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AUTOMORPHIC REPRESENTATIONS AND NUMBER THEORY*

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An excellent introduction to the theory of automorphic representations and the relations with number theory is the Corvallis proceedings, *Automorphic Forms, Representations and L-Functions*, Parts 1 and 2, Proc. Sympos. Pure Math., vol. 33, 1979. Although composed mainly of survey articles, the proceedings are already rather formidable. They are a measure of the breadth of the field. They will be most useful to mathematicians who are already experts in some branch of the subject.

Our purpose here is to give a modest introduction to the Corvallis proceedings. More precisely, our goal is to describe the Langlands functoriality conjecture, a mathematical insight of great beauty and simplicity. We will try to show both why it is a compelling question, and how it arose historically from Langlands' work on Eisenstein series. We hope that mathematicians from diverse – or at least neighboring – fields will find these notes accessible and will be encouraged to read other survey articles [2], [6], or to plunge directly into the Corvallis proceedings.

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We will discuss only the global functoriality conjecture, and only that part of it which corresponds to the unramified primes. It is then a statement about families of conjugacy classes in complex Lie groups. The point of view is essentially that of Tate's introduction to global class field theory [38, §1-5].

§1. A problem in number theory

Suppose that $f(x)$ is a monic polynomial of degree n with integral coefficients. Let E be the splitting field of $f(x)$ over $\mathbb{Q}$. If we were to order the roots of $f(x)$ we would obtain an embedding of $\mathrm{Gal}(E/\mathbb{Q})$, the Galois group of E over $\mathbb{Q}$, into a subgroup of S_n, the symmetric group on n letters. There is no canonical way to do this, so we obtain only a conjugacy class of subgroups of S_n. If p is a prime number, we can reduce $f(x) \bmod p$. It decomposes into a product of irreducible polynomials mod p of degrees $n_1, n_2, \ldots, n_r$, where $n_1 + \cdots + n_r = n$. These numbers determine a conjugacy class in S_n: the set of permutations which decompose into disjoint cycles of lengths $n_1, n_2, \ldots$ and n_r. The intersection of this class with any of the images of $\mathrm{Gal}(E/\mathbb{Q})$ in S_n may give several conjugacy classes in $\mathrm{Gal}(E/\mathbb{Q})$. However, if p does not divide the discriminant of $f(x)$, there is a distinguished class among these, called the *Frobenius class* of p. Incidentally, any finite Galois extension E can be realized as the splitting field of such an $f(x)$. However, the Frobenius class in $\mathrm{Gal}(E/\mathbb{Q})$ depends only on E and p, and not on $f(x)$. The prime p is said to *split completely in* E if its Frobenius class is 1; that is, if p does not divide the discriminant of $f(x)$ and

$f(x)$ splits into linear factors mod p. Let $S(E)$ be the set of primes that split completely.

<u>THEOREM 1.1</u>: $S : E \to S(E)$ is an *injective*, order reversing map from finite Galois extensions of $\mathbb{Q}$ into subsets of prime numbers.

The difficult part is the injectivity. It is a consequence of the Tchebotarev density theorem (see [38, p. 165]).

<u>PROBLEM</u>: What is the image of this map? i.e. what sets of prime numbers are of the form $S(E)$?

A reasonable solution to this problem would constitute nonabelian class field theory. One could parametrize the finite Galois extensions E by the collections $S(E)$. Any kind of independent characterization of one of the sets $S(E)$ is often called a *reciprocity law* for E.

<u>Example</u>: $f(x) = x^2 + 1$, $E = \mathbb{Q}(\sqrt{-1})$, Disc $f(x) = -4$, and $S(E) = \{p : p \equiv 1 \pmod 4\}$.

The problem has a solution in terms of such congruence conditions if $\mathrm{Gal}(E/\mathbb{Q})$ is abelian. On the other hand, if $\mathrm{Gal}(E/\mathbb{Q})$ is a simple group $S(E)$ cannot be described by congruence conditions alone.

It is convenient to embed the Galois group in $GL_n(\mathbb{C})$, since the conjugacy classes in $GL_n(\mathbb{C})$ are particularly easy to parametrize. Suppose then that

$$r : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_n(\mathbb{C})$$

is a continuous homomorphism. $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is the projective limit $\varprojlim_{E} \mathrm{Gal}(E/\mathbb{Q})$ over all finite Galois extensions E of $\mathbb{Q}$, so it is a compact totally disconnected group. Continuity means that r has finite image. The kernel of r equals $\mathrm{Gal}(\bar{\mathbb{Q}}/E_r)$ for a finite Galois extension E_r of $\mathbb{Q}$, and r embeds $\mathrm{Gal}(E_r/\mathbb{Q})$ into $GL_n(\mathbb{C})$. Any finite Galois extension E of $\mathbb{Q}$ equals E_r for some r. If p is unramified in E_r we have the Frobenius conjugacy class in $\mathrm{Gal}(E_r/\mathbb{Q})$, which embeds into a unique semisimple conjugacy class $\Phi_p(r)$ in $GL_n(\mathbb{C})$. Thus

$$S(E_r) = \{p : \Phi_p(r) = I\}.$$

A semisimple conjugacy class in $GL_n(\mathbb{C})$ is completely determined by its characteristic polynomial, so the characteristic polynomial deserves to have special notation. In fact, one defines

$$L_p(z,r) = \det\left(I - \Phi_p(r)z\right)^{-1},$$

the *local* L-*function of* r. If S is a finite set of primes which includes the ones that ramify, one defines a *global* L-*function*,

$$L_S(s,r) = \prod_{p \notin S} L_p(p^{-s},r), \qquad s \in \mathbb{C}.$$

It can be shown that this infinite product converges for s in some right half plane of $\mathbb{C}$. An elementary lemma on Dirichlet series implies the factors $L_p(p^{-s},r)$ are uniquely determined by the analytic function $L_S(s,r)$. So, therefore, are the semisimple conjugacy classes $\{\Phi_p(r) : p \notin S\}$. It is also known that $L_S(s,r)$ can be analytically continued with a functional equation relating $L_S(s,r)$ with $L_S(1-s,\tilde{r})$, where

$$\tilde{r}(g) = r(g^{-1})^t.$$

The location and residues of the poles of $L_S(s,r)$ could be determined from a conjecture of Artin, (see [16]. Artin's conjecture is referred to on p. 225).

Example: Fix $N \in \mathbb{N}$, and let $f(x) = \prod_{r \in (\mathbb{Z}/N\mathbb{Z})^*} \cdot \left(x - e^{\frac{2\pi i \nu}{N}}\right)$.

Then $f(x)$ is irreducible and $\mathrm{Gal}\left(\mathbb{Q}\left(e^{\frac{2\pi i}{N}}\right)\Big/\mathbb{Q}\right) \cong (\mathbb{Z}/N\mathbb{Z})^*$.

A prime p is unramified if it does not divide N. The Frobenius class is the image of p in $(\mathbb{Z}/N\mathbb{Z})^*$ [24]. To fit in with discussion above we can take a character χ of the $(\mathbb{Z}/N\mathbb{Z})^*$. The functions $L_S(s,\chi)$ are called *Dirichlet* L-*series*.

At this point, we perhaps should recall the precise definition of the Frobenius class. Let $\mathbb{Q}_p$ be the field of p-adic numbers. Then $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ is naturally embedded in $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ up to conjugacy in $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. There is a normal subgroup T_p of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ (the *inertia group*) such that the quotient group is isomorphic to $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ and, in particular, has a cyclic generator Φ. ($\mathbb{F}_p$ is a field with p elements.) Suppose that H is a closed normal subgroup of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. It corresponds to a Galois extension E of $\mathbb{Q}$. The prime p is called *unramified* if the images of T_p in $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ are contained in H. Then Φ maps to a well defined conjugacy class in

$$\mathrm{Gal}(E/\mathbb{Q}) \cong \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})/H.$$

This is the Frobenius class of p. (The reader unfamiliar with the facts from algebraic number theory discussed so far might read [38], §1,2, and then go to [7], [10] or [24] for the details.)

One possible way to obtain collections in the image of S is through algebraic geometry. Suppose that X is a nonsingular projective algebraic variety defined over $\mathbb{Q}$. Fix a nonnegative integer i. Grothendieck has defined for every prime number ℓ the ℓ-adic cohomology group $H^i(X,\mathbb{Q}_\ell)$, a vector space over $\mathbb{Q}_\ell$ whose dimension, n, equals the i^{th} Betti number of $X(\mathbb{C})$. It comes equipped with a continuous map

$$\rho_\ell : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL\big(H^i(X, \mathbb{Q}_\ell)\big).$$

Any choice of basis of $H^i(X,\mathbb{Q}_\ell)$ identifies $GL\big(H^i(X, \mathbb{Q}_\ell)\big)$ with $GL_n(\mathbb{Q}_\ell)$. Since $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is compact, it is possible to choose a basis such that $\rho_\ell\big(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})\big)$ lies in $GL_n(\mathbb{Z}_\ell)$, where $\mathbb{Z}_\ell$ is the ring of ℓ-adic integers [33, p. 1]. It follows from the work of Deligne on the Weil conjectures that the collection $\rho = \{\rho_\ell : \ell \text{ prime}\}$ is a compatible family of ℓ-adic representations, in the sense that the following two properties hold:

(i) There is a finite set S_ρ of primes such that if $p \notin S_\rho \cup \{\ell\}$, p is unramified in $\rho_\ell\big(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})\big)$ $\big($regarding this group as a quotient of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})\big)$, so there is a Frobenius conjugacy class $\Phi_p(\rho_\ell)$ in $GL_n(\mathbb{Z}_\ell)$.

(ii) For $p \notin S_\rho \cup \{\ell\}$, the characteristic polynomial $\det\big(I - \Phi_p(\rho_\ell)z\big)$ has coefficients in $\mathbb{Z} \subseteq \mathbb{Z}_\ell$, and is independent of ℓ.

If $p \notin S_\rho$, let $\Phi_p(\rho)$ be the unique *semisimple* conjugacy class in $GL_n(\mathbb{C})$ whose characteristic polynomial is $\det(I - \Phi_p(\rho_\ell)z)$, for any $\ell \neq p$. As before, one can define L-functions

$$L_p(z,\rho) = \det(I - \Phi_p(\rho)z)^{-1} \quad , \quad L_S(s,\rho) = \prod_{p \notin S} L_p(p^{-s},\rho).$$

Again, the global L-function converges in some right half plane. However, it is not known to have analytic continuation. It is expected that the conjugacy classes $\Phi_p(\rho_\ell)$ are semisimple, although this is also not known. If it were so one could construct a number of collections $S(E)$ simply from a knowledge of $L_S(s,\rho)$. For suppose

$$N = \Pi_p \, p^{N_p}, \qquad N_p \geq 0,$$

is a positive integer. Set

$$K_p(N) = \{k \in GL_n(\mathbb{Z}_p) : k \equiv 1 \pmod{p^{N_p}\mathbb{Z}_p}\},$$

and

$$K(N) = \Pi_{\ell \text{ prime}} K_\ell(N).$$

It can be shown that $K(N)$ is a normal subgroup of

$$K(1) = \Pi_{\ell \text{ prime}} GL_n(\mathbb{Z}_\ell),$$

and that $K(1)/K(N)$ is naturally isomorphic to $GL_n(\mathbb{Z}/N\mathbb{Z})$ (see [36], Lemma 1.38). By composing the maps

$$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \xrightarrow{\otimes\rho_\ell} K(1) \rightarrow K(1)/K(N)$$

one obtains a homomorphism

$$\rho(N) : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_n(\mathbb{Z}/N\mathbb{Z}).$$

The image of $\rho(N)$ corresponds to a finite Galois extension $E_{\rho(N)}$ of $\mathbb{Q}$; there is an injection

$$\mathrm{Gal}(E_{\rho(N)}/\mathbb{Q}) \to GL_n(\mathbb{Z}/N\mathbb{Z}).$$

If the conjugacy classes $\Phi_p(\rho_\ell)$ are all semisimple, the image of the Frobenius class in $GL_n(\mathbb{Z}/N\mathbb{Z})$ will be determined by the reduction modulo N of the characteristic polynomial of $\Phi_p(\rho)$. In particular,

$$S(E_{\rho(N)}) = \{p \notin S_\rho, p \nmid N : L_p(z,\rho)^{-1} \equiv (1-z)^n \pmod N\}.$$

This section is meant to serve as motivation for what follows. The main theme has been that interesting data from number theory or algebraic geometry can be encapsulated in a family $\{\Phi_p : p \notin S\}$ of semisimple conjugacy class in $GL_n(\mathbb{C})$.

§2. Automorphic representations of GL_n

In this section G will stand for the group GL_n. Then if A is any ring (commutative, with identity), $G(A)$ is the group of $(n \times n)$ matrices over A whose determinant is a unit in A. One such ring is the adèles, $\mathbb{A}$, the restricted direct product

$$\Pi_v \mathbb{Q}_v = \mathbb{R} \times \Pi_p \mathbb{Q}_p .$$

(We shall write v for a valuation over $\mathbb{Q}$, and p for the valuation associated to a finite prime.) Then $G(\mathbb{A})$ is a locally compact group. Embedded diagonally, $G(\mathbb{Q})$ is a discrete subgroup of $G(\mathbb{A})$. One studies the coset space $G(\mathbb{Q})\backslash G(\mathbb{A})$, with the quotient topology. The volume of this space, with respect to the $G(\mathbb{A})$ invariant measure, is not finite. To rectify this, define

$$Z = \left\{ \begin{pmatrix} z & & 0 \\ & \ddots & \\ 0 & & z \end{pmatrix} : z > 0 \right\},$$

a central subgroup of $G(\mathbb{R})$. Then modulo the subgroup $Z \cdot G(\mathbb{Q})$, $G(\mathbb{A})$ does have finite invariant volume. Let R be the regular representation of $G(\mathbb{A})$ on the Hilbert space

$$L = L^2\big(Z \cdot G(\mathbb{Q})\backslash G(\mathbb{A})\big),$$

so

$$\big(R(y)\phi\big)(x) = \phi(xy), \qquad \phi \in L,\ x,y \in G(\mathbb{A}).$$

This is a unitary representation of $G(\mathbb{A})$. One tries to decompose it as a direct integral of irreducible representations. Incidentally *any* irreducible representation π of $G(\mathbb{A})$ can be written as a restricted tensor product

$$\bigotimes_v \pi_v = \pi_{\mathbb{R}} \otimes \bigotimes_p \pi_p$$

of irreducible representations of the local groups [8].

At first glance the space $Z\cdot G(\mathbb{Q})\backslash G(\mathbb{A})$ might seem unduly abstract in comparison with, say, a quotient of a real Lie group. However, it is really a very natural object. In §1 we defined, for each $N = \Pi_p p^{N_p}$, the compact subgroup

$$K(N) = \Pi_p K_p(N)$$

of $G(\mathbb{A})$. Set

$$\mathcal{L}^{K(N)} = L^2\big(Z\cdot G(\mathbb{Q})\backslash G(\mathbb{A})/K(N)\big),$$

the Hilbert space of $K(N)$ fixed functions under R. Then

$$\mathcal{L} = \varinjlim_{N} \mathcal{L}^{K(N)},$$

so to understand $\mathcal{L}$ it is necessary and sufficient to understand each of the spaces $\mathcal{L}^{K(N)}$. Notice that $\mathcal{L}^{K(N)}$ is invariant under the normalizer of $K(N)$ in $G(\mathbb{A})$, which certainly contains $SL_n(\mathbb{R})$. Now the determinant fibres $Z\cdot G(\mathbb{Q})\backslash G(\mathbb{A})/K(N)$ over

$$\mathbb{R}^+\mathbb{Q}^*\backslash \text{Idèles}/\Pi_p\{\gamma \in \mathbb{Z}_p^* : \gamma \equiv 1 \pmod{p^{N_p}\mathbb{Z}_p}\},$$

an abelian group which is isomorphic to $(\mathbb{Z}/N\mathbb{Z})^*$. The fibre of any point is $SL_n(\mathbb{R})$-invariant, and as an $SL_n(\mathbb{R})$ space is isomorphic to

$$SL_n(\mathbb{Q})\backslash SL_n(\mathbb{A})/\big(SL_n(\mathbb{A}) \cap K(N)\big).$$

However, by strong approximation for SL_n,

$$SL_n(\mathbb{A}) = SL_n(\mathbb{Q})\big(SL_n(\mathbb{A}) \cap SL_n(\mathbb{R})\cdot K(N)\big),$$

(see [36, Lemma 6.15]. The proof in general is the same as for $n = 2$). It follows that as an $SL_n(\mathbb{R})$ space, the fibre is diffeomorphic with $\Gamma(N)\backslash SL_n(\mathbb{R})$, where

$$\Gamma(N) = \{\gamma \in SL_n(\mathbb{Z}) : \gamma \equiv 1 \pmod N\}.$$

Thus, as an $SL_n(\mathbb{R})$-module, $L^{K(N)}$ is isomorphic to $\#[(\mathbb{Z}/N\mathbb{Z})^*]$ copies of $L^2(\Gamma(N)\backslash SL_n(\mathbb{R}))$.

It is in the definition of Hecke operators that we can see the utility of the adèle picture most clearly. If $0 \leq i \leq n$, define an element

$$t_{p,i} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & \emptyset & \\ & & 1 & & & \\ & & & p & & \\ & \emptyset & & & \ddots & \\ & & & & & p \end{pmatrix} \begin{matrix} \\ \\ \\ \\ \left.\vphantom{\begin{matrix} p \\ \ddots \\ p \end{matrix}}\right\} i \end{matrix}$$

in $G(\mathbb{Q}_p)$. Let $f_{p,i}$ be the characteristic function in $G(\mathbb{Q}_p)$ of $K_p(N)\cdot t_{p,i}\cdot K_p(N)$. Define the *$i^{th}$ Hecke operator*, $T_{p,i}$, on $L^{K(N)}$ by

$$T_{p,i}\phi = \phi * f_{p,i} = \int_{G(\mathbb{Q}_p)} f_{p,i}(y)\big(R(y)\phi\big)dy,$$

for $\phi \in L^{K(N)}$. It is clear that $T_{p,i}\phi$ belongs to $L^{K(N)}$. How does $T_{p,i}$ behave on the irreducible constituents of R? Suppose that

$$(\pi, U) = (\otimes_v \pi_v, \otimes_v U_v)$$

is an irreducible representation of $G(\mathbb{A})$ which is equivalent to the subrepresentation of R on a closed invariant subspace L_π of L. Then

$$L_\pi^{K(N)} = L_\pi \cap L^{K(N)}$$

corresponds to the subspace

$$U^{K(N)} = U_{\mathbb{R}} \otimes \tilde{\otimes}_p U_p^{K_p(N)} .$$

$\left(U_p^{K_p(N)}\right.$ is the space of $K_p(N)$-invariant vectors in $\left.U_p.\right)$ If p does not divide N, $K_p(N) = GL_n(\mathbb{Z}_p)$, a maximal compact subgroup of $GL_n(\mathbb{Q}_p)$. Functions on $GL_n(\mathbb{Q}_p)$, such as $f_{p,i}$, which are bi-invariant under $GL_n(\mathbb{Z}_p)$ are p-adic analogues of the spherical functions for real groups [17], discussed in Helgason's lectures. The theory is similar. In particular, the space $U_p^{GL_n(\mathbb{Q}_p)}$ has dimension at most one. If $L_\pi^{K(N)} \neq \{0\}$, and p does not divide N, the dimension of $U_p^{K_p(N)}$ must be exactly one. Now $T_{p,i}$ leaves invariant the space $L_\pi^{K(N)}$; relative to the equivalence of L_π with U, it acts through the one-dimensional space $U_p^{K_p(N)}$. Therefore, the restriction of $T_{p,i}$ to $L_\pi^{K(N)}$ is a multiple of the identity operator by a complex number $c_{p,i}(\pi)$.

The smooth vectors are dense in L. It follows that for the representation (π, U) of $G(\mathbb{A})$ there is an N such that $U^{K(N)} \neq \{0\}$. Let N_π be the minimal such N, and let S_π be the set of prime divisors of N_π. For any prime $p \notin S_\pi$, we obtain $(n+1)$ complex numbers $\{c_{p,i}(\pi) : 0 \leq i \leq n\}$. Define a semisimple conjugacy $\Phi_p(\pi)$ in $GL_n(\mathbb{C})$ by constructing its characteristic polynomial from these numbers:

$$\det\left(1 - \Phi_p(\pi) z\right) = \Sigma_{i=0}^{n} (-1)^i \, p^{\frac{1}{2}i(n-i)} \, c_{p,i}(\pi) z^i .$$

THEOREM 2.1 (Strong multiplicity one). The representation π is uniquely determined by the family $\{\Phi_p(\pi) : p \notin S_\pi\}$ of conjugacy classes in $GL_n(\mathbb{C})$.

(See [21], [32]).

We have only defined the conjugacy classes $\{\Phi_p(\pi)\}$ for representations that occur discretely in R. This restriction is not necessary. In fact, if $\lambda \in \mathbb{C}$, define the representation

$$R_\lambda(x) = R(x)|\det x|^\lambda, \qquad x \in G(\mathbb{A}).$$

If λ is purely imaginary, R_λ is unitary and it, also, has a decomposition into a direct integral of irreducible representations of $G(\mathbb{A})$. For the present we shall define an *automorphic representation* informally as an irreducible representation of $G(\mathbb{A})$ which "occurs in" the direct integral decomposition of R_λ, for some λ. One can define the finite set S_π and the conjugacy classes $\{\Phi_p(\pi) : p \notin S_\pi\}$ as above for *any* automorphic representation π. We will not do so for we will give a more general definition in §5. We do note, howevever, that Jacquet, Piatetski-Shipiro and Shalika have recently shown that with certain obvious exceptions, Theorem 2.1 holds for any automorphic representation of $G(\mathbb{A})$. If π is any automorphic representation of $G(\mathbb{A})$, define

$$L_p(z,\pi) = \det(1 - \Phi_p(\pi)z)^{-1} = \left(\Sigma_{i=0}^n (-1)^i p^{\frac{1}{2}i(n-i)} c_{p,i}(\pi) z^i\right)^{-1}$$

if $p \nmid S_\pi$, and for any finite set of primes $S \supset S_\pi$, set

$$L_S(s,\pi) = \prod_{p \notin S} L_p(p^{-s},\pi).$$

As in the examples of §1, the right hand side converges for s in some right half plane.

Example: Let $n = 1$. Then $G(\mathbb{Q})\backslash G(\mathbb{A})$ is just the idèle class group of $\mathbb{Q}$. For any N,

$$K_p(N) = \{\gamma \in \mathbb{Z}_p^*: \gamma \equiv 1 \pmod{p^{N_p}\mathbb{Z}_p}\}.$$

An automorphic representation $\chi = \otimes_v \chi_p$ is just a character on the idèle class group and is known as a Grössencharakter. N_χ is the smallest positive integer such that χ is trivial on $K(N_\chi)$. If p does not divide N_χ there is a complex number s_p such that

$$\chi_p(v) = |v|_p^{s_p}, \qquad v \in \mathbb{Q}_p^*.$$

It is clear that $c_{p,0}(\chi) = 1$ and $c_{p,1}(\chi) = |p|_p^{s_p} = p^{-s_p}$. Therefore

$$L_p(\chi,z) = \left(1 - p^{-s_p}z\right)^{-1}$$

and

$$L_S(\chi,z) = \prod_{p \nmid S}\left(1 - p^{-(s+s_p)}\right)^{-1}.$$

THEOREM 2.2: If π is an automorphic representation of $G(\mathbb{A})$, the function $L_S(s,\pi)$ can be analytically continued with functional equation. Moreover, the location and residues of all poles can be determined.

See [18].

We could say a word about the converse to this theorem. Suppose that $\{\Phi_p : p \nmid S\}$ is a family of semisimple conjugacy classes in $GL_n(\mathbb{C})$. Suppose in addition that

$$\Pi_{p \notin S} \det(1 - \Phi_p p^{-s})^{-1}$$

converges in a right half plane to an analytic function which has all the properties established in Theorem 2.2 (i.e. analytic continuation, functional equation, predicted location of poles). Is there an automorphic representation π of $G(\mathbb{A})$ such that $\Phi_p(\pi) = \Phi_p$ for each $p \notin S$? The answer is no in general, but a partial solution is given for GL_2 in [19], for GL_3 in [20] and indeed is expected for GL_n. Rather than give precise statements, let us simply say that these analytic conditions on the family $\{\Phi_p\}$ are very strong, if not quite strong enough to insure the existence of π.

<u>CONJECTURE 2.3</u> (Langlands). Given a continuous representation

$$r : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_n(\mathbb{C})$$

there is an automorphic representation π of $GL_n(\mathbb{A})$ such that S_π is the set of primes that ramify for r, and such that $\Phi_p(r) = \Phi_p(\pi)$ for all primes $p \notin S_\pi$. In particular

$$S(E_r) : \{p : \Phi_p(\pi) = I\}.$$

This conjecture reduces the problem of §1 to the study of automorphic representations of $GL_n(\mathbb{A})$. The collections $S(E_r)$, which classify Galois extensions of $\mathbb{Q}$, could be recovered from data obtained analytically from the decomposition of R into irreducibles. Notice that $L_S(s,r) = L_S(s,\pi)$ for any finite set $S \supset S_\pi$, so the location and residues of the poles of any function $L_S(s,r)$ could be computed by Theorem 2.2. This is the conjecture of Artin.

<u>REMARK 1</u>: It is easy to restate everything we have done so far with the ground field $\mathbb{Q}$ replaced by an arbitrary number field F.

<u>REMARK 2</u>: Suppose that $n=1$. For any N, $L^{K(N)}$ is isomorphic to the space of functions on $(\mathbb{Z}/N\mathbb{Z})^*$. If p does not divide N, $T_{p,0}$ is the identity operator while $T_{p,1}$ corresponds to multiplication in $(\mathbb{Z}/N\mathbb{Z})^*$ by p. However the image of p in $(\mathbb{Z}/N\mathbb{Z})^*$ also corresponds to the Frobenius class in $\mathrm{Gal}\left(\mathbb{Q}\left(e^{\frac{2\pi i}{N}}\right)/\mathbb{Q}\right)$. Thus, the conjecture is true for any one dimensional representation of $\mathrm{Gal}\left(\mathbb{Q}\left(e^{\frac{2\pi i}{N}}\right)/\mathbb{Q}\right)$. So far, this is quite elementary. But the conjecture asks more, even for $n = 1$. *Any* one dimensional representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is to be associated to an automorphic representation of GL_1; by what we have just observed, this in turn corresponds to a one dimensional representation of $\mathrm{Gal}\left(\mathbb{Q}\left(e^{\frac{2\pi i}{N}}\right)/\mathbb{Q}\right)$, for some N. Thus, for any finite abelian extension E of $\mathbb{Q}$ the character group of $\mathrm{Gal}(E/\mathbb{Q})$ is naturally a subgroup of the character group of $\mathrm{Gal}\left(\mathbb{Q}\left(e^{\frac{2\pi i}{N}}\right)/\mathbb{Q}\right)$ for some N. This means that $\mathrm{Gal}(E/\mathbb{Q})$ is a quotient of $\mathrm{Gal}\left(\mathbb{Q}\left(e^{\frac{2\pi i}{N}}\right)/\mathbb{Q}\right)$. In other words E is contained in $\mathbb{Q}\left(e^{\frac{2\pi i}{N}}\right)$. This is Kronecker's theorem, and is certainly not elementary. If $\mathbb{Q}$ is replaced by a number field F, the conjecture for $n = 1$ is also known. It is just the Artin reciprocity law which is the heart of class field theory (see [38], §3, 4, 5).

Remark 3: Suppose that $n = 2$. Then for certain irreducible representations r, Langlands solved the conjecture by extending the work of Saito and Shintani on the base change problem for GL_2. See [28], [12].

This is a good point to describe the base change problem for it is closely related to Conjecture 2.3. We shall state it for GL_n although it has been proved completely only for $n = 2$. Suppose that $E \supset F$ is a cyclic extension of number fields of prime order ℓ. Let δ be a generator of $Gal(E/F)$. If r is an n dimensional representation of $Gal(\bar{F}/F)$, let R be the restriction of r to the subgroup $Gal(\bar{F}/E)$. Any n dimensional representation R of $Gal(\bar{F}/E)$ will be of this form if and only if $R^\delta \cong R$, where

$$R^\delta(g) = R(\delta^{-1}g\delta), \qquad g \in Gal(\bar{F}/E).$$

Now as we have remarked the notions discussed in this section all make sense if $\mathbb{Q}$ is replaced by F. The conjecture suggests there should be a map $\pi \to \Pi$ from automorphic representations of $GL_n(\mathbb{A}_F)$ ($\mathbb{A}_F$ being the adèles of F) to automorphic representations of $GL_n(\mathbb{A}_E)$. If Π occurs discretely in the regular representation it is uniquely determined by a family of conjugacy classes in $GL_n(\mathbb{C})$. We need only observe how the map $r \to R$ behaves on Frobenius conjugacy classes. Therefore the base change problem is:

(a) For each automorphic representation π of $GL_n(\mathbb{A}_F)$ prove that there is an automorphic representation Π of $GL_n(\mathbb{A}_E)$ such that S_Π is the set of prime ideals that divide the primes in S_π and such that for any prime $\mathfrak{P}$ of E which divides the prime $\mathfrak{p} \notin S_\pi$ of F,

$$\Phi_{\mathfrak{P}}(\Pi) = \begin{cases} \Phi_{\mathfrak{p}}(\pi)^{\ell} & \text{if } \mathfrak{p} \text{ splits in } E \\ \Phi_{\mathfrak{p}}(\pi) & \text{if } \mathfrak{p} \text{ remains prime in } E. \end{cases}$$

(b) Show that an automorphic representation Π of $GL_n(\mathbb{A}_E)$ is of this form if and only if $\Pi^\delta \cong \Pi$.

The solution of this problem for GL_2 used the trace formula for GL_2. For $n = 3$ progress has been made for Flicker [9].

The L functions of algebraic geometry should also correspond to L-functions for GL_n. Suppose that X, $n = \dim H^i(X, \mathbb{Q}_\ell)$, and

$$\rho = \{\rho_\ell : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL(H^i(X,\mathbb{Q}_\ell))\}$$

are as in §1. Then there should be an automorphic representation π of $GL_n(\mathbb{A})$ such that $S_\pi = S_\rho$ and such that $\Phi_p(\rho) = \Phi_p(\pi)$ for all primes $p \notin S$. In particular, for any N,

$$S(E_{\rho(N)}) = \{p \notin S_\pi,\ p \nmid N : L_p(z,\pi)^{-1} \equiv (1-z)^n \pmod N\}.$$

There is a statement of this last conjecture, in more general form, at the end of §2 of [30]. For the solution in case $n = 1$, see [33].

§3. Eisenstein series

The purpose of Eisenstein series is to describe the direct integral of that part of R which decomposes continuously The theory was begun by Selberg and completed by Langlands [25]. We shall give a brief description of the main results (see also [1] and [11]).

We shall state the results for a reductive algebraic group G. The reader unfamiliar with algebraic groups could skip to the next paragraph where objects defined for general G are described for GL_n. Let P_0 be a fixed minimal parabolic subgroup of G, defined over $\mathbb{Q}$, and let M_0 be a fixed Levi component also defined over $\mathbb{Q}$. If P is a standard parabolic subgroup of G, we shall write N for the unipotent radical of P and M for the Levi component of P which contains M_0. Let $X(M)_{\mathbb{Q}}$ be the abelian group (written additively) of maps from M to GL_1 defined over $\mathbb{Q}$. If $m \in M(\mathbb{A})$ and $\chi \in X(M)_{\mathbb{Q}}$, the value of χ at m, m^{χ}, is an idèle so it has an absolute value. Define a map H_M from $M(\mathbb{A})$ to the real vector space

$$\mathfrak{a}_P = \mathrm{Hom}\left(X(M)_{\mathbb{Q}}, \mathbb{R}\right)$$

by

$$e^{\langle \chi, H_M(x) \rangle} = |m^{\chi}|, \qquad \chi \in X(M)_{\mathbb{Q}}, \quad m \in M(\mathbb{A}).$$

Let A_P be the split component of the center of M, and let Z_M be the connected component of 1 in $A_P(\mathbb{R})^0$. Then $M(\mathbb{A})$ is the direct product of the kernel of H_M with Z_M. Finally, let $K = \Pi_v K_v$ be a maximal compact subgroup of $G(\mathbb{A})$, admissible relative to M_0.

If $G = GL_n$, we can take

$$P_0 = \left\{ \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \right\}, \quad M_0 = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix},$$

$$P = \left\{ \begin{pmatrix} \boxed{*} & \cdots & * \\ & \boxed{*} & \vdots \\ & \ddots & \\ & & \boxed{*} \end{pmatrix} \begin{matrix} \}n_1 \\ \\ \\ \}n_r \end{matrix} \right\}, \quad M = \left\{ \begin{pmatrix} \boxed{*} & & 0 \\ & \ddots & \\ 0 & & \boxed{*} \end{pmatrix} \begin{matrix} \}n_1 \\ \\ \}n_r \end{matrix} \right\},$$

and

$$N = \left\{ \begin{pmatrix} \boxed{I} & * & \cdots & * \\ & \ddots & & \vdots \\ & & & * \\ & 0 & & \boxed{I} \end{pmatrix} \right\},$$

where $n_1 + \cdots + n_r = n$. Then $X(M)_{\mathbb{Q}}$ is the group of maps

$$\chi_\nu : \begin{pmatrix} \boxed{m_1} & & 0 \\ & \ddots & \\ 0 & & \boxed{m_r} \end{pmatrix} \to \Pi_{i=1}^{r} (\det m_i)^{\nu_i},$$

where $\nu = (\nu_1, \ldots, \nu_r)$ ranges over $\mathbb{Z}^r$. The space $\mathfrak{a}_P^*$ is isomorphic to $\mathbb{R}^r$, and

$$\left\langle \chi_\nu , H_M \begin{pmatrix} \boxed{m_1} & & 0 \\ & \ddots & \\ 0 & & \boxed{m_r} \end{pmatrix} \right\rangle = \sum_{i=1}^{r} \nu_i \log|\det m_i|, \quad \nu \in \mathbb{Z}^r.$$

Also

$$Z_M = \left\{ \begin{pmatrix} \boxed{z_1 I} & & 0 \\ & \ddots & \\ 0 & & \boxed{z_r I} \end{pmatrix} : z_i > 0 \right\}.$$

Finally, we can take

$$K = O_n(\mathbb{R}) \times \Pi_p GL_n(\mathbb{Z}_p).$$

If x is any element in $G(\mathbb{A})$, we can write

$$x = nmk, \qquad n \in N(\mathbb{A}),\ m \in M(\mathbb{A}),\ k \in K.$$

Define

$$H_P(x) = H_M(m).$$

If $\rho = \rho_P \in \mathfrak{a}_P^*$ is one half the sum (with multiplicity) of the roots of (P, A_P) then

$$p \to e^{\left(2\rho(H_P(p))\right)}, \qquad p \in P(\mathbb{A}),$$

is the modular function of $P(\mathbb{A})$. Finally let $\Delta_P \subset X(M)_{\mathbb{Q}} \subset \mathfrak{a}_P^*$ be the *simple* roots of (P, A_P). Every $\alpha \in \Delta_P$ is the restriction to $\mathfrak{a}_P$ of a unique simple root β in Δ_{P_0}. Let $\alpha^\vee$ be the projection of the co-root $\beta^\vee$ onto $\mathfrak{a}_P$. (If $G = GL_n$,

$$\Delta_P = \left\{\alpha_i = \chi_{\underbrace{0,\ldots,0}_{i},1,-1,0,\ldots,0)} : 1 \leq i < n\right\}$$

and for each i, $\alpha_i^\vee$ is the element in $\mathfrak{a}_P$ such that

$$\chi_\nu(\alpha_i^\vee) = \nu_i n_i^{-1} - \nu_{i+1} n_{i+1}^{-1}. \Big)$$

Let $R_{M,\mathrm{disc}}$ be the subrepresentation of the regular representation $M(\mathbb{A})$ on $L^2\left(Z_M \cdot M(\mathbb{Q}) \backslash M(\mathbb{A})\right)$ that decomposes discretely. It acts on a closed invariant subspace, $L^2(Z_M\cdot M(\mathbb{Q})\backslash M(\mathbb{A}))_{disc}$, of $L^2(Z_M\cdot M(\mathbb{Q})\backslash M(\mathbb{A}))$. If σ is any representation of $M(\mathbb{A})$, and λ belongs to $\mathfrak{a}^*_{P,\mathbb{C}}$ (the complexification of $\mathfrak{a}^*_P$), set

$$\sigma_\lambda(m) = \sigma(m)e^{\lambda(H_M(m))}, \qquad m \in M(\mathbb{A}).$$

Then $R_{M,disc,\lambda}$ is a representation of $M(\mathbb{A}) \cong P(\mathbb{A})/N(\mathbb{A})$, which we can lift to $P(\mathbb{A})$. Let $I_P(\lambda)$ be this representation of $P(\mathbb{A})$ induced to $G(\mathbb{A})$. It acts on the Hilbert space of complex valued functions, ϕ, on $N(\mathbb{A})Z_M M(\mathbb{Q})\backslash G(\mathbb{A})$ such that

(i) the function $m \to \phi(mx)$, $m \in M(\mathbb{A})$, belongs to $L^2(Z_M M(\mathbb{Q})\backslash M(\mathbb{A}))_{disc}$ for each $x \in G(\mathbb{A})$,

and

(ii) $$\|\phi\|^2 = \int_K \int_{Z_M M(\mathbb{Q})\backslash M(\mathbb{A})} |\phi(mk)|^2 dm dk < \infty.$$

Then

$$(I_P(\lambda,y)\phi)(x) = \phi(xy)e^{(\lambda+\rho)(H_P(xy))}e^{-(\lambda+\rho)(H_P(x))}.$$

If λ is purely imaginary, $I_P(\lambda)$ is unitary.

We would expect to find intertwining operators between these induced representations. Let Ω be the restricted Weyl group of G. It acts on $A_0 = A_{P_0}$ and also on $\mathfrak{a}_0 = \mathfrak{a}_{P_0}$. If P and P' are standard parabolic subgroups, $\mathfrak{a}_P$ and $\mathfrak{a}_{P'}$ are both contained in $\mathfrak{a}_0$. Let $\Omega(\mathfrak{a}_P, \mathfrak{a}_{P'})$ be the set of distinct isomorphisms from $\mathfrak{a}_P$ onto $\mathfrak{a}_{P'}$ obtained by restricting elements in Ω to $\mathfrak{a}_P$. The groups P and P' are said to be *associated* if $(\mathfrak{a}_P, \mathfrak{a}_{P'})$ is not empty.

{If $G = GL_n$, Ω is isomorphic to the symmetric group S_n, by

$$\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \longrightarrow \begin{pmatrix} a_{\sigma(1)} & & 0 \\ & \ddots & \\ 0 & & a_{\sigma(n)} \end{pmatrix}, \qquad \sigma \in S_n .$$

The groups P and P' are associated if and only if the corresponding partitions are such that $r = r'$ and $(n_1', \ldots, n_r') = (n_{\tau(1)}, \ldots, n_{\tau(r)})$ for some $\tau \in S_r$.}

For each $s \in \Omega(\mathfrak{a}_P, \mathfrak{a}_{P'})$ let w_s be a representative of s in the normalizer of A_0 in $G(\mathbb{Q})$. Define

$$\big(M(s,\lambda)\phi\big)(x) = \int_{N'(\mathbb{A}) \cap w_s N(\mathbb{A}) w_s^{-1} \backslash N'(\mathbb{A})} \phi(w_s^{-1}nx) e^{(\lambda+\rho)\left(H_P(w_s^{-1}nx)\right)} e^{-(s\lambda+\rho')\left(H_{P'}(x)\right)} dn,$$

for $\phi \in H_P$, $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ and $\rho' = \rho_{P'}$.

<u>LEMMA 3.1</u>: There is a dense subspace H_P^0 of H_P (the space of functions in H_P whose right translates by K and left translates by the center of the universal enveloping algebra of $M(\mathbb{R})$ span a finite dimensional space) such that if $\phi \in H_P^0$ and

$$\big(\mathrm{Re}(\lambda) - \rho\big)(\alpha^\vee) > 0, \qquad \alpha \in \Delta_P,$$

then the integral defining $M(s,\lambda)\phi$ converges absolutely. For λ in this range, $M(s,\lambda)$ is an analytic function with values in $\mathrm{Hom}(H_P^0, H_{P'}^0)$ which intertwines $I_P(\lambda)$ and $I_{P'}(s\lambda)$. In other words, if f belongs to $C_c^\omega\big(G(\mathbb{A})\big)^K$, the space of functions such that $f(k^{-1}xk) = f(x)$ for each $k \in K$, and if

$$I_P(\lambda,f) = \int_{G(\mathbb{A})} f(x) I_P(\lambda,x)\,dx,$$

then

$$M(s,\lambda) I_P(\lambda,f) = I_{P'}(s\lambda,f) M(s,\lambda).$$

See [25].

Next, define

$$E(x,\phi,\lambda) = \sum_{\delta\in P(\mathbb{Q})\backslash G(\mathbb{Q})} \phi(\delta x) e^{(\lambda+\rho)\left(H_P(\delta x)\right)}$$

for $\phi \in \mathcal{H}_P$, $x \in G(\mathbb{A})$ and $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$.

LEMMA 3.2: If $\phi \in \mathcal{H}^0_P$ and

$$\left(\mathrm{Re}(\lambda) - \rho\right)(\alpha^\vee) > 0, \qquad \alpha \in \Delta_P,$$

then the series converges absolutely. It defines an analytic function of λ in this range.

[See [25].

We can now state the fundamental theorem of Eisenstein series.

THEOREM 3.3: (a) Suppose that $\phi \in \mathcal{H}^0_P$. Then $E(x,\phi,\lambda)$ and $M(s,\lambda)\phi$ can be analytically continued as meromorphic functions to $\mathfrak{a}^*_{P,\mathbb{C}}$. On $i\mathfrak{a}^*_P$, $E(x,\phi,\lambda)$ is regular and $M(s,\lambda)$ is unitary. If $f \in C^\infty_c\left(G(\mathbb{A})\right)^K$ and $t \in \Omega(\mathfrak{a}_{P'}, \mathfrak{a}_{P''})$ the following functional equations hold:

(i) $E(x, I_P(\lambda,f)\phi,\lambda) = \int_{C(\mathbb{A})} f(y)E(xy,\phi,\lambda)dy$

(ii) $E(x, M(s,\lambda)\phi,s\lambda) = E(x,\phi,\lambda)$

(iii) $M(ts,\lambda)\phi = M(t,s\lambda)M(s,\lambda)\phi.$

(b) Let $\mathcal{P}$ be an equivalence class of associated standard parabolic subgroups. Let $\hat{L}_{\mathcal{P}}$ be the set of collections $\{F_P : P \in \mathcal{P}\}$ of measurable functions

$$F_P : i\mathfrak{a}_P^* \to \mathcal{H}_P$$

such that

(i) $F_{P'}(s\lambda) = M(s,\lambda)F_P(\lambda), \qquad s \in \Omega(\mathfrak{a}_P,\mathfrak{a}_{P'}),$

(ii) $\|F\|^2 = \sum_{P\in\mathcal{P}} \int_{i\mathfrak{a}_P^*} \|F_P(\lambda)\|^2 d\lambda < \infty.$

Then the map which sends F to the function

$$\Sigma_{P\in\mathcal{P}} \int_{i\mathfrak{a}_P^*} E(x, F_P(\lambda),\lambda)d\lambda,$$

defined for F in a certain dense subspace of $\hat{L}_{\mathcal{P}}$, extends to a unitary map from $\hat{L}_{\mathcal{P}}$ onto a closed $G(\mathbb{A})$-invariant subspace $L^2_{\mathcal{P}}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. Moreover there is an orthogonal decomposition

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A})) = \bigoplus_{\mathcal{P}} L^2_{\mathcal{P}}(G(\mathbb{Q})\backslash G(\mathbb{A}))$$

The proof of this theorem is very difficult. See [25, §7 and Appendix II].

The theorem implies that the regular representation of $G(\mathbb{A})$ on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ decomposes as the direct integral over all (P,λ), where P is a standard parabolic subgroup and λ belongs to the positive chamber in $i\mathfrak{a}_P^*$, of the representations $I_P(\lambda)$. These representations, remember, were obtained by induction from the discrete spectrum of $M(\mathbb{A})$. The discrete spectrum of $G(\mathbb{A})$ corresponds to the case that $P = G$.

We have not burdened the reader with a discussion of normalizations of Haar measures. All the measures used in this section (and the following ones) are Haar measures which have been normalized in natural (but unspecified) ways.

We can now give a precise definition of automorphic representation. An irreducible representation of $G(\mathbb{A})$ is said to be *automorphic* if it is equivalent to an irreducible subquotient of one of the representations

$$I_P(\lambda), \qquad \lambda \in \mathfrak{a}_{P,\mathbb{C}}^*.$$

(See [4], [29]. Our definition is equivalent to the two equivalent conditions of [29, Proposition 2]).

§4. Global intertwining operators

Langlands proved the fundamental results on Eisenstein series before defining the functions $L_S(s,\pi)$ of §2. In fact the definition was suggested by the properties of Eisenstein series, and in particular the global intertwining operators $M(s,\lambda)$. A careful examination of these operators revealed a whole family of new L-functions, some of which were seen to have analytic continuation and functional equations [26].

Fix a standard parabolic P. Then $R_{M,\mathrm{disc}}$, regarded as a representation of $M(\mathbb{A})$, is equivalent to a direct sum

$$\mathfrak{f}(\sigma,U) \cong \mathfrak{E}(\otimes_v \sigma_v, \otimes_v U_v),$$

where each σ_v is an irreducible unitary representation of $M(\mathbb{Q}_v)$ on the Hilbert space U_v. Then

$$\left(I_P(\lambda),\ \mathcal{H}_P\right) \cong \oplus\left(\otimes_v I_P(\sigma_{v,\lambda}),\ \otimes_v \mathcal{H}_P(\sigma_v)\right),$$

where $I_P(\sigma_{v,\lambda})$ is the representation $G(\mathbb{Q})$ induced from the representation

$$\sigma_{v,\lambda}(nm) = \sigma_v(m)e^{\lambda\left(H_M(m)\right)}, \quad n \in N(\mathbb{Q}_v),\ m \in M(\mathbb{Q}_v),$$

of $P(\mathbb{Q}_v)$. It acts on $\mathcal{H}_P(\sigma_v)$, the Hilbert space of measurable functions

$$\Psi_v : N(\mathbb{Q}_v)\backslash G(\mathbb{Q}_v) \to U_v$$

such that

(i) $\Psi_v(mx) = \sigma_v(m)\Psi_v(x)$,

(ii) $\|\Psi_v\|^2 = \int_{K_v} \|\Psi_v(k)\|^2_{U_v} dk < \infty$.

Suppose that a function $\Psi = \otimes_v \Psi_v$ in $\otimes \mathcal{H}_P(\sigma_v)$ is right K-finite. This means that each Ψ_v is right K_v-finite and almost all Ψ_v are right K_v-invariant. Then for each $x \in G(\mathbb{A})$, the vector $\Psi(x) \in U$ corresponds to a smooth function in $L^2\left(Z_M \cdot M(\mathbb{Q})\backslash M(\mathbb{A})\right)_{\text{disc}}$. Let $i\left(\Psi(x)\right)$ be the value of this function at 1. Then

$$\psi(x) = i\left(\Psi(x)\right), \qquad x \in G(\mathbb{A}),$$

belongs to $\mathcal{H}_P^0$. Suppose that $s \in \Omega(\mathfrak{a}_P, \mathfrak{a}_{P'})$, that $w = w_s$, and that $x = \otimes x_v$. Then $\left(M(s,\lambda)\psi\right)(x)$ equals

$$\int_{N'(\mathbb{A})\cap wN(\mathbb{A})w^{-1}\backslash N'(\mathbb{A})} i\left(\Psi(w^{-1}nx)\right)e^{(\lambda+\rho)\left(H_P(w^{-1}nx)\right)} e^{-(s\lambda+\rho')\left(H_{P'}(x)\right)} dn$$

$$= i\left(\otimes_v \left(R_v(w,\lambda)\Psi_v\right)(x_v)\right),$$

where $\left(R_v(w,\lambda)\Psi_v\right)(x_v)$ equals

$$\int_{N'(\mathbb{Q}_v)\cap wN(\mathbb{Q}_v)w^{-1}\backslash N'(\mathbb{Q}_v)} \Psi_v(w^{-1}n_vx_v)e^{(\lambda+\rho)\left(H_P(w^{-1}n_vx_v)\right)}e^{-(s\lambda+\rho')\left(H_{P'}(x_v)\right)}dn_v.$$

THEOREM 4.1: If Ψ_v is right K_v-finite and

$$\left(\mathrm{Re}(\lambda)-\rho\right)(\alpha^\vee) > 0, \qquad \alpha \in \Delta_P,$$

the integral defining $R_v(w,\lambda)\Psi_v$ converges absolutely. For λ in this range, $R_v(w,\lambda)$ is an analytic function which intertwines $I_P(\sigma_{v,\lambda})$ and $I_{P'}(w\sigma_{v,\lambda})$. It can be analytically continued as a meromorphic function to $\mathfrak{a}^*_{P,\mathbb{C}}$. Moreover, if $w' = w_{s'}$, for $s' \in \Omega(\mathfrak{a}_{P'}, \mathfrak{a}_{P''})$, there is a meromorphic scalar valued function $\mu_v(w, w', \lambda, \sigma_v)$, which equals 1 if

$$\mathrm{length}(s's) = \mathrm{length}(s') + \mathrm{length}(s),$$

such that

$$R_v(w'w, \lambda) = \mu_v(w, w', \lambda, \sigma_v)R_v(w', s\lambda)R_v(w,\lambda).$$

For proofs of these statements see [35], [22], [23], [14] and [15]. The functions μ_v can be expressed in terms of Plancherel densities.

The space H^0_P is spanned by the functions $\psi(x) = i\left(\Psi(x)\right)$ described above. For almost all v, $\Psi_v(x_v)$ belongs to $H_P(\sigma_v)^{K_v}$, the space of K_v invariant functions. As we suggested in §2, the structure of this space is similar to that of real groups. It has dimension at most one. (This follows from the fact that the convolution algebra of K_v bi-invariant functions in $C_c(G(\mathbb{Q}_v))$ is abelian, proved under general conditions in [31].) There is a finite set of valuations, S_σ, including the real one, such that if p is not in S_σ, $\mathcal{H}_p(\sigma_p)^{K_p}$ has dimension exactly one. For such a p, $R_p(w,\lambda)$ clearly maps $\mathcal{H}_p(\sigma_p)^{K_p}$ to $\mathcal{H}_{p'}(w\sigma_p)^{K_p}$. The composition with the map which sends $\Psi'_p \in \mathcal{H}_{p'}(w\sigma_p)^{K_p}$ to the function

$$n_p m_p k_p \to \Psi'_p(w m_p k_p), \quad n_p \in N(\mathbb{Q}_p), \; m_p \in M(\mathbb{Q}_p), \; k_p \in K_p,$$

in $\mathcal{H}_P(\sigma_p)^{K_p}$ gives a scalar $m_p(w,\lambda,\sigma)$. Define

$$m_S(w,\lambda,\sigma) = \Pi_{p\notin S}\, m_p(w,\lambda,\sigma)$$

and

$$\mu_S(w,w',\lambda,\sigma) = \Pi_{v\in S}\, \mu_v(w,w',\lambda,\sigma).$$

Then Theorems 3.3 and 4.1 immediately yield

<u>THEOREM 4.2</u>: If the representation σ of $M(\mathbb{A})$ occurs in $R_{m,\mathrm{disc}}$, there is a finite set S such that if

$$\big(\mathrm{Re}(\lambda) - \rho\big)(\alpha^\vee) > 0, \qquad \alpha \in \Delta_P,$$

the infinite product defining $m_S(w,\lambda,\sigma)$ converges absolutely. It can be analytically continued to a meromorphic complex valued function of λ in $\mathfrak{a}^*_{P,\mathbb{C}}$. If $w' = w_{s'}$, $s \in \Omega(\mathfrak{a}_{P'}, \mathfrak{a}_{P''})$, then

$$m_S(w'w,\lambda,\sigma) = \mu_S(w,w',\lambda,\sigma)\, m_S(w',s\lambda,s\sigma)\, m_S(w,\lambda,\sigma).$$

Thus the theory of Eisenstein series leads to some interesting meromorphic functions. Let us show how to express them more explicitly. They are already quite intriguing for their analytic continuation and functional equations rely on the very deep Theorem 3.3. For simplicity, assume that G splits over $\mathbb{Q}$. Then $B = P_0$ is a Borel subgroup of G. Also

$$M_0 = A_0 = T$$

is a maximal torus in G and

$$X(M_0)_{\mathbb{Q}} = \mathrm{Hom}(T, GL_1) = L$$

is the dual module. (Recall that an algebraic torus is an algebraic group which is isomorphic to $(GL_1)^n$. The functor

$$T \to L = \mathrm{Hom}(T, GL_1)$$

defines an anti-isomorphism between the category of algebraic tori defined over a field F and the category of $\mathrm{Gal}(\bar{F}/F)$-modules which are finite and free over $\mathbb{Z}$. Given L, $T(\bar{F})$ is $\mathrm{Gal}(\bar{F}/F)$ isomorphic to $\mathrm{Hom}(L,\bar{F}^*)$. The case here is even simpler since T splits over $\mathbb{Q}$, so $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts trivially on L.) We will not define "splits over $\mathbb{Q}$". We need only know that (after having fixed a Chevalley lattice) it is possible to speak of the group of integral points; as in the case of GL_n, the groups $G(\mathbb{Z}_p)$, $T(\mathbb{Q}_p)$ etc. are defined for any p. We can take $K_p = G(\mathbb{Z}_p)$. Consider the map

$$H_0 = H_{P_0} : G(\mathbb{A}) \to \mathfrak{a}_0 .$$

For any prime p, set

$$H_{0,p}(x_p) = \log p \cdot H_0(x_p), \qquad x_p \in G(\mathbb{Q}_p).$$

Then

$$|t^{\chi}|_p = p^{\langle \chi, H_{0,p}(t)\rangle}, \qquad t \in T(\mathbb{Q}_p), \quad \chi \in X(T)_{\mathbb{Q}}.$$

The range of the valuation $|\ |_p$ is the set of powers of p, so $\langle \chi, H_{0,p}(t)\rangle$ is always an integer. Therefore $H_{0,p}$ is a homomorphism from the multiplicative group $T(\mathbb{Q}_p)$ to the additive group $L^\vee = \mathrm{Hom}(L, \mathbb{Z})$. The group $T(\mathbb{Z}_p)$ is compact. It follows easily that $H_{0,p}$ maps $T(\mathbb{Q}_p)/T(\mathbb{Z}_p)$ isomorphically onto $L^\vee$.

Suppose that $\sigma = \otimes_v \sigma_v$ is as above and that $p \notin S_\sigma$. Then $\mathcal{H}_p(\sigma_p)^{K_p} \neq \{0\}$. This means that $I_p(\sigma_p)$ is a constituent of a so-called class one principle series – a representation of $G(\mathbb{Q}_p)$ induced from the pull-back to $B(\mathbb{Q}_p)$ of a quasi-character on $T(\mathbb{Q}_p)$ which is trivial on $T(\mathbb{Z}_p)$. Such a quasi-character can be written

$$a \to p^{\lambda_p(H_{0,p}(a))}, \qquad a \in T(\mathbb{Q}_p),$$

where λ_p is an element in

$$\mathfrak{a}^*_{0,\mathbb{C}} = \mathrm{Hom}(L^\vee, \mathbb{C}).$$

To construct $\sigma_{p,\lambda}$ instead of σ_p, we simply replace λ_p by $\lambda_p + \lambda$. It follows that

$$m_p(w,\lambda,\sigma) = \int_{N'(\mathbb{Q}_p)\cap wN(\mathbb{Q}_p)w^{-1}\backslash N'(\mathbb{Q}_p)} p^{(\lambda_p+\lambda+\rho_0)(H_{0,p}(w^{-1}n))}\, dn,$$

for λ in the domain of absolute convergence of the integral, and

$$\rho_0 = \rho_{P_0} = \rho_{P_0 \cap M} + \rho_P.$$

Incidentally, the measure dn is the quotient of those Haar measures on $N'(\mathbb{Q}_p)$ and $N'(\mathbb{Q}_p) \cap wN(\mathbb{Q}_p)w^{-1}$ for which the intersection with $G(\mathbb{Z}_p)$ of each group has volume 1.

Consider the special case that $G = GL_2$, $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $P = P_0$. Then σ will be an automorphic representation of $GL_1 \times GL_1$;

$$\sigma\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} = \chi_1(a_1)\chi_2(a_2),$$

with χ_1 and χ_2 Grössencharakters. If p is unramified for both χ_1 and χ_2, there will be complex numbers $s_{1,p}$ and $s_{2,p}$ such that

$$\chi_i(a) = |a|_p^{s_{i,p}} \qquad a \in \mathbb{Q}_p^*.$$

Identify $L^\vee$ with $\mathbb{Z}^2$ in the canonical way. Then

$$H_{0,p}\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} = \left(\log_p|a_1|_p, \ \log_p|a_2|_p\right).$$

The quasi-character λ_p corresponds to the pair $(s_{1,p}, s_{2,p})$, which acts on $L^\vee$ by the dot product. The dual space $\mathfrak{a}^*_{P,\mathbb{C}}$ is identified with $\mathbb{C}^2$, and ρ_0 becomes $(\frac{1}{2}, -\frac{1}{2})$. Set $\lambda = s\rho_0$ for a fixed complex number s. Then

$$p^{(\lambda_p+\lambda+\rho)\left(H_{0,p}\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\right)} = |a_1|_p^{s_{1,p}+\frac{1}{2}+\frac{1}{2}s}|a_2|_p^{s_{2,p}+\frac{1}{2}+\frac{1}{2}s}.$$

If

$$n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \qquad x \in \mathbb{Q}_p,$$

let

$$wn(x) = n_1 \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} k, \quad n_1 \in N(\mathbb{Q}_p), \quad r \in \mathbb{Q}_p^*, \quad k \in GL_2(\mathbb{Z}_p).$$

In order to evaluate the integral it is necessary to express $|r|_p$ as a function of x. Notice that if

$$v = (u_1, u_2), \qquad u_1, u_2 \in \mathbb{Q}_p,$$

and

$$\|v\| = \mathrm{Max}\{|u_1|_p, |u_2|_p\},$$

then $\|vk\| = \|v\|$ for each $k \in GL_2(\mathbb{Z}_p)$. Therefore

$$|r|_p^{-1} = \|(0,1)wn(x)\| = \mathrm{Max}\{1, |x|_p\}.$$

Thus,

$$\begin{aligned} m_p(w,\lambda,\sigma) &= \int_{N(\mathbb{Q}_p)} e^{(\lambda_p+\lambda+\rho_0)(H_{0,p}(wn))} dn \\ &= \int_{\mathbb{Q}_p} (\mathrm{Max}\{1, |x|_p\})^{-(t+1)} dx \\ &= \int_{\mathbb{Z}_p} 1\, dx + \int_{\mathbb{Q}_p - \mathbb{Z}_p} |x|^{-(t+1)} dx \end{aligned}$$

where

$$t = s_{1,p} - s_{2,p} + s.$$

We want the Haar measure on $\mathbb{Q}_p$ such that $\mathrm{vol}(\mathbb{Z}_p)$ equals one. $\mathbb{Q}_p$ is the disjoint union of the sets $\{p^n U_p : -\infty < n < \infty\}$, where

$$U_p = \{x \in \mathbb{Q}_p : \ |x|_p = 1\},$$

so

$$\begin{aligned} 1 = \mathrm{vol}(\mathbb{Z}_p) &= \textstyle\sum_{n=0}^{\infty} \mathrm{vol}(p^n U_p) \\ &= \mathrm{vol}(U_p) \textstyle\sum_{n=0}^{\infty} \left(\frac{1}{p}\right)^n \\ &= \mathrm{vol}(U_p)\left(1 - \frac{1}{p}\right)^{-1}. \end{aligned}$$

In other words, $\mathrm{vol}(U_p)$ equals $\left(1 - \frac{1}{p}\right)$. It follows that $m_p(w,\lambda,\sigma)$ equals

$$1 + \textstyle\sum_{n=1}^{\infty} p^{-n(t+1)} \, p^n\left(1 - \frac{1}{p}\right).$$

If $\mathrm{Re}(t) > 0$ this converges and equals

$$1 + \left(1 - \frac{1}{p}\right)\frac{p^{-t}}{1 - p^{-t}}$$

$$= \left(1 - p^{-(s_{1,p} - s_{2,p})} p^{-(s+1)}\right)\left(1 - p^{-(s_{1,p} - s_{2,p})} p^{-s}\right)^{-1}$$

$$= \frac{L_p(s, \ \chi_1\chi_2^{-1})}{L_p(1+s, \ \chi_1\chi_2^{-1})}.$$

We have shown that the scalar $m_S(w,\lambda,\sigma)$ equals $\dfrac{L_S(s, \ \chi_1\chi_2^{-1})}{L_S(1+s, \ \chi_1\chi_2^{-1})}$.

Theorem 4.2 implies that $\dfrac{L_S(s, \ \chi_1\chi_2^{-1})}{L_S(1+s, \ \chi_1\chi_2^{-1})}$ can be analytically continued as a mermorphic function. It also yields the functional equation

$$\frac{L_S(s, \ \chi_1\chi_2^{-1})}{L_S(1+s, \ \chi_1\chi_2^{-1})} = \mu_S(w, \ w^{-1}, \ s\rho_0, \ \sigma)\frac{L_S(1-s, \ \chi_2\chi_1^{-1})}{L_S(-s, \ \chi_2\chi_1^{-1})}.$$

(μ_S is a product of Plancherel densities, and is an elementary function.) It follows that $L_S(s,\chi_1\chi_2^{-1})$ can also be analytically continued. This was well known as was the functional equation. (The usual function equation is of course stronger, for it relates the functions $L_S(s, \chi_1\chi_2^{-1})$ and $L_S(1-s, \chi_2\chi_1^{-1})$. It is the case $n=1$ in Theorem 2.2, which was established many years ago by Hecke.) However, Theorem 3.3 applies to arbitrary G where it certainly leads to nonclassical results.

In general, the integral

$$\int_{N'(\mathbb{Q}_p)\cap wN(\mathbb{Q}_p)w^{-1}\backslash N'(\mathbb{Q}_p)} p^{(\lambda_P+\lambda+\rho_0)\left(H_{0,p}(w^{-1}n)\right)}\, dn$$

is evaluated by the method of Gindinkin and Karpelevic ([13], [26]). Identify each $s \in \Omega(\mathfrak{a}_P, \mathfrak{a}_{P'})$ with the unique element in Ω which maps each simple root of (M,A_0) to a simple root. We can certainly assume that w acts on A_0 by this element in Ω. Then

$$N'(\mathbb{Q}_p) \cap wN(\mathbb{Q}_p)w^{-1}\backslash N'(\mathbb{Q}_p) \simeq N_0(\mathbb{Q}_p) \cap wN_0(\mathbb{Q}_p)w^{-1}\backslash N_0(\mathbb{Q}_p).$$

For almost p, each element w belongs to K_p. We can therefore suppose this is so for $p \notin S$. Then $m_p(w,\lambda,\sigma)$ equals

$$\int_{\bar{N}(s)} p^{(\lambda_P+\lambda+\rho_0)\left(H_{0,p}(\bar{n})\right)}\, d\bar{n},$$

where $\bar{N}(s)$ is the intersection of $wN_0(\mathbb{Q}_p)w^{-1}$ with $\bar{N}_0(\mathbb{Q}_p)$, the unipotent radical of the parabolic subgroup opposite to P_0. Suppose that $s = s_1 s_\gamma$, where s_1 is some other element in Ω, and s_γ is the reflection about a simple root γ of (G, A_0) such that

$$\text{length}(s) = \text{length}(s_1) + 1.$$

A simple change of variables exhibits the integral as the product of

$$\int_{\bar{N}(s_1)} p^{\left(s_\gamma(\lambda_p+\lambda)+\rho_0\right)\left(H_{0,p}(\bar{n})\right)}\, d\bar{n}$$

with

$$\int_{\bar{N}(s_\gamma)} p^{(\lambda_p+\lambda+\rho_0)\left(H_{0,p}(\bar{n})\right)}\, d\bar{n}.$$

Since $\bar{N}(s_\gamma)$ is one dimensional, this last integral reduces to the one on GL_2, which we have just calculated. It equals

$$\frac{1 - p^{-\left((\lambda_p+\lambda)(\gamma^\vee)+1\right)}}{1 - p^{-(\lambda_p+\lambda)(\gamma^\vee)}} .$$

Now if Σ_+ is the set of positive roots of (G,A_0),

$$\{\beta \in \Sigma_+ : s\beta < 0\} = s_\gamma^{-1}\{\beta \in \Sigma_+ : s_1\beta < 0\}.$$

It follows by induction on the length of s that the integral equals

$$\Pi_{\{\beta\in\Sigma_+ : s\beta<0\}} \frac{1 - p^{-\left((\lambda_p+\lambda)(\beta^\vee)+1\right)}}{1 - p^{-(\lambda_p+\lambda)(\beta^\vee)}} .$$

This therefore equals $m_p(w,\lambda,\sigma)$.

§5. L-groups and functoriality

The history of representations of reductive groups has often been of phenomena which generalize from GL_2 to arbitrary groups. The functions $m_S(w,\lambda,\sigma)$ could hardly be otherwise. They must surely be quotients of L-functions which generalize the functions $L_S(s, \chi_1\chi_2^{-1})$ obtained for GL_2. We would expect them to arise from a family of semisimple conjugacy classes in some complex general linear group. The connection is made through the notion of an L-group, introduced by Langlands in [27].

For the moment, we shall continue to take G to be a reductive group defined and split over $\mathbb{Q}$. Suppose that $\pi = \otimes\pi_v$ is an automorphic representation of G. It is said to be unramified at a prime p if π_p has a K_p-fixed vector. Then π_p is a constituent of the class one principle series associated to a quasi-character

$$a \to p^{\lambda_p(H_{0,p}(a))}, \qquad a \in T(\mathbb{Q}_p)/T(\mathbb{Z}_p).$$

The quasi-character is not uniquely determined by π_p. The situation is similar to that of real groups. The unramified representations π_p of $G(\mathbb{Q}_p)$ are in bijective correspondence with the orbits of the Weyl group Ω acting on quasi-characters of $T(\mathbb{Q}_p)/T(\mathbb{Z}_p)$ [3]. Now $L^\vee$ is a finite free $\mathbb{Z}$-module. It is associated to a unique algebraic torus over $\mathbb{C}$ whose group of complex points, denoted by ${}^LT^0$, is canonically isomorphic to $\mathrm{Hom}(L^\vee, \mathbb{C}^*)$. The map

$$H \to p^{-\lambda_p(H)}, \qquad H \in L^\vee,$$

belongs to $\mathrm{Hom}(L^\vee, \mathbb{C}^*)$ so it defines a point t_p in ${}^LT^0$; that is

$$t_p^H = p^{-\lambda_p(H)}, \qquad H \in L^\vee.$$

The Weyl group Ω acts on ${}^LT^0$ through its action on $L^\vee$. There is thus a bijective correspondence between irreducible class one representations of $G(\mathbb{Q}_p)$ and orbits of Ω in ${}^LT^0$.

The set of roots, Σ, of (G,T) is contained in L. Similarly the co-roots, $\Sigma^\vee$, are contained in $L^\vee$. Thus associated to the pair (G,T), regarded only as algebraic groups over $\mathbb{C}$, there is the *root datum* $(L, \Sigma, L^\vee, \Sigma^\vee)$. Associated to the triple (G,B,T) there is the *based root datum* $(L, \Delta, L^\vee, \Delta^\vee)$, where $\Delta = \Delta_{P_0}$ is the set of simple roots and $\Delta_{P_0}^\vee = \Delta^\vee$ the simple co-roots, defined by B. (For formal definitions of root data and based root datum see [37].) Conversely any root datum comes from a pair $(\bar{G}, \bar{T})$ and any based root datum comes from a triple $(\bar{G}, \bar{B}, \bar{T})$, where $\bar{G}$ is a reductive group over $\mathbb{C}$, $\bar{T}$ is a maximal torus, and $\bar{B}$ is a Borel subgroup containing $\bar{T}$ [37]. Both $(\bar{G}, \bar{T})$ and $(\bar{G}, \bar{B}, \bar{T})$ are uniquely determined up to isomorphism. Now $(L^\vee, \Delta^\vee, L, \Delta)$ is a second based root datum. It therefore comes from a triple $({}^LG^0, {}^LB^0, {}^LT^0)$. By convention, ${}^LG^0$ will denote the set of *complex points* of a reductive algebraic group over $\mathbb{C}$, and ${}^LB^0$ will be a Borel subgroup of ${}^LG^0$. As above, ${}^LT^0$ equals $\mathrm{Hom}(L^\vee, \mathbb{C}^*)$ and can be regarded as a Cartan subgroup of ${}^LG^0$. The set of simple roots of $({}^LG^0, {}^LT^0)$ is $\Delta^\vee$, and the set of simple co-roots is Δ. There is a bijective correspondence $P \longleftrightarrow {}^LP^0$ between standard parabolic subgroups of G and of ${}^LG^0$. The Weyl group of $({}^LG^0, {}^LT^0)$ is isomorphic to Ω, with its natural action on ${}^LT^0$. Some examples are in the following brief table

G	${}^LG^0$
GL_n	$GL_n(\mathbb{C})$
PGL_n	$SL_n(\mathbb{C})$
SL_n	$PGL_n(\mathbb{C})$
Sp_{2n}	$SO_{2n+1}(\mathbb{C})$
PSp_{2n}	$Spin_{2n+1}(\mathbb{C})$
SO_{2n+1}	$Sp_{2n}(\mathbb{C})$
$Spin_{2n+1}$	$PSp_{2n}(\mathbb{C})$
$Spin_{2n}$	$SO_{2n}(\mathbb{C})/\{\pm 1\}$
$SO_{2n}/\{\pm 1\}$	$Spin_{2n}(\mathbb{C})$

The semisimple conjugacy classes in ${}^LG^0$ are in one-to-one correspondence with the orbits of Ω in ${}^LT^0$. It follows that for every automorphic representation π of the split group G there is a finite set S_π of primes, and for every $p \notin S_\pi$ there is a semisimple conjugacy class $\Phi_p(\pi)$ in ${}^LG^0$. How does this compare to the conjugacy classes defined in an ad hoc manner for GL_n in §2? The *Hecke algebra* for $G(\mathbb{Q}_p)$, $H(G(\mathbb{Q}_p))$, is the space of compactly supported functions on $G(\mathbb{Q}_p)$ which are left and right invariant by $K_p = G(\mathbb{Z}_p)$. It is an algebra under convolution. If $f \in H(G(\mathbb{Q}_p))$, and (π_p, U_p) is an (irreducible) unramified representation of $G(\mathbb{Q}_p)$, the operator

$$\pi_p(f) = \int_{G(\mathbb{Q}_p)} f(x)\pi_p(x)dx$$

on U_p leaves invariant the subspace $U_p^{G(\mathbb{Z}_p)}$. It vanishes on the complement of $U_p^{G(\mathbb{Z}_p)}$ in U_p. Since $\dim U_p^{G(\mathbb{Z}_p)} = 1$, the restriction of $\pi_p(f)$ to this subspace is a scalar, say $c_{p,f}(\pi_p)$. It is clear that

$$c_{p,f_1*f_2}(\pi_p) = c_{p,f_1}(\pi_p)c_{p,f_2}(\pi_p).$$

We regard $c_{p,f}$ as a function on the semisimple conjugacy classes in ${}^LG^0$. It can be shown that the map $f \to c_{p,f}$ is an isomorphism from $\mathcal{H}(G(\mathbb{Q}_p))$ onto the algebra of class functions on ${}^LG^0$ generated by the characters of finite dimensional representations [see [3] and either [5] or [31]). In the case of $G = GL_n$, to what class function on $GL_n(\mathbb{C})$ does $f_{p,i}$, defined in §2, correspond? It turns out to be just the multiple by $p^{-\frac{1}{2}i(n-i)}$ of the character of the action of $GL_n(\mathbb{C})$ on the $i^{\underline{th}}$ exterior power of $\mathbb{C}^n$ [39]. These characters, evaluated on a conjugacy class in $GL_n(\mathbb{C})$, are the coefficients of the characteristic polynomial. We obtain the formula for $\Phi_p(\pi)$ given in §2.

Now, return to the problem of §4. Then $\sigma = \bigotimes_v \sigma_v$ is a representation of $M(\mathbb{A})$ that occurs in $R_{M,\mathrm{disc}}$. If $p \notin S_\sigma$,

$$\begin{aligned} m_p(w,\lambda,\sigma) &= \Pi_{\{\beta\in\Sigma_+ : s\beta<0\}} \frac{1 - p^{-\left((\lambda_p+\lambda)(\beta^\vee)+1\right)}}{1 - p^{-(\lambda_p+\lambda)(\beta^\vee)}} \\ &= \Pi_{\{\beta\in\Sigma_+ : s\beta<0\}} \frac{1 - (t_p)^{\beta^\vee}\cdot p^{-\left(\lambda(\beta^\vee)+1\right)}}{1 - (t_p)^{\beta^\vee}\cdot p^{-\lambda(\beta^\vee)}}. \end{aligned}$$

Let ${}^LP^0 = {}^LN^0 \cdot {}^LM^0$ be the standard parabolic subgroup of ${}^LG^0$ which corresponds to P. Then ${}^LM^0$ acts on ${}^L\mathfrak{n}^0$, the Lie algebra of ${}^LN^0$. Let ${}^LA_P^0$ be the center of ${}^LM^0$. It is contained in ${}^LT^0$. The weights on ${}^LA_P^0$ of the representation of ${}^LM^0$ on ${}^L\mathfrak{n}^0$ are just the restrictions, $\alpha^\vee$, to ${}^LA_P^0$ of roots $\beta^\vee$ of $({}^LG^0, {}^LT^0)$; that is,

$$ {}^L\mathfrak{n}^0 = \bigoplus_{\alpha^\vee > 0} {}^L\mathfrak{n}^0_{\alpha^\vee} $$

where

$$ {}^L\mathfrak{n}^0_{\alpha^\vee} = \{X \in {}^L\mathfrak{n}^0 : \mathrm{Ad}(a)X = a^{\alpha^\vee} X \text{ for all } a \in {}^LA^0_P\}. $$

Each subspace ${}^L\mathfrak{n}^0_{\alpha^\vee}$ of ${}^L\mathfrak{n}^0$ is invariant under ${}^LM^0$. Let $r_{\alpha^\vee}$ be the restriction of ${}^LM^0$ to this subspace. The weights of $r_{\alpha^\vee}$ on ${}^LT^0$ are the set of $\beta^\vee$ in Σ_+ whose restriction to ${}^LA^0_P$ is $\alpha^\vee$. Now λ is a linear function $\mathfrak{a}_P$. Therefore, if $\beta^\vee$ restricts to $\alpha^\vee$,

$$ e^{-\lambda(\beta^\vee)} = e^{-\lambda(\alpha^\vee)}. $$

It follows that

$$ m_p(w,\lambda,\sigma) = \prod_{\{\alpha^\vee>0,\, s\alpha^\vee<0\}} \frac{\det\left(I - r_{\alpha^\vee}(t_p)p^{-(\lambda(\alpha^\vee)+1)}\right)}{\det\left(I - r_{\alpha^\vee}(t_p)p^{-\lambda(\alpha^\vee)}\right)} $$

$$ = \prod_{\{\alpha^\vee>0,\, s\alpha^\vee<0\}} \frac{\det\left(I - r_{\alpha^\vee}(\Phi_p(\pi))p^{-(\lambda(\alpha^\vee)+1)}\right)}{\det\left(I - r_{\alpha^\vee}(\Phi_p(\pi))p^{-\lambda(\alpha^\vee)}\right)}. $$

We are finally in a position to appreciate the definition of a Langlands' L-function. Suppose that $\pi = \bigotimes_v \pi_v$ is an automorphic representation of $G(\mathbb{A})$. Then if r is any finite dimensional, complex analytic representation of ${}^LG^0$, define

$$ L_p(z,\pi,r) = \det\left(I - r(\Phi_p(\pi))z\right)^{-1}, \qquad p \notin S_\pi, $$

and

$$L_S(s,\pi,r) = \Pi_{p\notin S} L_p(p^{-s},\pi,r),$$

for any finite set of valuations, S, which contains S_π. If $G = GL_n$ and r is the standard n-dimensional representation of $GL_n(\mathbb{C})$, the L-functions are just those defined in §2. They are called *standard* (or *principal*) L-functions.

Returning once again to the case that σ is a representation of $M(\mathbb{A})$ which occurs in $R_{M,\mathrm{disc}}$, we see that

$$m_S(w,\lambda,\sigma) = \Pi_{\{\alpha^\vee>0,s\alpha^\vee<0\}} \frac{L_S(\lambda(\alpha^\vee),\ \sigma,\ r_{\alpha^\vee})}{L_S(\lambda(\alpha^\vee)+1,\ \sigma,\ r_{\alpha^\vee})}.$$

According to Theorem 4.2 this function has analytic continuation and functional equation. If P is a maximal parabolic, and there is only one root of (P,A), there will then be only one $\alpha^\vee$ in the product. It will follow that $L_S(s,\ \sigma,\ r_{\alpha^\vee})$ has analytic continuation. It also will satisfy a functional equation, albeit weaker than the ones satisfied by classical L-functions.

<u>CONJECTURE 5.1</u> (Langlands): Suppose that π is an automorphic representation of $G(\mathbb{A})$ and that r is a finite dimensional analytic representation of ${}^LG^0$. Then $L_S(s,\pi,r)$ can be analytically continued with a functional equation relating $L_S(s,\pi,r)$ with $L_S(1-s,\ \pi,\ \tilde{r})$.

See [27, Question 1].

This conjecture suggests another one. Suppose one were handed a split group G and a family $\{\Phi_p : p \notin S\}$ of semi-simple conjugacy classes in ${}^LG^0$. Suppose it happened that for every finite dimensional representation r of ${}^L G^0$ the function

$$s \to \Pi_{p \notin S} \det\left(1 - r(\Phi_p) p^{-s}\right)^{-1}$$

was defined in a half plane and could be analytically continued with functional equation. With the converse theorems for GL_2 and GL_3 in mind, one would strongly suspect that the family $\{\Phi_p\}$ arose from an automorphic representation of G.

Now, let $\bar{G}$ be another split group, and let $\rho : {}^L G^0 \to {}^L \bar{G}^0$ be an analytic homomorphism. If $\bar{r}$ is a finite dimensional analytic representation of ${}^L \bar{G}^0$,

$$r = \bar{r} \circ \rho$$

is a finite dimensional analytic representation of ${}^L G^0$. Take an automorphic representation π of G, and for each $p \notin S_\pi$ let $\bar{\Phi}_p$ be the semisimple conjugacy class in ${}^L \bar{G}^0$ which contains $\rho\left(\Phi_p(\pi)\right)$. It is clear that

$$L_S(s,\pi,r) = \Pi_{p \notin S} \det\left(I - \bar{r}(\bar{\Phi}_p) p^{-s}\right)^{-1} .$$

The analytic continuation and functional equation of the function on the right would follow from Conjecture 5.1. The family $\{\bar{\Phi}_p\}$ surely ought to be associated with an automorphic representation of $\bar{G}$.

<u>CONJECTURE 5.2</u> (Langlands): Given an analytic homomorphism $\rho : {}^L G^0 \to {}^L \bar{G}^0$ and an automorphic representation π of G there is an automorphic representation $\bar{\pi}$ of $\bar{G}$ such that $S_\pi = S_{\bar{\pi}}$ and such that for each $p \notin S_\pi$, $\Phi_p(\bar{\pi})$ is the

conjugacy class in ${}^L\bar{G}^0$ which contains $\rho(\Phi_p(\pi))$.

See [27], [2].

If $\bar{G} = GL_n$ we could take $\bar{r}$ to be the standard representation of $GL_n(\mathbb{C})$. Then r equals ρ and $L_S(s,\pi,r)$ clearly equals $L_S(s,\bar{\pi})$, a standard L-function for GL_n. Its analytic continuation and functional equation would be assured by Theorem 2.2. Therefore Conjecture 5.2 implies Conjecture 5.1.

For all of §5 and much of §4 we have taken G to be a split group. This was only for simplicity. What happens if G is an arbitrary reductive group defined over $\mathbb{Q}$? We can always choose a maximal torus T defined over $\mathbb{Q}$ and a Borel subgroup (not necessarily defined over $\mathbb{Q}$) such that $A_0 \subset T \subset B \subset P_0$. The based root datum, $(L, \Delta, L^\vee, \Delta^\vee)$, associated as it is to the group $G(\mathbb{C})$, is defined as above. So is the triple $({}^LG^0, {}^LB^0, {}^LT^0)$. The theory of algebraic groups yields in addition a homomorphism from $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ to $\mathrm{Aut}(G)/\mathrm{Int}(G)$, the group of automorphisms of G defined over $\bar{\mathbb{Q}}$, modulo the group of inner automorphisms. It follows from a theorem of Chevalley that this quotient group is isomorphic to the group of automorphisms of the based root datum $(L, \Delta, L^\vee, \Delta^\vee)$. Every automorphism of $(L, \Delta, L^\vee, \Delta^\vee)$ is clearly an automorphism of $(L^\vee, \Delta^\vee, L, \Delta)$, so therefore gives an element in $\mathrm{Aut}({}^LG^0)/\mathrm{Int}({}^LG^0)$. It is easily seen that the exact sequence

$$1 \to \mathrm{Int}({}^LG^0) \to \mathrm{Aut}({}^LG^0) \to \mathrm{Aut}({}^LG^0)/\mathrm{Int}({}^LG^0) \to 1$$

splits. This gives a homomorphism of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ into $\mathrm{Aut}({}^LG^0)$. In other words, $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on ${}^LG^0$. Define ${}^LG$ to be the semidirect product

$$^{L}G^{0} \rtimes \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}).$$

It is a locally compact group, called the L-*group* of G. The exact sequence above can be split in such a way that $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ normalizes $^{L}B^{0}$ and $^{L}T^{0}$. Then $^{L}B = {}^{L}B^{0} \rtimes \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and $^{L}T = {}^{L}T^{0} \rtimes \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ are closed subgroups of ^{L}G.

Suppose that $\pi = \bigotimes_{v} \pi_{v}$ is an automorphic representation of G. Then π is said to be *unramified* at a prime p if π_p contains a K_p-fixed vector and if in addition, G is quasi-split over $\mathbb{Q}_p$ and splits over an unramified extension of $\mathbb{Q}_p$. One shows that for each unramified p there corresponds a natural conjugacy class $\Phi_p(\pi)$ in ^{L}G whose projection onto $^{L}G^{0}$ contains only semisimple elements [3]. For finite dimensional representations r of ^{L}G, the L-functions $L_S(s,\pi,r)$ are defined exactly as above. The computation of the functions $m_p(w,\lambda,\sigma)$ can be carried out for general G. The resulting formula is similar to the one for split G (see [34]).

Suppose that G and $\bar{G}$ are reductive groups over $\mathbb{Q}$. An L-*homomorphism* is a continuous homomorphism

$$\rho : {}^{L}G \to {}^{L}\bar{G}$$

which is compatible with the projections of each group onto $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and whose restriction to $^{L}G^{0}$ is a complex analytic homomorphism of $^{L}G^{0}$ to $^{L}\bar{G}^{0}$, ([3]). The generalization of Conjecture 5.2 is

CONJECTURE 5.3 (Langlands): Suppose G and $\bar{G}$ are reductive groups over $\mathbb{Q}$ which are quasi-split. Suppose that $\rho : {}^L G \to {}^L\bar{G}$ is an L-homomorphism. Then for every automorphic representation π of G there is an automorphic representation $\bar{\pi}$ of $\bar{G}$ such that $S_\pi = S_{\bar{\pi}}$ and such that for each $p \notin S_\pi$, $\Phi_p(\bar{\pi})$ is the conjugacy class in ${}^L\bar{G}$ which contains $\rho(\Phi_p(\pi))$.

See [27], [2].

This conjecture is known as the functoriality principle, and is very far from being solved. It implies that all the L-functions $L_S(s,\pi,r)$ can be analytically continued with functional equation, and that the location and residues of all poles can be determined. For an important special case, take $G = \{1\}$ and $\bar{G} = GL_n$. Then an L-homomorphism is just a continuous homomorphism

$$\sigma : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_n(\mathbb{C}).$$

The functoriality principle in this case reduces to Conjecture 2.3.

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