Vol. 49, No. 1 DUKE MATHEMATICAL JOURNAL © March 1982

ON THE INNER PRODUCT OF TRUNCATED EISENSTEIN SERIES

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CONTENTS

Introduction. One can think of Eisenstein series as the spectral kernels for the Laplace–Beltrami operator on a certain class of noncompact Riemannian manifolds. They are the eigenfunctions corresponding to the continuous spectrum. In particular they are not square integrable. However, there is a natural way to truncate these functions so that they are square integrable. The object of this paper is to investigate the inner product of two such truncated functions. Our main result is an asymptotic formula for the inner product, as the variable of truncation approaches infinity. The formula, which is based on an inner product formula of Langlands for cuspidal Eisenstein series, is rather simple. It is given in terms of certain operators which are analogues of the classical scattering matrix.

The most efficient way to work with Eisenstein series is through adèle groups. The close connection between the analysis on adèle groups and that on locally symmetric Riemannian manifolds is well known and will not be discussed here. Let G be an algebraic group defined over $\mathbb{Q}$, which for the introduction we take to be semisimple, and let $P = N_P M_P$ be a standard parabolic subgroup of G. If A_P is the split component of the center of the Levi component M_P, let $\mathcal{Q}_P$ be the space of square integrable automorphic forms on

$$N_P(\mathbb{A})M_P(\mathbb{Q})A_P(\mathbb{R})^0\backslash G(\mathbb{A}).$$

Received September 15, 1981. Partially supported by NSERC Grant A3483.

It has an important subspace $\mathcal{A}_{P,\mathrm{cusp}}$ consisting of those ϕ in $\mathcal{A}_P$ for which each function

$$\phi_x : \to \phi(mx), \qquad m \in M_P(\mathbb{A}), \quad x \in G(\mathbb{A}),$$

is a cusp form on $M_P(\mathbb{A})$. The theory of Eisenstein series associates to every $\phi \in \mathcal{A}_P$ and every λ in a real vector space $i\mathfrak{a}_P^*$, a smooth function $E(\phi,\lambda)$ on $G(\mathbb{Q})\backslash G(\mathbb{A})$. It is not square integrable. However, for every point T in a certain chamber, $\mathfrak{a}_0^+$, there is a truncation operator Λ^T which acts on functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$. The truncated Eisenstein series, $\Lambda^T E(\phi,\lambda)$, is square integrable. Now suppose that P' is another standard parabolic subgroup and that $\lambda' \in i\mathfrak{a}_{P'}^*$. Suppose that ϕ and ϕ' are restricted to lie in the respective subspaces $\mathcal{A}_{P,\mathrm{cusp}} \subset \mathcal{A}_P$ and $\mathcal{A}_{P',\mathrm{cusp}} \subset \mathcal{A}_{P'}$. Then Langlands has established the elegant inner product formula

$$\left(\Lambda^T E(\phi,\lambda), \Lambda^T E(\phi',\lambda')\right) = \omega^T(\lambda,\lambda',\phi,\phi'), \tag{1}$$

where for any $\phi \in \mathcal{A}_P$ and $\phi' \in \mathcal{A}_{P'}$, $\omega^T(\lambda,\lambda',\phi,\phi')$ is defined as the sum over all standard parabolic subgroups P_1 and all elements t and t' in the Weyl sets $W(\mathfrak{a}_P,\mathfrak{a}_{P_1})$ and $W(\mathfrak{a}_{P'},\mathfrak{a}_{P_1})$, of the functions

$$\mathrm{vol}\left(\mathfrak{a}_{P_1}/L_{P_1}\right)\frac{\left(M(t,\lambda)\phi, M(t',\lambda')\phi'\right)e^{(t\lambda - t'\lambda')(T)}}{\Pi_{\alpha\in\Delta_{P_1}}(t\lambda - t'\lambda')(\alpha^\vee)}$$

(For any unfamiliar notation we refer the reader to the text.) In particular, if P and P' are not associated, the inner product is zero.

It is important to study the inner product when the vectors $\phi \in \mathcal{A}_P$ and $\phi' \in \mathcal{A}_{P'}$ are arbitrary. Then, unfortunately, the formula is false. In fact, for general ϕ and ϕ' the inner product of $\Lambda^T E(\phi,\lambda)$ and $\Lambda^T E(\phi',\lambda')$ is apparently quite complicated. The main result of this paper (Theorem 9.1) is an *asymptotic* formula. We will show that

$$\left(\Lambda^T E(\phi,\lambda), \Lambda^T E(\phi',\lambda')\right) \sim \omega^T(\lambda,\lambda',\phi,\phi') \tag{1*}$$

as T approaches infinity away from the walls in $\mathfrak{a}_0^+$. In particular, if P and P' are not associated, the inner product is asymptotic to zero.

There does not seem to be any direct way to prove the formula (1*). This is perhaps not surprising for there is at present no direct way to study the function $E(\phi,\lambda)$. Langlands was able to establish the analytic continuation and functional equations only by taking residues of *cuspidal* Eisenstein series $E(F_B(\Lambda), \Lambda + \lambda)$, where B is a standard parabolic subgroup which is contained in P, Λ is a point in $\mathfrak{a}_{B,\mathbb{C}}^*$, and F_B is an analytic function from $\mathfrak{a}_{B,\mathbb{C}}^*$ to $\mathcal{A}_{B,\mathrm{cusp}}$. We must proceed this way also. By formula (1), we have

$$\begin{aligned}&\left(\Lambda^T E(F_B(\Lambda), \Lambda + \lambda), \Lambda^T E\left(F'_{B'}(\overline{\Lambda}'), \overline{\Lambda}' + \lambda'\right)\right)\\ &\qquad = \omega^T\left(\Lambda + \lambda, -\Lambda' + \lambda', F_B(\Lambda), F'_{B'}(\overline{\Lambda}')\right).\end{aligned} \tag{2}$$

Thanks to Langlands' work, we know that the inner product

$$\left(\Lambda^T E(\phi, \lambda), \Lambda^T E(\phi', \lambda')\right),$$

which is what we are after, can be obtained as a sum of residues in Λ and Λ' of inner products on the left hand side of (2). Consequently, we are reduced to studying the corresponding sum of residues of the right hand side of (2). We will examine it as a function of T. In §3 we shall show that it equals a sum

$$\sum_{(t, t')} \sum_{X \in \mathfrak{S}(t, t')} p_X^T(\lambda, \lambda', \phi, \phi') e^{(t\lambda - t'\lambda' + X)(T)} \tag{3}$$

where (t, t') ranges over certain maps with domain $i\mathfrak{a}_P^* \times i\mathfrak{a}_{P'}^*$, $\mathfrak{S}(t, t')$ is a finite set of points in $\mathfrak{a}_0^*$, and $p_X^T(\lambda, \lambda', \phi, \phi')$ is a polynomial function of T.

The elements (t, t') in the sum (3) will include pairs from products

$$W(\mathfrak{a}_P, \mathfrak{a}_{P_1}) \times W(\mathfrak{a}_{P'}, \mathfrak{a}_{P_1}) \tag{4}$$

of Weyl sets. If (t, t') is not of this form, we will use a property of the truncation operator to show that each $X \in \mathfrak{S}(t, t')$ is a nonzero point in the closure of the negative dual chamber of $\mathfrak{a}_0^+$ (Lemmas 7.1 and 7.2). Thus, as T approaches infinity, the contribution of any such pair can be ignored.

Suppose then that (t, t') belongs to the product (4). Lemma 7.2 will still tell us that each $X \in \mathfrak{S}(t, t')$ lies in the closure of the negative dual chamber. However, X could equal 0. The remaining problem is to calculate the coefficient $p_0^T(\lambda, \lambda', \phi, \phi')$. Lemma 4.1 will enable us to relate $p_0^T(\lambda, \lambda', \phi, \phi')$ to a coefficient

$$p_0^{T, P_1}(t\lambda, t'\lambda', M(t, \lambda)\phi, M(t', \lambda')\phi'), \tag{5}$$

in which the group G has been replaced by P_1. But $M(t, \lambda)\phi$ and $M(t', \lambda')\phi'$ are both functions in $\mathcal{Q}_{P_1}$, a space associated with the square integrable automorphic forms on $M_{P_1}(\mathbf{Q}) \backslash M_{P_1}(\mathbf{A})$. It will not be hard to show that the inner product of truncated *square integrable* automorphic forms is asymptotic to the ordinary inner product. It will follow that the coefficient (5) actually equals

$$(M(t, \lambda)\phi, M(t', \lambda')\phi').$$

(See Lemma 8.1.) It is in particular independent of T. This will enable us finally to prove our asymptotic formula in §9.

Inner products of truncated Eisenstein series occur in the trace formula (see §3 of [1(b)]). They are the main ingredients of the distributions which were denoted by J_χ^T in [1(b)]. In order to put the trace formula to use, it is essential to evaluate these distributions. The main result of this paper turns out to be just what is needed. We shall use it in another paper to establish an explicit formula for the distributions J_χ^T.

As we have already suggested, this paper relies heavily on Langlands' work on Eisenstein series. We have summarized the results we require in §2. They can be extracted, with patience, from Chapter 7 of Langlands' book [2(b)]. For more details, the reader can refer to [3].

§1. The problem. Let G be a reductive algebraic group defined over $\mathbf{Q}$. We shall fix a minimal parabolic subgroup P_0 of G and a Levi component M_0 of P_0, both defined over $\mathbf{Q}$. In this paper, a parabolic subgroup will mean a parabolic subgroup of G, defined over $\mathbf{Q}$, which contains P_0. Suppose that P is a parabolic subgroup. Let N_P denote the unipotent radical of P, and let M_P denote the unique Levi component of P which contains M_0. Let A_P be the split component of the center of M_P. If $X(M_P)_{\mathbf{Q}}$ is the group of characters of M_P defined over $\mathbf{Q}$,

$$\mathfrak{a}_P = \operatorname{Hom}(X(M_P)_{\mathbf{Q}}, \mathbf{R})$$

is a real vector space whose dimension equals that of A_P. Its dual space is

$$\mathfrak{a}_P^* = X(M_P)_{\mathbf{Q}} \otimes \mathbf{R}.$$

We shall write $A_0 = A_{P_0}$, $\mathfrak{a}_0 = \mathfrak{a}_{P_0}$ and $\mathfrak{a}_0^* = \mathfrak{a}_{P_0}^*$. Let W_0 be the restricted Weyl group of (G, A_0). It acts on $\mathfrak{a}_0$ and $\mathfrak{a}_0^*$ in the usual way. If P and P' are parabolic subgroups we write $W(\mathfrak{a}_P, \mathfrak{a}_{P'})$ for the set of isomorphisms from $\mathfrak{a}_P$ onto $\mathfrak{a}_{P'}$ that can be obtained by restricting elements in W_0 to $\mathfrak{a}_P$. If $t \in W(\mathfrak{a}_P, \mathfrak{a}_{P'})$ we will let w_t stand for a fixed representative of t in $G(\mathbf{Q})$.

Let K be a fixed maximal compact subgroup of the adèlized group $G(\mathbf{A})$, which satisfies the usual conditions. More precisely, we assume that K is admissible relative to M_0 in the sense of [1(c)]. This is just the framework of the papers [1(a)] and [1(b)] on the trace formula. We shall freely adopt the notation and conventions from these two papers, often with additional reminders, but sometimes without further comment.

Before discussing Eisenstein series we should agree on our choices of Haar measures. Fix a Euclidean norm $\| \;\|$ on the space $\mathfrak{a}_0$ which is invariant under W_0. For any parabolic subgroup P, take the Euclidean measure on $\mathfrak{a}_P$ associated to the restriction of $\| \;\|$ to $\mathfrak{a}_P$. This normalizes Haar measures on all the spaces $\{\mathfrak{a}_P\}$. We can then normalize the Haar measures on the groups K, $G(\mathbf{A})$, $N_P(\mathbf{A})$, $M_P(\mathbf{A})$, $A_P(\mathbf{R})^0$, $M_P(\mathbf{A})^1$ etc. by following the prescriptions of §1 of [1(a)].

Let $\mathfrak{U}(G(\mathbf{R}))$ be the universal enveloping algebra of the complexification of the Lie algebra of $G(\mathbf{R})$, and let $\mathfrak{Z}$ be the center of $\mathfrak{U}(G(\mathbf{R}))$. Suppose that P is a parabolic subgroup. We shall write $\mathcal{Q}^2(N_P(\mathbf{A})M_P(\mathbf{Q})A_P(\mathbf{R})^0\backslash G(\mathbf{A}))$ for the space of $(\mathfrak{Z}, K)$ finite functions on $N_P(\mathbf{A})M_P(\mathbf{Q})A_P(\mathbf{R})^0\backslash G(\mathbf{A})$ which are square integrable. It is, in other words, the space of smooth functions

$$\phi : N_P(\mathbf{A})M_P(\mathbf{Q})A_P(\mathbf{R})^0\backslash G(\mathbf{A}) \to \mathbf{C}$$

which satisfy the following two conditions.

(i) The span of the set of functions

$$x \to (z\phi)(xk), \qquad x \in G(\mathbf{A}),$$

indexed by $k \in K$ and $z \in \mathfrak{Z}$, is finite dimensional.

(ii)

$$\|\phi\|^2 = \int_K \int_{M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1} |\phi(mk)|^2 \, dm \, dk < \infty.$$

(This definition is slightly different from that of [1(a)]. In the notation of [1(a)], $\mathcal{Q}^2(N_P(\mathbb{A})M_P(\mathbb{Q})A_P(\mathbb{R})^0\backslash G(\mathbb{A}))$ is the direct sum $\bigoplus_\pi \mathcal{H}_P^0(\pi)$, where π ranges over all classes of irreducible unitary representations of $M_P(\mathbb{A})$ which are trivial on $A_P(\mathbb{R})^0$.) Built into the definition is the property that for any $x \in G(\mathbb{A})$ and ϕ as above, the function

$$m \to \phi(mx), \qquad m \in M_P(\mathbb{A}),$$

belongs to the subspace of $L^2(M_P(\mathbb{Q})A_P(R)^0\backslash M_P(\mathbb{A}))$ on which the regular representation of $M_P(\mathbb{A})$ decomposes discretely. In fact the closure of $\mathcal{Q}^2(N_P(\mathbb{A})M_P(\mathbb{Q})A_P(\mathbb{R})^0\backslash G(\mathbb{A}))$ is just the Hilbert space on which the corresponding induced representation of $G(\mathbb{A})$ acts. If ϕ belongs to $\mathcal{Q}^2(N_P(\mathbb{A})M_P(\mathbb{Q})$ $A_P(R)^0\backslash G(\mathbb{A}))$ we have the Eisenstein series

$$E(x, \phi, \lambda) = \sum_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \phi(\delta x) e^{(\lambda + \rho_P)(H_P(\delta x))}.$$

It converges for $\mathrm{Re}(\lambda)$ in a certain chamber, and continues analytically to a meromorphic function of $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$.

In §3 of [1(a)] we described an orthogonal direct sum decomposition

$$L^2\big(M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1\big) = \bigoplus_{\chi \in \mathfrak{X}} L^2\big(M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1\big)_\chi$$

indexed by a certain set $\mathfrak{X}$. (One purpose for introducing this notation was to keep track of the source of any Eisenstein series as a cuspidal Eisenstein series.) Suppose that $\chi \in \mathfrak{X}$. We shall denote simply by $\mathcal{Q}_{P,\chi}$, the subspace of vectors ϕ in $\mathcal{Q}^2(N_P(\mathbb{A})M_P(\mathbb{Q})A_P(\mathbb{R})^0\backslash G(\mathbb{A}))$ such that for all x in $G(\mathbb{A})$ the function

$$m \to \phi(mx), \qquad m \in M_P(\mathbb{A})^1,$$

belongs to $L^2(M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1)_\chi$. This subspace is infinite dimensional. It is often appropriate to further single out a finite dimensional subspace. To do so, let Γ be a finite set of equivalence classes of irreducible unitary representations of K. Let $\mathcal{Q}_{P,\chi,\Gamma}$ be the set of vectors ϕ in $\mathcal{Q}_{P,\chi}$ such that for all x in $G(\mathbb{A})$, the function

$$k \to \phi(xk), \qquad k \in K,$$

is a sum of matrix coefficients of classes in Γ. Then this is a subspace of $\mathcal{Q}_{P,\chi}$ which is finite dimensional.

For the rest of this paper T will denote a point in $\mathfrak{a}_0^+$ which is suitably regular; recall that this means that $\alpha(T)$ is large for every root α of (P_0, A_0). Then we have the truncation operator

$$(\Lambda^T h)(x) = \sum_{\{P : P \supset P_0\}} (-1)^{\dim(A_P/A_G)}$$
$$\sum_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \int_{N_P(\mathbb{Q})\backslash N_P(\mathbb{A})} h(n\delta x)\hat{\tau}_P(H_P(\delta x) - T)\, dn,$$

acting on any continuous function h on $G(\mathbb{Q})\backslash G(\mathbb{A})$ ([1(b)], §1). Fix parabolic subgroups P, P', and also elements $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$, $\lambda' \in \mathfrak{a}^*_{P',\mathbb{C}}$ and $\phi \in \mathcal{Q}^2(N_P(\mathbb{A})M_P(\mathbb{Q})A_P(\mathbb{R})^0\backslash G(\mathbb{A}))$, $\phi' \in \mathcal{Q}^2(N_{P'}(\mathbb{A})M_{P'}(\mathbb{Q})A_{P'}(\mathbb{R})^0\backslash G(\mathbb{A}))$. We propose to investigate the inner product

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \Lambda^T E(x,\phi,\lambda)\, \overline{\Lambda^T E(x,\phi',\lambda')}\, dx. \tag{1.1}$$

If ϕ belongs to $\mathcal{Q}_{P,\chi}$ the function

$$\Lambda^T E(x,\phi,\lambda), \qquad x \in G(\mathbb{A})^1,$$

will belong to $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})^1)_\chi$. Therefore, if $\phi' \in \mathcal{Q}_{P',\chi'}$, with χ' distinct from χ, the inner product (1.1) will vanish. We can therefore fix $\chi \in \mathfrak{X}$ and assume that $\phi \in \mathcal{Q}_{P,\chi,\Gamma}$ and $\phi' \in \mathcal{Q}_{P',\chi,\Gamma}$ for some fixed finite set Γ of representations of K.

Corresponding to χ is a class $\mathcal{P}_\chi$ of associated (standard) parabolic subgroups. Recall that $\mathfrak{X}$ is the set of W_0 conjugacy classes of pairs (M_B, r_B), where B is a group in $\mathcal{P}_\chi$ and r_B is an irreducible unitary representation of $M_B(\mathbb{A})^1$ which occurs in the space of cuspidal functions on $M_B(\mathbb{Q})\backslash M_B(\mathbb{A})^1$. If $B \in \mathcal{P}_\chi$ the space $L^2(M_B(\mathbb{Q})\backslash M_B(\mathbb{A})^1)_\chi$ is easy to characterize. It is just the space of cuspidal functions in $L^2(M_B(\mathbb{Q})\backslash M_B(\mathbb{A})^1)$ which transform under $M_B(\mathbb{A})^1$ as a sum of representations r_B for which (M_B, r_B) belongs to χ. This makes the definition of $\mathcal{Q}_{B,\chi}$ somewhat more concrete. For the general group P, the space $\mathcal{Q}_{P,\chi}$ could vanish. In order for this not to happen, χ will have to satisfy special conditions. First of all, there will need to be a group $B \in \mathcal{P}_\chi$ which is contained in P. Secondly, any Eisenstein series

$$E(x,\phi,\lambda), \qquad \phi \in \mathcal{Q}_{P,\chi,\Gamma},$$

must be obtainable as a sum of residues of Eisenstein series $E(x,\Phi,\Lambda)$ in which $\Phi \in \mathcal{Q}_{B,\chi,\Gamma}$ and Λ is a linear function on $\mathfrak{a}_{B,\mathbb{C}}$ whose restriction to $\mathfrak{a}_{P,\mathbb{C}}$ is λ. This point is one of the main results in Chapter 7 of Langlands' book [2(b)]. We shall discuss it in greater detail in the next section.

§2. Residues of cuspidal Eisenstein series. Langlands' method for studying general Eisenstein series is to take residues of cuspidal Eisenstein series. We shall recall the features of his theory that are needed for this paper.

Let $B \subset P$ be parabolic subgroups. Suppose that

$$\mathfrak{t}_0 \subset \mathfrak{t}_1 \subset \cdots \subset \mathfrak{t}_r = \mathfrak{a}^*_B$$

is a sequence of affine spaces

$$\mathfrak{t}_i = \Lambda_i + \tilde{\mathfrak{t}}_i,$$

where $\tilde{\mathfrak{t}}_i$ is a linear subspace of $\mathfrak{a}_B^*$ and Λ_i is a vector in $\mathfrak{a}_B^*$ which is orthogonal to $\tilde{\mathfrak{t}}_i$. We assume that $\tilde{\mathfrak{t}}_0 = \mathfrak{a}_P^*$ and that for any i,

$$\tilde{\mathfrak{t}}_{i-1} = \{\lambda \in \tilde{\mathfrak{t}}_i : \lambda(\beta^\vee) = 0\}$$

for some root β of (B, A_B) such that $\beta^\vee$ does not vanish on $\tilde{\mathfrak{t}}_i$. Suppose that for each i we have also chosen a unit vector ν_i in $\tilde{\mathfrak{t}}_i$ which is orthogonal to $\tilde{\mathfrak{t}}_{i-1}$. It is uniquely determined up to sign. Let us denote the sequence of affine subspaces, together with the choices of unit normals, by the letter S; we shall denote the smallest space $\mathfrak{t}_0$ by

$$\mathfrak{t}_S = \Lambda_S + \mathfrak{a}_P^*.$$

Suppose that Ψ is a meromorphic function on $\mathfrak{a}_{B,\mathbf{C}}^*$ whose singularities lie along hyperplanes. Then we can obtain a meromorphic function on $\mathfrak{t}_{S,\mathbf{C}} = \Lambda_S + \mathfrak{a}_{P,\mathbf{C}}^*$ by taking successive residues with respect to S. More precisely, let Λ_0 be a point in $\mathfrak{t}_{S,\mathbf{C}}$ with the property that any singular hyperplane of Ψ which contains Λ_0 also contains $\mathfrak{t}_{S,\mathbf{C}}$. Set

$$\Lambda_0(u) = \Lambda_0 + u_1\nu_1 + \cdots + u_r\nu_r,$$

for

$$u = (u_1, \ldots, u_r)$$

in $\mathbf{C}^r$. Let $\Gamma_1, \ldots, \Gamma_r$ be small positively oriented circles about the origin in the complex plane such that for each i, the radius of Γ_{i+1} is much smaller than that of Γ_i. Then

$$(2\pi i)^{-r}\int_{\Gamma_1} \cdots \int_{\Gamma_r} \Psi(\Lambda_0(u))\, du_r \ldots du_1$$

is a meromorphic function of Λ_0. We will denote it by $\operatorname{Res}_{S,\Lambda\to\Lambda_0}\Psi(\Lambda)$ or sometimes just $\operatorname{Res}_S \Psi(\Lambda_0)$.

Suppose that the class χ and the finite set Γ are fixed, as in §1. In order to allow for induction arguments, Langlands treats Eisenstein series

$$E^Q(x, \phi, \lambda) = \sum_{\delta \in P(\mathbf{Q})\backslash Q(\mathbf{Q})} \phi(\delta x) e^{(\lambda + \rho_P^Q)(H_P(\delta x))}, \qquad \phi \in \mathcal{Q}_{P,\chi,\Gamma},$$

for any parabolic subgroup $Q \supset P$. (In [1(a)] and [1(b)] this series would be denoted by $\delta_P(x)^{-1/2}E_Q(x, \phi, \lambda)$. It can be analytically continued in λ; for any λ in general position,

$$E^Q(\phi, \lambda) : x \to E^Q(x, \phi, \lambda)$$

is a smooth function on $N_Q(A)M_Q(\mathbf{Q})\backslash G(\mathbf{A})$.) Now suppose that B is any group in $\mathcal{P}_X$ which is contained in P. Let F_B be a meromorphic function from $\mathfrak{a}^*_{B,\mathbf{C}}$ to $\mathcal{A}_{B,X,\Gamma}$ whose singularities lie along hyperplanes. Then for fixed x, $E^Q(x, F_B(\Lambda), \Lambda)$ is a meromorphic function on $\mathfrak{a}^*_{B,\mathbf{C}}$ whose singularities also lie along hyperplanes. For S as above,

$$\mathrm{Res}_S E^Q(x, F_B(\Lambda), \Lambda), \qquad \Lambda \in \mathfrak{t}_{S,\mathbf{C}},$$

is a meromorphic function on $\mathfrak{t}_{S,\mathbf{C}}$. Let $S((\mathfrak{a}^P_B)^*_{\mathbf{C}})$ be the symmetric algebra on $(\mathfrak{a}^P_B)^*_{\mathbf{C}}$, the orthogonal complement of $\mathfrak{a}^*_{P,\mathbf{C}}$ in $\mathfrak{a}^*_{B,\mathbf{C}}$. If F_B is regular at a point Λ in $\mathfrak{t}_{S,\mathbf{C}}$, let

$$dF_B(\Lambda) = d_S F_B(\Lambda)$$

be the linear map from $S((\mathfrak{a}^P_B)^*_{\mathbf{C}})$ to $\mathcal{A}_{B,X,\Gamma}$ obtained by expanding the analytic function

$$\eta \to F_B(\eta + \Lambda), \qquad \eta \in (\mathfrak{a}^P_B)^*_{\mathbf{C}},$$

as a Taylor series about $\eta = 0$. Then $\mathrm{Res}_S E^Q(x, F_B(\Lambda), \Lambda)$ depends only on the vector $dF_B(\Lambda)$.

Embedded in the proof of Theorem 7.1 of [2(b)] is the construction of all Eisenstein series as residues of this kind. For each $B \in \mathcal{P}_X$, with $B \subset P$, Langlands constructs a finite set of affine subspaces

$$\mathfrak{t} = \Lambda_{\mathfrak{t}} + \mathfrak{a}^*_P$$

of $\mathfrak{a}^*_B$, and for each $\mathfrak{t}$, a finite set $\mathbb{S}_B(P, \mathfrak{t})$ of sequences of affine spaces (with distinguished unit normals) as above, such that $\mathfrak{t}_S = \mathfrak{t}$ for each $S \in \mathbb{S}_B(P, \mathfrak{t})$. If

$$\Phi_{\mathfrak{t}} = dF_B(\Lambda),$$

for F_B as above and $\Lambda \in \mathfrak{t}_{\mathbf{C}}$, set

$$E^Q_{\mathfrak{t}}(x, \Phi_{\mathfrak{t}}, \Lambda) = \Sigma_{S \in \mathbb{S}_B(P,\mathfrak{t})} \mathrm{Res}_S E^Q(x, F_B(\Lambda), \Lambda),$$

for each group $Q \supset P$. Then $E^Q_{\mathfrak{t}}$ is a function on

$$N_Q(\mathbf{A})M_Q(\mathbf{Q})\backslash G(\mathbf{A}) \times \mathrm{Hom}\Big(S\big((\mathfrak{a}^P_B)^*_{\mathbf{C}}\big), \mathcal{A}_{B,X,\Gamma}\Big) \times \mathfrak{t}_{\mathbf{C}}.$$

The collection

$$\big\{ E^Q_{\mathfrak{t}}(x, \Phi_{\mathfrak{t}}, \Lambda) : Q \supset P \big\}$$

is called an *Eisenstein system* associated to $\mathfrak{t}$. Langlands shows that the affine spaces $\{\mathfrak{t}\}$ and the associated Eisenstein systems are canonical. On the other hand, the sets $\mathbb{S}_B(P, \mathfrak{t})$ are not canonical. That is, the Eisenstein systems can be expressed in different ways as residues. In this paper we shall simply fix a choice of sets $\mathbb{S}_B(P,\mathfrak{t})$ for each B and $\mathfrak{t}$.

For any group $B \in \mathcal{P}_\chi$, $B \subset P$, let $\mathbb{S}_B(P)$ be the disjoint union over all $\mathfrak{t}$ of the sets $\mathbb{S}_B(P,\mathfrak{t})$. Suppose that for each such B, F_B is a meromorphic function from $\mathfrak{a}^*_{B,\mathbb{C}}$ to $\mathcal{Q}_{B,\chi,\Gamma}$, whose singularities lie along hyperplanes, and which is regular at each point Λ_S, $S \in \mathbb{S}_B(P)$. Then

$$\Phi = \bigoplus_B \bigoplus_{S \in \mathbb{S}_B(P)} (d_S F_B)(\Lambda_S)$$

belongs to the vector space

$$\bigoplus_B \bigoplus_{S \in \mathbb{S}_B(P)} \mathrm{Hom}\big(S\big(\mathfrak{a}_B^P\big)^*_{\mathbb{C}}\big), \mathcal{Q}_{B,\chi,\Gamma}\big).$$

Let $L_{P,\chi,\Gamma}$ be the subspace consisting of vectors of this form. It is a consequence of Theorem 7.1 of [2(b)] that the function

$$\phi(x) = \sum_{\{B \in \mathcal{P}_\chi : B \subset P\}} \sum_{S \in \mathbb{S}_B(P)} \mathrm{Res}_{S,\Lambda \to \Lambda_S} E^P(x, F_B(\Lambda), \Lambda) \tag{2.1}$$

belongs to $\mathcal{Q}_{P,\chi,\Gamma}$ and that $\mathcal{Q}_{P,\chi,\Gamma}$ is generated by functions of this form. $\phi(x)$ depends only on the vector Φ, so that

$$\Phi \to \phi$$

is a surjective linear map from $L_{P,\chi,\Gamma}$ onto $\mathcal{Q}_{P,\chi,\Gamma}$. (Also implicit in [2(b)] is the construction of a positive, semidefinite inner product on $L_{P,\chi,\Gamma}$ which coincides with the natural pairing

$$(\phi, \phi') = \int_K \int_{M_P(\mathbb{Q}) \backslash M_P(\mathbb{A})^1} \phi(mk) \overline{\phi'(mk)}\, dm\, dk$$

on $\mathcal{Q}_{P,\chi,\Gamma}$. However, we will not need to use the inner product here.) More generally, suppose that $Q \supset P$ and that Φ and ϕ are related as above. Then if λ is a point in general position in $\mathfrak{a}^*_{P,\mathbb{C}}$, Theorem 7.1 of [2(b)] tells us that the Eisenstein series $E^Q(x,\phi,\lambda)$ equals

$$\sum_{\{B \in \mathcal{P}_\chi : B \subset P\}} \sum_{S \in \mathbb{S}_B(P)} \mathrm{Res}_{S,\Lambda \to \Lambda_S} E^Q(x, F_B(\Lambda), \Lambda + \lambda). \tag{2.2}$$

Of course $L_{P,\chi,\Gamma}$ is naturally isomorphic to

$$\bigoplus_B \bigoplus_{\mathfrak{t}} \mathrm{Hom}\big(S\big(\big(\mathfrak{a}_B^P\big)^*_{\mathbb{C}}\big), \mathcal{Q}_{B,\chi,\Gamma}\big).$$

If

$$\Phi = \bigoplus_B \bigoplus_{\mathfrak{t}} \Phi_{\mathfrak{t}}$$

is the corresponding decomposition of Φ,

$$E^Q(x,\phi,\lambda) = \sum_B \sum_{\mathfrak{t}} E_{\mathfrak{t}}^Q(x,\Phi_{\mathfrak{t}},\Lambda_{\mathfrak{t}}+\lambda). \tag{2.3}$$

Thus the general Eisenstein series can be expressed in terms of the canonical Eisenstein systems. However, for this paper we must continue to express them in the noncanonical fashion as residues.

Suppose that P_1 is a group which is associated to P and that $t \in W(\mathfrak{a}_P, \mathfrak{a}_{P_1})$. There is a unique element $t_B \in W(\mathfrak{a}_B, \mathfrak{a}_{B_1})$, for some group $B_1 \in \mathcal{P}_\chi$ with $B_1 \subset P_1$, such that the restriction of t_B to $\mathfrak{a}_P$ equals t and such that $t_B(\alpha)$ is a simple root of (B_1, A_{B_1}) for every simple root α of (B, A_B) which vanishes on $\mathfrak{a}_P$. Suppose that $S \in \mathcal{S}_B(P)$. We can clearly transform S by t_B into a sequence tS of affine spaces (with distinguished unit normals) associated to B_1 and P_1. It follows from the symmetry of the situation that we may take for $\mathcal{S}_{B_1}(P_1)$ the collection

$$t\mathcal{S}_B(P) = \{tS : S \in \mathcal{S}_B(P)\}.$$

Assume that the collection

$$\{F_B(\Lambda) : B \in \mathcal{P}_\chi, B \subset P\}$$

of functions is given as above. Let λ be a point in general position in $\mathfrak{a}_{P,\mathbf{C}}^*$. Then

$$\tilde{F}_{B_1}(\Lambda_1) = M(t_B, t_B^{-1}\Lambda_1 + \lambda) F_B(t_B^{-1}\Lambda_1), \qquad \Lambda_1 \in \mathfrak{a}_{B_1,\mathbf{C}}^*,$$

is a meromorphic function from $\mathfrak{a}_{B_1,\mathbf{C}}^*$ to $\mathcal{Q}_{B_1,\chi,\Gamma}$. By choosing λ properly we can be sure that $\tilde{F}_{B_1}$ is regular at each point Λ_{S_1}, $S_1 \in t\mathcal{S}_B(P)$. (Recall that $M(t_B, \Lambda)$, $\Lambda \in \mathfrak{a}_{B,\mathbf{C}}^*$, is a meromorphic function on $\mathfrak{a}_{B,\mathbf{C}}^*$. It takes values in the space of linear maps from $\mathcal{Q}_{B,\chi,\Gamma}$ to $\mathcal{Q}_{B_1,\chi,\Gamma}$. Similarly, $M(t,\lambda)$ is a meromorphic function on $\mathfrak{a}_{P,\mathbf{C}}^*$ with values in $\mathrm{Hom}(\mathcal{Q}_{P,\chi,\Gamma}, \mathcal{Q}_{P_1,\chi,\Gamma})$.)

LEMMA 2.1. *Suppose that the vector*

$$\bigoplus_B \bigoplus_{S \in \mathcal{S}_B(P)} (d_S F_B)(\Lambda_S)$$

in $L_{P,\chi,\Gamma}$ *corresponds to the function* ϕ *in* $\mathcal{Q}_{P,\chi,\Gamma}$. *Then the vector*

$$\bigoplus_{B_1} \bigoplus_{S_1 \in t\mathcal{S}_B(P)} (d_{S_1} \tilde{F}_{B_1})(\Lambda_{S_1})$$

in $L_{P_1,\chi,\Gamma}$ *corresponds to the function* $M(t,\lambda)\phi$ *in* $\mathcal{Q}_{P_1,\chi,\Gamma}$.

Proof. Suppose that

$$(\mathrm{Re}(\lambda) - \rho_P)(\alpha^\vee) > 0$$

for every simple root α of (P, A_P). Then $(M(t,\lambda)\phi)(x)$ equals the integral over n

in $N_{P_1}(\mathbb{A}) \cap w_t N_P(\mathbb{A}) w_t^{-1} \backslash N_{P_1}(\mathbb{A})$ of

$$\phi(w_t^{-1}nx)\exp\{(\lambda + \rho_P)(H_P(w_t^{-1}nx))\}\exp\{-(t\lambda + \rho_{P_1})(H_{P_1}(x))\}.$$

But $\phi(w_t^{-1}nx)$ is a sum of residues of Eisenstein series

$$E^P(w_t^{-1}nx, F_B(\Lambda), \Lambda).$$

Substitute this latter function for ϕ in the integral over n. For Λ in the domain of convergence of the Eisenstein series we obtain the integral over n of

$$\sum_{\delta \in B(\mathbb{Q}) \cap M_P(\mathbb{Q}) \backslash M_P(\mathbb{Q})} \Phi(\delta w_t^{-1}nx)\exp\{(\Lambda + \lambda + \rho_B)(H_B(\delta w_t^{-1}nx))\}$$
$$\times \exp\{-(t\lambda + \rho_{P_1})(H_{P_1}(x))\}.$$

A routine change of variables yields

$$E^{P_1}(x, M(t_B, \Lambda + \lambda)F_B(\Lambda), t\Lambda).$$

If $\Lambda_1 = t_B\Lambda$, the last expression is just

$$E^{P_1}(x, \tilde{F}_{B_1}(\Lambda_1), \Lambda_1).$$

The lemma follows by summing the residues in Λ_1 with respect to all S_1 in $t\mathbb{S}_B(P)$. $\square$

§3. Exponents. We can now return to our study of the inner product (1.1). The class $\chi \in \mathfrak{X}$, the finite set Γ of representations of K, and the parabolic subgroups P and P' will remain fixed for the rest of this paper. We do assume that P and P' each contain groups in $\mathcal{P}_\chi$.

It is convenient to study a slightly more general inner product than (1.1). For the rest of this paper, Q will be a parabolic subgroup which contains P and P'. If U is any point in $\mathfrak{a}_Q$ define

$$M_Q(\mathbb{A}, U) = \{x \in M_Q(\mathbb{A}): H_Q(x) = U\}.$$

It is a closed subset of $M_Q(\mathbb{A})$. Notice that $M_Q(\mathbb{A}, 0) = M_Q(A)^1$. In fact, $M_Q(\mathbb{A}, U)$ is the translate of $M_Q(\mathbb{A})^1$ by a unique point in $A_Q(\mathbb{R})^0$. We can accordingly translate our Haar measure on $M_Q(\mathbb{A})^1$ to a measure on $M_Q(\mathbb{A}, U)$. With its left action on $M_Q(\mathbb{A}, U)$, $M_Q(\mathbb{Q})$ becomes a properly discontinuous group of measure preserving transformations of $M_Q(\mathbb{A}, U)$. The space of orbits, $M_Q(\mathbb{Q}) \backslash M_Q(\mathbb{A}, U)$, has finite volume. Let T_Q be the projection of the point T onto $\mathfrak{a}_Q$. If h and h' are functions on $N_Q(\mathbb{A})M_Q(\mathbb{Q}) \backslash G(\mathbb{A})$, define

$$(h, h')_{Q,T} = \int_K \int_{M_Q(\mathbb{Q}) \backslash M_Q(\mathbb{A}, T_Q)} h(mk)\overline{h'(mk)}\, dm\, dk,$$

provided that both integrals exist. This is actually the inner product which is best suited for us. Let $\Lambda^{T,Q}$ be the partial truncation operator on functions on $Q(\mathbf{Q})\backslash G(\mathbf{A})$ introduced on p. 97 of [1(b)]. We are going to study the inner product

$$\left(\Lambda^{T,Q}E^{Q}(\phi,\lambda),\Lambda^{T,Q}E^{Q}(\phi',-\bar{\lambda}')\right)_{Q,T}, \tag{3.1}$$

for $\lambda \in \mathfrak{a}^*_{P,\mathbf{C}}$, $\lambda' \in \mathfrak{a}^*_{P',\mathbf{C}}$, $\phi \in \mathcal{Q}_{P,\chi,\Gamma}$ and $\phi' \in \mathcal{Q}_{P',\chi,\Gamma}$. Our starting point will be an explicit formula of Langlands' for the special case that P and P' belong to the associated class $\mathcal{P}_{\chi}$.

Suppose that P_1 is any parabolic subgroup which is contained in Q. As in [1(c), §2] we write

$$\theta^{Q}_{P_1}(\zeta) = \mathrm{vol}\left(\mathfrak{a}^{Q}_{P_1}/L^{Q}_{P_1}\right)^{-1}\prod_{\alpha\in\Delta^{Q}_{P_1}}\zeta(\alpha^{\vee}), \qquad \zeta \in \mathfrak{a}^*_{P_1,\mathbf{C}},$$

where $L^{Q}_{P_1}$ is the lattice in $\mathfrak{a}^{Q}_{P_1}$ generated by the co-roots $\{\alpha^{\vee} : \alpha \in \Delta^{Q}_{P_1}\}$. (For the definition of the elements $\alpha^{\vee}$ see [1(a), p. 916].) Let $W^{Q}(\mathfrak{a}_P,\mathfrak{a}_{P_1})$ be the set of maps in $W(\mathfrak{a}_P,\mathfrak{a}_{P_1})$ which leave the space $\mathfrak{a}_Q$ pointwise fixed. Define a function $\omega^{T,Q}(\lambda,\lambda',\phi,\phi')$ to be the sum

$$\sum_{P_1}\ \sum_{t\in W^{Q}(\mathfrak{a}_P,\mathfrak{a}_{P_1})}\ \sum_{t'\in W^{Q}(\mathfrak{a}_{P'},\mathfrak{a}_{P_1})} \omega^{T,Q}_{t,t'}(\lambda,\lambda',\phi,\phi'),$$

where $\omega^{T,Q}_{t,t'}(\lambda,\lambda',\phi,\phi')$ is the expression

$$e^{(t\lambda-t'\lambda')(T)}\left(M(t,\lambda)\phi, M(t',-\bar{\lambda}')\phi'\right)\theta^{Q}_{P_1}(t\lambda-t'\lambda')^{-1}.$$

This function will be of basic interest for the rest of the paper. It is meromorphic in $\lambda \in \mathfrak{a}^*_{P,\mathbf{C}}$ and $\lambda' \in \mathfrak{a}^*_{P',\mathbf{C}}$; it is linear in $\phi \in \mathcal{Q}_{P,\chi,\Gamma}$ and conjugate linear in $\phi' \in \mathcal{Q}_{P',\chi,\Gamma}$. Then if P and P' belong to $\mathcal{P}_{\chi}$, Langlands' inner product formula is

$$\left(\lambda^{T,Q}E^{Q}(\phi,\lambda),\Lambda^{T,Q}E^{Q}(\phi',-\bar{\lambda}')\right)_{Q,T} = \omega^{T,Q}(\lambda,\lambda',\phi,\phi'). \tag{3.2}$$

(See [2(a), §9] and [1(b), Lemma 4.2]. Actually the formula proved in [1(b)] is the special case of (3.2) in which $Q = G$ and the points λ and λ' are orthogonal to $\mathfrak{a}^*_{Q,\mathbf{C}}$. (We carelessly omitted these conditions on λ and λ' in [1(b)].) However it is trivial to extend the formula from G to arbitrary Q. Moreover, if λ and λ' are translated by points λ_Q and λ'_Q in $\mathfrak{a}^*_{Q,\mathbf{C}}$, both sides of (3.2) are multiplied by $e^{(\lambda_Q-\lambda'_Q)(T)}$. In other words, (3.2) is valid as originally stated.) If P and P' do not belong to $\mathcal{P}_{\chi}$, (3.2) is no longer true. Our goal in this paper is to show that it nevertheless holds asymptotically in T.

From now on, assume that the vector $\phi \in \mathcal{Q}_{P,\chi,\Gamma}$ is the image under the map (2.1) of a vector

$$\Phi = \bigoplus_{\{B\in\mathcal{P}_{\chi}:B\subset P\}}\ \bigoplus_{S\in\mathcal{S}_B(P)} (d_S F_B)(\Lambda_S)$$

in $L_{P,\chi,\Gamma}$, for a collection $\{F_B\}$ of functions as in §2. Similarly assume that $\phi' \in \mathcal{Q}_{P',\chi,\Gamma}$ is obtained in the same way from the vector

$$\Phi' = \bigoplus_{\{B' \in \mathfrak{P}_\chi : B' \subset P'\}} \bigoplus_{S' \in \mathfrak{S}_{B'}(P')} (d_{S'} F'_{B'})(\Lambda_{S'})$$

in $L_{P',\chi,\Gamma}$.

LEMMA 3.1. *Let λ and λ' be points in general position in $\mathfrak{a}^*_{P,\mathbb{C}}$ and $\mathfrak{a}^*_{P',\mathbb{C}}$ respectively. Then the inner product* (3.1) *equals the sum over $\{B, B' \in P_\chi : B \subset P, B' \subset P'\}$, over $S \in \mathfrak{S}_B(P)$ and over $S' \in \mathfrak{S}_{B'}(P')$ of*

$$\mathrm{Res}_{S,\Lambda \to \Lambda_S} \mathrm{Res}_{S',\Lambda' \to \Lambda_{S'}} \omega^{T,Q}\big(\Lambda + \lambda, -\Lambda' + \lambda', F_B(\Lambda), F'_{B'}(\overline{\Lambda}')\big). \tag{3.3}$$

Proof. If $\Lambda \in \mathfrak{a}^*_{B,\mathbb{C}}$ and $\Lambda' \in \mathfrak{a}^*_{B',\mathbb{C}}$, the function

$$\omega^{T,Q}\big(\Lambda + \lambda, -\Lambda' + \lambda', F_B(\Lambda), F'_{B'}(\overline{\Lambda}')\big)$$

equals

$$\int_K \int_{M_Q(\mathbb{Q}) \backslash M_Q(\mathbb{A}, T_Q)} \Lambda^{T,Q} E^Q\big(mk, F_B(\Lambda), \Lambda + \lambda\big) \cdot \overline{\Lambda^{T,Q} E^Q\big(mk, F'_{B'}(\overline{\Lambda}'), \overline{\Lambda}' - \overline{\lambda}'\big)}\, dm\, dk.$$

These functions are clearly meromorphic in Λ and Λ'. The lemma will follow from (2.2). We need only show that the residue operators can be interchanged with the integrals over m and k and then with the truncation operators.

We observed how to estimate truncated Eisenstein series in [1(b), p. 108]. For any positive integer N there is a locally bounded function $c(\Lambda, \lambda)$ on the set of points (Λ, λ) at which $E^Q(x, F_B(\Lambda), \Lambda + \lambda)$ is regular, such that

$$\big|\Lambda^{T,Q} E^Q\big(mk, F_B(\Lambda), \Lambda + \lambda\big)\big| \leqslant c(\Lambda, \lambda) \|m\|^{-N}$$

for all k and m in a given Siegel set in $M_Q(\mathbb{A})^1$. (Here $\| \;\|$ is a suitable "norm" function on $G(\mathbb{A})$ defined as, for example, in [1(a), §1].) The interchange of the residue operators with the integrals over m and k then follows from Fubini's theorem.

In the formula

$$\sum_{\{Q_1 : P_0 \subset Q_1 \subset Q\}} (-1)^{\dim(A_{Q_1}/A_Q)} \cdot \sum_{\delta \in Q_1(\mathbb{Q}) \backslash Q(\mathbb{Q})} \int_{N_{Q_1}(\mathbb{Q}) \backslash N_{Q_1}(\mathbb{A})} h(n\delta x) \hat{\tau}^Q_{Q_1}\big(H_{Q_1}(\delta x) - T\big)\, dn$$

for the partial truncation operator

$$\Lambda^{T,Q} h(x), \qquad h \in C\big(Q(\mathbb{Q}) \backslash G(\mathbb{A})\big),$$

the sum over δ is really over a finite set ([1(a), Lemma 5.1]). The set depends on x but is independent of h. Since the sets $N_{Q_1}(\mathbf{Q})\backslash N_{Q_1}(\mathbf{A})$ are compact, the residue operators can indeed be interchanged with the truncation operators. The lemma follows. $\square$

If $\lambda \in \mathfrak{a}^*_{P,\mathbf{C}}$ and $\lambda' \in \mathfrak{a}^*_{P',\mathbf{C}}$ are points in general position, let $\Omega^{T,Q}(\lambda,\lambda',\phi,\phi')$ denote the sum over $\{B,B' \in \mathcal{P}_\chi : B \subset P, B' \subset P'\}$ and $\{S,S' : S \in \mathbb{S}_B(P), S' \in \mathbb{S}_{B'}(P')\}$ of the expression (3.3). Then Lemma 3.1 becomes the formula

$$\left(\Lambda^{T,Q}E^Q(\phi,\lambda), \Lambda^{T,Q}E^Q(\phi', -\bar{\lambda}')\right)_{Q,T} = \Omega^{T,Q}(\lambda,\lambda',\phi,\phi').$$

It is the analogue of (3.2) when P and P' are not restricted to lie in $\mathcal{P}_\chi$. Unfortunately, $\Omega^{T,Q}(\lambda,\lambda',\phi,\phi')$ does not have as explicit a formula as $\omega^{T,Q}(\lambda,\lambda',\phi,\phi')$. However, Lemma 3.1 does tell us that $\Omega^{T,Q}(\lambda,\lambda',\phi,\phi')$ is a genuine function of ϕ and ϕ'; it is independent of the choice of functions $\{F_B\}$ and $\{F'_{B'}\}$.

We would like to study $\Omega^{T,Q}(\lambda,\lambda',\phi,\phi')$ as a function of T. It equals the sum over B, B', S, S' and also over $B_1 \in \mathcal{P}_\chi$, $s \in W^Q(\mathfrak{a}_B,\mathfrak{a}_{B_1})$ and $s' \in W^Q(\mathfrak{a}_{B'},\mathfrak{a}_{B_1})$ of

$$\operatorname{Res}_{S,\Lambda\to\Lambda_S}\operatorname{Res}_{S',\Lambda'\to\Lambda_{S'}}\omega^{T,Q}_{s,s'}\left(\Lambda+\lambda, -\Lambda'+\lambda', F_B(\Lambda), F'_{B'}(\bar{\Lambda}')\right). \tag{3.4}$$

This last expression is clearly the product of a polynomial in T with the exponential $e^{(s\lambda - s'\lambda' + X)(T)}$, where

$$X = s\Lambda_S + s'\Lambda_{S'},$$

a point in $(\mathfrak{a}^Q_{B_1})^*$ which is independent of λ and λ'. If $\mathcal{P}$ is any class of associated (standard) parabolic subgroups, let $W^Q(\mathfrak{a}_P \oplus \mathfrak{a}_{P'}, \mathcal{P})$ be the set of linear transformations

$$(t,t') : \mathfrak{a}_P \oplus \mathfrak{a}_{P'} \to t\mathfrak{a}_P \oplus t'\mathfrak{a}_{P'}$$

obtained by restricting the maps

$$(s,s') : \mathfrak{a}_R \oplus \mathfrak{a}_{R'} \to \mathfrak{a}_{R_1} \oplus \mathfrak{a}_{R_1}$$

in any of the sets

$$W^Q(\mathfrak{a}_R,\mathfrak{a}_{R_1}) \times W^Q(\mathfrak{a}_{R'},\mathfrak{a}_{R_1}), \qquad R,R',R_1 \in \mathcal{P}, \quad R \subset P, \quad R' \subset P',$$

to $\mathfrak{a}_P \oplus \mathfrak{a}_{P'}$. Take $\mathcal{P} = \mathcal{P}_\chi$ and choose $(t,t') \in W^Q(\mathfrak{a}_P \oplus \mathfrak{a}_{P'}, \mathcal{P}_\chi)$. Let $\Omega^{T,Q}_{t,t'}(\lambda,\lambda',\phi,\phi')$ be the contribution to $\Omega^{T,Q}(\lambda,\lambda',\phi,\phi')$ from those terms in the multiple sum above for which the restriction to $\mathfrak{a}_P \oplus \mathfrak{a}_{P'}$ of (s,s') equals (t,t'). Then by definition we have

$$\Omega^{T,Q}(\lambda,\lambda',\phi,\phi') = \sum_{(t,t') \in W^Q(\mathfrak{a}_P \oplus \mathfrak{a}_{P'}, \mathcal{P}_\chi)} \Omega^{T,Q}_{t,t'}(\lambda,\lambda',\phi,\phi'). \tag{3.5}$$

Incidentally, if P and P' are associated, and $\mathcal{P}$ is the class which contains these groups, we also have

$$\omega^{T,Q}(\lambda,\lambda',\phi,\phi') = \sum_{(t,t')\in W^Q(\mathfrak{a}_P\oplus\mathfrak{a}_{P'},\mathcal{P})} \omega_{t,t'}^{T,Q}(\lambda,\lambda',\phi,\phi'). \tag{3.6}$$

(If P and P' are not associated, $\omega^{T,Q}(\lambda,\lambda',\phi,\phi')$ is by definition zero.) It is clear that there is a finite set $\mathcal{E}^Q(t,t')$ of points in $(\mathfrak{a}_0^Q)^*$, and for each $X\in\mathcal{E}^Q(t,t')$ a nonzero function

$$p_X^{T,Q}(\lambda,\lambda',\phi,\phi'),$$

which is a polynomial in T, such that

$$\Omega_{t,t'}^{T,Q}(\lambda,\lambda',\phi,\phi') = \sum_{X\in\mathcal{E}^Q(t,t')} p_X^{T,Q}(\lambda,\lambda',\phi,\phi')e^{(t\lambda-t'\lambda'+X)(T)}. \tag{3.7}$$

Thus, as a function of T, $\Omega^{T,Q}(\lambda,\lambda',\phi,\phi')$ is a finite sum of exponentials with polynomial coefficients. The polynomial

$$p_X^{T,Q}(\lambda,\lambda',\phi,\phi'), \qquad X\in\mathcal{E}^Q(t,t'),$$

is the coefficient of $e^{(t\lambda-t'\lambda'+X)(T)}$, so as the notation suggests, it depends only on ϕ and ϕ'. It is independent of the choice of functions $\{F_B\}$ and $\{F'_{B'}\}$. Like $\Omega^{T,Q}(\lambda,\lambda',\phi,\phi')$ it is meromorphic in λ and λ', linear in ϕ and conjugate linear in ϕ'.

§4. A comparison between two groups. For this section we assume that P and P' belong to the same associated class $\mathcal{P}$. Suppose that P_1 is another group in $\mathcal{P}$ which is also contained in Q. If t is in $W^Q(\mathfrak{a}_P,\mathfrak{a}_{P_1})$ we have defined an element $t_B\in W^Q(\mathfrak{a}_B,\mathfrak{a}_{B_1})$ for each group $B\in\mathcal{P}_\chi$ with $B\subset P$. The restriction of t_B to $\mathfrak{a}_P$ equals t. Similarly, given $t'\in W^Q(\mathfrak{a}_{P'},\mathfrak{a}_{P_1})$ and $B'\in\mathcal{P}_\chi$, $B'\subset P'$, there is an element $t'_{B'}\in W^Q(\mathfrak{a}_{B'},\mathfrak{a}_{B'_1})$ whose restriction to $\mathfrak{a}_{P'}$ equals t'. In fact, it is possible to choose B' so that $B'_1=B_1$. It follows that the map

$$(t,t'):\mathfrak{a}_P\oplus\mathfrak{a}_{P'}\to t\mathfrak{a}_P\oplus t'\mathfrak{a}_{P'}$$

belongs to the set $W^Q(\mathfrak{a}_P\oplus\mathfrak{a}_{P'},\mathcal{P}_\chi)$ defined in §3. Thus, $W^Q(\mathfrak{a}_P\oplus\mathfrak{a}_{P'},\mathcal{P})$ is naturally embedded as a subset of $W^Q(\mathfrak{a}_P\oplus\mathfrak{a}_{P'},\mathcal{P}_\chi)$.

Fix a group $P_1\in\mathcal{P}$, $P_1\subset Q$, and elements $t\in W^Q(\mathfrak{a}_P,\mathfrak{a}_{P_1})$, t' in $W^Q(\mathfrak{a}_{P'},\mathfrak{a}_{P_1})$. In this section a formula which relates $\Omega_{t,t'}^{T,Q}$ to the function

$$\Omega^{T,P_1}:\mathfrak{a}_{P_1,\mathbf{C}}^*\times\mathfrak{a}_{P_1,\mathbf{C}}^*\times\mathcal{A}_{P_1,\chi,\Gamma}\times\mathcal{A}_{P_1,\chi,\Gamma}\to\mathbf{C}.$$

will be found. The group B_1 for which t_B belongs to $W^Q(\mathfrak{a}_B,\mathfrak{a}_{B_1})$ is uniquely determined by B. In fact

$$B\to B_1$$

is a bijection between the sets $\{B \in \mathcal{P}_\chi : B \subset P\}$ and $\{B_1 \in \mathcal{P}_\chi : B_1 \subset P_1\}$. Suppose that B and B_2 are groups in these respective sets. Then the set of elements in $W^Q(\mathfrak{a}_B, \mathfrak{a}_{B_2})$ whose restriction to $\mathfrak{a}_P$ equals t is just the set

$$st_B, \qquad s \in W^{P_1}(\mathfrak{a}_{B_1}, \mathfrak{a}_{B_2}).$$

It follows that $\Omega^{T,Q}_{t,t'}(\lambda, \lambda', \phi, \phi')$ can be written as the sum over $\{B, B' \in \mathcal{P}_\chi : B \subset P, B' \subset P'\}$, $S \in \mathbb{S}_P(B)$, $S' \in \mathbb{S}_{P'}(B')$ and also over $\{B_2 \in \mathcal{P}_\chi : B_2 \subset P_1\}$, $s \in W^{P_1}(\mathfrak{a}_{B_1}, \mathfrak{a}_{B_2})$ and $s' \in W^{P_1}(\mathfrak{a}_{B_1'}, \mathfrak{a}_{B_2})$ of

$$\operatorname{Res}_{S,\Lambda \to \Lambda_S} \operatorname{Res}_{S',\Lambda' \to \Lambda_{S'}} \left\{ \omega^{T,Q}_{st_B, s't'_{B'}}\left(\Lambda + \lambda, -\Lambda' + \lambda', F_B(\Lambda), F'_{B'}(\overline{\Lambda}')\right)\right\}.$$

(B_1 and B_1' are of course the groups such that $t_B \in W^Q(\mathfrak{a}_B, \mathfrak{a}_{B_1})$ and $t'_{B'} \in W^Q(\mathfrak{a}_{B'}, \mathfrak{a}_{B_1'})$.) The last expression in the brackets we will write as the product of

$$\exp\{(st_B(\Lambda + \lambda) - s't'_{B'}(-\Lambda' + \lambda'))(T)\}, \tag{4.1}$$

$$\left(M(st_B, \Lambda + \lambda)F_B(\Lambda), M\left(s't'_{B'}, \overline{\Lambda}' - \bar{\lambda}'\right)F'_{B'}(\overline{\Lambda}')\right), \tag{4.2}$$

and

$$\theta^Q_{B_2}(st_B(\Lambda + \lambda) - s't'_{B'}(-\Lambda' + \lambda'))^{-1}. \tag{4.3}$$

We are going to take a certain derivative of $\Omega^{T,Q}_{t,t'}(\lambda, \lambda', \phi, \phi')$ with respect to T. It will simply entail differentiating (4.1). Suppose that R is any parabolic subgroup contained in Q. Recall that Δ^R_0 is the set of simple roots of the Levi component M_R. It is a subset of Δ^Q_0. If ψ is any smooth function on $\mathfrak{a}_0$, define

$$(D_\beta \psi)(T) = \lim_{x \to 0} \frac{d}{dx} \psi(T + x\beta^\vee)$$

for each root β in the complement of Δ^R_0 in Δ^Q_0, and set

$$(D_{Q|R}\psi)(T) = D_{Q|R,T}\psi(T) = \left(\left(\prod_{\beta \in \Delta^Q_0 \setminus \Delta^R_0} D_\beta\right)\psi\right)(T).$$

In this section we will let R be the group P_1. Let $\Lambda_1 = t_B\Lambda$ and $\Lambda_1' = t'_{B'}\Lambda'$. Then (4.1) equals

$$\exp\{(s(\Lambda_1 + t\lambda) - s'(-\Lambda_1' + t'\lambda'))(T)\}. \tag{4.1$'$}$$

Operating on this function by $D_{Q|P_1,T}$ serves only to multiply it by

$$\prod_{\beta \in \Delta^Q_0 \setminus \Delta^{P_1}_0} (s(\Lambda_1 + t\lambda) - s'(-\Lambda_1' + t'\lambda'))(\beta^\vee). \tag{4.4}$$

If β belongs to $\Delta_0^Q \backslash \Delta_0^{P_1}$, its restriction β_1 to $\mathfrak{a}_{B_2}^Q$ is a root in $\Delta_{B_2}^Q$. In fact $\beta \to \beta_1$ is a bijection from $\Delta_0^Q \backslash \Delta_0^{P_1}$ onto $\Delta_{B_2}^Q \backslash \Delta_{B_2}^{P_1}$. If ζ is any point in $\mathfrak{a}_{B_2,\mathbf{C}}^*$, then $\zeta(\beta^\vee)$ equals $\zeta(\beta_1^\vee)$. The product (4.4) can therefore be taken over $\Delta_{B_2}^Q \backslash \Delta_{B_2}^{P_1}$. Next consider the expression (4.3). It equals

$$\operatorname{vol}\big(\mathfrak{a}_{B_2}^Q / L_{B_2}^Q\big)\Bigg(\prod_{\alpha \in \Delta_{B_2}^Q} (s(\Lambda_1 + t\lambda) - s'(-\Lambda_1' + t'\lambda'))(\alpha^\vee)\Bigg)^{-1}.$$

Since

$$\operatorname{vol}\big(\mathfrak{a}_{B_2}^Q / L_{B_2}^Q\big) = \operatorname{vol}\big(\mathfrak{a}_{P_1}^Q / L_{P_1}^Q\big)\operatorname{vol}\big(\mathfrak{a}_{B_2}^{P_1} / L_{B_2}^{P_1}\big),$$

(4.3) can be written as the product of

$$\operatorname{vol}\big(\mathfrak{a}_{P_1}^Q / L_{P_1}^Q\big),$$

$$\prod_{\alpha \in \Delta_{B_2}^Q \backslash \Delta_{B_2}^{P_1}} \big((s(\Lambda_1 + t\lambda) - s'(-\Lambda_1' + t'\lambda'))(\alpha^\vee)\big)^{-1},$$

and

$$\operatorname{vol}\big(\mathfrak{a}_{B_2}^{P_1} / L_{B_2}^{P_1}\big)\Bigg(\prod_{\alpha \in \Delta_{B_2}^{P_1}} (s(\Lambda_1 + t\lambda) - s'(-\Lambda_1' + t'\lambda'))(\alpha^\vee)\Bigg)^{-1}.$$

The second of these three terms will cancel (4.4). The third one just equals

$$\theta_{B_2}^{P_1}(s(\Lambda_1 + t\lambda) - s'(-\Lambda_1' + t'\lambda'))^{-1}. \tag{4.3'}$$

Finally, the basic functional equations of the M functions allow us to write (4.2) as

$$\big(M(s, t_B(\Lambda + \lambda))M(t_B, \Lambda + \lambda)F_B(\Lambda), M\big(s', t_{B'}'(\overline{\Lambda}' - \overline{\lambda}')\big)M\big(t_{B'}', \overline{\Lambda}' - \overline{\lambda}'\big)F_{B'}'(\overline{\Lambda}')\big).$$

This in turn can be written

$$\big(M(s, \Lambda_1 + t\lambda)\tilde{F}_{B_1}(\Lambda_1), M\big(s', \overline{\Lambda}_1' - t'\overline{\lambda}'\big)\tilde{F}_{B_1'}'(\overline{\Lambda}_1')\big), \tag{4.2'}$$

if

$$\tilde{F}_{B_1}(\Lambda_1) = M\big(t_B, t_B^{-1}\Lambda_1 + \lambda\big)F_B\big(t_B^{-1}\Lambda_1\big)$$

and

$$\tilde{F}_{B_1'}'(\Lambda_1') = M\big(t_{B'}', (t_{B'}')^{-1}\Lambda_1' - \overline{\lambda}'\big)F_{B'}'\big((t_{B'}')^{-1}\Lambda_1'\big).$$

We have shown that if $D_{Q|P_1,T}$ is made to operate on the product of (4.1), (4.2) and (4.3) the result is the product of $\operatorname{vol}(\mathfrak{a}_{P_1}^Q / L_{P_1}^Q)$ with (4.1′), (4.2′) and (4.3′).

Now the product of (4.1′), (4.2′) and (4.3′) equals

$$\omega_{s,s'}^{T,P_1}\big(\Lambda_1 + t\lambda, -\Lambda_1' + t'\lambda', \tilde{F}_{B_1}(\Lambda_1), \tilde{F}_{B_1'}'(\overline{\Lambda}_1')\big).$$

We have to take a residue of this expression. If

$$S_1 = tS, \qquad S \in \mathbb{S}_P(B),$$

the residue operators $\mathrm{Res}_{S,\Lambda\to\Lambda_S}$ and $\mathrm{Res}_{S_1,\Lambda_1\to\Lambda_{S_1}}$ are equal. Similarly

$$\mathrm{Res}_{S',\Lambda'\to\Lambda_{S'}} = \mathrm{Res}_{S_1',\Lambda_1'\to\Lambda_{S_1'}}$$

if $S_1' = t'S'$. It follows that

$$D_{Q\,|\,P_1,T}\Omega_{t,t'}^{T,Q}(\lambda,\lambda',\phi,\phi')\mathrm{vol}\big(\mathfrak{a}_{P_1}^Q/L_{P_1}^Q\big)^{-1}$$

equals the sum over $\{B_1, B_1' \in \mathcal{P}_\chi : B_1 \subset P_1, B_1' \subset P_1\}$, $S_1 \in t\mathbb{S}_P(B)$, $S_1' \in t'\mathbb{S}_{P'}(B')$ and also over $\{B_2 \in \mathcal{P}_\chi : B_2 \subset P_1\}$, $s \in W^{P_1}(\mathfrak{a}_{B_1}, \mathfrak{a}_{B_2})$ and $s' \in W^{P_1}(\mathfrak{a}_{B_1'}, \mathfrak{a}_{B_2})$ of

$$\mathrm{Res}_{S_1,\Lambda_1\to\Lambda_{S_1}}\mathrm{Res}_{S_1',\Lambda_1'\to\Lambda_{S_1'}}\Big\{\omega_{s,s'}^{T,P_1}\big(\Lambda_1 + t\lambda, -\Lambda_1' + t'\lambda', \tilde{F}_{B_1}(\Lambda_1), \tilde{F}_{B_1'}'(\overline{\Lambda}_1')\big)\Big\}.$$

This is just the sum over B_1, B_1', S_1 and S_1' of

$$\mathrm{Res}_{S_1,\Lambda_1\to\Lambda_{S_1}}\mathrm{Res}_{S_1',\Lambda_1'\to\Lambda_{S_1'}}\Big\{\omega^{T,P_1}\big(\Lambda_1 + t\lambda, -\Lambda_1' + t'\lambda', \tilde{F}_{B_1}(\Lambda_1), \tilde{F}_{B_1'}'(\overline{\Lambda}_1')\big)\Big\}.$$

Appealing to Lemma 2.1 and the definition in §3 we see that this last multiple sum equals

$$\Omega^{T,P_1}\big(t\lambda, t'\lambda', M(t,\lambda)\phi, M(t',-\overline{\lambda}')\phi'\big).$$

We summarize this as a lemma.

LEMMA 4.1. *Suppose there is a parabolic subgroup P_1 which is associated to both P and P', such that $t \in W^Q(\mathfrak{a}_P, \mathfrak{a}_{P'})$ and $t' \in W^Q(\mathfrak{a}_{P'}, \mathfrak{a}_{P_1})$. Then*

$$D_{Q\,|\,P_1,T}\Omega_{t,t'}^{T,Q}(\lambda,\lambda',\phi,\phi')$$

equals the product of

$$\mathrm{vol}\big(\mathfrak{a}_{P_1}^Q/L_{P_1}^Q\big)$$

with

$$\Omega^{T,P_1}\big(t\lambda, t'\lambda', M(t,\lambda)\phi, M(t',-\overline{\lambda}')\phi'\big)$$

for $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$, $\lambda' \in \mathfrak{a}_{P,\mathbb{C}}^*$, $\phi \in \mathcal{Q}_{P,\chi,\Gamma}$ and $\phi' \in \mathcal{Q}_{P',\chi,\Gamma}$. $\square$

The only element in $W^{P_1}(\mathfrak{a}_{P_1} \oplus \mathfrak{a}_{P_1}, \mathcal{P}_X)$ is the pair $(1,1)$. Then $\Omega^{T,P_1} = \Omega^{T,P_1}_{1,1}$.

COROLLARY 4.2. *Under the assumptions of the lemma,* $\mathfrak{S}^Q(t,t') = \mathfrak{S}^{P_1}(1,1)$. *If X is a point in* $\mathfrak{S}^Q(t,t')$,

$$\exp\{-(t\lambda - t'\lambda' + X)(T)\} D_{Q|P_1,T}\left(p_X^{T,Q}(\lambda,\lambda',\phi,\phi')\exp\{(t\lambda - t'\lambda' + X)(T)\}\right)$$

equals

$$\operatorname{vol}\left(\mathfrak{a}_{P_1}^Q / L_{P_1}^Q\right) p_X^{T,P_1}\left(t\lambda, t'\lambda', M(t,\lambda)\phi, M(t', -\bar{\lambda}')\phi'\right).$$

Proof. The lemma tells us that

$$\sum_{X \in \mathfrak{S}^Q(t,t')} \tilde{p}_X^{T,Q}(\lambda,\lambda',\phi,\phi')\exp\{(t\lambda - t'\lambda' + X)(T)\}$$

equals

$$\sum_{X \in \mathfrak{S}^{P_1}(1,1)} \operatorname{vol}\left(\mathfrak{a}_{P_1}^Q / L_{P_1}^Q\right) p_X^{T,P_1}\left(t\lambda, t'\lambda', M(t,\lambda)\phi, M(t', -\bar{\lambda}')\phi'\right)$$
$$\cdot \exp\{(t\lambda - t'\lambda' + X)(T)\},$$

where $\tilde{p}_X^{T,Q}(\lambda,\lambda',\phi,\phi')$ equals

$$\exp\{-(t\lambda - t'\lambda' + X)(T)\} D_{Q|P_1,T}\left(p_X^{T,Q}(\lambda,\lambda',\phi,\phi')\exp\{(t\lambda - t'\lambda' + X)(T)\}\right).$$

The function $\tilde{p}_X^{T,Q}(\lambda,\lambda',\phi,\phi')$ is also a polynomial in T. The corollary will follow if we can show that it does not vanish. The polynomial $\tilde{p}_X^{T,Q}(\lambda,\lambda',\phi,\phi')$ will actually have the same total degree as $p_X^{T,Q}(\lambda,\lambda',\phi,\phi')$, and will in particular not vanish, provided that the expression

$$\exp\{-(t\lambda - t'\lambda' + X)(T)\}\left(D_{Q|P_1,T}\exp\{(t\lambda - t'\lambda' + X)(T)\}\right)$$
$$= \prod_{\alpha \in \Delta_0^Q \setminus \Delta_0^{P_1}} (t\lambda - t'\lambda' + X)(\alpha^\vee)$$

is not zero. We can clearly arrange this by choosing λ and λ' so that $t\lambda - t'\lambda'$ is a point in $\mathfrak{a}_{P_1,\mathbf{C}}^*$ in general position. $\square$

It follows from the corollary (and its proof) that the functions $p_X^{T,Q}(\lambda,\lambda',\phi,\phi')$ and $p_X^{T,P_1}(t\lambda, t'\lambda', M(t,\lambda), M(t', -\bar{\lambda}')\phi')$, as polynomials in T, have the same total degree.

§5. A property of the truncation operator. The last lemma suggests that a differential operator $D_{Q|P_1}$ in T has something to do with our inner product. In the next three sections we shall pursue this lead, but along a different path. Only in §8 will we return to take advantage of Lemma 4.1. In the place of the particular group P_1, we will take an arbitrary parabolic subgroup R contained in Q. For any pair h and h' of rather general functions on $N_Q(\mathsf{A})M_Q(\mathsf{Q})\backslash G(\mathsf{A})$, a simple formula for the action of $D_{Q|R,T}$ on $(\Lambda^{T,Q}h, \Lambda^{T,Q}h')_{Q,T}$ will be found.

The result is actually quite easy to derive, at least formally. Under general conditions on h and h' (we will be more specific a little later),

$$(\Lambda^{T,Q}h, \Lambda^{T,Q}h')_{Q,T} = \left((\Lambda^{T,Q})^2 h, h'\right)_{Q,T} = (\Lambda^{T,Q}h, h')_{Q,T}.$$

If we were able to differentiate inside the inner product, we could just let $D_{Q|R,T}$ act on

$$(\Lambda^{T,Q}h)(x) = \sum_{\{Q_1 : P_0 \subset Q_1 \subset Q\}} (-1)^{\dim(A_{Q_1}/A_Q)}$$
$$\cdot \sum_{\delta \in Q_1(\mathsf{Q})\backslash Q(\mathsf{Q})} \int_{N_{Q_1}(\mathsf{Q})\backslash N_{Q_1}(\mathsf{A})} h(n\delta x)\hat{\tau}_{Q_1}^{Q}(H_{Q_1}(\delta x) - T)\, dn.$$

Now

$$D_{Q|R,T}\hat{\tau}_{Q_1}^{Q}(H - T), \qquad H \in \mathfrak{a}_0,$$

is a distribution on $\mathfrak{a}_0$. It is the translation by T of the distribution

$$(-1)^{\dim(A_R/A_Q)} D_{Q|R}\hat{\tau}_{Q_1}^{Q}.$$

LEMMA 5.1. *If Q_1 is not contained in R, $D_{Q|R}\hat{\tau}_{Q_1}^{Q}$ equals zero. If Q_1 is contained in R, it equals*

$$\mathrm{vol}\left(\mathfrak{a}_R^Q / L_R^Q\right) \cdot \hat{\tau}_{Q_1}^{R} \cdot (\mathrm{Dir})_R^Q,$$

where $(\mathrm{Dir})_R^Q$ is the distribution on

$$\mathfrak{a}_0 = \mathfrak{a}_0^R \oplus \mathfrak{a}_R^Q \oplus \mathfrak{a}_Q$$

which is constant in the $\mathfrak{a}_0^R$ and $\mathfrak{a}_Q$ directions and equals the Dirac distribution (with respect to our fixed measure on $\mathfrak{a}_R^Q$) in the $\mathfrak{a}_R^Q$ direction.

Proof. Suppose that Q_1 is not contained in R. Let β be a root in the complement of Δ_0^R in $\Delta_0^{Q_1}$. Then

$$\hat{\tau}_{Q_1}^{Q}(H + x\beta^\vee) = \hat{\tau}_{Q_1}^{Q}(H), \qquad H \in \mathfrak{a}_{Q_1}, \quad x \in \mathbf{R}.$$

It follows that $D_\beta\hat{\tau}_{Q_1}^{Q} = 0$. Therefore $D_{Q|R}\hat{\tau}_{Q_1}^{Q} = 0$.

Suppose then that Q_1 is contained in R. It is a consequence of the definitions that $D_{Q|R}\hat{\tau}_{Q_1}^Q$ is the product of $D_{Q|R}\hat{\tau}_R^Q$ with the characteristic function of

$$\{H \in \mathfrak{a}_{Q_1} : \varpi(H) > 0, \varpi \in \hat{\Delta}_{Q_1}^Q \backslash \hat{\Delta}_R^Q\}. \tag{5.1}$$

To evaluate $D_{Q|R}\hat{\tau}_R^Q$, we write an arbitrary point in $\mathfrak{a}_R^Q$ as

$$\sum_{\alpha \in \Delta_R^Q} u_\alpha \alpha^\vee, \qquad u_\alpha \in \mathbf{R}.$$

In these co-ordinates, $D_{Q|R}\hat{\tau}_R^Q$ is just a product of Dirac distributions in each variable u_α. Modulo the Jacobian factor $\mathrm{vol}(\mathfrak{a}_R^Q/L_R^Q)$, it equals the Dirac distribution on $\mathfrak{a}_R^Q$. In particular, the distribution $D_{Q|R}\hat{\tau}_R^Q$ is supported on $\mathfrak{a}_0^R \oplus \mathfrak{a}_Q$. The restriction to this subspace of the characteristic function of (5.1) is just $\hat{\tau}_{Q_1}^R$. It follows that

$$D_{Q|R}\hat{\tau}_{Q_1}^Q = \mathrm{vol}(\mathfrak{a}_R^Q/L_R^Q) \cdot \hat{\tau}_{Q_1}^R \cdot (\mathrm{Dir})_R^Q. \quad \square$$

Let us argue heuristically for a moment. We see from the lemma that $D_{Q|R,T}(\Lambda^{T,Q}h)(x)$ equals the product of $\mathrm{vol}(\mathfrak{a}_R^Q/L_R^Q)$ with the sum over $\{Q_1 : P_0 \subset Q_1 \subset R\}$ of

$$(-1)^{\dim(A_{Q_1}/A_R)} \sum_{\delta \in Q_1(\mathbf{Q})\backslash Q(\mathbf{Q})} \int_{N_{Q_1}(\mathbf{Q})\backslash N_{Q_1}(\mathbf{A})} h(n\delta x)\hat{\tau}_{Q_1}^R(H_{Q_1}(\delta x) - T)$$
$$\cdot (\mathrm{Dir})_R^Q(H_R(\delta x) - T)\, dn.$$

The result is

$$\mathrm{vol}(\mathfrak{a}_R^Q/L_R^Q) \sum_{\delta \in R(\mathbf{Q})\backslash Q(\mathbf{Q})} (\Lambda^{T,R}h)(\delta x) \cdot (\mathrm{Dir})_R^Q(H_R(\delta x) - T).$$

This suggests a simple formula for $D_{Q|R,T}(\Lambda^{T,Q}h, h')_{Q,T}$. It is just the formula we want; however, we need to make our argument rigorous.

First we should be more precise about the functions h and h'. Suppose that N is a positive integer. If h is a smooth function on $G(\mathbf{A})$ and $u \in \mathfrak{U}(G(\mathbf{R}))$, let

$$\|h\|_{u,N} = \sup_{x \in G(\mathbf{A})} |(uh)(x)|\, \|x\|^{-N},$$

where uh is the left invariant derivative of h with respect to u. Define $SI(N_Q(\mathbf{A})M_Q(\mathbf{Q})\backslash G(\mathbf{A}), N)$ to be the space of all smooth functions h on $N_Q(\mathbf{A})M_Q(\mathbf{Q})\backslash G(\mathbf{A})$ such that the seminorm $\|h\|_{u,N}$ is finite for each u in $\mathfrak{U}(G(\mathbf{R}))$. Let $SI(N_Q(\mathbf{A})M_Q(\mathbf{Q})\backslash G(\mathbf{A}))$ denote the union over all N of these spaces. We shall take our functions h and h' from this latter space. Lemma 1.4 of [1(b)], applied to the group M_Q, then tells us that $\Lambda^{T,Q}h$ is rapidly decreasing on any Siegel set in $M_Q(\mathbf{A})^1$. In particular, the inner product $(\Lambda^{T,Q}h, h')_{Q,T}$ is defined by an absolutely convergent integral. We would like to show that it is a smooth function of T.

We shall let U be another point in $\mathfrak{a}_0$ and study $(\Lambda^{T+U,Q}h)(x)$ as a function of U. If we constrain U to lie in a small neighborhood of the origin in $\mathfrak{a}_0$, $T+U$ will remain a suitably regular point in $\mathfrak{a}_0^+$. We will be led through a discussion which parallels that of [1(c), §2]. In particular, if $Q_1 \subset Q$, we define a function

$$\Gamma_{Q_1}^Q(H, U), \qquad U, H \in \mathfrak{a}_0,$$

inductively on $\dim(A_{Q_1}/A_Q)$ by demanding that

$$\hat{\tau}_{Q_1}^Q(H-U) = \sum_{\{R_1 : Q_1 \subset R_1 \subset Q\}} (-1)^{\dim(A_{R_1}/A_Q)} \hat{\tau}_{Q_1}^{R_1}(H) \Gamma_{R_1}^Q(H, U),$$

for all $Q_1 \subset Q$. (In §2 of [1(c)] we gave this definition, but only with $Q = G$. We denoted the function $\Gamma'_{Q_1}(H, U)$ instead of $\Gamma_{Q_1}^G(H, U)$. The properties established for $\Gamma'_{Q_1}(H, U)$ will all hold for $\Gamma_{Q_1}^Q(H, X)$; we would have only to replace G in any verification by the group M_{Q_1}.) The functions $\Gamma_{Q_1}^Q(H, U)$ depend only on the projections of H and U onto $\mathfrak{a}_{Q_1}^Q$.

LEMMA 5.2. *The function* $(\Lambda^{T+U,Q}h)(x)$ *equals*

$$\sum_{\{R_1 : P_0 \subset R_1 \subset Q\}} \sum_{\delta \in R_1(\mathbf{Q}) \backslash Q(\mathbf{Q})} (\Lambda^{T,R_1}h)(\delta x) \cdot \Gamma_{R_1}^Q(H_{R_1}(\delta x) - T, U).$$

Proof. In the defining formula for $\Lambda^{T+U,Q}h$, express

$$\hat{\tau}_{Q_1}^Q(H_{Q_1}(\delta x) - (T+U)) = \hat{\tau}_{Q_1}^Q((H_{Q_1}(\delta x) - T) - U)$$

in terms of the functions $\Gamma_{Q_1}^Q$. Then $(\Lambda^{T+U,Q}h)(x)$ becomes the sum over $\{Q_1, R_1 : P_0 \subset Q_1 \subset R_1 \subset Q\}$ and $\delta \in Q_1(\mathbf{Q}) \backslash Q(\mathbf{Q})$ of the product of $\Gamma_{R_1}^Q(H_{R_1}(\delta x) - T, U)$ with

$$(-1)^{\dim(A_{Q_1}/A_{R_1})} \hat{\tau}_{Q_1}^{R_1}(H_{Q_1}(\delta x) - T) \int_{N_{Q_1}(\mathbf{Q}) \backslash N_{Q_1}(\mathbf{A})} h(n \delta x)\, dn.$$

We obtain the required formula by replacing the sum over $Q_1(\mathbf{Q}) \backslash Q(\mathbf{Q})$ by a double sum over $Q_1(\mathbf{Q}) \backslash R_1(\mathbf{Q})$ and $R_1(\mathbf{Q}) \backslash Q(\mathbf{Q})$. $\square$

Lemma 2.1 of [1(c)] tells us that the support of the function

$$H \to \Gamma_{R_1}^Q(H, U), \qquad H \in \mathfrak{a}_{R_1}^Q,$$

is contained in a compact subset of $\mathfrak{a}_{R_1}^Q$. As long as U remains bounded, this subset can be taken to be independent of U. It follows that we may evaluate $(\Lambda^{T+U,Q}h, h')_{Q,T}$ by the last lemma, taking the double integral (in the definition of the inner product) inside the sum over R_1. Since $R_1(\mathbf{Q}) \backslash Q(\mathbf{Q})$ equals

$R_1(\mathrm{Q}) \cap M_Q(\mathrm{Q})\backslash M_Q(\mathrm{Q})$, $(\Lambda^{T+U,Q}h, h')_{Q,T}$ equals the sum over $\{R_1 : P_0 \subset R_1 \subset Q\}$ of

$$\int_K \int_{R_1(\mathrm{Q}) \cap M_Q(\mathrm{Q})\backslash M_Q(\mathrm{A}, T_Q)} (\Lambda^{T,R_1}h)(mk)\overline{h'(mk)}$$

$$\cdot \Gamma^Q_{R_1}(H_{R_1}(m) - T, U)\, dm\, dk. \tag{5.2}$$

If $H \in \mathfrak{a}_0$, let $a(H)$ denote the unique element in $A_0(\mathrm{R})^0$ such that $H_{P_0}(a(H)) = H$. Also, set

$$h'_{R_1}(x) = \int_{N_{R_1}(\mathrm{Q})\backslash N_{R_1}(\mathrm{A})} h'(nx)\, dx.$$

Then (5.2) equals the integral over $k \in K$, $m \in M_{R_1}(\mathrm{Q})\backslash M_{R_1}(\mathrm{A})^1$ and $H \in \mathfrak{a}^Q_{R_1}$ of

$$(\Lambda^{T,R_1}h)(ma(H + T_Q)k)\overline{h'_{R_1}(ma(H + T_Q)k)}\,\Gamma^Q_{R_1}(H - T, U)\exp\{-2\rho^Q_{R_1}(H)\},$$

where $\rho^Q_{R_1}$ is the usual vector in $\mathfrak{a}^*_{R_1}$ such that

$$p \to \exp\{-2\rho^Q_{R_1}(H_{R_1}(p))\}, \qquad p \in R_1(\mathrm{A}) \cap M_Q(\mathrm{A}),$$

is the modular function of $R_1(\mathrm{A}) \cap M_Q(\mathrm{A})$. The absolute convergence of this triple integral follows from Lemma 1.4 of [1(b)] (applied to the group M_{R_1}) and the compactness of the support of $\Gamma^Q_{R_1}(\cdot, U)$.

We would like to show that (5.2) is a smooth function of U. To do so we use the inversion formula

$$\Gamma^Q_{R_1}(H, U) = \sum_{\{R_2 : R_1 \subset R_2 \subset Q\}} (-1)^{\dim(A_{R_2}/A_Q)} \tau^{R_2}_{R_1}(H)\hat{\tau}^Q_{R_2}(H - U),$$

a simple consequence of the definition of $\Gamma^Q_{R_1}$. (See [1(c), §2].) Suppose that ψ is any smooth function on $\mathfrak{a}^Q_{R_1}$. Let ψ_0 be a smooth, compactly supported function on $\mathfrak{a}^Q_{R_1}$ which equals ψ on a large ball about the origin in $\mathfrak{a}^Q_{R_1}$. Then

$$\int_{\mathfrak{a}^Q_{R_1}} \psi(H)\Gamma^Q_{R_1}(H - T, U)\, dH = \int_{\mathfrak{a}^Q_{R_1}} \psi_0(H)\Gamma^Q_{R_1}(H - T, U)\, dH,$$

which in turn equals

$$\sum_{\{R_2 : R_1 \subset R_2 \subset Q\}} (-1)^{\dim(A_{R_2}/A_Q)} \int_{\mathfrak{a}^Q_{R_2}} \psi_0(H)\tau^{R_2}_{R_1}(H - T)\hat{\tau}^Q_{R_2}(H - T - U)\, dH. \tag{5.3}$$

The smoothness of this function of U follows from the fundamental theorem of calculus. If $D_{Q|R,U}$ is allowed to act on (5.3), the value at $U = 0$ of the resulting function will be given by Lemma 3.1. It will be zero if R_1 is not contained in R. If R_1 is contained in R, the value is given by taking only those R_2 in the sum in

(5.3) which are contained in R, and replacing $\hat{\tau}^Q_{R_2}(H-T-U)$ by

$$(-1)^{\dim(A_R/A_Q)}\mathrm{vol}(\mathfrak{a}^Q_R/L^Q_R)\hat{\tau}^R_{R_2}(H-T)(\mathrm{Dir})^Q_R(H-T).$$

(We are writing the Dirac distribution as if it were a function.) Now

$$\sum_{\{R_2:R_1\subset R_2\subset R\}}(-1)^{\dim(A_{R_2}/A_R)}\tau^{R_2}_{R_1}(H-T)\hat{\tau}^R_{R_2}(H-T)$$

vanishes if $R_1\neq R$ and of course equals 1 if $R_1=R$. (See the remark in [1(a)] following Corollary 6.2.) Let T^Q_R be the projection of T onto $\mathfrak{a}^Q_R$. Then the required value at $U=0$ will be zero if $R_1\neq R$, and will be

$$\mathrm{vol}(\mathfrak{a}^Q_R/L^Q_R)\psi_0(T^Q_R)=\mathrm{vol}(\mathfrak{a}^Q_R/L^Q_R)\psi(T^Q_R)$$

if $R_1=R$.

Thus (5.2) is a smooth function of U. The same is therefore true of $(\Lambda^{T+U,Q}h,h')_{Q,T}$. Moreover the value at $U=0$ of $D_{Q\mid R,U}(\Lambda^{T+U,Q}h,h')_{Q,T}$, which of course is the same as $D_{Q\mid R,T}(\Lambda^{T,Q}h,h')_{Q,T}$, equals the product of $\mathrm{vol}(\mathfrak{a}^Q_R/L^Q_R)$ with

$$\int_K\int_{M_R(\mathbf{Q})\backslash M_R(\mathbf{A})^1}(\Lambda^{T,R}h)\big(ma(T^Q_R+T_Q)k\big)$$
$$\cdot\overline{h'_{R_1}\big(ma(T^Q_R+T_Q)k\big)}\exp\{-2\rho^Q_R(T^Q_R)\}\,dm\,dk.$$

This double integral is just

$$\exp\{-2\rho^Q_R(T)\}(\Lambda^{T,R}h,h'_R)_{R,T}.$$

We have essentially proved

LEMMA 5.3. *If h and h' are functions in $SI(N_Q(\mathbf{A})M_Q(\mathbf{Q})\backslash G(\mathbf{A}))$, the inner product $(\Lambda^{T,Q}h,\Lambda^{T,Q}h')_{Q,T}$ is a smooth function of T, and*

$$D_{Q\mid R,T}(\Lambda^{T,Q}h,\Lambda^{T,Q}h')_{Q,T}=\mathrm{vol}(\mathfrak{a}^Q_R/L^Q_R)\exp\{-2\rho^Q_R(T)\}(\Lambda^{T,R}h,\Lambda^{T,R}h')_{R,T}.$$

Proof. We have shown that $(\Lambda^{T,Q}h,h')_{Q,T}$ is smooth and that

$$D_{Q\mid R,T}(\Lambda^{T,Q}h,h')_{Q,T}=\mathrm{vol}(\mathfrak{a}^Q_R/L^Q_R)\exp\{-2\rho^Q_R(T)\}(\Lambda^{T,R}h,h'_R)_{R,T}.$$

Corollary 1.2 and Lemma 1.3 of [1(b)] imply that

$$(\Lambda^{T,Q}h,h')_{Q,T}=(\Lambda^{T,Q}h,\Lambda^{T,Q}h')_{Q,T}$$

and

$$(\Lambda^{T,R}h,h'_R)_{R,T}=(\Lambda^{T,R}h,\Lambda^{T,R}h'_R)_{R,T}.$$

The lemma follows from the fact that $\Lambda^{T,R}h'_R=\Lambda^{T,R}h'$. $\square$

§6. The constant terms of Eisenstein series. The formula of Lemma 5.3 becomes more concrete if we specialize to $h = E^Q(\phi,\lambda)$ and $h' = E^Q(\phi', -\bar{\lambda}')$. The reason is that these functions are actually automorphic forms. Let us write simply $\mathcal{Q}(Q)$ for the subspace of functions h in $SI(N_Q(\mathbf{A})M_Q(\mathbf{Q})\backslash G(\mathbf{A}))$ which are $(\mathfrak{Z}, K)$ finite; that is, such that the span of the set of functions

$$x \to (zh)(xk), \qquad x \in G(\mathbf{A}),$$

indexed by $k \in K$ and $z \in \mathfrak{Z}$, is finite dimensional. Any such function is also $A_Q(\mathbf{R})^0$ finite; the space spanned by the set of functions

$$x \to h(ax), \qquad x \in G(\mathbf{A}),$$

indexed by $a \in A_Q(\mathbf{R})^0$, is finite dimensional [2(b), Chapter 4]. It is well known that the functions $E^Q(\phi,\lambda)$ and $E^Q(\phi', -\bar{\lambda}')$ belong to $\mathcal{Q}(Q)$.

Fix a parabolic subgroup $R \subset Q$ and let h be a function in $\mathcal{Q}(Q)$. Then the function

$$h_R(x) = \int_{N_R(\mathbf{Q})\backslash N_R(\mathbf{A})} h(nx)\,dn, \qquad x \in G(\mathbf{A}),$$

belongs to $\mathcal{Q}(R)$. In particular, h_R is $A_R(\mathbf{R})^0$ finite. It follows that

$$h_R(x) = \sum_{i=1}^{n} \exp\{(\Lambda_i + \rho_R^Q)(H_R(x))\} h_i(x), \tag{6.1}$$

where $\{\Lambda_i\}$ is a set of distinct points in $\mathfrak{a}_{R,\mathbf{C}}^*$ and each h_i is a nonzero function in $\mathcal{Q}(R)$ such that

$$h_i(ax), \qquad a \in A_R(\mathbf{R})^0,$$

is a polynomial in $H_R(a)$. (See [2(b), Lemma 4.2].) It is an immediate consequence of the definition of the partial truncation operator that $\Lambda^{T,R}h = \Lambda^{T,R}h_R$. Therefore if h' is a second function in $\mathcal{Q}(Q)$, and

$$h'_R(x) = \sum_{j=1}^{n'} \exp\{(\Lambda'_j + \rho_R^Q)(H_R(x))\} h'_j(x)$$

is the corresponding decomposition (6.1), the formula of Lemma 5.3 becomes

$$D_{Q|R,T}(\Lambda^{T,Q}h, \Lambda^{T,Q}h')_{T,Q} = \mathrm{vol}(\mathfrak{a}_R^Q/L_R^Q)\sum_{i,j} \exp\{(\Lambda_i + \bar{\Lambda}_j)(T_R)\} \times \left(\Lambda^{T,R}h_i, \Lambda^{T,R}h'_j\right)_{R,T}. \tag{6.2}$$

Remember that T_R is the projection of T onto $\mathfrak{a}_R$. Let T_0^R be the projection of T onto $\mathfrak{a}_{P_0}^R$. Then $T = T_R + T_0^R$. The partial truncation operator $\Lambda^{T,R}$ depends only on T_0^R. Therefore

$$\left(\Lambda^{T,R}h_i, \Lambda^{T,R}h'_j\right)_{R,T}$$

is a polynomial in T_R. Thus, as a function of T_R, the right hand side of (6.2) is rather transparent.

We shall presently use (6.2) to extract information about the exponents $\mathcal{E}^Q(t,t')$. We must first prove

LEMMA 6.1. *Let*

$$h = E^Q(\phi,\lambda),$$

where $\phi \in \mathcal{A}_{P,\chi,\Gamma}$ *and* λ *is a point in general position in* $i\mathfrak{a}_P^*$. *Suppose that* $h_R \neq 0$, *and let* Λ_i *be any of the points in the decomposition* (6.1). *Then*

$$\mathrm{Re}\big(\Lambda_i(\varpi^\vee)\big) \leqslant 0$$

for each $\varpi \in \hat{\Delta}_R^Q$.

Proof. Let $\mathcal{A}_{\mathrm{cusp}}(Q)$ be the space of functions g in $\mathcal{A}(Q)$ such that $g_{Q_1} = 0$ for every parabolic subgroup Q_1 which is strictly contained in Q. Recall that a function $f \in \mathcal{A}(Q)$ is said to have *cuspidal component zero* if

$$(f, g)_{Q,T} = 0$$

for all T and all $g \in \mathcal{A}_{\mathrm{cusp}}(Q)$. A basic result (Lemma 3.7 of [2(b)]) asserts that if the cuspidal component of f_{Q_1} is zero for every $Q_1 \subset Q$ then f itself is zero.

Now

$$h(x) = E^Q(x,\phi,\lambda)$$

is a sum of residues of cuspidal Eisenstein series

$$f(x) = E^Q(x, F_B(\Lambda), \Lambda).$$

If $Q_1 \subset Q$, the cuspidal component of f_{Q_1} is zero unless Q_1 belongs to $\mathcal{P}_\chi$. The same is therefore true of h. If Q_1 is contained in R, we have

$$h_{Q_1}(x) = (h_R)_{Q_1}(x) = \sum_{i=1}^{n} \exp\big\{(\Lambda_i + \rho_R^Q)(H_R(x))\big\}(h_i)_{Q_1}(x).$$

The exponents $\{\Lambda_i + \rho_R^Q\}$ are all distinct. Therefore the cuspidal component of each function $(h_i)_{Q_1}$ will also be zero unless $Q_1 \in \mathcal{P}_\chi$. Fix i_0, $1 \leqslant i_0 \leqslant n$. The function h_{i_0} is not zero so there must be a group $B_1 \in \mathcal{P}_\chi$ which is contained in R, such that $(h_{i_0})_{B_1} \neq 0$. Now for each i we have

$$(h_i)_{B_1}(x) = \sum_l \exp\big\{\big(\mu_{il} + \rho_{B_1}^R\big)\big(H_{B_1}(x)\big)\big\}\, g_{il}(x),$$

for a set $\{\mu_{il} : 1 \leqslant l \leqslant n_i\}$ of distinct points in $(\mathfrak{a}_{B_1}^R)_{\mathbb{C}}^*$ and a set $\{g_{il}\}$ of nonzero functions in $\mathcal{A}(B_1)$ such that

$$g_{il}(ax), \qquad a \in A_{B_1}(\mathbf{R})^0,$$

is a polynomial in $H_{B_1}(a)$. Since $\rho_{B_1}^R + \rho_R^Q = \rho_{B_1}^Q$, the function $h_{B_1}(x)$ equals

$$\sum_i \sum_l \exp\left\{\left(\Lambda_i + \mu_{il} + \rho_{B_1}^Q\right)\left(H_{B_1}(x)\right)\right\} g_{il}(x). \tag{6.3}$$

Keep in mind that if $i = i_0$, the inner sum over l is not zero; that is, the set $\{\mu_{i_0 l}\}$ is not empty.

On the other hand, there is a general prescription from [2(b)] for writing $h_{B_1}(x)$. As in (2.3) we can write $E^Q(x, \phi, \lambda)$ as

$$\sum_B \sum_{\mathfrak{t}} E_{\mathfrak{t}}^Q(x, \Phi_{\mathfrak{t}}, \Lambda_{\mathfrak{t}} + \lambda).$$

According to formula (7.b) of [2(b)],

$$\int_{N_{B_1}(\mathbf{Q})\backslash N_{B_1}(A)} E_{\mathfrak{t}}^Q(nx, \Phi_{\mathfrak{t}}, \Lambda_{\mathfrak{t}} + \lambda)\, dn$$

equals

$$\sum_s \exp\left\{\left(s(\Lambda_{\mathfrak{t}} + \lambda) + \rho_{B_1}^Q\right)\left(H_{B_1}(x)\right)\right\}\left(N(s, \Lambda_{\mathfrak{t}} + \lambda)\Phi_{\mathfrak{t}}\right)(x), \tag{6.4}$$

where s ranges over the set of maps obtained by restricting elements in $W^Q(\mathfrak{a}_B, \mathfrak{a}_{B_1})$ to $\mathfrak{t}$. We needn't say anything about $(N(s, \Lambda_{\mathfrak{t}} + \lambda)\Phi_{\mathfrak{t}})(x)$ beyond noting that it is a function in $\mathcal{Q}(B_1)$ which is a polynomial in $H_{B_1}(x)$. Let s be an element occurring in the sum above. Let $\mathfrak{b}$ be the set of points $\Lambda \in \mathfrak{a}_0^*$ such that $\Lambda(\varpi^\vee) = 0$ for every ϖ in the intersection of $\hat{\Delta}_0^Q$ and $s\mathfrak{a}_P^*$. In the terminology of [2(b)], $\mathfrak{b}$ is the orthogonal complement of the distinguished subspace of $s\mathfrak{a}_P^*$ of largest dimension. Let $^+\mathfrak{b}$ be the set of points $\Lambda \in \mathfrak{b}$ such that $\Lambda(\varpi^\vee)$ is positive for every ϖ in the complement of $s\mathfrak{a}_P^*$ in $\hat{\Delta}_0^Q$. Now suppose that the function $(N(s, \Lambda_{\mathfrak{t}} + \lambda)\Phi_{\mathfrak{t}})(x)$ does not vanish. Then Lemma 7.5 of [2(b)] asserts that $(-s\mathfrak{t})$ belongs to a certain collection of canonical affine subspaces of $\mathfrak{a}_{B_1}^*$, a collection which by Theorem 7.1 of [2(b)] satisfies the geometric assumptions of the just quoted Lemma 7.5. The upshot is that the point $(-s\Lambda_{\mathfrak{t}})$ belongs to $^+\mathfrak{b}$.

Let us group together all those terms in the sum over B and $\mathfrak{t}$ of (6.4) for which the restrictions of s to $\mathfrak{a}_P$ are equal. The result is a sum over all maps

$$t : \mathfrak{a}_P \to \mathfrak{a}_{B_1}$$

which can be obtained by restricting elements in any of the sets

$$W^Q(\mathfrak{a}_B, \mathfrak{a}_{B_1}), \qquad B \in \mathcal{P}_X, \quad B \subset P,$$

to $\mathfrak{a}_P$. For every such t we obtain a finite set $\mathcal{E}^Q(t)$ of points in $\mathfrak{a}_{B_1}^*$ and for each $\xi \in \mathcal{E}^Q(t)$ a nonzero function

$$x \to q_\xi^Q(x, \phi, \lambda)$$

in $\mathcal{Q}(B_1)$, which is a polynomial in $H_{B_1}(x)$, such that

$$\int_{N_{B_1}(\mathbf{Q})\backslash N_{B_1}(\mathbf{A})} E^Q(nx,\phi,\lambda)\,dn$$

equals

$$\sum_t \sum_{\xi\in\mathfrak{S}^Q(t)} \exp\left\{\left(t\lambda+\xi+\rho_{B_1}^Q\right)\left(H_{B_1}(x)\right)\right\} q_\xi^Q(x,\phi,\lambda). \tag{6.5}$$

The points in $\mathfrak{S}^Q(t)$ are all orthogonal to $t\mathfrak{a}_P^*$. Any $\xi\in\mathfrak{S}^Q(t)$ has the property that $\xi(\varpi^\vee)<0$ for all ϖ in the complement of $t\mathfrak{a}_P^*$ in $\hat{\Delta}_0^Q$.

The expressions (6.3) and (6.5) are equal. Comparing exponents we see that for every l there is a t and ξ such that

$$\Lambda_{i_0}+\mu_{i_0l}=t\lambda+\xi.$$

Suppose that $\varpi\in\hat{\Delta}_R^Q$. Then

$$\begin{aligned}\operatorname{Re}\left(\Lambda_{i_0}(\varpi^\vee)\right) &= \operatorname{Re}\left((\Lambda_{i_0}+\mu_{i_0l})(\varpi^\vee)\right)\\ &= \operatorname{Re}\left((t\lambda+\xi)(\varpi^\vee)\right)\\ &= \xi(\varpi^\vee),\end{aligned}$$

since μ_{i_0l} vanishes on $\mathfrak{a}_R^Q$ and λ is purely imaginary. Now $\hat{\Delta}_R^Q$ is a subset of $\hat{\Delta}_0^Q$, so ϖ belongs to either $\hat{\Delta}_0^Q\cap t\mathfrak{a}_P^*$ or $\hat{\Delta}_0^Q\backslash t\mathfrak{a}_P^*$. In the first instance, $\xi(\varpi^\vee)=0$, and in the second, $\xi(\varpi^\vee)<0$. This proves the lemma. □

If the $\dim\mathfrak{a}_R>\dim\mathfrak{a}_P$, the set $\hat{\Delta}_R^Q$ will contain more elements than $\hat{\Delta}_0^Q\cap t\mathfrak{a}_P^*$. It will have to contain an element which does not lie in $t\mathfrak{a}_P^*$. A similar statement holds if $\dim\mathfrak{a}_R>\dim\mathfrak{a}_{P'}$. From the last part of the proof of the lemma we obtain

COROLLARY 6.2. *Suppose in addition to the assumptions of the lemma that* $\dim\mathfrak{a}_R>\min\{\dim\mathfrak{a}_P,\dim\mathfrak{a}_{P'}\}$. *Then there is an element* ϖ *in* $\hat{\Delta}_R^Q$ *such that* $\operatorname{Re}(\Lambda_i(\varpi^\vee))$ *is strictly negative.* □

§7. The negative dual chamber. Our goal in this section is to show that all the exponents in $\mathfrak{S}^Q(t,t')$ lie in the closure of the negative dual chamber in $(\mathfrak{a}_0^Q)^*$. Otherwise said, each exponent will be a linear combination, with nonpositive coefficients, of the roots Δ_0^Q.

Set

$$h(x)=E^Q(x,\phi,\lambda),\qquad \phi\in\mathcal{Q}_{P,\chi,\Gamma},\quad \lambda\in i\mathfrak{a}_P^*,$$

and

$$h'(x)=E^Q(x,\phi',\lambda'),\qquad \phi'\in\mathcal{Q}_{P',\chi,\Gamma},\quad \lambda'\in i\mathfrak{a}_{P'}^*.$$

Take $R \subset Q$ and consider the function

$$D_{Q\,|\,R,T}\,(\Lambda^{T,Q}h, \Lambda^{T,Q}h')_{Q,T}\,.$$

It appears on the one hand in formula (6.2). However, it also equals

$$D_{Q\,|\,R,T}\Omega^{T,Q}(\lambda, \lambda', \phi, \phi'),$$

which is in turn equal to the sum over (t,t') in $W^Q(\mathfrak{a}_P \oplus \mathfrak{a}_{P'}, \mathcal{P}_\chi)$ and X in $\mathcal{E}^Q(t,t')$ of

$$D_{Q\,|\,R,T}\big(p^{T,Q}(\lambda,\lambda',\phi,\phi')\exp\{(t\lambda - t'\lambda' + X)(T)\}\big).$$

As in the special case treated in Corollary 4.2, we write this last expression as the product of $\exp\{(t\lambda - t'\lambda' + X)(T)\}$ with another function $\tilde{p}_X^{T,Q}(\lambda,\lambda',\phi,\phi')$, which is also a polynomial in T. This second polynomial would conceivably vanish. In any case its total degree is at most that of $p_X^{T,Q}(\lambda,\lambda',\phi,\phi')$. The total degrees will be equal if and only if the expression

$$\prod_{\alpha \in \Delta_0^Q \setminus \Delta_0^R} (t\lambda - t'\lambda' + X)(\alpha^\vee) \tag{7.1}$$

does not vanish. In particular, if (7.1) does not vanish, $\tilde{p}_X^{T,Q}(\lambda,\lambda',\phi,\phi')$ is not zero.

Thus for any $R \subset Q$, we have an equality between the function

$$\sum_{(t,t')}\ \sum_{X \in \mathcal{E}^Q(t,t')} \tilde{p}_X^{T,Q}(\lambda,\lambda',\phi,\phi')\exp\{(t\lambda - t'\lambda' + X)(T)\}$$

and

$$\mathrm{vol}(\mathfrak{a}_R^Q/L_R^Q)\sum_{i,j}\exp\big\{(\Lambda_i + \overline{\Lambda}_j)(T_R)\big\}\big(\Lambda^{T,R}h_i, \Lambda^{T,R}h_j\big)_{R,T},$$

the right hand side of (6.2). Consider these expressions as functions of T_R. Suppose that X is any exponent in $\mathcal{E}^Q(t,t')$ for which (7.1) does not vanish. Then there will be an i and j such that the projection of $t\lambda - t'\lambda' + X$ onto $\mathfrak{a}_{R,\mathbf{C}}^*$ equals $\Lambda_i + \overline{\Lambda}_j$. (See the remarks preceeding Lemma 6.1.) If ϖ is any element in $\hat{\Delta}_R^Q$, $X(\varpi^\vee)$ equals

$$\mathrm{Re}\big(\Lambda_i(\varpi^\vee)\big) + \mathrm{Re}\big(\Lambda_j(\varpi^\vee)\big),$$

since λ and λ' are purely imaginary. Lemma 6.1 tells us that $X(\varpi^\vee) \leqslant 0$. If $\dim \mathfrak{a}_R > \min\{\dim \mathfrak{a}_P, \dim \mathfrak{a}_{P'}\}$ its corollary affirms that there is a $\varpi \in \hat{\Delta}_R^Q$ such that $X(\varpi^\vee) < 0$.

Lemma 7.1. *Suppose that X is an element in $\mathcal{E}^Q(t,t')$, where (t,t') is in $W^Q(\mathfrak{a}_P \oplus \mathfrak{a}_{P'}, \mathcal{P}_\chi)$. Then $X(\varpi^\vee) \leqslant 0$ for each $\varpi \in \hat{\Delta}_0^Q$.*

Proof. If $\alpha \in \Delta_0^Q$, let ϖ_α be the vector in $\hat{\Delta}_0^Q$ which is dual to $\alpha^\vee$. Set $c_\alpha = X(\varpi_\alpha^\vee)$. We must show that no c_α is positive. Let

$$\Delta_1 = \left\{\alpha_1 \in \Delta_0^Q : c_{\alpha_1} > 0\right\}$$

and

$$\Delta_2 = \left\{\alpha_2 \in \Delta_0^Q : c_{\alpha_2} < 0\right\}.$$

Then

$$X = \sum_{\alpha_1 \in \Delta_1} c_{\alpha_1}\alpha_1 + \sum_{\alpha_2 \in \Delta_2} c_{\alpha_2}\alpha_2.$$

We shall assume that the lemma is false. Then the set Δ_1 is not empty.

Suppose that

$$\Big(\sum_{\alpha_1 \in \Delta_1} c_{\alpha_1}\alpha_1\Big)(\alpha^\vee) \leqslant 0$$

for each $\alpha \in \Delta_1$; that is, the vector $\sum_{\alpha_1 \in \Delta_1} c_{\alpha_1}\alpha_1$ lies in the closure of the negative chamber in the space spanned by Δ_1. It is well known that the negative chamber is contained in the negative dual chamber. This means that

$$c_\alpha = \Big(\sum_{\alpha_1 \in \Delta_1} c_{\alpha_1}\alpha_1\Big)(\varpi_\alpha^\vee) \leqslant 0,$$

which is a contradiction. Thus there is an $\alpha_1 \in \Delta_1$ such that $(\sum_{\alpha_1 \in \Delta_1} c_{\alpha_1}\alpha_1)(\alpha^\vee)$ is positive. Now $\alpha_2(\alpha^\vee) \leqslant 0$ for each $\alpha_2 \in \Delta_2$. Since each $c_{\alpha_2} \leqslant 0$, the number $(\sum_{\alpha_2 \in \Delta_2} c_{\alpha_2}\alpha_2)(\alpha^\vee)$ is nonnegative. It follows that $X(\alpha^\vee)$ is positive. (We will only need to use the fact that $X(\alpha^\vee)$ is nonzero.)

Define a parabolic subgroup $R \subset Q$ by letting Δ_0^R be the complement of α in Δ_0^Q. Since $X(\alpha^\vee)$ is a nonzero real number and λ and λ' are purely imaginary, the expression (7.1) does not vanish. Moreover, ϖ_α lies in $\mathfrak{a}_R^Q$; it is, in fact, the unique element in $\hat{\Delta}_R^Q$. According to the discussion preceeding the lemma, $X(\varpi_\alpha^\vee) \leqslant 0$. This is a contradiction, since

$$X(\varpi_\alpha^\vee) = c_\alpha,$$

a positive number. □

A natural question is suggested by this last lemma. Can there be any exponent in $\mathfrak{S}^Q(t,t')$ which is zero? This is possible, it turns out, only in the situation discussed in §4.

LEMMA 7.2. *Suppose that for $(t,t') \in W^Q(\mathfrak{a}_P \oplus \mathfrak{a}_{P'}, \mathcal{P}_\chi)$ as in the last lemma, there is no parabolic subgroup P_1 associated to both P and P' such that $t \in W^Q(\mathfrak{a}_P, \mathfrak{a}_{P_1})$ and $t' \in W^Q(\mathfrak{a}_{P'}, \mathfrak{a}_{P_1})$. Then no exponent in $\mathfrak{S}^Q(t,t')$ is zero.*

Proof. Let R be the parabolic subgroup of largest dimension which is contained in Q and such that $\mathfrak{a}_R$ contains both $t\mathfrak{a}_P$ and $t'\mathfrak{a}_{P'}$. Then Δ_0^R is the set of roots in Δ_0^Q that vanish on both $t\mathfrak{a}_P$ and $t'\mathfrak{a}_{P'}$. Our hypothesis on t and t' is precisely that $\dim \mathfrak{a}_R > \min\{\dim \mathfrak{a}_P, \dim \mathfrak{a}_{P'}\}$. Fix $X \in \mathfrak{S}^Q(t,t')$. We can choose points $\lambda \in i\mathfrak{a}_P^*$ and $\lambda' \in \mathfrak{a}_{P'}^*$ such that for each α in the complement of Δ_0^R in Δ_0^Q,

$$(t\lambda - t'\lambda' + X)(\alpha^\vee) \neq 0.$$

In other words, the expression (7.1) does not vanish. According to the remark preceeding the last lemma, there is a $\varpi \in \Delta_R^Q$ such that $X(\varpi^\vee) < 0$. In particular, X is not zero. □

§8. Coefficients of the zero exponents. We want to consider the consequences of letting T approach infinity. We shall say that T approaches infinity *strongly in* $\mathfrak{a}_0^+$ if there is a $\delta > 0$ such that as $\|T\|$ approaches infinity,

$$\alpha(T) \geqslant \delta\|T\|,$$

for each $\alpha \in \Delta_0$. Any $T \in \mathfrak{a}_0$ can be written

$$T = \sum_{\alpha \in \Delta_0} r_\alpha \varpi_\alpha^\vee, \tag{8.1}$$

where for each $\alpha \in \Delta_0$, r_α is a real number and ϖ_α is the vector in $\hat{\Delta}_0$ which is dual to $\alpha^\vee$. If T approaches infinity strongly in $\mathfrak{a}_0^+$, each r_α will be greater than $\delta\|T\|$. The contributions to $\Omega^{T,Q}(\lambda,\lambda',\phi,\phi')$ from the nonzero exponents will all be negligible. Thus, to have a precise asymptotic formula for $\Omega^{T,Q}(\lambda,\lambda',\phi,\phi')$ we need only calculate the coefficients of any zero exponent. According to Lemma 7.2, the set

$$\mathfrak{S}^Q(t,t'), \qquad (t,t') \in W^Q(\mathfrak{a}_P \oplus \mathfrak{a}_{P'}, \mathcal{P}_X),$$

contains a zero exponent only if P and P' belong to the same associated class $\mathcal{P}$, and (t,t') is an element in $W^Q(\mathfrak{a}_P \oplus \mathfrak{a}_{P'}, \mathcal{P})$. For the rest of §8 we shall assume that this is so. That is, $t \in W^Q(\mathfrak{a}_P, \mathfrak{a}_{P_1})$ and $t' \in W^Q(\mathfrak{a}_{P'}, \mathfrak{a}_{P_1})$ for some group $P_1 \in \mathcal{P}$.

At this point we shall make a minor change in our notation. For t and t' as above, let us enlarge $\mathfrak{S}^Q(t,t')$ by adding the point 0, with the understanding that the coefficient $p_0^{T,Q}(\lambda,\lambda',\phi,\phi')$ might vanish. We would like to calculate it. With Corollary 4.2 in mind, we first consider the special case in which $Q = P_1$. In this case we also have $P = P' = Q$ and $(t,t') = (1,1)$.

LEMMA 8.1. *Suppose that $P = P' = P_1 = Q$. Then*

$$p_0^{T,Q}(\lambda,\lambda',\phi,\phi') = (\phi,\phi'),$$

the inner product on the right, of course, being that of the finite dimensional Hilbert space $\mathcal{Q}_{P,\chi,\Gamma} = \mathcal{Q}_{P',\chi,\Gamma}$.

Proof. Since $P = P' = Q$, the sets $W^Q(\mathfrak{a}_P \oplus \mathfrak{a}_{P'}, \mathfrak{P}_\chi)$ and $W^Q(\mathfrak{a}_P \oplus \mathfrak{a}_{P'}, \mathfrak{P})$ are equal; they both consist of the single pair $(1, 1)$. Therefore

$$\Omega^{T,Q}(\lambda, \lambda', \phi, \phi') = \sum_{X \in \mathcal{E}^Q(1,1)} p_X^{T,Q}(\lambda, \lambda', \phi, \phi') \exp\{(\lambda - \lambda' + X)(T)\}.$$

As we know, this equals

$$\int_K \int_{M_Q(\mathbf{Q}) \backslash M_Q(\mathbf{A}, T_Q)} \Lambda^{T,Q} E^Q(mk, \phi, \lambda) \overline{\Lambda^{T,Q} E^Q(mk, \phi', -\bar{\lambda}')}\, dm\, dk.$$

Now if $k \in K$ and $m \in M_P(\mathbf{A}, T_P)$,

$$\begin{aligned} \Lambda^{T,P} E^P(mk, \phi, \lambda) &= \Lambda^{T,P} \phi(mk) \cdot \exp\{\lambda(H_P(m))\} \\ &= \Lambda^{T,P} \phi(mk) \cdot \exp\{\lambda(T)\}. \end{aligned}$$

Since $Q = P = P'$, we obtain an equality between

$$\int_K \int_{M_Q(\mathbf{Q}) \backslash M_Q(\mathbf{A})^1} \Lambda^{T,Q} \phi(mk) \cdot \overline{\Lambda^{T,Q} \phi'(mk)}\, dm\, dk \tag{8.2}$$

and

$$\sum_{X \in \mathcal{E}^Q(1,1)} p_X^{T,Q}(\lambda, \lambda', \phi, \phi') e^{X(T)}. \tag{8.3}$$

In particular, the polynomials $p_X^{T,Q}(\lambda, \lambda', \phi, \phi')$ are independent of λ and λ'.

Now (8.2) also equals

$$\int_K \int_{M_Q(\mathbf{Q}) \backslash M_Q(\mathbf{A})^1} \Lambda^{T,Q} \phi(mk) \phi'(mk)\, dm\, dk. \tag{8.4}$$

Our justification for the convergence of integrals of this sort has always been Lemma 1.4 of [1(b)]. If we look at the proof of this lemma we see that (8.4) actually equals

$$\int_K \int_{M_Q(\mathbf{Q}) \backslash M_Q(\mathbf{A})^1} F^Q(mk, T) \phi(mk) \overline{\phi'(mk)}\, dm\, dk,$$

modulo a term which approaches zero as T approaches infinity strongly in $\mathfrak{a}_0^+$. Here

$$F^Q(mk, T), \qquad m \in M_Q(\mathbf{Q}) \backslash M_Q(\mathbf{A})^1, \quad k \in K,$$

is the characteristic function of a compact subset of $M_Q(\mathbf{Q}) \backslash M_Q(\mathbf{A})^1 \times K$. This compact subset can be made arbitrarily large by taking T to be sufficiently regular. In particular, if T approaches infinity strongly in $\mathfrak{a}_0^+$, $F^P(mk, T)$

approaches 1 pointwise in mk. It follows from the dominated convergence theorem that (8.3) approaches

$$\int_K \int_{M_Q(\mathbf{Q})\backslash M_Q(\mathbf{A})^1} \phi(mk)\overline{\phi'(mk)}\,dm\,dk = (\phi, \phi')$$

Suppose that X is a nonzero element in $\mathfrak{E}^Q(1,1)$. Then

$$e^{X(T)} = \prod_{\alpha \in \Delta_0} \exp\{r_\alpha X(\varpi_\alpha^\vee)\}.$$

By Lemma 7.1, each $X(\varpi_\alpha^\vee)$ is less than or equal to zero. Since $X \neq 0$, some $X(\varpi_\alpha^\vee)$ is strictly negative. Therefore if T approaches infinity strongly in $\mathfrak{a}_0^+$,

$$e^{X(T)} \leqslant e^{-\epsilon\|T\|}$$

for some $\epsilon > 0$. The polynomial

$$T \to p_X^{T,Q}(\lambda, \lambda', \phi, \phi')$$

certainly has absolute value bounded by a power of $1 + \|T\|$, so

$$p_X^{T,Q}(\lambda, \lambda', \phi, \phi')e^{X(T)}$$

approaches zero. All that remains of (8.3) is $p_0^{T,Q}(\lambda, \lambda', \phi, \phi')$. This must approach (ϕ, ϕ'). Remember that T can approach infinity anywhere within the set

$$\{T \in \mathfrak{a}_0 : \alpha(T) \geqslant \delta\|T\|, \alpha \in \Delta_0\}.$$

It follows that $p_0^{T,Q}(\lambda, \lambda', \phi, \phi')$, a polynomial in T, is actually independent of T. It equals (ϕ, ϕ'). □

We return to the general case, in which Q is any parabolic subgroup which contains P, P' and P_1. Combining Corollary 4.2 with the lemma just proved, we see that

$$\exp\{-(t\lambda - t'\lambda')(T)\}D_{Q|P_1,T}\left(p_0^{T,Q}(\lambda, \lambda', \phi, \phi')\exp\{(t\lambda - t'\lambda')(T)\}\right)$$

equals

$$\mathrm{vol}\left(\mathfrak{a}_{P_1}^Q / L_{P_1}^Q\right)\left(M(t,\lambda)\phi, M(t', -\bar{\lambda}')\phi'\right).$$

Moreover $p_0^{T,Q}(\lambda, \lambda', \phi, \phi')$ is independent of T. (See the remark following Corollary 4.2.) It therefore equals

$$\left(\prod_{\alpha \in \Delta_0^Q \backslash \Delta_0^{P_1}} (t\lambda - t'\lambda')(\alpha^\vee)\right)^{-1} \mathrm{vol}\left(\mathfrak{a}_{P_1}^Q / L_{P_1}^Q\right)\left(M(t,\lambda)\phi, M(t', -\bar{\lambda}')\phi'\right).$$

If $\alpha \in \Delta_0^Q \backslash \Delta_0^{P_1}$, the projection of $\alpha^\vee$ onto $\mathfrak{a}_{P_1}$ equals $\alpha_1^\vee$ for a uniquely determined root $\alpha_1 \in \Delta^Q_{P_1}$. Since $t\lambda - t'\lambda'$ belongs to $\mathfrak{a}^*_{P_1,\mathbf{C}}$,

$$\Big(\prod_{\alpha \in \Delta^Q_0 \backslash \Delta^{P_1}_0} (t\lambda - t'\lambda')(\alpha^\vee)\Big)^{-1} \operatorname{vol}\big(\mathfrak{a}^Q_{P_1}/L^Q_{P_1}\big)$$

$$= \Big(\prod_{\alpha_1 \in \Delta^Q_{P_1}} (t\lambda - t'\lambda')(\alpha_1^\vee)\Big)^{-1} \operatorname{vol}\big(\mathfrak{a}^Q_{P_1}/L^Q_{P_1}\big)$$

$$= \theta^Q_{P_1}(t\lambda - t'\lambda')^{-1}.$$

It follows that

$$p_0^{T,Q}(\lambda,\lambda',\phi,\phi') = \big(M(t,\lambda)\phi, M(t',-\bar{\lambda}')\phi'\big)\theta^Q_{P_1}(t\lambda - t'\lambda')^{-1}. \tag{8.5}$$

§9. Conclusion. We can now put all our results together. We know that

$$\big(\Lambda^{T,Q}E^Q(\phi,\Lambda), \Lambda^{T,Q}E^Q(\phi',-\bar{\lambda}')\big)_{Q,T} \tag{9.1}$$

equals

$$\sum_{(t,t') \in W^Q(\mathfrak{a}_P \oplus \mathfrak{a}_{P'}, \mathcal{P}_X)} \sum_{X \in \mathfrak{S}^Q(t,t')} p_X^{T,Q}(\lambda,\lambda',\phi,\phi')\exp\{(t\lambda - t'\lambda' + X)(T)\}.$$

Let

$$X_0 = 0, X_1, \ldots, X_n$$

be the set of distinct points in the union over (t,t') of the sets $\mathfrak{S}^Q(t,t')$. Then (9.1) equals the sum, over $0 \leqslant k \leqslant n$, of the product of $e^{X_k(T)}$ with the function

$$q_k^{T,Q}(\lambda,\lambda',\phi,\phi') = \sum_{(t,t')} \sum_{\{X \in \mathfrak{S}^Q(t,t')\,:\, X = X_k\}} p_X^{T,Q}(\lambda,\lambda',\phi,\phi')\exp\{(t\lambda - t'\lambda')(T)\}.$$

It is known from [2(b)] that the Eisenstein series $E^Q(x,\phi,\lambda)$ is regular on the imaginary space $i\mathfrak{a}^*_P$. Consequently (9.1), which is a meromorphic function of $(\lambda,\lambda') \in \mathfrak{a}^*_{P,\mathbf{C}} \times \mathfrak{a}^*_{P',\mathbf{C}}$, is regular on $i\mathfrak{a}^*_P \times i\mathfrak{a}^*_{P'}$. Since it is the coefficient of the real exponential $e^{X_k(T)}$ in the decomposition of (9.1), $q_k^{T,Q}(\lambda,\lambda',\phi,\phi')$ is also regular for $(\lambda,\lambda') \in i\mathfrak{a}^*_P \times i\mathfrak{a}^*_{P'}$. The individual functions $p_X^{T,Q}(\lambda,\lambda',\phi,\phi')$ may not be regular on $i\mathfrak{a}^*_P \times i\mathfrak{a}^*_{P'}$. However the poles of $p_X^{T,Q}(\lambda,\lambda',\phi,\phi')$ which meet $i\mathfrak{a}^*_P \times i\mathfrak{a}^*_{P'}$ are all of bounded order. From this it is easy to show that there is an integer n such that

$$\big|q_k^{T,Q}(\lambda,\lambda',\phi,\phi')\big|\big((1 + \|T\|)^n \|\phi\|\,\|\phi'\|\big)^{-1}$$

is bounded for all ϕ, ϕ', T and for (λ,λ') in any compact subset of $i\mathfrak{a}^*_P \times i\mathfrak{a}^*_{P'}$.

Let δ and N be fixed positive numbers, with N large. Consider all T in the set

$$\{T \in \mathfrak{a}_0^+ : \alpha(T) > \delta\|T\| > N, \alpha \in \Delta_0\}. \tag{9.2}$$

Then T is suitably regular in $\mathfrak{a}_0^+$. Moreover,

$$e^{X_k(T)} \leqslant \min_{\alpha \in \Delta_0} \left\{\exp\left\{\delta X_k(\varpi_\alpha^\vee)\|T\|\right\}\right\},$$

since by Lemma 7.1 each number $X_k(\varpi_\alpha^\vee)$ is less than or equal to zero. If $k \geqslant 1$ there is an α such that $X_k(\varpi_\alpha^\vee)$ is strictly negative. Therefore there is an $\epsilon > 0$ such that

$$\left|q_k^{T,Q}(\lambda, \lambda', \phi, \phi')e^{X_k(T)}\right|\left(e^{-\epsilon\|T\|}\|\phi\|\,\|\phi'\|\right)^{-1}$$

is bounded for all ϕ, ϕ', $k \geqslant 1$, T in the set (9.2), and (λ, λ') in any compact subset of $i\mathfrak{a}_P^* \times i\mathfrak{a}_{P'}^*$.

The only remaining term corresponds to $k = 0$. According to Lemma 7.2 and formula (9.2), $q_0^{T,Q}(\lambda, \lambda', \phi, \phi')$ is the sum over P_1, $t \in W^Q(\mathfrak{a}_P, \mathfrak{a}_{P_1})$, and t' in $W^Q(\mathfrak{a}_{P'}, \mathfrak{a}_{P_1})$ of

$$\exp\{(t\lambda - t'\lambda')(T)\}\left(M(t,\lambda)\phi, M(t', -\bar{\lambda}')\phi'\right)\theta_{P_1}^Q(t\lambda - t'\lambda')^{-1}.$$

This is just

$$\sum_{P_1}\sum_{t}\sum_{t'}\omega_{t,t'}^{T,P}(\lambda, \lambda', \phi, \phi') = \omega^{T,Q}(\lambda, \lambda', \phi, \phi').$$

We have proved the following, which is our main theorem.

THEOREM 9.1. *There is a positive number* ϵ *and a locally bounded function* ρ *on* $i\mathfrak{a}_P^* \times i\mathfrak{a}_{P'}^*$ *such that*

$$\left|\left(\Lambda^{T,Q}E^Q(\phi,\lambda), \Lambda^{T,Q}E^Q(\phi',\lambda')\right)_{Q,T} - \omega^{T,Q}(\lambda, \lambda', \phi, \phi')\right|$$

is bounded by

$$\rho(\lambda, \lambda')\|\phi\|\,\|\phi'\|e^{-\epsilon\|T\|},$$

for all $\phi \in \mathcal{Q}_{P,\chi,\Gamma}$, $\phi' \in \mathcal{Q}_{P',\chi,\Gamma}$, $\lambda \in i\mathfrak{a}_P^*$, $\lambda' \in i\mathfrak{a}_{P'}^*$ *and all* T *in the set* (9.2). □

We will use a special case of this theorem in studying the trace formula. It is worth stating separately as a corollary. It is the case that $Q = G$, $P = P'$, and $\lambda = \lambda'$. As in the theorem, λ will be taken to be purely imaginary. Then as a function of λ,

$$\Lambda^T E(x, \phi, \lambda) \cdot \overline{\Lambda^T E(x, \phi', \lambda)}$$

is invariant by $i\mathfrak{a}_G^*$. It follows that

$$\left(\Lambda^T E(\phi,\lambda), \Lambda^T E(\phi',\lambda)\right)_{G,T}$$

equals the ordinary inner product

$$\int_{G(\mathbf{Q})\backslash G(\mathbf{A})^1} \Lambda^T E(x,\phi,\lambda)\,\overline{\Lambda^T E(x,\phi',\lambda)}\,dx.$$

We obtain

COROLLARY 9.2. *There is a positive number ϵ and a locally bounded function ρ on $i\mathfrak{a}_P^*$ such that*

$$\left|\int_{G(\mathbf{Q})\backslash G(\mathbf{A})^1} \Lambda^T E(x,\phi,\lambda)\,\overline{\Lambda^T E(x,\phi',\lambda)}\,dx - \omega^T(\lambda,\lambda,\phi,\phi')\right|$$

is bounded by

$$\rho(\lambda)\|\phi\|\,\|\phi'\|e^{-\epsilon\|T\|},$$

for all $\phi,\phi' \in \mathcal{Q}_{P,\chi,\Gamma}$, $\lambda \in i\mathfrak{a}_P^*$ and all T in the set (9.2).

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