The Selberg trace formula for groups of F-rank one

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Introduction

An important tool for the study of automorphic forms is a non-abelian analogue of the Poisson summation formula, generally known as the Selberg trace formula. There have been a number of publications on the subject following Selberg's original paper [10], the most recent being [2] and [7, § 16]. With the exception of Selberg's brief account [11], however, most authors have restricted themselves to the groups SL(2) and GL(2). In this paper we develop the formula for a wider class of groups.

We shall work in an adèlic framework so our group G will be a reductive algebraic group defined over a number field F. We require that the F-rank of the semisimple component of G be one. To simplify our introduction, let us assume that G itself is semisimple. If $\mathbf{A}$ is the adèle ring of F, let λ be the regular representation of $G_{\mathbf{A}}$ on $L^2(G_{\mathbf{A}}/G_F)$. It is important to try to decompose λ into irreducible representations.

To begin with, λ splits into a sum of two representions λ_0 and λ_1 such that λ_0 is a direct sum of irreducible representations while λ_1 decomposes continuously. The theory of Eisenstein series provides us with a fairly good understanding of the decomposition of λ_1. However, virtually nothing is known about how to pick out the irreducible components of λ_0. It is at this problem that the Selberg trace formula is aimed.

λ may be regarded as a representation of $L^1(G_{\mathbf{A}})$. Our first aim will be to prove that the operator $\lambda_0(f)$ is of trace class when f is a suitably regular function on $G_{\mathbf{A}}$. Besides imposing the usual conditions on f we shall be forced to make an additional assumption. If G_∞ is a product of groups of real rank one this assumption is harmless, but in general it is unsatisfactory. After two preliminary sections, we state our assumptions in § 3, where we also establish the desired properties of $\lambda_0(f)$.

Once we have proved that the operator $\lambda_0(f)$ is of trace class, we can go about calculating its trace. To do this we must take the kernel, K_0, of $\lambda_0(f)$, and integrate it over the diagonal. K_0 can be expressed as the difference

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of the kernels K and K_1 of $\lambda(f)$ and $\lambda_1(f)$ respectively. To study K_1 we shall need to quote a number of results from the theory of Eisenstein series. The basic references are [5], [8], and [9] where the results are proved for discrete subgroups of real Lie groups. This covers our situation because any automorphic form on $G_{\mathbf{A}}/G_F$ can be regarded as a finite sum of functions, each of which is an automorphic form on the quotient of G_∞ by some arithmetic subgroup.

In Sections 4 through 8 we analyze the functions $K(x, x)$ and $K_1(x, x)$, breaking each one up into a number of components. Although neither of these functions is integrable, we find in § 6 that the non-integrable components of each function cancel. All the remaining terms turn out to be integrable, although in § 8 we need to appeal to the integrability of $K_0(x, x)$ itself to verify this. We integrate each term as we go along, leaving the results to be collected in § 9 in our final formula. The treatment of these last five sections is strongly motivated by [7, pp. 526–546].

Most of the methods used in this paper originate with Selberg (see [11]), including the ideas behind the proof of Theorem 3.2 and the convergence of the integrals in § 8. These were described to me by Robert Langlands whom I would like to thank for his encouragement. I am also grateful for the comments of Stephen Gelbart, who read through the original manuscript.

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1. Preliminaries

Let G be a connected reductive algebraic matrix group defined over a number field F. For any place v of F we shall write G_v for G_{F_v}, the group of F_v-rational points of G. We shall denote the adèles of F by $\mathbf{A}$, and we write $G_{\mathbf{A}}$ for the corresponding adèlized group. If ∞ and f stand for the set of infinite and finite places of F respectively, we can write

$$G_{\mathbf{A}} = G_\infty G_f .$$

Our concern will be the study of certain complex-valued functions on

$G_{\mathbf{A}}$. If H is any F-subgroup of G, we write $C_c^\infty(H_{\mathbf{A}})$ for the space of linear combinations of functions

$$f = \prod_v f_v$$

that satisfy the following conditions:

(i) If v is infinite, $f_v \in C_c^\infty(H_v)$;

(ii) If v is finite, f_v is locally constant and has compact support;

(iii) For almost all finite places v, f_v is the characteristic function of G_{O_v}.

We shall also sometimes write $C^\infty(H_{\mathbf{A}})$ for the space of linear combinations of functions of the form

$$f = f_\infty f_f\ ,$$

where f_∞ is a differentiable function H_∞ and f_f is a locally constant function on H_f.

The radical of G is a torus which is defined over F. Let Z be its F-split component. Let $X(G)$ be the group of rational characters on G, and let $X(G)_F$ be those characters in $X(G)$ which are defined over F. Define

$$\mathfrak{z} = \mathrm{Hom}\,(X(G)_F, \mathbf{R})\ ,$$
$$\mathfrak{z}^* = X(G)_F \otimes \mathbf{R}\ .$$

Then

$$\dim Z = \dim \mathfrak{z} = \dim \mathfrak{z}^*\ .$$

We define a map

$$H_G\colon G_{\mathbf{A}} \longrightarrow \mathfrak{z}$$

by

$$e^{\langle \chi, H_G(x)\rangle} = |\,\chi(x)\,|,\ \chi \in X(G)_F,\ x \in G_{\mathbf{A}}\ .$$

Let $G^1_{\mathbf{A}}$ be the kernel of H_G.

We shall define a subgroup Z_∞^+ of Z_∞. Fix a basis $\chi_1, \cdots, \chi_r$ of $X(G)_F$. The restriction of these characters defines an F-homomorphism ϕ from Z to $\mathrm{GL}(1)^r$. This in turn defines a homomorphism ϕ_∞ from the identity component of Z_∞ onto the identity component of $\mathrm{GL}\,(1, \infty)^r$. For any positive real number λ we let $\xi(\lambda)$ be the idèle such that $\xi(\lambda)_v = 1$ for every finite place v of F and $\xi(\lambda)_w = \lambda$ for every infinite place w of F. The collection $\{\xi(\lambda)\colon \lambda > 0\}$ defines a subgroup, $\mathrm{GL}^+\,(1, \infty)$, of the identity component of $\mathrm{GL}\,(1, \infty)$. Let Z_∞^+ be the inverse image of $\mathrm{GL}^+\,(1, \infty)^r$ under ϕ_∞.

The restriction of H_G maps Z_∞^+ bijectively onto $\mathfrak{z}$. Therefore $G_{\mathbf{A}}$ is the direct product of Z_∞^+ and $G^1_{\mathbf{A}}$. The group Z_∞^+ is independent of the basis $\chi_1, \cdots, \chi_r$.

We would like to specify Haar measures on certain subgroups of $G_{\mathbf{A}}$. On any discrete group we will use the Haar measure that assigns to any point the measure 1. On any quotient of unimodular groups to which we have assigned Haar measures we will use the corresponding quotient measure.

Let dx be the Tamagawa measure on $G_{\mathbf{A}}$. We recall the definition. The representation σ of the Galois group $\mathfrak{g}(\bar{F}/F)$ on the vector space $X(G)\otimes F$ defines an Euler product

$$L(s, \sigma) = \prod_v L_v(s, \sigma) .$$

The order of the pole of $L(s, \sigma)$ at 1 equals the multiplicity of the identity representation in σ, which is

$$\dim\left(X(G)_F \otimes F\right) = r.$$

Let ψ be a nontrivial character on $\mathbf{A}$ which is trivial on F. At each place v of F, ψ defines a nontrivial character ψ_v of F_v. Let $d\xi_v$ be the measure on F_v, self-dual with respect to ψ_v, and let $d\xi = \prod_v d\xi_v$. Let ω be a left invariant form of highest degree on G, defined over F. For each v define the measure dx_v on G_v to be

$$L(1, \sigma_v)\,|\,\omega\,|_v ,$$

where $|\,\omega\,|_v$ is the measure defined by the form ω and $d\xi_v$. If Δ_F is the discriminant of F, dx is the Haar measure on $G_{\mathbf{A}}$ which equals

$$|\,\Delta_F\,|^{-\dim G/2} \lim_{s\to 1} \frac{1}{(s-1)^r L(s, \sigma)} \prod_v dx_v .$$

This measure is independent of the choice of ψ and ω.

Any basis $\chi_1, \cdots, \chi_r$ of $X(G)_F$ defines an isomorphism between Z_∞^+ and $(\mathbf{R}_+^*)^r$. We take as measure on Z_∞^+ that which corresponds to the Euclidean measure on $(\mathbf{R}_+^*)^r$. This measure is independent of the choice of $\chi_1, \cdots, \chi_r$. Our measures on $G_{\mathbf{A}}$ and Z_∞^+ define a measure on $G_{\mathbf{A}}^1$ which we also denote by dx. It is well known that the number

$$\tau(G) = \int_{G_{\mathbf{A}}^1/G_F} dx = \int_{G_{\mathbf{A}}/G_F Z_\infty^+} dx$$

is finite. $\tau(G)$ is the Tamagawa number of G.

Let P be a parabolic subgroup of G defined over F. Let N be the unipotent radical of P. N is connected and defined over F. Fix a Levi component, M, of P. M is connected. It is known that M is defined over F and that P is the semi-direct product of M and N. In particular the maps

$$M \times N \longrightarrow P ,$$
$$N \times M \longrightarrow P ,$$

regarded as morphisms of algebraic sets, are isomorphisms defined over F. It follows that

$$P_F = M_F N_F = N_F M_F \ .$$

Let A be the F-split component of the radical of M. A contains Z. If we replace (G, Z) by (M, A) we can define the vector space $\mathfrak{a}$, the map H_M, the groups $M_{\mathbf{A}}^1$ and A_∞^+, and the measures on $M_{\mathbf{A}}$, A_∞^+ and $M_{\mathbf{A}}^1$ as above.

There is an isomorphism of affine varieties

$$\exp\colon \mathfrak{n} \longrightarrow N$$

defined over F, from $\mathfrak{n}$, the Lie algebra of N, onto N. On $\mathfrak{n}_{\mathbf{A}}$ choose the Haar measure dX that makes the measure of $\mathfrak{n}_{\mathbf{A}}/\mathfrak{n}_F$ equal to 1. Let dn be the Haar measure on $N_{\mathbf{A}}$ which is the image of dX under the above map. We define left and right Haar measures on $P_{\mathbf{A}}$ by

$$\int_{P_{\mathbf{A}}} \phi(p) d_l p = \int_{M_{\mathbf{A}}} \int_{N_{\mathbf{A}}} \phi(mn)\, dn\, dm \ ,$$

$$\int_{P_{\mathbf{A}}} \phi(p) d_r p = \int_{M_{\mathbf{A}}} \int_{N_{\mathbf{A}}} \phi(nm)\, dn\, dm \ , \qquad \phi \in C_c^\infty(P_{\mathbf{A}}) \ .$$

There is a homomorphism, $\delta_P\colon P_{\mathbf{A}} \rightarrow \mathbf{R}_+^*$, such that

$$d_r p = \delta_P(p) d_l p \ .$$

We shall write $P_{\mathbf{A}}^1$ for the group $M_{\mathbf{A}}^1 N_{\mathbf{A}}$. $P_{\mathbf{A}}^1$ is unimodular and a Haar measure dp on $P_{\mathbf{A}}^1$ is defined by our choices of Haar measures on $M_{\mathbf{A}}^1$ and $N_{\mathbf{A}}$. P_F is a discrete subgroup of $P_{\mathbf{A}}^1$, and the volume of $P_{\mathbf{A}}^1/P_F$ is $\tau(M)$, the Tamagawa number of M. Finally, it is obvious that the group $P_{\mathbf{A}}$ is a semi-direct product of A_∞^+ and $P_{\mathbf{A}}^1$.

Suppose that ${}^oP$ is a fixed minimal parabolic subgroup defined over F. It is known ([1, Theorem 4.13]) that ${}^oP$ is unique up to conjugation under F. Now for any finite place v, G_{O_v} is an open compact subgroup of G_v. It is known ([3, p. 10]) that the double coset space

$$\left(\textstyle\prod_{v \in f} G_{O_v}\right) \backslash G_f / {}^oP_f$$

is finite. It follows that for almost all finite v, $G_v = G_{O_v} \cdot {}^oP_v$.

A recent unpublished theorem of Bruhat and Tits states that for any finite place v there is an open compact subgroup K_v of G_v which, among other things, has the property that

$$G_v = K_v \cdot {}^oP_v \ . \tag{1.1}$$

For a statement of this theorem see [6, Theorem 5].

At any finite place v we shall define K_v to be G_{O_v} if $G_v = G_{O_v} \cdot {}^oP_v$. For the other finite places we take K_v to be any open compact subgroup of G_v that satisfies (1.1). At each infinite place v, we take K_v to be any maximal compact subgroup of G_v such that the Lie algebras of K_v and A_v are orthogonal under the Killing form. If we define

$$K = \prod_v K_v$$

then

$$G_{\mathbf{A}} = K \cdot {}^oP_{\mathbf{A}} .$$

Let dk be the normalized Haar measure on K. There is a positive constant c_G such that for all $f \in C_c^\infty(G_{\mathbf{A}})$

$$\int_{G_{\mathbf{A}}} f(x)dx = c_G \int_K \int_{{}^oP_{\mathbf{A}}} f(kp)\delta_P(p)d_l p\, dk = c_G \int_K \int_{{}^oP_{\mathbf{A}}} f(kp)d_r p\, dk .$$

We shall assume from now on that the F-rank of G/Z is 1, and we shall write P for the minimal parabolic subgroup ${}^oP$. The dimension of A/Z is 1, and $M_{\mathbf{A}}^1/M_F$ is compact. We fix for once and for all an isomorphism

$$t \longrightarrow h_t , \qquad t \in \mathbf{R}$$

from $\mathbf{R}$ onto a subgroup T of A_∞^+ such that A_∞^+ is the direct product of Z_∞^+ and T and for any $\phi \in C_c^\infty(A_\infty^+)$,

$$\int_{A_\infty^+} \phi(a_\infty^+)da_\infty^+ = \int_{Z_\infty^+} \int_{-\infty}^{\infty} \phi(z_\infty^+ h_t)dt\, dz_\infty^+ .$$

Any element of $x \in G_{\mathbf{A}}$ has the decomposition

$$x = kmh_t nz ,$$

for $k \in K$, $m \in M_{\mathbf{A}}^1$, $t \in \mathbf{R}$, $n \in N_{\mathbf{A}}$, and $z \in Z_\infty^+$. The number t is uniquely determined. We shall denote it by $H(x)$. There is a real number ρ such that

$$\delta_P(p) = e^{2\rho H(p)} , \qquad p \in P_{\mathbf{A}} .$$

We can assume that ρ is positive. For any $f \in C_c^\infty(G_{\mathbf{A}}/Z_\infty^+)$ we have the formula

$$\int_{G_{\mathbf{A}}/Z_\infty^+} f(x)dx = c_G \int_K \int_{M_{\mathbf{A}}/Z_\infty^+} \int_{N_{\mathbf{A}}} \int_{-\infty}^{\infty} f(kmh_t n)e^{2\rho t}dt\, dn\, dm\, dk .$$

Let us agree upon some additional notation that we shall later need. Suppose that $\gamma \in G_F$ and that H is a connected F-subgroup of G. We shall write $H^+(\gamma)$ for the centralizer of γ in H. It is clear that $H^+(\gamma)$ is defined over F. It is also obvious that $H^+(\gamma)_F$ is the centralizer of γ in H_F. We reserve the notation $H(\gamma)$ for the identity component of $H^+(\gamma)$. $H(\gamma)$ is also defined over F and it is a normal subgroup of finite index in $H^{+}(\gamma)$. In particular, the group of rational points, $H(\gamma)_F$, has finite index in the group $H^{+}(\gamma)_F$. We shall write $n_{\gamma,H}$ for this latter index.

We will be interested primarily in the case where the group H is reductive, and γ is a semisimple element in H_F. Then it is known ([1, p. 70]) that $H(\gamma)$ is reductive.

An important tool in analyzing the elements of G_F is the Bruhat decomposition. $N(A)$, the normalizer of A in G, is defined over F. $N(A)/M$ is a group of order 2, and the nontrivial coset of $N(A)/M$ has a representative w which is rational over F. w normalizes M, and for $a \in A_{\mathbf{A}}$,

$$waw^{-1} = a^{-1} .$$

Then according to the Bruhat decomposition, G_F is the disjoint union of P_F and $N_F \cdot w \cdot P_F$.

We shall need to appeal to some results from reduction theory. We refer to [3] where the results are proved for the case $F = \mathbf{Q}$. The results can be applied to our situation by restriction of the ground field F to $\mathbf{Q}$.

For any number $c > 0$, let

$$S(c) = \{x \in G_{\mathbf{A}} \colon H(x) \leqq \log c\} .$$

LEMMA 1.1. *There is a constant $c > 0$ such that $G_{\mathbf{A}} = S(c) \cdot G_F$.*

For a proof see [3, p. 16]. □

From [3, Theorem 9] we also have

LEMMA 1.2. *For any $c > 0$ the set of all γ in G_F such that*

$$S(c) \cap S(c)\gamma \neq \emptyset$$

is finite modulo P_F. □

We write $A_\infty^{+}(c)$ for $S(c) \cap A_\infty^{+}$. If ω is a relatively compact subset of $P_{\mathbf{A}}^1$ the set

$$\mathfrak{S}(c) = K \cdot A_\infty^{+}(c) \cdot \omega$$

is called a *Siegel domain for* $G_{\mathbf{A}}$. By Lemma 1.2 we may choose a Siegel domain $\mathfrak{S}(c)$ such that

$$G_{\mathbf{A}} = \mathfrak{S}(c) \cdot G_F .$$

In order to prove the above results one employs a strongly F-rational representation ρ of G. For a general account of such representations see [1, § 12]. We shall also need to use a strongly F-rational representation, which we now describe.

Let $\tilde{A}$ be a maximal torus of G which is contained in M. Let B be a Borel subgroup of G such that

$$\tilde{A} \subseteqq B \subseteqq P .$$

If Δ is the set of simple roots of $(G, \tilde{A})$ associated to B let $\{\Lambda_\alpha : \alpha \in \Delta\}$ be the corresponding set of fundamental weights. These are elements in $X(\tilde{A}) \otimes \mathbf{Q}$ which lie in $X(\tilde{A})$ if G happens to be simply connected.

Any linear combination

$$\Lambda = \sum_{\alpha \in \Delta} c_\alpha \Lambda_\alpha , \qquad c_\alpha \geqq 0,\ c_\alpha \in \mathbf{Z}$$

such that Λ lies in $X(\tilde{A})$ is the highest weight of an irreducible representation of G. This representation is strongly rational if and only if Λ is the restriction to $\tilde{A}$ of a character in $X(M)_F$.

We shall now take Λ to be any fixed character that satisfies this property. Let ρ be the strongly F-rational representation of G whose highest weight is Λ and let ρ act on the vector space V defined over F. If $\beta \in X(A)$ is the simple F-root of (G, A), then the restriction of the character 2Λ to A equals $n_\beta \cdot \beta$ for a positive integer n_β. We have a decomposition

$$V = V^0 \oplus V^1 \oplus \cdots \oplus V^{n_\beta}$$

of V into a direct sum of subspaces defined over F such that for $0 \leqq j \leqq n_\beta$,

$$\rho(a)v_j = a^{(\Lambda - j\beta)} v_j , \qquad v_j \in V^j,\ a \in A .$$

The spaces V^0 and V^{n_β} are one dimensional and V^0 is stable under the restriction of ρ to P.

ρ defines a representation of $G_{\mathbf{A}}$ on $V_{\mathbf{A}}$. There is a positive rational number b such that

$$|\Lambda(a)| = e^{bH(a)} , \qquad a \in A_{\mathbf{A}} .$$

Fix a basis $\{e_0, \cdots, e_d\}$ of V_F such that

(i) each basis element lies in one of the spaces V^j,

(ii) $e_0 \in V^0$, $e_d \in V^{n_\beta}$, and

(iii) $\rho(w)e_0 = e_d$.

If v is any finite place and $\xi_v \in V_{F_v}$, define

$$\|\xi_v\|_v = \max_i |\xi_v^i|_v ,$$

where (ξ_v^i) are the co-ordinates of ξ_v with respect to the above basis. If v is any infinite place, we make V_{F_v} into a Hilbert space over F_v by demanding that $\{e_0, \cdots, e_d\}$ be an orthonormal basis. An element $\xi \in V_{\mathbf{A}}$ is said to be primitive if $\|\xi_v\|_v$ equals 1 for almost all v, in which case we write

$$\|\xi\| = \prod_v \|\xi_v\|_v .$$

The function $||\cdot||$ is called the height function associated to the basis $\{e_0, \cdots, e_d\}$.

It is easy to verify that if $v \in V_{\mathbf{A}}$ is primitive then $\rho(x)v$ is also primitive for any $x \in G_{\mathbf{A}}$. The map

$$x \longrightarrow \rho(x)v \;, \qquad x \in G_{\mathbf{A}}$$

is continuous with respect to the topology defined by $||\cdot||$. It is also clear that for any $x \in G_{\mathbf{A}}$ and $p \in P_{\mathbf{A}}$,

$$||\rho(xp)e_0|| = e^{bH(p)}||\rho(x)e_0|| \;.$$

LEMMA 1.3. *There is a positive number* $\varepsilon_0 \leqq 1$ *such that*

$$e^{H(nw)} \geqq \varepsilon_0 \;, \qquad n \in N_{\mathbf{A}} \;.$$

Proof. Define ε to be the supremum over $k \in K$ of $||\rho(k)e_0||$. For $n \in N_{\mathbf{A}}$ we may write

$$nw = kp \;, \qquad k \in K,\; p \in P_{\mathbf{A}} \;.$$

Then

$$||\rho(nw)e_0|| = e^{bH(nw)}||\rho(k)e_0||$$

so that

$$e^{bH(nw)} \geqq \varepsilon_1^{-1}||\rho(nw)e_0|| \;.$$

On the other hand

$$||\rho(nw)e_0|| = ||\rho(n)e_d|| \;.$$

Now e_d is a lowest weight vector for ρ, and

$$\rho(n)e_d - e_d$$

is contained in the span of $\{e_0, \cdots, e_{d-1}\}$. From the definition of our height function,

$$||\rho(n)e_d|| \geqq ||e_d|| = 1 \;.$$

The lemma follows for ε_0 equal to $(\varepsilon_1)^{-1/b}$. $\square$

LEMMA 1.4. *If* ε_0 *is the constant of the last lemma, suppose that* $e^{H(x)} < \varepsilon_0$ *for some* $x \in G_{\mathbf{A}}$. *Fix* $\gamma \in G_F$. *Then* $H(x\gamma) < 1$ *if and only if* γ *is in* P_F.

Proof. For $\gamma \in P_F$ it is obvious that

$$e^{H(x\gamma)} = e^{H(x)} < \varepsilon_0 \leqq 1 \;.$$

On the other hand, suppose that γ is not in P_F. By the Bruhat decomposition

$$\gamma = \nu w \pi, \;\; \nu \in N_F, \;\; \pi \in P_F \;.$$

We can write

$$x = knmh_t z,\ k \in K,\ n \in N_{\mathbf{A}},\ m \in M^1_{\mathbf{A}},\ t \in \mathbf{R} \text{ and } z \in Z^+_\infty .$$

Then

$$x\nu w = k \cdot n_1 w \cdot w^{-1} m w \cdot h_t^{-1} z ,$$

where

$$n_1 = n \cdot m h_t \nu h_t^{-1} m^{-1}$$

lies in $N_{\mathbf{A}}$. It follows that

$$e^{H(x\gamma)} = e^{H(x\nu w)} = e^{H(n_1 w)} e^{-t} .$$

Now

$$e^{-t} = e^{-H(x)} > \varepsilon_0^{-1} ;$$

while by the last lemma

$$e^{H(n_1 w)} \geqq \varepsilon_0 .$$

Therefore $e^{H(x\gamma)} > 1$. □

COROLLARY 1.5. *For any $c > 0$ there is an ε with $0 < \varepsilon < \varepsilon_0$ such that if $\gamma \in G_F$ and*

$$S(\varepsilon) \cap S(c)\gamma \neq \emptyset$$

then γ lies in P_F.

Proof. The corollary follows from Lemmas 1.2 and 1.4. □

Suppose that $\mathfrak{S}$ is a Siegel domain. A function f on $G_{\mathbf{A}}$ is said to be *slowly increasing* on $\mathfrak{S}$ if there are constants C and N such that

$$|f(x)| \leqq Ce^{-NH(x)} , \qquad x \in \mathfrak{S} .$$

f is said to be *rapidly decreasing* on $\mathfrak{S}$ if for every N there is a constant C_N such that

$$|f(x)| \leqq C_N e^{NH(x)} , \qquad x \in \mathfrak{S} .$$

Suppose that h is a continuous function on $G_{\mathbf{A}}/G_F Z^+_\infty$. The function defined on $G_{\mathbf{A}}/P_F Z^+_\infty$ by

$$h_P(x) = \int_{N_{\mathbf{A}}/N_F} h(xn)dn$$

is called the *constant term* of h.

Let $\mathfrak{Z}$ be the center of the universal enveloping algebra of the Lie algebra of G_∞. It is clear how to define the action of $\mathfrak{Z}$ on $C^\infty(G_{\mathbf{A}})$. A function h in $C^\infty(G_{\mathbf{A}}/G_F Z^+_\infty)$ is called an *automorphic form* if

(i) h is left K-finite,

(ii) h is $\mathfrak{Z}$-finite,

(iii) h is slowly increasing on any Siegel domain.

The following well-known principle is basic to the theory of automorphic forms.

LEMMA 1.6. *Suppose that h is an automorphic form on $G_{\mathbf{A}}/G_F Z_\infty^+$. Then the function $h - h_P$ is rapidly decreasing on any Siegel domain.*

A proof of this lemma can be extracted from [5, Lemma 10]. $\square$

2. The spectral decomposition

The left regular representation, λ, of $G_{\mathbf{A}}$ on the Hilbert space $L^2(G_{\mathbf{A}}/G_F Z_\infty^+)$ is unitary. A fundamental problem in the theory of automorphic forms is to decompose this representation into a direct integral of irreducible representation of $G_{\mathbf{A}}$.

Let $L^2(\{G\})$ be the space of functions h in $L^2(G_{\mathbf{A}}/G_F Z_\infty^+)$ such that for almost all x

$$\int_{N_{\mathbf{A}}/N_F} f(xn)dn = 0 .$$

It is clear that $L^2(\{G\})$ is a closed λ-invariant subspace of $L^2(G_{\mathbf{A}}/G_F Z_\infty^+)$. It is called the space of *cusp forms*. Analysis of this space is the deepest part of the above problem.

The theory of Eisenstein series provides an intertwining operator between the restriction of λ to the orthogonal complement of $L^2(\{G\})$ in $L^2(G_{\mathbf{A}}/G_F Z_\infty^+)$ and a direct integral of certain induced representations. In this section we describe this intertwining operator.

Let $\mathfrak{Z}_M$ be the center of the universal enveloping algebra of the Lie algebra of M_∞. Suppose that τ is an irreducible representation of K and that χ is a homomorphism of $\mathfrak{Z}_M$ into C. Let L be the vector space of functions ϕ in $C^\infty(G_{\mathbf{A}}/M_F A_\infty^+ N_{\mathbf{A}})$ such that for any $x \in G_{\mathbf{A}}$,

(i) $\phi(x; z) = \chi(z)\phi(x)$, $z \in \mathfrak{Z}_M$,

and

(ii) the function $\beta(k) = \phi(k^{-1}x)$ is contained in a subspace of $L^2(K)$ on which the right regular representation of K is equivalent to τ.

It is known that L is finite-dimensional. We define an inner product on L by

$$(\phi_1, \phi_2) = c_G \int_K \int_{M_{\mathbf{A}}/M_F A_\infty^+} \phi_1(k^{-1}m)\overline{\phi_2(k^{-1}m)}dm\, dk .$$

We shall refer to L as the simple $(K, \mathfrak{Z}_M)$-type asociated to (τ, χ). We write $V(P)$ for the collection of all simple $(K, \mathfrak{Z}_M)$ types.

For any complex homomorphism χ of $\mathfrak{Z}_M$ let $V(P, \chi)$ be the collection of simple $(K, \mathfrak{Z}_M)$ types associated to χ. It is easy to see that $V(P, \chi)$ is not empty for only countably many χ. We fix for once and for all an indexing of these homomorphisms by the natural numbers. Then

$$V(P) = \bigcup_{n=1}^{\infty} V(P, \chi_n) .$$

Let $\mathcal{H}(n)$ and $\mathcal{H}$ be the orthogonal direct sums defined by

$$\mathcal{H}(n) = \bigoplus_{L \in V(P, \chi_n)} L ,$$
$$\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}(n) .$$

Then $\mathcal{H}$ may be identified with the space of measurable functions

$$\phi: G_{\mathbf{A}}/M_F A_\infty^+ N_{\mathbf{A}} \longrightarrow \mathbf{C}$$

such that

$$\| \phi \|^2 = c_G \int_K \int_{M_{\mathbf{A}}/M_F A_\infty^+} | \phi(km) |^2 dm\, dk < \infty .$$

For any $z \in \mathbf{C}$ there is a representation $\pi(z)$ of $G_{\mathbf{A}}/Z_\infty^+$ on $\mathcal{H}$ defined by

$$(\pi(z: y)\phi)(x) = \phi(y^{-1}x) e^{(z-\rho)H(y^{-1}x)} e^{-(z-\rho)H(x)} ,$$

for $\phi \in \mathcal{H}$ and $x, y \in G_{\mathbf{A}}/Z_\infty^+$. $\pi(z)$ is just an induced representation. Each space $\mathcal{H}(n)$ is invariant under $\pi(z)$.

For f a measurable function of compact support on $G_{\mathbf{A}}/Z_\infty^+$, define as usual

$$\pi(z: f) = \int_{G_{\mathbf{A}}/Z_\infty^+} f(y) \pi(z: y) dy .$$

$\pi(z: f)$ is a bounded operator on $\mathcal{H}$.

LEMMA 2.1. *For $z \in \mathbf{C}$ and $y \in G_{\mathbf{A}}$, the adjoint operator $\pi(z: y)^*$ equals $\pi(-\bar{z}: y^{-1})$.*

Proof. Let Q be the group $M_F \cdot A_\infty^+ \cdot N_{\mathbf{A}}$. Our earlier normalization of Haar measures defines a left Haar measure $d_l q$ on Q. Since $G_{\mathbf{A}}/Q$ is compact, we can find a real valued function $\beta \in C_c^\infty(G_{\mathbf{A}})$ such that

$$\int_Q \beta(xq) d_l q = 1 , \qquad x \in G_{\mathbf{A}} .$$

Then it is clear that for $\phi, \psi \in \mathcal{H}$,

$$(\phi, \psi) = \int_{G_{\mathbf{A}}} e^{-2\rho H(x)} \beta(x) \phi(x) \overline{\psi(x)} dx .$$

In particular, $(\pi(z: y)\phi, \psi)$ equals

$$\int_{G_{\mathbf{A}}} e^{-2\rho H(x)}\beta(x)\phi(y^{-1}x)e^{(z-\rho)H(y^{-1}x)}e^{-(z-\rho)H(x)}\overline{\psi(x)}dx\ .$$

Changing variables we can see that this expression also equals $(\phi, \pi(-\bar{z}: y^{-1})\psi)$. □

COROLLARY 2.2. *For any compactly supported measurable function f on $G_{\mathbf{A}}/Z_\infty^+$,*

$$\pi(z: f)^* = \pi(-\bar{z}: f^*)$$

where $f^*(y) = \overline{f(y^{-1})}$, $y \in G_{\mathbf{A}}/Z_\infty^+$. □

For any $f \in C_c^\infty(G_{\mathbf{A}}/Z_\infty^+)$ and $z \in \mathbf{C}$, the function

$$P(z: f: x, y) = e^{(-z+\rho)H(x)}e^{(z+\rho)H(y)}\sum_{\mu \in M_F}\int_{N_{\mathbf{A}}}\int_{-\infty}^{\infty} f(x\mu h_t n y^{-1})e^{(-z+\rho)t}dt\, dn$$

is continuous on the compact space $(G_{\mathbf{A}}/M_F A_\infty^+ N_{\mathbf{A}}) \times (G_{\mathbf{A}}/M_F A_\infty^+ N_{\mathbf{A}})$. For fixed x, y and f it is a Schwartz function of z in $\mathbf{R}$.

LEMMA 2.3. *For $f \in C_c^\infty(G_{\mathbf{A}}/Z_\infty^+)$, $z \in \mathbf{C}$, and $\phi \in \mathcal{H}$, $\big(\pi(z: f)\phi\big)(x)$ equals*

$$c_G\int_K \int_{M_{\mathbf{A}}/M_F A_\infty^+} P(z: f: x, km)\phi(km)dm\, dk\ .$$

Proof.

$$\begin{aligned}
\big(\pi(z: f)\phi\big)(x) &= \int_{G_{\mathbf{A}}/Z_\infty^+} \big(f(y)\pi(z: y)\phi\big)(x)dy \\
&= \int_{G_{\mathbf{A}}/Z_\infty^+} f(y)\phi(y^{-1}x)e^{(z-\rho)H(y^{-1}x)}e^{-(z-\rho)H(x)}dy \\
&= \int_{G_{\mathbf{A}}/Z_\infty^+} f(xy^{-1})\phi(y)e^{(z-\rho)H(y)}e^{-(z-\rho)H(x)}dy \\
&= c_G\int_K \int_{M_{\mathbf{A}}/M_F A_\infty^+} \Big\{\sum_{\mu \in M_F}\int_{-\infty}^{\infty}\int_{N_{\mathbf{A}}} f(x\mu h_t n m^{-1}k^{-1}) \\
&\qquad\qquad e^{(-z+\rho)t}dn\, dt\, e^{(-z+\rho)H(x)}\Big\}\phi(km)dm\, dk \\
&= c_G\int_K \int_{M_{\mathbf{A}}/M_F A_\infty^+} P(z: f: x, km)\phi(km)dm\, dk\ .
\end{aligned}$$

□

COROLLARY 2.4. *The operator $\pi(z: f)$ of the lemma is of Hilbert-Schmidt class.*

Proof. This is clear, since the kernel of $\pi(z: f)$ is square-integrable over $(K \cdot M_{\mathbf{A}}/M_F A_\infty^+) \times (K \cdot M_{\mathbf{A}}/M_F A_\infty^+)$. □

We shall later come across some functions

$$R(z\colon f\colon x,\, y)\,, \qquad\qquad x,\, y \in G_{\mathrm{A}}/M_F A_\infty^+ N_{\mathrm{A}}\,,$$

such that for every $\phi \in \mathcal{H}$ the integral

$$c_G \int_K \int_{M_{\mathrm{A}}/M_F A_\infty^+} R(z\colon f\colon x,\, km)\phi(km)dm\, dk$$

equals $(\pi(z\colon f)\phi)(x)$ for almost all x. We will be able to conclude from the last lemma that $R(z\colon f\colon x,\, y)$ and $P(z\colon f\colon x,\, y)$ are equal for almost all x and y. If $R(z\colon f\colon x,\, y)$ happens to be continuous separately in x and y the two functions will be equal for all x and y.

Fix L in $V(P)$. For $\phi \in L$ it is known ([5, Lemma 23, Cor. 3]) that the series

$$E(\phi\colon z\colon x) = \sum\nolimits_{\delta \in G_F/P_F} \phi(x\delta)e^{(z-\rho)H(x\delta)}$$

converges uniformly for x in compact subsets of $G_{\mathrm{A}}/Z_\infty^+$ and z in compact subsets of $D_\rho = \{z\colon \mathrm{Re}\, z < -\rho\}$. In fact it can be shown ([5, Lemma 24]) that for any Siegel domain $\mathfrak{S}$ there is a locally bounded function c on $(-\infty,\, -\rho)$ and an integer N such that for $x \in \mathfrak{S}$ and z in D_ρ,

$$\sum\nolimits_{\delta \in G_F/P_F} |\, \phi(x\delta)e^{(z-\rho)H(x\delta)}\,| \leqq c\,(\mathrm{Re}\, z)e^{(-N+\mathrm{Re}\, z+\rho)H(x)}\,.$$

$E(\phi\colon z\colon x)$ is called the *Eisenstein series* associated to ϕ. Let us review its basic properties.

First of all, $E(\phi\colon z\colon x)$ is a left K-finite eigenfunction of $\mathfrak{Z}$ on $G_{\mathrm{A}}/G_F Z_\infty^+$, so it is an automorphic form. The constant term, $E_P(\phi\colon z\colon x)$, equals

$$\phi(x)e^{(z-\rho)H(x)} + \bigl(M(z)\phi\bigr)(x)e^{(-z-\rho)H(x)}\,,$$

where $M(z)$ is a uniquely defined analytic function which maps D_ρ into the space of linear operators on L ([5, Theorem 5]). $M(z)^*$, the adjoint of $M(z)$, equals $M(\bar{z})$ ([5, Lemma 48]).

For any $x \in G_{\mathrm{A}}$ and $\phi \in L$, $E(\phi\colon z\colon x)$ and $M(z)\phi$ can be continued to meromorphic functions on C which are regular on the imaginary axis. Any poles which lie to the left of the imaginary axis are simple, and must occur on the interval $[-\rho,\, 0)$. All poles must occur simultaneously for $E(\phi\colon z\colon x)$ and $M(z)\phi$. If D is the set of points in C where $M(z)$ is holomorphic, $E(\phi\colon z\colon x)$ is continuous on $D \times G_{\mathrm{A}}$. In addition, for any Siegel domain $\mathfrak{S}$, any compact subset ω of D, and any ϕ in L, there are numbers C and N such that

$$(2.1) \qquad |\, E(\phi\colon z\colon x)\,| \leqq C\,||\,\phi\,||\, e^{-NH(x)}\,, \qquad z \in \omega\,,\ x \in \mathfrak{S}\,.$$

Finally $E(\phi\colon z\colon x)$ and $M(z)$ satisfy the following functional equations:

$$M(z)M(-z)\phi = \phi\,,$$
$$E(\phi\colon z\colon x) = E\bigl(M(z)\phi\colon -z\colon x\bigr)\,, \qquad \phi \in L\,.$$

(See [5, Theorem 7].)

There is a useful formula for $M(z)$. For $m \in M_{\mathbf{A}}$, let $m^w = w^{-1}mw$.

LEMMA 2.5. *Fix $L \in V(P)$ and $\phi \in L$. Then for $k \in K$, $m \in M^1_{\mathbf{A}}$, and for* $\operatorname{Re} z < -\rho$, *the integral*

$$\int_{N_{\mathbf{A}}} \phi(knwm^w)e^{(z-\rho)H(nw)}dn$$

is absolutely convergent. It equals

$$(M(z)\phi)(km) .$$

Proof. For $x \in G_{\mathbf{A}}$ we have

$$\int_{N_{\mathbf{A}}/N_F} E(\phi: z: xn)dn = \int_{N_{\mathbf{A}}/N_F} \sum\nolimits_{\gamma \in G_F/P_F} \phi(xn\gamma)e^{(z-\rho)H(xn\gamma)}dn .$$

By the Bruhat decomposition, $\{e\} \cup \{N_F \circ w\}$ is a set of representatives for G_F/P_F. Therefore the above integral equals

$$\int_{N_{\mathbf{A}}/N_F} \phi(xn)e^{(z-\rho)H(xn)}dn + \int_{N_{\mathbf{A}}/N_F} \sum\nolimits_{\gamma \in N_F} \phi(xn\gamma w)e^{(z-\rho)H(xn\gamma w)}dn$$
$$= \phi(x)e^{(z-\rho)H(x)} + \int_{N_{\mathbf{A}}} \phi(xnw)e^{(z-\rho)H(xnw)}dn .$$

Suppose that

$$x = kmh_tnz , \qquad k \in K , \; m \in M^1_{\mathbf{A}} , \; n \in N_{\mathbf{A}} , \; z \in Z^+_\infty .$$

Then

$$\int_{N_{\mathbf{A}}} \phi(xnw)e^{(z-\rho)H(xnw)}dn = \int_{N_{\mathbf{A}}} \phi(knmh_tw)e^{(z-\rho)H(knmh_tw)}e^{-2\rho t}dn$$
$$= e^{(-z-\rho)t}\int_{N_{\mathbf{A}}} \phi(knwm^w)e^{(z-\rho)H(nw)}dn .$$

Since $t = H(x)$, the integral

$$\int_{N_{\mathbf{A}}} \phi(knwm^w)e^{(z-\rho)H(nw)}dn$$

must equal $(M(z)\phi)(km)$. The absolute convergence of this integral is an immediate consequence of the absolute convergence of the Eisenstein series. $\square$

For any $L \in V(P)$ let $\mathcal{H}(L)$ be the space of entire functions with values in L which are Fourier-Laplace tranforms of functions in $C_c^\infty(i\mathbf{R}) \otimes L$. Then on any vertical strip, functions in $\mathcal{H}(L)$ decrease at infinity faster than any polynomial. For any $a \in \mathcal{H}(L)$ define

$$E_a(x) = \frac{1}{2\pi}\int_{\mathrm{Re}\,z=z_0} E(a(z): z: x)d|z| ,$$

where z_0 is any real number smaller than $-\rho$. By an estimate quoted earlier the integral is absolutely convergent, and it is independent of z_0. It is known ([5, Lemma 26, Cor. 1 and Lemma 40]) that for a and b in $\mathcal{H}(L)$, E_a and E_b are in $L^2(G_{\mathbf{A}}/G_F Z_\infty^-)$ and that

$$(2.2)\ \int_{G_{\mathbf{A}}/G_F Z_\infty^-} E_a(x)\overline{E_b(x)}dx = \frac{1}{2\pi}\int_{\mathrm{Re}\,z=z_0}\{(a(z), b(-\bar{z})) + (M(z)a(z), b(\bar{z}))\}d|z| ,$$

for any $z_0 < -\rho$.

Let $L^2(\{P\}, \{L\})$ be the closure in $L^2(G_{\mathbf{A}}/G_F Z_\infty^+)$ of the vector space

$$\{E_a: a \in \mathcal{H}(L)\} .$$

If $L^2(\{P\})$ is the orthogonal complement of $L^2(\{G\})$ in $L^2(G_{\mathbf{A}}/G_F Z_\infty^+)$, it is known that $L^2(\{P\})$ is the orthogonal direct sum

$$\bigoplus_{L\in V(P)} L^2(\{P\}, \{L\}) .$$

Fix L in $V(P)$. For $a, b \in \mathcal{H}(L)$ we shall examine the integral

$$\frac{1}{2\pi}\int_{\mathrm{Re}\,z=z_0}\{(a(z), b(-\bar{z})) + (M(z)a(z), b(\bar{z}))\}d|z| .$$

The integrand is clearly meromorphic in z. Now it is known ([5, Lemma 101]) that the norm of $M(z)$ is bounded at infinity in the strip $\{z: z_0 \leqq \mathrm{Re}\,z \leqq 0\}$. Therefore we can use the residue theorem to shift our contour of integration to the imaginary axis.

For $z \in [-\rho, 0)$ let $\mu(z)$ be the residue of $-2\pi M(\zeta)$ at $\zeta = z$. $\mu(z)$ vanishes for all but a finite number of z. Our integral becomes

$$\frac{1}{2\pi}\sum_{z\in[-\rho,0)} (\mu(z)a(z), b(z))$$
$$+ \frac{1}{4\pi}\int_{-i\infty}^{i\infty} (a(z) + M(-z)a(-z), b(z) + M(-z)b(-z))d|z| .$$

The expression defines a positive semi-definite inner product on $\mathcal{H}(L)$. It follows from a simple approximation argument, which we leave to the reader, that the linear operators

$$\mu(z),\ z \in [-\rho, 0) ,$$

are all positive semi-definite.

Let V be the vector space of functions from $[-\rho, 0)$ to L. Define a positive semi-definite inner product on V to be

$$\frac{1}{2\pi}\sum_{z\in[-\rho,0)}\big(a_0(z),\ \mu(z)b_0(z)\big)\ , \qquad a_0,\ b_0\in V\ .$$

Factoring out by the space of null vectors, we obtain a finite-dimensional Hilbert space which we denote by $\hat{L}_0^2(\{P\}, \{L\})$.

Let $\hat{L}_1^2(\{P\}, \{L\})$ be the space of square integrable functions a_1 from $i\mathbf{R}$ to L such that

$$a_1(-z) = M(z)a_1(z)\ , \qquad z\in i\mathbf{R}\ .$$

We define our inner product on this space to be

$$\frac{1}{\pi}\int_{\mathrm{Re}z=0}\big(a_1(z),\ b_1(z)\big)d\,|\,z\,|\ , \qquad a_1,\ \mathrm{b}_1\in\hat{L}_1^2(\{P\},\ \{L\})\ .$$

Let $\hat{L}^2(\{P\}, \{L\})$ be the orthogonal direct sum of $\hat{L}_0^2(\{P\}, \{L\})$ and $\hat{L}_1^2(\{P\}, \{L\})$.

Any function a in $\mathcal{H}(L)$ obviously defines a vector a_0 in $\hat{L}_0^2(\{P\}, \{L\})$. We define a function a_1 in $\hat{L}_1^2(\{P\}, \{L\})$ by

$$a_1(z) = \frac{1}{2}\big(a(z) + M(-z)a(-z)\big)\ , \qquad z\in i\mathbf{R}\ .$$

Then the correspondence

$$(a_0,\ a_1)\longleftrightarrow E_a\ , \qquad a\in\mathcal{H}(L)$$

is a linear isometry between dense subspaces of $\hat{L}^2(\{P\}, \{L\})$ and $L^2(\{P\}, \{L\})$. We extend this isometry to an isomorphism

$$E\colon \hat{L}^2(\{P\},\ \{L\})\longrightarrow L^2(\{P\},\ \{L\})\ .$$

Let us denote the restrictions of E to $\hat{L}_0^2(\{P\}, \{L\})$ and $\hat{L}_1^2(\{P\}, \{L\})$ by E_0 and E_1 respectively and we will write $L_0^2(\{P\}, \{L\})$ and $L_1^2(\{P\}, \{L\})$ for the corresponding ranges in $L^2(\{P\}, \{L\})$. For $i = 0$ or 1 and $a_i\in\hat{L}_i^2(\{P\}, \{L\})$ we need a formula for the function

$$(E_ia_i)(x)\ , \qquad x\in G_{\mathbf{A}}\ .$$

For $\phi\in L$, $z\in[-\rho, 0)$, and $x\in G_{\mathbf{A}}$, define $E_0(\phi\colon z\colon x)$ to be the residue,

$$-2\pi\cdot\mathrm{Res}_{\zeta=z}\,E(\phi\colon \zeta\colon x)\ ,$$

of $E(\phi\colon \zeta\colon x)$ at $\zeta = z$. $E_0(\phi\colon z\colon x)$ is clearly an automorphic form. Its constant term equals

$$-2\pi\cdot\mathrm{Res}_{\zeta=z}\int_{N_{\mathbf{A}}/N_F}E(\phi\colon \zeta\colon xn)dn\ ,$$

which is

$$\big(\mu(z)\phi\big)(x)e^{(-z-\rho)H(x)}\ .$$

This latter function is square-integrable over any Siegel domain so it follows from Lemma 1.6 that $E_0(\phi\colon z\colon x)$ is square-integrable on $G_{\mathbf{A}}/G_F Z_\infty^+$.

Another poperty of $E_0(\phi\colon z\colon x)$ is that it is orthogonal to $L^2(\{G\})$. To see this, choose any automorphic form h in $L^2(\{G\})$. By Lemma 1.6 the function

$$E(\zeta\colon \phi\colon x)\overline{h(x)}$$

is integrable for $\zeta \in \mathbf{C}$. If $\zeta < -\rho$, its integral equals

$$\int_{G_{\mathbf{A}}/P_F Z_\infty^+} \left(\phi(x)e^{(\zeta-\rho)H(x)}\right)\overline{h(x)}dx\ .$$

This expression vanishes, since

$$\int_{N_{\mathbf{A}}/N_F} h(xn)dn = 0\ , \qquad x \in G_{\mathbf{A}}\ .$$

By analytic continuation,

$$\int_{G_{\mathbf{A}}/G_F Z_\infty^+} E_0(z\colon \phi\colon x)\overline{h(x)}dx = 0\ .$$

Our assertion then follows from the well-known fact that there is an orthonormal basis of $L^2(\{G\})$ consisting of automorphic forms.

LEMMA 2.6. *Suppose that we are given a pair* (a_0, a_1) *such that*
(i) $a_0 \in \hat{L}_0^2(\{P\}, \{L\})$,
(ii) a_1 *is a function of compact support in* $\hat{L}_1^2(\{P\}, \{L\})$.
Then for almost all x *the function*

$$(E_0 a_0)(x) + (E_1 a_1)(x)$$

equals

$$\frac{1}{2\pi}\sum\nolimits_{z\in[-\rho,0]} E_0\big(a_0(z)\colon z\colon x\big) + \frac{1}{2\pi}\int_{-i\infty}^{i\infty} E\big(a_1(z)\colon z\colon x\big)d\,|\,z\,|\ . \tag{2.3}$$

Proof. Let $f(x)$ be the function defined by (2.3). Its constant term, $f_P(x)$, equals

$$\frac{1}{2\pi}\sum\nolimits_{z\in[-\rho,0)} \big(\mu(z)a_0(z)\big)(x)e^{(-z-\rho)H(x)} + \frac{1}{\pi}\int_{-i\infty}^{i\infty} \big(a_1(z)\big)(x)e^{(z-\rho)H(x)}d\,|\,z\,|\ .$$

This formula, together with a strengthened version of Lemma 1.6 ([8, Lemma 3.4]), insures that f is square-integrable. It is clear that f is orthogonal to $L^2(\{G\})$.

Choose a sequence $\{b^n\}$ of functions in $\mathcal{H}(L)$ such that as n approaches ∞, (b_0^n, b_1^n) approaches (a_0, a_1) in $\hat{L}^2(\{P\}, \{L\})$. The constant term of E_{b^n} equals

$$\frac{1}{2\pi}\int_{N_{\mathbf{A}}/N_F}\int_{\mathrm{Re}\,z=z_0} E(b^n(z)\colon z\colon xn)d\,|\,z\,|\,dn$$

for any $z_0 < -\rho$. Changing the order of integration we obtain

$$\frac{1}{2\pi}\int_{\mathrm{Re}\,z=z_0}\{(b^n(z))(x)e^{(z-\rho)H(x)} + (M(z)b^n(z))(x)e^{(-z-\rho)H(x)}\}d\,|\,z\,|\ ,$$

which equals

$$\frac{1}{2\pi}\sum\nolimits_{z\in[-\rho,0)}(\mu(z)b^n(z))(x)e^{(-z-\rho)H(x)}$$
$$+\ \frac{1}{2\pi}\int_{\mathrm{Re}\,z=0}\{(b^n(z))(x)e^{(z-\rho)H(x)} + (M(z)b^n(z))(x)e^{(-z-\rho)H(x)}\}d\,|\,z\,|\ ,$$

by the residue theorem. We rewrite this formula as

$$\frac{1}{2\pi}\sum\nolimits_{z\in[-\rho,0)}(\mu(z)b_0^n(z))(x)e^{(-z-\rho)H(x)} + \frac{1}{\pi}\int_{\mathrm{Re}\,z=0}(b_1^n(z))(x)e^{(z-\rho)H(x)}d\,|\,z\,|\ .$$

By choosing the sequence $\{b^n\}$ suitably we can force this last expression to approach $f_P(x)$ for all x, as n approaches infinity. On the other hand, since E_{b^n} converges to $E_0a_0 + E_1a_1$ in the mean, the constant terms of $E_0a_0 + E_1a_1$ and f are equal almost everywhere. Now the function

$$f - E_0a_0 - E_1a_1$$

lies in $L^2(G_{\mathbf{A}}/G_FZ_\infty^+)$ and is orthogonal to $L^2(\{G\})$. Since its constant term equals 0 almost everywhere, the function itself must vanish almost everywhere. □

Fix $i = 0, 1$. Define

$$\hat{L}_i^2(\{P\}) = \bigoplus\nolimits_{L\in V(P)}\hat{L}_i^2(\{P\}, \{L\})\ ,$$
$$L_i^2(\{P\}) = \bigoplus\nolimits_{L\in V(P)}L_i^2(\{P\}, \{L\})\ .$$

By taking the direct sums over $V(P)$ of the maps

$$E_i\colon \hat{L}_i^2(\{P\}, \{L\}) \longrightarrow L^2(G_{\mathbf{A}}/G_FZ_\infty^+)\ , \qquad L\in V(P)\ ,$$

we obtain a map from $\hat{L}_i^2(\{P\})$ into $L^2(G_{\mathbf{A}}/G_FZ_\infty^+)$ which we denote again by E_i. The image of this map is $L_i^2(\{P\})$. Consider the adjoint map

$$E_i^*\colon L^2(G_{\mathbf{A}}/G_FZ_\infty^+) \longrightarrow \hat{L}_i^2(\{P\})\ .$$

Then the map $E_iE_i^*$ is the orthogonal projection of $L^2(G_{\mathbf{A}}/G_FZ_\infty^+)$ onto $L_i^2(\{P\})$. The space $L^2(\{P\})$, which is the orthogonal complement of $L^2(\{G\})$ in $L^2(G_{\mathbf{A}}/G_FZ_\infty^+)$, equals

$$L_0^2(\{P\}) \oplus L_1^2(\{P\})\ .$$

For any L, and $z\in[-\rho, 0)$, define an operator $r(z)$ on L to equal zero on the kernel of $\mu(z)$ and to equal $\mu(z)^{-1}$ on the orthogonal complement in L of the kernel of $\mu(z)$.

LEMMA 2.7 *Suppose that* $L \in V(P)$ *and* $\phi \in L$. *Then for any* $h \in C_c^\infty(G_{\mathbf{A}}/G_F Z_\infty^+)$,

(i) $\big((E_0^* h)(z), \phi\big) = \int_{G_{\mathbf{A}}/G_F Z_\infty^+} h(x)\overline{E_0(r(z)\phi: z: x)}dx$, *for* $z \in [-\rho, 0)$, *and*

(ii) $\big((E_1^* h)(z), \phi\big) = (1/2)\int_{G_{\mathbf{A}}/G_F Z_\infty^+} h(x)\overline{E(\phi: z: x)}dx$

for almost all z *in* $i\mathbf{R}$.

Proof. These formulas follow easily from the last lemma. Let us prove only (ii).

Let a_1 be any continuous function of compact support in $\hat{L}_1^2(\{P\}, \{L\})$. Form the inner product of $E_1^* h$ and a_1. On the one hand this equals

$$\frac{1}{\pi}\int_{-i\infty}^{i\infty} \big((E_1^* h)(z), a_1(z)\big)d|z|,$$

while on the other hand we obtain

$$\int_{G_{\mathbf{A}}/G_F Z_\infty^+} h(x)\overline{(E_1 a_1)(x)}dx,$$

which is the same as

$$\frac{1}{2\pi}\int_{-i\infty}^{i\infty}\int_{G_{\mathbf{A}}/G_F Z_\infty^+} h(x)\overline{E(a_1(z): z: x)}dx\, d|z|,$$

by Lemma 2.6. Since a_1 is arbitrary, we get the required formula. □

Our final task for this section is to verify the intertwining property of the operators E_0 and E_1.

LEMMA 2.8. *Suppose that* f *is a left and right* K*-finite function in* $C_c^\infty(G_{\mathbf{A}}/Z_\infty^+)$. *Fix* $L \in V(P)$ *and* $\phi \in L$. *Then*

$$\lambda(f)E(\phi: z: x) = E\big(\pi(z: f)\phi: z: x\big)$$

as meromorphic functions of z.

Proof. For $\operatorname{Re} z < -\rho$ the formula is a direct consequence of the definitions of $\pi(z{:}f)$ and $E(\phi{:}z{:}x)$. The lemma follows in general by analytic continuation. □

COROLLARY 2.9. *For* f *as above,*

$$\pi(z: f)M(z) = M(z)\pi(-z: f)$$

as meromorphic functions in z.

Proof. Fix L and ϕ. Using the functional equation for $E(\phi: z: x)$ we observe that

$$\begin{aligned} E\big(M(z)\pi(-z: f)\phi: z: x\big) &= E\big(\pi(-z: f)\phi: -z: x\big) \\ &= \lambda(f)E(\phi: -z: x) = \lambda(f)E\big(M(z)\phi: z: x\big) \\ &= E\big(\pi(z: f)M(z)\phi: z: x\big) . \end{aligned}$$

However, for $\psi \neq 0$, $\psi \in L$, $E(\psi: z: x)$ is not identically 0 in z and x. Therefore

$$M(z)\pi(f: -z)\phi - \pi(f: z)M(z)\phi = 0 ,$$

which proves the required result. □

COROLLARY 2.10. *$L_0^2(\{P\})$ and $L_1^2(\{P\})$ are λ-invariant subspaces of $L^2(G_{\mathbf{A}}/G_F Z_\infty^+)$.*

Proof. This is obvious. □

We shall sometimes write $L_1^2(G_{\mathbf{A}}/G_F Z_\infty^+)$ for $L_1^2(\{P\})$. We shall denote the direct sum

$$L^2(\{G\}) \oplus L_0^2(\{P\})$$

by $L_0^2(G_{\mathbf{A}}/G_F Z_\infty^+)$. (This notation is different from [5] and [9] where the symbol L_0^2 is used for the space of cusp forms.) For any x in $G_{\mathbf{A}}/Z_\infty^+$ we denote the restrictions of $\lambda(x)$ to $L^2(\{G\})$, $L_0^2(\{P\})$, $L_1^2(G_{\mathbf{A}}/G_F Z_\infty^+)$, and $L_0^2(G_{\mathbf{A}}/G_F Z_\infty^+)$ by $\lambda_0(\{G\}: x)$, $\lambda_0(\{P\}: x)$, $\lambda_1(x)$, and $\lambda_0(x)$ respectively. For $f \in C_c^\infty(G_{\mathbf{A}}/Z_\infty^+)$ we can define the operators $\lambda_0(\{G\}: f)$, $\lambda_0(\{P\}: f)$, $\lambda_1(f)$, and $\lambda_0(f)$. $\lambda_0(f)$ is the operator we shall be most interested in. It is the sum of $\lambda_0(\{G\}: f)$ and $\lambda_0(\{P\}: f)$.

It should be noted that the results of this section are but special cases of [8, § 7], where the spectral decomposition is carried out for groups of arbitrary rank.

3. The operator $\lambda_0(f)$

Suppose that f is a complex-valued function on $G_{\mathbf{A}}/Z_\infty^+$ which is the convolution over $G_{\mathbf{A}}/Z_\infty^+$ of two left and right K-finite functions f' and f'' in $C_c^\infty(G_{\mathbf{A}}/Z_\infty^+)$. $\lambda_0(f)$ is a bounded operator on $L^2(G_{\mathbf{A}}/G_F Z_\infty^+)$. It seems likely that $\lambda_0(f)$ is of trace class. Our objective will be to prove this fact under an additional assumption on f and then to find a formula for the trace.

In calculating the trace of $\lambda_0(f)$ we shall integrate its kernel over the diagonal. This necessitates finding the integral kernels of $\lambda(f)$ and $\lambda_1(f)$. While we are at it, we may as well find a formula for the kernel of $\lambda_0(\{P\}: f)$. This will perhaps have to be studied to prove that $\lambda_0(f)$ is of trace class without using the additional condition (Assumption 3.5) that we shall impose.

For any function h in $C_c^\infty(G_A/G_F Z_\infty^+)$

$$(\lambda(f)h)(x) = \int_{G_A/Z_\infty^+} f(y)h(y^{-1}x)dy = \int_{G_A/Z_\infty^+} f(xy^{-1})h(y)dy$$
$$= \int_{G_A/G_F Z_\infty^+} \{\textstyle\sum_{\gamma\in G_F} f(x\gamma y^{-1})\}h(y)dy \ .$$

The series

$$K(x, y) = \textstyle\sum_{\gamma\in G_F} f(x\gamma y^{-1})$$

is finite for x and y lying in fixed compact subsets of G_A/Z_∞^+. Therefore $\lambda(f)$ is an integral operator with kernel K.

It is clear that K is a smooth function. In addition K is slowly increasing. In fact, given any Siegel domain $\mathfrak{S}$, and a K-finite function g in $C_c^\infty(G_A/Z_\infty^+)$, it is known ([5, Lemma 9]) that there are constants C and M such that

$$|\textstyle\sum_{\gamma\in G_F} g(x\gamma y^{-1})| \leqq Ce^{-MH(x)} , \qquad x, y \in \mathfrak{S} \ . \tag{3.1}$$

In studying the operators $\lambda_0(\{P\}: f)$ and $\lambda_1(f)$ we must examine certain λ-invariant subspaces of $L_0^2(\{P\})$ and $L_1^2(\{P\})$. For any positive integer N define

$$\hat{L}_i^2(N) = \bigoplus_{n=1}^N \bigoplus_{L\in V(P,\chi_n)} \hat{L}_i^2(\{P\}, \{L\}) , \qquad i = 0, 1 \ .$$

For $T > 0$ let $\hat{L}_1^2(N, T)$ be the subspace of elements a_1 in $\hat{L}_1^2(N)$ such that the projection of a_1 onto any of the direct summands

$$\{\hat{L}_1^2(\{P\}, \{L\}): L \in \textstyle\bigcup_{n=1}^N V(P, \chi_n)\}$$

is a function supported on the interval $[-iT, iT]$. Let $L_0^2(N)$ and $L_1^2(N, T)$ be the images of $\hat{L}_0^2(N)$ and $\hat{L}_1^2(N, T)$ under the maps E_0 and E_1. $L_0^2(N)$ and $L_1^2(N, T)$ are λ-invariant subspaces of $L^2(G_A/G_F Z_\infty^+)$. Let $\lambda_0(N: f)$ and $\lambda_1(N, T: f)$ be the compositions of $\lambda(f)$ with the projections of $L^2(G_A/G_F Z_\infty^+)$ onto these subspaces.

In order to get a formula for the integral kernels of these operators we shall fix for once and for all an orthonormal basis

$$\{\phi_\alpha: \alpha \in I\}$$

of $\mathcal{H}$. We may assume that each basis vector lies in some space $L \in V(P)$.

For each positive integer n let I_n be the set of indices α for which there is an $L \in V(P, \chi_n)$ such that ϕ_α lies in L. Then

$$I = \textstyle\bigcup_{n=1}^\infty I_n \ .$$

For $\alpha, \beta \in I$ define

$$\pi_{\alpha\beta}(z\colon f) = \big(\pi(z\colon f)\phi_\beta,\ \phi_\alpha\big)\ .$$

Then for any positive integer n we define functions on $(G_{\mathbf{A}}/G_F Z_\infty^+) \times (G_{\mathbf{A}}/G_F Z_\infty^+)$ by

$$K_0(n\colon z\colon f\colon x,\ y)$$
$$= \frac{1}{2\pi} \sum\nolimits_{\alpha,\beta \in I_n} \pi_{\alpha\beta}(z\colon f) E_0(\phi_\alpha\colon z\colon x)\overline{E_0\big(r(z)\phi_\beta\colon z\colon y\big)}\ , \qquad z \in [-\rho,\ 0)\ ,$$

and

$$K_1(n\colon z\colon f\colon x,\ y) = \frac{1}{4\pi} \sum\nolimits_{\alpha,\beta \in I_n} \pi_{\alpha\beta}(z\colon f) E(\phi_\alpha\colon z\colon x)\overline{E(\phi_\beta\colon z\colon y)}\ , \qquad z \in i\mathbf{R}\ .$$

Since f is left and right K-finite, $\pi_{\alpha\beta}(z\colon f)$ vanishes for all but finitely many α and β in I_n. It follows that the above sums are finite. $K_0(n\colon z\colon f\colon x,\ y)$ vanishes for all but finitely many z in $[-\rho,\ 0)$.

LEMMA 3.1. *$\lambda_0(N\colon f)$ and $\lambda_1(N,\ T\colon f)$ are integral operators whose kernels are*

$$\sum\nolimits_{n=1}^{N} \sum\nolimits_{z \in [-\rho, 0)} K_0(n\colon z\colon f\colon x,\ y)$$

and

$$\sum\nolimits_{n=1}^{N} \int_{-iT}^{iT} K_1(n\colon z\colon f\colon x,\ y)d\,|\,z\,|$$

respectively.

Proof. The result follows from a routine use of Lemmas 2.6, 2.7, and 2.8. ☐

We remark that the kernels defined by this lemma are both continuous functions on $(G_{\mathbf{A}}/G_F Z_\infty^+) \times (G_{\mathbf{A}}/G_F Z_\infty^+)$.

THEOREM 3.2. *Given any Siegel domain $\mathfrak{S}$ we can choose constants C and M such that for all x and y in $\mathfrak{S}$ the expressions*

$$\sum\nolimits_{n=1}^{\infty} \sum\nolimits_{z \in [-\rho, 0)} |\,K_0(n\colon z\colon f\colon x,\ y\,| \tag{3.2}$$

and

$$\sum\nolimits_{n=1}^{\infty} \int_{-i\infty}^{i\infty} |\,K_1(n\colon z\colon f\colon x,\ y)\,|\,d\,|\,z\,| \tag{3.3}$$

are both bounded by $Ce^{-MH(x)}e^{-MH(y)}$.

Proof. Recall that $f = f' * f''$ where f' and f'' are both K-finite. For any n we fix a finite-dimensional subspace $\mathcal{H}^f(n)$ of $\mathcal{H}(n)$ which contains the ranges, and the orthogonal complements of the kernels, of the restrictions of both $\pi(z\colon f')$ and $\pi(z\colon f'')$ to $\mathcal{H}(n)$. For $x \in G_{\mathbf{A}}/Z_\infty^+$ and $z \in [-\rho,\ 0)$, define a vector $E_0^f(z\colon x)$ in $\mathcal{H}^f(n)$ by

$$\big(\phi,\ E_0^f(z\colon x)\big) = E(\phi\colon z\colon x)\ , \qquad \phi \in \mathcal{H}^f(n)\ .$$

Then

$$|K_0(n\colon z\colon f\colon x,\ y)| = \frac{1}{2\pi}\Big|\sum\nolimits_{\alpha,\beta\in I_n} \pi_{\alpha\beta}(z\colon f)E_0(\phi_\alpha\colon z\colon x)\overline{E_0\big(r(z)\phi_\beta\colon z\colon y\big)}\Big|$$

$$= \frac{1}{2\pi}\Big|\sum\nolimits_{\beta\in I_n} E_0\big(\pi(z\colon f)\phi_\beta\colon z\colon x\big)E_0\big(r(z)\phi_\beta\colon z\colon y\big)\Big|$$

$$= \frac{1}{2\pi}\Big|\sum\nolimits_{\beta\in I_n} \big(E_0^f(z\colon y),\ r(z)\phi_\beta\big)\overline{\big(\pi(z\colon f)\phi_\beta,\ E_0^f(z\colon x)\big)}\Big|$$

$$= \frac{1}{2\pi}\Big|\big(r(z)E_0^f(z\colon y),\ \pi(z\colon f)^* E_0^f(z\colon x)\big)\Big|$$

$$= \frac{1}{2\pi}\Big|\big(\pi(z\colon f'')r(z)E_0^f(z\colon y),\ \pi(z\colon f')^* E_0^f(z\colon x)\big)\Big|\ .$$

We have used the facts that $r(z)$ is self-adjoint and that

$$\pi(z\colon f)^* = \pi(z\colon f' * f'')^* = \pi(z\colon f'')^* \circ \pi(z\colon f')^*\ .$$

It is readily seen from Corollary 2.9 that

$$\pi(z\colon f'')r(z) = r(z)\pi(-z\colon f'')\ .$$

Since z is real, we have

$$\pi(-z\colon f'') = \pi\big(z\colon (f'')^*\big)^*\ ,$$

by Corollary 2.2. Therefore $|K_0(n\colon z\colon f\colon x,\ y)|$ equals

$$\frac{1}{2\pi}\Big|\big(r(z)\pi\big(z\colon (f'')^*\big)^* E_0^f(z\colon y),\ \pi(z\colon f')^* E_0^f(z\colon x)\big)\Big|. \tag{3.4}$$

If we apply Schwarz' inequality to the positive semi-definite form

$$\langle\phi,\ \chi\rangle = \big(r(z)\phi,\ \chi\big)\ , \qquad \phi,\ \chi \in \mathcal{H}^f(n)\ ,$$

we see that (3.4) is bounded by the product of

$$\frac{1}{\sqrt{2\pi}}\big(r(z)\pi\big(z\colon (f'')^*\big)^* E_0^f(z\colon y),\ \pi\big(z\colon (f'')^*\big)^* E_0^f(z\colon y)\big)^{1/2}$$

and

$$\frac{1}{\sqrt{2\pi}}\big(r(z)\pi(z\colon f')^* E_0^f(z\colon x),\ \pi(z\colon f')^* E_0^f(z\colon x)\big)^{1/2}\ .$$

Define

$${}^1f = f' * (f')^*\ ,\quad {}^2f = (f'')^* * f''\ .$$

Then the above product equals

$$K_0(n\colon z\colon {}^2f\colon y,\ y)^{1/2}K_0(n\colon z\colon {}^1f\colon x,\ x)^{1/2}\ .$$

Applying Schwarz' inequality to the sum over n and z in (3.2), we observe that it is only necessary to establish the bound for (3.2) in the case that $f = {}^1f$ and $y = x$.

Now we have

$$|K_0(n: z: {}^1f: x, x)| = K_0(n: z: {}^1f: x, x) .$$

By Lemma 3.1

$$\sum_{n=1}^{N} \sum_{z \in [-\rho, 0)} K_0(n: z: {}^1f: x, x)$$

is the value on the diagonal of the kernel of $\lambda_0(N, {}^1f)$. But $\lambda_0(N, {}^1f)$ is the restriction of the positive, semi-definite operator $\lambda({}^1f)$ to the invariant subspace $L_0^2(N)$. Therefore its kernel, which is continuous, can be bounded on the diagonal by $K(x, x)$. However, according to (3.1) this last function can in turn be bounded on a Siegel domain by a function $Ce^{MH(x)}$. But this function is independent of N. Consequently it majorizes

$$\sum_{n=1}^{\infty} \sum_{z \in [-\rho, 0)} K_0(n: z: {}^1f: x, x) \qquad \text{for all } x .$$

We deal with (3.3) in exactly the same manner. First of all we show that for z imaginary,

$$|K_1(n: z: f: x, y)|$$

is bounded by

$$K_1(n: z: {}^1f: x, x)^{1/2} K_1(n: z: {}^2f: y, y)^{1/2} .$$

Consequently, it is enough to establish the theorem when f is replaced by ${}^1f$ and $y = x$. We then resort to Lemma 3.1, verifying the required result as above. $\square$

COROLLARY 3.3. *The functions*

$$K_0(\{P\}: x, y) = \sum_{n=1}^{\infty} \sum_{z \in [-\rho, 0)} K_0(n: z: f: x, y)$$

and

$$K_1(\{P\}: x, y) = K_1(x, y) = \sum_{n=1}^{\infty} \int_{-i\infty}^{i\infty} K_1(n: z: f: x, y) d|z|$$

are the integral kernels of $\lambda_0(\{P\}: f)$ *and* $\lambda_1(f)$ *respectively.*

Proof. This follows from Lemma 3.1 and Theorem 3.2. $\square$

It follows from the corollary that the kernel of $\lambda_0(f)$ equals

$$K_0(x, y) = K(x, y) - K_1(x, y) .$$

Now we turn to the task of proving that $\lambda_0(f)$ is of trace class. Let $\mathcal{E}_M$ be the set equivalence classes of irreducible unitary representations of

$M_{\mathbf{A}}/A_\infty^+$. Let λ_M be the regular representation of $M_{\mathbf{A}}/A_\infty^+$ on $L^2(M_{\mathbf{A}}/M_F A_\infty^+)$. It is a well known that there is a λ_M-decomposition

$$L^2(M_{\mathbf{A}}/M_F A_\infty^+) = \bigoplus_{\sigma \in \mathcal{E}_M} V(\sigma)$$

where for each σ in $\mathcal{E}_M$, the representation of $M_{\mathbf{A}}/A_\infty^+$ obtained by restricting λ_M to $V(\sigma)$ belongs to a nonnegative integral multiple of the class σ. Let $\mathcal{H}(\sigma)$ be the space of functions ϕ in $\mathcal{H}$ such that for any $x \in G_{\mathbf{A}}$ the function

$$\phi_x(m) = \phi(xm), \qquad m \in M_{\mathbf{A}}/M_F A_\infty^+,$$

lies in $V(\sigma)$. It is clear that

$$\mathcal{H} = \bigoplus_{\sigma \in \mathcal{E}_M} \mathcal{H}(\sigma).$$

We assume from now on that the basis $\{\phi_\alpha\}_{\alpha \in I}$, chosen earlier, is compatible with this decomposition.

The element w, representing the nontrivial Weyl group element, defines a coset in the group of automorphisms of $M_{\mathbf{A}}$ modulo the group of inner automorphisms. In this way w defines an involution on $\mathcal{E}_M$. We say that a class $\sigma \in \mathcal{E}_M$ is *unramified* if $\sigma^w \neq \sigma$, and *ramified* if $\sigma^w = \sigma$.

LEMMA 3.4. *Suppose that $\sigma \in \mathcal{E}_M$ is unramified. For any $L \in V(P)$ let $L(\sigma) = L \cap \mathcal{H}(\sigma)$. Then for any $z \in \mathbf{C}$ the space $L(\sigma) + L(\sigma^w)$ is invariant under $M(z)$. Furthermore the restriction of $M(z)$ to this space is regular for $z \in [-\rho, 0)$.*

Proof. It is clear from the formula in Lemma 2.5 that $M(z)$ maps $L(\sigma)$ into $L(\sigma^w)$. Of course the formula is only true for $\operatorname{Re} z < -\rho$, but our assertion follows by analytic continuation. Similarly $M(z)$ maps $L(\sigma^w)$ into $L(\sigma)$, so that $L(\sigma) + L(\sigma^w)$ is an invariant subspace of $M(z)$.

For $z \in [-\rho, 0)$ let $\mu(z)$ be the residue of $-2\pi M(\zeta)$ at $\zeta = z$. Let I_σ be the subset of indices in I such that $\{\phi_\beta\}_{\beta \in I_\sigma}$ is an orthonormal basis for $L(\sigma) + L(\sigma^w)$. The trace of the restriction of $\mu(z)$ to $L(\sigma) + L(\sigma^w)$ equals

$$\sum_{\beta \in I_\sigma} (\mu(z)\phi_\beta, \phi_\beta).$$

We have assumed that each ϕ_β is either in $L(\sigma)$ or in $L(\sigma^w)$, so since $L(\sigma)$ is orthogonal to $L(\sigma^w)$, this expression equals 0. On the other hand, we saw in § 2 that $\mu(z)$ was positive semi-definite. Therefore $\mu(z) = 0$, so M is regular at z. □

If $\sigma \in \mathcal{E}_M$ and g is any function in $C_c^\infty(G_{\mathbf{A}}/Z_\infty^+)$, we shall denote the restriction of $\pi(z\colon g)$ to $\mathcal{H}(\sigma)$ by $\pi(\sigma\colon z\colon g)$. At this point it is necessary to place an additional restriction on our function f. We summarize all the requirements in the following:

ASSUMPTION 3.5. *f is the convolution of two functions f' and f'' in $C_c^\infty(G_{\mathbf{A}}/Z_\infty^+)$. f' and f'' are left and right K-finite and in addition*

$$\pi(\sigma\colon z\colon f') = \pi(\sigma\colon z\colon f'') = 0\ , \qquad z \in \mathbf{C}\ ,$$

for almost all ramified classes $\sigma \in \mathcal{E}_M$.

We remark that this last condition is always true if the Lie group M_∞/A_∞ is compact.

THEOREM 3.6. *Suppose that f satisfies Assumption 3.5. Then $\lambda_0(f)$ is of trace class.*

Proof. $\lambda_0(f)$ is the sum of $\lambda_0(\{G\}\colon f)$ and $\lambda_0(\{P\}\colon f)$. $\lambda_0(\{G\}\colon f)$ equals

$$\lambda_0(\{G\}\colon f')\lambda_0(\{G\}\colon f'')\ .$$

It is known ([5, p. 14]) that the operators $\lambda_0(\{G\}\colon f')$ and $\lambda_0(\{G\}\colon f'')$ are of Hilbert-Schmidt class. Therefore $\lambda_0(\{G\}\colon f)$ is of trace class.

Since f' and f'' are left and right K-finite, the operators $\pi(\sigma\colon z\colon f')$ and $\pi(\sigma\colon z\colon f'')$ are of finite rank for any $z \in \mathbf{C}$ and $\sigma \in \mathcal{E}_M$. Therefore, by Assumption 3.5 and Lemma 3.4, both $\lambda_0(\{P\}\colon f')$ and $\lambda_0(\{P\}\colon f'')$ are of finite rank. It follows that $\lambda_0(\{P\}\colon f)$ is of trace class. □

Now that we have proved that $\lambda_0(f)$ is of trace class, we would like to be able to say that $K_0(x, y)$ is integrable over the diagonal, and that its integral yields the trace of $\lambda_0(f)$. However, we have not shown that the kernel $K_0(x, y)$ is continuous, so we must proceed cautiously.

LEMMA 3.7. *The function $K_0(x, y)$ is continuous in each variable separately.*

Proof. Following the notation of the proof of Theorem 3.2, we know that

$$|\,K_1(n\colon z\colon f\colon x, y)\,|$$

is bounded by

$$K_1(n\colon z\colon {}^1f\colon x, x)^{1/2}K_1(n\colon z\colon {}^2f\colon y, y)^{1/2}\ .$$

Then for any N and T the integral

$$I(f\colon x, y) = \sum_{n=N}^\infty \int_{\substack{z \in i\mathbf{R} \\ |z| \geqq T}} |\,K_1(n\colon z\colon f\colon x, y)\,|\, d\,|\,z\,|$$

is no greater than

$$I({}^1f\colon x, x)^{1/2} I({}^2f\colon y, y)^{1/2}\ .$$

For y lying in any Siegel domain this last expression can be bounded by

$$CI({}^1f\colon x,\, x)^{1/2}e^{-MH(y)}\ ,$$

by Theorem 3.2. It follows that for any fixed x the integral defining $K_1(x, y)$ converges uniformly for y in compact subsets of $G_{\mathbf{A}}/G_F Z_\infty^+$. Therefore $K_1(x, y)$ is continuous in y. Since $K(x, y)$ is continuous,

$$K_0(x, y) = K(x, y) - K_1(x, y)$$

is also continuous in y. Similarly, $K_0(x, y)$ is continuous in x. $\square$

Our operator $\lambda_0(f)$ is the product of the two Hilbert-Schmidt operators $\lambda_0(f')$ and $\lambda_0(f'')$. Recall that for any Siegel domain $\mathfrak{S}$ we may choose constants C and M such that for x and v in $\mathfrak{S}$,

$$\left|\sum\nolimits_{\gamma \in G_F} f'(x\gamma v^{-1})\right| \leqq Ce^{-MH(x)}\ .$$

If follows that for any x there is a unique function $h'_x(v)$ in $L_0^2(G_{\mathbf{A}}/G_F Z_\infty^+)$, such that for any $\phi \in L_0^2(G_{\mathbf{A}}/G_F Z_\infty^+)$,

$$\int_{G_{\mathbf{A}}/G_F Z_\infty^+} \sum\nolimits_{\gamma \in G_F} f(x\gamma v^{-1})\phi(v)dv = \int_{G_{\mathbf{A}}/G_F Z_\infty^+} h'_x(v)\phi(v)dv\ .$$

Combining the dominated convergence theorem with the above inequality we see that for any $\phi \in L_0^2(G_{\mathbf{A}}/G_F Z_\infty^+)$ the function

$$x \longrightarrow \int_{G_{\mathbf{A}}/G_F Z_\infty^+} h_x(v)\phi(v)dv$$

is continuous. Let

$$H'(x, v) = h'_x(v)\ .$$

Then $H'(x, v)$ is a Hilbert-Schmidt kernel for $\lambda_0(f')$.

Suppose that $H''(v, y)$ is a fixed Hilbert-Schmidt kernel for our second operator $\lambda_0(f'')$. We may assume that for every y, $H''(v, y)$ is square-integrable in v. Then

$$H(x, y) = \int_{G_{\mathbf{A}}/G_F Z_\infty^+} H'(x, v)H''(v, y)dv$$

is well defined for each x and y. It is a Hilbert-Schmidt kernel for $\lambda_0(f)$, and for any fixed y it is continuous in x.

LEMMA 3.8. *The kernel H is integrable over the diagonal and its integral equals the trace of $\lambda_0(f)$.*

Proof. The integrability follows from Schwarz' inequality. Now suppose that $\{\psi_\alpha\}$ is an orthonormal basis for $L_0^2(G_{\mathbf{A}}/G_F Z_\infty^+)$.

Define

$$h'_{\alpha\beta} = \left(\lambda_0(f')\psi_\beta,\, \psi_\alpha\right)$$

and

$$h''_{\gamma\delta} = \left(\lambda_0(f'')\psi_\delta,\ \psi_\gamma\right) .$$

Then

$$\int H(x,\ x)dx$$

equals

$$\int\int \left(\sum_{\alpha,\beta} h'_{\alpha\beta}\psi_\alpha(x)\overline{\psi_\beta(v)}\right)\sum_{\gamma,\delta} h''_{\gamma\delta}\psi_\gamma(v)\overline{\psi_\delta(x)}\big)dv\,dx$$

where the convergence of the infinite sums is in the mean. We may interpret this integral as as the inner product on $L^2(G_{\mathbf{A}}/G_F Z_\infty^+ \times G_{\mathbf{A}}/G_F Z_\infty^-)$. Since the inner product is continuous on any Hilbert space we may interchange the integral and summation signs. The result is

$$\sum_{\alpha,\beta} h'_{\alpha\beta}h''_{\beta\alpha} ,$$

which is just the trace of $\lambda_0(f)$. □

THEOREM 3.9. *The kernel $K_0(x, y)$ is integrable over the diagonal and its integral equals the trace of $\lambda_0(f)$.*

Proof. The functions $H(x, y)$ and $K_0(x, y)$ are both Hilbert-Schmidt kernels for $\lambda_0(f)$ so they must be equal almost everywhere on $(G_{\mathbf{A}}/G_F Z_\infty^+) \times (G_{\mathbf{A}}/G_F Z_\infty^+)$. For any positive integer n let S_n be the set of points y such that the measure of the set

$$T_y = \{x\colon K_0(x,\ y) \neq H(x,\ y)\}$$

is greater than $1/n$. Then measure of S_n is 0. Therefore the measure of

$$S = \bigcup_{n=1}^{\infty} S_n$$

is also zero.

For any y not in S, $H(x, y)$ equals $K_0(x, y)$ for almost all x. But these two functions are continuous in x, so they must be equal for all x. In particular, the set of points y such that

$$H(y,\ y) \neq K_0(y,\ y)$$

has measure 0. Our theorem now follows from Lemma 3.8. □

4. An arrangement of the terms in the kernel

From now on we require that f satisfy Assumption 3.5. We have just shown that

$$K_0(x,\ x) = K(x,\ x) - K_1(x,\ x)$$

is integrable and that

$$\operatorname{tr} \lambda_0(f) = \int_{G_{\mathbf{A}}/G_F Z_\infty^+} K_0(x, x)dx .$$

Before we calculate this integral, we must group the terms in the integrand in a suitable manner.

Recall that $K(x, x)$ equals

$$\sum_{\gamma \in G_F} f(x\gamma x^{-1}) .$$

An element in G_F is said to be *elliptic* if it is not G_F-conjugate to any element in P_F. Any such element is semisimple. Let G_e be the collection of elliptic elements in G_F. Before classifying the remaining elements of G_F we shall first prove a few simple lemmas.

Suppose $\mu \in M_F$. Then μ is semisimple. Recall that $G^+(\mu)$, $P^+(\mu)$, $M^+(\mu)$, and $N^+(\mu)$ are the centralizers of μ in G, P, M, and N respectively.

LEMMA 4.1. *For any* $\mu \in M_F$

$$P^+(\mu) = M^+(\mu)N^+(\mu) .$$

Proof. It is obvious that $M^+(\mu)N^+(\mu)$ is contained in $P^+(\mu)$. Suppose that

$$p = mn , \qquad m \in M , \; n \in N ,$$

is in $P^+(\mu)$. Then

$$mn = \mu p \mu^{-1} .$$

Since μ normalizes both M and N,

$$\mu m \mu^{-1} = m \text{ and } \mu n \mu^{-1} = n . \qquad \square$$

For any such μ, $G(\mu)$ is reductive, and $P(\mu)$ is a minimal parabolic subgroup defined over F. From the lemma,

$$P(\mu) = M(\mu)N(\mu)$$

is a Levi decomposition for $P(\mu)$. In the discussion of § 1 we may replace (G, P, M, N) by $\big(G(\mu), P(\mu), M(\mu), N(\mu)\big)$ and make all the corresponding definitions. We use them without further comment.

It is known $\big([1, § 11]\big)$ that $N^+(\mu)$ is connected. It follows that $N(\mu)_F$ is the centralizer of μ in N_F.

LEMMA 4.2. *Fix* $\mu \in M_F$. *Suppose that* ϕ *is a compactly supported function on* $N_{\mathbf{A}}$. *Then*

$$\sum_{\delta \in N_F/N(\mu)_F} \sum_{\nu \in N(\mu)_F} \phi(\mu^{-1}\delta\mu\nu\delta^{-1}) = \sum_{\eta \in N_F} \phi(\eta) .$$

Proof. Suppose that $\beta(\mu)$ is the simple F-root of $\big(G(\mu), A\big)$ relative to our ordering on $X(A) \otimes \mathbf{Q}$. Fix $j = 1, 2$. Let $\mathfrak{n}(\mu)^j$ be the set of all X in $\mathfrak{n}(\mu)$, the Lie algebra of $N(\mu)$, such that

$$\mathrm{Ad}\,(a)X = a^{j\beta(\mu)}X\,, \qquad a \in A\,.$$

$\mathfrak{n}(\mu)^j$ is a subspace of $\mathfrak{n}(\mu)$, defined over F. Let us write $\mathfrak{n}^j$ for $\mathfrak{n}(e)^j$. Then $\mathrm{Ad}\,(\mu^{-1})$ is a semisimple linear operator on $\mathfrak{n}^j$ which is defined over F. Let $\widetilde{\mathfrak{n}(\mu)}^j$ be a complementary subspace of $\mathfrak{n}(\mu)^j$ in $\mathfrak{n}^j$ which is invariant under $\mathrm{Ad}\,(\mu^{-1})$. $\widetilde{\mathfrak{n}(\mu)}^j$ is also defined over F. Notice that the linear operator

$$\mathrm{Ad}\,(\mu) - \mathrm{id}$$

is invertible on $\widetilde{\mathfrak{n}(\mu)}^j$. It is clear that

$$N(\mu)^j = \exp \mathfrak{n}(\mu)^j$$

and

$$\widetilde{N(\mu)}^j = \exp \widetilde{\mathfrak{n}(\mu)}^j$$

are F-closed subsets of N.

Define

$$N^j = \exp \mathfrak{n}^j\,, \qquad j = 1, 2.$$

Then $\widetilde{N(\mu)}^1_F$, $\widetilde{N(\mu)}^2_F$, and $\widetilde{N(\mu)}^1_F \cdot \widetilde{N(\mu)}^2_F$ are sets of representatives for $N_F/N(\mu)_F \cdot N^2_F$, $N^2_F/N(\mu)^2_F$ and $N_F/N(\mu)_F$ respectively.

We have

$$\begin{aligned}
&\sum_{\delta \in N_F/N(\mu)_F} \sum_{\nu \in N(\mu)_F} \phi(\mu^{-1}\delta\mu\nu\delta^{-1}) \\
&= \sum_{(\delta_1,\delta_2) \in (\widetilde{N(\mu)}^1_F \times \widetilde{N(\mu)}^2_F)} \sum_{(\nu_1,\nu_2) \in (N(\mu)^1_F \times N(\mu)^2_F)} \phi(\mu^{-1}\delta_1\delta_2\mu\nu_1\nu_2\delta_2^{-1}\delta_1^{-1}) \\
&= \sum_{\delta_1,\nu_1,\delta_2,\nu_2} \phi(\mu^{-1}\delta_1\mu\nu_1\delta_1^{-1}\mu^{-1}\delta_2\mu\delta_2^{-1}\nu_2)
\end{aligned}$$

since N^1_F and N^2_F commute. This in turn equals

$$\begin{aligned}
&\sum_{\delta_1,\nu_1,\nu_2} \sum_{\Gamma \in \widetilde{\mathfrak{n}(\mu)}^2_F} \phi\big(\mu^{-1}\delta_1\mu\nu_1\delta_1^{-1} \exp\big(\mathrm{Ad}\,(\mu^{-1})\Gamma - \Gamma\big)\nu_2\big) \\
&= \sum_{\delta_1,\nu_1,\nu_2} \sum_{\Gamma \in \widetilde{\mathfrak{n}(\mu)}^2_F} \phi\big(\mu^{-1}\delta_1\mu\nu_1\delta_1^{-1} \exp(\Gamma)\nu_2\big) \\
&= \sum_{\delta_1,\nu_1} \sum_{\eta_2 \in N^2_F} \phi(\mu^{-1}\delta_1\mu\nu_1\delta_1^{-1}\eta_2) \\
&= \sum_{\Gamma \in \widetilde{\mathfrak{n}(\mu)}^1_F} \sum_{\nu_1 \in N(\mu)^1_F} \sum_{\eta_2 \in N^2_F} \phi\,\big(\exp\big(\mathrm{Ad}\,(\mu^{-1})\Gamma - \Gamma\big)\nu_1\eta_2\big) \\
&= \sum_{\Gamma,\nu_1,\eta_2} \phi\,\big(\exp(\Gamma)\nu_1\eta_2\big) \\
&= \sum_{\eta \in N_F} \phi(\eta)\,. \qquad \square
\end{aligned}$$

COROLLARY 4.3. *Suppose that* $\{N\}_\mu$ *is any set of representatives of* $N_F/N(\mu)_F$ *in* N_F. *Then the map*

$$(\delta, \nu) \longrightarrow \mu^{-1}\delta\mu\nu\delta^{-1}\,, \qquad \delta \in \{N\}_\mu\,,\ \nu \in N(\mu)_F\,,$$

is a bijection from $\{N\}_\mu \times N(\mu)_F$ *onto* N_F.

This corollary is just a restatement of the lemma. □

While we are at it, we may as well state an adèlic version of the last lemma.

LEMMA 4.4. *For any compactly supported measurable function ϕ on $N_{\mathbf{A}}$ we have*

$$\int_{N_{\mathbf{A}}/N(\mu)_{\mathbf{A}}}\int_{N(\mu)_{\mathbf{A}}}\phi(\mu^{-1}n^*\mu nn^{*-1})dn\,dn^* = \int_{N_{\mathbf{A}}}\phi(n)dn\ .$$

Proof. Fix $j = 1, 2$. Our Haar measures on $\mathfrak{n}_{\mathbf{A}}^j$ and $\mathfrak{n}(\mu)_{\mathbf{A}}^j$ define a Haar measure $d\tilde{X}^j$ on $\widetilde{\mathfrak{n}(\mu)}_{\mathbf{A}}^j$. It is an easy consequence of the product formula for F that

$$\int_{\widetilde{\mathfrak{n}(\mu)}_{\mathbf{A}}^j}\phi^j(\exp(\mathrm{Ad}(\mu^{-1})\tilde{X}^j - \tilde{X}^j))d\tilde{X}^j = \int_{\widetilde{\mathfrak{n}(\mu)}_{\mathbf{A}}^j}\phi^j(\exp\tilde{X}^j)d\tilde{X}^j$$

for any compactly supported measurable function ϕ^j on $\widetilde{N(\mu)}_{\mathbf{A}}^j$.

To prove our lemma we just repeat the argument of Lemma 4.2, replacing each sum over a set of F-rational points with an integral over the corresponding adèle space. □

LEMMA 4.5. *Any element in P_F is P_F-conjugate to an element $\mu\nu$ with $\mu \in M_F$ and $\nu \in N(\mu)_F$.*

Proof. Any element in P_F can be written as $\mu\eta$ for $\mu \in M_F$ and $\eta \in N_F$. By Corollary 4.3 there are elements $\delta \in N_F$ and $\nu \in N(\mu)_F$ such that

$$\eta = \mu^{-1}\delta\mu\nu\delta^{-1}\ .$$

In other words

$$\mu\eta = \delta\mu\nu\delta^{-1}\ . \qquad \square$$

Let M_r be the set of elements μ in M_F such that $N(\mu)$ is trivial. Let M_s be the complement of M_r in M_F.

Recall that M is a subgroup of index 2 in $N(A)$, the normalizer of A. M_r and M_s are both stable under conjugation by elements in $N(A)_F$.

LEMMA 4.6. *Suppose that δ_1 and δ_2 are elements in G_F and that*

$$\delta_1\mu_1\delta_1^{-1} = \delta_2\mu_2\delta_2^{-1}$$

for two elements μ_1 and μ_2 in M_r. Then δ_1 is in the same $N(A)_F$-coset as δ_2.

Proof. Let $\varepsilon = \delta_2^{-1}\delta_1$. Then $\mu_2^{-1}\varepsilon\mu_1 = \varepsilon$. Either ε is in P_F or it is in $N_F\cdot w\cdot P_F$. In the first case

$$\varepsilon = \mu\nu\ , \qquad \mu \in M_F\ ,\ \nu \in N_F\ ,$$

and

$$\mu\nu = \mu_2^{-1}\mu\nu\mu_1 = \mu_2^{-1}\mu\mu_1\mu_1^{-1}\nu\mu_1 \;..$$

It follows that $\nu \in N(\mu_1)_F$, so that $\nu = \varepsilon$. Therefore $\delta_1 = \delta_2\mu$.

On the other hand, suppose that

$$\varepsilon = \nu w\pi\,, \qquad \nu \in N_F\,,\ \pi \in P_F\,.$$

Then

$$\nu w\pi = \mu_2^{-1}\nu w\pi\mu_1 = \mu_2^{-1}\nu\mu_2 w\cdot w^{-1}\mu_2^{-1}w\pi\mu_1\,.$$

By the Bruhat decomposition,

$$\mu_2^{-1}\nu\mu_2 = \nu$$

and

$$w^{-1}\mu_2^{-1}w\pi\mu_1 = \pi\,.$$

Since $N(\mu_2)_F = \{e\}$, ν must equal e, so that

$$\delta_1 = \delta_2 w\pi\,.$$

However, by the same argument as above π must belong to M_F. $\square$

LEMMA 4.7. *Suppose that δ_1 and δ_2 are elements in G_F and that*

$$\delta_1\mu_1\nu_1\delta_1^{-1} = \delta_2\mu_2\nu_2\delta_2^{-1}$$

for μ_1, $\mu_2 \in M_s$, and ν_1 and ν_2 nontrivial elements in $N(\mu_1)_F$ and $N(\mu_2)_F$ respectively. Then δ_1 and δ_2 are in the same P_F-conjugacy class.

Proof. Let $\varepsilon = \delta_2^{-1}\delta_1$. Then

$$\nu_2^{-1}\mu_2^{-1}\varepsilon\mu_1\nu_1 = \varepsilon.$$

If ε is not in P_F,

$$\varepsilon = \nu w\pi\,, \qquad \nu \in N_F\,,\ \pi \in P_F\,.$$

Then

$$\nu w\pi = \nu_2^{-1}\mu_2^{-1}\cdot\nu w\pi\cdot\mu_1\nu_1 = \nu_2^{-1}\mu_2^{-1}\nu\mu_2\cdot w\cdot w^{-1}\mu_2^{-1}w\pi\mu_1\nu_1\,.$$

By the Bruhat decomposition, we have

$$\nu_2^{-1}\mu_2^{-1}\nu\mu_2 = \nu\,.$$

This implies that $\nu_2 = \mu_2^{-1}\nu\mu_2\nu^{-1}$. By Corollary 4.3, ν_2 equals the identity element. This is a contradiction, since we assumed that ν_2 was nontrivial. Therefore $\delta_2^{-1}\delta_1 \in P_F$. $\square$

We shall write $\{G_e\}$ for a fixed set of representatives of G_F-conjugacy classes in G_e. Let $\{M_r\}$ and $\{M_s\}$ denote fixed sets of representatives of M_F-conjugacy classes in M_r and M_s respectively. Finally, fix a set $\{\{M_s\}\}$ of representatives in $\{M_s\}$ of those G_F-conjugacy classes in G_F which intersect

M_s. It is clear that the contribution to $K(x, x)$ from elements which are G_F-conjugate to an element in M_s is

$$I^M(f: x) = \sum_{\mu \in \{\{M_s\}\}} (n_{\mu,G})^{-1} \sum_{\delta \in G_F/G(\mu)_F} f(x\delta\mu\delta^{-1}x^{-1}) .$$

In this formula we have had to include the integer $n_{\mu,G}$ which, as we recall, is the index of $G(\mu)_F$ in $G^+(\mu)_F$. The contribution from the elliptic elements is

$$I_e(f: x) = \sum_{\gamma \in \{G_e\}} (n_{\gamma,G})^{-1} \sum_{\delta \in G_F/G(\gamma)_F} f(x\delta\gamma\delta^{-1}x^{-1}) .$$

Lemmas 4.6 and 4.7 account for the contribution from the remaing elements. $K(x, x)$ becomes the sum of

$$I_e(f: x) + I^M(f: x)$$

together with

$$\sum_{\delta \in G_F/N(A)_F} \sum_{\mu \in M_r} f(x\delta\mu\delta^{-1}x^{-1}) \tag{4.1}$$

and

$$\sum_{\delta \in G_F/P_F} \sum_{\mu \in M_s} \sum_{\substack{\nu \in N(\mu)_F \\ \nu \neq e}} f(x\delta\mu\nu\delta^{-1}x^{-1}) . \tag{4.2}$$

Suppose that ε_0 is the positive number defined by Lemma 1.3. Fix a number ε between 0 and ε_0. Let χ_ε be the characteristic function of the set $S(\varepsilon)$. Since M_r is stable under conjugation by w, the term (4.1) equals

$$\begin{aligned} &\frac{1}{2} \sum_{\delta \in G_F/M_F} \sum_{\mu \in M_r} f(x\delta\mu\delta^{-1}x^{-1}) \\ &= \frac{1}{2} \sum_{\mu \in \{M_r\}} (n_{\mu,M})^{-1} \sum_{\delta \in G_F/M(\mu)_F} f(x\delta\mu\delta^{-1}x^{-1}) \end{aligned}$$

which we decompose into the sum of

$$J_r^P(f: x: \varepsilon) = \frac{1}{2} \sum_{\mu \in \{M_r\}} (n_{\mu,M})^{-1} \sum_{\delta \in G_F/M(\mu)_F} f(x\delta\mu\delta^{-1}x^{-1})\big(\chi_\varepsilon(x\delta) + \chi_\varepsilon(x\delta w)\big)$$

and

$$I_r^P(f: x: \varepsilon) = \frac{1}{2} \sum_{\mu \in \{M_r\}} (n_{\mu,M})^{-1} \sum_{\delta \in G_F/M(\mu)_F} f(x\delta\mu\delta^{-1}x^{-1})\big(1 - \chi_\varepsilon(x\delta) - \chi_\varepsilon(x\delta w)\big) .$$

The term (4.2) equals

$$\sum_{\mu \in \{M_s\}} (n_{\mu,M})^{-1} \sum_{\delta \in G_F/P(\mu)_F} \sum_{\substack{\nu \in N(\mu)_F \\ \nu \neq e}} f(x\delta\mu\nu\delta^{-1}x^{-1})$$

which we decompose into the sum of

$$\begin{aligned} &J_s^P(f: x: \varepsilon) \\ &= \sum_{\mu \in \{M_s\}} (n_{\mu,M})^{-1} \sum_{\delta \in G_F/P(\mu)_F} \sum_{\substack{\nu \in N(\mu)_F \\ \nu \neq e}} f(x\delta\mu\nu\delta^{-1}x^{-1})\chi_\varepsilon(x\delta) \end{aligned}$$

and

$$I_s^P(f\colon x\colon \varepsilon)$$
$$= \sum_{\mu \in \{M_s\}} (n_{\mu.M})^{-1} \sum_{\delta \in G_F/P(\mu)_F} \sum_{\substack{\nu \in N(\mu)_F \\ \nu \neq e}} f(x\delta\mu\nu\delta^{-1}x^{-1})(1 - \chi_\varepsilon(x\delta)) .$$

We have used the equality of the integers $n_{\mu.M}$ and $n_{\mu.P}$, which follows from Lemma 4.1 and the fact that $N^+(\mu)$ is connected.

Next we shall break up the function $K_1(x, x)$. Suppose that ϕ belongs to some $L \in V(P)$. Recall that $E_P(\phi\colon z\colon x)$, the constant term of $E(\phi\colon z\colon x)$, equals

$$\phi(x)e^{(z-\rho)H(x)} + \big(M(z)\phi\big)(x)e^{(-z-\rho)H(x)} .$$

For any number ε between 0 and ε_0 we define $E_\varepsilon'(\phi\colon z\colon x)$ to be

$$\sum_{\delta \in G_F/P_F} E_P(\phi\colon z\colon x\delta)\chi_\varepsilon(x\delta) ,$$

where, as before, χ_ε is the characteristic function of the set $S(\varepsilon)$. For any x the sum is finite by Lemma. 1.2. We set $E_\varepsilon''(\phi\colon z\colon x)$ equal to $E(\phi\colon z\colon x) - E_\varepsilon'(\phi\colon z\colon x)$.

For convenience, set $H_P(n\colon z\colon f\colon x)$ equal to

$$\frac{1}{4\pi} \sum_{\alpha,\beta \in I_n} \pi_{\alpha\beta}(z\colon f)E_P(\phi_\alpha\colon z\colon x)\overline{E_P(\phi_\beta\colon z\colon x)} .$$

Now $E_P(\phi\colon z\colon x)$ is obtained by integrating the function

$$h(n) = E(\phi\colon z\colon xn) , \qquad n \in N_{\mathbf{A}}$$

over the compact set $N_{\mathbf{A}}/N_F$. Therefore by Torelli's theorem and Theorem 3.2 we can associate constants C and N to any Siegel domain $\mathfrak{S}$ such that the inequality

$$\sum_{n=1}^{\infty} \int_{-i\infty}^{i\infty} | H_P(n\colon z\colon f\colon x) |\, d\,| z | \leqq Ce^{-NH(x)} \tag{4.3}$$

holds for all x in $\mathfrak{S}$.

We define $K'(f\colon x\colon \varepsilon)$ to be

$$\sum_{\delta \in G_F/P_F} \sum_{n=1}^{\infty} \int_{-i\infty}^{i\infty} H_P(n\colon z\colon f\colon x\delta)\chi_\varepsilon(x\delta)d\,| z | .$$

It is easy to see from Lemma 1.4 that $K'(f\colon x\colon \varepsilon)$ also equals

$$\frac{1}{4\pi} \sum_{\alpha,\beta \in I} \int_{-i\infty}^{i\infty} \pi_{\alpha\beta}(z\colon f)E_\varepsilon'(\phi_\alpha\colon z\colon x)\overline{E_\varepsilon'(\phi_\beta\colon z\colon x)}d\,| z | .$$

LEMMA 4.8. *Given any Siegel domain $\mathfrak{S}$ there are constants C and N such that for all $x \in \mathfrak{S}$ the expression*

$$\frac{1}{4\pi} \sum_{n=1}^{\infty} \int_{-i\infty}^{i\infty} \Big| \sum_{\alpha,\beta \in I_n} \pi_{\alpha\beta}(z\colon f)E_\varepsilon'(\phi_\alpha\colon z\colon x)\overline{E_\varepsilon'(\phi_\beta\colon z\colon x)} \Big|\, d\,| z |$$

is bounded by $Ce^{-NH(x)}$.

Proof. It is clear from Lemma 1.4 that our expression equals

$$\sum_{\delta \in G_F/P_F} \sum_{n=1}^{\infty} \int_{-i\infty}^{i\infty} | H_P(n: z: f: x\delta) | \chi_\varepsilon(x\delta) d | z | .$$

By Lemma 1.2 the number of terms in the sum over G_F/P_F is no greater than some integer which is independent of $x \in \mathfrak{S}$. The lemma then follows from (4.3) and Corollary 1.5. □

Let us define $K''(f: x: \varepsilon)$ to be

$$K_1(x, x) - K'(f: x: \varepsilon).$$

Then we may write $K(x, x) - K_1(x, x)$ as the sum of the following five terms:

$$\begin{aligned} &I_e(f: x) , \\ &I^M(f: x) , \\ &J_r^P(f: x: \varepsilon) + J_s^P(f: x: \varepsilon) - K'(f: x: \varepsilon) , \\ &I_r^P(f: x: \varepsilon) + I_s^P(f: x: \varepsilon) , \end{aligned}$$

and

$$-K''(f: x: \varepsilon) .$$

We shall refer to these terms respectively as the *elliptic*, *singular*, and *first*, *second*, and *third parabolic* terms.

We would like to evaluate the integrals over $G_{\mathbf{A}}/G_F Z_\infty^+$ of each of these five terms. However, integrals arise in the third parabolic term whose convergence is not at all obvious. One way to surmount this difficulty is to prove that each of the first four terms is integrable over $G_{\mathbf{A}}/G_F Z_\infty^+$. This would verify the integrability of the fifth term.

It will be sufficient to prove a weaker result. Let us say that a function h is *weakly integrable* over $G_{\mathbf{A}}/G_F Z_\infty^+$ if

(i) it is locally integrable,

and

(ii) for some $c > 0$ the integral

$$\int_{-\infty}^{\log c} \left| \int_K \int_{P_{\mathbf{A}}^1/P_F} h(kh_t p) dp\, dk \right| e^{2\rho t} dt$$

is finite.

When we come to the first parabolic term it will be easier to prove only that it is weakly integrable. Of course this will weaken our conclusion on the integrability of the third parabolic term, but that will not matter.

5. The elliptic and singular terms

Our first concern will be to prove that the elliptic term is integrable over $G_{\mathbf{A}}/G_F Z_\infty^+$. The integral

$$\int_{G_{\mathbf{A}}/G_F Z_\infty^+} | I_e(f: x) | \, dx$$

is bounded by the integral over $G_{\mathbf{A}}/G_F Z_\infty^+$ of

$$\sum_{\gamma \in G_e} | f(x\gamma x^{-1}) | \, . \tag{5.1}$$

Choose $c > 0$ and a Siegel domain

$$\mathfrak{S}(c) = K \cdot A_\infty^+(c) \cdot \omega$$

such that

$$G_{\mathbf{A}} = \mathfrak{S}(c) \cdot G_F \, .$$

ω is a relatively compact subset of $P_{\mathbf{A}}^1$. It is a simple matter to check that

$$\omega_c = \{h_t v h_t^{-1} \colon v \in \omega, \, t \leqq \log c\}$$

is also a relatively compact subset of $P_{\mathbf{A}}^1$.

Suppose that $S \subseteqq G_{\mathbf{A}}$ is the support of f. S is compact modulo Z_∞^+. Let C be the closure in $G_{\mathbf{A}}$ of the set

$$\omega_c^{-1} K \cdot S \cdot k \omega_c \, .$$

C is compact modulo Z_∞^+.

LEMMA 5.1. *Suppose that C is a compact subset of $G_{\mathbf{A}}$ modulo Z_∞^+. Then there is a number $\varepsilon > 0$ such that if $\gamma \in G_F$ and $h_t \gamma h_t^{-1}$ lies in C for some $t < \log \varepsilon$, then γ is in P_F.*

Proof. Let ρ be the strongly F-rational representation of G on the vector space V defined in § 1. We use the basis $\{e_0, \cdots, e_d\}$ and the height function on V introduced in § 1. Now ρ is trivial Z_∞^+. It follows that

$$\sup_{x \in C} || \rho(x) e_0 ||$$

is finite. We set this supremum equal to ε^{-2b}, where b is the positive rational number defined in § 1.

If γ is not in P_F,

$$\gamma = \nu w \pi \, , \qquad \nu \in N_F \, , \ \pi \in P_F \, ,$$

by the Bruhat decomposition. Then

$$\| \rho(h_t \gamma h_t^{-1})e_0 \| = e^{-bH(h_t)} \| \rho(h_t \nu)e_d \| \geqq e^{-2bH(h_t)} \| e_d \| = e^{-2bt} .$$

If $h_t \gamma h_t^{-1}$ lies in C,

$$e^{-2bt} \leqq \varepsilon^{-2b} ,$$

so that $t \geqq \log \varepsilon$. The lemma is proved. $\square$

It follows from this lemma that the function on $G_{\mathbf{A}}/G_F Z_\infty^+$ defined by (5.1) has compact support. Therefore the elliptic term is integrable over $G_{\mathbf{A}}/G_F Z_\infty^+$. Its integral equals

$$\sum\nolimits_{\gamma \in \{G_e\}} (n_{\gamma,G})^{-1} \int_{G(\gamma)_{\mathbf{A}}/G(\gamma)_F Z_\infty^+} dx_1 \int_{G_{\mathbf{A}}/G(\gamma)_{\mathbf{A}}} f(x\gamma x^{-1}) dx .$$

Now for $\gamma \in \{G_e\}$, Z is the split component of the radical of $G(\gamma)$. We have agreed to use the Tamagawa measure on $G(\gamma)_{\mathbf{A}}$. However, we cannot immediately insert the Tamagawa number of $G(\gamma)$ in the above formula because our measure on Z_∞^+ does not define the appropriate quotient measure. We must correct by a factor $\Gamma_{\gamma,G}$, which we define to be the index in $X(G(\gamma))_F$ of the group obtained by restricting the characters in $X(G)_F$ to $G(\gamma)$. If we write $\tilde{\tau}(\gamma, G)$ for the number

$$(n_{\gamma,G})^{-1} (\Gamma_{\gamma,G})^{-1} \tau(G(\gamma)) ,$$

the integral of the elliptic term becomes

$$\sum\nolimits_{\gamma \in \{G_e\}} \tilde{\tau}(\gamma, G) \int_{G_{\mathbf{A}}/G(\gamma)_{\mathbf{A}}} f(x\gamma x^{-1}) dx .$$

Before discussing the singular term we shall prove two more lemmas.

LEMMA 5.2. *Let C be a subset of $G_{\mathbf{A}}$ which is compact modulo Z_∞^+. Then there is only a finite number of elements μ in $\{M_r\} \cup \{M_s\}$ such that there is an x in $G_{\mathbf{A}}$ and an n in $N_{\mathbf{A}}$ for which $x\mu n x^{-1}$ lies in C.*

Proof. Let $C_1 = \{kck^{-1} : c \in C, k \in K\}$. Since $P_{\mathbf{A}}$ is closed in $G_{\mathbf{A}}$, $C_1 \cap P_{\mathbf{A}}$ is compact modulo Z_∞^+. We can choose a subset C_M of $M_{\mathbf{A}}$ which is compact modulo Z_∞^+ such that

$$C_1 \cap P_{\mathbf{A}} \subseteqq C_M N_{\mathbf{A}} .$$

Suppose that $x\mu n x^{-1}$ lies in C. Then if

$$x = kp , \qquad k \in K, \ p \in P_{\mathbf{A}} ,$$

$p\mu n p^{-1}$ lies in $C_M N_{\mathbf{A}}$.

Let ω be a relatively compact set of representatives in $P_{\mathbf{A}}$ for the compact double coset space $A_\infty^+ \backslash P_{\mathbf{A}} / P_F$. If

$$p = a v \pi , \qquad a \in A_\infty^+, \ v \in \omega , \ \pi \in P_F ,$$

then the element $v\pi \cdot \mu n \cdot \pi^{-1}v^{-1}$ lies in the set

$$a^{-1} \cdot C_M N_{\mathbf{A}} \cdot a = C_M N_{\mathbf{A}} .$$

Choose a subset C'_M of $M_{\mathbf{A}}$, compact modulo Z_∞^+, such that $\omega^{-1} \circ C_M N_{\mathbf{A}} \circ \omega$ is contained in $C'_M N_{\mathbf{A}}$. Then $\pi \mu n \pi^{-1}$ lies in $C'_M N_{\mathbf{A}}$. In particular μ is M_F-conjugate to an element in C'_M. However, the projection of M_F onto $M_{\mathbf{A}}/Z_\infty^+$ is a discrete subgroup of $M_{\mathbf{A}}/Z_\infty^+$ so that $M_F \cap C'_M$ is finite. Certainly only finitely many M_F-conjugacy classes in M_F meet C'_M. The lemma is proved. □

LEMMA 5.3. *Fix $\mu \in \{M_r\} \cup \{M_s\}$. Suppose that C is a compact subset of $P_{\mathbf{A}}^1$. Then there is a compact subset C_1 of $P_{\mathbf{A}}^1/P(\mu)_{\mathbf{A}}^1$ such that if $p \in P_{\mathbf{A}}^1/P(\mu)_{\mathbf{A}}^1$ and*

$$\left(p \cdot \mu N(\mu)_{\mathbf{A}} \cdot p^{-1}\right) \cap C \neq \varnothing ,$$

then p lies in C_1.

Proof. Let ω be a relatively compact fundamental set in $P_{\mathbf{A}}^1$ for $P_{\mathbf{A}}^1/P_F$. Denote the closure in $P_{\mathbf{A}}^1$ of $\omega^{-1}C\omega$ by C'. Let $\mathcal{F}$ be the collection of cosets δ in $P_F/P(\mu)_F$ such that

$$\left(\delta \cdot \mu N(\mu)_{\mathbf{A}} \cdot \delta^{-1}\right) \cap C' \neq \varnothing .$$

The main point of the lemma is to show that $\mathcal{F}$ is finite. Assuming this fact for the moment, we let C'_1 be the closure in $P_{\mathbf{A}}^1/P(\mu)_F$ of the set

$$\bigcup_{\delta \in \mathcal{F}} \omega\delta .$$

Then if

$$\left(p \cdot \mu N(\mu)_{\mathbf{A}} \cdot p^{-1}\right) \cap C \neq \varnothing$$

for some $p \in P_{\mathbf{A}}^1/P(\mu)_F$, p must lie in C'_1. If C_1 is the projection of C'_1 onto $P_{\mathbf{A}}^1/P(\mu)_{\mathbf{A}}^1$, then C_1 is the required set.

It remains to show that $\mathcal{F}$ is finite. For $j = 1, 2$, define $N(\mu)^j$, $\widetilde{N(\mu)}^j$, and N^j as in the proof of Lemma 4.2. If $\{M\}_\mu$ is a set of representatives of $M_F/M(\mu)_F$ in M_F, it is clear that

$$\{M\}_\mu \cdot \widetilde{N(\mu)}_F^1 \cdot \widetilde{N(\mu)}_F^2$$

is a set of representatives of $P_F/P(\mu)_F$ in P_F.

Now there is a compact subset C_M of $M_{\mathbf{A}}^1$ such that

$$C' \subseteqq C_M N_{\mathbf{A}} .$$

Let $\mathcal{F}_M$ be the set of all elements δ in $\{M\}_\mu$ such that $\delta\mu\delta^{-1}$ lies in C_M. Keeping in mind that $M(\mu)_F$ is of finite index in $M^+(\mu)_F$, and using the fact that $M_F \cap C_M$ is finite, we conclude that $\mathcal{F}_M$ is finite. It follows that the union over all $\delta \in \mathcal{F}_M$ of the sets $\delta^{-1}C'\delta$ is a compact subset of $P^1_{\mathbf{A}}$ and is certainly contained in

$$M^1_{\mathbf{A}} \cdot \tilde{C}^1_N \cdot N(\mu)^1_{\mathbf{A}} \cdot N^2_{\mathbf{A}}$$

for some compact subset $\tilde{C}^1_N$ of $\widetilde{N(\mu)}^1_{\mathbf{A}}$.

Let $\mathcal{F}^1_N$ be the set of all elements η_1 in $\widetilde{N(\mu)}^1_F$ such that

$$\mu^{-1}\eta_1\mu\eta_1^{-1} \in \tilde{C}^1_N \cdot N(\mu)^1_{\mathbf{A}} \cdot N^2_{\mathbf{A}} .$$

$\mathcal{F}^1_N$ is finite. The union over all $\delta \in \mathcal{F}_M$ and $\eta_1 \in \mathcal{F}^1_N$ of the sets $\eta_1^{-1}\delta^{-1}C'\delta\eta_1$ is compact and is certainly contained in

$$M^1_{\mathbf{A}} \cdot N^1_{\mathbf{A}} \cdot \tilde{C}^2_N \cdot N(\mu)^2_{\mathbf{A}}$$

for some compact subset $\tilde{C}^2_N$ of $N(\mu)^2_{\mathbf{A}}$. Let $\mathcal{F}^2_N$ be the finite set of elements $\eta_2 \in \widetilde{N(\mu)}^2_F$ such that

$$\mu^{-1}\eta_2\mu\eta_2^{-1} \in \tilde{C}^2_N \cdot N(\mu)^2_{\mathbf{A}} .$$

Then the finite set

$$\mathcal{F}_M \cdot \mathcal{F}^1_N \cdot \mathcal{F}^2_N$$

contains a set of representatives of our collection $\mathcal{F}$ of cosets. □

Let us now deal with the singular term. The integral

$$\int_{G_{\mathbf{A}}/G_F Z^+_\infty} | I^M(f: x) | \, dx$$

is bounded by

$$(5.1) \qquad \sum\nolimits_{\mu \in \{\{M_s\}\}} (n_{\mu,G})^{-1} \int_{G_{\mathbf{A}}/G(\mu)_F Z^+_\infty} | f(x\mu x^{-1}) | \, dx .$$

The function f is compactly supported on $G_{\mathbf{A}}/Z^+_\infty$ so by Lemma 5.2 the sum over μ is finite.

For any $\mu \in \{M_s\}$, $G(\mu)$ is a reductive group defined over F. $G(\mu)$ contains A, but since μ lies in M_s, A is not contained in the center of $G(\mu)$. Therefore the F-split component of the radical of $G(\mu)$ is Z. In particular, the volume of $G(\mu)_{\mathbf{A}}/G(\mu)_F Z^+_\infty$ with respect to the Tamagawa measure on $G(\mu)$ is the quotient of $\tau(G(\mu))$ and $\Gamma_{\mu,G}$, the correction factor introduced in our discussion of the elliptic term.

The integral (5.1) equals

$$\sum\nolimits_{\mu \in \{\{M_s\}\}} \tilde{\tau}(\mu, G) \int_{G_{\mathbf{A}}/G(\mu)_{\mathbf{A}}} | f(x\mu x^{-1}) | \, dx .$$

It follows from Lemma 5.3 that for a fixed μ the function on $P_{\mathbf{A}}/P(\mu)_{\mathbf{A}}$ defined by

$$p \longrightarrow \int_K |f(kp\mu p^{-1}k^{-1})|\,dk\,, \qquad p \in P_{\mathrm{A}}/P(\mu)_{\mathrm{A}}\,,$$

is of compact support. We conclude that the singular term is integrable over $G_{\mathrm{A}}/G_F Z_\infty^+$. Its integral equals

$$\sum\nolimits_{\mu \in \{(M_s)\}} \tilde{\tau}(\mu, G) \int_{G_{\mathrm{A}}/G(\mu)_{\mathrm{A}}} f(x\mu x^{-1})dx\,.$$

6. The first parabolic term

The first parabolic term equals

$$J_r^P(f\colon x\colon \varepsilon) + J_s^P(f\colon x\colon \varepsilon) - K'(f\colon x\colon \varepsilon)\,.$$

In this section we shall prove that this term is weakly integrable over $G_{\mathrm{A}}/G_F Z_\infty^+$ and that its integral approaches 0 as ε approaches 0.

$J_r^P(f\colon x\colon \varepsilon)$ equals

$$\frac{1}{2}\sum\nolimits_{\mu \in \{M_r\}} (n_{\mu,M})^{-1} \sum\nolimits_{\delta \in G_F/M(\mu)_F} f(x\delta\mu\delta^{-1}x^{-1})\big(\chi_\varepsilon(x\delta) + \chi_\varepsilon(x\delta w)\big)\,.$$

Now

$$\begin{aligned} &\frac{1}{2}\sum\nolimits_{\mu \in \{M_r\}} (n_{\mu,M})^{-1} \sum\nolimits_{\delta \in G_F/M(\mu)_F} f(x\delta\mu\delta^{-1}x^{-1})\chi_\varepsilon(x\delta w) \\ &\qquad = \frac{1}{2}\sum\nolimits_{\delta \in G_F/M_F} \sum\nolimits_{\mu \in M_r} f(x\delta\mu\delta^{-1}x^{-1})\chi_\varepsilon(x\delta w) \\ &\qquad = \frac{1}{2}\sum\nolimits_{\delta \in G_F/M_F} \sum\nolimits_{\mu \in M_r} f(x\delta\mu\delta^{-1}x^{-1})\chi_\varepsilon(x\delta) \\ &\qquad = \frac{1}{2}\sum\nolimits_{\mu \in \{M_r\}} (n_{\mu,M})^{-1} \sum\nolimits_{\delta \in G_F/M(\mu)_F} f(x\delta\mu\delta^{-1}x^{-1})\chi_\varepsilon(x\delta)\,. \end{aligned}$$

Therefore $J_r^P(f\colon x\colon \varepsilon)$ equals

$$\sum\nolimits_{\mu \in \{M_r\}} (n_{\mu,M})^{-1} \sum\nolimits_{\delta \in G_F/M(\mu)_F} f(x\delta\mu\delta^{-1}x^{-1})\chi_\varepsilon(x\delta)\,.$$

For $\mu \in \{M_r\}$, the group $N(\mu)$ is trivial. It follows from Lemma 4.2 that $J_r^P(f\colon x\colon \varepsilon)$ equals

$$\sum\nolimits_{\mu \in \{M_r\}} (n_{\mu,M})^{-1} \sum\nolimits_{\delta \in G_F/M(\mu)_F N_F} \sum\nolimits_{\nu \in N_F} f(x\delta\mu\nu\delta^{-1}x^{-1})\chi_\varepsilon(x\delta)\,..$$

Now $J_s^P(f\colon x\colon \varepsilon)$ equals the difference between

$$\sum\nolimits_{\mu \in \{M_s\}} (n_{\mu,M})^{-1} \sum\nolimits_{\delta \in G_F/P(\mu)_F} \sum\nolimits_{\nu \in N(\mu)_F} f(x\delta\mu\nu\delta^{-1}x^{-1})\chi_\varepsilon(x\delta) \tag{6.1}$$

and

$$\sum\nolimits_{\mu \in \{M_s\}} (n_{\mu,M})^{-1} \sum\nolimits_{\delta \in G_F/P(\mu)_F} f(x\delta\mu\delta^{-1}x^{-1})\chi_\varepsilon(x\delta)\,. \tag{6.2}$$

The integral over $G_{\mathrm{A}}/G_F Z_\infty^+$ of the absolute value of (6.2) is bounded by

$$\sum_{\mu \in \{M_s\}} (n_{\mu,M})^{-1} \int_{G_{\mathbf{A}}/P(\mu)_F Z_\infty^+} |f(x\mu x^{-1})| \chi_\varepsilon(x) dx ,$$

which we may write as the product of

$$\int_{-\infty}^{\log \varepsilon} e^{2\rho t} dt$$

with

$$c_G \sum_{\mu \in \{M_s\}} (n_{\mu,M})^{-1} \tau(M(\mu)) \int_K \int_{P_{\mathbf{A}}^1/P(\mu)_{\mathbf{A}}^1} |f(kp\mu p^{-1}k^{-1})| \, dp \, dk .$$

It follows directly from Lemmas 5.2 and 5.3 that this last expression is finite. The integral

$$\int_{-\infty}^{\log \varepsilon} e^{2\rho t} dt$$

is obviously finite and approaches 0 as ε approaches 0.

By Lemmas 4.1 and 4.2 the term (6.1) equals

$$\sum_{\mu \in \{M_s\}} (n_{\mu,M})^{-1} \sum_{\delta \in G_F/M(\mu)_F N_F} \sum_{\nu \in \mathfrak{N}_F} f(x\delta\mu\nu\delta^{-1}x^{-1}) \chi_\varepsilon(x\delta) .$$

Since $\{M_r\} \cup \{M_s\}$ is a set of representatives of the conjugacy classes in M_F, we have shown that

$$J_r^P(f: x: \varepsilon) + J_s^P(f: x: \varepsilon)$$

equals the sum of

$$\sum_{\delta \in G_F/P_F} \sum_{\mu \in M_F} \sum_{\nu \in \mathfrak{N}_F} f(x\delta\mu\nu\delta^{-1}x^{-1}) \chi_\varepsilon(x\delta) \tag{6.3}$$

and an expression whose integral over $G_{\mathbf{A}}/G_F Z_\infty^+$ approaches 0 as ε approaches 0.

The space $\mathfrak{n}_{\mathbf{A}}$ is a locally compact abelian group under addition which contains $\mathfrak{n}_F$ as a discrete subgroup. Let $X_{\mathbf{A}}$ be the unitary dual group of $\mathfrak{n}_{\mathbf{A}}$ and let X_F be the subgroup of characters in $X_{\mathbf{A}}$ which are trivial on $\mathfrak{n}_F$.

Let $\|\cdot\|$ be the height function on $X_{\mathbf{A}}$ associated to some fixed basis of X_F. It is easy to verify that there is an N such that

$$\sum_{\substack{\xi \in X_F \\ \xi \neq 0}} \|\xi\|^{-N} < \infty .$$

For $\xi \in X_{\mathbf{A}}$ and $t \in \mathbf{R}$, define

$$\xi^t(Y) = \xi(\mathrm{Ad}\,(h_t)Y) , \qquad Y \in \mathfrak{n}_{\mathbf{A}} .$$

It is clear that there is a number $d > 0$ such that if ξ is primitive and $t \geqq 0$,

$$\|\xi^t\| \geqq e^{dt} \|\xi\| .$$

For fixed $y \in G_{\mathbf{A}}$ and $\mu \in M_F$ the function

$$f(y\cdot\mu\exp Y\cdot y^{-1})\,, \qquad Y\in\mathfrak{n}_{\mathbf{A}}$$

is of Schwartz-Bruhat type on $\mathfrak{n}_{\mathbf{A}}$. For $\xi\in X_{\mathbf{A}}$, define

$$\Psi(\xi,\mu,y)=\int_{N_{\mathbf{A}}}f(y\cdot\mu\exp Y\cdot y^{-1})\xi(Y)dY\,.$$

Then by the Poisson summation formula,

$$\sum_{\nu\in N_F}f(x\delta\mu\nu\delta^{-1}x^{-1})\chi_\varepsilon(x\delta)$$

is the sum of

$$\Psi(0,\mu,x\delta)\chi_\varepsilon(x\delta) \tag{6.4}$$

and

$$\sum_{\substack{\xi\in X_F\\ \xi\neq 0}}\Psi(\xi,\mu,x\delta)\chi_\varepsilon(x\delta)\,. \tag{6.5}$$

If we sum the absolute value of (6.5) over $\mu\in M_F$ and $\delta\in G_F/P_F$, and then integrate over $G_{\mathbf{A}}/G_FZ_\infty^+$, the result is bounded by

$$\int_{G_{\mathbf{A}}/P_FZ_\infty^+}\sum_{\mu\in M_F}\sum_{\substack{\xi\in X_F\\ \xi\neq 0}}|\Psi(\xi,\mu,x)|\chi_\varepsilon(x)dx\,.$$

If ω is a relatively compact fundamental domain for $P_{\mathbf{A}}^1/P_F$ in $P_{\mathbf{A}}^1$, this integral equals

$$c_G\int_K\int_{-\infty}^{\log\varepsilon}\int_\omega\sum_{\mu\in M_F}\sum_{\xi\neq 0}|\Psi(\xi,\mu,kh_tv)e^{2\rho t}dv\,dt\,dk\,.$$

We may assume that $h_t\omega h_t^{-1}$ is contained in ω for every $t\leqq 0$. Then the above integral is bounded by

$$c_G\int_{K\times\omega}\int_{-\infty}^{\log\varepsilon}\sum_{\mu\in M_F}\sum_{\xi\neq 0}|\Psi(\xi,\mu,kvh_t)|\,dt\,dv\,dk\,. \tag{6.6}$$

Notice that

$$\Psi(\xi,\mu,kvh_t)=e^{-2\rho t}\Psi(\xi^{-t},\mu,kv)\,.$$

Keep in mind that $\Psi(\cdot,\mu,kv)$ is the Fourier transform of a Schwartz-Bruhat function and is continuous in kv. We observe by a slight restatement of Lemma 5.2 that there are only finitely many $\mu\in M_F$ such that

$$\Psi(\xi,\mu,kv)\neq 0$$

for some $\xi\in X_{\mathbf{A}}$ and some $kv\in K\times\omega$. Therefore, for any N there is a constant Γ_N such that for any primitive $\xi\in X_{\mathbf{A}}$,

$$\sum_{\mu\in M_F}|\Psi(\xi,\mu,kv)|\leqq\Gamma_N\|\xi\|^{-N}\,, \qquad kv\in K\times\omega\,.$$

It follows that for every N, (6.6) is bounded by

$$c_G \Gamma_N \tau(M) \int_{-\infty}^{\log \varepsilon} e^{-2\rho t} \left(\sum_{\xi \neq 0} \| \xi^{-t} \|^{-N} \right) dt$$

which is in turn majorized by

$$c_G \Gamma_N \tau(M) \int_{-\infty}^{\log \varepsilon} e^{-2\rho t} e^{dNt} dt \cdot \sum_{\xi \neq 0} \| \xi \|^{-N} .$$

For sufficiently large N this last expression is finite and approaches 0 as ε approaches 0.

If we sum (6.4) over μ and δ we arrive at the expression

$$\sum_{\mu \in M_F} \sum_{\delta \in G_F/P_F} \Psi(0, \mu, x\delta) \chi_\varepsilon(x\delta) . \tag{6.7}$$

For fixed x there are only finitely many $\delta \in G_F/P_F$ such that $\chi_\varepsilon(x\delta) \neq 0$. Therefore the inner sum is finite. From this fact it is easily seen that the outer sum is also finite.

To summarize what we have shown so far, the expression

$$J_r^P(f\colon x\colon \varepsilon) + J_s^P(f\colon x\colon \varepsilon)$$

equals the sum of (6.7) and a function whose integral over $G_{\mathbf{A}}/G_F Z_\infty^+$ approaches 0 as ε approaches 0. The function (6.7) is not integrable. We leave it for the moment.

In the first parabolic term we still have to consider the contribution from $-K'(f\colon x\colon \varepsilon)$. The function $K'(f\colon x\colon \varepsilon)$ equals

$$\frac{1}{4\pi} \sum_{\delta \in G_F/P_F} \sum_{n=1}^{\infty} \int_{-i\infty}^{i\infty} \left\{ \sum_{\beta \in I_n} E_P(\pi(z\colon f)\phi_\beta\colon z\colon x\delta) \overline{E_P(\phi_\beta\colon z\colon x\delta)} \right\} d\,|z|\, \chi_\varepsilon(x\delta) .$$

We may formally write this expression as the sum of the following four terms:

$$\frac{1}{4\pi} \sum_{\delta \in G_F/P_F} \sum_{n=1}^{\infty} \int_{-i\infty}^{i\infty} \left\{ \sum_{\beta \in I_n} (\pi(z\colon f)\phi_\beta)(x\delta) \overline{\phi_\beta(x\delta)} \right\} d\,|z|\, e^{-2\rho H(x\delta)} \chi_\varepsilon(x\delta) , \tag{6.8}$$

$$\frac{1}{4\pi} \sum_{\delta \in G_F/P_F} \sum_{n=1}^{\infty} \int_{-i\infty}^{i\infty} \left\{ \sum_{\beta \in I_n} (M(z)\pi(z\colon f)\phi_\beta)(x\delta) \overline{(M(z)\phi_\beta)(x\delta)} \right\} d\,|z|\, e^{-2\rho H(x\delta)} \chi_\varepsilon(x\delta) , \tag{6.9}$$

$$\frac{1}{4\pi} \sum_{\delta \in G_F/P_F} \sum_{n=1}^{\infty} \int_{-i\infty}^{i\infty} \left\{ \sum_{\beta \in I_n} (M(z)\pi(z\colon f)\phi_\beta)(x\delta) \overline{\phi_\beta(x\delta)} \right\} e^{-2zH(x\delta)} d\,|z|\, \chi_\varepsilon(x\delta) , \tag{6.10}$$

and

$$(6.11)\quad \frac{1}{4\pi}\sum_{\delta\in G_F/P_F}\sum_{n=1}^{\infty} \int_{-i\infty}^{i\infty}\{\sum_{\beta\in I_n}(\pi(z\colon f)\phi_\beta)(x\delta)\overline{(M(z)\phi_\beta)(x\delta)}e^{2zH(x\delta)}\}d\,|\,z\,|\,e^{-2\rho H(x\delta)}\chi_\varepsilon(x\delta)\ .$$

In order to justify this step we need to prove the following,

LEMMA 6.1. *For any $y \in G_{\mathbf{A}}$ the expressions*

$$(6.12)\quad \int_{-i\infty}^{i\infty}\sum_{n=1}^{\infty}|\sum_{\beta\in I_n}(\pi(z\colon f)\phi_\beta)(y)\overline{\phi_\beta(y)}\,|\,d\,|\,z\,|\ ,$$

$$(6.13)\quad \int_{-i\infty}^{i\infty}\sum_{n=1}^{\infty}|\sum_{\beta\in I_n}(M(z)\pi(z\colon f)\phi_\beta)(y)\overline{(M(z)\phi_\beta)(y)}\,|\,d\,|\,z\,|\ ,$$

$$(6.14)\quad \int_{-i\infty}^{i\infty}\sum_{n=1}^{\infty}|\sum_{\beta\in I_n}(M(z)\pi(z\colon f)\phi_\beta)(y)\overline{\phi_\beta(y)}\,|\,d\,|\,z\,|\ ,$$

and

$$(6.15)\quad \int_{-i\infty}^{i\infty}\sum_{n=1}^{\infty}|\sum_{\beta\in I_n}(\pi(z\colon f)\phi_\beta)(y)\overline{(M(z)\phi_\beta)(y)}\,|\,d\,|\,z\,|$$

are all finite.

Proof. Fix a positive integer n and an imaginary number z. It is clear that the function

$$R(n\colon z\colon f\colon y,\, v) = \sum_{\beta\in I_n}(\pi(z\colon f)\phi_\beta)(y)\overline{\phi_\beta(v)}\ ,$$

which is continuous in y and v, is the kernel of the restriction of $\pi(z\colon f)$ to $\mathcal{H}(n)$. Therefore, if we define ${}^1f$ and ${}^2f$ as in the proof of Theorem 3.2, the absolute value

$$|\,R(n\colon z\colon f\colon y,\, y)\,|$$

is bounded by

$$|\,R(n\colon z\colon {}^1f\colon y,\, y)\,|^{1/2}\,|\,R(n\colon z\colon {}^2f\colon y,\, y)\,|^{1/2}\ .$$

By Schwartz' inequality we need only show that (6.12) is finite when f is replaced by ${}^1f$.

Now

$$R(n\colon z\colon {}^1f\colon y,\, y) \geqq 0\ .$$

Therefore, for every N,

$$\sum_{n=1}^{N}|\,R(n\colon z\colon {}^1f\colon y,\, y)\,|$$

is bounded by the function $P(z\colon {}^1f\colon y,\, y)$ defined in § 2. It follows that the series

$$\sum_{n=1}^{\infty} R(n\colon z\colon {}^1f\colon y,\, v)$$

is absolutely convergent and defines a function $R(z\colon {}^1f\colon y,\, v)$ which is the kernel of $\pi(z\colon {}^1f)$. By an argument similar to that used in the proof of Lemma 3.7, $R(z\colon {}^1f\colon y,\, v)$ is continuous in y and v separately. Therefore, as we remarked in § 2, $R(z\colon {}^1f\colon y,\, v)$ equals $P(z\colon {}^1f\colon y,\, v)$ for all y and v. The formula for $P(z\colon {}^1f\colon y,\, y)$ is given in § 2. It is clear that the integral

$$\int_{-i\infty}^{i\infty} P(z\colon {}^1f\colon y,\, y) d\,|\,z\,|$$

is finite. Therefore the expression (6.12) is finite.

For any z and n, the set

$$\{M(z)\phi_\beta\}_{\beta\in I_n}$$

is an orthonormal basis for $\mathcal{H}(n)$. It follows that the function

$$\sum_{\beta\in I_n} \big(M(z)\pi(z\colon f)\phi_\beta\big)(y)\overline{\big(M(z)\phi_\beta\big)(v)}$$

is the kernel of the restriction of $\pi(-z\colon f)$ to $\mathcal{H}(n)$. The finiteness of (6.13) follows by the above argument.

The function

$$\sum_{\beta\in I_n} \big(M(z)\pi(z\colon f)\phi_\beta\big)(y)\overline{\phi_\beta(v)} \tag{6.16}$$

is the kernel of the restriction of $M(z)\pi(z\colon f)$ to $\mathcal{H}(n)$. We recall that $\pi(z\colon f) = \pi(z\colon f')\pi(z\colon f'')$. We have

$$\begin{aligned} M(z)\pi(z\colon f')\big(M(z)\pi(z\colon f')\big)^* &= M(z)\pi(z\colon f')\pi\big(z\colon (f')^*\big)M(z)^{-1} \\ &= M(z)\pi(z\colon {}^1f)M(z)^{-1} = \pi(-z\colon {}^1f)\,. \end{aligned}$$

Therefore the absolute value of the function (6.16) at $v = y$ is bounded by

$$|\,R(n\colon -z\colon {}^1f\colon y,\, y)\,|^{1/2}\cdot|\,R(n\colon z\colon {}^2f\colon y,\, y)\,|^{1/2}\,.$$

It follows that (6.14) is finite. Similarly (6.15) is also finite. □

The term (6.8) equals

$$\frac{1}{4\pi}\sum_{\delta\in G_F/P_F}\int_{-i\infty}^{i\infty} P(z\colon f\colon x\delta,\, x\delta)d\,|\,z\,|\cdot e^{-2\rho H(x\delta)}\cdot\chi_\varepsilon(x\delta)\,,$$

for all x. This expression is just

$$\frac{1}{4\pi}\sum_{\delta\in G_F/P_F}\int_{-i\infty}^{i\infty}\Big(\sum_{\mu\in M_F}\int_{N_{\mathbf{A}}}\int_{-\infty}^{\infty} f(x\delta\mu h_t n\delta^{-1}x^{-1})e^{(z-\rho)t}dt\,dn\Big)d\,|\,z\,|\cdot\chi_\varepsilon(x\delta)\,,$$

which in turn can be written as

$$\frac{1}{2}\sum_{\delta\in G_F/P_F}\sum_{\mu\in M_F}\int_{N_{\mathbf{A}}} f(x\delta\mu n\delta^{-1}x^{-1})dn\cdot\chi_\varepsilon(x\delta) \tag{6.17}$$

by the Fourier inversion formula. Similarly, the term (6.9) equals

$$\frac{1}{4\pi}\sum_{\delta\in G_F/P_F}\int_{-i\infty}^{i\infty}P(-z: f: x\delta, x\delta)d\,|\,z\,|\cdot e^{-2\rho H(x\delta)}\cdot\chi_\varepsilon(x\delta)$$

for all x. This expression also equals (6.17).

Therefore, the contribution to $-K'(f: x: \varepsilon)$ from the terms (6.8) and (6.9) is the product of (6.17) with (-2). The result exactly cancels out the term (6.7).

LEMMA 6.2. *The functions defined by* (6.10) *and* (6.11) *are weakly integrable over $G_{\mathbf{A}}/G_F Z_\infty^+$, and their integrals approach* 0 *as ε approaches* 0.

Proof. It is easily seen from the proof of the last lemma that these functions are locally integrable. Let c be any positive number smaller than ε_0 and let $h(x)$ be the function on $G_{\mathbf{A}}/G_F Z_\infty^+$ defined by (6.10). Then

$$c_G\int_{-\infty}^{\log c}\left|\int_K\int_{P_{\mathbf{A}}^1/P_F}h(kh_tp)dp\,dk\right|e^{2\rho t}dt$$

equals

$$\frac{1}{4\pi}\int_{-\infty}^{\log\varepsilon}\left|\int_{-i\infty}^{i\infty}\sum_{\beta\in I}\big(M(z)\pi(z: f)\phi_\beta, \phi_\beta\big)^{-2zt}d\,|\,z\,|\right|dt\ .$$

Our use of Fubini's theorem is justified by the compactness of K and $P_{\mathbf{A}}^1/P_F$. By Assumption 3.5 and the proof of Lemma 3.4 the function

$$\big(M(z)\pi(z: f)\phi_\beta, \phi_\beta\big)\ , \qquad z\in i\mathbf{R}$$

is zero for all but finitely many $\beta\in I$.

Therefore, in the above integral over z, we can change the contour to a line $\{z: \operatorname{Re} z = \delta\}$, for $\delta < 0$. The assertions of the lemma for the function (6.10) follow immediately. The result for (6.11) is proved the same way. □

This lemma accounts for the last of the components in the first parabolic term. We have completed the proof promised at the beginning of this section.

7. The second parabolic term

The second parabolic term equals the sum of $I_r^P(f: x: \varepsilon)$ and $I_s^P(f: x: \varepsilon)$. In this section we shall prove that both these functions are integrable over $G_{\mathbf{A}}/G_F Z_\infty^+$. We shall then calculate their integrals.

The function $I_r^P(f: x: \varepsilon)$ equals

$$\frac{1}{2}\sum_{\mu\in\{M\}}(n_{\mu,M})^{-1}\sum_{\delta\in G_F/M(\mu)_F}f(x\delta\mu\delta^{-1}x^{-1})\big(1-\chi_\varepsilon(x\delta)-\chi_\varepsilon(x\delta w)\big)\ .$$

The integral

$$\int_{G_{\mathbf{A}}/G_F Z_\infty^+} | I_r^P(f: x: \varepsilon) | \, dx$$

is bounded by the expression

$$\frac{1}{2} \sum_{\mu \in \{M_r\}} (n_{\mu,M})^{-1} \int_{G_{\mathbf{A}}/M(\mu)_F Z_\infty^+} | f(x\mu x^{-1}) | \big(1 - \chi_\varepsilon(x) - \chi_\varepsilon(xw)\big) dx ,$$

which may be written as

$$\frac{c_G}{2} \sum_{\mu \in \{M_r\}} (n_{\mu,M})^{-1} \int_K \int_{P_{\mathbf{A}}/M(\mu)_F A_\infty^+} \int_{-\infty}^{\infty} | f(kp\mu p^{-1}k^{-1}) |$$
$$\cdot \big(1 - \chi_\varepsilon(ph_t) - \chi_\varepsilon(ph_t w)\big) dt \, d_r p \, dk .$$

For any $\mu \in M_F$ let $\Gamma_{\mu,M}$ be the index in $X(M(\mu))_F$ of the group obtained by restricting the characters in $X(M)_F$ to $M(\mu)$. We define

$$\tilde{\tau}(\mu, M) = (n_{\mu,M})^{-1} \cdot (\Gamma_{\mu,M})^{-1} \cdot \tau(M(\mu)) .$$

The above integral becomes

$$\frac{c_G}{2} \sum_{\mu \in \{M_r\}} \tilde{\tau}(\mu, M) \int_K \int_{P_{\mathbf{A}}/M(\mu)_{\mathbf{A}}} | f(kp\mu p^{-1}k^{-1}) |$$
$$\cdot \int_{-\infty}^{\infty} \big(1 - \chi_\varepsilon(ph_t) - \chi_\varepsilon(ph_t w)\big) dt \, d_r p \, dk .$$

By Lemma 5.2 the sum over μ is finite. Since the function

$$f^K(p) = \int_K f(kpk^{-1}) dk , \qquad p \in P_{\mathbf{A}} ,$$

has compact support on $P_{\mathbf{A}}/Z_\infty^+$, the integral over $P_{\mathbf{A}}/M(\mu)_{\mathbf{A}}$ can be taken over a compact set, by Lemma 5.3. Finally, it is clear that for any p the function

$$t \longrightarrow 1 - \chi_\varepsilon(ph_t) - \chi_\varepsilon(ph_t w) , \qquad t \in (-\infty, \infty) ,$$

has compact support. Therefore $I_r^P(f: x: \varepsilon)$ is integrable over $G_{\mathbf{A}}/G_F Z_\infty^+$. Its integral equals

$$\frac{c_G}{2} \sum_{\mu \in \{M_r\}} \tilde{\tau}(\mu, M) \int_K \int_{N_{\mathbf{A}}} \int_{M_{\mathbf{A}}/M(\mu)_{\mathbf{A}}} f(knm\mu m^{-1}n^{-1}k^{-1})$$
$$\cdot \int_{-\infty}^{\infty} \big(1 - \chi_\varepsilon(knmh_t) - \chi_\varepsilon(knmh_t w)\big) dt \, dm \, dn \, dk .$$

For fixed m and n,

$$\big(1 - \chi_\varepsilon(knmh_t) - \chi_\varepsilon(knmh_t w)\big)$$

is the characteristic function of the interval

$$[\log \varepsilon - H(m), H(nw) - \log \varepsilon - H(m)] .$$

Our integral is therefore the sum of

$$(7.1)\qquad \frac{c_G}{2}\sum_{\mu\in\{M_r\}}\tilde{\tau}(\mu, M)\int_K\int_{N_{\mathbf{A}}}\int_{M_{\mathbf{A}}/M(\mu)_{\mathbf{A}}} f(knm\mu m^{-1}n^{-1}k^{-1}) \cdot H(nw)dm\,dn\,dk\,,$$

and

$$(7.2)\qquad -\log\varepsilon\cdot c_G\sum_{\mu\in\{M_r\}}\tilde{\tau}(\mu, M) \cdot\int_K\int_{N_{\mathbf{A}}}\int_{M_{\mathbf{A}}/M(\mu)_{\mathbf{A}}} f(knm\mu m^{-1}n^{-1}k^{-1})dm\,dn\,dk\,.$$

After changing the variable of integration on $N_{\mathbf{A}}$ we may appeal to Lemma 4.4, rewriting (7.2) as

$$-\log\varepsilon\cdot c_G\sum_{\mu\in\{M_r\}}\tilde{\tau}(\mu, M)\int_K\int_{M_{\mathbf{A}}/M(\mu)_{\mathbf{A}}}\int_{N_{\mathbf{A}}} f(km\mu n m^{-1}k^{-1}) \cdot e^{2\rho H(m)}dn\,dm\,dk\,.$$

This in turn equals

$$-\log\varepsilon\cdot c_G\sum_{\mu\in\{M_r\}}(n_{\mu,M})^{-1}\int_K\int_{M_{\mathbf{A}}/M(\mu)_F A_\infty^+}\int_{N_{\mathbf{A}}} f(km\mu n m^{-1}k^{-1}) \cdot e^{2\rho H(m)}dn\,dm\,dk\,,$$

which is the same as

$$(7.3)\qquad -\log\varepsilon\cdot c_G\int_K\int_{M_{\mathbf{A}}/M_F A_\infty^+}\sum_{\mu\in M_r}\int_{N_{\mathbf{A}}} f(km\mu n m^{-1}k^{-1}) \cdot e^{2\rho H(m)}dn\,dm\,dk\,.$$

We now consider the function $I_s^P(f\colon x\colon\varepsilon)$. Our discussion will include integrals over the groups $P(\mu)_{\mathbf{A}}$ and $P(\mu)_{\mathbf{A}}^1$, for elements μ in $\{M_s\}$. We note that according to our understanding on the choice of Haar measures, the product measure on $P(\mu)_{\mathbf{A}}^1\times A_\infty^+$ is a multiple of our right Haar measure on $P(\mu)_{\mathbf{A}}$ by $\Gamma_{\mu,M}$. Define

$$\delta_{P(\mu)}(p) = e^{2\rho(\mu)H(p)}\,,\qquad p\in P(\mu)_{\mathbf{A}}\,,$$

to be the modular function of $P(\mu)_{\mathbf{A}}$.

LEMMA 7.1. *For $\mu\in\{M_s\}$ and $\phi\in C_c^\infty(P_{\mathbf{A}})$,*

$$\int_{P_{\mathbf{A}}}\phi(p)\delta_P(p)d_lp = \Gamma_{\mu,M}\int_{P_{\mathbf{A}}^1/P(\mu)_{\mathbf{A}}^1}\int_{P(\mu)_{\mathbf{A}}}\phi(p^*p)\delta_{P(\mu)}(p)d_lp\,dp^*\,,$$

where dp^* is the invariant measure on $P_{\mathbf{A}}^1/P(\mu)_{\mathbf{A}}^1$ defined by our Haar measures on $P_{\mathbf{A}}^1$ and $P(\mu)_{\mathbf{A}}^1$.

Proof.
$$\int_{P_{\mathbf{A}}} \phi(p)\delta_P(p)d_lp = \int_{P_{\mathbf{A}}} \phi(p)d_rp = \int_{P^1_{\mathbf{A}}}\int_{A^+_\infty} \phi(pa^+_\infty)da^+_\infty\, dp$$
$$= \int_{P^1_{\mathbf{A}}/P(\mu)^1_{\mathbf{A}}}\int_{P(\mu)^1_{\mathbf{A}}}\int_{A^+_\infty} \phi(p^*pa^+_\infty)da^+_\infty\, dp\, dp^*$$
$$= \Gamma_{\mu,M}\int_{P^1_{\mathbf{A}}/P(\mu)^1_{\mathbf{A}}}\int_{P(\mu)_{\mathbf{A}}} \phi(p^*p)\delta_{P(\mu)}(p)d_lp\, dp^* . \qquad \square$$

Recall that $I^P_s(f\colon x\colon \varepsilon)$ equals
$$\sum_{\mu\in\{M_s\}} (n_{\mu,M})^{-1}\sum_{\delta\in G_F/P(\mu)_F}\sum_{\substack{\nu\in N(\mu)_F\\ \nu\neq e}} f(x\delta\mu\nu\delta^{-1}x^{-1})\big(1-\chi_\varepsilon(x\delta)\big) .$$
The integral
$$\int_{G_{\mathbf{A}}/G_F Z^+_\infty} |I^P_s(f\colon x\colon \varepsilon)|\, dx$$
is bounded by
$$\sum_{\mu\in\{M_s\}} (n_{\mu,M})^{-1}\int_{G_{\mathbf{A}}/P(\mu)_F Z^+_\infty} \sum_{\substack{\nu\in N(\mu)_F\\ \nu\neq e}} | f(x\mu\nu x^{-1}) |\, \big(1 - \chi_\varepsilon(x)\big)dx .$$
This expression can be written as
$$c_G\sum_{\mu\in\{M_s\}} (n_{\mu,M})^{-1}\int_K\int_{P_{\mathbf{A}}/P(\mu)_F Z^+_\infty} \sum_{\substack{\nu\in N(\mu)_F\\ \nu\neq e}} | f(kp\mu\nu p^{-1}k^{-1}) |$$
$$\cdot\big(1 - \chi_\varepsilon(p)\big)\delta_P(p)d_lp\, dk ,$$
which by the last lemma equals

(7.4)
$$c_G\sum_{\mu\in\{M_s\}} (n_{\mu,M})^{-1}(\Gamma_{\mu,M})\int_K\int_{P^1_{\mathbf{A}}/P(\mu)^1_{\mathbf{A}}}\int_{P(\mu)_{\mathbf{A}}/P(\mu)_F Z^+_\infty}$$
$$\cdot\sum_{\substack{\nu\in N(\mu)_F\\ \nu\neq e}} | f(kp^*\cdot p\mu\nu p^{-1}\cdot p^{*-1}k^{-1}) |\, \big(1 - \chi_\varepsilon(p)\big)\delta_{P(\mu)}(p)d_lp\, dp^*\, dk .$$

By Lemma 5.3 the integral over $P^1_{\mathbf{A}}/P(\mu)^1_{\mathbf{A}}$ can be taken over a compact subset C_1 of $P^1_{\mathbf{A}}/P(\mu)^1_{\mathbf{A}}$ or equivalently, over a fixed compact set $C(\mu)$ of representatives of C_1 in $P^1_{\mathbf{A}}$.

For $\mu\in\{M_s\}$ and $n\in N(\mu)_{\mathbf{A}}$, define $\Phi_\mu(f\colon n)$ to be

(7.5)
$$c_G(n_{\mu,M})^{-1}\int_K\int_{C(\mu)} f(kp\mu np^{-1}k^{-1})dp\, dk .$$

The support, $U(\mu)$, of this function is a compact subset of $N(\mu)_{\mathbf{A}}$. The expression (7.4) equals
$$\sum_{\mu\in\{M_s\}}\int_{\log\varepsilon}^{\infty} e^{2\rho(\mu)t}\int_{P(\mu)^1_{\mathbf{A}}/P(\mu)_F} \sum_{\substack{\nu\in N(\mu)_F\\ \nu\neq e}} \Phi_\mu(| f |\colon h_tp\nu p^{-1}h_t^{-1})dp\, dt .$$

Let $\omega(\mu)$ be a relatively compact set of representatives of $P(\mu)^1_{\mathbf{A}}/P(\mu)_F$ in

$P(\mu)^1_{\mathbf{A}}$. We can choose a positive number t_0 large enough so that the intersection of the set

$$\{v^{-1}h_t^{-1}\cdot n\cdot h_t v\colon v\in\omega(\mu),\ t\geqq t_0,\ n\in U(\mu)\}$$

with $N(\mu)_F$ is just $\{e\}$. Then our integral equals

$$\sum_{\mu\in\{M_s\}}\int_{\log\varepsilon}^{t_0} e^{2\rho(\mu)t}\int_{\omega(\mu)}\sum_{\substack{\nu\in N(\mu)_F\\ \nu\neq e}}\Phi_\mu(|f|\colon h_t v\nu v^{-1}h_t^{-1})dv\,dt\ .$$

It follows that $I_s^P(f\colon x\colon \varepsilon)$ is integrable over $G_{\mathbf{A}}/G_F Z_\infty^+$.

For any $z\in\mathbf{C}$ we define $I_\varepsilon(z)$ to be

$$\sum_{\mu\in\{M_s\}}\int_{\log\varepsilon}^{\infty} e^{(2\rho(\mu)t)(1+z)}\int_{P(\mu)^1_{\mathbf{A}}/P(\mu)_F}\sum_{\substack{\nu\in N(\mu)_F\\ \nu\neq e}}\Phi_\mu(f\colon h_t p\nu p^{-1}h_t^{-1})dp\,dt\ .$$

LEMMA 7.2. *For any $z\in\mathbf{C}$ the integral defining $I_\varepsilon(z)$ is absolutely convergent. $I_\varepsilon(z)$ is an entire function whose value at $z=0$ equals the integral*

$$\int_{G_{\mathbf{A}}/G_F Z_\infty^+} I_s^P(f\colon x\colon\varepsilon)dx\ .$$

Proof. All statements of the lemma are obvious conseqences of the above discussion. □

It remains to calculate $I_\varepsilon(0)$. Specifically, we shall express $I_\varepsilon(0)$ as a sum of an expression which is independent of ε, and a term whose dependence on ε is quite transparent. The idea, which we take from [7], is to replace the integral over $\{t\geqq\log\varepsilon\}$ which appears in the definition of $I_\varepsilon(z)$ by the difference of an integral over $\mathbf{R}$ and an integral over $\{t\leqq\log\varepsilon\}$.

Fix $\mu\in\{M_s\}$. Let $X(\mu)_{\mathbf{A}}$ be the unitary dual group of $\mathfrak{n}(\mu)_{\mathbf{A}}$ and let $X(\mu)_F$ be those characters in $X(\mu)_{\mathbf{A}}$ which are trivial on $\mathfrak{n}(\mu)_F$. For any $p\in P(\mu)_{\mathbf{A}}$ and $\xi\in X(\mu)_{\mathbf{A}}$ define

$$\Psi_\mu(\xi,p)=\int_{\mathfrak{n}(\mu)_{\mathbf{A}}}\Phi_\mu(f\colon p\cdot\exp Y\cdot p^{-1})\xi(Y)dY\ .$$

By the Poisson summation formula,

$$\sum_{\substack{\nu\in N(\mu)_F\\ \nu\neq e}}\Phi_\mu(f\colon p\nu p^{-1})$$

equals

$$\sum_{\substack{\xi\in X(\mu)_F\\ \xi\neq e}}\Psi_\mu(\xi,p)+\Psi_\mu(0,p)-\Phi_\mu(f\colon e)\ .$$

By choosing a height function on $X(\mu)_{\mathbf{A}}$ we can repeat the argument of § 6 to show that for any $z\in\mathbf{C}$ the expression

$$\int_{-\infty}^{\log \varepsilon}\int_{P(\mu)^1_{\mathbf{A}}/P(\mu)_F} \sum_{\substack{\xi \in X(\mu)_F \\ \xi \neq e}} |\Psi_\mu(\xi, h_t p)| \cdot |e^{(2\rho(\mu)t)(1+z)}| \, dp\, dt$$

is finite and approaches 0 as ε approaches 0.

The integral

$$\int_{-\infty}^{\log \varepsilon}\int_{P(\mu)^1_{\mathbf{A}}/P(\mu)_F} \Phi_\mu(f\colon e) e^{(2\rho(\mu)t)(1+z)} dp\, dt$$

is absolutely convergent for $\operatorname{Re} z > -1$. It equals

$$\varepsilon^{(2\rho(\mu))(1+z)}(2\rho(\mu))^{-1}(1+z)^{-1}\cdot \tau(M(\mu))\cdot \Phi_\mu(f\colon e)\ .$$

At $z = 0$ this function approaches 0 as ε approaches 0.

Finally, the integral

$$\int_{-\infty}^{\log \varepsilon}\int_{P(\mu)^1_{\mathbf{A}}/P(\mu)_F} \Psi_\mu(0, h_t p) e^{(2\rho(\mu))(1+z)t} dp\, dt$$

is absolutely convergent for $\operatorname{Re} z > 0$. It equals

$$\varepsilon^{2\rho(\mu)z}(2\rho(\mu)z)^{-1}\cdot \tau(M(\mu))\cdot \int_{N(\mu)_{\mathbf{A}}} \Phi_\mu(f\colon n) dn\ .$$

For $\operatorname{Re} z > 0$ we define $\tilde{\theta}(\mu\colon z\colon f)$ to be

$$\int_{-\infty}^{\infty}\int_{P(\mu)^1_{\mathbf{A}}/P(\mu)_F} \sum_{\substack{\nu \in N(\mu)_F \\ \nu \neq e}} \Phi_\mu(f\colon h_t p \nu p^{-1} h_t^{-1}) e^{(2\rho(\mu)t)(1+z)} dp\, dt\ .$$

LEMMA 7.3. *The integral defining $\tilde{\theta}(\mu\colon z\colon f)$ is absolutely convergent for* $\operatorname{Re} z > 0$. *It can be analytically continued to a meromorphic function on* C *whose only singularities are simple poles at $z = 0$ and $z = -1$. Finally, modulo a term which approaches* 0 *as ε approaches* 0, *$I_\varepsilon(0)$ equals the limit as z approaches* 0 *of*

$$\sum_{\mu \in \{M_S\}} \left\{\tilde{\theta}(\mu\colon z\colon f) - \varepsilon^{2\rho(\mu)z}(2\rho(\mu)z)^{-1}\cdot \tau(M(\mu))\cdot \int_{N(\mu)_{\mathbf{A}}} \Phi_\mu(f\colon n) dn\right\}\ .$$

Proof. This lemma follows from the above discussion and Lemma 7.2. □

The constant term of the Laurent expansion about $z = 0$ of the function

$$\sum_{\mu \in \{M_S\}} \tilde{\theta}(\mu\colon z\colon f)$$

we write simply as

$$\sum_{\mu \in \{M_S\}} \lim_{z\to 0} \frac{d}{dz}\{z\tilde{\theta}(\mu\colon z\colon f)\}\ . \tag{7.6}$$

The constant term of the Laurent expansion of

$$-\sum_{\mu \in \{M_s\}} \varepsilon^{2\rho(\mu)z}(2\rho(\mu)z)^{-1} \cdot \tau(M(\mu)) \cdot \int_{N(\mu)_{\mathbf{A}}} \Phi_\mu(f: n)dn$$

equals

$$-\log \varepsilon \sum_{\mu \in \{M_s\}} \tau(M(\mu)) \cdot \int_{N(\mu)_{\mathbf{A}}} \Phi_\mu(f: n)dn \ . \tag{7.7}$$

It is clear that

$$\int_{N(\mu)_{\mathbf{A}}} \Phi_\mu(f: n)dn$$

equals

$$c_G(n_{\mu,M})^{-1}\int_K \int_{P^1_{\mathbf{A}}/P(\mu)^1_{\mathbf{A}}} \int_{N(\mu)_{\mathbf{A}}} f(kp \cdot \mu n \cdot p^{-1}k^{-1})dn\, dp\, dk \ .$$

If we write the integral over $P^1_{\mathrm{A}}/P(\mu)^1_{\mathrm{A}}$ as an iterated integral over $P^1_{\mathrm{A}}/P(\mu)^1_{\mathrm{A}}N_{\mathrm{A}}$ and $P(\mu)^1_{\mathrm{A}}N_{\mathrm{A}}/P(\mu)^1_{\mathrm{A}}$ we arrive at the expression

$$c_G(n_{\mu,M})^{-1}\int_K \int_{M^1_{\mathbf{A}}/M(\mu)^1_{\mathbf{A}}} \int_{N_{\mathbf{A}}} f(km\mu n m^{-1}k^{-1})\, dn\, dm\, dk \ ,$$

by virtue of Lemmas 4.1 and 4.4. It follows that (7.7) may be written as

$$-\log \varepsilon \cdot c_G \int_K \int_{M^1_{\mathbf{A}}/M_F} \sum_{\mu \in M_s} \int_{N_{\mathbf{A}}} f(km\mu n m^{-1}k^{-1})dn\, dm\, dk \ ,$$

which is the same as

$$-\log \varepsilon \cdot c_G \int_K \int_{M_{\mathbf{A}}/M_F A^+_\infty} \sum_{\mu \in M_s} \int_{N_{\mathbf{A}}} f(km\mu n m^{-1}k^{-1})e^{2\rho H(m)}dn\, dm\, dk \ . \tag{7.8}$$

Our discussion of the second parabolic term is now complete. We have shown that the integral over $G_{\mathrm{A}}/G_F Z^+_\infty$ of this term equals the sum of the expressions (7.1) and (7.6), the term

$$-\log \varepsilon \cdot c_G \int_K \int_{M_{\mathbf{A}}/M_F A^+_\infty} \sum_{\mu \in M_F} \int_{N_{\mathbf{A}}} f(km\mu n m^1 k^{-1})e^{2\rho H(m)}dn\, dm\, dk \ , \tag{7.9}$$

obtained by combining (7.8) with (7.3), and an expression which approaches 0 as ε approaches 0.

8. The third parabolic term

We have proved that the first four terms in the kernel of $\lambda_0(f)$ are weakly integrable. It follows that the final term, $-K''(f: x: \varepsilon)$, is also weakly integrable over $G_{\mathrm{A}}/G_F Z^+_\infty$. In this section we shall calculate its integral.

For convenience we set $H(n: z: f: x)$ equal to

$$\frac{1}{4\pi}\sum_{\alpha,\beta\in I_n}\pi_{\alpha\beta}(z\colon f)E(\phi_\alpha\colon z\colon x)\overline{E(\phi_\beta\colon z\colon x)}\ .$$

We also define $H'_\varepsilon(n\colon z\colon f\colon x)$ and $H''_\varepsilon(n\colon z\colon f\colon x)$ by replacing all the functions $E(\phi\colon z\colon x)$ in this definition by $E'_\varepsilon(\phi\colon z\colon x)$ and $E''_\varepsilon(\phi\colon z\colon x)$ respectively. Then $K''(f\colon x\colon \varepsilon)$ equals

$$\sum_{n=1}^{\infty}\int_{-i\infty}^{i\infty}(H(n\colon z\colon f\colon x)-H'_\varepsilon(n\colon z\colon f\colon x))d\,|z|\ .$$

By Theorem 3.2 and Lemma 4.8 we can associate to any Siegel domain $\mathfrak{S}$ constants C and N such that for all $x\in\mathfrak{S}$ the inequality

$$\sum_{n=1}^{\infty}\int_{-i\infty}^{i\infty}|H(n\colon z\colon f\colon x)-H'_\varepsilon(n\colon z\colon f\colon x)|\,d\,|z|\leqq Ce^{-NH(x)} \tag{8.1}$$

is valid.

For $0<t<\varepsilon_0$ let $\widetilde{S(t)}$ be the projection of $S(t)$ onto $G_{\mathbf{A}}/G_FZ_\infty^+$. Let $G(t)$ be the closure of the complement of $\widetilde{S(t)}$ in $G_{\mathbf{A}}/G_FZ_\infty^+$. $G(t)$ is a compact subset of $G_{\mathbf{A}}/G_FZ_\infty^+$ and our integral

$$-\int_{G_{\mathbf{A}}/G_FZ_\infty^+}K''(f\colon x\ \varepsilon)dx$$

equals

$$-\lim_{t\searrow 0}\int_{G(t)}K''(f\colon x\colon \varepsilon)dx\ .$$

By (8.1) and Fubini's theorem this second expression equals

$$-\lim_{t\searrow 0}\sum_{n=1}^{\infty}\int_{-i\infty}^{i\infty}\int_{G(t)}(H(n\colon z\colon f\colon x)-H'_\varepsilon(n\colon z\colon f\colon x))dx\,d\,|z|\ . \tag{8.2}$$

The integrand is the sum of

$$H''_\varepsilon(n\colon z\colon f\colon x)$$

and

$$\frac{1}{4\pi}\sum_{\alpha,\beta\in I_n}\pi_{\alpha\beta}(z\colon x)\{E'_\varepsilon(\phi_\alpha\colon z\colon x)\overline{E''_\varepsilon(\phi_\beta\colon z\colon x)}+E''_\varepsilon(\phi_\alpha\colon z\colon x)\overline{E'_\varepsilon(\phi_\beta\colon z\colon x)}\}\ .$$

LEMMA 8.1. *For any α, $\beta\in I$,*

$$\int_{G(t)}E'_\varepsilon(\phi_\alpha\colon z\colon x)\overline{E''_\varepsilon(\phi_\beta\colon z\colon x)}dx=0$$

Proof. By Lemma 1.4,

$$E'_\varepsilon(\phi_\beta\colon z\colon x)=E_P(\phi_\beta\colon z\colon x)$$

whenever $x\in S(\varepsilon)$. Since $E(\phi_\beta\colon z\colon x)$ is an automorphic form, $E''_\varepsilon(\phi_\beta\colon z\colon x)$ is rapidly decreasing on any Siegel domain, by Lemma 1.6. But $E'_\varepsilon(\phi_\alpha\colon z\colon x)$ is slowly increasing on any Siegel domain, so that the function

$$E'_\varepsilon(\phi_\alpha\colon z\colon x)\overline{E''_\varepsilon(\phi_\beta\colon z\colon x)}$$

is integrable over $G_{\mathbf{A}}/G_F Z_\infty^-$. Its integral equals

$$\int_{G_{\mathbf{A}}/P_F Z_\infty^+} E_P(\phi_\alpha\colon z\colon x)\overline{E''_\varepsilon(\phi_\beta\colon z\colon x)}\chi_\varepsilon(x)dx$$

which in turn can be written

$$\int_{G_{\mathbf{A}}/P_F N_{\mathbf{A}} Z_\infty^+} E_P(\phi_\alpha\colon z\colon x)\left\{\chi_\varepsilon(x)\int_{N_{\mathbf{A}}/N_F} \overline{E''_\varepsilon(\phi_\beta\colon z\colon xn)}dn\right\}dx\ .$$

The expression in the brackets equals zero identically in x.

On the other hand, since $t < \varepsilon_0$, the integral

$$\int_{\widetilde{S(t)}} E'_\varepsilon(\phi_\alpha\colon z\colon x)\overline{E''_\varepsilon(\phi_\beta\colon z\colon x)}dx$$

equals

$$\int_{S(t)} E_P(\phi_\alpha\colon z\colon x)\overline{E''_\varepsilon(\phi_\beta\colon z\colon x)}\chi_\varepsilon(x)dx\ ,$$

by Lemma 1.4. This second integral is equal to

$$\int_{S(t)} E_P(\phi_\alpha\colon z\colon x)\left\{\chi_\varepsilon(x)\int_{N_{\mathbf{A}}/N_F} \overline{E''_\varepsilon(\phi_\beta\colon z\colon xn)}dn\right\}dx\ ,$$

an expression which equals 0. Since $G(t)$ is the complement of $\widetilde{S(t)}$ in $G_{\mathbf{A}}/G_F Z_\infty^+$, our lemma follows. □

It follows from the lemma that (8.2) equals

$$(8.3)\qquad -\lim_{t\searrow 0}\sum_{n=1}^{\infty}\int_{-i\infty}^{i\infty}\int_{G(t)} H''_\varepsilon(n\colon z\colon f\colon x)dx\, d\,|z|\ .$$

LEMMA 8.2. *The integral*

$$(8.4)\qquad \sum_{n=1}^{\infty}\int_{-i\infty}^{i\infty}\int_{G_{\mathbf{A}}/G_F Z_\infty^+} |\,H''_\varepsilon(n\colon z\colon f\colon x)\,|\,dx\, d\,|z|$$

is finite.

Proof. The proof follows the same idea as that of Theorem 3.2. For ${}^1f$ and ${}^2f$ as in the proof of Theorem 3.2, we can easily see that when z is imaginary

$$|\,H''_\varepsilon(n\colon z\colon f\colon x)\,|$$

is bounded by

$$|\,H''_\varepsilon(n\colon z\colon {}^1f\colon x)\,|^{1/2}\cdot|\,H''_\varepsilon(n\colon z\colon {}^2f\colon x)\,|^{1/2}\ .$$

By Schwartz' inequality we need only prove the lemma when f is replaced by ${}^1f$. From the fact that the operator $\pi(z\colon {}^1f)$ is positive semi-definite it follows that

$$H_\varepsilon''(n\colon z\colon {}^1f\colon x) \geqq 0 \ .$$

On the other hand ${}^1f$ also satisfies Assumption 3.5, so that

$$\lim_{t\searrow 0} \sum_{n=1}^{\infty} \int_{-i\infty}^{i\infty} \int_{G(t)} H_\varepsilon''(n\colon z\colon {}^1f\colon x)dx < \infty \ ,$$

by (8.3). It follows that

$$\sum_{n=1}^{\infty} \int_{-i\infty}^{i\infty} \int_{G_{\mathbf{A}}/G_F Z_\infty^+} H_\varepsilon''(n\colon z\colon {}^1f\colon x)dx < \infty \ ,$$

which completes the proof. □

This lemma enables us to conclude that the integral over $G_{\mathbf{A}}/G_F Z_\infty^+$ of $-K''(f\colon x\colon \varepsilon)$ equals

$$(8.5) \qquad -\frac{1}{4\pi} \sum_{\alpha,\beta\in I} \int_{-i\infty}^{i\infty} \int_{G_{\mathbf{A}}/G_F Z_\infty^+} \pi_{\alpha\beta}(z\colon f) E_\varepsilon''(\phi_\alpha\colon z\colon x)\overline{E_\varepsilon''(\phi_\beta\colon z\colon x)}dx\, d|z| \ .$$

LEMMA 8.3. *For $\alpha, \beta \in I$ and z a nonzero imaginary number, the inner product*

$$\int_{G_{\mathbf{A}}/G_F Z_\infty^+} E_\varepsilon''(\phi_\alpha\colon z\colon x)\overline{E_\varepsilon''(\phi_\beta\colon z\colon x)}dx$$

is the sum of the following three terms:

$$(8.6) \qquad -2\log\varepsilon\cdot(\phi_\alpha, \phi_\beta) \ ,$$

$$(8.7) \qquad -\frac{1}{2}\Big\{\Big(M(-z)\cdot\frac{d}{dz}M(z)\phi_\alpha, \phi_\beta\Big) + \Big(\phi_\alpha, M(-z)\cdot\frac{d}{dz}M(z)\phi_\beta\Big)\Big\} \ ,$$

and

$$(8.8) \qquad -\frac{1}{2z}\Big\{\varepsilon^{2z}\big(\phi_\alpha, M(z)\phi_\beta\big) - \varepsilon^{-2z}\big(M(z)\phi_\alpha, \phi_\beta\big)\Big\} \ .$$

Proof. First of all, suppose that λ and $\bar{\mu}$ are distinct complex numbers whose real parts are both less than $-\rho$. Then it is known that

$$(8.9) \qquad \int_{G_{\mathbf{A}}/G_F Z_\infty^+} E_\varepsilon''(\phi_\alpha\colon \lambda\colon x)\overline{E_\varepsilon''(\phi_\beta\colon \mu\colon x)}dx$$

equals the sum of

$$(8.10) \qquad (\lambda + \bar{\mu})^{-1}\{\varepsilon^{-(\lambda+\bar{\mu})}(\phi_\alpha, \phi_\beta) - \varepsilon^{(\lambda+\bar{\mu})}\big(M(\lambda)\phi_\alpha, M(\mu)\phi_\beta\big)\} \ ,$$

and

$$(\lambda - \bar{\mu})^{-1}\{\varepsilon^{-(\lambda-\bar{\mu})}(\phi_\alpha, M(\mu)\phi_\beta) - \varepsilon^{(\lambda-\bar{\mu})}(M(\lambda)\phi_\alpha, \phi_\beta)\} . \tag{8.11}$$

This formula is stated in [8] and proved in [9]. It follows from a straightforward argument, which we will not reproduce, based on the formula (2.2).

The functions defined by (8.9), (8.10), and (8.11) are all meromorphic in λ and $\bar{\mu}$. We set μ equal to z and let λ approach z. The limit of (8.11) equals (8.8). The limit of (8.10) is the limit as t approaches 0 of

$$\frac{1}{t}\{\varepsilon^{-t}(\phi_\alpha, \phi_\beta) - \varepsilon^{t}(M(t + z)\phi_\beta, M(z)\phi_\beta)\} ,$$

which is the sum of (8.6) and (8.7). On the other hand, the limit of (8.9) is the required inner product, so the lemma follows from analytic continuation. □

LEMMA 8.4. *For z imaginary the operator*

$$M(-z)\cdot\frac{d}{dz}M(z)$$

is self-adjoint.

Proof. The adjoint of this operator equals

$$\left(\frac{d}{d\bar{z}}M(z)^*\right)M(-z)^* = -\left(\frac{d}{dz}M(\bar{z})\right)M(-\bar{z}) = -\left(\frac{d}{dz}M(-z)\right)M(z) ,$$

since z is imaginary. On the other hand

$$-\left(\frac{d}{dz}M(-z)\right)M(z) = M(-z)\cdot\frac{d}{dz}M(z) ,$$

by virtue of the fact that $M(-z)M(z)$ equals the identity. □

Let us write $\pi(n: z: f)$ for the restriction of the operator $\pi(z: f)$ to $\mathcal{H}(n)$.

LEMMA 8.5. *The integral*

$$\frac{1}{4\pi}\sum_{n=1}^{\infty}\int_{-i\infty}^{i\infty}\left|\operatorname{tr}\left\{M(-z)\cdot\left(\frac{d}{dz}M(z)\right)\cdot\pi(n: z: f)\right\}\right| d|z|$$

is finite.

Proof. For any ε between 0 and ε_0 the given integral is bounded by the sum of (8.4) and the expression

$$\frac{\log\varepsilon}{2\pi}\sum_{n=1}^{\infty}\int_{-i\infty}^{i\infty}|\operatorname{tr}\pi(n: z: f)|\, d|z| \tag{8.12}$$

together with the integral over the imaginary axis of

$$\text{(8.13)}\qquad \frac{1}{4\pi}\sum_{\beta\in I}\left|\frac{\varepsilon^{2z}}{2z}(M(-z)\pi(z:f)\phi_\beta,\ \phi_\beta) - \frac{\varepsilon^{-2z}}{2z}(M(z)\pi(z:f)\phi_\beta,\ \phi_\beta)\right|.$$

The expression (8.4) is of course finite by Lemma 8.2. (8.12) is bounded by

$$\frac{\log\varepsilon}{2\pi}\left(\int_{-i\infty}^{i\infty}\operatorname{tr}\pi(z:{}^1f)d|z|\right)^{1/2}\left(\int_{-i\infty}^{i\infty}\operatorname{tr}\pi(z:{}^2f)d|z|\right)^{1/2}$$

where ${}^1f$ and ${}^2f$ are defined in the proof of Theorem 3.2. A glance at Lemma 2.3 and the formula preceding it confirms that this expression is also finite.

Finally, by Assumption 3.5 and the proof of Lemma 3.4, the sum over β in (8.13) is finite. For any β the function

$$\frac{\varepsilon^{2z}}{2z}(M(-z)\pi(z:f)\phi_\beta,\ \phi_\beta) - \frac{\varepsilon^{-2z}}{2z}(M(z)\pi(z:f)\phi_\beta,\ \phi_\beta)$$

is regular at $z = 0$, and is in fact integrable over the imaginary axis. This concludes the proof. □

To complete our calculation we substitute each of the three terms of Lemma 8.3 into the expression (8.5). The first one yields

$$\frac{\log\varepsilon}{2\pi}\int_{-i\infty}^{i\infty}\operatorname{tr}\pi(z:f)d|z|$$

which equals

$$c_G\frac{\log\varepsilon}{2\pi}\int_{-i\infty}^{i\infty}\int_K\int_{M_{\mathbf{A}}/M_FA_\infty^+}P(z:f:km,\ km)dm\,dk\,d|z|\,,$$

since our kernel P is continuous. After inserting the formula for P given in § 2 and applying the Fourier inversion formula we obtain

$$c_G\log\varepsilon\int_K\int_{M_{\mathbf{A}}/M_FA_\infty^+}\sum_{\mu\in M_F}\int_{N_{\mathbf{A}}}f(km\mu nm^{-1}k^{-1})e^{2\rho H(m)}dn\,dm\,dk\ .$$

This expression cancels (7.9).

The contribution of (8.7) to the formula (8.5) is

$$\text{(8.13)}\qquad \frac{1}{4\pi}\sum_{n=1}^{\infty}\int_{-i\infty}^{i\infty}\operatorname{tr}\left\{M(-z)\cdot\left(\frac{d}{dz}M(z)\right)\cdot\pi(n:z:f)\right\}d|z|\ .$$

The contribution of (8.8) equals the sum of

$$\frac{1}{4\pi}\sum_{\beta=1}^{\infty}\int_{-i\infty}^{i\infty}\varepsilon^{2z}\frac{1}{2z}\{(M(-z)\pi(z:f)\phi_\beta,\ \phi_\beta) - (M(z)\pi(z:f)\phi_\beta,\ \phi_\beta)\}d|z|\,,$$

and

$$\frac{1}{4\pi}\sum_{\beta=1}^{\infty}\int_{-i\infty}^{i\infty}\frac{\varepsilon^{2z}-\varepsilon^{-2z}}{2z}(M(z)\pi(z:f)\phi_\beta,\ \phi_\beta)d|z|\ .$$

In both these terms the sum over β is finite. As ε approaches 0 the first expression approaches 0 by the Riemann-Lebesgue lemma, while the second term approaches

$$-\frac{1}{4}\operatorname{tr}\{M(0)\pi(0: f)\}\ . \tag{8.14}$$

9. Concluding remarks

Our computation is now complete. We have shown that the trace of $\lambda_0(f)$ is the sum of a certain number of terms, each of which is independent of ε, and an expression which approaches 0 as ε approaches 0. Since we started off by letting ε be any number between 0 and ε_0 this latter expression must vanish.

The remaining terms are scattered throughout the earlier sections. They are

(i) *The elliptic term,*

$$\sum_{\gamma\in\{G_e\}}\tilde{\tau}(\gamma, G)\int_{G_{\mathbf{A}}/G(\gamma)_{\mathbf{A}}} f(x\gamma x^{-1})dx\ ;$$

(ii) *The singular term,*

$$\sum_{\mu\in\{\{M_s\}\}}\tilde{\tau}(\mu, G)\int_{G_{\mathbf{A}}/G(\mu)_{\mathbf{A}}} f(x\mu x^{-1})dx\ ;$$

and

(iii) The total *parabolic term,* which is the sum of

$$\begin{aligned}&\frac{c_G}{2}\sum_{\mu\in\{M_r\}}\tilde{\tau}(\mu, M)\int_K\int_{N_{\mathbf{A}}}\int_{M_{\mathbf{A}}/M(\mu)_{\mathbf{A}}} f(knm\mu m^{-1}n^{-1}k^{-1})\\ &\quad\cdot H(nw)dm\,dn\,dk\ ,\end{aligned} \tag{9.1}$$

$$\sum_{\mu\in\{M_s\}}\lim_{z\to 0}\frac{d}{dz}\{z\tilde{\theta}(\mu: z: f)\}\ , \tag{9.2}$$

$$\frac{1}{4\pi}\sum_{n=1}^{\infty}\int_{-i\infty}^{i\infty}\operatorname{tr}\left\{M(-z)\cdot\left(\frac{d}{dz}M(z)\right)\cdot\pi(n: z: f)\right\}d|z|\ , \tag{9.3}$$

and

$$-\frac{1}{4}\operatorname{tr}\{M(0)\pi(0: f)\}\ . \tag{9.4}$$

We need hardly remark that our formula is not yet in a reasonable form. Considerably more work is required in several directions before we might hope to obtain information about the space of cusp forms.

In the first place we have left the term (9.2) in an unsatisfactory state. What is needed is some sort of analysis on the orbit structure of

$P(\mu)$ in $N(\mu)$, for elements μ in M_s.

Once (9.2) has been put into a more tractable form we can start analyzing the various terms of the trace formula as distributions on $G_{\mathbf{A}}$. In view of Harish-Chandra's work on the Schwartz space, it makes sense to ask whether a distribution on any of the local groups G_v is tempered. Every term in the trace formula is, in all probability, a linear combination of products of tempered distributions on the groups G_v. The problem would be to calculate the Fourier transforms of these distributions.

If v is a place of F, a distribution T_v on G_v is said to be *invariant* if for any pair of functions f and g in $C_c^\infty(G_v)$

$$T_v(f*g) = T_v(g*f) .$$

The distributions defined by the elliptic and singular terms are all invariant, as are those defined by (9.4). On the other hand, the distributions defined by the remaining terms are not invariant. This complicates the problem of calculating the Fourier transforms. Of course the sum of the terms (9.1), (9.2), and (9.3) defines an invariant linear functional on f. However, it will not be possible to see how the noninvariant components of these terms cancel without calculating the appropriate Fourier transforms.

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