ON A FAMILY OF DISTRIBUTIONS OBTAINED FROM EISENSTEIN SERIES I: APPLICATION OF THE PALEY-WIENER THEOREM

By JAMES ARTHUR*

Introduction. This is the first of two papers aimed at finding explicit formulas for certain distributions. The distributions are obtained from Eisenstein series and are important ingredients in the trace formula. In this paper we shall resolve some analytic difficulties that center around an interchange of two limits. The next paper will be devoted to the calculations which will culminate in the formulas.

Let G be a reductive algebraic group defined over $\mathbf{Q}$. As usual, we will write $G(\mathbf{A})^1$ for the intersection of the kernels of the maps

$$x \rightarrow |\xi(x)|, \qquad x \in G(\mathbf{A}),$$

in which ξ ranges over the group $X(G)_{\mathbf{Q}}$ of characters of G defined over $\mathbf{Q}$. The trace formula is an identity

$$\sum_{\mathfrak{o}\in\mathfrak{O}} J_{\mathfrak{o}}(f) = \sum_{\chi\in\mathfrak{X}} J_\chi(f), \qquad f \in C_c^\infty(G(\mathbf{A})^1),$$

between distributions on $G(\mathbf{A})^1$. The distributions on the left are parametrized by semisimple conjugacy classes in $G(\mathbf{Q})$ and are closely related to weighted orbital integrals on $G(\mathbf{A})^1$. Although they need to be better understood for any general applications of the trace formula, the remaining problems are primarily local. We will not discuss them here. The distributions on the right are defined in terms of truncated Eisenstein series. They are parametrized by the set $\mathfrak{X}$ of Weyl group orbits of pairs (M_B, r_B), where M_B is the Levi component of a standard parabolic subgroup B and r_B is an irreducible cuspidal automorphic representation of $M_B(A)^1$. Here the situation is worse. It does not seem to be possible to

---

Manuscript received November 20, 1981.

*Partially supported by NSERC Grant A3483.

0002-9327/82/1046-1243 $01.50

apply the trace formula without explicit formulas for these distributions. The problem is a global one.

Let us recall how $J_\chi(f)$ was defined. The convolution on $L^2(G(\mathbf{Q})\backslash G(\mathbf{A})^1)$ by a function $f \in C_c^\infty(G(\mathbf{A})^1)$ is an integral operator with a kernel $K(x, y)$. The theory of Eisenstein series gives a natural formula for $K(x, y)$ as a (continuous) sum of certain components, which we indexed in [1(a)] by the standard parabolic subgroups P of G and the irreducible unitary representations of $M_P(\mathbf{A})$. (The component associated to a given representation of $M_P(\mathbf{A})$ was defined to be zero unless the restriction of the representation to $M_P(\mathbf{A})^1$ occurred discretely in $L^2(M_P(\mathbf{Q})\backslash M_P(\mathbf{A})^1)$.) Langlands' results in Chapter 7 of [3] allowed us to decompose the discrete spectrum of $L^2(M_P(\mathbf{Q})\backslash M_P(\mathbf{A})^1)$ into a direct sum of subrepresentations, indexed by the classes in $\mathfrak{X}$. This led directly to a natural decomposition

$$K(x,y) = \sum_{\chi\in\mathfrak{X}} K_\chi(x,y), \qquad x,y \in G(\mathbf{A})\backslash G(\mathbf{A})^1),$$

of the kernel ([1(a), Section 4]). The functions $K_\chi(x, y)$ are in general not integrable over the diagonal. However, we introduced a truncation operator Λ^T in [1(b)] in which T was a suitably regular point in a chamber, $\mathfrak{a}_0^+$, associated to the minimal standard parabolic subgroup. We allowed Λ^T to act on each variable in $K_\chi(x, y)$ separately, and then showed that for suitably regular T a certain distribution $J_\chi^T(f)$ could be expressed as the integral of the resulting function over the diagonal ([1(b), Theorem 3.2]). Subsequently, in [1(c)], we found that $J_\chi^T(f)$ was a polynomial function of T. The distribution $J_\chi(f)$ was defined as the value of this polynomial at a certain point T_0.

Suppose that $\chi \in \mathfrak{X}$ is fixed. The formula we have just described allows us, for suitably regular T, to express $J_\chi^T(f)$ as a (continuous) sum over the standard parabolic subgroups P and the irreducible unitary representations of $M_P(\mathbf{A})$. Each such representation of $M_P(\mathbf{A})$ can be identified with a unique pair (π, λ), where π is an irreducible unitary representation of $M_P(\mathbf{A})^1$ and λ belongs to the real vector space

$$i\mathfrak{a}_P^* = i(X(M_P)_{\mathbf{Q}} \otimes \mathbf{R}).$$

We will therefore have a formula

$$J_\chi^T(f) = \sum_P \sum_\pi \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \Psi_\pi^T(\lambda, f)\,d\lambda, \tag{1}$$

for a certain function $\Psi_\pi^T(\lambda, f)$, which is defined in terms of the inner product of truncated Eisenstein series. In a recent paper [1(e)] we gave a rather explicit asymptotic formula for the inner product of truncated Eisenstein series. It will provide an asymptotic formula for $\Psi_\pi^T(\lambda, f)$, as T approaches infinity away from the walls in $\mathfrak{a}_0^+$. We might hope to substitute this formula for $\Psi_\pi^T(\lambda, f)$ in (1), and then let T approach infinity. Instead of $\Psi_\pi^T(\lambda, f)$ we would be left with a much more concrete function. We will not write it down here, but let us just say that with some version of the Riemann-Lebesque lemma, we might reasonably hope to then evaluate $J_\chi^T(f)$ as a polynomial in T. However, this is too optimistic. There are apparently no uniform estimates for λ in $i\mathfrak{a}_P^*/i\mathfrak{a}_G^*$ that would justify such a use of the Riemann-Lebesque lemma. Moreover, the asymptotic formula for $\Psi_\pi^T(\lambda, f)$ is uniform only for λ in compact sets. Since λ is to be integrated over the real vector space $i\mathfrak{a}_P^*/i\mathfrak{a}_G^*$, the substitution for $\Psi_\pi^T(\lambda, f)$ in (1) is not valid. The purpose of this paper is to resolve these difficulties.

The problems hinge on the fact that the multiple integral on the right hand side of (1) is over a noncompact domain. The sum over the standard parabolic subgroups P is certainly finite. As we will observe in Section 2, f can be taken to be a K-finite function. It then follows from the results in [3, Chapter 7] that the sum over π in (1) reduces also to a finite sum. Therefore, only the integral over λ has noncompact domain. For each $\epsilon > 0$ and every representation π we will define a function B_π^ϵ in $C_c^\infty(i\mathfrak{a}_P^*/i\mathfrak{a}_G^*)$ such that

$$\sum_P \sum_\pi \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \Psi_\pi^T(\lambda, f) B_\pi^\epsilon(\lambda) d\lambda \tag{2}$$

is asymptotic to a polynomial $P^T(B^\epsilon)$ as T approaches infinity away from the walls in $\mathfrak{a}_0^+$. We shall show, moreover, that

$$\lim_{\epsilon \to 0} P^T(B^\epsilon) = J_\chi^T(f).$$

These two results will be stated together as Theorem 6.3. It is the principal theorem of the paper, and it provides an escape from the difficulties discussed above.

The function B_π^ϵ will be defined in terms of the infinitesimal character of π. (By this we mean the infinitesimal character of the component of π at the real valuation.) In Section 3 we will choose a Cartan subalgebra $\mathfrak{h}$ of the split real form of the Lie algebra of $G(\mathbf{C})$. It will be invariant under the complex Weyl group W of $G(\mathbf{R})$. There will also be natural embeddings $\mathfrak{a}_P^* \subset \mathfrak{h}^*$ of the dual spaces. The infinitesimal character of π provides a Weyl orbit of linear functions

$$X_\pi + iY_\pi, \qquad X_\pi, Y_\pi \in \mathfrak{h}^*/\mathfrak{a}_G^*,$$

on $\mathfrak{h}_{\mathbf{C}}$. Suppose that B is a W-invariant function in $C_c^\infty(i\mathfrak{h}^*/i\mathfrak{a}_G^*)$ such that $B(0) = 1$. If $\epsilon > 0$ we will define

$$B^\epsilon(\nu) = B(\epsilon\nu), \qquad \nu \in i\mathfrak{h}^*/i\mathfrak{a}_G^*,$$

and

$$B_\pi^\epsilon(\lambda) = B^\epsilon(iY_\pi + \lambda) = B(\epsilon(iY_\pi + \lambda)), \qquad \lambda \in i\mathfrak{a}_P^*/i\mathfrak{a}_G^*.$$

Then B_π^ϵ is a cutoff function in the "imaginary part" of the infinitesimal character of a certain representation of $G(\mathbf{A})$. (It is the representation induced from $P(\mathbf{A})$ which is associated to the pair (π, λ).) Theorem 6.3 tells us that for the purpose of calculating the polynomial $J_\chi^T(f)$, we can insert B_π^ϵ into the formula (1). The theorem could also be interpreted as a justification of the interchange of limits as T approaches infinity and ϵ approaches zero.

The proof of Theorem 6.3 is indirect. The main ingredients are two rather deep results from other papers. The first, already mentioned, is the fact that the right hand side of (1) is a polynomial for suitably regular T. Although this was actually proved in [1(c)], most of the work was done in [1(b)]. In the present paper it is necessary to know how the suitable regularity of T depends on the function f. We will state this quantitative version of the result as Proposition 2.2, and we will leave for the appendix the task of showing how it follows from the work of [1(b)]. Our second ingredient is a multiplier theorem for the Hecke algebra on $G(\mathbf{R})^1 = G(\mathbf{R}) \cap G(\mathbf{A})^1$. It was proved in [1(d), Section III.4] as a corollary of the Paley-Wiener theorem. The theorem describes the multipliers in a form which is remarkably similar to the cutoff functions $\{B_\pi^\epsilon(\lambda)\}$. Like the families $\{B_\pi^\epsilon(\lambda)\}$, the multipliers are parametrized by functions on the "imaginary parts" of infinitesimal characters, or what is the same thing, by W-invariant functions on $i\mathfrak{h}^*/i\mathfrak{a}_G^*$. Instead of being compactly supported, however, the functions which parametrize multipliers are taken from the classical Paley-Wiener space on $i\mathfrak{h}^*/i\mathfrak{a}_G^*$. In Section 3 we will recall this result and show how it applies to the K finite functions in $C_c^\infty(G(\mathbf{A})^1)$.

We will combine these two results in Section 4. Taken together they will tell us that if

$$\psi_\pi^T(H) = \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \Psi_\pi^T(\lambda, f) e^{\lambda(H)} d\lambda, \qquad H \in \mathfrak{h}^1,$$

then the expression

$$|W|^{-1} \sum_{s \in W} \sum_P \sum_\pi \psi_\pi^T(s^{-1}H) e^{(X_\pi + iY_\pi)(s^{-1}H)} \tag{3}$$

is a polynomial in T, whenever the minimum distance from T to the walls of $\mathfrak{a}_0^+$ is greater than a constant multiple of $1 + \|H\|$. (Here $\mathfrak{h}^1$ is the annihilator of $\mathfrak{a}_G^*$ in $\mathfrak{h}$.) On the other hand, the formula (1) can be written

$$J_\chi^T(f) = \sum_P \sum_\pi \psi_\pi^T(0).$$

Notice that the sum on the right equals the value of (3) at $H = 0$. Now, consider a function $B \in C_c^\infty(i\mathfrak{h}^*/i\mathfrak{a}_G^*)$ as above. Then B^ϵ is the Fourier transform of a W-invariant Schwartz function β_ϵ on $\mathfrak{h}^1$, and the functions $\{\beta_\epsilon : \epsilon > 0\}$ form an approximate identity on $\mathfrak{h}^1$. The temptation is to integrate the function (3) against β_ϵ. It would lead formally to something very close to the required Theorem 6.3. However, because of the presence of the real linear function X_π, the expression (3) is not a tempered function of H. This is serious, since β_ϵ is *not* compactly supported.

We will deal with the nontempered nature of (3) in Section 5. There we will prove an elementary but somewhat complicated proposition on polynomials. The proposition will permit us, roughly speaking, to remove the real functionals from the expression (3). We will then be able to integrate against β_ϵ, as proposed, in Section 6. This will lead us to a proof of Theorem 6.3.

Having proved Theorem 6.3, we will be able to invoke the asymptotic formula for $\Psi_\pi^T(\lambda, f)$. After discussing it in Section 7, we will prove Theorem 7.1, the final result of the paper. It states that if $\Psi_\pi^T(\lambda, f)$ is replaced in (2) by its asymptotic formula, the resulting expression is still asymptotic to the polynomial $P^T(B^\epsilon)$. The way will be clear for finding explicit formulas for $P^T(B^\epsilon)$ and $J_\chi^T(f)$, as we shall see in the next paper.

Notational Convention. If H is any locally compact group, we shall let $\Pi(H)$ denote the set of equivalence classes of irreducible unitary representations of H.

1. The distributions J_χ^T. Let G be a reductive algebraic group defined over $\mathbf{Q}$. We shall fix a minimal parabolic subgroup P_0 of G and a Levi component M_0 of P_0, both defined over $\mathbf{Q}$. In this paper a parabolic subgroup will mean a parabolic subgroup of G, defined over $\mathbf{Q}$, which contains P_0. Suppose that P is such a subgroup. We shall write N_P for the unipotent radical of P, and M_P for the unique Levi component of P which contains M_0. Let A_P be the split component of the center of M_P. If $X(M_P)_{\mathbf{Q}}$ is the group of characters of M_P defined over $\mathbf{Q}$,

$$\mathfrak{a}_P = \mathrm{Hom}(X(M_P)_{\mathbf{Q}}, \mathbf{R})$$

is a real vector space whose dimension equals that of A_P. Its dual space is

$$\mathfrak{a}_P^* = X(M_P)_{\mathbf{Q}} \otimes \mathbf{R}.$$

We shall write $A_0 = A_{P_0}$, $\mathfrak{a}_0 = \mathfrak{a}_{P_0}$ and $\mathfrak{a}_0^* = \mathfrak{a}_{P_0}^*$. For all of this paper, K will be a fixed maximal compact subgroup of the adèlized group $G(\mathbf{A})$. We will want K to satisfy the natural conditions that were summarized in Section 1 of [1(c)] by defining K to be admissible with respect to M_0.

Suppose that P is a parabolic subgroup. Let H_P be the associated function from $G(\mathbf{A})$ to $\mathfrak{a}_P$ ([1(a), Section 1]). The kernel, $M_P(\mathbf{A})^1$, of H_P in $M_P(\mathbf{A})$ is a closed subgroup of $M_P(\mathbf{A})$. We should point out that the representations $\Pi(M_P(\mathbf{A})^1)$ can be naturally identified with the orbits of $i\mathfrak{a}_P^*$ on $\Pi(M_P(\mathbf{A}))$ under the action

$$\pi \to \pi_\lambda = e^{\lambda(H_P(\cdot))}\pi, \qquad \pi \in \Pi(M_P(\mathbf{A})), \qquad \lambda \in i\mathfrak{a}_P^*.$$

Now let Q be a parabolic subgroup which contains P. Then there is a canonical surjection from $\mathfrak{a}_P$ to $\mathfrak{a}_Q$ and a canonical injection from $\mathfrak{a}_Q^*$ to $\mathfrak{a}_P^*$. The kernel, $\mathfrak{a}_P^Q$, of the first map is a vector space whose dual space is $\mathfrak{a}_P^*/\mathfrak{a}_Q^*$. Observe that H_Q is just the composition of H_P with the map from $\mathfrak{a}_P$ onto $\mathfrak{a}_Q$.

We shall fix a Euclidean norm $\|\cdot\|$ on the space $\mathfrak{a}_0$ which is invariant under the action on $\mathfrak{a}_0$ of the Weyl group of (G, A_0). On each space $\mathfrak{a}_P^Q$, $P \subset Q$, we take as Haar measure the Euclidean measure associated to the restriction of $\|\cdot\|$ to $\mathfrak{a}_P^Q$. The real vector space $i\mathfrak{a}_P^*/i\mathfrak{a}_Q^*$ is isomorphic to the character group of $\mathfrak{a}_P^Q$, so we can take the Haar measure which is dual to the one on $\mathfrak{a}_P^Q$. We can then normalize the Haar measures on the groups K, $G(\mathbf{A})$, $N_P(\mathbf{A})$, $M_P(\mathbf{A})$, $A_P(\mathbf{R})^0$ (the identity component of $A_P(\mathbf{R})$), $M_P(\mathbf{A})^1$ etc. by following the prescription in Section 1 of [1(a)].

Suppose that P is a parabolic subgroup. We shall write $\mathcal{A}^2(P)$ for the space of automorphic forms on $N_P(\mathbf{A})M_P(\mathbf{Q})\backslash G(\mathbf{A})$ which are square integrable on $M_P(\mathbf{Q})\backslash M_P(\mathbf{A})^1 \times K$. It is the space of functions

$$\phi : N_P(\mathbf{A})M_P(\mathbf{Q})\backslash G(\mathbf{A}) \to \mathbf{C}$$

which satisfy the following two conditions.

(i) The set of functions

$$x \to (z\phi)(xk),$$

indexed by the left and right invariant differential operators z on $G(\mathbf{R})$, and the elements $k \in K$, spans a finite dimensional space.

(ii) $$\int_K \int_{M_P(\mathbf{Q})\backslash M_P(\mathbf{A})^1} |\phi(mk)|^2 dm dk < \infty.$$

For any $\phi \in \mathcal{A}^2(P)$ there is the Eisenstein series

$$E(x, \phi, \lambda) = \sum_{\delta \in P(\mathbf{Q})\backslash G(\mathbf{Q})} \phi(\delta x) e^{(\lambda + \rho_P)(H_P(\delta x))}.$$

It converges for $\mathrm{Re}(\lambda)$ in a certain chamber and continues analytically to a meromorphic function of $\lambda \in \mathfrak{a}_{P,\mathbf{C}}^*$. (See [3]). As always, ρ_P is the vector in $\mathfrak{a}_P^*$ such that $e^{2\rho_P(H_P(\cdot))}$ is the modular function on $P(\mathbf{A})$.)

In [1(a)] we introduced a decomposition

$$L^2(M_P(\mathbf{Q})\backslash M_P(\mathbf{A})^1) = \bigoplus_{\chi \in \mathfrak{X}} L^2(M_P(\mathbf{Q})\backslash M_P(\mathbf{A}))_\chi,$$

indexed by a certain set $\mathfrak{X}$. (Recall that $\mathfrak{X}$ is the set of orbits, under the Weyl group of (G, A_0), of pairs (M_B, r_B), where B is a parabolic subgroup and r_B is an irreducible cuspidal automorphic representation of $M_B(\mathbf{A})^1$.) Suppose that $\chi \in \mathfrak{X}$ and that $\pi \in \Pi(M_P(\mathbf{A}))$. Let $\mathcal{A}^2_{\chi,\pi}(P)$ be the subspace of functions ϕ in $\mathcal{A}^2(P)$ such that for each $x \in G(\mathbf{A})$, the function

$$\phi_x(m) = \phi(mx), \qquad m \in M_P(\mathbf{A}),$$

transforms under $M_P(\mathbf{A})$ according to the representation π and in addition, the restriction of ϕ_x to $M_P(\mathbf{A})^1$ belongs to $L^2(M_P(\mathbf{Q})\backslash M_P(\mathbf{A})^1)_\chi$. The inner product

$$(\phi, \phi') = \int_K \int_{M_P(\mathbf{Q})\backslash M_P(\mathbf{Q})^1} \phi(mk)\, \overline{\phi'(mk)}\, dm\, dk$$

is positive definite on $\mathfrak{A}^2_{\chi,\pi}(P)$. Let $\overline{\mathfrak{A}}^2_{\chi,\pi}(P)$ be the Hilbert space completion of $\mathfrak{A}^2_{\chi,\pi}(P)$. For each $\lambda \in \mathfrak{a}^*_{P,\mathbf{C}}$ we have an induced representation $\rho_{\chi,\pi}(P, \lambda)$ of $G(\mathbf{A})$ on $\overline{\mathfrak{A}}^2_{\chi,\pi}(P)$ defined by

$$(\rho_{\chi,\pi}(P, \lambda, y)\phi)(x) = \phi(xy)e^{(\lambda+\rho_P)(H_P(xy))}e^{-(\lambda+\rho_P)(H_P(x))},$$

for elements $x, y \in G(\mathbf{A})$ and $\phi \in \overline{\mathfrak{A}}^2_{\chi,\pi}(P)$. It is unitary if λ is purely imaginary. (In [1(a)] we denoted the space $\overline{\mathfrak{A}}^2_{\chi,\pi}(P)$ by $\mathcal{H}_P(\pi)_\chi$ and the representation $\rho_{\chi,\pi}(P, \lambda)$ by $I_P(\pi_\lambda)_\chi$.)

The distribution

$$J^T_\chi(f), \qquad f \in C^\infty_c(G(\mathbf{A})^1),$$

was introduced in Section 3 of [1(b)]. It depends on a class $\chi \in \mathfrak{X}$, which will remain fixed for the rest of this paper. It also depends on a suitably regular point T in $\mathfrak{a}_0^+$. Recall that if Δ_0 is the set of simple roots of (P_0, A_0), and

$$d_{P_0}(T) = \min_{\alpha\in\Delta_0} \{\alpha(T)\},$$

then suitably regular means that the number $d_{P_0}(T)$ is sufficiently large. For any such T we expressed $J^T_\chi(f)$ in [1(b)] by two different formulas. The formula which actually served as the definition for $J^T_\chi(f)$ will not be needed here. It was exploited in Proposition 2.3 of [1(c)] to show that

$$T \to J^T_\chi(f)$$

is a polynomial function of T. This is the case, a priori, only when $J^T_\chi(f)$ is defined; that is, whenever T is suitably regular in $\mathfrak{a}_0^+$. However, the polynomial certainly extends uniquely to all T. Thus, $J_\chi^T(f)$ is defined as a polynomial function of T for all $T \in \mathfrak{a}_0$.

The second formula for $J_\chi^T(f)$ is the one we will use here. It is given in terms of the truncation operator Λ^T introduced in Section 1 of [1(b)]. Recall that Λ^T operates on functions on $G(\mathbf{Q})\backslash G(\mathbf{A})$ and is defined for suitably regular points T in $\mathfrak{a}_0^+$. Given P, $\pi \in \Pi(M_P(\mathbf{A}))$ and $\lambda \in i\mathfrak{a}_P^*$, define an operator $\Omega_{\chi,\pi}^T(P, \lambda)$ on $\mathcal{Q}_{\chi,\pi}^2(P)$ by setting $(\Omega_{\chi,\pi}^T(P, \lambda)\phi, \phi')$ equal to

$$(1.1) \qquad \int_{G(\mathbf{Q})\backslash G(\mathbf{A})^1} \Lambda^T E(x, \phi, \lambda) \cdot \overline{\Lambda^T E(x, \phi', \lambda)} dx,$$

for each pair of vectors ϕ and ϕ' in $\mathcal{Q}_{\chi,\pi}^2(P)$. (In [1(b)], the operator $\Omega_{\chi,\pi}^T(P, \lambda)$ was denoted by $M_P^T(\pi_\lambda)_\chi$.) If $|\mathcal{P}(M_P)|$ denotes the number of chambers in $\mathfrak{a}_P$, set

$$(1.2) \qquad \Psi_\pi^T(\lambda, f) = |\mathcal{P}(M_P)|^{-1} \operatorname{tr}(\Omega_{\chi,\pi}^T(P, \lambda)\rho_{\chi,\pi}(P, \lambda, f))$$

for any $f \in C_c^\infty(G(\mathbf{A})^1)$. Then the second formula for our distribution is

$$(1.3) \qquad J_\chi^T(f) = \sum_P \sum_{\pi \in \Pi(M_P(\mathbf{A})^1)} \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \Psi_\pi^T(\lambda, f) d\lambda.$$

(The domain of $\Psi_\pi^T(\lambda, f)$, as a function of π, is actually $\Pi(M_P(\mathbf{A}))$. However the integral over λ in (1.3) depends only on the orbit of π, so it does represent a function on $\Pi(M_P(\mathbf{A})^1)$.) Formula (1.3) is just Theorem 3.2 of [1(b)]. It is valid whenever T is a suitably regular point in $\mathfrak{a}_0^+$. The convergence of (1.3), as well as the existence of the trace in (1.2), is a consequence of Theorem 3.1 of [1(b)].

2. Three important properties. The distribution $J_\chi^T(f)$, as given by formula (1.3), is our main object of study. It has three properties, all related to formula (1.3), which will be crucial to this paper. Each of them is implicit in the paper [1(b)]. Unfortunately we did not keep track of the dependence on T of many of our earlier results. For example, in [1(a)] and [1(b)] we simply agreed at the beginning to let T be a suitably regular point in $\mathfrak{a}_0^+$. We did not say how the "suitable regularity" of T, required for results on $J_\chi^T(f)$, actually depended on f. We didn't really need to, for f

was fixed throughout most of [1(a)] and [1(b)]. In this paper, however, we must allow f to vary and this necessitates a re-examination of some arguments from [1(b)]. We shall save the details for an appendix, being content here to just state the properties in the form we shall use.

PROPOSITION 2.1. *We can find positive integers C_0 and d_0 such that for any $f \in C_c^\infty(G(\mathbf{A})^1)$, any $n \geq 0$ and any $T \in \mathfrak{a}_0$ for which $d_{P_0}(T) > C_0$, the expression*

$$\sum_P \sum_{\pi \in \Pi(M_P(\mathbf{A})^1)} \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} |\Psi_\pi^T(\lambda, f)|(1 + \|\lambda\|)^n d\lambda$$

is bounded by

$$c_{n,f}(1 + \|T\|)^{d_0},$$

where $c_{n,f}$ depends on n and f, but not on T. (Here $\| \ \|$ is the norm on $i\mathfrak{a}_P^*/i\mathfrak{a}_G^*$ which is dual to the one on $\mathfrak{a}_P^G$.)

If it were not for the dependence on T, this proposition would be an immediate consequence of Theorem 3.1 of [1(b)]. For a proof of the proposition, see the appendix. □

The proposition tells us that

$$\sum_P \sum_{\pi \in \Pi(M_P(\mathbf{A})^1)} \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \Psi_\pi^T(\lambda, f) d\lambda$$

converges for all T in the translate of $\mathfrak{a}_0^+$ by a point which is independent of f. By (1.3), this function equals $J_\chi^T(f)$, and is in particular a polynomial in T, for all T in some other translate of $\mathfrak{a}_0^+$. However, this second translate may well depend on f. It turns out to depend only on the support of f.

To keep track of the support of f we need to define a suitable function $\| \ \|$ on $G(\mathbf{A})$. In Section 1 of [1(a)] we discussed height functions

$$\|\Lambda(x)\| = \prod_v \|\Lambda(x_v)\|_v, \qquad x \in G(\mathbf{A}),$$

associated to rational representations

$$\Lambda : G \to GL_n,$$

defined over $\mathbf{Q}$. We then selected a Λ and wrote the corresponding height function as

$$\|x\| = \prod_{v} \|x_v\|_v. \tag{2.1}$$

(As is usual in products of this form, we let v range over the valuations of $\mathbf{Q}$ and write x_v for the component of $x \in G(\mathbf{A})$ in $G(\mathbf{Q}_v)$.) In this paper we will not single out Λ. We will instead fix any function $\|\ \|$ on $G(\mathbf{A})$ of the form (2.1) which satisfies the following three properties.

(i) $\|x\| \geq 1, \qquad x \in G(\mathbf{A})$.

(ii) For any rational representation Λ there are constants c_1 and n_1 such that

$$\|\Lambda(x)\| \leq c_1 \|x\|^{n_1}, \qquad x \in G(\mathbf{A}).$$

(iii) There exists a rational representation Λ_0 and constants c_0 and n_0 such that

$$\|x\| \leq c_0 \|\Lambda_0(x)\|^{n_0}, \qquad x \in G(\mathbf{A}).$$

In the next section we shall impose more conditions on $\|\ \|_{\mathbf{R}}$, the component of $\|\ \|$ at infinity. It is not hard to show that for any positive number N the set

$$G(\mathbf{A}, N) = \{x \in G(\mathbf{A}) : \log\|x\| \leq N\}$$

is compact.

If N is any positive number, let $C_N^\infty(G(\mathbf{A})^1)$ be the space of smooth functions on $G(\mathbf{A})^1$ which are supported on $G(\mathbf{A})^1 \cap G(\mathbf{A}, N)$. The second key property is given by

Proposition 2.2. *There is a positive number C_0 with the following property. For any $N > 0$ and any $f \in C_N^\infty(G(\mathbf{A})^1)$, the function*

$$\sum_P \sum_{\pi \in \Pi(M_P(\mathbf{A})^1)} \int_{i\mathfrak{a}_P^*/\mathfrak{a}_G^*} \Psi_\pi^T(\lambda, f)\, d\lambda$$

equals $J_\chi^T(f)$, and is in particular a polynomial in T, whenever

$$d_{P_0}(T) > C_0(1 + N).$$

For a proof, see the appendix. ■

The third property of $J_\chi^T(f)$ will allow us to restrict our attention to K finite functions. If f belongs to $C_c^\infty(G(\mathbf{A})^1)$ and τ_1 and τ_2 are irreducible representations in $\Pi(K)$, the function

$$f_{\tau_1,\tau_2}(x) = \deg(\tau_1)\cdot\deg(\tau_2)\int_K\int_K \operatorname{tr}(\tau_1(k_1))f(k_1^{-1}xk_2^{-1})\operatorname{tr}(\tau_2(k_2))dk_1dk_2$$

is also in $C_c^\infty(G(\mathbf{A}))$.

PROPOSITION 2.3.

$$J_\chi^T(f) = \sum_{\tau_1,\tau_2} J_\chi^T(f_{\tau_1,\tau_2}).$$

For a proof, see the appendix. □

Let $C_c^\infty(G(\mathbf{A})^1, K)$ be the space of functions in $C_c^\infty(G(\mathbf{A})^1)$ whose left and right translates by K each span a finite dimensional space. Set

$$C_N^\infty(G(\mathbf{A})^1, K) = C_N^\infty(G(\mathbf{A})^1) \cap C_c^\infty(G(\mathbf{A})^1, K),$$

for any N. Any finite sum of functions f_{τ_1,τ_2} belongs to $C_c^\infty(G(\mathbf{A})^1, K)$. The last proposition tells us that the value of J_χ^T at an arbitrary function can be approximated by its value at a K finite function. We will therefore be able to assume that f is K finite. If this is the case, the operators $\rho_{\chi,\pi}(P, \lambda, f)$ are all of finite rank. The subspace of functions in

$$\bigoplus_{\pi\in\Pi(M_P(\mathbf{A}))} \mathfrak{A}_{\chi,\pi}^2(P)$$

which are $A_P(\mathbf{R})^0$ invariant and transform under K according to a fixed finite set Γ of irreducible representations, is finite dimensional. (This subspace was denoted by $\mathfrak{A}_{P,\chi,\Gamma}$ in [1(e)].) It follows that when f is K finite, the sum over π in (1.3) reduces to a *finite* sum. (In a future paper we will actually show that (1.3) is a finite sum even when f is not K finite.)

3. The multiplier theorem. Included in our assumptions is a decomposition of K as $\Pi_v K_v$, where K_v is a maximal compact subgroup of

$G(\mathbf{Q}_v)$ for each valuation v. Let $\mathfrak{h}_K$ be a fixed Cartan subalgebra of the Lie algebra of $K_{\mathbf{R}} \cap M_0(\mathbf{R})$. Let $\mathfrak{h}_0$ be the Lie algebra of a fixed maximal real split torus in $M_0(\mathbf{R})$, and set

$$\mathfrak{h} = i\mathfrak{h}_K \oplus \mathfrak{h}_0.$$

If $\mathfrak{g}$ is the Lie algebra of $G(\mathbf{R})$, $\mathfrak{h}_{\mathbf{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbf{C}}$. The real subspace $\mathfrak{h} \subset \mathfrak{h}_{\mathbf{C}}$ is invariant under the Weyl group W of $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$. Observe that for any parabolic subgroup P there is a natural surjective map

$$h_P : \mathfrak{h} \to \mathfrak{a}_P$$

which is trivial on $i\mathfrak{h}_K$ and which is given by

$$h_P(Y) = H_P(\exp Y), \qquad Y \in \mathfrak{h}_0,$$

on $\mathfrak{a}_0$. The dual of this map gives us an embedding of $\mathfrak{a}_P^*$ into $\mathfrak{h}^*$. We shall denote the kernel of h_G by $\mathfrak{h}^1$. It is a subspace of $\mathfrak{h}$ which is also invariant under W.

Let us fix a positive definite inner product $(\ ,\ )$ on $\mathfrak{h}$ which is invariant under the Weyl group W. Let $\|\cdot\|$ be the associated Euclidean norm on $\mathfrak{h}$. We already have Euclidean norms on the spaces $\mathfrak{a}_P$; we can certainly choose $(\ ,\ )$ so that each of the maps h_P is a partial isometry. We will take the Euclidean measure associated to $\|\ \|$ for our Haar measure on $\mathfrak{h}$. If x is any element in $G(\mathbf{R})$, write

$$x = k_1 \exp X k_2, \qquad k_1, k_2 \in K_{\mathbf{R}}, \qquad X \in \mathfrak{h}_0,$$

and define

$$\|x\|_{\mathbf{R}} = e^{\|X\|}.$$

Then $\|\ \|_{\mathbf{R}}$ is a function from $G(\mathbf{R})$ to $\{t \in \mathbf{R} : t \geq 1\}$. We assume from now on that the "norm" (2.1) on $G(\mathbf{A})$ has this function as its component at infinity. This assumption is made only for convenience, so that our notation is consistent with that of [1(d)].

We are going to use Theorem III.4.2 of [1(d)], a result which pertains to the subalgebra $C_c^\infty(G(\mathbf{R}), K_{\mathbf{R}})$ of left and right $K_{\mathbf{R}}$ finite functions in $C_c^\infty(G(\mathbf{R}))$. Let $\mathcal{E}(\mathfrak{h})^W$ be the algebra of compactly supported distributions on $\mathfrak{h}$ which are invariant under W. The theorem states that for every $\gamma \in \mathcal{E}(\mathfrak{h})^W$ and $f_{\mathbf{R}} \in C_c^\infty(G(\mathbf{R}), K_{\mathbf{R}})$ there is a unique function $f_{\mathbf{R},\gamma}$ in $C_c^\infty(G(\mathbf{R}), K_{\mathbf{R}})$ with the following property. If $\Pi_{\mathbf{R}}$ is any representation in $\Pi(G(\mathbf{R}))$ then

$$\Pi_{\mathbf{R}}(f_{\mathbf{R},\gamma}) = \hat{\gamma}(\nu_{\Pi_{\mathbf{R}}})\Pi_{\mathbf{R}}(f_{\mathbf{R}}),$$

where $\{\nu_{\Pi_{\mathbf{R}}}\}$ is the W-orbit in $\mathfrak{h}_{\mathbf{C}}^*$ associated to the infinitesimal character of $\Pi_{\mathbf{R}}$ and $\hat{\gamma}$ is the Fourier-Laplace transform of γ.

Returning to the global situation, we suppose that f is a function in $C_c^\infty(G(\mathbf{A})^1, K)$. Then f is the restriction to $G(\mathbf{A})^1$ of a function on $G(\mathbf{A})$ which is a finite sum

$$\Sigma(f_{\mathbf{R}} \otimes (\otimes_{v \neq \mathbf{R}} f_v)),$$

where each $f_{\mathbf{R}} \in C_c^\infty(G(\mathbf{R}), K_{\mathbf{R}})$, and each $f_v \in C_c^\infty(G(\mathbf{Q}_v))$. Suppose that γ belongs to the subspace, $\mathcal{E}(\mathfrak{h}^1)^W$, of distributions in $\mathcal{E}(\mathfrak{h})^W$ which are supported on $\mathfrak{h}^1$. Define f_γ to be the restriction to $G(\mathbf{A})^1$ of the finite sum

$$\Sigma(f_{\mathbf{R},\gamma} \otimes (\otimes_{v \neq \mathbf{R}} f_v)).$$

We shall show that it depends only on f. Any representation $\Pi \in \Pi(G(\mathbf{A}))$ is a restricted tensor product

$$\otimes_v \Pi_v, \qquad \Pi_v \in \Pi(G(\mathbf{Q}_v)).$$

(See [2].) Define

$$\{\nu_\Pi\} = \{\nu_{\Pi_{\mathbf{R}}}\}.$$

It is clear that

$$\{\nu_{\Pi_\lambda}\} = \{\nu_\Pi + \lambda\}$$

for each point $\lambda \in i\mathfrak{a}_G^*$. Since γ is supported on $\mathfrak{h}^1$, we have

$$\hat{\gamma}(\nu_{\Pi_\lambda}) = \hat{\gamma}(\nu_\Pi + \lambda) = \hat{\gamma}(\nu_\Pi).$$

In other words, $\hat{\gamma}(\nu_\Pi)$ depends only on the restriction of Π to $G(\mathbf{A})^1$. Therefore

$$\Pi(f_\gamma) = \int_{i\mathfrak{a}_G^*} \hat{\gamma}(\nu_{\Pi_\lambda})\Pi_\lambda(\Sigma(\otimes_v f_v))d\lambda$$

$$= \hat{\gamma}(\nu_\Pi)\int_{i\mathfrak{a}_G^*} \Pi_\lambda(\Sigma(\otimes_v f_v))$$

$$= \hat{\gamma}(\nu_\Pi)\Pi(f).$$

Since Π is arbitrary, f_γ does depend only on f.

If γ belongs to $\mathcal{E}(\mathfrak{h}^1)^W$, let N_γ be any positive number such that γ is supported on the set

$$\{H \in \mathfrak{h}^1 : \|H\| \le N_\gamma\}.$$

It follows from Corollary III.4.3 of [1(d)] and our definitions that if f belongs to $C_N^\infty(G(\mathbf{A})^1, K)$ then f_γ belongs to $C_{N+N_\gamma}^\infty(G(\mathbf{A})^1, K)$. We therefore have

PROPOSITION 3.1. *For any $\gamma \in \mathcal{E}(\mathfrak{h}^1)^W$ there is a map*

$$f \to f_\gamma, \qquad f \in C_N^\infty(G(\mathbf{A})^1, K),$$

from $C_N^\infty(G(\mathbf{A})^1, K)$ to $C_{N+N_\gamma}^\infty(G(\mathbf{A})^1, K)$ such that

$$\Pi(f_\gamma) = \hat{\gamma}(\nu_\Pi)\Pi(f)$$

for every $\Pi \in \Pi(G(\mathbf{A})^1)$. □

Suppose that P is a parabolic subgroup and that $\pi \in \Pi(M_P(\mathbf{A}))$. By the definition above we have an orbit $\{\nu_\pi\}$ of the complex Weyl group of $M_P(\mathbf{R})$ in $\mathfrak{h}_\mathbf{C}^*$. Let λ be a point in $i\mathfrak{a}_P^*$. Then a well known formula for the infinitesimal character of an induced representation gives an equality

$$\{\nu_{\rho_{\chi,\pi}(P,\lambda)}\} = \{\nu_\pi + \lambda\}$$

of W-orbits in $\mathfrak{h}_\mathbf{C}^*$. We obtain

COROLLARY 3.2. *If f and γ are as in the proposition and $\Psi_\pi^T(\lambda, \cdot)$ is the function defined by* (1.2), *then*

$$\Psi_\pi^T(\lambda, f_\gamma) = \hat{\gamma}(\nu_\pi + \lambda)\Psi_\pi^T(\lambda, f). \qquad \square$$

4. The main step. We are now ready to unveil the central calculation of the paper. It is really quite simple. Take any function $f \in C_c^\infty(G(\mathbf{A})^1, K)$. We will keep f fixed for the rest of the paper (except for the appendix) so we might as well write

$$\Psi_\pi^T(\lambda) = \Psi_\pi^T(\lambda, f)$$

for the function defined by (1.2). We shall look at the formula for

$$J_\chi^T(f_\gamma), \qquad \gamma \in \mathcal{E}(\mathfrak{h}^1)^W.$$

By (1.3) and Corollary 3.2, $J_\chi^T(f_\gamma)$ equals

$$\sum_P \sum_{\pi \in \Pi(M_P(\mathbf{A})^1)} \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \Psi_\pi^T(\lambda, f_\gamma) d\lambda$$

$$= \sum_P \sum_{\pi \in \Pi(M_P(\mathbf{A})^1)} \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \hat{\gamma}(\nu_\pi + \lambda)\Psi_\pi^T(\lambda) d\lambda.$$

Treating γ in our notation as if it were a function, we write the last expression as

$$\sum_P \sum_{\pi \in \Pi(M_P(\mathbf{A})^1)} \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \Psi_\pi^T(\lambda) \int_{\mathfrak{h}^1} \gamma(H) e^{(\nu_\pi + \lambda)(H)} dH\, d\lambda$$

$$= \int_{\mathfrak{h}^1} \Big(\sum_P \sum_{\pi \in \Pi(M_P(\mathbf{A})^1)} \psi_\pi^T(H) e^{\nu_\pi(H)}\Big) \gamma(H) dH,$$

where

$$\psi_\pi^T(H) = \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \Psi_\pi^T(\lambda) e^{\lambda(H)} d\lambda, \tag{4.1}$$

for $\pi \in \Pi(M_P(\mathbf{A}))$ and $H \in \mathfrak{h}^1$. These manipulations make sense so long as Proposition 2.1 applies; that is, whenever $d_{P_0}(T) > C_0$. We shall assume that this is the case. Then ψ_π^T is a smooth function on $\mathfrak{h}^1$. In fact for every differential operator D with constant coefficients on $\mathfrak{h}^1$ there is a constant c_D such that

$$\sup_{H \in \mathfrak{h}^1} |D\psi_\pi^T(H)| \le c_D(1 + \|T\|)^{d_0}, \tag{4.2}$$

provided, of course, that $d_{P_0}(T) > C_0$. Notice also that ψ_π^T is constant on the cosets of the kernel of h_P in $\mathfrak{h}^1$.

There is certainly a number $N_f > 0$ such that f belongs to $C_{N_f}^\infty(G(\mathbf{A})^1, K)$. We define a constant

$$C = C_0(1 + N_f)$$

for use in the following discussion. The definitions and estimates we have just made are certainly all valid if C_0 is replaced by C. Now, we fix $H \in \mathfrak{h}^1$, and let γ_H be the discrete measure in $\mathfrak{h}^1$ at the point H. Define

$$\gamma = |W|^{-1} \sum_{s \in W} \gamma_{s^{-1}H}.$$

Then $J_\chi^T(f_\gamma)$ equals

$$|W|^{-1} \sum_{s \in W} \sum_P \sum_{\pi \in \Pi(M_P(\mathbf{A})^1)} \psi_\pi^T(s^{-1}H)e^{\nu_\pi(s^{-1}H)}.$$

We can take $N_\gamma = \|H\|$, so by Proposition 3.1 the function f_γ belongs to $C_{N_f + \|H\|}^\infty(G(\mathbf{A})^1, K)$. It follows from Proposition 2.2 that this last expression is a polynomial in T whenever

$$d_{P_0}(T) > C(1 + \|H\|).$$

If H is taken to be the origin in $\mathfrak{h}^1$, then γ is just the Dirac measure on $\mathfrak{h}^1$. The function f_γ equals f. This gives us

$$J_\chi^T(f) = \sum_P \sum_{\pi \in \Pi(M_P(\mathbf{A})^1)} \psi_\pi^T(0),$$

again provided that $d_{P_0}(T) > C$.

It follows from what we have just seen that there is a unique function

$$p^T(H), \qquad T \in \mathfrak{a}_0, \qquad H \in \mathfrak{h}^1,$$

which is a polynomial in T, and such that

$$p^T(H) = |W|^{-1} \sum_{s\in W} \sum_P \sum_{\pi\in\Pi(M_P(\mathbf{A})^1)} \psi_\pi^T(s^{-1}H)e^{\nu_\pi(s^{-1}H)} \tag{4.3}$$

whenever

$$d_{P_0}(T) > C(1 + \|H\|).$$

Since each $\psi_\pi^T(H)$ is a smooth function of H, $p^T(H)$ must also be smooth in H. From the estimate (4.2) (with $D = 1$) we see that the total degree of $p^T(H)$, as a polynomial in T, is at most d_0. Finally, the formula for $J_\chi^T(f)$ becomes

$$J_\chi^T(f) = p^T(0), \tag{4.4}$$

for $d(T) > C$, and hence for all $T \in \mathfrak{a}_0$.

The point of this discussion has been to embed the polynomial

$$T \to J_\chi^T(f)$$

in a family

$$T \to p^T(H), \qquad H \in \mathfrak{h}^1,$$

of polynomials. Our plan is to extract information about $p^T(0)$ from the average behaviour of the polynomials $p^T(H)$ near $H = 0$. Ideally we would like to integrate $p^T(H)$ against an arbitrary Schwartz function on $\mathfrak{h}^1$. Unfortunately this is not possible, for $p^T(H)$ is not a tempered function of H. We shall change our notation slightly to highlight the difficulty.

Given P and $\pi \in \Pi(M_P(\mathbf{A}))$, let

$$\nu_\pi = X_\pi + iY_\pi, \qquad X_\pi, Y_\pi \in \mathfrak{h}^*,$$

be the decomposition of ν_π into real and imaginary parts. These points actually stand for orbits in $\mathfrak{h}^*$ of the Weyl group of M_P, but from now on we shall take them to be fixed representatives of the corresponding orbits. Suppose that π is identified with its restriction to the group $M_P(\mathbf{A})^1$. Then ν_π is determined only modulo $i\mathfrak{a}_P^*$. The real part, X_π, is uniquely determined in $\mathfrak{h}^*$, but the imaginary part, Y_π, becomes only a point in $\mathfrak{h}^*/\mathfrak{a}_P^*$. Now the right hand side of (4.3) is a sum over pairs

$$(s, \pi), \qquad s \in W, \qquad \pi \in \Pi(M_P(\mathbf{A})^1), \qquad P \supset P_0.$$

Only finitely many summands are nonzero. Let us define two pairs

$$(s_\iota, \pi_\iota), \qquad s_\iota \in W, \qquad \pi_\iota \in \Pi(M_{P_\iota}(\mathbf{A})^1), \qquad \iota = 1, 2,$$

to be equivalent if

$$s_1 X_{\pi_1} = s_2 X_{\pi_2}.$$

Let $\mathcal{E}$ denote the set of equivalence classes of pairs. For any class $\Gamma \in \mathcal{E}$ we shall write X_Γ for the common point

$$sX_\pi, \qquad (s, \pi) \in \Gamma.$$

Then (4.3) can be written

$$(4.3') \qquad p^T(H) = \sum_{\Gamma \in \mathcal{E}} \psi_\Gamma^T(H) e^{X_\Gamma(H)}, \qquad d_{P_0}(T) > C(1 + \|H\|),$$

where

$$(4.5) \qquad \psi_\Gamma^T(H) = |W|^{-1} \sum_{(s,\pi)\in\Gamma} \psi_\pi^T(s^{-1}H) e^{iY_\pi(s^{-1}H)},$$

a smooth function of H. As in (4.2), we can choose a constant c_D for every D such that

$$(4.2') \qquad |D\psi_\Gamma^T(H)| \leq c_D(1 + \|T\|)^{d_0},$$

for all Γ, H and all T such that $d_{P_0}(T) > C$. In particular, $\psi_\Gamma^T(H)$ is a tempered function of H.

Suppose that Γ is an element in $\mathcal{E}$ such that $X_\Gamma \neq 0$ and such that ψ_Γ^T does not vanish. Is it possible for

$$\psi_\Gamma^T(H) e^{X_\Gamma(H)},$$

the contribution of Γ to (4.3′), to be a tempered function of H? Could $\psi_\Gamma^T(H)$ decrease sufficiently to compensate for the exponential increase of

$e^{X_\Gamma(H)}$? The answer is no! To see this let H_Γ be the unique vector in $\mathfrak{h}^1$ such that

$$X_\Gamma(H) = (H_\Gamma, H)$$

for each $H \in \mathfrak{h}^1$. Suppose that

$$(s, \pi), \qquad s \in W, \qquad \pi \in \Pi(M_P(\mathbf{A})^1),$$

belongs to Γ. Then

$$X_\pi(H) = X_\Gamma(sH) = (s^{-1}H_\Gamma, H), \qquad H \in \mathfrak{h}^1.$$

It follows readily that $s^{-1}H_\Gamma$ belongs to the kernel of h_p in $\mathfrak{h}^1$. Since ψ_π^T is invariant under translations from the kernel of h_P,

$$\psi_\Gamma^T(tH_\Gamma + H) = \sum_{(s,\pi)\in\Gamma} (|W|^{-1}\psi_\pi^T(s^{-1}H)e^{iY_\pi(s^{-1}H)})e^{itY_\pi(s^{-1}H_\Gamma)}.$$

Thus as t ranges over the real numbers, $\psi_\Gamma^T(tH_\Gamma + H)$ spans a finite dimensional space V_Γ of functions of H and T. In fact as a finite linear combination in V_Γ, $\psi_\Gamma^T(tH_\Gamma + H)$ has coefficients of the form $e^{it\mu_\pi}$, $\mu_\pi \in \mathbf{R}$. Therefore $\psi_\Gamma^T(tH_\Gamma + H)$ cannot approach zero as t approaches infinity. The function $\psi_\Gamma^T(H)e^{X_\Gamma(H)}$ cannot be tempered in the direction of H_Γ. Thus (4.3′) gives a rather precise description of the failure of $P^T(H)$ to be tempered.

5. More polynomials. In this section we will construct a family of functions $\{p_\Gamma^T(H):\Gamma \in \mathcal{E}\}$ which are polynomials in T and *tempered* functions of H. The construction does not require anything further from Eisenstein series or the trace formula. It is an exercise in real variables, which requires nothing more than elementary analysis and the formal properties of the collection $\{\psi_\Gamma^T(H)\}$. We shall recapitulate these properties in such a way as to display the elementary nature of the construction, and also to allow for induction arguments. To simplify our notation, we shall denote the function $d_{P_0}(T)$ by $d(T)$ in this section.

We are given a finite set

$$\{\psi_\Gamma^T(H):\Gamma \in \mathcal{E}\}$$

of functions on a product $\mathfrak{h}^1 \times \mathfrak{a}_0$ of two Euclidean spaces. The domain of each function is

$$\{H \in \mathfrak{h}^1\} \times \{T \in \mathfrak{a}_0 : d(T) > C_0\},$$

the product of $\mathfrak{h}^1$ with a translate of a certain chamber in $\mathfrak{a}_0$. We are also given a finite set

$$\{X_\Gamma : \Gamma \in \mathcal{E}\}$$

of distinct linear functions on $\mathfrak{h}^1$. If X_Γ is a nonzero function from this set and H_Γ is the point in $\mathfrak{h}^1$ such that

$$X_\Gamma(H) = (H_\Gamma, H), \qquad H \in \mathfrak{h}^1,$$

then the set

$$\{\psi_\Gamma^T(tH_\Gamma + H) : t \in \mathbf{R}\}$$

spans a finite dimensional space of functions of H and T. Finally, we are given a function $p^T(H)$, which is smooth in H and a polynomial in T of degree at most d_0, as well as positive constants C and ϵ, which satisfy the following property: for every differential operator D with constant coefficients on $\mathfrak{h}^1$ there is a constant c_D such that

$$(5.1) \quad \left| D\left(\left(\sum_{\Gamma \in \mathcal{E}} \psi_\Gamma^T(H) e^{X_\Gamma(H)}\right) - p^T(H)\right)\right| \leq c_D e^{-\epsilon d(T)}(1 + \|T\|)^{d_0},$$

for all H and T with $d(T) > C(1 + \|H\|)$, and

$$(5.2) \qquad |D\psi_\Gamma^T(H)| \leq c_D(1 + \|T\|)^{d_0}$$

for all Γ, H and all T with $d(T) > C$. (We saw in Section 4 that the first inequality held with $c_D = 0$. We have stated it this way to allow for induction arguments. The second inequality is just (4.2′).)

PROPOSITION 5.1. *We can find a collection*

$$\{p_\Gamma^T(H) : \Gamma \in \mathcal{E}\}$$

of functions which are smooth in H and are polynomials in T of total degree at most d_0 such that

$$p^T(H) = \sum_\Gamma p^T_\Gamma(H)e^{\chi_\Gamma(H)}, \tag{5.3}$$

and such that for positive constants C and ϵ the following condition is satisfied: for every differential operator D with constant coefficients on $\mathfrak{h}^1$ there is a constant c_D such that

$$|D(\psi^T_\Gamma(H) - p^T_\Gamma(H))| \leq c_D e^{-\epsilon d(T)}(1 + \|T\|)^{d_0} \tag{5.4}$$

for all Γ, H and all T with $d(T) > C(1 + \|H\|)$, and

$$|Dp^T_\Gamma(H)| \leq c_D(1 + \|H\|)^{d_0}(1 + \|T\|)^{d_0} \tag{5.5}$$

for all H and T.

Notice that (5.4) determines the polynomial $p^T_\Gamma(H)$ uniquely. Before beginning the proof of the proposition we will also observe that the inequality (5.5) follows from (5.4) and (5.2). For if $d(T) > C(1 + \|H\|)$, then $|Dp^T_\Gamma(H)|$ will be bounded by a constant multiple of $(1 + \|T\|)^{d_0}$. The following lemma tells us that we can choose c_D so that (5.5) is valid.

LEMMA 5.2. *Given d_0 we can choose constants a_1 and a_2 with the following property. Let $q(T)$ be a polynomial on $\mathfrak{a}_0$ of degree at most d_0 such that for numbers $A > 0$ and $B > 1$,*

$$|q(T)| \leq A(1 + \|T\|)^{d_0}$$

for all T with $d(T) > B$. Then each coefficient of $q(T)$ is bounded by $a_1 AB^{d_0}$, and

$$|q(T)| \leq a_2 AB^{d_0}(1 + \|T\|)^{d_0}$$

for all T.

The first estimate of the lemma follows from an interpolation argument. One solves for the coefficients of q in terms of the values of q at a finite number of points T with $d(T) > B$. We leave the details to the reader. The second estimate of the lemma follows from the first. □

Proof of Proposition 5.1. We shall prove the proposition by induction on the number of elements in $\mathcal{E}$. Suppose first of all that there is no element $\Gamma \in \mathcal{E}$ with $X_\Gamma \neq 0$. The functionals $\{X_\Gamma\}$ are distinct, so if $\mathcal{E}$ is not empty it will consist of one element Γ, with $X_\Gamma = 0$. In this case we set

$$p_\Gamma^T(H) = p^T(H).$$

Then (5.3) is trivial and (5.4) becomes the same estimate as (5.1). As we have observed, (5.5) is a consequence of (5.2) and (5.4).

We can therefore suppose that there are elements $\Gamma \in \mathcal{E}$ with $X_\Gamma \neq 0$. Choose $\Gamma' \in \mathcal{E}$ that $\|H_{\Gamma'}\|$ is as large as possible. It follows from the definition of $H_{\Gamma'}$ that if $\Gamma \in \mathcal{E}$ is distinct from Γ', the number

$$(X_{\Gamma'} - X_\Gamma)(H_{\Gamma'}) = \|H_{\Gamma'}\|^2 - X_\Gamma(H_{\Gamma'})$$

is strictly positive. Now, consider the function

$$\psi_{\Gamma'}^T(H) - p^T(H)e^{-X_{\Gamma'}(H)}.$$

It equals

$$\Big[\sum_\Gamma \psi_\Gamma^T(H)e^{X_\Gamma(H)} - P^T(H)\Big]e^{-X_{\Gamma'}(H)} + \sum_{\Gamma \neq \Gamma'} [\psi_\Gamma^T(H)]e^{-(X_{\Gamma'} - X_\Gamma)(H)}.$$

This last expression is a sum of terms, each of which is a product of an exponential with a second function of H (which is enclosed in the square brackets). The result of letting a differential operator D act on one of the terms is the product of the exponential with the derivative of the second function with respect to some other differential operator. We may therefore use (5.1) and (5.2) to estimate the expression

$$|D(\psi_{\Gamma'}^T(H) - p^T(H)e^{-X_{\Gamma'}(H)})|.$$

There is a constant c_D' such that it is bounded by

$$c_D'\Big(e^{-X_{\Gamma'}(H) - \epsilon d(T)} + \sum_{\Gamma \neq \Gamma'} e^{-(X_{\Gamma'} - X_\Gamma)(H)}\Big)(1 + \|T\|)^{d_0},$$

whenever $d(T) > C(1 + \|H\|)$. Our next step is to replace the variable H in the estimate by $tH_{\Gamma'} + H$. In the resulting expressions, $H \in \mathfrak{h}^1$, $T \in \mathfrak{a}_0$

and $t \in \mathbf{R}$ will be subjected to the constraints $t > C_1(1 + \|H\|)$ and $d(T) > C_2 t$, for constants C_1 and C_2. We will take C_1 and C_2 to be arbitrarily large. Then $d(T)$ will be larger than $C(1 + \|tH_{\Gamma'} + H\|)$, so the estimate will remain in force. In addition, there will be a positive constant ϵ' such that

$$e^{-X_{\Gamma'}(tH_{\Gamma'}+H)-\epsilon d(T)} < e^{-X_{\Gamma'}(tH_{\Gamma'}+H)} \leq e^{-\epsilon' t},$$

and

$$\sum_{\Gamma \neq \Gamma'} e^{-(X_{\Gamma'}-X_{\Gamma})(tH_{\Gamma'}+H)} \leq e^{-\epsilon' t}.$$

It follows that for every D there is a constant c'_D such that

(5.6)

$$|D(\psi^T_{\Gamma'}(tH_{\Gamma'}+H) - p^T(tH_{\Gamma'}+H)e^{-X_{\Gamma'}(tH_{\Gamma'}+H)})| \leq c'_D e^{-\epsilon' t}(1 + \|T\|)^{d_0},$$

for H, T and t constrained as above. (In this expression, D can act either through the argument $tH_{\Gamma'} + H$ or just through H.)

Let $t_1 = 0, t_2, \ldots, t_m$ be fixed real numbers such that

$$\{\psi^T_{\Gamma'}(t_i H_{\Gamma'} + H) : 1 \leq i \leq n\}$$

is a basis of the space $V_{\Gamma'}$ spanned by

$$\{\psi^T_{\Gamma'}(tH_{\Gamma'} + H) : t \in \mathbf{R}\}.$$

We shall let $\Psi(T, H)$ and $P(T, H)$ be the vectors in $\mathbf{C}^n$ whose i^{th} components are

$$\psi^T_{\Gamma'}(t_i H_{\Gamma'} + H)$$

and

$$e^{-X_{\Gamma'}(t_i H_{\Gamma'}+H)} p^T(t_i H_{\Gamma'} + H)$$

respectively. Now translation by $tH_{\Gamma'}$, $t \in \mathbf{R}$, is a representation of the real numbers on $V_{\Gamma'}$. It follows that there is an $(m \times m)$-matrix R such that

$$\Psi(T, tH_{\Gamma'} + H) = e^{tR}\Psi(T, H), \qquad t \in \mathbf{R}.$$

Combined with (5.2), this equation yields the fact that all the eigenvalues of R are purely imaginary. In particular there is a constant c_R such that

$$\|e^{-tR}\| \leq c_R(1 + |t|)^{m-1},$$

for all $t \in \mathbf{R}$. (The operator norm here is that associated to the standard Hermitian inner product on $\mathbf{C}^n$.) We would like to estimate

$$\|D(\Psi(T, H) - e^{-tR}P(T, tH_{\Gamma'} + H))\|.$$

It is bounded by

$$\|e^{-tR}\| \, \|D(e^{tR}\Psi(T, H) - P(T, tH_{\Gamma'} + H))\|,$$

which is in turn bounded by

$$c_R(1 + |t|)^{m-1}\|D(\Psi(T, tH_{\Gamma'} + H) - P(T, tH_{\Gamma'} + H)\|.$$

This last expression contains the norm of a vector in $\mathbf{C}^n$, each of whose components can be estimated by (5.6). It follows that we may choose the positive constants C_1, C_2, ϵ' and c_D' so that

$$\|D(\Psi(T, H) - e^{-tR}P(T, tH_{\Gamma'} + H))\| \leq c_D' e^{-\epsilon' t}(1 + \|T\|)^{d_0}, \tag{5.7}$$

whenever $t > C_1(1 + \|H\|)$ and $d(T) > C_2 t$.

Set

$$P_n(T, H) = e^{-nR}P(T, nH_{\Gamma'} + H),$$

for each positive integer n. Then by (5.7),

$$\|D(P_{n+1}(T, H) - P_n(T, H))\| \leq 2c_D' e^{-\epsilon' n}(1 + \|T\|)^{d_0},$$

whenever $n > C_1(1 + \|H\|)$ and $d(T) > C_2(n + 1)$. Now

$$P_{n+1}(T, H) - P_n(T, H)$$

is a polynomial in T of degree at most d_0. We can apply Lemma 5.2. We have

$$\|D(P_{n+1}(T,H) - P_n(T,H))\| \leq a_2 \cdot 2c'_D e^{-\epsilon' n}(C_2(n+1))^{d_0}(1 + \|T\|)^{d_0},$$

for $n > C_1(1 + \|H\|)$ and for *all* T. Observe that

$$\sum_{k=0}^{\infty} e^{-\epsilon'(n+k)}(n+k+1)^{d_0}$$

is bounded by a constant multiple of $e^{-(\epsilon' n/2)}$. In particular the sequence

$$\{DP_n(T, H)\}_{n=1}^{\infty}$$

converges uniformly for H in compact sets. Therefore there is a $\mathbf{C}^n$ valued function $P_\infty(T, H)$, which is smooth in H and is a polynomial in T of degree at most d_0, such that for any D, H, n and T, with $n > C_1(1 + \|H\|)$,

$$\|D(P_\infty(T, H) - P_n(T, H))\|$$

is bounded by a constant multiple of

$$e^{-(\epsilon' n/2)}(1 + \|T\|)^{d_0}.$$

For our final estimate, we will combine this last inequality with (5.7). We will require that $d(T)$ be greater than $C'(1 + \|H\|)$, where C' is the constant $C_2(1 + C_1)$. Given T, set both n and t equal to the greatest integer in $C_2^{-1}d(T)$. Then the constraints $d(T) > C_2 t$, $t > C_1(1 + \|H\|)$ and $n > C_1(1 + \|H\|)$ all hold. Moreover,

$$e^{-t} = e^{-n} \leq e \cdot e^{-C_2^{-1}d(T)}.$$

It follows that we can find absolute constants C' and ϵ', and a constant c'_D for every D, such that

$$\|D(\Psi(T, H) - P_\infty(T, H))\| \leq c'_D e^{-\epsilon' d(T)}(1 + \|T\|)^{d_0},$$

whenever $d(T) > C'(1 + \|H\|)$. The projection of the vector $\Psi(T, H)$ onto the first component in $\mathbf{C}^n$ is just the function $\psi_{\Gamma'}^T(H)$. We define the function $p_{\Gamma'}^T(H)$ demanded by the proposition to be simply the projection of $P_\infty(t, H)$ onto the first component in $\mathbf{C}^n$. Then

$$|D(\psi_{\Gamma'}^T(H) - p_{\Gamma'}^T(H))| \leq c_D' e^{-\epsilon' d(T)}(1 + \|T\|)^{d_0},$$

for all H and T with $d(T) > C'(1 + \|H\|)$. This is just the required inequality (5.4). We have already observed that (5.5) is a consequence of (5.2) and (5.4).

Let $\tilde{\mathcal{E}}$ be the complement of Γ' in $\mathcal{E}$. To complete the induction step we must define the function $\tilde{p}^T(H)$ associated to the set $\tilde{\mathcal{E}}$. We set

$$\tilde{p}^T(H) = p^T(H) - p_{\Gamma'}^T(H)e^{X_{\Gamma'}(H)}.$$

The function is smooth in H and a polynomial in T of degree at most d_0. Our last task is to verify (5.1) for the set $\tilde{\mathcal{E}}$. The function we must estimate,

$$\left|D\Big(\sum_{\Gamma \in \tilde{\mathcal{E}}} \psi_\Gamma^T(H)e^{X_\Gamma(H)} - \tilde{p}^T(H)\Big)\right|, \tag{5.8}$$

is bounded by the sum of

$$\left|D\Big(\sum_{\Gamma \in \mathcal{E}} \psi_\Gamma^T(H)e^{X_\Gamma(H)} - p^T(H)\Big)\right|$$

and

$$|D(\psi_{\Gamma'}^T(H)e^{X_{\Gamma'}(H)} - p_{\Gamma'}^T(H)e^{X_{\Gamma'}(H)})|.$$

Suppose that $\tilde{C}$ is a constant which is greater than both C and C', and that $d(T) > \tilde{C}(1 + \|H\|)$. Then (5.8) is bounded by a constant multiple of

$$(e^{-\epsilon d(T)} + e^{X_{\Gamma'}(H)}e^{-\epsilon' d(T)})(1 + \|T\|)^{d_0}.$$

If $\tilde{C}$ is large enough, there will be a positive number $\tilde{\epsilon}$ so that for H and T as above,

$$e^{X_{\Gamma'}(H)}e^{-\epsilon' d(T)} \leq e^{-\tilde{\epsilon} d(T)}.$$

If $\tilde{\epsilon}$ is also taken to be no greater than ϵ, there will be a constant $\tilde{c}_D$ such that (5.8) is bounded by

$$\tilde{c}_D e^{-\tilde{\epsilon} d(T)}(1 + \|T\|)^{d_0},$$

for all H and t with $d(T) > \tilde{C}(1 + \|H\|)$. This establishes (5.1) for the smaller set $\tilde{\mathcal{E}}$. By induction, the proposition holds for $\tilde{\mathcal{E}}$. In particular, (5.3) is valid. In view of the definition of $\tilde{p}^T(H)$, (5.3) is also valid for $\mathcal{E}$. Thus, all the assertions of the proposition hold also for the original set $\mathcal{E}$.

The bottom step of the induction is when $\mathcal{E}$ contains only one element, Γ. The case that $X_\Gamma = 0$ was dispatched at the beginning. If $X_\Gamma \neq 0$ we can take $\Gamma' = \Gamma$ and proceed as above. Then the sum over Γ in (5.8) is empty, and we have a strong inequality for the polynomial $\tilde{p}^T(H)$. So strong, in fact, that it forces $\tilde{p}^T(H)$ to vanish. In other words,

$$p^T(H) = p^T_{\Gamma'}(H) e^{X_{\Gamma'}(H)}.$$

This is just (5.3). We have completed the proof of the proposition in case $\mathcal{E}$ contains one element, and hence for all $\mathcal{E}$. □

6. New test functions. Our ultimate goal is to calculate $J^T_\chi(f)$. In view of formulas (5.3) and (4.4), we have

$$J^T_\chi(f) = p^T(0) = \sum_{\Gamma \in \mathcal{E}} p^T_\Gamma(0),$$

for all T in $\mathfrak{a}_0$. We have just seen that $p^T_\Gamma(H)$ is a tempered function of H, so if β belongs to the Schwartz space $\mathcal{S}(\mathfrak{h}^1)$ of $\mathfrak{h}^1$ we can define

$$p^T_\Gamma(\beta) = \int_{\mathfrak{h}^1} p^T_\Gamma(H)\beta(H)dH.$$

It is a polynomial in T of degree at most d_0. To approximate $p^T_\Gamma(0)$ we will replace β by

$$\beta_\epsilon(H) = (\epsilon)^{-\dim(\mathfrak{h}^1)}\beta(\epsilon^{-1}H), \qquad H \in \mathfrak{h}^1, \qquad \epsilon > 0,$$

and let ϵ approach zero.

LEMMA 6.1. *Suppose that β is a function in $\mathcal{S}(\mathfrak{h}^1)$ such that*

$$\int_{\mathfrak{h}^1} \beta(H)dH = 1.$$

Then for each $T \in \mathfrak{a}_0$,

$$\lim_{\epsilon \to 0} p_\Gamma^T(\beta_\epsilon) = p_\Gamma^T(0).$$

Proof. Results of this kind are well known. Following the usual argument we observe that

$$\begin{aligned} |p_\Gamma^T(\beta_\epsilon) - p_\Gamma^T(0)| & \\ &\leq \left| \int_{\mathfrak{h}^1} p_\Gamma^T(H)\beta_\epsilon(H)dH - \int_{\mathfrak{h}^1} p_\Gamma^T(0)\beta_\epsilon(H)dH \right| \\ &\leq \int_{\mathfrak{h}^1} |p_\Gamma^T(H) - p_\Gamma^T(0)|\,|\beta_\epsilon(H)|\,dH, \end{aligned}$$

since β_ϵ also has integral equal to 1. To estimate $|p_\Gamma^T(H) - p_\Gamma^T(0)|$ we combine the mean value theorem with the inequality (5.5) for $|Dp_\Gamma^T(H)|$. (In this case D will be a first order differential operator.) There will be a constant c such that

$$|p_\Gamma^T(H) - p_\Gamma^T(0)| \leq c(1 + \|T\|)^{d_0}\|H\|(1 + \|H\|)^{d_0}.$$

With a change of variable the integral

$$\int_{\mathfrak{h}^1} \|H\|(1 + \|H\|)^{d_0}|\beta_\epsilon(H)|\,dH$$

becomes

$$\epsilon \int_{\mathfrak{h}^1} \|H\|(1 + \|\epsilon H\|)^{d_0}|\beta(H)|\,dH,$$

which as long as $\epsilon \leq 1$, is certainly bounded by

$$\epsilon \int_{\mathfrak{h}^1} \|H\|(1 + \|H\|)^{d_0}|\beta(H)|\,dH.$$

This last expression approaches 0 with ϵ. The lemma follows. □

The integral $p_\Gamma^T(\beta)$ cannot be calculated directly since $p_\Gamma^T(H)$ is, after all, not given explicitly. It is only $\psi_\Gamma^T(H)$ for which we have some semblance of a formula. We need a lemma to relate $p_\Gamma^T(\beta)$ with this function.

As in [1(e)] we shall say that T approaches infinity strongly in $\mathfrak{a}_0^+$ if $\|T\|$ approaches infinity but T remains within a region

$$\{T \in \mathfrak{a}_0 : d_{P_0}(T) > \delta\|T\|\}$$

for some fixed positive constant δ.

LEMMA 6.2. *For any $\beta \in \mathcal{S}(\mathfrak{h}^1)$ the expression*

$$\int_{\mathfrak{h}^1} \psi_\Gamma^T(H)\beta(H)dH - p_\Gamma^T(\beta)$$

approaches zero as T approaches infinity strongly in $\mathfrak{a}_0^+$.

Proof. The given expression has absolute value bounded by

$$\int_{\mathfrak{h}^1} |\psi_\Gamma^T(H) - p_\Gamma^T(H)|\,|\beta(H)|\,dH. \tag{6.1}$$

By (5.4) there are constants C, ϵ and c such that

$$|\psi_\Gamma^T(H) - p_\Gamma^T(H)| \le ce^{-\epsilon d_{P_0}(T)}(1 + \|T\|)^{d_0},$$

whenever $d_{P_0}(T) > C(1 + \|H\|)$. However, we are letting T approach infinity strongly in $\mathfrak{a}_0^+$, so $\|T\|$ will be bounded by a constant multiple of $d_{P_0}(T)$. We can therefore choose the constants C, ϵ and c so that

$$|\psi_\Gamma^T(H) - p_\Gamma^T(H)| \le ce^{-\epsilon\|T\|}$$

whenever $\|T\| > C(1 + \|H\|)$. It follows that the contribution to the integral (6.1) from the set of H with $\|T\| > C(1 + \|H\|)$ is bounded by a constant multiple of $e^{-\epsilon\|T\|}$.

To deal with the remaining contribution to (6.1) we appeal to (5.2) and (5.5). These inequalities tell us that $|\psi_\Gamma^T(H) - p_\Gamma^T(H)|$ is bounded by a constant multiple of $(1 + \|H\|)^{d_0}(1 + \|T\|)^{d_0}$ for all H and T (as long as $d_{P_0}(T)$ is greater than some absolute constant). We are now assuming that

$$\|T\| \leq C(1 + \|H\|).$$

Then there is a constant c_1 such that

$$|\psi_\Gamma^T(H) - p_\Gamma^T(H)| \leq c_1(1 + \|H\|)^{2d_0}.$$

Choose any $n > 0$. The remaining contribution to (6.1) is bounded by the product of

$$c_1 C^n \|T\|^{-n}$$

and the integral of

$$|\beta(H)|(1 + \|H\|)^{2d_0+n}.$$

Since β is a Schwartz function, the integral of this last expression is finite.

We have shown that for any n, (6.1) is bounded by a constant multiple of $\|T\|^{-n}$. In particular, it does approach zero as T approaches infinity strongly in $\mathfrak{a}_0^+$. □

We are now ready for the main result of this paper. It concerns test functions in $\mathcal{S}(i\mathfrak{h}^*/i\mathfrak{a}_G^*)^W$, the space of Schwartz functions on the real vector space $i\mathfrak{h}^*/i\mathfrak{a}_G^*$ which are symmetric under W. In the present context the space $\mathcal{S}(i\mathfrak{h}^*/i\mathfrak{a}_G^*)^W$ has a natural representation theoretic interpretation. As we saw in Section 3, there is associated to any $\Pi \in \Pi(G(\mathbf{A}))$ a W-orbit $\{\nu_\Pi\}$ in $\mathfrak{h}_{\mathbf{C}}^*$. Each ν_Π has a decomposition

$$\nu_\Pi = X_\Pi + iY_\Pi, \qquad X_\Pi, Y_\Pi \in \mathfrak{h}^*.$$

If Π is identified with its restriction to $G(\mathbf{A})^1$, the imaginary part, iY_Π is only determined modulo $i\mathfrak{a}_G^*$. It becomes a W-orbit in $i\mathfrak{h}^*/i\mathfrak{a}_G^*$. Thus the "imaginary part" of the infinitesimal character gives a fibering of $\Pi(G(\mathbf{A})^1)$ over the space of W-orbits in $i\mathfrak{h}^*/i\mathfrak{a}_G^*$. Our test functions will then come from the space of Schwartz functions on the base space of this fibration.

Suppose that $B \in \mathcal{S}(i\mathfrak{z}^*/i\mathfrak{a}_G^*)^W$. For any $\pi \in \Pi(M_P(\mathbf{A}))$ we shall write

$$B_\pi(\lambda) = B(iY_\pi + \lambda), \qquad \lambda \in i\mathfrak{a}_P^*/i\mathfrak{a}_G^*.$$

It is a Schwartz function on $i\mathfrak{a}_P^*/i\mathfrak{a}_G^*$. We shall also write

$$B^\epsilon(\nu) = B(\epsilon\nu), \qquad \nu \in i\mathfrak{h}^*/i\mathfrak{a}_G^*,$$

for every positive number ϵ.

THEOREM 6.3. (i) *For every function $B \in \mathcal{S}(i\mathfrak{h}^*/i\mathfrak{a}_G^*)^W$ there is a unique polynomial $P^T(B)$ in T such that*

$$\sum_P \sum_{\pi\in\Pi(M_P(\mathbf{A})^1)} \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \psi_\pi^T(\lambda)B_\pi(\lambda)d\lambda - P^T(B)$$

approaches zero as T approaches infinity strongly in $\mathfrak{a}_0^+$.

(ii) *If $B(0) = 1$ then*

$$J_\chi^T(f) = \lim_{\epsilon\to 0} P^T(B^\epsilon).$$

Proof. Fix $B \in \mathcal{S}(i\mathfrak{h}^*/i\mathfrak{a}_G^*)^W$. There is a unique function β in $\mathcal{S}(\mathfrak{h}^1)$ such that

$$B(\nu) = \int_{\mathfrak{h}^1} \beta(H)e^{\nu(H)}dH, \qquad \nu \in i\mathfrak{h}^*/i\mathfrak{a}_G^*.$$

It is symmetric under W. We shall define

$$P^T(B) = \sum_{\Gamma\in\mathcal{E}} p_\Gamma^T(\beta).$$

This is certainly a polynomial in T. According to Lemma 6.2, $P^T(B)$ differs from

$$\sum_{\Gamma\in\mathcal{E}} \int_{\mathfrak{h}^1} \psi_\Gamma^T(H)\beta(H)dH \tag{6.2}$$

by a function which approaches zero as T approaches infinity strongly in $\mathfrak{a}_0^+$. However, (6.2) equals

$$\sum_{\Gamma\in\mathcal{E}} \int_{\mathfrak{h}^1} |W|^{-1} \sum_{(s,\pi)\in\Gamma} \psi_\pi^T(s^{-1}H)e^{iY_\pi(s^{-1}H)}\beta(H)dH$$

$$= |W|^{-1} \sum_{s \in W} \sum_{P} \sum_{\pi \in \Pi(M_P(\mathbf{A})^1)} \int_{\mathfrak{h}^1} \psi_\pi^T(s^{-1}H) e^{iY_\pi(s^{-1}H)} \beta(H) dH,$$

by the definition (4.5) of ψ_Γ^T. Since β is symmetric, this can be written

$$\sum_P \sum_\pi \int_{\mathfrak{h}^1} \psi_\pi^T(H) e^{iY_\pi(H)} \beta(H) dH.$$

Recalling the definition (4.1) of ψ_π^T, we see that this last expression equals

$$\sum_P \sum_\pi \int_{\mathfrak{h}^1} \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \Psi_\pi^T(\lambda) e^{(iY_\pi + \lambda)(H)} \beta(H) d\lambda dH,$$

which is the same as

$$\sum_P \sum_{\pi \in \Pi(M_P(\mathbf{A})^1)} \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \Psi_\pi^T(\lambda) B_\pi(\lambda) d\lambda. \tag{6.3}$$

Therefore the difference between (6.3) and $P^T(B)$ approaches zero as T approaches infinity strongly in $\mathfrak{a}_0^+$, as required. The polynomial $P^T(B)$ is clearly uniquely determined by this property.

We still have to prove (ii). Suppose that B and β are related as above and that $B(0) = 1$. Then

$$\int_{\mathfrak{h}^1} \beta(H) dH = 1.$$

Moreover,

$$B^\epsilon(\nu) = \int_{\mathfrak{h}^1} \beta_\epsilon(H) e^{\nu(H)} dH$$

for any $\epsilon > 0$. It follows from Lemma 6.1 that

$$\begin{aligned} &\lim_{\epsilon \to 0} P^T(B^\epsilon) \\ &\qquad = \lim_{\epsilon \to 0} \sum_{\Gamma \in \mathcal{E}} p_\Gamma^T(\beta_\epsilon) \\ &\qquad = \sum_{\Gamma \in \mathcal{E}} p_\Gamma^T(0). \end{aligned}$$

We have already seen that $\sum_{\Gamma} p_{\Gamma}^{T}(0)$ equals $J_{\chi}^{T}(f)$. The theorem is proved. □

Suppose that $B(0) = 1$ as in part (ii) of the theorem. Set

$$B_{\pi}^{\epsilon}(\lambda) = (B^{\epsilon})_{\pi}(\lambda) = B(\epsilon(iY_{\pi} + \lambda)), \qquad \lambda \in i\mathfrak{a}_P^*/i\mathfrak{a}_G^*.$$

Then the dominated convergence theorem tells us that

$$\lim_{\epsilon \to 0} \sum_P \sum_\pi \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \Psi_\pi^T(\lambda) B_\pi^\epsilon(\lambda) d\lambda$$

equals

$$\sum_P \sum_\pi \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \Psi_\pi^T(\lambda) d\lambda.$$

This equals $J_{\chi}^{T}(f)$ for all points T which are suitably regular in $\mathfrak{a}_0^+$. Therefore the last theorem is the assertion of the equality of

$$\lim_{\epsilon \to 0} \lim_{T \to \infty} \sum_P \sum_\pi \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \Psi_\pi^T(\lambda) B_\pi^\epsilon(\lambda) d\lambda$$

and

$$\lim_{T \to \infty} \lim_{\epsilon \to 0} \sum_P \sum_\pi \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \Psi_\pi^T(\lambda) B_\pi^\epsilon(\lambda) d\lambda$$

where in each case the limit in T is interpreted as the polynomial which is asymptotic to the given function as T approaches infinity strongly in $\mathfrak{a}_0^+$. Thus Theorem 6.3, which is the principal result of this paper, really does concern the interchange of two limits.

7. Towards an explicit formula. The last theorem is an important step in the direction of an explicit formula for $J_{\chi}^{T}(f)$. This may not be apparent to the reader. Indeed, the only formula we have at the moment for $J_{\chi}^{T}(f)$ is in terms of $\Omega_{\chi,\pi}^{T}(P, \lambda)$. This operator is defined by the inner product of truncated Eisenstein series, for which, to be sure, there is no explicit formula. However, if we combine Theorem 6.3 with the main result of a previous paper [1(e)], we will be able to replace $\Omega^T_{\chi,\pi}(P, \lambda)$ by something more transparent.

Suppose that P is a parabolic subgroup. Recall that $\Pi(M_P(\mathbf{A})^1)$ is canonically identified with the set of orbits of $i\mathfrak{a}_P^*$ in $\Pi(M_P(\mathbf{A}))$. We can, and from now on will, further identify $\Pi(M_P(\mathbf{A})^1)$ with a certain set of representatives in $\Pi(M_P(\mathbf{A}))$ of these orbits. Since $M_P(\mathbf{A})$ is the direct product of $M_P(\mathbf{A})^1$ and $A_P(\mathbf{R})^0$, any representation of $M_P(\mathbf{A})^1$ corresponds to a representation of $M_P(\mathbf{A})$ which is trivial on $A_P(\mathbf{R})^0$. We identify these two representations, thereby embedding $\Pi(M_P(\mathbf{A})^1)$ in $\Pi(M_P(\mathbf{A}))$. This allows us to speak properly of $\mathcal{A}^2_{\chi,\pi}(P)$ for any $\pi \in \Pi(M_P(\mathbf{A})^1)$. It is a space of functions on $N_P(\mathbf{A})M_P(\mathbf{Q})A_P(\mathbf{R})^0\backslash G(\mathbf{A})$. The direct sum

$$\bigoplus_{\pi\in\Pi(M_P(\mathbf{A})^1)} \mathcal{A}^2_{\chi,\pi}(P)$$

is just the space we denoted by $\mathcal{A}_{P,\chi}$ in [1(e)].

Let W_0 be the Weyl group of $(G, \mathbf{A}_0)$. If P_1 is another parabolic subgroup, we define the Weyl set $W(\mathfrak{a}_P, \mathfrak{a}_{P_1})$ and the functions

$$M(t, \lambda), \qquad t \in W(\mathfrak{a}_P, \mathfrak{a}_{P_1}),$$

as in [1(e)]. $M(t, \lambda)$ is an analytic function of $\lambda \in i\mathfrak{a}_P^*$ which for any $\pi \in \Pi(M_P(\mathbf{A})^1)$ takes values in the space of linear maps from $\mathcal{A}^2_{\chi,\pi}(P)$ to $\mathcal{A}^2_{\chi,t\pi}(P_1)$. Finally, also following [1(e)], we define

$$\theta_{P_1}(\zeta) = \mathrm{vol}(\mathfrak{a}^G_{P_1}/\mathbf{Z}(\Delta^\vee_{P_1}))^{-1} \prod_{\alpha\in\Delta_{P_1}} \zeta(\alpha^\vee), \qquad \zeta \in i\mathfrak{a}^*_{P_1},$$

where Δ_{P_1} is the set of simple roots of (P_1, A_{P_1}) and $\mathbf{Z}(\Delta^\vee_{P_1})$ is the lattice in $\mathfrak{a}^G_{P_1}$ generated by the co-roots

$$\{\alpha^\vee : \alpha \in \Delta_{P_1}\}.$$

Let T be a suitably regular point in $\mathfrak{a}_0^+$ and consider the expression

(7.1)

$$\sum_{P_1\supset P_0} \sum_{t,t'\in W(\mathfrak{a}_P,\mathfrak{a}_{P_1})} M(t, \lambda)^{-1}M(t', \lambda')e^{(t'\lambda'-t\lambda)(T)}\theta_{P_1}(t'\lambda' - t\lambda)^{-1},$$

for λ, $\lambda' \in i\mathfrak{a}_P^*$. It is an operator which for any $\pi \in \Pi(M_P(\mathbf{A})^1)$ maps $\mathcal{A}^2_{\chi,\pi}(P)$ to itself. Evaluate the operator at a vector $\phi' \in \mathcal{A}^2_{\chi,\pi}(P)$ and then take the inner product with another vector $\phi \in \mathcal{A}^2_{\chi,\pi}(P)$. Since $M(t, \lambda)$ is unitary, the result is

$$\sum_{P_1} \sum_{t,t'} (M(t', \lambda')\phi', M(t, \lambda)\phi) e^{(t'\lambda' - t\lambda)(T)} \theta_{P_1}(t'\lambda' - t\lambda)^{-1}.$$

This is just the function we denoted by $\omega^T(\lambda', \lambda, \phi', \phi)$ in [1(e)]. We observed in Section 9 of that paper that it was regular for purely imaginary λ' and λ. It follows that the operator (7.1) is regular at $\lambda' = \lambda$. We will denote its value at $\lambda' = \lambda$ by $\omega^T_{\chi,\pi}(P, \lambda)$.

Let $C_c^\infty(i\mathfrak{h}^*/i\mathfrak{a}_G^*)^W$ be the space of smooth, compactly supported functions on $i\mathfrak{h}^*/i\mathfrak{a}_G^*$ which are symmetric under W. If B belongs to $C_c^\infty(i\mathfrak{h}^*/i\mathfrak{a}_G^*)^W$, it is also contained in $\mathcal{S}(i\mathfrak{h}^*/i\mathfrak{a}_G^*)^W$, and by Theorem 6.3 has an associated polynomial $P^T(B)$. In this case the functions B_π, $\pi \in \Pi(M_P(\mathbf{A}))$, all belong to $C_c^\infty(i\mathfrak{a}_P^*/i\mathfrak{a}_G^*)$.

THEOREM 7.1. *Suppose that $B \in C_c^\infty(i\mathfrak{h}^*/i\mathfrak{a}_G^*)^W$. Then $P^T(B)$ is the unique polynomial which differs from*

(7.2)

$$\sum_{P \supset P_0} \sum_{\pi \in \Pi(M_P(\mathbf{A})^1)} |\mathcal{P}(M_P)|^{-1} \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \operatorname{tr}(\omega^T_{\chi,\pi}(P, \lambda)\rho_{\chi,\pi}(P, \lambda, f))B_\pi(\lambda)d\lambda$$

by an expression which approaches zero as T approaches infinity strongly in $\mathfrak{a}_0^+$.

Proof. The expression

$$\sum_P \sum_{\pi \in \Pi(M_P(\mathbf{A})^1)} \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \Psi^T_\pi(\lambda)B_\pi(\lambda)d\lambda$$

of Theorem 6.3 equals

(7.3)

$$\sum_{P \supset P_0} \sum_{\pi \in \Pi(M_P(\mathbf{A})^1)} |\mathcal{P}(M_P)|^{-1} \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \operatorname{tr}(\Omega^T_{\chi,\pi}(P, \lambda)\rho_{\chi,\pi}(P, \lambda, f))B_\pi(\lambda)d\lambda.$$

We have only to show that the difference between (7.2) and (7.3) approaches zero as T approaches infinity strongly in $\mathfrak{a}_0^+$. Remember that f belongs to the space $C_c^\infty(G(\mathbf{A})^1, K)$. Therefore the two operators $\Omega_{\chi,\pi}^T(P, \lambda)\rho_{\chi,\pi}(P, \lambda, f)$ and $\omega_{\chi,\pi}^T(P, \lambda)\rho_{\chi,\pi}(P, \lambda, f)$ act through a finite dimensional subspace of $\mathcal{A}_{\chi,\pi}^2(P)$. Moreover, they both vanish for all but finitely many π. We need only estimate the integral of the function obtained by multiplying $B_\pi(\lambda)$ with the difference between

$$(\Omega_{\chi,\pi}^T(P, \lambda)\rho_{\chi,\pi}(P, \lambda, f)\phi, \phi)$$

and

$$(\omega_{\chi,\pi}^T(P, \lambda)\rho_{\chi,\pi}(P, \lambda, f)\phi, \phi),$$

for any $\phi \in \mathcal{A}_{\chi,\pi}^2(P)$. By definition, the first inner product equals

$$\int_{G(\mathbf{Q})\backslash G(\mathbf{A})^1} \Lambda^T E(x, \rho_{\chi,\pi}(P, \lambda, f)\phi, \lambda)\, \overline{\Lambda^T E(x, \phi, \lambda)}\, dx,$$

while the second one is just

$$\omega^T(\lambda, \lambda, \rho_{\chi,\pi}(P, \lambda, f)\phi, \phi).$$

According to Corollary 9.2 of [5(e)], the difference between the two is bounded in absolute value by

$$r(\lambda)\|\phi\|^2 e^{-\epsilon\|T\|},$$

where ϵ is a positive number, $r(\lambda)$ is a locally bounded function on $i\mathfrak{a}_P^*$ and T remains within a region

$$\{T \in \mathfrak{a}_0^+ : d_{P_0}(T) > \delta\|T\|\}.$$

When the difference is multiplied by $B_\pi(\lambda)$ and then integrated over λ, the result certainly approaches zero as T approaches infinity strongly in $\mathfrak{a}_0^+$. The theorem follows. □

The theorem tells us that in calculating the polynomial $P^T(B)$ we may work with (7.1) instead of the inner product of truncated Eisenstein series.

However, the situation is not yet under control, for it is not so easy to see what happens to (7.1) as λ' approaches λ. Then it is necessary to integrate the result against a smooth, compactly supported function of λ, and determine the behaviour as T approaches infinity strongly in $\mathfrak{a}_0^+$. These questions have a combinatorial flavour, and will be tackled in the next paper.

Appendix. We must derive the three propositions of Section 2 from the results of [1(b)]. Unfortunately the paper [1(b)] is quite difficult to read, partly because the exposition is too brief, and also because there are a large number of proof-reading errors. However, there are no serious errors, and it is our hope that the paper can still be understood by a sufficiently tolerant reader.

Proof of Proposition 2.2. The assertion of Proposition 2.2 is just the formula (1.3), but with the domain for T described quantitatively in terms of the support of f. The formula is essentially Lemma 2.4 of [1(b)]. Although we cited [1(b), Theorem 3.2] in Section 1 as our justification of (1.3), the main ingredient of the theorem is just this lemma. In fact, a glance at the proof of Theorem 3.2 of [1(b)] reveals that it is valid for any T for which Lemma 2.4 holds. Our task, then, is to find conditions on T for which the lemma is valid.

Referring to the proof of Lemma 2.4 of [1(b)], we see that the lemma, and hence formula (1.3), is valid for any T such that

(A.1)

$$\int_{P_1(\mathbf{Q})\backslash G(\mathbf{A})^1} \sum_{\gamma \in F(P_1,P_2)} \sigma_1^2(H_0(x) - T)\Lambda_2^{T,P_1} K_{P_1,\chi}(\gamma x, x)dx$$

vanishes for each pair $P_1 \subset P_2$ of distinct (standard) parabolic subgroups. The symbols here are all defined in [1(b)]. It does not matter exactly what they are. The main point is that the expression (A.1), as a double integral over x and γ, is absolutely convergent. This was established in the proof of Theorem 2.1 of [1(b)]. An inspection of the proof of this theorem reveals that T can be any point for which the properties of the truncation operator, derived in Section 1 of [1(b)], hold. In particular, for the absolute convergence of (A.1), T does not depend on f.

The support of f intervenes in the proof of the lemma through a constant C, introduced on page 106 of [1(b)]. We shall write $C = C_f$ to denote the dependence on f. The constant has its origins in the proof of Theorem

2.1 of [1(b)]. It is characterized as follows. For each ϖ in $\hat{\Delta}_{P_1}$, the basis of $(\mathfrak{a}_{P_1}^G)^*$ dual to

$$\{\alpha^\vee : \alpha \in \Delta_{P_1}\},$$

let Λ be a fixed rational representation of G with highest weight $d\varpi$, for some $d > 0$. Choose a height function $\|\cdot\|$ for Λ, as in [1(a)], and let v be a highest weight vector. Then C_f is any constant which for each ϖ is greater than the supremum, as x ranges over the support of f in $G(\mathbf{A})^1$, of

$$d \log\left(\frac{\|\Lambda(x)v\|}{\|v\|}\right).$$

Now we can choose constants c_1 and n_1 such that for each ϖ,

$$\|\Lambda(x)v\| \leq c_1\|x\|^{n_1}\|v\|, \qquad x \in G(\mathbf{A})^1.$$

Suppose that f belongs to $C_N^\infty(G(\mathbf{A})^1)$. Then if x lies in the support of f,

$$\begin{aligned}\log\left(\frac{\|\Lambda(x)v\|}{\|v\|}\right) &\leq \log c_1 + n_1 \log\|x\| \\ &\leq \log c_1 + n_1 N,\end{aligned}$$

for each ϖ and the associated Λ. It follows that there is a constant C_0, independent of f, for which we can set

$$C_f = C_0(1 + N),$$

for any $f \in C_N^\infty(G(\mathbf{A})^1)$ and $N > 0$.

It is shown on page 107 of [1(b)] how to choose T in order to make the expression (A.1) vanish. T need only be chosen so that for every element s in $\Omega(P_1, P_2)$ there is a function ϖ in $\hat{\Delta}_{P_1}^{P_2}$ such that

$$\varpi_0(T - sT) > C_f.$$

As in [1(b)], $\Omega(P_1, P_2)$ is a certain set of elements in the complement of the Weyl group of (M_{P_1}, A_0) in the Weyl group of (M_{P_2}, A_0). It is easy to check that this condition on T will hold for all $P_1 \subsetneq P_2$ whenever $\alpha(T) > C_f$ for every α in Δ_0; that is, whenever

$$d_{P_0}(T) > C_f = C_0(1 + N).$$

It is for these T that the formula (1.3) holds. This gives Proposition 2.2. $\square$

Remarks. 1. This proof does not require that f be infinitely differentiable. As in Lemma 2.4 of [1(b)], f need only be differentiable of sufficiently high order.

2. Nowhere did the proof depend on the class $\chi \in \mathfrak{X}$. Therefore the constant C_0 in Proposition 2.2 can be chosen independently of χ.

The other propositions of Section 2 will follow from a quantitative version of Theorem 3.1 of [1(b)]. Suppose that K_0 is a subgroup of finite index in

$$K_f = \prod_{v \neq \mathbf{R}} K_v.$$

Suppose also that W is an irreducible representation in $\Pi(K_{\mathbf{R}})$. Given $P \supset P_0$ and $\pi \in \Pi(M_P(\mathbf{A})^1)$, let $\mathfrak{A}^2_{\chi,\pi}(P)_{K_0}$ be the space of K_0 invariant functions in $\mathfrak{A}^2_{\chi,\pi}(P)$, and let $\mathfrak{A}^2_{\chi,\pi}(P)_{K_0,W}$ be the subspace of functions in $\mathfrak{A}^2_{\chi,\pi}(P)_{K_0}$ which transform under $K_{\mathbf{R}}$ according to W. This second space is finite dimensional. If A is any operator on $\mathfrak{A}^2_{\chi,\pi}(P)$ for which $\mathfrak{A}^2_{\chi,\pi}(P)_{K_0}$ (resp. $\mathfrak{A}^2_{\chi,\pi}(P)_{K_0,W}$) is an invariant subspace, let A_{K_0} (resp. $A_{K_0,W}$) denote the restriction of A to $\mathfrak{A}^2_{\chi,\pi}(P)_{K_0}$ (resp. $\mathfrak{A}^2_{\chi,\pi}(P)_{K_0,W}$). Now, choose a left invariant differential operator Δ on

$$G(\mathbf{R})^1 = G(\mathbf{A})^1 \cap G(\mathbf{R})$$

as on pages 108 and 109 of [1(b)]. Then Δ acts on $\mathfrak{A}^2_{\chi,\pi}(P)$ through each of the representations $\rho_{\chi,\pi}(P, \lambda)$. The operators

$$\rho_{\chi,\pi}(P, \lambda, \Delta), \qquad \lambda \in i\mathfrak{a}_P^*,$$

have $\mathfrak{A}^2_{\chi,\pi}(P)_{K_0}$ and $\mathfrak{A}^2_{\chi,\pi}(P)_{K_0,W}$ as invariant subspaces. If $\mathfrak{A}^2_{\chi,\pi}(P)_{K_0,W}$ is not zero, $\rho_{\chi,\pi}(P, \lambda, \Delta)_{K_0,W}$ is the product of the identity with a real number greater than 1.

Up to this point, χ has been a fixed class in $\mathfrak{X}$. For the next lemma χ will not be fixed, but will instead index a sum over all the elements of $\mathfrak{X}$.

LEMMA A.1. *We can find positive integers C_0, d_0 and m which satisfy the following property. For any subgroup $K_0 \subset K_f$ of finite index and any $T \in \mathfrak{a}_0$ with $d_{P_0}(T) > C_0$, the expression*

(A.2)

$$\sum_{\chi \in \mathfrak{X}} \sum_{P} \sum_{\pi \in \Pi(M_P(\mathbf{A})^1)} |\mathcal{P}(M_P)|^{-1} \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \|\Omega_{\chi,\pi}^T(P,\lambda)_{K_0} \cdot \rho_{\chi,\pi}(P,\lambda,\Delta^m)_{K_0}^{-1}\|_1 d\lambda$$

is bounded by

$$c_{K_0}(1 + \|T\|)^{d_0},$$

where c_{K_0} is a constant which depends only on K_0 and $\| \ \|_1$ denotes the trace class norm.

Proof. Except for the dependence on T, this lemma is just Theorem 3.1 of [1(b)]. We shall combine the proof of this theorem with an argument that was used in Section 4 of [1(a)].

The operator

$$\Omega_{\chi,\pi}^T(P,\lambda), \qquad \lambda \in i\mathfrak{a}_P^*,$$

is positive definite. It leaves the spaces $\mathfrak{A}_{\chi,\pi}^2(P)_{K_0,W}$ invariant, and commutes with $\rho_{\chi,\pi}(P,\lambda,\Delta)$. Consequently, each operator

$$\Omega_{\chi,\pi}^T(P,\lambda)_{K_0,W} \cdot \rho_{\chi,\pi}(P,\lambda,\Delta^m)_{K_0,W}^{-1}, \qquad \lambda \in i\mathfrak{a}_P^*, \qquad W \in \Pi(K_{\mathbf{R}}),$$

is positive definite. It follows that the expression (A.2) equals the sum over $\chi \in \mathfrak{X}$, $P \supset P_0$, $\pi \in \Pi(M_P(\mathbf{A})^1)$ and $W \in \Pi(K_{\mathbf{R}})$, of the integral over $\lambda \in i\mathfrak{a}_P^*/i\mathfrak{a}_G^*$, of the product of $|\mathcal{P}(M_P)|^{-1}$ with

(A.3) $$\mathrm{tr}(\Omega_{\chi,\pi}^T(P,\lambda)_{K_0,W} \cdot \rho_{\chi,\pi}(P,\lambda,\Delta^m)_{K_0,W}^{-1}).$$

Define functions g_1 and g_2 on $G(\mathbf{A})^1$ as on page 111 of [1(b)]. Then

$$\Delta^m g_1 + g_2$$

equals the product of the Dirac distribution at 1 in $G(\mathbf{R})^1$, with the characteristic function of K_0 in $G(\mathbf{A}_f)$ divided by the volume of K_0 in $G(\mathbf{A}_f)$.

(Δ^m acts through the Archimedean valuation as a left invariant differential operator.) The functions g_1 and g_2 are both invariant under conjugation by $K_{\mathbf{R}}$. The support in $G(\mathbf{A})^1$ of each function is bounded independently of K_0. Moreover, by taking m to be large, we can insure that the functions are differentiable of high order. For simplicity set

$$\Omega = \Omega^T_{\chi,\pi}(P, \lambda)_{K_0,W},$$

$$D = \rho_{\chi,\pi}(P, \lambda, \Delta^m)^{-1}_{K_0,W},$$

and

$$G_i = \rho_{\chi,\pi}(P, \lambda, g_i)_{K_0,W}, \qquad i = 1, 2,$$

with χ, π, P, W and λ being as in (A.3). Then

$$D = G_1 + DG_2.$$

The expression (A.3) equals $\mathrm{tr}(\Omega D)$, which is bounded by

$$|\mathrm{tr}(\Omega G_1)| + |\mathrm{tr}(\Omega D G_2)|.$$

Since D is a positive scalar which is less than 1, this is in turn bounded by

$$|\mathrm{tr}(\Omega G_1)| + |\mathrm{tr}(\Omega G_2)|.$$

According to Corollary 4.2 of [1(a)], g_1 and g_2 can each be expressed as a finite sum of convolutions $h * h'$, where h and h' are functions on $G(\mathbf{A})^1$ which are differentiable of high order, are bi-invariant under K_0, and whose supports are bounded independently of K_0. Then (A.3) is bounded by a finite sum

$$\Sigma\ \mathrm{tr}(\Omega H H'),$$

in which

$$H = \rho_{\chi,\pi}(P, \lambda, h)_{K_0,W}$$

and

$$H' = \rho_{\chi,\pi}(P, \lambda, h')_{K_0,W}.$$

Let $\Omega^{1/2}$ be the positive definite square root of Ω. We have

$$\begin{aligned}
&|\operatorname{tr}(\Omega H H')| \\
&\qquad = |\operatorname{tr}(\Omega^{1/2} H H' \Omega^{1/2})| \\
&\qquad \leq \operatorname{tr}(\Omega^{1/2} H (\Omega^{1/2} H)^*)^{1/2} \cdot \operatorname{tr}((H'\Omega^{1/2})^* H' \Omega^{1/2})^{1/2} \\
&\qquad = \operatorname{tr}(\Omega H H^*)^{1/2} \cdot \operatorname{tr}(\Omega (H')^* H')^{1/2},
\end{aligned}$$

by the Schwartz inequality. Applying the Schwarz inequality again we see that (A.2) is bounded by a finite sum, each term of which is the product of the square root of

(A.4)

$$\sum_{\chi} \sum_{P} \sum_{\pi} \sum_{W} \int |\mathcal{P}(M_P)|^{-1} \operatorname{tr}(\Omega^T_{\chi,\pi}(P, \lambda)_{K_0, W} \cdot \rho_{\chi,\pi}(P, \lambda, h * h^*)_{K_0, W}) d\lambda$$

with the square root of the corresponding expression in $(h')^* * h'$. Now (A.4) equals the sum over $\chi \in \mathfrak{X}$ of

$$\sum_{P} \sum_{\pi} \int |\mathcal{P}(M_P)|^{-1} \operatorname{tr}(\Omega^T_{\chi,\pi}(P, \lambda) \rho_{\chi,\pi}(P, \lambda, h * h^*)) d\lambda.$$

This last expression is just

$$\sum_{P} \sum_{\pi} \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \Psi^T_{\pi}(\lambda, h * h^*) d\lambda,$$

an expression to which we can apply Proposition 2.2. (See the two remarks following the proof of the proposition.) We obtain a constant C_0 such that (A.4) equals

$$\sum_{\chi \in \mathfrak{X}} J^T_{\chi}(h * h^*)$$

whenever $d_{P_0}(T) > C_0$.

We have shown that m and C_0 can be chosen such that (A.2) is bounded by a finite sum of functions of the form

$$\Big(\sum_{\chi\in\mathfrak{X}} J_\chi^T(h * h^*)\Big)^{1/2}\Big(\sum_{\chi\in\mathfrak{X}} J_\chi^T((h')^* * h')\Big)^{1/2}$$

for all T with $d_{P_0}(T) > C_0$. However, $J_\chi^T(h * h^*)$ and $J_\chi^T((h')^* * h')$ are polynomials in T whose degrees are independent of χ. Moreover, the sums over χ converge. The required estimate for (A.2) follows. ∎

We return to the setting prior to the lemma, in which the class $\chi \in \mathfrak{X}$ was fixed. Suppose for a moment that $P \supset P_0$ and $\pi \in \Pi(M_P(\mathbf{A})^1)$ are also fixed, and that ϕ is a vector in $\mathcal{Q}_{\chi,\pi}^2(P)$. There is certainly a K_0 such that ϕ belongs to $\mathcal{Q}_{\chi,\pi}^2(P)_{K_0}$. Moreover, we can find integers n_0 and c_0 such that

$$(1 + \|\lambda\|)^{-n_0} \leq c_0\|\rho_{\chi,\pi}(P, \lambda, \Delta^m)_{K_0}^{-1}\|$$

for all $\lambda \in i\mathfrak{a}_P^*$. It follows from the lemma that there is a constant c_ϕ such that

$$\int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} (\Omega_{\chi,\pi}^T(P, \lambda)\phi, \phi)(1 + \|\lambda\|)^{-n_0}d\lambda \leq c_\phi(1 + \|T\|)^{d_0}$$

for all T with $d_{P_0}(T) > C_0$. This estimate is a weaker version of the lemma and will be used in a subsequent paper.

Proof of Proposition 2.1. The constants C_0 and d_0 required by the proposition will be those given by Lemma A.1. Fix $f \in C_c^\infty(G(\mathbf{A})^1)$. There is certainly a K_0 such that

$$\rho_{\chi,\pi}(P, \lambda, f) = \rho_{\chi,\pi}(P, \lambda, f)_{K_0}$$

for each P, π and λ. We have

$$\begin{aligned}
&|\Psi_\pi^T(\lambda, f)| \\
&\quad = |\mathcal{P}(M_P)|^{-1}|\mathrm{tr}(\Omega_{\chi,\pi}^T(P, \lambda)\rho_{\chi,\pi}(P, \lambda, f))| \\
&\quad = |\mathcal{P}(M_P)|^{-1}|\mathrm{tr}(\Omega_{\chi,\pi}^T(P, \lambda)_{K_0}\cdot\rho_{\chi,\pi}(P, \lambda, f)_{K_0})| \\
&\quad \leq |\mathcal{P}(M_P)|^{-1}\|\Omega_{\chi,\pi}^T(P, \lambda)_{K_0}\cdot\rho_{\chi,\pi}(P, \lambda, f)_{K_0}\|_1 \\
&\quad \leq |\mathcal{P}(M_P)|^{-1}\|\Omega_{\chi,\pi}^T(P, \lambda)_{K_0}\cdot\rho_{\chi,\pi}(P, \lambda, \Delta^m)_{K_0}^{-1}\|_1\cdot\|\rho_{\chi,\pi}(P, \lambda, \Delta^m f)\|.
\end{aligned}$$

Suppose that n is a positive integer. It is certainly possible to find a constant $c'_{n,f}$ such that

$$\|\rho_{\chi,\pi}(P, \lambda, \Delta^m f)\| \leq c'_{n,f}(1 + \|\lambda\|)^{-n},$$

for all $P \supset P_0$, $\pi \in \Pi(M_P(\mathbf{A})^1)$ and $\lambda \in i\mathfrak{a}_P^*$. Proposition 2.1 follows from Lemma A.1 with

$$c_{n,f} = c_{K_0} \cdot c'_{n,f}. \qquad \square$$

Proof of Proposition 2.3. Fix $f \in C_c^\infty(G(\mathbf{A})^1)$. The functions

$$\{f_{\tau_1,\tau_2} : \tau_1, \tau_2 \in \Pi(K)\}$$

have uniformly bounded support. Therefore, by Proposition 2.2 there is an integer C (depending on f) such that

$$J_\chi^T(f_{\tau_1,\tau_2}) = \sum_P \sum_\pi \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \Psi_\pi^T(\lambda, f_{\tau_1,\tau_2}) d\lambda,$$

for all $\tau_1, \tau_2 \in \Pi(K)$ and all T with $d_{P_0}(T) > C$. The formula

$$J_\chi^T(f) = \sum_{\tau_1,\tau_2} J_\chi^T(f_{\tau_1,\tau_2})$$

then follows from Lemma A.1 by an argument which is similar to the proof of Proposition 2.1. This completes the proof of the last of the propositions of Section 2. ∎

UNIVERSITY OF TORONTO

BIBLIOGRAPHY

[1a] Arthur, J., A trace formula for reductive groups I: terms associated to classes in $G(\mathbf{Q})$, *Duke Math. J.* **45** (1978), 911-952.

[1b] ______, A trace formula for reductive groups II: applications of a truncation operator, *Comp. Math.* **40** (1980), 87-121.

[1c] ______, The trace formula in invariant form, *Ann. of Math.* **114** (1981), 1-74.

[1d] ______, A Paley-Wiener theorem for real reductive groups, to appear in *Acta Math.*

[1e] ______, On the inner product of truncated Eisenstein series, *Duke Math. J.* **49** (1982), 35-70.

[2] Flath, D., Decomposition of representations into tensor products, in *Automorphic Forms, Representatives and L-Functions, Proc. Sympos, Pure Math.*, **33** (1979), Part I, 179-184.

[3] Langlands, R. P., On the functional equations satisfied by Eisenstein series, *Lecture Notes in Math.*, **544** (1976).