# GLOBAL UNIQUENESS THEOREMS FOR LINEAR AND NONLINEAR WAVES

SPYROS ALEXAKIS AND ARICK SHAO

Abstract. We prove a unique continuation from infinity theorem for regular waves of the form $[\Box + \mathcal{V}(t, x)]\phi = 0$. Under the assumption of no incoming and no outgoing radiation on specific halves of past and future null infinities, we show that the solution must vanish everywhere. The "no radiation" assumption is captured in a specific, *finite* rate of decay which in general depends on the $L^\infty$-profile of the potential $\mathcal{V}$. We show that the result is optimal in many regards. These results are then extended to certain power-law type nonlinear wave equations, where the order of decay one must assume is independent of the size of the nonlinear term. These results are obtained using a new family of global Carleman estimates on the exterior of a null cone. A companion paper to this one will explore further applications of these new estimates to such nonlinear waves.

Contents

| | |
|---|---|
| 1. Introduction | 1 |
| 1.1. The Main Results | 4 |
| 1.2. The Global Carleman Estimates | 8 |
| 2. Carleman Estimates | 9 |
| 2.1. The Preliminary Estimate | 9 |
| 2.2. The Linear Estimate | 13 |
| 2.3. The Nonlinear Estimate | 16 |
| 3. Proofs of the Main Results | 17 |
| 3.1. Special Domains | 18 |
| 3.2. Boundary Limits | 21 |
| 3.3. Proof of Theorem 1.1 | 24 |
| 3.4. Proofs of Theorems 1.5 and 1.6 | 26 |
| 3.5. Optimality via Counterexamples | 28 |
| Acknowledgments. | 29 |
| References | 29 |

## 1. Introduction

This paper presents certain global unique continuation results for linear and nonlinear wave equations. The motivating challenge is to investigate the extent to which globally regular waves can be reconstructed from the radiation they emit towards (suitable portions of) null infinity. We approach this in the sense of uniqueness: if a regular wave emits no radiation towards appropriate portions of null infinity, then it must vanish.

The belief that a lack of radiation emitted towards infinity should imply the triviality of the underlying solution has been implicit in the physics literature for many classical fields. For instance, in the case of linear Maxwell equations, early results in this direction go back at least to [17]. Moreover, in general relativity, the question whether non-radiating gravitational fields must be trivial (i.e., stationary) goes back at least to [14], in connection with the possibility of time-periodic solutions of Einstein's equations. The presumption that the answer must be affirmative under suitable assumptions underpins many of the central stipulations in the field; see for example the issue of the final state in [7].

We deal here with self-adjoint wave equations over the Minkowski spacetime,

$$\mathbb{R}^{n+1} = \{(t, x) \mid t \in \mathbb{R},\, x = (x^1, \dots, x^n) \in \mathbb{R}^n\}.$$

Our analysis in performed in the exterior region $\mathcal{D}$ of a light cone. Roughly, we show that if a solution $\phi$ of such a wave equation decays faster toward infinity in $\mathcal{D}$ than the rate enjoyed by free waves (with smooth and rapidly decaying initial data),[1] then $\phi$ itself must vanish on $\mathcal{D}$.

Straightforward examples in the Minkowski spacetime show that if one does not assume regularity of a wave in a suitably large portion of spacetime, then then unique continuation from infinity will fail *unless* one assumes vanishing to infinite order. In the context of the early and later investigations in the physics literature, one always made sufficient regularity assumptions at infinity to *derive* this vanishing to infinite order. The vanishing/stationarity of the field can then be derived under the additional assumption of analyticity near a portion of future of null infinity; see, for instance, [4, 5, 14, 15, 16]. However, the assumption of analyticity can not be justified on physical grounds; in fact, the recent work [10] suggests that the local argument near a piece of null infinity fails without that assumption.

In earlier joint work with V. Schlue, [2], the authors were able to prove that the assumption of vanishing to infinite order at (suitable parts) of null infinities *does* imply the vanishing of the solution near null infinity. [2] This earlier result can thus be seen as a satisfactory answer to the above question in the physics literature, whenever the assumption of vanishing to infinite order can be derived (either by assuming sufficient regularity towards infinity, or by the nature of the problem).

However, this leaves open the question of whether the infinite-order vanishing assumption can be relaxed, if *in addition* one assumes the solution to be globally regular (which rules out the aforementioned counterexamples). Obviously, one must assume faster decay towards past and future null infinity ($\mathcal{I}^-$ and $\mathcal{I}^+$, respectively) than that enjoyed by free linear waves. The main theorems of this paper derive precisely such results, for self-adjoint wave operators over Minkowski space.

Our first result in this direction applies to linear wave equations of the form

$$[\Box + \mathcal{V}(t, x)]\phi = 0. \tag{1.1}$$

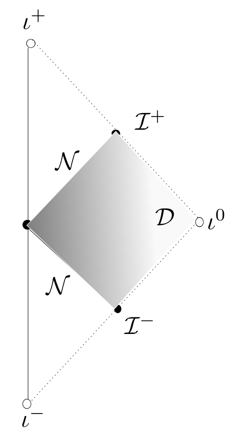
Informally speaking, we show that if the potential $\mathcal{V}$ decays and satisfies suitable $L^\infty$-bounds, and if the solution $\phi$ decays faster [3] on $\mathcal{D}$ toward null infinity than generic solutions of the free wave equation, then $\phi$ must vanish everywhere on $\mathcal{D}$.

[1] In other words, $\phi$ vanishes at infinity to a given *finite* order.

[2] In fact, it was shown that the parts of null infinity where one must make this assumption depend strongly on the mass of the background spacetime.

[3] The rate of decay required for $\phi$ depends on the profile of $\mathcal{V}$.

FIGURE 1. The null cone $\mathcal{N}$ and the exterior domain $\mathcal{D}$ in the Penrose diagram




A second result deals with the same equations, but shows that if $\mathcal{V}$ satisfies a suitable monotonicity property, then it suffices to only assume a specific decay for $\phi$, which is independent of the size of $\mathcal{V}$. Furthermore, this latter result generalizes immediately to a class of nonlinear wave equations that includes the usual power-law (defocusing and focusing) nonlinear wave equations.

The method of proof replies on new Carleman-type estimates for linear and nonlinear wave operators over the entire domain $\mathcal{D}$. In a companion paper, [3], we will develop these estimates further to derive localized estimates for these classes of wave equations inside time cones. These estimates are then applied toward understanding the profile of energy concentration near singularities.

The precise statements of our unique continuation results are in Section 1.1. We also discuss in Sections 1.1 and 3.5 how the results here are essentially optimal.

Both the results here and in [2] can be compared with results in the literature on decay properties of eigenfunctions of elliptic operators. As discussed extensively in the introduction of [2], the results there on (local) unique continuation from infinity assuming vanishing of infinite order can be considered as a strengthening of classical results on non-existence of positive or zero $L^2$-eigenvalues of elliptic operators

$$L := -\Delta - \mathcal{V},$$

which go back to [1, 11, 12, 13, 18]. Indeed such an eigenfunction $u(x)$, with eigenvalue $\lambda \geq 0$, would correspond to time-periodic (or static) solutions

$$v(t,x) := e^{i\sqrt{\lambda}t}u(x)$$

of the corresponding wave equation

$$[\Box + \mathcal{V}]v = 0.$$

The condition of infinite order vanishing can be derived for positive eigenvalues and must be imposed for zero eigenvalues. For the latter case, there exist straightforward examples of static solutions $L\phi = 0$ over $\mathbb{R}^n$ which vanish to any prescribed order $M > 0$, yet are not zero. (We review such examples in brief in Section 3.5). Yet, these examples make $\mathcal{V}$ large in proportion to the assumed order of vanishing $M$. In particular, these static examples show that in order to use the decay of a solution to (1.1) to derive the global vanishing of the solution, the $L^\infty$-profile of $\mathcal{V}$ must in general depend on the assumed order of vanishing.

We can also see our results as a strengthening of known work on negative eigenvalues of elliptic operators $L$. Indeed, [13, Theorem 4.2] strengthens a result of Agmon for $L$ with $\mathcal{V} = O(r^{-1-\epsilon})$, where $\epsilon > 0$ and $r$ is the Euclidean distance from the origin, to derive that a non-trivial solution $w$ of

$$Lu = -k^2 u$$

must satisfy

$$u = e^{-kr} r^{-(n-1)/2} [f(\omega) + \varphi(r, \omega)],$$

where

$$f \in L^2(\mathbb{S}^{n-1}), \qquad f \not\equiv 0, \qquad \int_{\mathbb{S}^{n-1}} |\varphi(r,\omega)|^2 d\omega = O(r^{-2\gamma}), \qquad \gamma \in (0, \epsilon).$$

These yield solutions $v(t,x)$ of the corresponding wave equation defined via

$$v(t,x) := e^{\pm k \cdot t} u(x).$$

Note that such solutions will decay exponentially one of the halves

$$\mathcal{I}_0^+ := \{u \leq 0, \, v = +\infty\}, \qquad \mathcal{I}_0^- := \{v \geq 0, \, u = -\infty\},$$

of future and past null infinities $\mathcal{I}^+, \mathcal{I}^-$, while having a finite, non-zero radiation field $e^{-2v} f(\omega)$, $e^{-2u} f(\omega)$ on the other. The condition $f \neq 0$ can precisely be seen as a unique continuation statement: if the eigenfunction corresponded to a wave of vanishing radiation, it would have to vanish itself. We remark that this discussion also shows that one cannot hope to show uniqueness of solutions to wave equations in the form (1.1) for general smooth potentials $\mathcal{V}$ by even assuming infinite order-vanishing on the entire past null infinity.

1.1. **The Main Results.** Recall the Minkowski metric on $\mathbb{R}^{n+1}$, given by

$$g := -dt^2 + dr^2 + r^{n-1} \mathring{\gamma}, \qquad t \in \mathbb{R}, \quad r \in [0, \infty), \tag{1.2}$$

where $r = |x|$, and where $\mathring{\gamma}$ is the round metric on $\mathbb{S}^{n-1}$. In terms of null coordinates,

$$u := \frac{1}{2}(t - r), \qquad v := \frac{1}{2}(t + r), \tag{1.3}$$

the Minkowski metric takes the form

$$g = -4 du dv + r^{n-1} \mathring{\gamma}. \tag{1.4}$$

We use the usual notations—$\partial_t, \partial_r, \partial_u, \partial_v$—to denote derivatives with respect to these coordinates. For the remaining spherical directions, we use $\nabla\!\!\!\!/$ to denote the induced connections for the level spheres of $(t, r)$. In particular, we let $|\nabla\!\!\!\!/ \phi|^2$ denote the squared ($g$-)norm of the spherical derivatives of $\phi$:

$$|\nabla\!\!\!\!/ \phi|^2 := g(\nabla\!\!\!\!/ \phi, \nabla\!\!\!\!/ \phi) = r^2 \cdot \mathring{\gamma}(\nabla\!\!\!\!/ \phi, \nabla\!\!\!\!/ \phi). \tag{1.5}$$

Our main results deal with solutions of wave equations in the exterior of the double null cone about the origin,

$$\mathcal{D} := \{\xi \in \mathbb{R}^{n+1} \mid |t(\xi)| < |r(\xi)|\}. \tag{1.6}$$

In our geometric descriptions, we will often refer to the standard Penrose compactification of Minkowski spacetime, see Figure 1, as this provides our basic intuition of the structure of infinity. With this in mind, we note the following:

- $\mathcal{D}$ is the diamond-shaped region in Minkowski spacetime bounded by the null cone about the origin and by the outer half of null infinity $\mathcal{I}^{\pm}$.
- This boundary of $\mathcal{D}$ has four corners: the origin, spacelike infinity $\iota^0$, and the "midpoints" of future and past null infinity.

1.1.1. *Linear Wave Equations.* Recall if $\phi$ is a free wave, i.e., $\Box\phi \equiv 0$, then the radiation field of $\phi$ at null and spacelike infinities is captured by the limit of

$$\mathcal{R}(\phi) := (1+|u|)^{\frac{n-1}{2}}(1+|v|)^{\frac{n-1}{2}}\phi. \tag{1.7}$$

This can be seen, for example, via Penrose compactification by solving the corresponding wave equation on the Einstein cylinder. These asymptotics also hold for many linear waves with suitably decaying potentials, and suitable nonlinear waves with small initial data; see [6].

Our first result deals with solutions of linear wave equations of the form

$$\Box\phi + \mathcal{V}\phi = 0, \tag{1.8}$$

with $\mathcal{V}$ satisfying suitable $L^\infty$-type bounds. We require decay of a power $\delta > 0$ faster than in (1.7) for solutions $\phi$ of (1.8), with $\mathcal{V}$ satisfying $\delta$-dependent bounds. On the other hand, we make no assumptions on the sign or the monotonicity of $\mathcal{V}$.

**Theorem 1.1.** *Fix $0 < p < \delta$, and let $\phi \in \mathcal{C}^2(\mathcal{D})$ such that:*

- *$\phi$ satisfies the following differential inequality on $\mathcal{D}$,*

$$|\Box\phi| \leq |\mathcal{V}||\phi|, \tag{1.9}$$

*where $\mathcal{V} \in \mathcal{C}^0(\mathcal{D})$, and where $\mathcal{V}$ satisfies*

$$|\mathcal{V}| \leq \varepsilon \cdot p^{\frac{3}{2}}\sqrt{\min(\delta-p, p)} \cdot \min(|uv|^{-1+\frac{p}{2}}, |uv|^{-1-\frac{p}{2}}), \tag{1.10}$$

*for some universal small constant $\varepsilon$ (that can be determined from the proof).*

- *$\phi$ satisfies the following decay conditions:* [4]

$$\begin{aligned} &\sup_{\mathcal{D}}\left\{[(1+|u|)(1+|v|)]^{\frac{n-1+\delta}{2}}(|u\cdot\partial_u\phi| + |v\cdot\partial_v\phi|)\right\} < \infty, \\ &\sup_{\mathcal{D}\cap\{|uv|<1\}}\left\{[(1+|u|)(1+|v|)]^{\frac{n-1+\delta}{2}}|uv|^{\frac{1}{2}}|\nabla\!\!\!/\phi|\right\} < \infty, \\ &\sup_{\mathcal{D}}\left\{[(1+|u|)(1+|v|)]^{\frac{n-1+\delta}{2}}|\phi|\right\} < \infty. \end{aligned} \tag{1.11}$$

*Then, $\phi$ vanishes everywhere on $\mathcal{D}$.*

**Remark.** *We note that if one knows that $\phi$ is $\mathcal{C}^2$-regular on all of $\mathbb{R}^{1+n}$ and thus vanishes on $\overline{\mathcal{D}}$, one can derive that $\phi$ vanishes in the entire spacetime by standard energy estimates. We also note that the assumed decay (1.11) of $\phi$ in $\mathcal{D}$ is a quantitative assumption of no incoming radiation from half of past null infinity, and no outgoing radiation in half of future null infinity.*

**Remark.** *Note that (1.10) implies $\mathcal{V}$ decays like $r^{-2-p}$ at spatial infinity. Yet the condition is weaker at null infinities, where $\mathcal{V}$ decays like $r^{-1-\frac{p}{2}}$.*

[4] The assumption (1.11) can be replaced by corresponding conditions for weighted fluxes through level sets of $f$. This will become apparent from the proof of the theorem.

The necessity of assuming this decay for $\mathcal{V}$ to obtain any sort of uniqueness theorem is already well-known in the elliptic setting, which corresponds to time-independent solutions of (1.8); see [2] for a discussion. Moreover, the smallness of the potential $\mathcal{V}$ in Theorem 1.1 is *necessary.* This is proved in Section 3.5 by construction of a counterexample.

On the other hand, if $\mathcal{V}$ in (1.8) has a certain monotonicity (and we return to the setting of a differential *equation* rather an inequality), then in fact no smallness is required of $\mathcal{V}$. The precise statement is given in the subsequent theorem:

**Theorem 1.2.** *Fix $\delta > 0$, and let $\phi \in \mathcal{C}^2(\mathcal{D})$ such that:*

- *$\phi$ satisfies*

$$\Box\phi + V\phi = 0, \tag{1.12}$$

  *where $V \in \mathcal{C}^1(\mathcal{D}) \cap L^\infty(\mathcal{D})$ satisfies*

$$V > 0, \qquad (u\partial_u + v\partial_v)(\log V) > -2 + \mu, \tag{1.13}$$

  *everywhere on $\mathcal{D}$ for some constant $\mu > 0$.*
- *$\phi$ satisfies* (1.11)*, as well as the following decay condition:*

$$\sup_{\mathcal{D}\cap\{|uv|>1\}} \left\{ [(1+|u|)(1+|v|)]^{\frac{n-1+\delta}{2}} |uv|^{\frac{1}{2}} V^{\frac{1}{2}} |\phi| \right\} < \infty. \tag{1.14}$$

*Then, $\phi$ vanishes everywhere on $\mathcal{D}$.*

**Remark.** *Note that $u\partial_u + v\partial_v = t\partial_t + r\partial_r$ is precisely the dilation vector field on $\mathbb{R}^{n+1}$, which generates a conformal symmetry of Minkowski spacetime.*

**Remark.** *In particular, Theorem 1.2 applies to*

$$\Box\phi + \phi = 0, \tag{1.15}$$

*i.e., the negative-mass Klein-Gordon equations.*

The above two results are proven using Carleman-type estimates that we obtain in Section 2. While Theorem 1.1 is proved separately, Theorem 1.2 is a special case of uniqueness results for *nonlinear* wave equations, which we discuss below.

1.1.2. *Nonlinear Wave Equations.* The Carleman estimates that we obtain in Section 2 are of sufficient generality to be directly applicable to certain classes of nonlinear wave equations. In fact, they yield stronger results for these equations, compared to the general linear case. The stronger nature of the estimates is manifested in the fact that the order of vanishing at infinity needs only be slightly faster than a specific rate, irrespective of the size of the (nonlinear) potential. Surprisingly perhaps, it turns out that one obtains these improved estimates in either the focusing or the defocusing case, depending on the power of the nonlinearity as compared to the *conformal* power.

The class of equations that we will consider will be generalizations of the usual power-law focusing and defocusing nonlinear wave equations: [5]

$$\Box\phi \pm V(t,x)|\phi|^{p-1}\phi = 0, \qquad p \geq 1, \tag{1.16}$$

with $V \in \mathcal{C}^1(\mathcal{D})$ and $V(t,x) > 0$ for all points in $(t,x) \in \mathcal{D}$.

[5] The case $p = 1$ yields linear wave equations with potential.

**Definition 1.3.** *We call such equations* focusing *if the sign in* (1.16) *is* $+$, *and* defocusing *if the sign in* (1.16) *is* $-$.

**Definition 1.4.** *We refer to* (1.16) *as:*

- *"Subconformal", if* $p < 1 + \frac{4}{n-1}$.
- *"Conformal", if* $p = 1 + \frac{4}{n-1}$.
- *"Superconformal", if* $p > 1 + \frac{4}{n-1}$.

We now state the remaining unique continuation results, one applicable to subconformal focusing-type equations, and the other to both conformal and superconformal defocusing type equations of the form (1.16).

**Theorem 1.5.** *Fix any* $\delta > 0$, *and let* $\phi \in \mathcal{C}^2(\mathcal{D})$ *such that:*

- $\phi$ *satisfies*

$$\Box\phi + V|\phi|^{p-1}\phi = 0, \qquad 1 \leq p < 1 + \frac{4}{n-1}, \tag{1.17}$$

  *where* $V \in \mathcal{C}^1(\mathcal{D}) \cap L^\infty(\mathcal{D})$ *satisfies* [6]

$$V > 0, \qquad (u\partial_u + v\partial_v)(\log V) > -\frac{n-1}{2}\left(1 + \frac{4}{n-1} - p\right) + \mu, \tag{1.18}$$

  *everywhere on* $\mathcal{D}$ *for some constant* $\mu > 0$.
- $\phi$ *satisfies* (1.11), *as well as the following decay condition:*

$$\sup_{\mathcal{D}\cap\{|uv|>1\}} \left\{ [(1+|u|)(1+|v|)]^{\frac{n-1+\delta}{p+1}} |uv|^{\frac{1}{p+1}} V^{\frac{1}{p+1}} |\phi| \right\} < \infty. \tag{1.19}$$

*Then,* $\phi$ *vanishes everywhere on* $\mathcal{D}$.

**Remark.** *In particular, taking* $p = 1$ *in Theorem 1.5 results in Theorem 1.2.*

**Theorem 1.6.** *Fix* $\delta > 0$, *and let* $\phi \in \mathcal{C}^2(\mathcal{D})$ *such that:*

- $\phi$ *satisfies*

$$\Box\phi - V|\phi|^{p-1}\phi = 0, \qquad p \geq 1 + \frac{4}{n-1}, \tag{1.20}$$

  *where* $V \in \mathcal{C}^1(\mathcal{D}) \cap L^\infty(\mathcal{D})$ *satisfies, everywhere on* $\mathcal{D}$,

$$V > 0, \qquad (u\partial_u + v\partial_v)(\log V) \leq \frac{n-1}{2}\left(p - 1 - \frac{4}{n-1}\right). \tag{1.21}$$

- $\phi$ *satisfies* (1.11).

*Then,* $\phi$ *vanishes everywhere on* $\mathcal{D}$.

**Remark.** *Note that for defocusing-type equations,* (1.20), *one does not require the extra decay condition* (1.19) *needed for focusing-type equations.*

**Remark.** *The Carleman estimates we employ are robust enough so that Theorems 1.5 and 1.6 can be even further generalized to operators of the form*

$$\Box\phi \pm V(t,x) \cdot \dot{W}(\phi) = 0. \tag{1.22}$$

*Roughly, we can find analogous unique continuation results in the following cases:*

- *Focusing-type, with* $\dot{W}(\phi)$ *growing at a subconformal rate.*
- *Defocusing-type, with* $\dot{W}(\phi)$ *growing at a conformal or superconformal rate.*

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[6] Fixed sign error in monotonicity condition.

*For simplicity, though, we restrict our attention to equations of the form* (1.16).

For example, by taking $V(t, x) = 1$, we recover unique continuation results for the usual focusing and defocusing nonlinear wave equations:

**Corollary 1.7.** *Fix $\delta > 0$, and let $\phi \in \mathcal{C}^2(\mathcal{D})$ satisfy*

$$\square\phi + |\phi|^{p-1}\phi = 0, \qquad 1 \leq p < 1 + \frac{4}{n-1}, \tag{1.23}$$

*as well as the decay conditions* (1.11) *and* (1.19)*. Then, $\phi \equiv 0$ on $\mathcal{D}$.*

**Corollary 1.8.** *Fix $\delta > 0$, and let $\phi \in \mathcal{C}^2(\mathcal{D})$ satisfy*

$$\square\phi - |\phi|^{p-1}\phi = 0, \qquad p \geq 1 + \frac{4}{n-1}, \tag{1.24}$$

*as well as the decay conditions* (1.11)*. Then, $\phi \equiv 0$ on $\mathcal{D}$.*

The main results—Theorems 1.1, 1.5, and 1.6—are proved in Section 3.

1.2. **The Global Carleman Estimates.** The main tool for our uniqueness results is a new family of Carleman estimates which are global, in the sense that they apply to regular functions defined over the entire region $\mathcal{D}$. The precise estimates are presented in Theorems 2.13 and 2.18.

Here, we approach Carleman estimates from the perspective of energy estimates for geometric wave equations via the use of multipliers. As such, we mostly follow the notations developed in [8] and adopted in [2]. Similar local Carleman estimates, but from the null cone rather than from null infinity, were proved in [9].

From this geometric point of view, the basic process behind proving Carleman estimates can be summarized as follows:

- One applies multipliers and integrates by parts like for energy estimates, but the goal is now to obtain positive bulk terms.
- In order to achieve the above, we do not work directly with the solution $\phi$ itself. Rather, we undergo a conjugation by considering the wave equation for $\psi = e^{-F}\phi$, where $e^{-F}$ is a specially chosen weight function.

For further discussion on the geometric view of Carleman estimates and their proofs, the reader is referred to [2, Section 3.3].

The function $F$ we work with here is a reparametrization $F(f)$ of the Minkowski square distance function from the origin,

$$f \in \mathcal{C}^\infty(\mathcal{D}), \qquad f := -uv = \frac{1}{4}(r^2 - t^2). \tag{1.25}$$

Its level sets form a family of timelike hyperboloids having zero pseudoconvexity. As a result of this, the bulk terms we obtain unfortunately yields no first derivative terms. However, the specific nature of this $f$ crucially helps us, as *it allows us to generate (zero-order) bulk terms that can be made positive on all of $\mathcal{D}$.*

This global positivity of the bulk is important here primarily because *the weight $e^{-F}$ vanishes on the null cone $\mathcal{N}$ about the origin*, i.e., the left boundary of $\mathcal{D}$. In practice, this allows us to eliminate flux terms at this "inner" boundary; in particular, we do not introduce a cutoff function, which is commonly used in (local) unique continuation problems. This is ultimately responsible for us only needing to assume finite-order vanishing at null infinity for our uniqueness results. [7]

[7] More specifically, in our current context, the lack of a cutoff function removes the need to take $a \nearrow \infty$ in Theorems 2.13 and 2.18 when proving unique continuation.

The first Carleman estimate, Theorem 2.13, applies to the linear wave operator $\Box$. In this case, special care is required to generate the positive bulk. In particular, we implicitly utilize that our domain $\mathcal{D}$ is symmetric up to an inversion across a hyperboloid to construct two complementary reparametrizations $F_\pm$ which match up at the hyperboloid. While this conformal inversion (see Section 3.2.2) is not used explicitly in the proof, it is manifest in the idea that local Carleman estimates from infinity and from a null cone are dual to each other.

For the nonlinear equations that we deal with, our second Carleman estimate, Theorem 2.18, directly uses the nonlinearity to produce a positive bulk. This is in contrast to the usual method of treating nonlinear terms by seeking to absorb them into the positive bulk arising from the linear terms. In the context of unique continuation, this results in the improvement (for certain equations) from Theorem 1.1, which requires suitably small potentials (relative to the assumed order of vanishing), to Theorems 1.2, 1.5, and 1.6, for which no such smallness is required.

Finally, we remark that Theorem 2.18 will also be applied in the companion paper [3] to study nonlinear wave equations for different purposes. Thus, we present the main estimates for more general domains than needed in the present paper.

## 2. Carleman Estimates

In this section, we derive new Carleman estimates for functions $\phi \in \mathcal{C}^2(\mathcal{D})$, where $\mathcal{D}$ is the exterior of the double null cone (see (1.6)),

$$\mathcal{D} := \{Q \in \mathbb{R}^{n+1} \mid u(Q) < 0,\, v(Q) > 0\}.$$

Throughout, we will let $\nabla$ denote the Levi-Civita connection for $(\mathbb{R}^{1+n}, g)$, and we will let $\nabla^\sharp$ denote the gradient operator with respect to $g$. We also recall the function hyperbolic square distance function $f$ defined in (1.25).

### 2.1. The Preliminary Estimate.

In order to state the upcoming inequalities succintly, we make some preliminary definitions.

**Definition 2.1.** *We define a* reparametrization *of $f$ to be a function of the form $F \circ f$, where $F \in \mathcal{C}^\infty(0, \infty)$. For convenience, we will abbreviate $F \circ f$ by $F$. We use the symbol $'$ to denote differentiation of a reparametrization as a function on $(0, \infty)$, that is, differentiation with respect to $f$.*

**Definition 2.2.** *We say that an open, connected subset $\Omega \subseteq \mathcal{D}$ is* admissible *iff:*

- *The closure of $\Omega$ is a compact subset of $\mathcal{D}$.*
- *The boundary $\partial\Omega$ of $\Omega$ is piecewise smooth, with each smooth piece being either a spacelike or a timelike hypersurface of $\mathcal{D}$.*

*For an admissible $\Omega \subseteq \mathcal{D}$, we define the* oriented unit normal *$\mathcal{N}$ of $\partial\Omega$ as follows:*

- *$\mathcal{N}$ is the inward-pointing unit normal on each spacelike piece of $\partial\Omega$.*
- *$\mathcal{N}$ is the outward-pointing unit normal on each timelike piece of $\partial\Omega$.*

*Integrals over such an admissible region $\Omega$ and portions of its boundary $\partial\Omega$ will be with respect to the volume forms induced by $g$.*

**Definition 2.3.** *We say that a reparametrization $F$ of $f$ is* inward-directed *on an admissible region $\Omega \subseteq \mathcal{D}$ iff $F' < 0$ everywhere on $\Omega$.*

Lastly, we provide the general form of the wave operators we will consider: [8]

[8] These include all the wave operators arising from the main theorems throughout Section 1.1.

**Definition 2.4.** *Let $U \in \mathcal{C}^1(\mathcal{D} \times \mathbb{R})$, and let $\dot{U}$ be the partial derivative of $U$ in the last ($\mathbb{R}$-)component. Define the following (possibly) nonlinear wave operator:*

$$\Box_U \phi(Q) = \Box \phi(Q) + \dot{U}(Q, \phi(Q)), \qquad Q \in \mathcal{D}. \tag{2.1}$$

*Furthermore, derivatives $\nabla U$ of $U$ will be with respect to the first ($\mathcal{D}$-)component.*

We can now state our preliminary Carleman-type inequality:

**Proposition 2.5.** *Let $\phi \in \mathcal{C}^2(\mathcal{D})$, and let $\Omega \subseteq \mathcal{D}$ be an admissible region. Furthermore, let $U$ and $\Box_U$ be as in Definition 2.4. Then, for any inward-directed reparametrization $F$ of $f$ on $\Omega$, the following inequality holds,*

$$\begin{aligned} \frac{1}{8} \int_\Omega e^{-2F} |F'|^{-1} \cdot |\Box_U \phi|^2 \geq \int_\Omega e^{-2F} (f|F'| G_F - H_F) \cdot \phi^2 \\ - \int_\Omega \mathcal{B}^F_U - \int_{\partial\Omega} P^F_\beta \mathcal{N}^\beta , \end{aligned} \tag{2.2}$$

*where:*

- *$G_F$ and $H_F$ are defined in terms of $f$ as*

$$G_F := -(fF')', \qquad H_F := \frac{1}{2}(f G_F)', \tag{2.3}$$

- *$\mathcal{B}^F_U$ is the nonlinear bulk quantity,*

$$\begin{aligned} \mathcal{B}^F_U := e^{-2F} \left( \frac{n-1}{4} - fF' \right) \cdot \dot{U}(\phi)\phi - e^{-2F} \nabla^\alpha f \cdot \nabla_\alpha U(\phi) \\ - 2e^{-2F} \left( \frac{n+1}{4} - fF' \right) \cdot U(\phi), \end{aligned} \tag{2.4}$$

- *$\mathcal{N}$ is the oriented unit normal of $\partial\Omega$.*
- *$P^F$ is the current,*

$$\begin{aligned} P^F_\beta := e^{-2F} \left( \nabla^\alpha f \cdot \nabla_\alpha \phi \nabla_\beta \phi - \frac{1}{2} \nabla_\beta f \cdot \nabla^\mu \phi \nabla_\mu \phi \right) \\ + e^{-2F} \nabla_\beta f \cdot U(\phi) + e^{-2F} \left( \frac{n-1}{4} - fF' \right) \cdot \phi \nabla_\beta \phi \\ + e^{-2F} \left[ \left( fF' - \frac{n-1}{4} \right) F' - \frac{1}{2} G_F \right] \nabla_\beta f \cdot \phi^2 . \end{aligned} \tag{2.5}$$

The remainder of this subsection is dedicated to the proof of (2.2).

2.1.1. *Preliminaries.* We first collect some elementary computations regarding $f$.

**Lemma 2.6.** *The following identities hold:*

$$\partial_u f = -v, \qquad \partial_v f = -u. \tag{2.6}$$

*As a result,*

$$\nabla^\sharp f = \frac{1}{2}(u \cdot \partial_u + v \cdot \partial_v), \qquad \nabla^\alpha f \nabla_\alpha f = f. \tag{2.7}$$

**Lemma 2.7.** *The following identity holds:*

$$\nabla^2 f = \frac{1}{2} g. \tag{2.8}$$

*Moreover,*

$$\Box f = \frac{n+1}{2}, \qquad \nabla^\alpha f \nabla^\beta f \nabla_{\alpha\beta} f = \frac{1}{2} f. \tag{2.9}$$

**Remark.** *In particular, equation* (2.8) *implies that the level sets of $f$ have exactly zero pseudoconvexity.*

We also recall the (Lorentzian) divergence theorem in terms of our current language: *if $\Omega \subseteq \mathcal{D}$ be admissible, and if $P$ is a smooth* 1*-form on $\mathcal{D}$, then*

$$\int_\Omega \nabla^\beta P_\beta = \int_{\partial\Omega} P_\beta \mathcal{N}^\beta, \tag{2.10}$$

*where $\mathcal{N}$ is the oriented unit normal of $\partial\Omega$.*

2.1.2. *Proof of Proposition 2.5.* We begin by defining the following shorthands:

- Define $\psi \in \mathcal{C}^\infty(\mathcal{D})$ and the conjugated operator $\mathcal{L}_U$ by

$$\psi := e^{-F}\phi, \qquad \mathcal{L}_U \psi := e^{-F} \Box_U \phi = e^{-F} \Box_U (e^F \psi). \tag{2.11}$$

- Let $S$ and $S_*$ define the operators

$$S\psi := \nabla^\alpha f \nabla_\alpha \psi, \qquad S_* \psi := S\psi + \frac{n-1}{4} \cdot \psi. \tag{2.12}$$

- Recall the stress-energy tensor for the wave equation, applied to $\psi$:

$$\mathcal{Q}_{\alpha\beta}[\psi] := \mathcal{Q}_{\alpha\beta} := \nabla_\alpha \psi \nabla_\beta \psi - \frac{1}{2} g_{\alpha\beta} \nabla^\mu \psi \nabla_\mu \psi. \tag{2.13}$$

The proof will revolve around an energy estimate for the wave equation, but for $\psi$ rather than $\phi$. We also make note of the following relations between $\psi$ and $\phi$:

**Lemma 2.8.** *The following identities hold:*

$$\nabla_\alpha \psi = e^{-F} (\nabla_\alpha \phi - F' \nabla_\alpha f \cdot \phi), \qquad S\psi = e^{-F}(S\phi - fF' \cdot \phi). \tag{2.14}$$

*Furthermore, we have the expansion*

$$\mathcal{L}_U \psi = \Box \psi + 2F' \cdot S_* \psi + [f(F')^2 - G_F] \cdot \psi + e^{-F} \dot{U}(\phi). \tag{2.15}$$

*Proof.* First, (2.14) is immediate from definition and from (2.7). We next compute

$$\begin{aligned} \mathcal{L}_U \psi &= e^{-F} \nabla^\alpha (F' e^F \nabla_\alpha f \cdot \psi) + e^{-F} \nabla^\alpha (e^F \nabla_\alpha \psi) + e^{-F} \dot{U}(\phi) \\ &= \Box \psi + 2F' \cdot S\psi + f(F')^2 \cdot \psi + fF'' \cdot \psi + F' \Box f \cdot \psi + e^{-F} \dot{U}(\phi), \end{aligned} \tag{2.16}$$

where we again applied (2.7). Since (2.3) and (2.9) imply

$$\begin{aligned} f(F')^2 + fF'' + F' \Box f &= f(F')^2 + (fF'' + F') + \frac{n-1}{2} F' \\ &= f(F')^2 - G_F + \frac{n-1}{2} F', \end{aligned} \tag{2.17}$$

then (2.15) follows from applying (2.17) to (2.16). $\square$

The first step in proving Proposition 2.5 is to expand $\mathcal{L}_U \psi S_* \psi$:

**Lemma 2.9.** *The following identity holds,*

$$\mathcal{L}_U \psi S_* \psi = 2F' \cdot |S_* \psi|^2 + (fF' G_F + H_F) \cdot \psi^2 + \mathcal{B}_U + \nabla^\beta P^F_\beta, \tag{2.18}$$

*Proof.* From the stress-energy tensor (2.13), we can compute,

$$\begin{aligned}\nabla^\beta(\mathcal{Q}_{\alpha\beta}\nabla^\alpha f) &= \nabla^\beta \mathcal{Q}_{\alpha\beta}\nabla^\alpha f + \mathcal{Q}_{\alpha\beta}\nabla^{\alpha\beta} f \\ &= \Box\psi S\psi + \nabla^2_{\alpha\beta} f \cdot \nabla^\alpha\psi\nabla^\beta\psi - \frac{1}{2}\Box f \cdot \nabla^\beta\psi\nabla_\beta\psi, \\ \nabla^\beta(\psi\nabla_\beta\psi) &= \psi\Box\psi + \nabla^\beta\psi\nabla_\beta\psi.\end{aligned} \tag{2.19}$$

Summing the equations in (2.19) and recalling (2.8) and (2.9), we obtain

$$\nabla^\beta\left(\mathcal{Q}_{\alpha\beta}\nabla^\alpha f + \frac{n-1}{4}\cdot\psi\nabla_\beta\psi\right) = \Box\psi S_*\psi. \tag{2.20}$$

Multiplying (2.15) by $S_*\psi$ and applying (2.20) results in the identity

$$\begin{aligned}\mathcal{L}_U\psi S_*\psi &= 2F'\cdot|S_*\psi|^2 + [f(F')^2 - G_F]\cdot\psi S_*\psi + e^{-F}\dot{U}(\phi)S_*\psi \\ &\quad + \nabla^\beta\left(\mathcal{Q}_{\alpha\beta}\nabla^\alpha f + \frac{n-1}{4}\cdot\psi\nabla_\beta\psi\right).\end{aligned} \tag{2.21}$$

Next, letting $\mathcal{A} = f(F')^2 - G_F$, the product rule and (2.9) imply

$$\begin{aligned}\mathcal{A}\cdot\psi S_*\psi &= \frac{1}{2}\mathcal{A}\cdot\nabla^\beta f\nabla_\beta(\psi^2) + \frac{n-1}{4}\mathcal{A}\cdot\psi^2 \\ &= \frac{1}{2}\nabla^\beta(\mathcal{A}\nabla_\beta f\cdot\psi^2) - \frac{1}{2}\nabla^\beta f\nabla_\beta\mathcal{A}\cdot\psi^2 - \frac{1}{2}\mathcal{A}\cdot\psi^2 \\ &= \frac{1}{2}\nabla^\beta(\mathcal{A}\nabla_\beta f\cdot\psi^2) - \frac{1}{2}(f\mathcal{A})'\cdot\psi^2 \\ &= \frac{1}{2}\nabla^\beta(\mathcal{A}\nabla_\beta f\cdot\psi^2) + (fF'G_F + H_F)\cdot\psi^2.\end{aligned} \tag{2.22}$$

Moreover, recalling (2.14), we can write

$$e^{-F}\dot{U}(\phi)S_*\psi = e^{-2F}\dot{U}(\phi)S\phi + e^{-2F}\left(\frac{n-1}{4} - fF'\right)\dot{U}(\phi)\phi. \tag{2.23}$$

From the product and chain rules, (2.7), and (2.9), we see that

$$\begin{aligned}e^{-2F}\cdot\dot{U}(\phi)S\phi &= \nabla^\beta[e^{-2F}\nabla_\beta f\cdot U(\phi)] - e^{-2F}\cdot SU(\phi) \\ &\quad - \nabla^\beta(e^{-2F}\nabla_\beta f)\cdot U(\phi) \\ &= \nabla^\beta[e^{-2F}\nabla_\beta f\cdot U(\phi)] - e^{-2F}\cdot SU(\phi) \\ &\quad - 2e^{-2F}\left(fF' - \frac{n+1}{4}\right)\cdot U(\phi).\end{aligned} \tag{2.24}$$

Therefore, from (2.23) and (2.24), it follows that

$$e^{-F}\dot{U}(\phi)S_*\psi = \nabla^\beta[e^{-2F}\nabla_\beta f\cdot U(\phi)] + \mathcal{B}^F_U. \tag{2.25}$$

Combining (2.21) with (2.22) and (2.25) yields

$$\begin{aligned}\mathcal{L}_U\psi S_*\psi &= \Box\psi S_*\psi + 2F'\cdot|S_w\psi|^2 + (fF'G_F + H_F)\cdot\psi^2 + \mathcal{B}^F_U \\ &\quad + \nabla^\beta\left[e^{-2F}\nabla_\beta f\cdot U(\phi) + \frac{1}{2}\mathcal{A}\nabla_\beta f\cdot\psi^2\right] \\ &\quad + \nabla^\beta\left(\mathcal{Q}_{\alpha\beta}\nabla^\alpha f + \frac{n-1}{4}\cdot\psi\nabla_\beta\psi\right)\end{aligned} \tag{2.26}$$

Thus, to prove (2.18), it remains only to show that

$$P^F_\beta = \mathcal{Q}_{\alpha\beta}\nabla^\alpha f + \frac{n-1}{4}\cdot\psi\nabla_\beta\psi + \frac{1}{2}\mathcal{A}\nabla_\beta f\cdot\psi^2 + e^{-2F}\nabla_\beta f\cdot U(\phi). \tag{2.27}$$

Note we obtain from (2.7), (2.13), and (2.14) that

$$\begin{aligned}\mathcal{Q}_{\alpha\beta}\nabla^\alpha f &= e^{-2F}\nabla^\alpha f(\nabla_\alpha\phi - F'\nabla_\alpha f\cdot\phi)(\nabla_\beta\phi - F'\nabla_\beta f\cdot\phi)\\ &\quad - \frac{1}{2}e^{-2F}\nabla_\beta f(\nabla^\mu\phi - F'\nabla^\mu f\cdot\phi)(\nabla_\mu\phi - F'\nabla_\mu f\cdot\phi)\\ &= e^{-2F}\left(S\phi\nabla_\beta\phi - \frac{1}{2}\nabla_\beta f\cdot\nabla^\mu\phi\nabla_\mu\phi\right) - e^{-2F}fF'\cdot\phi\nabla_\beta\phi\\ &\quad + \frac{1}{2}e^{-2F}f\nabla_\beta f(F')^2\cdot\phi^2.\end{aligned} \tag{2.28}$$

Using (2.14), we also see that

$$\frac{n-1}{4}\cdot\psi\nabla_\beta\psi = \frac{n-1}{4}e^{-2F}(\phi\nabla_\beta\phi - F'\nabla_\beta f\cdot\phi^2). \tag{2.29}$$

Since the definition of $\mathcal{A}$ yields

$$\frac{1}{2}\mathcal{A}\nabla_\beta f\cdot\psi^2 = \frac{1}{2}e^{-2F}[f(F')^2 - G_F]\nabla_\beta f\cdot\phi^2. \tag{2.30}$$

then combining (2.28)-(2.30) yields (2.27) and completes the proof. □

**Lemma 2.10.** *The following pointwise inequality holds,*

$$\frac{1}{8}|F'|^{-1}|\mathcal{L}_U\psi|^2 \geq (f|F'|G_F - H_F)\cdot\psi^2 - \mathcal{B}^F_U - \nabla^\beta P^F_\beta. \tag{2.31}$$

*Proof.* From (2.18), we have

$$-\mathcal{L}\psi S_*\psi = 2|F'||S_*\psi|^2 + (f|F'|G_F - H_F)\psi^2 - \mathcal{B}^F_U - \nabla^\beta P^F_\beta, \tag{2.32}$$

The inequality (2.31) follows immediately from (2.32) and the basic inequality

$$-\mathcal{L}\psi S_*\psi \leq \frac{1}{8}|F'|^{-1}|\mathcal{L}\psi|^2 + 2|F'||S_*\psi|^2.$$

□

To complete the proof of Proposition 2.5, we integrate (2.31) over $\Omega$ and apply the divergence theorem, (2.10), to the last term on the right-hand side of (2.31).

2.2. **The Linear Estimate.** We now derive Carleman estimates for the wave operator $\Box$ (with no potential). In terms of the terminology presented in Proposition 2.5, we wish to consider the situation in which $U \equiv 0$, so that we have no positive bulk contribution from $U$, i.e., $\mathcal{B}^F_U \equiv 0$.

Ideally, the reparametrization of $f$ we would like to take is $F = -a\log f$ (corresponding to power law decay for the wave at infinity), where $a > 0$. However, for this $F$, we see that the quantities $G_F$ and $H_F$, defined in (2.3), vanish identically, so that (2.2) produces no positive bulk terms at all. Thus, we must add correction terms to the above $F$ in order to generate the desired positive bulk.

Furthermore, to ensure that these corrections remain everywhere lower order, we must construct separate reparametrizations for regions with $f$ small ($f < 1$) and with $f$ large ($f > 1$). We must also ensure that these two reparametrizations match at the boundary $f = 1$. These considerations motivate the definitions below:

**Definition 2.11.** *Fix constants $a, b, p \in \mathbb{R}$ satisfying the following conditions:*

$$a > 0, \qquad 0 < p < 2a, \qquad 0 \leq b < \frac{1}{4} \min(2a - p, 4p). \tag{2.33}$$

**Definition 2.12.** *With $a, b, p$ as in* (2.33)*, we define the reparametrizations*

$$F_\pm := -(a \pm b) \log f - \frac{b}{p} f^{\mp p}. \tag{2.34}$$

In particular, $F_-$ will be our desired reparametrization on $\{f < 1\}$, while $F_+$ will be applicable in the opposite region $\{f > 1\}$. By applying Proposition 2.5 with $F_\pm$ and $U \equiv 0$, we will derive the following inequalities:

**Theorem 2.13.** *Let $\phi \in \mathcal{C}^2(\mathcal{D})$, and fix $a, b, p \in \mathbb{R}$ satisfying* (2.33)*. Let $\Omega \subseteq \mathcal{D}$ be an admissible region, and partition $\Omega$ as*

$$\Omega_l := \{Q \in \Omega \mid f(Q) < 1\}, \qquad \Omega_h := \{Q \in \Omega \mid f(Q) > 1\}.$$

*Then, there exist constants $C, K > 0$ such that:*

$$Cbp^2 \int_{\Omega_l} f^{2(a-b)} f^{p-1} \phi^2 \leq K a^{-1} \int_{\Omega_l} f^{2(a-b)} \cdot f |\Box \phi|^2 + \int_{\partial \Omega_l} P^-_\beta \mathcal{N}^\beta, \tag{2.35}$$

$$Cbp^2 \int_{\Omega_h} f^{2(a+b)} f^{-p-1} \phi^2 \leq K a^{-1} \int_{\Omega_h} f^{2(a+b)} \cdot f |\Box \phi|^2 + \int_{\partial \Omega_h} P^+_\beta \mathcal{N}^\beta, \tag{2.36}$$

*where $\mathcal{N}$ denotes the oriented unit normals of $\partial \Omega_l$ and $\partial \Omega_h$, and where*

$$\begin{aligned} P^\pm_\beta &:= e^{-2F_\pm} \left( \nabla^\alpha f \cdot \nabla_\alpha \phi \nabla_\beta \phi - \frac{1}{2} \nabla_\beta f \cdot \nabla^\mu \phi \nabla_\mu \phi \right) \\ &\quad + e^{-2F_\pm} \left( \frac{n-1}{4} - f F_\pm' \right) \cdot \phi \nabla_\beta \phi \\ &\quad + e^{-2F_\pm} \left[ \left( f F_\pm' - \frac{n-1}{4} \right) F_\pm' - \frac{1}{2} bp f^{\mp p - 1} \right] \nabla_\beta f \cdot \phi^2. \end{aligned} \tag{2.37}$$

*Furthermore, on the middle boundary $\{f = 1\}$, we have that*

$$P^-|_{f=1} = P^+|_{f=1}. \tag{2.38}$$

2.2.1. *Special Reparametrizations.* We begin with some elementary computations.

**Proposition 2.14.** *The following inequalities hold:*

$$b < \frac{a}{2}, \qquad a \pm b \simeq a, \qquad a - b - \frac{1}{2} p > b. \tag{2.39}$$

*Proof.* The first inequality follows from (2.33), and the comparison $a \pm b \simeq a$ follows immediately from this. For the remaining inequality, we apply (2.33) twice:

$$a - b - \frac{p}{2} > a - \frac{1}{4}(2a - p) - \frac{p}{2} = \frac{1}{2} a - \frac{1}{4} p > b. \qquad \square$$

**Proposition 2.15.** *The following identities hold for $F_\pm$:*

$$F_\pm' = -(a \pm b) f^{-1} \pm b f^{\mp p - 1}, \tag{2.40}$$

*Furthermore, recalling the notations in* (2.3)*, we have that*

$$G_{F_\pm} = bp f^{\mp p - 1}, \qquad H_{F_\pm} = \mp \frac{1}{2} bp^2 f^{\mp p - 1}. \tag{2.41}$$

*In particular, on the level set $\mathcal{F}_1 = \{f = 1\}$, we have*

$$F_+|_{\mathcal{F}_1} = F_-|_{\mathcal{F}_1}, \qquad F_+'|_{\mathcal{F}_1} = F_-'|_{\mathcal{F}_1}, \qquad G_{F_+}|_{\mathcal{F}_1} = G_{F_-}|_{\mathcal{F}_1}. \tag{2.42}$$

*Proof.* These are direct computations. □

**Proposition 2.16.** *The following comparisons hold:*

- *If* $0 < f \leq 1$*, then*

$$f^{a-b} < e^{-F_-} \leq e f^{a-b}, \qquad - a f^{-1} \leq F_-' < -(a-b) f^{-1}. \tag{2.43}$$

- *If* $1 \leq f < \infty$*, then*

$$f^{a+b} < e^{-F_+} \leq e f^{a+b}, \qquad - (a+b) f^{-1} < F_+' \leq -a f^{-1}. \tag{2.44}$$

*In particular,* (2.43) *implies that* $F_-$ *is inward-directed whenever* $f < 1$*, while* (2.44) *implies* $F_+$ *is inward-directed whenever* $f > 1$*.*

*Proof.* The comparisons (2.43) and (2.44) follow immediately from (2.34), (2.40), and the trivial inequality $bp^{-1} < 1$, which is a consequence of (2.33). The remaining monotonicity properties follow from (2.39), (2.43), and (2.44). □

**Proposition 2.17.** *The following inequalities hold:*

- *If* $0 < f < 1$*, then*

$$f |F_-'| G_{F_-} - H_{F_-} > b^2 p f^{p-1} > 0. \tag{2.45}$$

- *If* $1 < f < \infty$*, then*

$$f |F_+'| G_{F_+} - H_{F_+} > b^2 p f^{-p-1} > 0. \tag{2.46}$$

*Proof.* First, in the case $0 < f < 1$, we have

$$\begin{aligned} f |F_-'| G_{F_-} - H_{F_-} &= (a - b + b f^p) b p f^{p-1} - \frac{1}{2} b p^2 f^{p-1} \\ &= b p f^{p-1} \left( a - b - \frac{1}{2} p + b f^p \right) \\ &\geq bp \left( a - b - \frac{1}{2} p \right) f^{p-1}. \end{aligned} \tag{2.47}$$

Similarly, when $1 < f < \infty$, we have

$$\begin{aligned} f |F_+'| G_{F_+} - H_{F_+} &= (a + b - b f^{-p}) b p f^{-p-1} + \frac{1}{2} b p^2 f^{-p-1} \\ &\geq bp \left( a + \frac{1}{2} p \right) f^{-p-1}. \end{aligned} \tag{2.48}$$

The desired inequalities now follow by applying (2.39) to (2.47) and (2.48). □

2.2.2. *Proof of Theorem 2.13.* First, for (2.35), we apply Theorem 2.5 with $F = F_-$ and $U \equiv 0$. Combining this with (2.39), (2.43), and (2.45), we obtain the inequality

$$C b^2 p \int_\Omega f^{2(a-b)} f^{p-1} \phi^2 \leq K a^{-1} \int_\Omega f^{2(a-b)} f |\Box \phi|^2 + \int_{\partial \Omega} P^{F_-}_\beta \mathcal{N}^\beta, \tag{2.49}$$

where $C$ and $K$ are constants, and where $\mathcal{N}$, $P^{F_-}$ are as defined in Theorem 2.5. Since $U \equiv 0$, then $P^{F_-}$ is precisely the one-form $P^-$ in (2.37), proving (2.35).

Similarly, for (2.36), we apply Theorem 2.5 with $F = F_+$ and $U \equiv 0$, and we combine the result with (2.39), (2.44), and (2.46), which yields

$$C b^2 p \int_\Omega f^{2(a+b)} f^{-p-1} \phi^2 \leq K a^{-1} \int_\Omega f^{2(a+b)} f |\Box \phi|^2 + \int_{\partial \Omega} P^{F_+}_\beta \mathcal{N}^\beta, \tag{2.50}$$

Since $P^{F_+}$ is precisely $P^+$, we obtain (2.36).

Finally, (2.38) is an immediate consequence of (2.37) and (2.42)

2.3. **The Nonlinear Estimate.** We next discuss Carleman estimates for nonlinear wave equations, in particular those found in Theorems 1.5 and 1.6. With respect to the terminology within Proposition 2.5, we consider $U \in \mathcal{C}^1(\mathcal{D} \times \mathbb{R})$ of the form

$$U(Q, \phi) = \pm \frac{1}{p+1} V(Q) \cdot |\phi|^{p+1}, \qquad p \geq 1, \tag{2.51}$$

where $V \in \mathcal{C}^1(\mathcal{D})$ is strictly positive. From Definition 2.4, this corresponds to

$$\Box_U \phi = \Box \phi \pm V \cdot |\phi|^{p-1} \phi, \qquad p \geq 1. \tag{2.52}$$

Since we will be expecting positive bulk terms arising solely from $U$ (that is, $-\mathcal{B}_U^F > 0$ in (2.2)), we no longer require the correction terms used throughout Section 2.2 for our reparametrizations of $f$. In other words, we can simply use

$$F_0 = -a \log f, \qquad a > 0. \tag{2.53}$$

In particular, we need not consider the regions $\{f > 1\}$ and $\{f < 1\}$ separately. This makes some aspects of the analysis much simpler compared to Theorem 2.13.

The Carleman estimate we will derive is the following:

**Theorem 2.18.** *Let $\phi \in \mathcal{C}^2(\mathcal{D})$, and let $\Omega \subseteq \mathcal{D}$ be an admissible region. Furthermore, let $p \geq 1$, and let $V \in \mathcal{C}^1(\mathcal{D})$ be strictly positive. Then,*

$$\pm \frac{1}{p+1} \int_\Omega f^{2a} \cdot V \Gamma_V \cdot |\phi|^{p+1} \leq \frac{1}{8a} \int_\Omega f^{2a} f \cdot |\Box_V^\pm \phi|^2 + \int_{\partial\Omega} P_\beta^{\pm V} \mathcal{N}^\beta, \tag{2.54}$$

*where $\mathcal{N}$ is the oriented unit normal to $\partial\Omega$, and where:*

$$\begin{aligned} \Box_V^\pm \phi &:= \Box \phi \pm V |\phi|^{p-1} \phi, \\ \Gamma_V &:= \nabla^\alpha f \nabla_\alpha (\log V) - \frac{n-1+4a}{4} \left( p - 1 - \frac{4}{n-1+4a} \right), \\ P_\beta^{\pm V} &:= f^{2a} \left( \nabla^\alpha f \cdot \nabla_\alpha \phi \nabla_\beta \phi - \frac{1}{2} \nabla_\beta f \cdot \nabla^\mu \phi \nabla_\mu \phi \right) \\ &\qquad \pm \frac{1}{p+1} f^{2a} \nabla_\beta f \cdot V |\phi|^{p+1} + \left( \frac{n-1}{4} + a \right) f^{2a} \cdot \phi \nabla_\beta \phi \\ &\qquad + a \left( \frac{n-1}{4} + a \right) f^{2a} f^{-1} \nabla_\beta f \cdot \phi^2. \end{aligned} \tag{2.55}$$

The remainder of this section is dedicated to the proof of Theorem 2.18.

2.3.1. *Positive Bulk Conditions.* The main new task is to examine the bulk term $\mathcal{B}_U^{F_0}$ (see (2.4)) arising from the $U$ defined in (2.51). From a direct computation using (2.4) and (2.51), we obtain the following:

**Proposition 2.19.** *Let $U$ and $F_0$ be as in* (2.51) *and* (2.53)*. Then,*

$$-\mathcal{B}_U^{F_0} = \pm \frac{1}{p+1} f^{2a} V \cdot \Gamma_V \cdot |\phi|^{p+1}, \tag{2.56}$$

*where $\mathcal{B}_U^{F_0}$ is as defined in* (2.4)*, and $\Gamma_V$ is as in* (2.55)*.*

*Proof.* For an arbitrary reparametrization, we compute, using (2.4) and (2.51),

$$\begin{aligned} \mathcal{B}^F_U &= \pm e^{-2F} \left( \frac{n-1}{4} - fF' \right) V \cdot |\phi|^{p+1} \mp \frac{1}{p+1} e^{-2F} \nabla^\alpha f \nabla_\alpha V \cdot |\phi|^{p+1} \\ &\quad \mp \frac{2}{p+1} e^{-2F} \left( \frac{n+1}{4} - fF' \right) V \cdot |\phi|^{p+1} \\ &= \mp \frac{1}{p+1} e^{-2F} [\nabla^\alpha f \nabla_\alpha V - p^* V + (p-1) fF'V] \cdot |\phi|^{p+1}, \end{aligned} \tag{2.57}$$

where

$$p^* = \frac{(p+1)(n-1)}{4} - \frac{n+1}{2}.$$

Substituting $F_0$ for $F$, and noting that

$$fF_0' \equiv -a, \qquad e^{-2F} = f^{2a},$$

we immediately obtain (2.56). □

As a result, $-\mathcal{B}^{F_0}_U$ is strictly positive $\mathcal{D}$ if and only if $\pm\Gamma_V > 0$.

**Remark.** *The computations in Proposition 2.19 can readily be generalized. For example, one can consider wave operators of the form*

$$U(Q, \phi) = \pm V(Q) W(\phi), \qquad \Box_U \phi = \Box\phi \pm V \cdot \dot{W}(\phi), \tag{2.58}$$

*where we also assume $V(Q) \cdot W(\phi) > 0$ for all $(Q, \phi)$. (These correspond to further generalizations of focusing and defocusing wave operators.) From analogous calculations, one sees that $-\mathcal{B}^{F_0}_U$ is everywhere positive if*

- *There is some $p \geq 1$ such that $\pm\Gamma_V > 0$.*
- *$W(\phi)$ grows at most as quickly as $|\phi|^{p+1}$ when $VW$ is positive.*
- *$W(\phi)$ grows at least as quickly as $|\phi|^{p+1}$ when $VW$ is negative.*

*Such statements can be even further extended to more general $U$, but precise formulations of these statements tend to be more complicated.*

2.3.2. *Proof of Theorem 2.18.* This follows immediately by applying (2.2)—with $F = F_0 = -a \log f$ and $U$ as in (2.51)—and then by expanding $\mathcal{B}^{F_0}_U$ using (2.56).

## 3. Proofs of the Main Results

The goal of this section is to prove the global uniqueness results—Theorems 1.1, 1.5, and 1.6—from Section 1.1. The main steps will be to apply the Carleman estimates from the preceding section: Theorems 2.13 for the proof of Theorem 1.1, and Theorem 2.18 for the proofs of Theorems 1.5 and 1.6.

Note first of all that the weight $(1 + |u|)(1 + |v|)$ can be written as

$$(1 + |u|)(1 + |v|) = (1 + r + f). \tag{3.1}$$

Thus, the decay conditions (1.11) can be more conveniently expressed as

$$\begin{aligned} \sup_{\mathcal{D}} \left[ (1 + r + f)^{\frac{n-1+\delta}{2}} (|u \cdot \partial_u \phi| + |v \cdot \partial_v \phi|) \right] &< \infty, \\ \sup_{\mathcal{D} \cap \{f < 1\}} \left[ (1 + r)^{\frac{n-1+\delta}{2}} f^{\frac{1}{2}} |\nabla\!\!\!/ \phi| \right] &< \infty, \\ \sup_{\mathcal{D}} \left[ (1 + r + f)^{\frac{n-1+\delta}{2}} |\phi| \right] &< \infty, \end{aligned} \tag{3.2}$$

while the special decay condition (1.19) is equivalent to

$$\sup_{\mathcal{D}\cap\{f>1\}} \left[(1+r+f)^{\frac{n-1+\delta}{p+1}} f^{\frac{1}{p+1}} V^{\frac{1}{p+1}} |\phi|\right] < \infty. \tag{3.3}$$

From now on, we will refer to (3.2) and (3.3) as our decay assumptions.

3.1. **Special Domains.** The first preliminary step is to define the admissible regions on which we apply our Carleman estimates. A natural choice for this $\Omega$ would be domains with level sets of $f$ as its boundary. We denote these level sets by

$$\mathcal{F}_\omega := \{Q \in \mathcal{D} \mid f(Q) = \omega\}. \tag{3.4}$$

Observe that the $\mathcal{F}_\omega$'s, for all $0 < \omega < \infty$, form a family of timelike hyperboloids terminating at the corners of $\mathcal{D}$ on future and past null infinity. [9]

The $\mathcal{F}_\omega$'s are useful here since they characterize the boundary of $\mathcal{D}$ in the limit. Indeed, in the Penrose-compactified sense, $\mathcal{F}_\omega$ tends toward the null cone about the origin as $\omega \searrow 0$, and $\mathcal{F}_\omega$ tends toward the outer half of null infinity as $\omega \nearrow \infty$.

However, one defect in the above is that the region between two $\mathcal{F}_\omega$'s fails to be bounded. As a result, we define an additional function

$$h \in \mathcal{C}^\infty(\mathcal{D}), \qquad h := -\frac{v}{u} = \frac{r+t}{r-t}, \tag{3.5}$$

whose level sets we denote by

$$\mathcal{H}^\tau := \{Q \in \mathcal{D} \mid h(Q) = \tau\}. \tag{3.6}$$

The $\mathcal{H}^\tau$'s, for $0 < \tau < \infty$, form a family of spacelike cones terminating at the origin and at spacelike infinity. Moreover, in the Penrose-compactified picture:

- As $\tau \nearrow \infty$, the $\mathcal{H}^\tau$'s tend toward both the future null cone about the origin and the outer half of future null infinity.
- As $\tau \searrow 0$, the $\mathcal{H}^\tau$'s tend toward both the past null cone about the origin and the outer half of past null infinity.

The regions we wish to consider are those bounded by level sets of $f$ and $h$. More specifically, given $0 < \rho < \omega < \infty$ and $0 < \sigma < \tau < \infty$, we define

$$\mathcal{D}^{\sigma,\tau}_{\rho,\omega} := \{Q \in \mathcal{D} \mid \rho < f(Q) < \omega,\, \sigma < h(Q) < \tau\}, \tag{3.7}$$

We also define corresponding cutoffs to the $\mathcal{F}_\omega$'s and $\mathcal{H}^\tau$'s:

$$\mathcal{F}^{\sigma,\tau}_\omega := \{Q \in \mathcal{F}_\omega \mid \sigma < h(Q) < \tau\}, \qquad \mathcal{H}^\tau_{\rho,\omega} := \{Q \in \mathcal{H}^\tau \mid \rho < f(Q) < \omega\}. \tag{3.8}$$

3.1.1. *Basic Properties.* We begin by listing some properties of $f$ and $h$ that will be needed in upcoming computations. First, the derivative of $h$ satisfy the following:

**Lemma 3.1.** *The following identities hold:*

$$\partial_u h = \frac{v}{u^2}, \qquad \partial_v h = -\frac{1}{u}. \tag{3.9}$$

*As a result,*

$$\nabla^\sharp h = \frac{1}{2} u^{-2} (u \cdot \partial_u - v \cdot \partial_v), \qquad \nabla^\alpha h \nabla_\alpha h = -u^{-4} f, \qquad \nabla^\alpha h \nabla_\alpha f = 0. \tag{3.10}$$

[9] See Figure 1.

In particular, observe that (2.7) implies the $\mathcal{F}_\omega$'s are timelike, while (3.10) implies the $\mathcal{H}^\tau$'s are spacelike. Furthermore, the last identity in (3.10) implies that the $\mathcal{F}_\omega$'s and $\mathcal{H}^\tau$'s are everywhere orthogonal to each other.

Next, observe the region $\mathcal{D}^{\sigma,\tau}_{\rho,\omega}$ has piecewise smooth boundary, with

$$\partial\mathcal{D}^{\sigma,\tau}_{\rho,\omega} = \mathcal{F}^{\sigma,\tau}_{\omega} \cup \mathcal{F}^{\sigma,\tau}_{\rho} \cup \mathcal{H}^{\tau}_{\rho,\omega} \cup \mathcal{H}^{\sigma}_{\rho,\omega}, \tag{3.11}$$

hence it is indeed an admissible region. Also, from (2.7) and (3.10), we see that:

- On $\mathcal{F}^{\sigma,\tau}_{\omega}$ and $\mathcal{F}^{\sigma,\tau}_{\rho}$, the outer unit normals with respect to $\mathcal{D}^{\sigma,\tau}_{\rho,\omega}$ are

$$\mathcal{N}(\mathcal{F}^{\sigma,\tau}_{\omega}) = N := f^{-\frac{1}{2}}\nabla^\sharp f, \qquad \mathcal{N}(\mathcal{F}^{\sigma,\tau}_{\rho}) = -N = -f^{-\frac{1}{2}}\nabla^\sharp f. \tag{3.12}$$

- On $\mathcal{H}^{\tau}_{\rho,\omega}$ and $\mathcal{H}^{\sigma}_{\rho,\omega}$, the inner unit normals with respect to $\mathcal{D}^{\sigma,\tau}_{\rho,\omega}$ are

$$\mathcal{N}(\mathcal{H}^{\tau}_{\rho,\omega}) = -T := u^2 f^{-\frac{1}{2}}\nabla^\sharp h, \qquad \mathcal{N}(\mathcal{H}^{\sigma}_{\rho,\omega}) = T = -u^2 f^{-\frac{1}{2}}\nabla^\sharp h. \tag{3.13}$$

In view of the above, we obtain the following:

**Lemma 3.2.** *If $P$ is a continuous 1-form on $\mathcal{D}$, and if $\mathcal{N}$ is the oriented unit normal for $\partial\mathcal{D}^{\sigma,\tau}_{\rho,\omega}$, then the following identity holds:*

$$\begin{aligned}\int_{\partial\mathcal{D}^{\sigma,\tau}_{\rho,\omega}} P_\beta \mathcal{N}^\beta &= \int_{\mathcal{F}^{\sigma,\tau}_{\omega}} f^{-\frac{1}{2}} P_\beta \nabla^\beta f - \int_{\mathcal{F}^{\sigma,\tau}_{\rho}} f^{-\frac{1}{2}} P_\beta \nabla^\beta f \\ &\quad + \int_{\mathcal{H}^{\tau}_{\rho,\omega}} u^2 f^{-\frac{1}{2}} P_\beta \nabla^\beta h - \int_{\mathcal{H}^{\sigma}_{\rho,\omega}} u^2 f^{-\frac{1}{2}} P_\beta \nabla^\beta h.\end{aligned} \tag{3.14}$$

Finally, we note that the level sets of $(f, h)$ are simply the level spheres of $(t, r)$, and the values of these functions can be related as follows:

**Lemma 3.3.** *Given $Q \in \mathcal{D}$, we have that $(f(Q), h(Q)) = (\omega, \tau)$ if and only if*

$$\begin{aligned} v(Q) &= \omega^{\frac{1}{2}}\tau^{\frac{1}{2}}, \qquad u(Q) = -\omega^{\frac{1}{2}}\tau^{-\frac{1}{2}}, \\ r(Q) &= \omega^{\frac{1}{2}}(\tau^{\frac{1}{2}} + \tau^{-\frac{1}{2}}), \qquad t(Q) = \omega^{\frac{1}{2}}(\tau^{\frac{1}{2}} - \tau^{-\frac{1}{2}}).\end{aligned} \tag{3.15}$$

3.1.2. *Boundary Expansions.* In light of Lemma 3.2 and the Carleman estimates from Section 2, we will need to bound integrands of the form $P_\beta \nabla^\beta f$ and $P_\beta \nabla^\beta h$, where $P$ is one of the currents $P^\pm$ (see (2.37)) or $P^{\pm V}$ (see (2.55)).

**Lemma 3.4.** *Let $P^\pm$ be as in (2.37). Then, there exists $K > 0$ such that:*

- *In the region $\{f < 1\}$,*

$$\begin{aligned} -P^-_\beta \nabla^\beta f &\leq K f^{2(a-b)}[f \cdot |\nabla\!\!\!/ \phi|^2 + (n+a)^2 \cdot \phi^2], \\ |u^2 P^-_\beta \nabla^\beta h| &\leq K f^{2(a-b)}[(u \cdot \partial_u \phi)^2 + (v \cdot \partial_v \phi)^2 + (n+a)^2 \cdot \phi^2].\end{aligned} \tag{3.16}$$

- *In the region $\{f > 1\}$,*

$$\begin{aligned} P^+_\beta \nabla^\beta f &\leq K f^{2(a+b)}[(u \cdot \partial_u \phi)^2 + (v \cdot \partial_v \phi)^2 + (n+a)^2 \cdot \phi^2], \\ |u^2 P^+_\beta \nabla^\beta h| &\leq K f^{2(a+b)}[(u \cdot \partial_u \phi)^2 + (v \cdot \partial_v \phi)^2 + (n+a)^2 \cdot \phi^2].\end{aligned} \tag{3.17}$$

*Proof.* Applying (2.6), (2.7), (3.9), and (3.10) to the definition (2.37) (and noting in particular that $\nabla^\sharp f$ and $\nabla^\sharp h$ are everywhere orthogonal), we expand

$$\begin{aligned} P^\pm_\beta \nabla^\beta f &= \frac{1}{4} e^{-2F_\pm}[(u \cdot \partial_u \phi)^2 + (v \cdot \partial_v \phi)^2] - \frac{1}{2} e^{-2F_\pm} f \cdot |\nabla\!\!\!/ \phi|^2 \\ &\quad + \frac{1}{2} e^{-2F_\pm} \left( \frac{n-1}{4} - f F'_\pm \right) \cdot \phi (u \cdot \partial_u \phi + v \cdot \partial_v \phi)\end{aligned} \tag{3.18}$$

$$- e^{-2F_\pm} \left[ \left( \frac{n-1}{4} - f F'_\pm \right) f F'_\pm - \frac{1}{2} bp f^{\mp p} \right] \cdot \phi^2,$$

$$u^2 P^\pm_\beta \nabla^\beta h = \frac{1}{4} e^{-2F_\pm} [(u \cdot \partial_u \phi)^2 - (v \cdot \partial_v \phi)^2] + \frac{1}{2} e^{-2F_\pm} \left( \frac{n-1}{4} - f F'_\pm \right) \cdot \phi (u \cdot \partial_u \phi - v \cdot \partial_v \phi).$$

Next, we note the inequality

$$\left| \frac{1}{2} e^{-2F_\pm} \left( \frac{n-1}{4} - f F'_\pm \right) \cdot \phi (u \cdot \partial_u \phi \pm v \cdot \partial_v \phi) \right| \leq \frac{1}{4} e^{-2F_\pm} [(u \cdot \partial_u \phi)^2 + (v \cdot \partial_v \phi)^2] + \frac{1}{2} e^{-2F_\pm} \left( \frac{n-1}{4} - f F'_\pm \right)^2 \cdot \phi^2. \tag{3.19}$$

Applying (3.19) to each of the identities in (3.18) and then dropping any purely nonpositive terms on the right-hand side, we obtain

$$\begin{aligned} P^\pm_\beta \nabla^\beta f &\leq K e^{-2F_\pm} [(u \cdot \partial_u \phi)^2 + (v \cdot \partial_v \phi)^2] \\ &\quad + K e^{-2F_\pm} [n^2 + (f F'_\pm)^2 + bp f^{\mp p}] \cdot \phi^2, \\ -P^\pm_\beta \nabla^\beta f &\leq K e^{-2F_\pm} f \cdot |\nabla\!\!\!/ \phi|^2 + K e^{-2F_\pm} [n^2 + (f F'_\pm)^2 + bp f^{\mp p}] \cdot \phi^2, \\ |u^2 P^\pm_\beta \nabla^\beta h| &\leq K e^{-2F_\pm} [(u \cdot \partial_u \phi)^2 + (v \cdot \partial_v \phi)^2] + K e^{-2F_\pm} [n^2 + (f F'_\pm)^2] \cdot \phi^2. \end{aligned} \tag{3.20}$$

Now, recall from Propositions 2.14-2.16 that

$$\begin{cases} f^2 (F'_-)^2 + bp f^p \lesssim a^2, \quad e^{-2F_-} \lesssim f^{2(a-b)} & f < 1, \\ f^2 (F'_+)^2 + bp f^{-p} \lesssim a^2, \quad e^{-2F_+} \lesssim f^{2(a+b)} & f > 1. \end{cases} \tag{3.21}$$

Combining (3.20) and (3.21) results in both (3.16) and (3.17). □

**Lemma 3.5.** *Let* $P^{\pm V}$ *be as in* (2.55)*. Then, there exists* $K > 0$ *such that:*

$$\begin{aligned} P^{\pm V}_\beta \nabla^\beta f &\leq K f^{2a} [(u \cdot \partial_u \phi)^2 + (v \cdot \partial_v \phi)^2 + (n+a)^2 \cdot \phi^2] \\ &\quad \pm (p+1)^{-1} f^{2a} f V \cdot |\phi|^{p+1}, \\ -P^{\pm V}_\beta \nabla^\beta f &\leq K f^{2a} [f \cdot |\nabla\!\!\!/ \phi|^2 + (n+a)^2 \cdot \phi^2] \mp (p+1)^{-1} f^{2a} f V \cdot |\phi|^{p+1}, \\ |u^2 P^{\pm V}_\beta \nabla^\beta h| &\leq K f^{2a} [(u \cdot \partial_u \phi)^2 + (v \cdot \partial_v \phi)^2 + (n+a)^2 \cdot \phi^2]. \end{aligned} \tag{3.22}$$

*Proof.* The proof is analogous to that of Lemma 3.4. Applying (2.6), (2.7), (3.9), and (3.10) to (2.55) results in the expansions

$$\begin{aligned} P^{\pm V}_\beta \nabla^\beta f &= \frac{1}{4} f^{2a} [(u \cdot \partial_u \phi)^2 + (v \cdot \partial_v \phi)^2] - \frac{1}{2} f^{2a} f \cdot |\nabla\!\!\!/ \phi|^2 \\ &\quad + \frac{1}{2} \left( \frac{n-1}{4} + a \right) f^{2a} \cdot \phi (u \cdot \partial_u \phi + v \cdot \partial_v \phi) \\ &\quad + a \left( \frac{n-1}{4} + a \right) f^{2a} \cdot \phi^2 \pm \frac{1}{p+1} f^{2a} f V \cdot |\phi|^{p+1}, \\ u^2 P^{\pm V}_\beta \nabla^\beta h &= \frac{1}{4} f^{2a} [(u \cdot \partial_u \phi)^2 - (v \cdot \partial_v \phi)^2] \\ &\quad + \frac{1}{2} \left( \frac{n-1}{4} + a \right) f^{2a} \cdot \phi (u \cdot \partial_u \phi - v \cdot \partial_v \phi). \end{aligned} \tag{3.23}$$

Handling the cross-terms in (3.23) using an analogue of (3.19) yields (3.22). □

3.2. **Boundary Limits.** In order to convert Carleman estimates over the $\mathcal{D}^{\sigma,\tau}_{\rho,\omega}$'s into a unique continuation result, we will need to eliminate the resulting boundary terms. To do this, we must take the limit of the boundary terms toward null infinity and the null cone about the origin, i.e., the boundary of $\mathcal{D}$. More specifically, we wish to let $(\sigma, \tau) \to (0, \infty)$, and then $(\rho, \omega) \to (0, \infty)$.

3.2.1. *Coarea Formulas.* To obtain these necessary limits, we will need to express integrals over the $\mathcal{F}_\omega$'s and $\mathcal{H}^\tau$'s more explicitly. For this, we derive coarea formulas below in order to rewrite these expressions in terms of spherical integrals.

In what follows, we will assume that integrals over $\mathbb{S}^{n-1}$ will always be with respect to the volume form associated with the (unit) round metric $\mathring{\gamma}$.

We can foliate $\mathcal{F}_\omega$ by level sets of $t$, which are $(n-1)$-spheres. In other words,

$$\mathcal{F}_\omega \simeq \mathbb{R} \times \mathbb{S}^{n-1}, \tag{3.24}$$

where the $\mathbb{R}$-component corresponds to the $t$-coordinate, while the $\mathbb{S}^{n-1}$-component is the spherical value (as an element of a level set of $(t, r)$). Furthermore, by (3.15), restricting the correspondence (3.24) to finite cutoffs yields

$$\mathcal{F}^{\sigma,\tau}_\omega \simeq (\omega^{\frac{1}{2}}(\sigma^{\frac{1}{2}} - \sigma^{-\frac{1}{2}}), \omega^{\frac{1}{2}}(\tau^{\frac{1}{2}} - \tau^{-\frac{1}{2}})) \times \mathbb{S}^{n-1}. \tag{3.25}$$

We now wish to split integrals over $\mathcal{F}_\omega$ as in (3.25): as an integral first over $\mathbb{S}^{n-1}$ and then over $t$. For convenience, we define, for any $\Psi \in \mathcal{C}^\infty(\mathcal{D})$, the shorthands

$$\begin{aligned} \int_{h=\sigma}^{h=\tau} \int_{\mathbb{S}^{n-1}} \Psi|_{f=\omega} dt &:= \int_{\omega^{\frac{1}{2}}(\sigma^{\frac{1}{2}}-\sigma^{-\frac{1}{2}})}^{\omega^{\frac{1}{2}}(\tau^{\frac{1}{2}}-\tau^{-\frac{1}{2}})} \left[ \int_{\mathbb{S}^{n-1}} \Psi|_{(f,t)=(\omega,s)} \right] ds, \\ \int_{-\infty}^{\infty} \int_{\mathbb{S}^{n-1}} \Psi|_{f=\omega} dt &:= \int_{-\infty}^{\infty} \left[ \int_{\mathbb{S}^{n-1}} \Psi|_{(f,t)=(\omega,s)} \right] ds, \end{aligned} \tag{3.26}$$

with the second equation defined only when $\Psi$ is sufficiently integrable. Note these are the integrals over $\mathcal{F}_\omega$ and $\mathcal{F}^{\sigma,\tau}_\omega$, respectively, in terms of level spheres of $t$.

**Proposition 3.6.** *For any $\Psi \in \mathcal{C}^\infty(\mathcal{D})$, we have the identity*

$$\int_{\mathcal{F}^{\sigma,\tau}_\omega} \Psi = 2\omega^{\frac{1}{2}} \int_{h=\sigma}^{h=\tau} \int_{\mathbb{S}^{n-1}} \Psi r^{n-2}|_{f=\omega} dt. \tag{3.27}$$

*Proof.* Let $Dt$ denote the gradient of $t$ on $\mathcal{F}_\omega$, with respect to the metric induced by $g$. By definition, $Dt$ is tangent to $\mathcal{F}_\omega$ and normal to the level spheres of $(t, r)$. The vector field $T$, defined in (3.13), satisfies these same properties due to (3.10), so $Dt$ and $T$ point in the same direction. Since $T$ is unit, it follows from (3.13) that

$$|Dt|_g = |g(T, Dt)| = \frac{1}{2}\left(\sqrt{\frac{-u}{v}} + \sqrt{\frac{v}{-u}}\right) = \frac{1}{2} f^{-\frac{1}{2}} r. \tag{3.28}$$

Applying the coarea formula and (3.15) yields

$$\int_{\mathcal{F}^{\sigma,\tau}_\omega} \Psi = \int_{\omega^{\frac{1}{2}}(\sigma^{\frac{1}{2}}-\sigma^{-\frac{1}{2}})}^{\omega^{\frac{1}{2}}(\tau^{\frac{1}{2}}-\tau^{-\frac{1}{2}})} \int_{\mathbb{S}^{n-1}} \Psi |Dt|_g^{-1} r^{n-1}|_{(f,t)=(\omega,s)} ds. \tag{3.29}$$

From (3.28) and (3.29), we obtain (3.27). □

We define analogous notations for level sets of $h$. Any $\mathcal{H}^\tau$ can be foliated as

$$\mathcal{H}^\tau \simeq (0, \infty) \times \mathbb{S}^{n-1}, \tag{3.30}$$

where the first component now represents the $r$-coordinate, while the second is again the spherical value. Moreover, by (3.15), the same correspondence yields

$$\mathcal{H}^{\tau}_{\rho,\omega} \simeq (\omega^{\frac{1}{2}}(\sigma^{\frac{1}{2}} - \sigma^{-\frac{1}{2}}), \omega^{\frac{1}{2}}(\tau^{\frac{1}{2}} - \tau^{-\frac{1}{2}})) \times \mathbb{S}^{n-1}. \tag{3.31}$$

We also wish to split integrals over $\mathcal{H}^{\tau}$ accordingly. Thus, for any $\Psi \in \mathcal{C}^{\infty}(\mathcal{D})$, we define, in a manner analogous to (3.26), the shorthands

$$\begin{aligned} \int_{f=\rho}^{f=\omega} \int_{\mathbb{S}^{n-1}} \Psi|_{h=\tau} dr &:= \int_{\rho^{\frac{1}{2}}(\tau^{\frac{1}{2}}+\tau^{-\frac{1}{2}})}^{\omega^{\frac{1}{2}}(\tau^{\frac{1}{2}}+\tau^{-\frac{1}{2}})} \left[ \int_{\mathbb{S}^{n-1}} \Psi|_{(h,r)=(\tau,s)} \right] ds, \\ \int_{0}^{\infty} \int_{\mathbb{S}^{n-1}} \Psi|_{h=\tau} dr &:= \int_{0}^{\infty} \left[ \int_{\mathbb{S}^{n-1}} \Psi|_{(h,r)=(\tau,s)} \right] ds, \end{aligned} \tag{3.32}$$

where the second equation is defined only when $\Psi$ is sufficiently integrable.

**Proposition 3.7.** *For any $\Psi \in \mathcal{C}^{\infty}(\mathcal{D})$, we have the identity*

$$\int_{\mathcal{H}^{\tau}_{\rho,\omega}} \Psi = 2 \int_{f=\rho}^{f=\omega} \int_{\mathbb{S}^{n-1}} f^{\frac{1}{2}} \Psi r^{n-2}|_{h=\tau} dr. \tag{3.33}$$

*Proof.* The proof is analogous to that of (3.27). Let $Dr$ denote the gradient of $r$ on the level sets of $h$. By (3.10), both $Dr$ and $N$, as defined in (3.12), are tangent to the level sets of $h$ and are normal to the level spheres of $(t, r)$. Thus,

$$|Dr|_g = |g(N, Dr)| = \frac{1}{2} \left( \sqrt{\frac{-u}{v}} + \sqrt{\frac{v}{-u}} \right) = \frac{1}{2} f^{-\frac{1}{2}} r. \tag{3.34}$$

The result now follows from the coarea formula, (3.15), and (3.34). □

3.2.2. *The Conformal Inversion.* While (3.27) provides a formula for integrals over level sets of $f$, it is poorly adapted for the limit $f \nearrow \infty$ toward null infinity. To handle this limit precisely, we make use of the standard conformal inversion of Minkowski spacetime to identify null infinity with the null cone about the origin.

Consider the conformally inverted metric,

$$\bar{g} := f^{-2} g = -4 f^{-2} du dv + f^{-2} r^2 \mathring{\gamma}. \tag{3.35}$$

Furthermore, define the inverted null coordinates

$$\bar{u} := -\frac{1}{v} = f^{-1} u, \qquad \bar{v} := -\frac{1}{u} = f^{-1} v, \tag{3.36}$$

as well as the inverted time and radial parameters

$$\bar{t} := \bar{v} + \bar{u} = f^{-1} t, \qquad \bar{r} := \bar{v} - \bar{u} = f^{-1} r, \tag{3.37}$$

The inverted counterparts of the hyperbolic functions $f$ and $h$ are simply

$$\bar{f} := -\bar{u}\bar{v} = f^{-1}, \qquad \bar{h} := -\frac{\bar{v}}{\bar{u}} = h. \tag{3.38}$$

From (3.35)-(3.37), we see that in terms of the inverted null coordinates,

$$\bar{g} = -4 d\bar{u} d\bar{v} + \bar{r}^2 \mathring{\gamma}. \tag{3.39}$$

In other words, $\bar{g}$, in these new coordinates, is once again the Minkowski metric.

Note that $\mathcal{D}$ has an identical characterization in this inverted setting:

$$\mathcal{D} = \{ Q \in \mathbb{R}^{n+1} \mid \bar{u}(Q) < 0, \, \bar{v}(Q) > 0 \}. \tag{3.40}$$

Moreover, by (3.38), the outer half of null infinity in the physical setting, $f \nearrow \infty$, corresponds to the null cone about the origin in the inverted setting, $\bar{f} \searrow 0$. Thus, this inversion provides a useful tool for discussing behaviors near or at infinity.

We now derive a "dual" coarea formula for level sets of $f$:

**Proposition 3.8.** *For any $\Psi \in \mathcal{C}^\infty(\mathcal{D})$, we have*

$$\int_{\mathcal{F}_\omega^{\sigma,\tau}} \Psi = 2\omega^{n-\frac{1}{2}} \int_{\bar{h}=\sigma}^{\bar{h}=\tau} \int_{\mathbb{S}^{n-1}} \Psi \bar{r}^{n-2}|_{\bar{f}=\omega^{-1}} d\bar{t}. \tag{3.41}$$

*Proof.* Recalling the relations (3.38), and observing that the volume forms induced by $g$ and $\bar{g}$ on $\mathcal{F}_\omega^{\sigma,\tau}$ differ by a factor of $f^n$, we obtain that

$$\int_{\mathcal{F}_\omega^{\sigma,\tau}} \Psi = \omega^n \int_{(\bar{\mathcal{F}}_{1/\omega}^{\sigma,\tau}, \bar{g})} \Psi, \tag{3.42}$$

where right-hand integral is with respect to the $\bar{g}$-volume form. Applying (3.27) to the right-hand side of (3.42) with respect to $\bar{g}$ yields (3.41). □

**Remark.** *In fact, one can obtain corresponding unique continuation results from the null cone by applying the main theorems, such as Theorem 1.1, to the inverted Minkowski spacetime $(\mathbb{R}^{1+n}, \bar{g})$, and then expressing all objects back in terms of $g$.*

3.2.3. *The Boundary Limit Lemmas.* We now apply the preceding coarea formulas to obtain the desired boundary limits. We begin with the level sets of $h$.

**Lemma 3.9.** *Fix $\delta > 0$ and $0 < \rho < \omega$, and suppose $\Psi \in \mathcal{C}^0(\mathcal{D})$ satisfies*

$$\sup_{\mathcal{D}} (r^{n-1+\delta}|\Psi|) < \infty. \tag{3.43}$$

*Then, the following limits hold:*

$$\lim_{\tau \nearrow \infty} \int_{\mathcal{H}_{\rho,\omega}^\tau} |\Psi| = \lim_{\sigma \searrow 0} \int_{\mathcal{H}_{\rho,\omega}^\sigma} |\Psi| = 0. \tag{3.44}$$

*Proof.* First, for $\tau \gg 1$, we have from Lemma 3.3 that

$$t|_{\mathcal{H}_{\rho,\omega}^\tau} \simeq_{\rho,\omega} \tau^{\frac{1}{2}}, \qquad r|_{\mathcal{H}_{\rho,\omega}^\tau} \simeq_{\rho,\omega} \tau^{\frac{1}{2}}. \tag{3.45}$$

Applying (3.33), (3.45), and the boundedness of $f$ on $\mathcal{H}_{\rho,\omega}^\tau$, we see that

$$\begin{aligned} \int_{\mathcal{H}_{\rho,\omega}^\tau} |\Psi| &\lesssim \tau^{-\frac{1+\delta}{2}} \int_{f=\rho}^{f=\omega} \int_{\mathbb{S}^{n-1}} (\tau^{\frac{n-1+\delta}{2}} |\Psi|)|_{h=\tau} dr \\ &\lesssim \tau^{-\frac{\delta}{2}} \sup_{\mathcal{D}} (r^{n-1+\delta}|\Psi|). \end{aligned} \tag{3.46}$$

Letting $\tau \nearrow \infty$ and recalling (3.43) results in the first limit in (3.44).

For the remaining limit, we note from Lemma 3.3 that for $0 < \sigma \ll 1$,

$$-t|_{\mathcal{H}_{\rho,\omega}^\sigma} \simeq_{\rho,\omega} \sigma^{-\frac{1}{2}}, \qquad r|_{\mathcal{H}_{\rho,\omega}^\sigma} \simeq_{\rho,\omega} \sigma^{-\frac{1}{2}}. \tag{3.47}$$

If $\sigma$ is small, then (3.33) and (3.47) yields

$$\begin{aligned} \int_{\mathcal{H}_{\rho,\omega}^\sigma} |\Psi| &\lesssim \sigma^{\frac{1+\delta}{2}} \int_{f=\rho}^{f=\omega} \int_{\mathbb{S}^{n-1}} (\sigma^{-\frac{n-1+\delta}{2}} |\Psi|)|_{h=\tau} dr \\ &\lesssim \sigma^{\frac{\delta}{2}} \sup_{\mathcal{D}} (r^{n-1+\delta}|\Psi|), \end{aligned} \tag{3.48}$$

which vanishes by (3.43) as $\sigma \searrow 0$. □

Next, we consider level sets of $f$, which we split into two statements.

**Lemma 3.10.** *Suppose $\Psi \in \mathcal{C}^0(\mathcal{D})$ satisfies*

$$\sup_{\mathcal{D}}[(1+r)^{n-1+\delta}|\Psi|] < \infty. \tag{3.49}$$

*Then, for any $\alpha > 0$, the following limit holds:*

$$\lim_{\rho \searrow 0} \lim_{(\sigma,\tau)\to(0,\infty)} \int_{\mathcal{F}_\rho^{\sigma,\tau}} f^{-\frac{1}{2}+\alpha}|\Psi| = 0. \tag{3.50}$$

*Proof.* Applying (3.27) and (3.49), we see that

$$\begin{aligned} \int_{\mathcal{F}_\rho^{\sigma,\tau}} f^{-\frac{1}{2}+\alpha}|\Psi| &\lesssim \rho^\alpha \int_{h=\sigma}^{h=\tau} \int_{\mathbb{S}^{n-1}} |\Psi| r^{n-2}|_{f=\omega} dt \\ &\lesssim \rho^\alpha \int_{-\infty}^{\infty} (1+r)^{-1-\delta} dt. \end{aligned} \tag{3.51}$$

Since $|t| < r$ on $\mathcal{D}$, then (3.50) follows by letting $(\sigma,\tau) \to (0,\infty)$ and $\rho \searrow 0$. □

**Lemma 3.11.** *Suppose $\Psi \in \mathcal{C}^0(\mathcal{D})$ satisfies*

$$\sup_{\mathcal{D}}[(r+f)^{n-1+\delta}|\Psi|] < \infty. \tag{3.52}$$

*Then, for any $0 < \beta < \delta$, the following limit holds:*

$$\lim_{\omega \nearrow \infty} \lim_{(\sigma,\tau)\to(0,\infty)} \int_{\mathcal{F}_\omega^{\sigma,\tau}} f^{-\frac{1}{2}+\beta}|\Psi| = 0. \tag{3.53}$$

*Proof.* The main idea is to convert to the inverted setting, in which the estimate becomes analogous to Lemma 3.10. From (3.41), we obtain

$$\begin{aligned} \int_{\mathcal{F}_\omega^{\sigma,\tau}} f^{-\frac{1}{2}+\beta}|\Psi| &\lesssim \omega^{n-1+\beta} \int_{\bar{h}=\sigma}^{\bar{h}=\tau} \int_{\mathbb{S}^{n-1}} |\Psi| \bar{r}^{n-2}|_{\bar{f}=\omega^{-1}} d\bar{t} \\ &\lesssim \omega^{n-1+\beta} \int_{-\infty}^{\infty} (r+f)^{-(n-1-\delta)} \bar{r}^{n-2} d\bar{t}. \end{aligned} \tag{3.54}$$

Recalling now (3.37) and (3.38) yields

$$\begin{aligned} \int_{\mathcal{F}_\omega^{\sigma,\tau}} f^{-\frac{1}{2}+\beta}|\Psi| &\lesssim \omega^{n-1+\beta} \int_{-\infty}^{\infty} (r+f)^{-(n-1+\delta)} \bar{r}^{n-2} d\bar{t} \\ &\lesssim \omega^{n-1+\beta} \int_{-\infty}^{\infty} \omega^{-(n-1+\delta)} (1+\bar{r})^{-(n-1+\delta)} \bar{r}^{n-2} d\bar{t} \\ &\lesssim \omega^{\beta-\delta} \int_{-\infty}^{\infty} (1+\bar{r})^{-1-\delta} d\bar{t}. \end{aligned} \tag{3.55}$$

Since $|\bar{t}| < \bar{r}$, then letting $(\sigma,\tau) \to (0,\infty)$ and $\omega \nearrow \infty$ results in (3.53). □

3.3. **Proof of Theorem 1.1.** Let $p$ be as in the theorem statement, and let

$$a = \frac{\delta}{4} + \frac{p}{4}, \qquad b = \frac{1}{16}\min(\delta - p, 8p). \tag{3.56}$$

Note in particular that $2a$ lies exactly halfway between $p$ and $\delta$, and that $a, b, p$ satisfy the conditions (2.33). In addition, we fix arbitrary

$$0 < \rho < 1 < \omega < \infty, \qquad 0 < \sigma < \tau < \infty.$$

The idea is apply Theorem 2.13 to the domain $\mathcal{D}^{\sigma,\tau}_{\rho,\omega}$. To split into $f<1$ and $f>1$ regions, we partition the above domain into two parts, $\mathcal{D}^{\sigma,\tau}_{\rho,1}$ and $\mathcal{D}^{\sigma,\tau}_{1,\omega}$.

First, applying (2.35) and then (3.14) to $\mathcal{D}^{\sigma,\tau}_{\rho,1}$ yields

$$\begin{aligned}
Cbp^2 \int\limits_{\mathcal{D}^{\sigma,\tau}_{\rho,1}} f^{2(a-b)} f^{p-1}\phi^2 \leq Ka^{-1} \int\limits_{\mathcal{D}^{\sigma,\tau}_{\rho,1}} f^{2(a-b)} f|\Box\phi|^2 + \int\limits_{\mathcal{F}^{\sigma,\tau}_{1}} f^{-\frac{1}{2}} P^-_\beta \nabla^\beta f \\
- \int\limits_{\mathcal{F}^{\sigma,\tau}_{\rho}} f^{-\frac{1}{2}} P^-_\beta \nabla^\beta f + \int\limits_{\mathcal{H}^{\tau}_{\rho,1}} u^2 f^{-\frac{1}{2}} P^-_\beta \nabla^\beta h \\
- \int\limits_{\mathcal{H}^{\sigma}_{\rho,1}} u^2 f^{-\frac{1}{2}} P^-_\beta \nabla^\beta h.
\end{aligned} \tag{3.57}$$

Observe that the assumption (1.10) for $\mathcal{V}$ and (3.56) imply that

$$|\Box\phi|^2 \leq |\mathcal{V}|^2\phi^2 \lesssim \varepsilon^2 \cdot bp^3 \cdot f^{-2+p}\phi^2, \tag{3.58}$$

whenever $f<1$. Applying (3.58) to (3.57) and noting that $p<2a$, it follows that if $\varepsilon$ is sufficiently small, then the first term on the right-hand side of (3.57) can be absorbed into the left-hand side, yielding, for some other $C>0$,

$$\begin{aligned}
Cbp^2 \int\limits_{\mathcal{D}^{\sigma,\tau}_{\rho,1}} f^{2(a-b)} f^{p-1}\phi^2 \leq - \int\limits_{\mathcal{F}^{\sigma,\tau}_{\rho}} f^{-\frac{1}{2}} P^-_\beta \nabla^\beta f + \int\limits_{\mathcal{H}^{\tau}_{\rho,1}} u^2 f^{-\frac{1}{2}} P^-_\beta \nabla^\beta h \\
- \int\limits_{\mathcal{H}^{\sigma}_{\rho,1}} u^2 f^{-\frac{1}{2}} P^-_\beta \nabla^\beta h + \int\limits_{\mathcal{F}^{\sigma,\tau}_{1}} f^{-\frac{1}{2}} P^-_\beta \nabla^\beta f.
\end{aligned} \tag{3.59}$$

In a similar manner, we apply (2.36) and (3.14) to $\mathcal{D}^{\sigma,\tau}_{1,\omega}$. In this region, the assumption (1.10) for $\mathcal{V}$ and (3.56) imply the estimate

$$|\Box\phi|^2 \leq |\mathcal{V}|^2\phi^2 \lesssim \varepsilon^2 \cdot bp^3 \cdot f^{-2-p}\phi^2, \tag{3.60}$$

so that for sufficiently small $\varepsilon$, we have

$$\begin{aligned}
Cbp^2 \int\limits_{\mathcal{D}^{\sigma,\tau}_{1,\omega}} f^{2(a+b)} f^{-p-1}\phi^2 \leq \int\limits_{\mathcal{F}^{\sigma,\tau}_{\omega}} f^{-\frac{1}{2}} P^+_\beta \nabla^\beta f + \int\limits_{\mathcal{H}^{\tau}_{1,\omega}} u^2 f^{-\frac{1}{2}} P^+_\beta \nabla^\beta h \\
- \int\limits_{\mathcal{H}^{\sigma}_{1,\omega}} u^2 f^{-\frac{1}{2}} P^+_\beta \nabla^\beta h - \int\limits_{\mathcal{F}^{\sigma,\tau}_{1}} f^{-\frac{1}{2}} P^+_\beta \nabla^\beta f.
\end{aligned} \tag{3.61}$$

Next, we sum (3.59) and (3.61). By (2.38), the two integrals over $\mathcal{F}^{\sigma,\tau}_{1}$, hence

$$\begin{aligned}
& Cbp^2 \int\limits_{\mathcal{D}^{\sigma,\tau}_{\rho,1}} f^{2(a-b)} f^{p-1}\phi^2 + Cbp^2 \int\limits_{\mathcal{D}^{\sigma,\tau}_{1,\omega}} f^{2(a+b)} f^{-p-1}\phi^2 \\
&\leq \int\limits_{\mathcal{F}^{\sigma,\tau}_{\omega}} f^{-\frac{1}{2}} P^+_\beta \nabla^\beta f - \int\limits_{\mathcal{F}^{\sigma,\tau}_{\rho}} f^{-\frac{1}{2}} P^-_\beta \nabla^\beta f + \int\limits_{\mathcal{H}^{\tau}_{\rho,1}} u^2 f^{-\frac{1}{2}} P^-_\beta \nabla^\beta h \\
&\quad + \int\limits_{\mathcal{H}^{\tau}_{1,\omega}} u^2 f^{-\frac{1}{2}} P^+_\beta \nabla^\beta h - \int\limits_{\mathcal{H}^{\sigma}_{\rho,1}} u^2 f^{-\frac{1}{2}} P^-_\beta \nabla^\beta h - \int\limits_{\mathcal{H}^{\sigma}_{1,\omega}} u^2 f^{-\frac{1}{2}} P^+_\beta \nabla^\beta h \\
&= I_1 + I_2 + J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{3.62}$$

It remains to show that each of the terms on the right vanishes in the limit.

First, for $J_1$, we apply (3.16), along with the fact that $f$ is bounded from both above and below on $\mathcal{H}^\tau_{\rho,1}$, in order to obtain

$$|J_1| \lesssim \int_{\mathcal{H}^\tau_{\rho,1}} [(u \cdot \partial_u \phi)^2 + (v \cdot \partial_v \phi)^2 + \phi^2]. \tag{3.63}$$

An analogous application of (3.17) yields

$$|J_2| \lesssim \int_{\mathcal{H}^\sigma_{\rho,1}} [(u \cdot \partial_u \phi)^2 + (v \cdot \partial_v \phi)^2 + \phi^2]. \tag{3.64}$$

Recalling our assumption (3.2) for $\phi$ and applying Lemma 3.9 yields

$$\lim_{\tau \nearrow \infty} J_1 = \lim_{\sigma \searrow 0} J_2 = 0. \tag{3.65}$$

By the same arguments, we also obtain

$$\lim_{\tau \nearrow \infty} J_3 = \lim_{\sigma \searrow 0} J_4 = 0. \tag{3.66}$$

For the remaining terms $I_1$ and $I_2$, we take limits first as $(\tau, \sigma) \to (\infty, 0)$, and then as $(\omega, \rho) \to (\infty, 0)$. First, using (3.16), we obtain

$$|I_2| \lesssim \int_{\mathcal{F}^{\sigma,\tau}_\rho} f^{-\frac{1}{2}+2(a-b)} (f \cdot |\nabla\!\!\!/ \phi|^2 + \phi^2) \tag{3.67}$$

Since $a - b > 0$, then Lemma 3.10 and the decay assumption (3.2) imply

$$\lim_{\rho \searrow 0} \lim_{(\sigma,\tau) \to (0,\infty)} I_2 = 0. \tag{3.68}$$

A similar application of (3.17) yields

$$|I_1| \lesssim \int_{\mathcal{F}^{\sigma,\tau}_\omega} f^{-\frac{1}{2}+2(a+b)} [(u \cdot \partial_u \phi)^2 + (v \cdot \partial_v \phi)^2 + \phi^2] \tag{3.69}$$

Since $2(a + b) < \delta$, then Lemma 3.11 and (3.2) yield

$$\lim_{\omega \nearrow \infty} \lim_{(\sigma,\tau) \to (0,\infty)} I_1 = 0. \tag{3.70}$$

Finally, combining (3.62) with the limits (3.65), (3.66), (3.68), (3.70) and then applying the monotone convergence theorem, we see that

$$\int_{\mathcal{D}} \alpha \cdot \phi^2 = 0, \tag{3.71}$$

for some strictly positive function $\alpha$ on $\mathcal{D}$. It follows that $\phi$ vanishes everywhere on $\mathcal{D}$, completing the proof of Theorem 1.1.

3.4. **Proofs of Theorems 1.5 and 1.6.** The idea is similar to before, except we apply Theorem 2.18 instead of Theorem 2.13. In particular, we let $\Omega = \mathcal{D}^{\sigma,\tau}_{\rho,\omega}$ (there is no need to partition into two regions in this case), and fix

$$0 < a < \frac{\delta}{2}, \tag{3.72}$$

whose precise value is to be determined later. Let $V$ be as in the statement of Theorem 1.5 or 1.6. Choosing the sign depending on situation ("+" for Theorem 1.5, or "−" for Theorem 1.6), we have that $\Box^\pm_V \phi \equiv 0$.

As a result, applying (2.54), (3.14), and (3.22), we obtain

$$\begin{aligned}
\int_{\mathcal{D}^{\sigma,\tau}_{\rho,\omega}} f^{2a} V(\pm\Gamma_V)|\phi|^{p+1} &\leq L \int_{\mathcal{F}^{\sigma,\tau}_{\omega}} f^{2a} f^{-\frac{1}{2}} [(u \cdot \partial_u \phi)^2 + (v \cdot \partial_v \phi)^2 + \phi^2] \\
&\quad + L \int_{\mathcal{F}^{\sigma,\tau}_{\rho}} f^{2a} f^{-\frac{1}{2}} [f \cdot |\nabla\!\!\!/ \phi|^2 + \phi^2] \\
&\quad + L \int_{\mathcal{H}^{\tau}_{\rho,\omega}} f^{2a} f^{-\frac{1}{2}} [(v \cdot \partial_v \phi)^2 + \phi^2] \\
&\quad + L \int_{\mathcal{H}^{\sigma}_{\rho,\omega}} f^{2a} f^{-\frac{1}{2}} [(u \cdot \partial_u \phi)^2 + \phi^2] \\
&\quad \pm \int_{\mathcal{F}^{\sigma,\tau}_{\omega}} f^{2a} f^{-\frac{1}{2}} (fV \cdot |\phi|^{p+1}) \\
&\quad \mp \int_{\mathcal{F}^{\sigma,\tau}_{\rho}} f^{2a} f^{-\frac{1}{2}} (fV \cdot |\phi|^{p+1}) \\
&= I_1 + I_2 + J_1 + J_2 + Z_1 + Z_2,
\end{aligned} \tag{3.73}$$

for some $L > 0$. Moreover, by the same arguments as in the proof of Theorem 1.1,

$$\lim_{(\rho,\omega)\to(0,\infty)} \lim_{(\sigma,\tau)\to(0,\infty)} (I_1 + I_2 + J_1 + J_2) = 0. \tag{3.74}$$

Suppose first that we are in the defocusing case of Theorem 1.6. Then, $Z_1$ is negative and hence can be discarded, and we need only consider $Z_2$. Since $V$ is uniformly bounded, and since $\phi$ satisfies (3.2), then applying (3.50) yields

$$\lim_{\rho\searrow 0} \lim_{(\sigma,\tau)\to(0,\infty)} |Z_2| \lesssim \lim_{\rho\searrow 0} \lim_{(\sigma,\tau)\to(0,\infty)} \int_{\mathcal{F}^{\sigma,\tau}_{\rho}} f^{2a} f^{\frac{1}{2}} \cdot |\phi|^{p+1} = 0. \tag{3.75}$$

On the other hand, in the focusing case of Theorem 1.5, we can discard $Z_2$ due to sign and consider only $Z_1$. Since $f \nearrow \infty$, the factor $f$ in $Z_1$ now becomes large, hence the extra decay for $\phi$ in (1.19) is invoked to show the vanishing of this limit. Indeed, applying the decay assumptions (3.2), (3.3) along with (3.53), we see that

$$\lim_{\omega\nearrow\infty} \lim_{(\sigma,\tau)\to(0,\infty)} |Z_1| \lesssim \lim_{\omega\nearrow\infty} \lim_{(\sigma,\tau)\to(0,\infty)} \int_{\mathcal{F}^{\sigma,\tau}_{\omega}} f^{2a} f^{\frac{1}{2}} V \cdot |\phi|^{p+1} = 0. \tag{3.76}$$

Consequently, in both cases, we can do away with all the boundary terms:

$$\lim_{(\rho,\omega)\to(0,\infty)} \lim_{(\sigma,\tau)\to(0,\infty)} (I_1 + I_2 + J_1 + J_2 + Z_1 + Z_2) = 0, \tag{3.77}$$

It remains only to examine the left-hand side of (3.73). In the defocusing case, the monotonicity assumption (1.21) and (2.7) imply that

$$\nabla^\sharp f(\log V) < \frac{n-1+4a}{4} \left( p - 1 - \frac{4}{n-1+4a} \right), \tag{3.78}$$

so that $-\Gamma_V$, as defined in (2.55), is strictly positive. Similarly, in the focusing case, we can further shrink $a > 0$ (depending on $\mu$ in (1.18)) to guarantee that

$$\nabla^\sharp f(\log V) > -\frac{n-1+4a}{4} \left( 1 + \frac{4}{n-1+4a} - p \right), \tag{3.79}$$

so that $\Gamma_V$ is strictly positive. Consequently, in both cases, the factor $\pm f^{2a} V \cdot \Gamma_V$ on the left-hand side of (3.73) is strictly positive.

Finally, we combine the above positivity of $\pm\Gamma_V$ with (3.73) and (3.77), and we apply the monotone convergence theorem. This yields

$$\int_{\mathcal{D}} \alpha|\phi|^{p+1} = 0, \tag{3.80}$$

where $\alpha$ is a strictly positive function on $\mathcal{D}$. It follows that $\phi$ vanishes everywhere on $\mathcal{D}$, which completes the proofs of both Theorems 1.5 and 1.6.

3.5. **Optimality via Counterexamples.** Finally, we show that the condition on the $L^\infty$-norm of $\mathcal{V}$ in Theorem 1.1 cannot be removed. More specifically, we show that the small constant $\varepsilon$ in (1.10) cannot be replaced by an arbitrary constant. While this is well-known in the elliptic setting (and hence extends to the current setting), for completeness, we provide an elementary example here.

This is achieved by constructing solutions $\phi$ of a wave equation

$$(\Box + \mathcal{V})\phi = 0,$$

which satisfy all the assumptions of Theorem 1.1 except for the factor $\varepsilon$ in (1.10), which vanish to arbitrarily high finite order, but which are not identically zero.

For this, we construct functions $\psi \in \mathcal{C}^2(\mathbb{R}^n)$ satisfying

$$(\Delta + \mathcal{U})\psi = 0, \tag{3.81}$$

where $\mathcal{U} \in \mathcal{C}^\infty(\mathbb{R}^n)$ is compactly supported, and where

$$|\psi(t,x)| \lesssim (1+|x|)^{-p}, \tag{3.82}$$

where $p > 0$ is fixed but can be made arbitrarily large. Assuming this $\psi$, taking

$$\phi(t,x) = \psi(x), \qquad \mathcal{V}(t,x) = \mathcal{U}(x)$$

results in the desired counterexample, as $\phi$ trivially satisfies (1.11), and $\mathcal{V}$ satisfies (1.10) except for the constant. Consequently, it remains only to construct $\psi$.

3.5.1. *Construction of $\psi$.* Let $P \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ denote a spherical harmonic, satisying

$$\Delta_{\mathbb{S}^{n-1}} P = -aP, \qquad a > 0,$$

and let $\beta \in \mathcal{C}^\infty(0,\infty)$ be an everywhere positive function satisfying

$$\beta(r) := \begin{cases} r^{q_+} & r < 1, \\ r^{q_-} & r > 2, \end{cases} \qquad q_\pm := \frac{-(n-2) \pm \sqrt{(n-2)^2 + 4a}}{2}.$$

Observe in particular that $q_- < 0 < q_+$, and that $|q_\pm|$ can be made arbitrarily large by making $a$ arbitrarily large. Thus, if we define, in polar coordinates, the function

$$\psi(r,\omega) := \beta(r) \cdot P(\omega),$$

then for large enough $a$, this defines a $\mathcal{C}^2$-function on $\mathbb{R}^n$ satisfying (3.82).

Finally, we define

$$\mathcal{U}(r,\omega) := \frac{-\Delta\psi(r,\omega)}{\psi(r,\omega)} = \frac{-\beta''(r) - (n-1)r^{-1} \cdot \beta'(r) + ar^{-2} \cdot \beta(r)}{\beta(r)},$$

which extends to a smooth function on $\mathbb{R}^n \setminus \{0\}$. Furthermore, the chosen exponents $q_\pm$ ensure that $\Delta\psi(r,\omega)$ vanishes whenever $r < 1$ or $r > 2$. As a result, $\mathcal{U}$ is actually smooth on all of $\mathbb{R}^n$ and has compact support, and (3.81) is satisfied.

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Department of Mathematics, University of Toronto, 40 St George Street Rm 6290, Toronto, ON M5S 2E4, Canada
*E-mail address*: alexakis@math.utoronto.ca

Department of Mathematics, South Kensington Campus, Imperial College London, London SW7 2AZ, United Kingdom
*E-mail address*: c.shao@imperial.ac.uk