain smooth, compactly supported function on $i(\mathfrak{a}_M^L)^*$. The integral

$$\int_{\mathfrak{a}_M^L} \phi_S(U)dU$$

is just the value of that function at zero. It equals

$$\int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} d_S'(\lambda,0)B_\pi(\lambda)d\lambda = \int d_S'(\lambda)B_\pi(\lambda)d\lambda.$$

Therefore (4.5) equals

$$c_L^S(T)\int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} d_S'(\lambda)B_\pi(\lambda)d\lambda.$$

Since the function $c_L^S(T)$ is a polynomial, this last expression is a polynomial in T. To calculate its contribution to the asymptotic value of

(4.1) we need only take the sum over $S \in \mathfrak{F}(L)$. Our net result is that (4.1) differs from the polynomial

$$(4.6) \qquad |\det(s-1)_{\mathfrak{a}_M^L}|^{-1} \int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} \Big(\sum_{S \in \mathfrak{F}(L)} c_L^S(T) d_S'(\lambda) \Big) B_\pi(\lambda) d\lambda$$

by a function which approaches zero as T approaches infinity strongly in $\mathfrak{a}_0(P_0)$.

Suppose that $\lambda \in i\mathfrak{a}_L^*$. Then

$$c_{Q_1}(T, \Lambda) d_{Q_1}(\lambda, \Lambda), \qquad \Lambda \in i\mathfrak{a}_L^*, \qquad Q_1 \in \mathcal{P}(L),$$

is a product of (G, L) families. It follows from (2.2) that

$$\sum_{S \in \mathfrak{F}(L)} c_L^S(T, \Lambda) d_S'(\lambda, \Lambda)$$

equals

$$(4.7) \qquad \sum_{Q_1 \in \mathcal{P}(L)} c_{Q_1}(T, \Lambda) d_{Q_1}(\lambda, \Lambda) \theta_{Q_1}(\Lambda)^{-1}.$$

The integrand in (4.6) is the product of $B_\pi(\lambda)$ with the value of this expression at $\Lambda = 0$. To obtain an explicit formula we need to look back at the definitions of Section 2. Since λ and Λ lie in the subspace $i\mathfrak{a}_L^*$ of $i\mathfrak{a}_M^*$, λ_L equals λ and

$$\zeta = \Lambda - (s\lambda - \lambda) = \Lambda.$$

It follows from the definition (2.4) and the functional equation (1.2) that

$$\begin{aligned} d_Q(\lambda, \Lambda) &= \operatorname{tr}(M_{Q|P}(\lambda)^{-1} M_{Q|P}(s, \lambda + \Lambda) \rho_{\chi,\pi}(P, \lambda, f)) \\ &= \operatorname{tr}(M_{Q|P}(\lambda)^{-1} M_{Q|P}(\lambda + \Lambda) M_{P|P}(s, \lambda + \Lambda) \rho_{\chi,\pi}(P, \lambda, f)) \end{aligned}$$

for any $Q \in \mathcal{P}(M)$. To get $d_{Q_1}(\lambda, \Lambda)$, $Q_1 \in \mathcal{P}(L)$, we have only to choose $Q \subset Q_1$. Notice that the projection of $\lambda + \Lambda$ onto $(\mathfrak{a}_M^L)_{\mathbb{C}}^*$ is zero, so by (1.3)

$$M_{P|P}(s, \lambda + \Lambda) = M_{P|P}(s, 0).$$

We shall usually denote this operator by $M(P, s)$.

If λ and Λ are general points in $i\mathfrak{a}_M^*$, define

$$\mathfrak{M}_Q(P, \lambda, \Lambda) = M_{Q|P}(\lambda)^{-1} M_{Q|P}(\lambda + \Lambda)$$

and

$$\mathfrak{M}_Q^T(P, \lambda, \Lambda) = e^{\Lambda(Y_Q(T))} M_{Q|P}(\lambda)^{-1} M_{Q|P}(\lambda + \Lambda)$$

$$= c_Q(T, \Lambda)\mathfrak{M}_Q(P, \lambda, \Lambda),$$

for any $Q \in \mathcal{P}(M)$. Suppose that Q and Q' are adjacent groups in $\mathcal{P}(M)$ and that Λ lies on the hyperplane spanned by the common wall of their chambers. Then

$$\begin{aligned}\mathfrak{M}_{Q'}(P, \lambda, \Lambda) &= M_{Q'|P}(\lambda)^{-1} M_{Q'|P}(\lambda + \Lambda)\\ &= M_{Q|P}(\lambda)^{-1} M_{Q'|Q}(\lambda)^{-1} M_{Q'|Q}(\lambda + \Lambda) M_{Q|P}(\lambda + \Lambda)\\ &= M_{Q|P}(\lambda)^{-1} M_{Q|P}(\lambda + \Lambda)\\ &= \mathfrak{M}_Q(P, \lambda, \Lambda)\end{aligned}$$

by (1.2) and (1.3). In other words

$$\{\mathfrak{M}_Q(P, \lambda, \Lambda) : \Lambda \in i\mathfrak{a}_M^*,\ Q \in \mathcal{P}(M)\}$$

is a (G, M) family with values in the space of operators on $\mathcal{A}^2(P)$. (The discussion of Section 2 applies equally well to vector-valued (G, M) families.) Since it is a product of (G, M) families,

$$\{\mathfrak{M}_Q^T(P, \lambda, \Lambda) : \Lambda \in i\mathfrak{a}_M^*,\ Q \in \mathcal{P}(M)\}$$

is also a (G, M) family. Now, again take λ and Λ to lie in the subspace $i\mathfrak{a}_L^*$. Then

$$c_{Q_1}(T, \Lambda) d_{Q_1}(\lambda, \Lambda)$$

equals

$$\operatorname{tr}(\mathfrak{M}_{Q_1}^T(P, \lambda, \Lambda) M(P, s) \rho_{\chi,\pi}(P, \lambda, f)).$$

Consequently (4.7) equals

$$\operatorname{tr}\Big(\Big\{\sum_{Q_1 \in \mathcal{P}(L)} \mathcal{M}_{Q_1}^T(P, \lambda, \Lambda)\theta_{Q_1}(\Lambda)^{-1}\Big\} M(P, s)\rho_{\chi,\pi}(P, \lambda, f)\Big).$$

Its value at $\Lambda = 0$, which we know is a polynomial in T, equals

$$\operatorname{tr}(\mathcal{M}_L^T(P, \lambda)M(P, s)\rho_{\chi,\pi}(P, \lambda, f)),$$

by definition.

We can now give our formula for $P^T(B)$. If $L \supset M$ is any pair of Levi subgroups, let $W^L(\mathfrak{a}_M)_{\text{reg}}$ be the set of elements $s \in W(\mathfrak{a}_M)$ such that

$$\{H \in \mathfrak{a}_M : sH = H\},$$

the space of fixed vectors, equals $\mathfrak{a}_L$. We have essentially proved

THEOREM 4.1. *The polynomial $P^T(B)$ equals the sum over $\{P \in \mathcal{F}(M_0) : P \supset P_0\}$, $\pi \in \Pi(M_P(\mathbf{A})^1)$, $L \in \mathcal{L}(M_P)$ and s $W^L(\mathfrak{a}_P)_{\text{reg}}$ of the product of*

$$|\mathcal{P}(M_P)|^{-1}|\det(s - 1)_{\mathfrak{a}_P/\mathfrak{a}_L}|^{-1}$$

with

$$\int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} \operatorname{tr}(\mathcal{M}_L^T(P, \lambda)M(P, s)\rho_{\chi,\pi}(P, \lambda, f))B_\pi(\lambda)d\lambda. \tag{4.8}$$

Proof. The expression (4.8) is a polynomial in T. It equals the integral in (4.6). We have seen that (4.6) differs from (4.1) by a function which approaches 0 as T approaches infinity strongly in $\mathfrak{a}_0(P_0)$. The theorem follows from Lemma 2.1 and the asymptotic formula (1.6) for $P^T(B)$. □

5. A formula for $J_\chi(f)$. The last theorem has given us an explicit formula for the polynomial $P^T(B)$. It leads immediately to a formula for $J_\chi^T(f)$. Choose the function $B \in C_c^\infty(i\mathfrak{h}^*/i\mathfrak{a}_G^*)^W$ so that $B(0) = 1$, and set

$$B^\epsilon(\nu) = B(\epsilon\nu), \qquad \nu \in i\mathfrak{h}^*/i\mathfrak{a}_G^*, \qquad \epsilon > 0,$$

as in [1(g)]. Then if T any point in $\mathfrak{a}_0(P_0)$,

$$J_\chi^T(f) = \lim_{\epsilon \to 0} P^T(B^\epsilon),$$

by Theorem 6.3 of [1(g)].

In [1(e)] we introduced a distribution J_χ. Its interest stems from the fact that it is independent of the minimal parabolic subgroup P_0. We know that $J_\chi^T(f)$ is a polynomial in T. Then $J_\chi(f)$ is defined to be the value of J_χ^T at $T = T_0$. It follows that

$$J_\chi(f) = \lim_{\epsilon \to 0} P^{T_0}(B^\epsilon),$$

for any B as above.

Lemma 5.1. *Suppose that $P \in \mathcal{P}(M_0)$ and $L \in \mathcal{L}(M_P)$. Then*

$$\mathcal{M}_L^{T_0}(P, \lambda) = \mathcal{M}_L(P, \lambda)$$

for any $\lambda \in i\mathfrak{a}_M^*$.

Proof. By definition, $\mathcal{M}_L^{T_0}(P, \lambda)$ is the value at $\Lambda = 0$ of the function

$$\sum_{Q_1 \in \mathcal{P}(L)} e^{\Lambda(Y_{Q_1}(T_0))} \mathcal{M}_{Q_1}(P, \lambda, \Lambda), \qquad \Lambda \in i\mathfrak{a}_L^*.$$

The point $Y_{Q_1}(T_0)$ is just equal to the projection of T_0 onto $\mathfrak{a}_L$. The function above therefore equals

$$\begin{aligned} &e^{\Lambda(T_0)} \sum_{Q_1 \in \mathcal{P}(L)} \mathcal{M}_{Q_1}(P, \lambda, \Lambda) \\ &= e^{\Lambda(T_0)} \mathcal{M}_L(P, \lambda, \Lambda). \end{aligned}$$

Its value at $\Lambda = 0$ is

$$\mathcal{M}_L(P, \lambda, 0) = \mathcal{M}_L(P, \lambda),$$

as required. □

If M is any Levi subgroup, let W_0^M be the Weyl group of (M, A_0).

Theorem 5.2. *The distribution $J_\chi(f)$ equals the limit as ϵ approaches zero of the expression obtained by taking the sum over $M \in \mathcal{L}(M_0)$, $L \in \mathcal{L}(M)$, $\pi \in \Pi(M(\mathbf{A})^1)$ and $s \in W^L(\mathfrak{a}_M)_{\mathrm{reg}}$ of the product of*

$$|W_0^M|\,|W_0|^{-1}|\det(s-1)_{\mathfrak{a}_M^L}|^{-1}$$

with

$$\int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} |\mathcal{P}(M)|^{-1} \sum_{P\in\mathcal{P}(M)} \operatorname{tr}(\mathcal{M}_L(P,\lambda)M(P,s)\rho_{\chi,\pi}(P,\lambda,f))B_\pi^\epsilon(\lambda)d\lambda.$$

Here, B is any function in $C_c^\infty(i\mathfrak{h}^*/i\mathfrak{a}_G^*)^W$ such that $B(0) = 1$.

Proof. The function $\mathcal{M}_L^T(P, \lambda)$ is a polynomial in T. Its value at T_0 equals $\mathcal{M}_L(P, \lambda)$. To obtain an explicit formula for

$$J_\chi(f) = \lim_{\epsilon\to 0} P^{T_0}(B^\epsilon)$$

we have only to replace $\mathcal{M}_L^T(P, \lambda)$ by $\mathcal{M}_L(P, \lambda)$ in the formula for $P^T(B^\epsilon)$ given by Theorem 4.1. The resulting formula will contain a sum over $\{P \in \mathcal{F}(M_0): P \supset P_0\}$. The minimal parabolic subgroup P_0 no longer plays a special role, so we can sum the formula over $P_0 \in \mathcal{P}(M_0)$, as long as we remember to divide by $|\mathcal{P}(M_0)| = |W_0|$. The resulting expression will contain a sum over $\{P_0, P: P_0 \subset P\}$. However, each summand will be independent of P_0. We can remove P_0 from the expression altogether, provided that we multiply by $|\mathcal{P}^{M_P}(M_0)| = |W_0^{M_P}|$. The theorem follows. □

With this last theorem we have attained a plateau. We have found an explicit formula for $J_\chi(f)$. There is certainly room for improvement. It would be nice to be able to eliminate the test function B (and the limit over ϵ). To do so, we would need to prove absolute convergence of the resulting integrals. In addition, we would like our formula to hold for an arbitrary function f in $C_c^\infty(G(\mathbf{A})^1)$, and not just a K finite one. To deal with these questions it is necessary to be able to normalize the intertwining operators on p-adic groups. Canonical normalizations are expected to exist, but so far have been established only for the group GL_n. (See [7(b)].) We shall discuss the problem in the next section, at the same time formulating the hypothesis that each group $G(\mathbf{Q}_v)$ be one on which suitable normalizations exist. This hypothesis will apply to the rest of the paper.

6. Normalized intertwining operators. We again fix a Levi subgroup $M \in \mathcal{L}(M_0)$. Let $\Sigma^r(G, A_M)$ denote the set of reduced roots of (G, A_M), and for any $P \in \mathcal{P}(M)$ write Σ_P^r for the reduced roots of (P, A_M).

Then

$$\Delta_P \subset \Sigma_P^r \subset \Sigma^r(G, A_M)$$

Let β be a root in $\Sigma^r(G, A_M)$. Then β belongs to Δ_P for some $P \in \mathcal{P}(M)$. If $P_0 \in \mathcal{P}(M_0)$ is a minimal parabolic subgroup which is contained in P, then β is the restriction to $\mathfrak{a}_M$ of a unique root β_0 in Δ_{P_0}. Given P and P_0 we defined $\beta^\vee$ in [1(c)] to be the projection of the co-root $\beta_0^\vee$ onto $\mathfrak{a}_M$. We leave the reader to check that the "co-root" $\beta^\vee$ depends only on β and not P or P_0.

Any representation $\pi \in \Pi(M(\mathbf{A}))$ is a restricted tensor product

$$\otimes_v \pi_v, \qquad \pi_v \in \Pi(M(\mathbf{Q}_v)),$$

of representations of the local groups. We shall fix a valuation v on $\mathbf{Q}$, a representation π_v in $\Pi(M(\mathbf{Q}_v))$ and a representation π in $\Pi(M(\mathbf{A}))$ whose component at v equals π_v. We shall also fix groups P and Q in $\mathcal{P}(M)$. Any vector $\phi \in \mathcal{A}^2_{\chi,\pi}(P)$ belongs to a closed linear subspace of $\bar{\mathcal{A}}^2_{\chi,\pi}(P)$ on which the representations

$$\{\rho_{\chi,\pi}(P, \lambda, y_v) : y_v \in G(\mathbf{Q}_v)\}$$

are equivalent to those induced from the representations

$$n_v m_v \to e^{\lambda(H_M(m_v))}\pi_v(m_v), \qquad n_v \in N(\mathbf{Q}_v), \qquad m_v \in M(\mathbf{Q}_v),$$

of $P(\mathbf{Q}_v)$. If $x \in G(\mathbf{A})$, define $(M_{Q|P}(\pi_v, \lambda)\phi)(x)$ to be

$$\int_{N_Q(\mathbf{Q}_v)\cap N_P(\mathbf{Q}_v)\backslash N_Q(\mathbf{Q}_v)} \phi(nx)e^{(\lambda+\rho_P)(H_P(nx))}e^{-(\lambda+\rho_Q)(H_Q(x))}\,dn.$$

The integral converges only for $\mathrm{Re}(\lambda)$ in a certain chamber, but $M_{Q|P}(\pi_v, \lambda)$ can be analytically continued to a meromorphic function of $\lambda \in \mathfrak{a}^*_{M,\mathbf{C}}$ with values in the space of linear maps from $\mathcal{A}^2_{\chi,\pi}(P)$ to $\mathcal{A}^2_{\chi,\pi}(Q)$. Indeed, $M_{Q|P}(\pi_v, \lambda)$ is equivalent to the usual (unnormalized) intertwining operator between induced representations of $G(\mathbf{Q}_v)$, for which the corresponding statement is well known. ([3], [7(a)]).

It is possible to normalize the intertwining operators on real groups (see [1(b)], [3]). Let v be the real valuation. The normalizing factors are built out of certain meromorphic functions

$$n_\beta(\pi_v, z), \qquad z \in \mathbf{C},$$

of one complex variable, indexed by the reduced roots β of (G, A_M). Set

$$n_{Q|P}(\pi_v, \lambda) = \prod_{\beta \in \Sigma_Q^r \cap \Sigma_{\bar{P}}^r} n_\beta(\pi_v, \lambda(\beta^\vee)), \tag{6.1}$$

where $\bar{P} \in \mathcal{P}(M)$ is the group opposite to P. Then the operators

$$N_{Q|P}(\pi_v, \lambda) : \mathcal{A}^2_{\chi,\pi}(P) \to \mathcal{A}^2_{\chi,\pi}(Q),$$

defined by

$$M_{Q|P}(\pi_v, \lambda) = n_{Q|P}(\pi_v, \lambda) N_{Q|P}(\pi_v, \lambda), \tag{6.2}$$

are normalized intertwining operators. If R is any other group in $\mathcal{P}(M)$,

$$N_{R|P}(\pi_v, \lambda) = N_{R|Q}(\pi_v, \lambda) N_{Q|P}(\pi_v, \lambda). \tag{6.3}$$

The adjoint is given by

$$N_{Q|P}(\pi_v, \lambda)^* = N_{P|Q}(\pi_v, -\bar{\lambda}). \tag{6.4}$$

If $L \in \mathcal{L}(M)$, $S \in \mathcal{P}(L)$ and $R, R' \in \mathcal{P}^L(M)$, then

$$(N_{S(R')|S(R)}(\pi_v, \lambda)\phi)_k = N_{R'|R}(\pi_v, \lambda)\phi_k, \tag{6.5}$$

for any $\phi \in \mathcal{A}^2_{\chi,\pi}(S(R))$ and $k \in K$. Finally, suppose that a function $\phi \in \mathcal{A}^2_{\chi,\pi}(P)$ is invariant under the group K_v. Then

$$N_{Q|P}(\pi_v, \lambda)\phi = \phi. \tag{6.6}$$

We shall assume from now on that for *any* valuation v, the intertwining operators can be normalized in this way. In other words, we assume that there are normalizing factors, defined by (6.1), such that the operators defined by (6.2) have the properties (6.3)–(6.6). Let ϕ be a function in $\mathcal{A}^2_{\chi,\pi}(P)$. Then ϕ is K_v invariant for almost all v. It follows from (6.6) that

$$\prod N_{Q|P}(\pi_v, \lambda)\phi$$

is actually a finite product. We can therefore define

$$N_{Q|P}(\pi, \lambda) = \prod_v N_{Q|P}(\pi_v, \lambda).$$

We obtain another meromorphic function of $\lambda \in \mathfrak{a}^*_{M,\mathbf{C}}$ with values in the space of linear maps from $\mathcal{A}^2_{\chi,\pi}(P)$ to $\mathcal{A}^2_{\chi,\pi}(Q)$. This normalized global intertwining operator satisfies the properties described by simply replacing every π_v in the formulas (6.3)-(6.5) by π.

It is clear that

$$M_{Q|P}(\lambda)\phi = \prod_v M_{Q|P}(\pi_v, \lambda)\phi, \qquad \phi \in \mathcal{A}^2_{\chi,\pi}(P),$$

whenever the infinite product on the right converges. It follows that the function

$$n_{Q|P}(\pi, \lambda) = \prod_v n_{Q|P}(\pi_v, \lambda),$$

defined a priori in the domain of absolute convergence of the infinite product for $M_{Q|P}(\lambda)$, can be analytically continued as a meromorphic function of $\lambda \in \mathfrak{a}^*_{M,\mathbf{C}}$ so that

$$M_{Q|P}(\lambda) = n_{Q|P}(\pi, \lambda) N_{Q|P}(\pi, \lambda). \tag{6.7}$$

Moreover, by looking at adjacent groups in $\mathcal{P}(M)$ we see that

$$n_{Q|P}(\pi, \lambda) = \prod_{\beta \in \Sigma^r_Q \cap \Sigma^r_{\bar{P}}} n_\beta(\pi, \lambda(\beta^\vee)), \tag{6.8}$$

where $n_\beta(\pi, z)$ is defined by analytic continuation of an infinite product

$$\prod_v n_\beta(\pi_v, z)$$

which converges in a half plane. It is clear that $n_{Q|P}(\pi, \lambda)$ has properties analogous to (6.3)-(6.5).

The functions above give us two new (G, M) families. Fix $P \in \mathcal{P}(M)$, $\pi \in \Pi(M(\mathbf{A}))$ and $\lambda \in i\mathfrak{a}^*_M$. Define

$$\mathfrak{N}_Q(P, \pi, \lambda, \Lambda) = N_{Q|P}(\pi, \lambda)^{-1} N_{Q|P}(\pi, \lambda + \Lambda), \qquad Q \in \mathcal{P}(M),$$

and

$$\nu_Q(P, \pi, \lambda, \Lambda) = n_{Q|P}(\pi, \lambda)^{-1} n_{Q|P}(\pi, \lambda + \Lambda), \qquad Q \in \mathcal{P}(M).$$

These two collections of functions of $\Lambda \in i\mathfrak{a}_M^*$ are each (G, M) families. The verification, which uses the properties analogous to (6.3) and (6.5), is the same as for the collection $\{\mathfrak{M}_Q(P, \lambda, \Lambda)\}$.

7. Logarithmic derivatives. The (G, M) family

$$\{\nu_Q(P, \pi, \lambda, \Lambda) : \Lambda \in i\mathfrak{a}_M^*, Q \in \mathcal{P}(M)\}$$

is of a special form. Each function is a product, over the reduced roots Σ_Q^r, of functions of one complex variable. In this section we shall study such (G, M) families.

Let M be a fixed Levi subgroup. Suppose for each reduced root β of (G, A_M) that c_β is an analytic function on a neighborhood of $i\mathbf{R}$ in $\mathbf{C}$ such that $c_\beta(0) = 1$. Define

$$c_Q(\Lambda) = \prod_{\beta \in \Sigma_Q^r} c_\beta(\Lambda(\beta^\vee)), \qquad \Lambda \in i\mathfrak{a}_M^*, \tag{7.1}$$

for each group $Q \in \mathcal{P}(M)$. Suppose that Q and Q' are adjacent groups in $\mathcal{P}(M)$ and that Λ lies on the hyperplane spanned by the common wall of their chambers. There is a unique root α in $\Sigma_{Q'}^r \cap \Sigma_{\bar{Q}}^r$. It is a simple root of (Q', A_M) and is orthogonal to Λ. We have

$$\begin{aligned} c_{Q'}(\Lambda) &= \prod_{\beta \in \Sigma_{Q'}^r} c_\beta(\Lambda(\beta^\vee)) \\ &= c_\alpha(0) \prod_{\beta \in \Sigma_{Q'}^r \cap \Sigma_Q^r} c_\beta(\Lambda(\beta^\vee)) \\ &= c_{-\alpha}(0) \prod_{\beta \in \Sigma_{Q'}^r \cap \Sigma_Q^r} c_\beta(\Lambda(\beta^\vee)) \\ &= \prod_{\beta \in \Sigma_Q^r} c_\beta(\Lambda(\beta^\vee)) = c_Q(\Lambda), \end{aligned}$$

since

$$c_\alpha(\Lambda(\alpha^\vee)) = c_\alpha(0) = 1 = c_{-\alpha}(0) = c_{-\alpha}(\Lambda(-\alpha^\vee)).$$

It follows that $\{c_Q(\Lambda)\}$ is a (G, M) family. In this case it is possible to express the number

$$c_M = \lim_{\Lambda \to 0} \sum_{Q \in \mathcal{P}(M)} c_Q(\Lambda)\theta_Q(\Lambda)^{-1}$$

rather more explicitly.

LEMMA 7.1.

$$c_M = \sum_F \mathrm{vol}(\mathfrak{a}_M^G/\mathbf{Z}(F^\vee))(\prod_{\beta \in F} c'_\beta(0)),$$

where the sum is taken over all subsets F of $\Sigma^r(G, A_M)$ for which

$$F^\vee = \{\beta^\vee : \beta \in F\}$$

is a basis of $\mathfrak{a}_M^G$, and $\mathbf{Z}(F^\vee)$ stands for the lattice in $\mathfrak{a}_M^G$ generated by $F^\vee$.

Proof. Set

$$t = (t_\beta)_{\beta \in \Sigma^r(G, A_M)},$$

where each t_β is a positive real variable. Define

$$c_Q^t(\Lambda) = \prod_{\beta \in \Sigma_Q^r} c_\beta(t_\beta \Lambda(\beta^\vee)), \qquad Q \in \mathcal{P}(M).$$

Then $\{c_Q^t(\Lambda) : Q \in \mathcal{P}(M)\}$ is a (G, M) family which is also of the form (7.1). In particular,

$$c_M^t(\Lambda) = \sum_{Q \in \mathcal{P}(M)} c_Q^t(\Lambda)\theta_Q(\Lambda)^{-1}$$

is a regular function of Λ. To calculate its value, set

$$\Lambda = z\xi, \qquad z \in \mathbf{R}, \qquad \xi \in i\mathfrak{a}_M^*,$$

and let z approach zero. We obtain

$$c_M^t = \frac{1}{m!} \sum_{Q \in \mathcal{P}(M)} (\partial(\xi)^m c_Q^t)(0) \cdot \theta_Q(\xi)^{-1},$$

where $m = \dim(A_M/A_G)$. The expression on the right is independent of ξ. It is also a homogeneous polynomial of $t = (t_\beta)$ of total degree m. Let

$$(m_\beta)_{\beta \in \Sigma^r(G, A_M)}$$

be a sequence of nonnegative integers which add up to m. Then the coefficient of $\Pi_\beta t_\beta^{m_\beta}$ in the polynomial c_M^t equals

$$(\Sigma'_Q \theta_Q(\xi)^{-1})\Big(\prod_\beta (m_\beta!)^{-1}(\xi(\beta^\vee))^{m_\beta} \cdot c_\beta^{(m_\beta)}(0)\Big), \tag{7.2}$$

where Σ'_Q stands for the sum over those groups $Q \in \mathcal{P}(M)$ for which every β with $m_\beta \neq 0$ is a root (Q, A_M), and $c_\beta^{(m_\beta)}$ is the $m_\beta{}^{th}$ derivative of c_β. The expression (7.2) is also independent of ξ.

Suppose that (m_β) is such that for some β_1, the integer m_{β_1} is greater than one. Since (7.2) is independent of ξ, we can set

$$\xi = z\beta_1 + \xi_1 \qquad z \in i\mathbf{R},$$

with β_1 orthogonal to ξ_1, and let z approach zero. We can certainly choose ξ_1 such that the set

$$\{\alpha \in \Sigma^r(G, A_M) : \xi_1(\alpha^\vee) = 0\}$$

consists only of β_1 and $-\beta_1$. It follows that each function

$$z^{-1}\theta_Q(z\beta_1 + \xi_1) = \mathrm{vol}(\mathfrak{a}_Q^G/\mathbf{Z}(\Delta_Q^\vee))^{-1} \cdot z^{-1} \sum_{\alpha \in \Delta_Q} (z\beta_1 + \xi_1)(\alpha^\vee)$$

is bounded away from zero as z approaches zero. On the other hand,

$$\prod_\beta ((m_\beta!))^{-1} \cdot (z\beta_1 + \xi_1)(\beta^\vee)^{m_\beta} \cdot c_\beta^{(m_\beta)}(0)) = O(z^{m_\beta})$$

as z approaches zero. It follows that the coefficient (7.2) approaches zero. Since it is independent of ξ, it must actually equal zero. Thus, we may restrict our consideration to sequences (m_β) for which m_β equals 0 or 1, or to what is the same thing, subsets $F \subset \Sigma^r(G, A_M)$ consisting of m elements. It is easy to see that if F is not linearly independent, the corresponding coefficient (7.2) will vanish. Indeed, set

$$\xi = \eta + \xi_1,$$

where η is a point in general position in the span of F, and ξ_1 is in general position in the orthogonal complement of the span of F. It is clear that (7.2) will approach zero if η approaches zero. The coefficient must then actually equal zero.

Now c_M is obtained from c_M^t by setting each $t_\beta = 1$. We have therefore shown that c_M equals

$$\sum_F (\Sigma'_Q \theta_Q(\xi)^{-1}) \Big(\prod_{\beta \in F} \xi(\beta^\vee) \Big) \Big(\prod_{\beta \in F} c'_\beta(0) \Big),$$

where the summations are over those subsets F of $\Sigma^r(G, A_M)$ for which $F^\vee$ is a basis of $\mathfrak{a}_M^G$ and those groups Q in $\mathcal{P}(M)$ such that Σ_Q^r contains F. Lemma 7.1 is an immediate consequence of the following amusing formula.

LEMMA 7.2. *Take any subset F of $\Sigma^r(G, A_M)$ such that $F^\vee$ is a basis of $\mathfrak{a}_M^G$. Then*

$$\Big(\sum_{\{Q \in \mathcal{P}(M) : F \subset \Sigma_Q^r\}} \theta_Q(\xi)^{-1} \Big) \Big(\prod_{\beta \in F} \xi(\beta^\vee) \Big) = \mathrm{vol}(\mathfrak{a}_M^G / \mathbf{Z}(F^\vee)).$$

Proof. If β is any root in $\Sigma^r(G, A_M)$, define $d_\beta(z)$ to be e^z if β belongs to F and to be 1 otherwise. Then

$$d_Q(\Lambda) = \prod_{\beta \in \Sigma_Q^r} d_\beta(\Lambda(\beta^\vee)), \qquad \Lambda \in i\mathfrak{a}_M^*, \qquad Q \in \mathcal{P}(M),$$

is a (G, M) family of the form (7.1). It follows from what we have just shown that

$$d_M = (\Sigma'_Q \theta_Q(\xi)^{-1}) \Big(\prod_{\beta \in F} \xi(\beta^\vee) \Big).$$

On the other hand,

$$d_Q(\Lambda) = e^{\Lambda(Y_Q)},$$

where

$$Y_Q = \sum_{\beta \in F \cap \Sigma_Q^r} \beta^\vee.$$

Recalling the remarks at the beginning of Section 3, we note that d_M also equals the volume in $\mathfrak{a}_M^G$ of the convex hull of

$$\{Y_Q : Q \in \mathcal{P}(M)\}.$$

However, $\{Y_Q : Q \in \mathcal{P}(M)\}$ is just the set of vertices of the parallogram in $\mathfrak{a}_M^G$ spanned by the basis vectors $F^\vee$. The volume of the convex hull is therefore the volume of the parallelogram, which equals $\mathrm{vol}(\mathfrak{a}_M^G/\mathbf{Z}(F^\vee))$. This gives the required formula of Lemma 7.2, thereby completing the proof of Lemma 7.1. ∎

Suppose that $\{c_Q(\Lambda) : Q \in \mathcal{P}(M)\}$ is of the form (7.1) and that $L \in \mathcal{L}(M)$. Then the associated (G, L) family is also of the form (7.1). For suppose that β_1 is a reduced root of (G, A_L). Define

$$c_{\beta_1}(z) = \Pi'_\beta c_\beta(k_\beta z), \tag{7.3}$$

where the product is extended over those $\beta \in \Sigma^r(G, A_M)$ such that the projection of $\beta^\vee$ onto $\mathfrak{a}_L$ is a positive multiple k_β of $\beta_1^\vee$. Suppose that Λ is a point in $i\mathfrak{a}_L^*$ and that $Q_1 \in \mathcal{P}(L)$. Taking any group $Q \subset Q_1$ in $\mathcal{P}(M)$, we have

$$\begin{aligned} c_{Q_1}(\Lambda) &= c_Q(\Lambda) \\ &= \prod_{\beta \in \Sigma_Q^r} c_\beta(\Lambda(\beta^\vee)) \\ &= \prod_{\beta_1 \in \Sigma_{Q_1}^r} (\Pi'_\beta c_\beta(\Lambda(k_\beta \beta_1^\vee))) \\ &= \prod_{\beta_1 \in \Sigma_{Q_1}^r} c_{\beta_1}(\Lambda(\beta_1^\vee)). \end{aligned}$$

This function is certainly of the form (7.1).

If β is any root in $\Sigma^r(G, A_M)$, write $\beta_L^\vee$ for the projection of $\beta^\vee$ onto $\mathfrak{a}_L$. If F is a subset of $\Sigma^r(G, A_M)$, let $F_L^\vee$ be the disjoint union of all the vectors $\beta_L^\vee$, $\beta \in F$. (In particular, if $F_L^\vee$ forms a basis of $\mathfrak{a}_L^G$, the vectors $\beta_L^\vee$, $\beta \in F$, must all be distinct.) The number c_L has a simple formula in terms of the original functions

$$c_\beta(\Lambda), \qquad \beta \in \Sigma^r(G, A_M).$$

COROLLARY 7.3.

$$c_L = \sum_F \operatorname{vol}(\mathfrak{a}_L^G/\mathbf{Z}(F_L^\vee))\Big(\prod_{\beta\in F} c'_\beta(0)\Big),$$

where the sum is taken over all subsets F of $\Sigma^r(G, A_M)$ such that $F_L^\vee$ is a basis of $\mathfrak{a}_L^G$.

Proof. According to the lemma,

$$c_L = \sum_{F_1} \operatorname{vol}(\mathfrak{a}_L^G/\mathbf{Z}(F_1^\vee))\Big(\prod_{\beta_1\in F_1} c'_{\beta_1}(0)\Big),$$

the sum being taken over subsets F_1 of $\Sigma^r(G, A_L)$. Suppose that β_1 is a root of $\Sigma^r(G, A_L)$. Applying Leibnitz' rule to (7.3), we obtain

$$c'_{\beta_1}(0) = \sum_\beta k_\beta c'_\beta(0),$$

where the sum is over those β which occur in the product (7.3). Let F_1 be a subset of $\Sigma^r(G, A_L)$ such that $F_1^\vee$ is a basis of $\mathfrak{a}_L^G$. Then

$$\prod_{\beta_1\in F_1} c'_{\beta_1}(0)$$

equals

$$\sum_F \prod_{\beta\in F} (k_\beta c'_\beta(0)),$$

where F ranges over the subsets of $\Sigma^r(G, A_M)$ obtained by choosing, for each $\beta_1 \in F_1$, a root $\beta \in \Sigma^r(G, A_M)$ such that

$$\beta_L^\vee = k_\beta \beta_1^\vee, \qquad k_\beta > 0.$$

Each set $F_L^\vee$ will also be a basis of $\mathfrak{a}_L^G$. Moreover,

$$\operatorname{vol}(\mathfrak{a}_L^G/\mathbf{Z}(F_1^\vee))\Big(\prod_{\beta\in F} k_\beta\Big) = \operatorname{vol}(\mathfrak{a}_L^G/\mathbf{Z}(F_L^\vee)).$$

The corollary follows. □

Suppose that $L_1 \in \mathcal{L}(L)$ and that $S \in \mathcal{P}(L_1)$. Then

$$c_{Q_1}^S(\Lambda), \qquad \Lambda \in i\mathfrak{a}_L^*, \qquad Q_1 \in \mathcal{P}^{L_1}(L),$$

is an (L_1, L) family. It is of the form (7.1).

COROLLARY 7.4.

$$c_L^S = \sum_F \mathrm{vol}(\mathfrak{a}_L^{L_1}/\mathbf{Z}(F_L^\vee)) \cdot \Big(\prod_{\beta \in F} c'_\beta(0)\Big),$$

where the sum is over all subsets F of $\Sigma^r(L_1, A_M)$ such that $F_L^\vee$ is a basis of $\mathfrak{a}_L^{L_1}$. In particular, c_L^S depends only on L_1 and not on the group $S \in \mathcal{P}(L_1)$.

Proof. For each $Q_1 \in \mathcal{P}^{L_1}(L)$, $c_{Q_1}^S(\Lambda)$ is the product of

$$\prod_{\beta_1 \in \Sigma_{Q_1}^r} c_{\beta_1}(\Lambda)$$

with a function which is independent of Q_1 and whose value at 0 equals 1. In calculating c_L^S, this last function can be ignored. The required formula then follows from Corollary 7.3. □

We return now to the setting of Section 6. Fix $P \in \mathcal{P}(M)$, $\pi \in \Pi(M(\mathbf{A})^1)$ and $\lambda \in i\mathfrak{a}_M^*$. Then if $Q \in \mathcal{P}(M)$ and $\Lambda \in i\mathfrak{a}_M^*$,

$$\nu_Q(P, \pi, \lambda, \Lambda) = \prod_{\beta \in \Sigma_Q^r \cap \Sigma_{\bar{P}}^r} n_\beta(\pi, \lambda(\beta^\vee))^{-1} n_\beta(\pi, \lambda(\beta^\vee) + \Lambda(\beta^\vee)).$$

This function is clearly of the form (7.1), with

$$c_\beta(z) = n_\beta(\pi, \lambda(\beta^\vee))^{-1} n_\beta(\pi, \lambda(\beta^\vee) + z)$$

if β belongs to $\Sigma_{\bar{P}}^r$ and $c_\beta(z) = 1$ otherwise. The last corollary translates to

PROPOSITION 7.5. *Suppose that $L \in \mathcal{L}(M)$, $L_1 \in \mathcal{L}(L)$ and $S \in \mathcal{P}(L_1)$. Then*

$$\nu_L^S(P, \pi, \lambda) = \nu_L^S(P, \pi, \lambda, 0)$$

equals

$$\sum_F \mathrm{vol}(\mathfrak{a}_L^{L_1}/\mathbf{Z}(F_L^\vee))\Big(\prod_{\beta \in F} n_\beta(\pi, \lambda(\beta^\vee))^{-1} n'_\beta(\pi, \lambda(\beta^\vee))\Big),$$

where the sum is over all subsets F of $\Sigma^r(L_1, A_M)$ such that $F_L^\vee$ is a basis of $\mathfrak{a}_L^{L_1}$. ∎

8. Absolute convergence. In this section we shall show how to eliminate the test functions B from our formula for $J_\chi(f)$. We shall also do away with the restriction that f be K-finite. From now on, f will be an arbitrary function in $C_c^\infty(G(\mathbf{A})^1)$ which need not be K finite. The operator $\rho_{\chi,\pi}(P, \lambda, f)$ will no longer act on the vector space $\mathcal{A}_{\chi,\pi}^2(P)$, but rather on its closure, $\bar{\mathcal{A}}_{\chi,\pi}^2(P)$. The operators

$$\mathcal{M}_L(P, \lambda), \qquad L \in \mathcal{L}(M_P),$$

which appear in Theorem 5.2, also act on $\bar{\mathcal{A}}_{\chi,\pi}^2(P)$. However, they are unbounded. If A is any operator on $\bar{\mathcal{A}}_{\chi,\pi}^2(P)$, let $\|A\|_1$ denote the trace class norm of A.

The proof of the following theorem will occupy us for the rest of the paper.

THEOREM 8.1. *Suppose that $M \in \mathcal{L}(M_0)$, $P \in \mathcal{P}(M)$, $L \in \mathcal{L}(M)$ and that $f \in C_c^\infty(G(\mathbf{A})^1)$. Then*

$$\sum_{\pi \in \Pi(M(\mathbf{A})^1)} \int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} \| \mathcal{M}_L(P, \lambda)\rho_{\chi,\pi}(P, \lambda, f)\|_1 d\lambda$$

is finite.

If we assume the proof of this theorem for the moment, we can derive the following refinement of Theorem 5.2.

THEOREM 8.2. *Suppose that $f \in C_c^\infty(G(\mathbf{A})^1)$. Then $J_\chi(f)$ equals the sum over $M \in \mathcal{L}(M_0)$, $L \in \mathcal{L}(M)$, $\pi \in \Pi(M(\mathbf{A})^1)$ and $s \in W^L(\mathfrak{a}_M)_{\mathrm{reg}}$ of the product of*

$$|W_0^M|\,|W_0|^{-1}|\det(s-1)_{\mathfrak{a}_M^L}|^{-1}$$

with

$$\int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} |\mathcal{P}(M)|^{-1} \sum_{P \in \mathcal{P}(M)} \mathrm{tr}(\mathcal{M}_L(P, \lambda)M(P, s)\rho_{\chi,\pi}(P, \lambda, f))d\lambda.$$

Remarks. 1. As an operator on $\bar{\mathcal{A}}_{\chi,\pi}^2(P)$,

$$M(P, s) = M_{P|P}(s, 0)$$

is unitary. If $\lambda \in i\mathfrak{a}_L^*$, we have $s\lambda = \lambda$. Therefore

$$\begin{aligned} |\operatorname{tr}(\mathfrak{M}_L(P, \lambda)M(P, s)\rho_{\chi,\pi}(P, \lambda, f))| \\ &\leq \| \mathfrak{M}_L(P, \lambda)M(P, s)\rho_{\chi,\pi}(P, \lambda, f)\|_1 \\ &= \| \mathfrak{M}_L(P, \lambda)\rho_{\chi,\pi}(P, \lambda, f)M(P, s)\|_1 \\ &= \| \mathfrak{M}_L(P, \lambda)\rho_{\chi,\pi}(P, \lambda, f)\|_1. \end{aligned}$$

The convergence of the sums and integrals in Theorem 8.2 follows immediately from Theorem 8.1.

2. The formula for $J_\chi(f)$ given by the theorem is quite concrete. Its dependence on f is through the operator $\rho_{\chi,\pi}(P, \lambda, f)$, which is really given in terms of the Fourier transform of f. The operator $\mathfrak{M}_L(P, \lambda)$ depends only on the M functions. It equals

(8.1)

$$\lim_{\Lambda\to 0}\Big(\sum_{Q_1\in\mathcal{P}(L)} \operatorname{vol}(\mathfrak{a}_{Q_1}^G/\mathbf{Z}(\Delta_{Q_1}^\vee))M_{Q|P}(\lambda)^{-1}M_{Q|P}(\lambda+\Lambda)\Big(\prod_{\alpha\in\Delta_{Q_1}}\Lambda(\alpha^\vee)\Big)^{-1}\Big),$$

where Λ and λ are constrained to lie in $i\mathfrak{a}_L^*$, and for each $Q_1 \in \mathcal{P}(L)$, Q is a group in $\mathcal{P}(M)$ which is contained in Q_1. Consider the special case that $L = M$ and $\dim \mathfrak{a}_L^G = 1$. If α is the unique root in Δ_P, let ϖ be the element in $(\mathfrak{a}_M^G)^*$ such that $\varpi(\alpha^\vee) = 1$, and set

$$\lambda = z\varpi, \qquad z \in i\mathbf{R}.$$

Then the function (8.1) equals

$$-\operatorname{vol}(\mathfrak{a}_P^G/\mathbf{Z}\alpha^\vee)M_{\bar{P}|P}(z\varpi)^{-1}\cdot\frac{d}{dz}M_{\bar{P}|P}(z\varpi).$$

This should be compared with the formulas from [6(a)] and [2].

3. The terms in the formula for which $L = G$ are the most simple. For then there is no integral over λ, and the operator $\mathfrak{M}_L(P, \lambda)$ reduces to

the identity. These terms will be of particular interest in the applications of the trace formula.

Proof of Theorem 8.2. First suppose that f is a K finite function in $C_c^\infty(G(\mathbf{A})^1)$. Then we can apply Theorem 5.2. Let B be a function in $C_c^\infty(i\mathfrak{h}^*/i\mathfrak{a}_G^*)^W$ such that $B(0) = 1$. The functions

$$B_\pi^\epsilon(\lambda) = B(\epsilon(iY_\pi + \lambda)), \qquad \lambda \in i\mathfrak{a}_M^*,$$

are bounded uniformly in $\epsilon > 0$ and $\pi \in \Pi(M(\mathbf{A})^1)$. Therefore, Theorem 8.1 allows us to apply dominated convergence to the formula of Theorem 5.2. In this formula we can take the limit in ϵ inside all the sums and integrals. The result is the required formula for $J_\chi(f)$.

Now take f to be an arbitrary function in $C_c^\infty(G(\mathbf{A})^1)$. If τ_1 and τ_2 are irreducible representations in $\Pi(K)$, the function

$$f_{\tau_1,\tau_2}(x) = \deg(\tau_1)\deg(\tau_2)\int_K\int_K \mathrm{tr}(\tau_1(k_1))f(k_1^{-1}xk_2^{-1})\mathrm{tr}(\tau_2(k_2))dk_1dk_2$$

is K finite. Therefore $J_\chi(f_{\tau_1,\tau_2})$ is given by the formula of Theorem 8.2. But according to Proposition 2.3 of [1(g)],

$$J_\chi^T(f) = \sum_{\tau_1,\tau_2} J_\chi^T(f_{\tau_1,\tau_2}).$$

Setting $T = T_0$, we obtain

$$J_\chi(f) = \sum_{\tau_1,\tau_2} J_\chi(f_{\tau_1,\tau_2}).$$

It follows that $J_\chi(f)$ can be obtained by substituting f_{τ_1,τ_2} into the formula of Theorem 8.2 and summing over (τ_1, τ_2). Theorem 8.1 tells us that the sum converges to the required formula for $J_\chi(f)$. $\square$

We must still prove Theorem 8.1. First we need a lemma.

LEMMA 8.3. *Suppose that* $f \in C_c^\infty(G(\mathbf{A})^1)$ *and* $M \in \mathcal{L}(M_0)$. *Then there are only finitely many* $\pi \in \Pi(M(\mathbf{A})^1)$ *such that the operators*

$$\rho_{\chi,\pi}(P, \lambda, f), \qquad P \in \mathcal{P}(M), \qquad \lambda \in i\mathfrak{a}_M^*,$$

do not all vanish.

Proof. This result is in a sense implicit in Chapter 7 of [4(b)]. Recall that χ is a W_0-orbit of pairs (M_B, r_B), where B is a parabolic subgroup and r_B is an irreducible cuspidal automorphic representation of $M_B(\mathbf{A})^1$. Langlands' construction gives rise to an intertwining map between any nonzero operator $\rho_{\chi,\pi}(P, \lambda, f)$ and a certain sum of residues in Λ of operators

$$M_{B'|B}(t, \Lambda)\rho_{\chi,r_B}(B, \Lambda, f), \tag{8.2}$$

in which (M_B, r_B) belongs to χ, B and B' are parabolic subgroups which are contained in P, t is an element in $W(\mathfrak{a}_B, \mathfrak{a}_{B'})$ which leaves $\mathfrak{a}_P$ pointwise fixed, and Λ is a point in $\mathfrak{a}^*_{B,\mathbf{C}}$ whose projection onto $\mathfrak{a}^*_{P,\mathbf{C}}$ equals λ. The residues are taken at points

$$\Lambda = X + \lambda,$$

where X belongs to a fixed compact subset of $(\mathfrak{a}^{M_P}_{M_B})^*$. As a function of Λ, (8.2) takes values in an infinite dimensional space. Only the K finite matrix coefficients are, a priori, meromorphic. It is in this sense that the residues are to be taken. However, we shall show that only finitely many of the singular hyperplanes of (8.2) meet any compact subset of $\mathfrak{a}^*_{B,\mathbf{C}}$. This will leave only finitely many choices for X. Since there are only finitely many functions (8.2), (χ being of course fixed), there will be only finitely many choices for π.

In view of (1.4), we can replace the operator $M_{B'|B}(t, \Lambda)$ by $M_{B_1|B}(\Lambda)$, where B_1 is the group $t^{-1}B'$. We can also assume that

$$f = \otimes_v f_v, \qquad f_v \in C_c^\infty(G(\mathbf{Q}_v)).$$

If

$$r_B = \otimes_v r_v, \qquad r_v \in \Pi(M_B(\mathbf{Q}_v)),$$

the operator

$$M_{B_1|B}(\Lambda)\rho_{\chi,r_B}(B, \Lambda, f)$$

is the product of the meromorphic scalar valued function $n_{B_1|B}(r_B, \Lambda)$ with

$$\prod_v N_{B_1|B}(r_v, \Lambda)\rho_{\chi,r_B}(B, \Lambda, f_v).$$

Almost all the operator valued functions in this last product are identically 1. We have only to show that each of the other ones has only finitely many singular hyperplanes meeting any compact subset of $\mathfrak{a}^*_{B,\mathbf{C}}$. If v is a p-adic valuation, f_v is K_v finite. Then the function

$$N_{B_1|B}(r_v, \Lambda)\rho_{\chi,r_B}(B, \Lambda, f_v)$$

takes values in a finite dimensional space, and there is nothing to prove. On the other hand, any irreducible representation of a real group can be embedded in one which is induced from a discrete series. Therefore if v is the real valuation, we may assume that r_v itself belongs to the discrete series (modulo the center of $M_B(\mathbf{R})$). But then any of the associated intertwining operators are meromorphic, as operator valued functions on the space of *smooth* vectors. (See [3].) In particular, only finitely many singular hyperplanes of

$$N_{B_1|B}(r_v, \Lambda)\rho_{\chi,r_B}(B, \Lambda, f_v),$$

and hence also of (8.2), meet any compact subset of $\mathfrak{a}^*_{B,\mathbf{C}}$. The lemma follows. □

Fix $\pi \in \Pi(M(\mathbf{A})^1)$. To prove Theorem 8.1 it is enough, given the last lemma, to show that

$$\int_{i\mathfrak{a}^*_L/i\mathfrak{a}^*_G} \| \mathcal{M}_L(P, \lambda)\rho_{\chi,\pi}(P, \lambda, f)\|_1 \, d\lambda \tag{8.3}$$

is finite. The restriction of the operator

$$\mathcal{M}_{Q_1}(P, \lambda, \Lambda), \qquad Q_1 \in \mathcal{P}(L), \qquad \Lambda \in i\mathfrak{a}^*_L,$$

to $\bar{\mathfrak{A}}^2_{\chi,\pi}(P)$ equals

$$\mathcal{N}_{Q_1}(P, \pi, \lambda, \Lambda)\nu_{Q_1}(P, \pi, \lambda, \Lambda).$$

Applying (2.2) to this product of (G, L) families, we see that

$$\mathcal{M}_L(P, \lambda)\rho_{\chi,\pi}(P, \lambda, f)$$

equals

$$\sum_{S\in\mathcal{F}(L)} \mathcal{N}_S'(P, \pi, \lambda)\nu_L^S(P, \pi, \lambda)\rho_{\chi,\pi}(P, \lambda, f).$$

Since $\nu_L^S(P, \pi, \lambda)$ is a scalar, (8.3) is bounded by the sum over $S \in \mathcal{F}(L)$ of

$$\int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} \| \mathcal{N}_S'(P, \pi, \lambda)\rho_{\chi,\pi}(P, \lambda, f)\|_1 |\nu_L^S(P, \pi, \lambda)| d\lambda.$$

Now $\mathcal{N}_S'(P, \pi, \lambda)$ is built out of normalized intertwining operators on the local groups $G(\mathbf{Q}_v)$. It can be estimated by the methods of local harmonic analysis. By copying the proof of Lemma 9.1 of [1(e)], we can find a constant c_N, for every positive integer N, such that

$$\| \mathcal{N}_S'(P, \pi, \lambda)\rho_{\chi,\pi}(P, \lambda, f)\|_1 \leq c_N(1 + \|\lambda\|)^{-N},$$

for all $\lambda \in i\mathfrak{a}_L^*$. Therefore Theorem 8.1 will be proved if we can find an N such that

$$\int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} |\nu_L^S(P, \pi, \lambda)|(1 + \|\lambda\|)^{-N} d\lambda \tag{8.4}$$

is finite for each $S \in \mathcal{F}(L)$.

Fix $S \in \mathcal{F}(L)$ and set $L_1 = M_S$. By Proposition 7.5 the integral (8.4) is bounded by the sum, over all subsets F of $\Sigma^r(L_1, A_M)$ such that $F_L^\vee$ is a basis of $\mathfrak{a}_L^{L_1}$, of the product of

$$\mathrm{vol}(\mathfrak{a}_L^{L_1}/\mathbf{Z}(F_L^\vee))$$

with

$$\int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} \Big|\prod_{\beta\in F} n_\beta(\pi, \lambda(\beta^\vee))^{-1} n_\beta'(\pi, \lambda(\beta^\vee))\Big|(1 + \|\lambda\|)^{-N} d\lambda. \tag{8.5}$$

Fix such a set F. We can write

$$\lambda = \sum_{\beta\in F} z_\beta \varpi_\beta + \lambda_1, \qquad z_\beta \in i\mathbf{R}, \qquad \lambda_1 \in i\mathfrak{a}_{L_1}^*/i\mathfrak{a}_G^*,$$

where

$$\{\varpi_\beta : \beta \in F\}$$

is the basis of $(\mathfrak{a}_L^{L_1})^*$ which is dual to $F_L^\vee$. The integral (8.5) becomes a product of integrals over $i\mathbf{R}$ and an integral over $i\mathfrak{a}_{L_1}^*/i\mathfrak{a}_G^*$. The finiteness of (8.5), and hence the proof of Theorem 8.1, will follow immediately from

LEMMA 8.4. *There is an integer n_0 such that for every root β in $\Sigma^r(G, A_M)$, the integral*

$$\int_{i\mathbf{R}} |n_\beta(\pi, z)^{-1} n_\beta'(\pi, z)|(1 + |z|)^{-n_0} d|z|$$

is finite.

We will prove this lemma in the next section.

9. Proof of Lemma 8.4. A part of the proof of Lemma 8.4 goes back to an idea of Selberg. He proved a similar lemma for groups of real rank one by using estimates obtained from the trace formula. (See [6(b)].) For general groups we established estimates of a similar sort. (See the appendix of [1(g)].) Fix a group $M \in \mathcal{L}(M_0)$ and a representation $\pi \in \Pi(M(\mathbf{A})^1)$. For the moment we shall also fix parabolic subgroups $P \in \mathcal{P}(M)$ and $P_0 \in \mathcal{P}(M_0)$ such that P_0 is contained in P. There are constants n_0 and d_0 such that for every vector ϕ in $\mathcal{A}^2_{\chi,\pi}(P)$,

$$(9.1) \qquad \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} (\Omega^T_{\chi,\pi}(P, \lambda)\phi, \phi)(1 + \|\lambda\|)^{-n_0} d\lambda \leq c_\phi(1 + \|T\|)^{d_0},$$

for a constant c_ϕ depending on ϕ and for all T which are sufficiently regular in the chamber $\mathfrak{a}_0(P_0)$. (See the remark following Lemma A.1 of the appendix of [1(g)].) This is the estimate we shall use. Recall that $\Omega^T_{\chi,\pi}(P, \lambda)$ is an operator on $\mathcal{A}^2_{\chi,\pi}(P)$. By definition, $(\Omega^T_{\chi,\pi}(P, \lambda)\phi, \phi)$ equals the value at $\lambda' = \lambda$ and $\phi' = \phi$ of

$$(9.2) \qquad \int_{G(\mathbf{Q})\backslash G(\mathbf{A})^1} \Lambda^T E(x, \phi, \lambda)\, \overline{\Lambda^T E(x, \phi', -\bar{\lambda}')}\, dx.$$

The proof of Lemma 8.4 is more difficult in general than in the case that G has rank one. The problem is that the vector ϕ need not be cuspidal. Then there is no simple formula for the inner product (9.2). It is necessary to use the results of [1(f)], where the inner product was studied in detail.

In the notation of Section 9 of [1(f)], the inner product (9.2) equals

$$\sum_{k=0}^{n} q_k^{T,G}(\lambda, \lambda', \phi, \phi')e^{X_k(T)},$$

where $q_k^{T,G}(\lambda, \lambda', \phi, \phi')$ is defined as a certain finite sum

$$\sum_{(t,t')} \sum_X p_X^{T,G}(\lambda, \lambda', \phi, \phi')e^{(t\lambda - t'\lambda')(T)}.$$

We recall that

$$X_0 = 0, X_1, \ldots, X_n$$

are distinct points which lie in the negative dual chamber of $\mathfrak{a}_0(P_0)$ and $p_X^{T,G}(\lambda, \lambda', \phi, \phi')$ is a polynomial in T. If $0 \leq k \leq n$, the function $q_k^{T,G}(\lambda, \lambda', \phi, \phi')$ is regular for all (λ, λ') in $i\mathfrak{a}_P^* \times i\mathfrak{a}_P^*$. We shall examine its dependence on T when $\lambda' = \lambda$ and $\phi' = \phi$. Let

$$L_{kj}: i\mathfrak{a}_P^* \to i\mathfrak{a}_0^*, \qquad 1 \leq j \leq m_k,$$

be the set of *distinct* linear maps

$$\lambda \to t\lambda - t'\lambda, \qquad \lambda \in i\mathfrak{a}_P^*,$$

for which (t, t') occurs in the sum above for $q_k^{T,G}$. Then there are functions

$$\sigma_{kj}^T(\lambda), \qquad 1 \leq j \leq m_k,$$

which are polynomials in T, and are analytic at all points $\lambda \in i\mathfrak{a}_P^*$ for which the vectors

$$L_{kj}(\lambda), \qquad 1 \leq j \leq m_k,$$

are all distinct, such that

$$q_k^{T,G}(\lambda, \lambda, \phi, \phi)e^{X_k(T)} = \sum_{j=1}^{m_k} \sigma_{kj}^T(\lambda)e^{(L_{kj}(\lambda)+X_k)(T)},$$

for all $T \in \mathfrak{a}_0$ and $\lambda \in i\mathfrak{a}_P^*$. The case that $k = 0$ is of particular interest. In the notation of [1(g)],

$$q_0^{T,G}(\lambda, \lambda, \phi, \phi) = (\omega_{\chi,\pi}^T(P, \lambda)\phi, \phi),$$

a function for which we have an explicit formula. We shall parlay the estimate (9.1) into an estimate for this latter function.

Let ξ be any vector in $\mathfrak{a}_0(P_0)$. Then $X_k(\xi) < 0$ for all $k \neq 0$. Let $\delta = \delta_T$ be the linear operator on the space of functions of $T \in \mathfrak{a}_0$ obtained by translating any function by the vector ξ. If σ^T is a polynomial in T, and Y is a vector in $\mathfrak{a}_{0,\mathbf{C}}^*$, the operator

$$\delta_T - e^{Y(\xi)}I$$

maps the function

$$\sigma^T e^{Y(T)}$$

to

$$e^{Y(\xi)}(\sigma^{(T+\xi)} - \sigma^T)e^{Y(T)}.$$

A power of the operator will clearly annihilate the function. Given $\lambda \in i\mathfrak{a}_P^*$, define

$$\Delta_T(\lambda) = \prod_{k=1}^{n} \prod_{j=1}^{m_k} (\delta_T - e^{(L_{kj}(\lambda)+X_k)(\xi)}I)^d,$$

for some large integer d. This operator will annihilate all the functions

$$\sigma_{kj}^T(\lambda)e^{(L_{kj}(\lambda)+X_k)(T)}, \qquad 1 \leq k \leq n, \qquad 1 \leq j \leq m_k.$$

We therefore have

$$\begin{aligned} &\Delta_T(\lambda)(\Omega_{\chi,\pi}^T(P, \lambda)\phi, \phi) \\ &\qquad = \Delta_T(\lambda)\left(\sum_{j=1}^{m_0} \sigma_{0,j}^T(\lambda)e^{(L_{0,j}(\lambda))(T)}\right) \\ &\qquad = \Delta_T(\lambda)(\omega_{\chi,\pi}^T(P, \lambda)\phi, \phi). \end{aligned}$$

We will operate on the function of T on the left hand side of the inequality (9.1). Since

$$|e^{(L_{kj}(\lambda)+X_k)(\xi)}| = e^{X_k(\xi)} \leq 1,$$

the integral

$$\int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} |(\delta_T - e^{(L_{kj}(\lambda)+X_k)(\xi)}I)(\Omega_{\chi,\pi}^T(P, \lambda)\phi, \phi)|(1 + \|\lambda\|)^{-n_0}d\lambda$$

is bounded by

$$c_\phi((1 + \|T + \xi\|)^{d_0} + (1 + \|T\|)^{d_0})$$

for all k, j and all T which are sufficiently regular in $\mathfrak{a}_0(P_0)$. Consequently, an estimate similar to (9.1) obtains for the integral of the function

$$|\Delta_T(\lambda)(\Omega_{\chi,\pi}^T(P, \lambda)\phi, \phi)|(1 + \|\lambda\|)^{-n_0}.$$

By combining these two facts we obtain a constant c_ϕ' such that

(9.3)

$$\int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} |\Delta_T(\lambda)(\omega_{\chi,\pi}^T(P, \lambda)\phi, \phi)|(1 + \|\lambda\|)^{-n_0}d\lambda \leq c_\phi'(1 + \|T\|)^{d_0},$$

for all T sufficiently regular in $\mathfrak{a}_0(P_0)$.

We can now begin the proof of Lemma 8.4. Fix a root β in $\Sigma^r(G, A_M)$. Let L be the Levi subgroup in $\mathcal{L}(M)$ such that

$$\mathfrak{a}_L = \{H \in \mathfrak{a}_M : \beta(H) = 0\}.$$

We shall use the estimate (9.3), but with (G, M) replaced by (L, M). Take P to be the group in $\mathcal{P}^L(M)$ for which β is the simple root of (P, A_M). It is a maximal parabolic subgroup of L. The only other group in $\mathcal{P}^L(M)$ is $\bar{P}$. Let ϖ be the element in $(\mathfrak{a}_M^L)^*$ such that $\varpi(\beta^\vee) = 1$, and set

$$\lambda = z\varpi, \qquad z \in i\mathbf{R}.$$

Then the restriction of the operator $M_{\bar{P}|P}(\lambda)$ to $\mathcal{Q}^2_{\chi,\pi}(P)$ equals

$$N_{\bar{P}|P}(\pi, z\varpi)n_\beta(\pi, z).$$

The explicit formula for $(\omega^T_{\chi,\pi}(P, \lambda)\phi, \phi)$, discussed in Section 1, is given as a sum over the elements $s \in W^L(\mathfrak{a}_M)$. The summand corresponding to $s = 1$ is the product of $\mathrm{vol}(\mathfrak{a}^L_M/\mathbf{Z}\beta^\vee)$ with

$$\lim_{\lambda' \to \lambda} \sum_{Q \in \{P, \bar{P}\}} (M_{Q|P}(\lambda)^{-1}M_{Q|P}(\lambda')\phi, \phi)e^{(\lambda'-\lambda)(Y_Q(T))}\theta_Q(\lambda' - \lambda)^{-1}.$$

One sees that this limit equals the sum of

$$-\left(N_{\bar{P}|P}(\pi, z\varpi)^{-1}\frac{d}{dz}N_{\bar{P}|P}(\pi, z\varpi)\phi, \phi\right), \tag{9.4}$$

$$-n_\beta(\pi, z)^{-1}n'_\beta(\pi, z)(\phi, \phi), \tag{9.5}$$

and

$$2\varpi(T - T_0)(\phi, \phi). \tag{9.6}$$

There is a most one nontrivial element s in $W^L(\mathfrak{a}_M)$. If it exists, it maps $z\varpi$ to $-z\varpi$. Its contribution to the formula for $(\omega^T_{\chi,\pi}(P, \lambda)\phi, \phi)$ is the product of $\mathrm{vol}(\mathfrak{a}^L_M/\mathbf{Z}\alpha^\vee)$ with the sum of

$$\frac{1}{2z}(M_{P|P}(s, z\varpi)^{-1}\phi, \phi)e^{2z\varpi(T)} \tag{9.7}$$

and

$$\frac{1}{-2z}(M_{P|P}(s, z\varpi)\phi, \phi)e^{-2z\varpi(T)}, \tag{9.8}$$

provided that $z \neq 0$. (See Section 6 of [4(a)].)

We have only to look at how $\Delta_T(z\varpi)$ acts on the five functions (9.4)-(9.8). For example, $\Delta_T(z\varpi)$ acts on (9.8) by multiplying it by

$$\prod_{k=1}^{n}\prod_{j=1}^{m_k}(e^{-2z\varpi(\xi)} - e^{(L_{kj}(z\varpi)+X_k)(\xi)})^d,$$

a bounded function of $z \in i\mathbf{R}$. Moreover, the operators

$$M_{P|P}(s, z\varpi), \qquad z \in i\mathbf{R},$$

are all unitary, so that the absolute value of (9.8) is bounded whenever $z \in i\mathbf{R}$ is bounded away from zero. Similar assertions apply to the function (9.7). If $\phi^T(z)$ is the sum of (9.7) and (9.8), $\phi^T(z)$ is regular at $z = 0$. It follows that

$$\int_{i\mathbf{R}} |\Delta_T(z\varpi)\phi^T(z)|(1 + |z|)^{-n_0} d|z| \leq c_\phi(1 + |z|)^{d_0},$$

for constants n_0, d_0 and c_ϕ. The integrals involving (9.4) and (9.6) have similar estimates. Combining these with (9.3), we find that there are constants n_0, d_0 and c_0 such that

$$\int_{i\mathbf{R}} |\Delta_T(z\varpi)(n_\beta(\pi, z)^{-1} n_\beta'(\pi, z))|(1 + |z|)^{-n_0} d|z|$$

is bounded by

$$c_0(1 + \|T\|)^{d_0}.$$

However, the function

$$n_\beta(\pi, z)^{-1} n_\beta'(\pi, z)$$

is independent of T. The operator $\Delta_T(z\varpi)$ simply multiplies it by

$$\prod_{k=1}^{n} \prod_{j=1}^{m_k} (1 - e^{(L_{kj}(z\varpi) + X_k)(\xi)})^d,$$

a function which, for $z \in i\mathbf{R}$, is bounded away from zero. It follows that

$$\int_{i\mathbf{R}} |n_\beta(\pi, z)^{-1} n_\beta'(\pi, z)|(1 + |z|)^{-n_0} d|z|$$

is finite. This is what was required in Lemma 8.4. □

UNIVERSITY OF TORONTO

BIBLIOGRAPHY

[1a] Arthur, J., The characters of discrete series as orbital integrals, *Inv. Math.* **32** (1976), 205-261.

[1b] ———, On the invariant distributions associated to weighted orbital integrals, preprint.

[1c] ———, A trace formula for reductive groups I: terms associated to classes in $G(\mathbf{Q})$, *Duke Math. J.* **45** (1978), 911-952.

[1d] ———, Eisenstein series and the trace formula, *Proc. Sympos. Pure Math.*, **33**, Part I, Amer. Math. Soc., Providence, R.I. (1979), 253-274.

[1e] ———, The trace formula in invariant form, *Ann. of Math.* **114** (1981), 1-74.

[1f] ———, On the inner product of truncated Eisenstein series, *Duke Math. J.* **49** (1982), 35-70.

[1g] ———, On a family of distributions obtained from Eisenstein series I: Application of the Paley-Wiener theorem, *Amer. J. Math.* **104** (1982),

[2] Jacquet, H., and R. P. Langlands, Automorphic Forms on $GL(2)$, *Lecture Notes in Math.*, **114** (1970).

[3] Knapp, A.W., and E. M. Stein, Intertwining operators for semisimple groups II, *Inv. Math.* **60** (1980), 9-84.

[4a] Langlands, R. P., Eisenstein series, *Proc. Sympos. Pure Math.*, **9**, Amer. Math. Soc., Providence, R.I. (1966), 235-252.

[4b] ———, On the Functional Equations Satisfied by Eisenstein Series, *Lecture Notes in Math.*, **544** (1976).

[5] Osborne, M. S., and G. Warner, *The Theory of Eisenstein Systems*, Academic Press, N.Y., 1981.

[6a] Selberg, A., Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series, *J. Indian Math. Soc.* **20** (1956), 47-87.

[6b] ———, Discontinuous groups and harmonic analysis, *Proc. Int. Cong. Math.* 1962, 177-189.

[7a] Shahidi, F., On certain L-functions, *Amer. J. Math.* **103**, 2 (1981), 297-355.

[7b] ———, Local coefficients and normalization of intertwining operators for $GL(n)$, to appear in *Comp. Math.*