Singular Ricci solitons and their stability under the Ricci flow.

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Abstract

We introduce certain spherically symmetric singular Ricci solitons and study their stability under the Ricci flow from a dynamical PDE point of view. The solitons in question exist for all dimensions $n+1 \geq 3$, and all have a point singularity where the curvature blows up; their evolution under the Ricci flow is in sharp contrast to the evolution of their smooth counterparts. In particular, the family of diffeomorphisms associated with the Ricci flow "pushes away" from the singularity causing the evolving soliton to open up immediately becoming an incomplete (but non-singular) metric. The main objective of this paper is to study the local-in time stability of this dynamical evolution, under spherically symmetric perturbations of the singular initial metric. We prove a local well-posedness result for the Ricci flow in suitably weighted Sobolev spaces, which in particular implies that the "opening up" of the singularity persists for the perturbations as well.

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1 Introduction

The question of defining solutions of geometric evolution equations with singular initial data is an interesting challenge and has been studied in recent years for a variety of parabolic geometric PDE. For the Ricci flow, a number of solutions have been proposed in various settings. Simon [17] obtained solutions for the Ricci flow for $\mathcal{C}^0$ initial metrics that can be uniformly approximated by smooth metrics with bounded sectional curvature. Koch and Lamm [16] showed existence and uniqueness for the Ricci-DeTurck flow for initial data that are L^∞-close to the Euclidean metric. Angenent, Caputo and Knopf [3] considered initial data of neck-pinch type.[1] They constructed a solution to the flow starting from this singular initial metric, for which the singularity is immediately smoothed out. This can be thought of as a (very weak) notion of surgery in that the method of proof relies on a gluing construction to show the existence of such a solution, but not uniqueness. Cabezas-Rivas and Wilking [5] have obtained solutions of the Ricci flow on open manifolds with nonnegative (and possibly unbounded) complex sectional curvature, using the Cheeger-Gromoll convex exhaustion of such manifolds.

More results have been obtained in the Kähler case and in dimension 2, where the Ricci flow equation reduces to a scalar heat equation; we list a few examples: Chen, Tian and Zhang [8] consider the Kähler-Ricci flow for initial data with $\mathcal{C}^{1,1}$ potentials and construct solutions to the Ricci flow which immediately smooth out. The argument is based on an approximation of the initial potential by smoother ones. Finally, more results have been obtained in dimension 2 (see [15] for a survey): Giesen and Topping [13] (building on earlier work by Topping [19]) have given a construction of Ricci flows on surfaces starting from any (incomplete) initial metric whose curvature is unbounded; these solutions become instantaneously complete and are unique in the *maximally stretched* class that they introduce. More recently yet [14], they constructed examples of immortal solutions of the flow (on surfaces) which start out with a smooth initial metric, then the supremum of the Gauss curvature becomes infinite for some finite amount of time before becoming finite again.

This paper considers a special class of singular initial metrics and produces examples of Ricci flow whose behavior is different from those listed above. Our initial metrics are *close* to certain singular gradient Ricci solitons that we introduce separately in the first part of this paper. The solitons exist in all dimensions $n + 1 \geq 3$. Our main result is that for small enough perturbations of the singular Ricci solitons, the Ricci flow admits a unique solution, up to some time $T > 0$, within a natural class of evolving metrics which stay close (as measured in a certain weighted Sobolev space) to the evolving Ricci

[1] In particular these initial data can form in the evolution of a smooth spherically symmetric initial metric, as demonstrated in [1, 2].

solitons. In other words, we obtain a local well-posedness result for the Ricci flow for initial data with the same singularity profile as our Ricci solitons.

The solitons that we introduce (and, in fact, their perturbations that we consider) all have $SO(n+1,\mathbb{R})$-symmetry. In particular, the soliton metric at the initial time $t=0$ can be written in the form:

$$g_{\mathrm{sol}} = dx^2 + \psi(x)^2 g_{\mathbb{S}^n},$$

where $x \in (0,+\infty)$ for stasy and $x \in (0,\delta), \delta < +\infty$ for non-steady solitons; here $g_{\mathbb{S}^n}$ denotes the canonical metric of the unit n-sphere. In all cases the function $\psi(x)$ is a positive smooth function and moreover $\psi(x) \to 0$ as $x \to 0^+$, with leading order behaviour $\psi \sim x^{\frac{1}{\sqrt{n}}}$. In particular, the (incomplete) metric above can be extended to a complete $\mathcal{C}^0$ (in fact $\mathcal{C}^{\frac{1}{\sqrt{n}}}$) metric at $x=0$, but the extended metric will not be of class $\mathcal{C}^1$. We remark that (in the steady case) our (singular) solitons are complete Riemannian manifolds towards $+\infty$, with an asymptotic profile there that matches the Bryant soliton. For the rest of this introduction we discuss only the steady case.

Our first observation is that the evolution of the singular solitons themselves under the Ricci flow is in sharp contrast with the behavior of their smooth counterparts. As for smooth solitons, there exists an evolution of g_{sol} under the Ricci flow given by a 1-parameter family of radial[2] diffeomorphisms $\rho_t : (0,+\infty)\times\mathbb{S}^n \to (0,+\infty)\times\mathbb{S}^n$, $t \geq 0$, where $\rho_0 = \mathrm{Id}$. The diffeomorphisms ρ_t are such that the pullback $g(t) = \rho_t^*(g_{\mathrm{sol}})$ solves the Ricci flow

$$\partial_t g(t) = -2\mathrm{Ric}\big(g(t)\big), \qquad\qquad g(0) := g_{\mathrm{sol}}.$$

However, the map ρ_t is *not* surjective in this case. In fact, for each $t>0$, $\rho_t(0,\infty) = (m(t),+\infty)$ where $m(t)>0$ is non-decreasing in t. In other words the flow ρ_t *pushes away* from the singular point $x=0$. Thus, for each $t>0$ $(M,g(t))$ can be extended to a smooth manifold with boundary, where the induced metric on the boundary is that of a round sphere of radius $\lim_{x\to 0^+}\psi(\rho_t(x)) > 0$. One can then visualize the evolving soliton metric $g(t)$ backwards in time: Starting at time $t=1$ it contains the portion of the original soliton corresponding to $x > m(t)$, and its boundary at $x = m(t)$ shrinks down, as $t \to 0^+$, to a point which yields the singular metric g_{sol}.

The perturbation problem that we consider is still within the spherically symmetric category. In particular, the initial metrics we consider are in the form

$$\tilde{g} = dx^2 + \tilde{\psi}^2(x) g_{\mathbb{S}^n}$$

A loose version of our main result can be written in the following form; the precise statement can be found in Theorem 3.1.

Theorem 1.1. *Let*

$$\xi = \frac{\tilde{\psi}}{\psi} - 1$$

and assume that

$$\int_0^1 \frac{\xi^2}{x^{2\alpha}} + \frac{\xi_x^2}{x^{2\alpha-2}}dx + \int_1^{+\infty} \xi^2 + \xi_x^2 dx \ll 1$$

[2] "Radial" here and furtherdown means that the diffeomorhpism, for each $t \geq 0$, depends only on the parameter $x \in (0,\infty)$.

for a large enough constant α. Then there exists a unique evolving spherically symmetric metric $\tilde{g}(t)$, $t \in [0, T]$, solving the Ricci flow equation

$$\partial_t \tilde{g}(t) = -2\mathrm{Ric}\big(\tilde{g}(t)\big), \qquad \tilde{g}_0 := \tilde{g}, \qquad \xi(0,t) = 0$$

and which stays close, measured in a suitable weighted H^1-space, to the evolving soliton metric exhibiting the same "opening up" behavior of the initial singularity.

We remark briefly here on the choice of the weight function α: The definition of ξ and the assumption that ξ belongs to the weighted Sobolev space above can be interpreted geometricaly as requiring the initial metric $\tilde{g}$ (encoded in the function $\tilde{\psi}$) and the solition inital metric g, ncoded in the function ψ to *agree* asymptoticaly to high order α at $x = 0$. We expand more on this below.

1.1 Applications

It should be stressed at this point that our work here *does not* have direct bearing on the issue of "flowing through singularities" that form in finite time under the Ricci flow, (as studied, for example, in [3]), at least for closed manifolds. Indeed, it is well known that for such manifolds the minimum of the scalar curvature is a non-decreasing function under the Ricci flow; however the scalar curvature of the solitons we consider (and of their perturbations) converges to $-\infty$ at the singular point ($x = 0$).

Nonetheless, there are many important instances in PDEs of a geometric nature for which one has initially singular solutions for which one would like to know whether the evolution is stable under perturbations of the (singular) initial data. One specific example that we wish to mention is that of the Einstein equations in the general theory of relativity: We recall that the maximally extended Schwarzschild solution contains a space-like singular hypersurface in the black hole region; this corresponds to $\{T^2 - X^2 = 1\}$ in the Kruskal coordinates, Chapter 6 in [20]. It is in fact not known whether for the vacuum Einstein equations this singularity is stable under *any perturbations at all* of the initial data that lead to its formation.[3] One possible approach to produce such perturbations of the data at $T = 0$ that lead to a space-like singularity formation is to solve the vacuum Einstein equations *backwards in time*, starting from singular initial data which would correspond to perturbations of the Schwarzschild metric on a singular hypersurface that contains at least part of the space-like singularity. Producing a solution that exists (backwards) until the hypersurface $T = 0$ will then yield perturbations of the Schwarzschild initial data that still develop singularities in the future. The resulting (hyperbolic) PDE that one obtains for this problem has some key resemblances to the (parabolic) PDE that we deal with here; the key common feature is the behavior of certain singular space-time coefficients. This suggests that some of the methods developed here will have a wider applicability.

While the above solitons have been constructed over the manifolds $\mathbb{R} \times \mathbb{S}^n$, it would perhaps be natural to seek similar examples in the more general cohomogeneity-1 category, studied by Dancer and Wang, [10, 11, 12].

1.2 Outline of the ideas

Now, we briefly outline the sections of the paper and the challenges that each addresses. In Section 2 we introduce the (singular) spherically symmetric Ricci solitons

[3] For the purposes of this discussion let us say that the initial data is prescribed on a hypersurface that corresponds to $T = 0$ in the maximally extended Schwarzschild space-time (in the Kruskal coordinates).

that we consider. The study of these solitons follows the method presented in [6, Chapter 1], originally developed by R. Bryant. In the class of spherically symmetric metrics, the gradient Ricci soliton equation reduces to a second order ODE system, which can be transformed into a more tractable first order system in parameters (W, X, Y) via a transformation that we review in (A.4). Knowledge of the variables W, X, Y in the parameter y allows us to recover the metric component ψ and the gradient ϕ_x of the potential function ϕ of (A.3) in the parameter x. In the case of steady solitons, the system (A.6) in fact reduces to a 2×2 system; see §A.2. We provide a description of the trajectories in the X, Y-plane that correspond to our singular solitons and compare them to the Bryant soliton. In particular, we show there exists a 1-parameter family of singular gradient steady Ricci solitons; they are all singular at $x = 0$ with the leading order asymptotics

$$\psi(x) \sim x^{\frac{1}{\sqrt{n}}} \qquad \phi_x(x) \sim \frac{\sqrt{n}-1}{x}, \qquad n > 1$$

and they are complete towards $x = +\infty$, with the same asymptotic profile as the Bryant soliton.

In Section 3 we introduce the perturbation problem we will be studying in the rest of the paper. We consider spherically symmetric initial metrics of the form

$$\tilde{g} = \tilde{\chi}^2(x)dx^2 + \tilde{\psi}^2(x)g_{\mathbb{S}^n}$$

For such initial data, the Ricci flow equation can be written (after a change of variables) in the equivalent form (3.4) of a PDE coupled to an ODE. The evolving Ricci soliton metric (defined via the diffeomorphisms ρ_t) remains spherically symmetric and is represented by coordinate components $\chi(x,t), \psi(x,t)$, while the stipulated Ricci flow that we wish to solve for corresponds to two functions $\tilde{\chi}(x,t), \tilde{\psi}(x,t)$. Since the singular nature of the initial data do not allow the system (3.4) to be attacked directly, we introduce new variables which measure the closeness of $\tilde{\chi}, \tilde{\psi}$ to χ, ψ.

More precisely, we define

$$\zeta = \frac{\tilde{\chi}}{\chi} - 1 \qquad \xi = \frac{\tilde{\psi}}{\psi} - 1.$$

Then the system reduces to (3.10), for which the Ricci soliton corresponds to the solution $\zeta = 0, \xi = 0$. The coefficients of this system refer to the variable ψ of the background evolving soliton, expressed with respect to its arc-length parameter s. What is critical here is that the coefficients are singular at $(x,t) = (0,0)$; the precise nature of this singularity is essential in our further analysis.

A first challenge appears at this point, which in fact is independent of the singularities of the coefficients. Indeed, it is related to the presence of the second order term ξ_{ss} on the RHS of the first equation in (3.10). Since the first equation is only of first order in ζ, this term would *not* make it possible to close the energy estimates for our system. We therefore introduce a new variable defined by

$$\eta = \frac{(\zeta+1)^2}{(\xi+1)^{2n}} - 1.$$

The new system (3.14) for η and ξ involves only first derivatives of ξ in the evolution equation of η and therefore can (in principle) be approached via energy estimates. It is not clear whether there is any geometric significance underlying this change of variables.

It is in fact not a priori obvious that such a simplification of the system should have been possible via a change of variables. It is at this point that the spherical symmetry of both the background soliton and of the perturbations that we study is used in an essential way.

Thus, matters are reduced to proving well-posedness of (3.14), in the appropriate spaces. We follow the usual approach of performing an iteration[4], by solving a sequence of linear equations for the unknows (η^{m+1}, ξ^{m+1}) in terms of the known functions (η^m, ξ^m) solved for in the previous step, and proving that the sequence $(\eta^m, \xi^m), m \in \mathbb{N}$ converges to a solution (η, ξ) of our original system.

We note that the usual approach would be to replace only the highest order terms in the RHSs of (3.14) by the unknown function ξ^{m+1} and replace all the lower-order ones by the previously-solved-for η^m, ξ^m. However in the case at hand this approach would fail for any function space, due to the nature of the singularities in the coefficients. For example, as we will see the coefficient $\frac{\psi_s^2}{\psi^2}$ in the potential terms contains a factor of $\frac{1}{s^2}$, where $s(x,t)$ is the arc-length parameter of the background evolving soliton. It turns out that the leading order in the asymptotic expansion of s^2 near $x = 0, t = 0$ is of the form

$$s^2 \sim x^2 + 2(\sqrt{n} - 1)t.$$

Consequently, the best L^∞_x bound for $\frac{1}{s^2}$ would be $\frac{1}{s^2} \leq \frac{C}{t}$; this would result in an energy estimate of the form $\partial_t \mathcal{E} \leq \mathcal{E} t^{-1}$ which cannot close. The remedy for this problem is to modify the iteration procedure according to (4.2). In this linear iteration the unknown functions ξ^{m+1}, η^{m+1} at the $(m+1)$-step also appear in certain lower-order terms associated to the most singular coefficients.

Finally, we solve the system (4.2) and prove that it defines a contraction mapping in certain (time-dependent) weighted Sobolev spaces $H^1_\alpha(s)$ containing all functions

$$u \in H^1(\mathbb{R}_+) \qquad \int_0^1 \frac{u^2}{(s^2 + \sigma t)^\alpha} + \frac{u_s^2}{(s^2 + \sigma t)^{\alpha - 1}} ds < +\infty,$$

where we note that the weights depend on both the spatial and time variables x, t. (We note here that we use the length element ds which corresponds to the arc-length parameter of the background evolving Ricci soliton. In particular $s(x,t) := \rho_t(x)$; thus for all $t > 0$ $s(x,t) > s(0,t) > 0, \forall x > 0$.)

The rather involved estimates in Section 4 aim precisely to show that the parameters α and $\sigma > 0$ *can be* chosen in a way to make the estimates close; as we will see, this mostly amounts to controlling the terms in the energy estimate that arise from the most singular coefficients in (3.14). We note here that choosing α to be large forces both the initial data and the evolution of the solution to stay close the evolving soliton. Choosing σ large allows the evolving solution to 'depart' from the evolving soliton. Thus the challenge is to balance these competing parameters to make the estimates close. We note that it is essential for this 'balancing' to work that we can first close the estimates for the L^2 norms, and *after* this has been done we can estimate the H^1 norms.

Finally, in Section 5 we provide a proof of the existence of solutions to (4.2) in the appropriate spaces, using a modification of the Galerkin iteration to this singular PDE-ODE system. This part is included for the sake of completeness, since coupled systems of this singular nature do not appear to have been treated in the literature.

[4] In reality a contraction mapping argument, although we find it more convenient to phrase our proof in terms of the standard Picard iteration.

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2 Singular spherically symmetric Ricci Solitons

We will be considering metrics over $M^{n+1} = (0, B) \times \mathbb{S}^n$ (where $B \in \mathbb{R}_+$ or $B = +\infty$), in the form

$$g = dx^2 + \psi^2(x) g_{\mathbb{S}^n}, \tag{2.1}$$

where ψ is a positive smooth function and $g_{\mathbb{S}^n}$ denotes the canonical metric on the unit sphere. Our first aim for this section is to obtain such metrics which satisfy the (gradient) Ricci soliton equation

$$Ric(g) + \nabla^2\phi + \lambda g = 0 \qquad\qquad \lambda \in \mathbb{R}, \tag{2.2}$$

for a smooth radial potential function $\phi : M \to \mathbb{R}$, and which are singular as $x \to 0^+$. In particular we wish to construct a soliton metric which will extend continuously to $x = 0$ with $\psi(x) \to 0$, as $x \to 0^+$, but will not close smoothly there.

Following known work on the complete case, an approach originally initiated in (unpublished) work of R. Bryant (see Appendix A and [6]), we construct the following singular solutions of the equation (2.2).

Proposition 2.1 (Existence of singular Ricci solitons)**.** *For all $\lambda \in \mathbb{R}, n > 1$ there exists a class of spherically symmetric solutions to the gradient Ricci soliton equation (2.2) with profile*

$$\psi(x) \sim a x^{\frac{1}{\sqrt{n}}},\ a > 0 \qquad\qquad \phi_x(x) \sim \frac{\sqrt{n}-1}{x} \qquad\qquad \text{as } x \to 0^+. \tag{2.3}$$

These solutions are a priori defined for $B = \delta < +\infty$, for some $\delta > 0$ small, such that ψ, ϕ_x have a smooth limit, as $x \to \delta^- < +\infty$.

In the steady case $\lambda = 0$, the preceding solutions exist up to $B = +\infty$ and their behavior at infinity reads

$$cx^{\frac{1}{2}} \le \psi(x) \le Cx^{\frac{1}{2}} \qquad -C(1-\frac{1}{x}) \le \phi_x(x) \le -c(1-\frac{1}{x}) \qquad c, C > 0,\ \ x \gg 1. \tag{2.4}$$

Further, the behaviors of the derivatives of the above variables are in each case the derivatives of the corresponding bounds and asymptotics, e.g.,

$$\psi_x(x) \sim \frac{a}{\sqrt{n}} x^{\frac{1}{\sqrt{n}}-1},\ \text{as } x \to 0^+ \qquad\qquad -\frac{C}{x^2} \le \phi_{xx}(x) \le -\frac{c}{x^2},\ x \gg 1$$

Proof. See Propositions A.1, A.2 in Appendix A. □

Remark 2.1. *It is worth noting that for $\lambda = 0$ in dimension five, (i.e., $n = 4$) the soliton metrics and associated diffeomorphisms can in fact be written out explicitly:*

$$\psi(x) = a\sqrt{x} \qquad\qquad \phi_x(x) = \frac{1}{x} - \frac{6}{a^2}, \qquad\qquad x \in (0, +\infty),\ a > 0. \tag{2.5}$$

Remark 2.2. *In view of the asymptotics, we conclude that the above Ricci solitons metrics are C^0 extendible at $x = 0$, but singular in C^1 norm for all dimensions $n+1 \geq 3$. In particular, one can readily check that the most singular curvature components blow up like $1/x^2$, as $x \to 0^+$.*

2.1 The evolving soliton metric $g(t)$: the action of the diffeomorphisms.

Since the metric g (2.1) satisfies the gradient Ricci soliton equation (2.2), it admits a Ricci flow

$$\partial_t g(t) = -2Ric\big(g(t)\big) \qquad\qquad g(0) = g, \tag{2.6}$$

evolving via diffeomorphisms

$$g(t) = \epsilon(t)\rho_t^*(g)$$

up to some time $T > 0$, where $\epsilon(t) := 1 + 2\lambda t > 0$, $t \in [0, T)$, and

$$\rho_t(x, p) = \rho_t(x) \qquad\qquad \rho_0 = id_M$$

is the flow generated by the (time dependent) vector field

$$\frac{1}{\epsilon(t)}\nabla_g \phi.$$

Thus, by definition of the pullback

$$g(t) = \epsilon(t)\big[d(\rho_t(x))^2 + \psi^2(\rho_t(x))g_{\mathbb{S}^n}\big] \tag{2.7}$$

We note that since our manifold (M^{n+1}, g) is not complete at $x = 0$, $\rho_t(x)$ is not necessarily defined for all time, but nevertheless it exists locally $t \in (-\varepsilon_x, \varepsilon_x), x > 0$. However, it easily follows from the asymptotics below that for the steady ($\lambda = 0$) solitons the flow exists for all $t \geq 0$.

Suppressing the sphere coordinates corresponding to different points (x, p), (x, q) in M^{n+1}, we may consider ρ_t to be a real function in x

$$\rho_t : (0, B) \to (0, +\infty)$$

and further we identify the time derivative of ρ_t with the (single) component of $\nabla_g \phi$ in the ∂_x direction, that is,

$$\partial_t \rho_t(x) = \frac{1}{\epsilon(t)}(\nabla_{\partial/\partial x}\phi)_{\rho_t(x)} = \frac{1}{\epsilon(t)}\phi_x(\rho_t(x)). \tag{2.8}$$

According to the asymptotics (2.3),

$$\partial_t \rho_t(x) \sim \frac{1}{\epsilon(t)}\frac{\sqrt{n}-1}{\rho_t(x)} \tag{2.9}$$

which after integrating yields the leading behavior

$$\rho_t^2(x) \sim x^2 + 2(\sqrt{n}-1)t, \qquad\qquad \text{as } x, t \to 0^+. \tag{2.10}$$

Remark 2.3. *From the preceding asymptotics it follows that*

$$\rho_t\big((0,B)\big) \subseteq (\rho_t(0),+\infty),$$

$\rho_t(0) > 0, t > 0$ *non-decreasing, and in particular* ρ_t *is not surjective. A geometric interpretation of the latter is that the flow* ρ_t *"pushes" the domain away from the singularity at* $x = 0$*, smoothing out the incomplete metric.*

Restricting now on the singular steady solitons, we integrate (2.8) once more to arrive at the following estimate at infinity for the flow

$$x - Ct \le \rho_t(x) \le x - ct \qquad\qquad x \gg 1 \gg t \ge 0. \tag{2.11}$$

In fact, in the steady case $\lambda = 0$ we can give a complete description of the evolution of the singular soliton metrics. Indeed, in this case we derive that there is a critical slice $\{x_{\text{crit}}\} \times \mathbb{S}^n$ of the manifold $M^{n+1} = (0,+\infty) \times \mathbb{S}^n$, which is invariant under $\rho_t(\cdot)$ and moreover an attractor of the flow:

$$\phi_x(x) > 0,\ x \in (0, x_{\text{crit}}) \qquad \phi_x(x_{\text{crit}}) = 0 \qquad \phi_x(x) < 0,\ (x_{\text{crit}},+\infty) \tag{2.12}$$

Whence, for any point $x \in (0,+\infty)$, the integral curve $\rho_t(x)$ will 'reach' x_{crit} as time tends to infinity

$$\lim_{t\to+\infty} \rho_t(x) = x_{\text{crit}} \qquad\qquad \lim_{t\to+\infty} \rho_t\big((0,+\infty)\big) = [x_{\text{crit}},+\infty).$$

We remark also that the scalar curvature R achieves its maximum at x_{crit}, which means that the manifold is deformed in this sense towards higher level sets of scalar curvature.

In order to prove the above picture, it suffices to show that (2.12) is valid. From the profiles (2.3), (2.4) we confirm that ϕ_x has a positive sign close to $x = 0$ and is negative near $+\infty$. Hence there exists a point x_{crit} where $\phi_x(x_{\text{crit}}) = 0$. It remains to show that this is the only zero of ϕ_x. We recall at this point a general identity for solutions to the gradient Ricci soliton equation (2.2) (see for instance [6, Proposition 1.15]).

Proposition 2.2. *Let* $(M^m, g, \nabla\phi)$ *be a gradient Ricci soliton, i.e., a solution of the equation (2.2). Then the following quantities are constant:*

$$\begin{aligned} &(i)\ \ R + \Delta_g\phi + m\lambda = 0 \qquad\qquad \textit{(tracing)} \\ &(ii)\ \ R + |\nabla_g\phi|^2 + 2\lambda\phi = C_0, \end{aligned}$$

where R *is the scalar curvature of* (M^m, g)*.*

The fact that the scalar curvature R attains its maximum (C_0) at x_{crit} is an immediate consequence of identity (ii) for $\lambda = 0$.

Subtracting the two identities of the preceding proposition we obtain

$$\Delta_g\phi - |\nabla_g\phi|^2 - 2\lambda\phi + m\lambda = -C_0.$$

Whence, in our context for $\lambda = 0$, the previous equation amounts to

$$\phi_{xx} + \frac{\psi_x}{\psi}\phi_x - \phi_x^2 = -C_0, \tag{2.13}$$

Claim: $C_0 > 0$. From the asymptotics of ϕ_x (2.3), (2.4), we easily deduce that ϕ tends to $-\infty$ at both ends of the manifold $x = 0, +\infty$. This implies that ϕ has a global maximum M, realized at some point $\tilde{x}$. By (2.13) we get $C_0 \ge 0$. However, the constant C_0 cannot be zero, otherwise we would have $\phi \equiv M$ (by uniqueness of ODEs), which of course is not possible. Our claim follows.

Thus, every critical point of ϕ is a *strict* local maximum. Therefore, ϕ can only have one critical point, $x_{\text{crit}} = \tilde{x}$. ■

3 The Stability problem

Our main goal in this paper is to prove (local in time) well-posedness of the Ricci flow for spherically symmetric metrics which start out close enough (in certain spaces we construct in §3.4) to the soliton metrics (Propositions 2.1) we constructed in the previous section. We recall below in §3.1 a useful form of the Ricci flow equation for spherically symmetric metrics and then proceed to introduce a transformation of our system into new variables ζ, ξ (3.9). These are designed to capture the closeness of the (putative) evolving solution under the Ricci flow to the evolution of the background Ricci soliton. The resulting system involves a second order parabolic equation in ξ coupled with a transport equation in ζ, both of them having certain singular coefficients. This forces us to study well-posedness of the system in certain weighted Sobolev spaces. Our main result in these variables is stated in Theorem 3.1.

However, this is not the system we derive energy estimates with, because of the fact that the transport equation in ζ contains a second order term in ξ, which makes it impossible for such estimates to close. After a further crucial change of variables (§3.3), replacing ζ with a new variable η, the resulting PDE in η, ξ (3.14) for which we derive an estimate is of similar nature, except now this problem has been eliminated; the equation of η containing only first derivatives of ξ.

The singularities in the coefficients of the system are determined fully by the background evolving soliton metric. The precise asymptotics of these coefficients are essential to our further pursuits, so we begin by studying those right after writing down the final system (3.14). Next, in §3.4 we set up formally the function spaces in which we will be proving our well-posedness result for the system of η, ξ and state the final version of our main result very precisely in Theorem 3.2. The proof of Theorem 3.2 is carried out in the next section §4.

One final convention: We will be considering the stability question for all the singular Ricci solitons (see Proposition 2.1). Since for $\lambda \neq 0$ our knowledge is restricted only on the bounded interval $(0, \delta)$, we will treat two versions of the resulting PDE problem. One will concern a bounded domain and the other, for the steady case $\lambda = 0$, will regard the whole half-line; i.e., initial domain $x \in (0, B)$, $B = \delta < +\infty$ or $B = +\infty$.

3.1 Ricci flow in spherical symmetry

Let $\tilde{g}(t)$, $t \in [0, T]$, be a 1-parameter family of smooth spherically symmetric metrics on $M^{n+1} = (0, B) \times \mathbb{S}^n$ ($B = \delta < +\infty$ or $B = +\infty$)

$$\tilde{g}(t) = \tilde{\chi}^2(x,t)dx^2 + \tilde{\psi}^2(x,t)g_{\mathbb{S}^n}, \tag{3.1}$$

where $\tilde{\chi}, \tilde{\psi}$ are positive smooth functions, and assume it satisfies the Ricci flow equation

$$\partial_t \tilde{g}(t) = -2Ric\big(\tilde{g}(t)\big) \qquad\qquad t \in [0, T]. \tag{3.2}$$

We now let $\tilde{s}(x,t)$ be the radial arc-length parameter for the above metric at any given time t, i.e.,

$$d\tilde{s} = \tilde{\chi}(x,t)dx. \tag{3.3}$$

Expressing $\tilde{\psi}(\cdot, t)$ relative to the parameter $\tilde{s}$ (and slightly abusing notation), $\tilde{g}(t)$ becomes

$$\tilde{g}(t) = d\tilde{s}^2 + \tilde{\psi}^2(\tilde{s}, t)g_{\mathbb{S}^n}.$$

For this type of warped product metrics the Ricci tensor is given by (e.g., [6, §1.3.2])

$$Ric\big(\tilde{g}(t)\big) = -n\frac{\tilde{\psi}_{\tilde{s}\tilde{s}}}{\tilde{\psi}}d\tilde{s}^2 + (n-1-\tilde{\psi}\tilde{\psi}_{\tilde{s}\tilde{s}} - (n-1)\tilde{\psi}_{\tilde{s}}^2)g_{\mathbb{S}^n}.$$

Plugging into (3.2) we get

$$\begin{cases} 2\tilde{\chi}\tilde{\chi}_t = -2(-n\frac{\tilde{\psi}_{\tilde{s}\tilde{s}}}{\tilde{\psi}})d\tilde{s}^2(\partial_x,\partial_x) = 2n\frac{\tilde{\psi}_{\tilde{s}\tilde{s}}}{\tilde{\psi}}\tilde{\chi}^2 \\ 2\tilde{\psi}\tilde{\psi}_t = -2(n-1-\tilde{\psi}\tilde{\psi}_{\tilde{s}\tilde{s}} - (n-1)\tilde{\psi}_{\tilde{s}}^2) \end{cases}$$

Thus, the Ricci flow equation (3.2) reduces to the coupled system

$$\begin{cases} \tilde{\chi}_t = n\frac{\tilde{\psi}_{\tilde{s}\tilde{s}}}{\tilde{\psi}}\tilde{\chi} \\ \tilde{\psi}_t = \tilde{\psi}_{\tilde{s}\tilde{s}} - (n-1)\frac{1-\tilde{\psi}_{\tilde{s}}^2}{\tilde{\psi}} \end{cases} \qquad t\in[0,T]. \tag{3.4}$$

Observe that the first equation involves the evolution of the radial distance function, while the second involves the evolution of the radii of the spheres, at a given radial distance.

Of course, the singular Ricci soliton metrics we studies in the previous section fall in the same framework. Indeed, returning to (2.7) we may write

$$g(t) = ds^2 + \psi^2(s,t), g_{\mathbb{S}^n} = \chi^2(x,t)dx^2 + \psi^2(x,t)g_{\mathbb{S}^n}, \tag{3.5}$$

where we have set

$$s(x,t) = \sqrt{\epsilon(t)}\,\rho_t(x), \quad s(x,0) = x \qquad\qquad ds = \sqrt{\epsilon(t)}\,\partial_x\rho_t(x)dx \tag{3.6}$$

and

$$\chi(x,t) := \sqrt{\epsilon(t)}\,\partial_x\rho_t(x) \qquad\qquad \psi(x,t) := \sqrt{\epsilon(t)}\,\psi(\rho_t(x)). \tag{3.7}$$

Note that $\psi(x,0) = \psi(x)$ corresponds to the component of the metric g (2.1). Arguing similarly to the case of $\tilde{g}(t)$, it follows that the (2.6) is equivalent to

$$\begin{cases} \chi_t = n\frac{\psi_{ss}}{\psi}\chi \\ \psi_t = \psi_{ss} - (n-1)\frac{1-\psi_s^2}{\psi} \end{cases} \qquad \chi(x,0) = 1,\ \psi(x,0) = \psi(x). \tag{3.8}$$

3.2 The main stability result: A transformed system for the Ricci flow of the perturbed metric

The goal is to construct a spherically symmetric Ricci flow (3.1), (3.2) for the appropriate spherically symmetric perturbed metric $\tilde{g} := \tilde{g}(0)$. We now take a first step towards transforming our system of equations by introducing new variables. Let

$$\zeta = \frac{\tilde{\chi}}{\chi} - 1 \qquad\qquad \xi = \frac{\tilde{\psi}}{\psi} - 1. \tag{3.9}$$

The above formulas are defined for all $x\in(0,B), t\in[0,T]$. In particular, these variables measure (in a refined way) the difference between the unknown functions $\tilde{\chi},\tilde{\psi}$ and the background variables χ,ψ. Note in addition that requiring $\xi = 0$ at the endpoint $x = 0, t = 0$ forces $\tilde{\psi}$ to have the same leading order asymptotics at $x = 0$ as the background component ψ.

We next wish to convert (3.4) into a system of equations for ζ, ξ, expressing the evolution equations in terms of t and the arc-length parameter s of the background evolving Ricci soliton. We are then forced to deal with the discrepancy between $\tilde{s}, s$. We calculate:

$$\partial_{\tilde{s}} \overset{(3.3)}{=} \frac{1}{\tilde{\chi}}\partial_x = \frac{\chi}{\tilde{\chi}}\frac{1}{\chi}\partial_x \overset{(3.6),(3.7)}{=} \frac{1}{\zeta+1}\partial_s$$
$$\partial_{\tilde{s}}\partial_{\tilde{s}} = \frac{1}{\zeta+1}\partial_s(\frac{1}{\zeta+1}\partial_s) = \frac{1}{(\zeta+1)^2}\partial_s\partial_s - \frac{\zeta_s}{(\zeta+1)^3}\partial_s,$$

and hence we write

$$\tilde{\psi}_{\tilde{s}} = \frac{1}{\zeta+1}\big(\psi(\xi+1)\big)_s$$
$$\tilde{\psi}_{\tilde{s}\tilde{s}} = \frac{1}{(\zeta+1)^2}\big(\psi(\xi+1)\big)_{ss} - \frac{\zeta_s}{(\zeta+1)^3}\big(\psi(\xi+1)\big)_s.$$

Taking time derivatives in (3.9) and combining (3.4), (3.8), we derive the following coupled system in the new variables ζ, ξ.

$$\begin{aligned}
\zeta_t =& n\frac{\psi_{ss}}{\psi}\big[\frac{1}{\zeta+1} - (\zeta+1)\big] + 2n\frac{\psi_s}{\psi}\frac{\xi_s}{(\zeta+1)(\xi+1)} + n\frac{\xi_{ss}}{(\zeta+1)(\xi+1)} - n\frac{\psi_s}{\psi}\frac{\zeta_s}{(\zeta+1)^2} \\
&- n\frac{\zeta_s\xi_s}{(\zeta+1)^2(\xi+1)} \\
\xi_t =& (\frac{\psi_{ss}}{\psi} + (n-1)\frac{\psi_s^2}{\psi^2})\big[\frac{\xi+1}{(\zeta+1)^2} - \xi - 1\big] + \frac{n-1}{\psi^2}(\xi+1-\frac{1}{\xi+1}) \\
&+ 2n\frac{\psi_s}{\psi}\frac{\xi_s}{(\zeta+1)^2} + \frac{\xi_{ss}}{(\zeta+1)^2} + (n-1)\frac{\xi_s^2}{(\zeta+1)^2(\xi+1)} - \frac{\psi_s}{\psi}\frac{\zeta_s(\xi+1)}{(\zeta+1)^3} - \frac{\zeta_s\xi_s}{(\zeta+1)^3}
\end{aligned} \tag{3.10}$$

Notice that the coefficients of the preceding system are expressed in terms of the components (metric, curvature etc.) of the background soliton, which are of course singular at $x = t = 0$. We will elaborate more on the nature of the singularities in the next subsection. We simply mention that this is basically the reason that forces us to study (3.10) in non-standard modified spaces. The following version of our main theorem regards the local existence of the system in the variables ζ, ξ (3.9).

Theorem 3.1. *There exist constants $\alpha, \sigma > 0$ appropriately large, such that the system (3.10) is locally well-posed in the (time-dependent) weighted Sobolev space*

$$E(t) := \int_{x=0}^{x=\delta} \frac{u^2}{(s^2+\sigma t)^\alpha} + \frac{u_s^2}{(s^2+\sigma t)^{\alpha-1}}ds + \int_{x=\delta}^{x=B} u^2 + u_s^2 ds < +\infty, \tag{3.11}$$

($B = \delta < +\infty, \lambda \neq 0$ or $B = +\infty, \lambda = 0$) assuming $E(0)$ sufficiently small and the Dirichlet boundary condition

$$\xi(x,t) = 0, \quad \{x = 0, B\} \times [0, T] \tag{3.12}$$

We remark that the smallness assumption on $E(0)$ is required to control the smallness in L^∞ of η, ξ which appear in the denominators in (3.10) by $E(t)$. It could possibly be removed if the initial data lied in a suitably weighted H^2 space, combined with an assumption of smallness in L^∞ of η, ξ.

3.3 A crucial change of variables: The features of the resulting PDE

Unfortunately, due to the term $n\frac{\xi_{ss}}{(\zeta+1)(\xi+1)}$ in the first equation of (3.10) we cannot derive energy estimates in L^2 for ζ, ξ. We remedy this problem by replacing the variable ζ with

$$\eta := \frac{(\zeta+1)^2}{(\xi+1)^{2n}} - 1. \tag{3.13}$$

The new system of η, ξ reads

$$\begin{aligned}
\eta_t =& -2n(n-1)\Big(\frac{\psi_s^2}{\psi^2}\Big[\frac{1}{(\xi+1)^{2n}}-1\Big]+2\frac{\psi_s}{\psi}\frac{\xi_s}{(\xi+1)^{2n+1}}+\frac{1-(\xi+1)^{-2}}{\psi^2}+\frac{\xi_s^2}{(\xi+1)^{2n+2}}\Big)\\
&-2n(n-1)\frac{1-(\xi+1)^{-2}}{\psi^2}\eta+2n(n-1)\frac{\psi_s^2}{\psi^2}\eta\\
\xi_t =& \Big(\frac{\psi_{ss}}{\psi}+(n-1)\frac{\psi_s^2}{\psi^2}\Big)\Big[\frac{1}{(\eta+1)(\xi+1)^{2n-1}}-(\xi+1)\Big]+\frac{n-1}{\psi^2}\Big(\xi+1-\frac{1}{\xi+1}\Big)\\
&+n\frac{\psi_s}{\psi}\frac{\xi_s}{(\eta+1)(\xi+1)^{2n}}+\frac{\xi_{ss}}{(\eta+1)(\xi+1)^{2n}}-\frac{\xi_s^2}{(\eta+1)(\xi+1)^{2n+1}}\\
&-\frac{1}{2}\frac{\psi_s}{\psi}\frac{\eta_s}{(\eta+1)^2(\xi+1)^{2n-1}}-\frac{1}{2}\frac{\eta_s\xi_s}{(\eta+1)^2(\xi+1)^{2n}}
\end{aligned} \tag{3.14}$$

It is important that we know the exact leading asymptotics of the coefficients in (3.14), as $x, t \to 0^+$. Recall the formulas (3.6), (3.7)

$$s(x,t) = \sqrt{\epsilon(t)}\rho_t(x) \qquad\qquad \psi(s,t) = \sqrt{\epsilon(t)}\,\psi\Big(\frac{s}{\sqrt{\epsilon(t)}}\Big)$$

and the profile of the background singular soliton at the two ends $x = 0, B$ (Proposition 2.1) to deduce the following estimates:

$$\frac{\psi_s}{\psi} = O\Big(\frac{1}{s}\Big) \qquad \frac{\psi_s^2}{\psi^2} = O\Big(\frac{1}{s^2}\Big) \qquad \frac{\psi_{ss}}{\psi} = O\Big(\frac{1}{s^2}\Big) \qquad x \in (0,B),\ t \in [0,T] \tag{3.15}$$

and separately for

$$\frac{1}{\psi^2} = O\Big(\frac{1}{s^{\frac{2}{\sqrt{n}}}}\Big),\ x \ll 1 \qquad \frac{1}{\psi^2} = O\Big(\frac{1}{s}\Big),\ x \gg 1 \qquad t \in [0,T],\ n > 1 \tag{3.16}$$

for small $T > 0$. Using the above we also derive

$$\partial_s\Big(\frac{\psi_s}{\psi}\Big) = O\Big(\frac{1}{s^2}\Big) \qquad \partial_s\Big(\frac{\psi_s^2}{\psi^2}\Big) = O\Big(\frac{1}{s^3}\Big) \qquad x \in (0,B),\ t \in [0,T]. \tag{3.17}$$

Also, directly from the asymptotics of the flow $\rho_t^2(x)$ (2.10),(2.11) the arc-length parameter s of the background soliton shows to behave like

$$s^2(x,t) := \epsilon(t)\rho_t^2(x) \sim x^2 + 2(\sqrt{n}-1)t \qquad \text{as } x, t \to 0^+ \tag{3.18}$$

and

$$x - Ct \le s \le x - ct \qquad x \gg 1,\ B = +\infty,\ t \in [0,T] \tag{3.19}$$

with an evolution estimated employing (2.8):

$$\partial_t s = \frac{\lambda}{\epsilon(t)}s + O\Big(\frac{1}{s}\Big),\ x \ll 1 \quad -C \le \partial_t s \le -c,\ x \gg 1,\ B = +\infty \quad t \in [0,T] \tag{3.20}$$

Remark 3.1. *Evidently from the above asymptotics, the best L^∞_x estimate that one could hope for the ratio $1/s^2$ is of the form*

$$\|\frac{1}{s^2}\|_{L^\infty(x)} \leq \frac{C}{t}, \tag{3.21}$$

which of course fails to be integrable in $[0,T]$, $T>0$. Note that $1/s^2$ is the leading behavior, suggested from the above estimates, of the most singular coefficients of the potential terms in (3.14). This is precisely the reason why the standard Gronwall argument would fail to yield an energy estimate in the usual H^k spaces for the system in question.

It will be useful furtherdown to write the *less* singular coefficients in (3.14), namely, $\frac{1}{\psi^2}$ as

$$\frac{1}{\psi^2} =: \frac{A(s,t)}{s}, \qquad \partial_s(\frac{A(s,t)}{s}) = -2\frac{1}{\psi^2}\frac{\psi_s}{\psi} = \frac{A(s,t)}{s}O(\frac{1}{s}), \tag{3.22}$$

where setting

$$A(t) := \|A(s,t)\|_{L^\infty(s)}, \qquad \int_0^t A^2(\tau)d\tau = o(\sqrt{t}), \qquad \text{as } t \to 0^+. \tag{3.23}$$

As stated in Theorem 3.1, the spaces we will be dealing with involve the coordinate vector field ∂_s and the volume form ds of the background soliton metric. The first issue we stress here is the fact that the vector fields ∂_s, ∂_t (the latter is defined so that $\partial_t x = 0$) do not commute. In fact, we find the commutator to be singular:

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial}{\partial s} =& \frac{\partial}{\partial t}\left(\frac{1}{\sqrt{\epsilon(t)}\,\partial_x\rho_t(x)}\frac{\partial}{\partial x}\right) && \text{(by definition of } s \text{ (3.6))}\\
=& -\frac{\lambda}{\epsilon(t)^{\frac{3}{2}}}\frac{1}{\partial_x\rho_t(x)}\frac{\partial}{\partial x} - \frac{1}{\sqrt{\epsilon(t)}}\frac{\partial_t(\partial_x\rho_t(x))}{(\partial_x\rho_t(x))^2}\frac{\partial}{\partial x} + \frac{1}{\sqrt{\epsilon(t)}\,\partial_x\rho_t(x)}\frac{\partial}{\partial t}\frac{\partial}{\partial x}\\
=& -\frac{\lambda}{\epsilon(t)}\frac{\partial}{\partial s} - \frac{\partial_x\partial_t\rho_t(x)}{\partial_x\rho_t(x)}\frac{\partial}{\partial s} + \frac{1}{\sqrt{\epsilon(t)}\,\partial_x\rho_t(x)}\frac{\partial}{\partial x}\frac{\partial}{\partial t}\\
=& -\frac{\lambda}{\epsilon(t)}\frac{\partial}{\partial s} - \frac{\partial_x\big[\frac{1}{\epsilon(t)}\phi_x(\rho_t(x))\big]}{\partial_x\rho_t(x)}\frac{\partial}{\partial s} + \frac{\partial}{\partial s}\frac{\partial}{\partial t} && \text{(plugging in (2.8))}\\
=& -\frac{\lambda+\phi_{xx}(s)}{\epsilon(t)}\frac{\partial}{\partial s} + \frac{\partial}{\partial s}\frac{\partial}{\partial t}
\end{aligned}$$

Consulting the asymptotics of the second derivative potential function (deduced from Proposition 2.1) we conclude that

$$[\partial_t, \partial_s] = O(\frac{1}{s^2})\partial_s \qquad x \in (0,B),\ t \in [0,T]. \tag{3.24}$$

We must also calculate the evolution of the volume form ds. The derivation is similar:

$$\begin{aligned}
\partial_t ds &= \partial_t(\sqrt{\epsilon(t)}\,\partial_x\rho_t(x)dx) = \frac{\lambda}{\sqrt{\epsilon(t)}}\partial_x\rho_t(x)dx + \sqrt{\epsilon(t)}\,\partial_t\partial_x\rho_t(x)dx\\
&= \frac{\lambda}{\epsilon(t)}ds + \sqrt{\epsilon(t)}\,\partial_x\big[\frac{1}{\epsilon(t)}\phi_x(\rho_t(x))\big]dx = \frac{\lambda+\phi_{xx}(s)}{\epsilon(t)}ds,
\end{aligned}$$

which as above gives

$$\partial_t ds = O(\frac{1}{s^2})ds. \tag{3.25}$$

3.4 The weighted Sobolev spaces and the final version of the main theorem

As explained the singularities in the coefficients of the system (3.14), along with the asymptotic behaviors we have derived force us to study well-posedness in *weighted* Sobolev spaces. The weights will be adapted to the singularity at $x=0, t=0$.

Definition 3.1. *Let $\sigma > 0$ (to be determined later). We define the weight*

$$\ell^2(x,t) = \begin{cases} s^2+\sigma t, & (x,t)\in(0,\delta)\times[0,T],\ \lambda\in\mathbb{R} \\ \varphi(s,t), & (x,t)\in[\delta,\delta+1)\times[0,T],\ \lambda=0,\ B=+\infty \\ 1, & (x,t)\in[\delta+1,+\infty)\times[0,T], \quad\quad ''\ '' \end{cases} \tag{3.26}$$

where $\varphi(\cdot,t)$ is a cut off function interpolating between $\ell^2(\delta,t)$ and 1, for each $t\in[0,T]$.

When we derive the main energy estimates in the next section we will need the following key properties of the weight ℓ. First, we estimate immediately by Definition 3.1 and (3.20) how ℓ changes along the directions ∂_s, ∂_t:

$$\partial_s\ell = O(1) \qquad\qquad \partial_t\ell = \big[\frac{O(1)}{\ell}+\frac{\sigma}{\ell}\big]\mathbf{1}_{(0,\delta)} + O(1)\mathbf{1}_{[\delta,B)}. \tag{3.27}$$

Also, from the asymptotics of s^2 (3.18),(3.19) we obtain the following comparison estimate of the functions s,ℓ.

$$0 < c \leq \frac{\ell^2}{s^2} = \begin{cases} 1+\frac{2\sigma t}{s^2} \\ \frac{O(1)}{s^2} \end{cases} \leq \begin{cases} 1+\frac{C}{\sqrt{n-1}}\sigma, & x\in(0,\delta) \\ C, & x\in[\delta,+\infty),\ B=+\infty \end{cases} \qquad n>1. \tag{3.28}$$

Now we may proceed to the formal definition of the modified H^k spaces.

Definition 3.2. *For any given $t\in[0,T]$ and $\alpha\geq 1$, we define the weighted space*

$$H^k_\alpha[t]: \quad u\in H^k\big((0,B)\big), \quad \|u\|^2_{H^k_\alpha[t]} = \int_{x=0}^{x=B} \frac{u^2}{\ell^{2\alpha}}+\cdots+\frac{(\partial_s^k u)^2}{\ell^{2\alpha-2k}}ds < +\infty. \tag{3.29}$$

In the case $k=0$, we denote $H^0_\alpha[t]$ by $L^2_\alpha[t]$. When it is clear, we will suppress t in the notation.

In this spirit, we define the energy

$$\mathcal{E}(u,v;T) = \|u\|^2_{C(0,T;H^1_\alpha)} + \|u\|^2_{L^2(0,T;H^1_{\alpha+1})} + \|v\|^2_{C(0,T;H^1_\alpha)} + \|v\|^2_{L^2(0,T;H^2_{\alpha+1})} \tag{3.30}$$

and for brevity let

$$\mathcal{E}_0 = \|\eta_0\|^2_{H^1_\alpha} + \|\xi_0\|^2_{H^1_\alpha}, \tag{3.31}$$

where $\eta_0 := \eta(x,0)$, $\xi_0 := \xi(x,0)$. We can formulate now a more precise version of our main result regarding the system (3.14).

Theorem 3.2. *There exist $\alpha>0, \sigma:=\sigma(\alpha)>0$ sufficiently large such that if $\mathcal{E}_0$ is sufficiently small, then the system (3.14), subject to*

$$\xi(x,t) = 0 \qquad\qquad \{x=0,B\}\times[0,T], \tag{3.32}$$

admits a unique solution up to some time $T:=T(\mathcal{E}_0,\alpha,\sigma)>0$ in the spaces

$$\begin{aligned} \eta &\in C(0,T;H^1_\alpha)\cap L^2(0,T;H^1_{\alpha+1}) & \xi &\in C(0,T;H^1_\alpha)\cap L^2(0,T;H^2_{\alpha+1}) \\ \eta_t &\in C(0,T;L^2_{\alpha-2})\cap L^2(0,T;H^1_{\alpha-1}) & \xi_t &\in L^2(0,T;L^2_{\alpha-1}) \end{aligned} \tag{3.33}$$

with initial data η_0, ξ_0.

We remark here the fact that once we have such a solution to (3.14), then we straightforwardly derive that this solution (η, ξ) corresponds to a solution of (3.4), which in fact will be smooth over $M^{n+1} \times (0, T]$, given the parabolicity of the Ricci flow.

4 The Contraction Mapping

We will prove Theorem 3.2 via an iteration scheme, which is essentially a contraction mapping argument. We note that throughout the subsequent estimates we will use the symbol C to denote a positive constant depending only on n. Further, the endpoints of any integration in the spatial variable, unless otherwise indicated, will be the two ends $x = 0, B$.

4.1 The iteration scheme and the contraction mapping

In order to derive energy estimates, it is very important how we define the Picard iteration for the system (3.14). We choose to keep in the unknowns at each step the linear lower order terms in the RHSs which are associated to the most singular coefficients in the system. We construct a sequence $\{\eta^m, \xi^m\}_{m=0}^{\infty}$ in the spaces

$$\begin{aligned} \eta^m &\in C(0,T;H^1_\alpha)\cap L^2(0,T;H^1_{\alpha+1}) \qquad & \xi^m &\in C(0,T;H^1_\alpha)\cap L^2(0,T;H^2_{\alpha+1}) \\ \eta^m_t &\in C(0,T;L^2_{\alpha-2})\cap L^2(0,T;H^1_{\alpha-1}) & \xi^m_t &\in L^2(0,T;L^2_{\alpha-1}), \end{aligned} \tag{4.1}$$

satisfying

$$\begin{aligned} \eta^{m+1}_t &= 2n(n-1)\bigg(\frac{\psi_s^2}{\psi^2}\frac{2n\xi^{m+1}+\sum_{j=2}^{2n}\binom{2n}{j}|\xi^m|^j}{(\xi^m+1)^{2n}} - 2\frac{\psi_s}{\psi}\frac{\xi^{m+1}_s}{(\xi^m+1)^{2n+1}} \\ &\quad - \frac{A(s,t)}{s}\xi^m\frac{\xi^m+2}{(\xi^m+1)^2}(1+\eta^m) - \frac{|\xi^m_s|^2}{(\xi^m+1)^{2n+2}} + \frac{\psi_s^2}{\psi^2}\eta^{m+1}\bigg) \\ \xi^{m+1}_t &= (\frac{\psi_{ss}}{\psi} + (n-1)\frac{\psi_s^2}{\psi^2})\left[\frac{-\eta^{m+1}-2n(\eta^m+1)\xi^{m+1}}{(\eta^m+1)(\xi^m+1)^{2n-1}} - \frac{\sum_{j=2}^{2n}\binom{2n}{j}|\xi^m|^j}{(\xi^m+1)^{2n-1}}\right] \\ &\quad + (n-1)\frac{A(s,t)}{s}\xi^m\frac{\xi^m+2}{\xi^m+1} + n\frac{\psi_s}{\psi}\frac{\xi^{m+1}_s}{(\eta^m+1)(\xi^m+1)^{2n}} + \frac{\xi^{m+1}_{ss}}{(\eta^m+1)(\xi^m+1)^{2n}} \\ &\quad - \frac{|\xi^m_s|^2}{(\eta^m+1)(\xi^m+1)^{2n+1}} - \frac{1}{2}\frac{\psi_s}{\psi}\frac{\eta^{m+1}_s}{(\eta^m+1)^2(\xi^m+1)^{2n-1}} - \frac{1}{2}\frac{\eta^m_s\xi^m_s}{(\eta^m+1)^2(\xi^m+1)^{2n}}, \end{aligned} \tag{4.2}$$

where we set $\eta^0 = \xi^0 = 0$ and initially

$$\eta^{m+1}\Big|_{t=0} = \eta_0 \qquad \xi^{m+1}\Big|_{t=0} = \xi_0 \qquad\qquad m = 0, 1, \dots \tag{4.3}$$

Further, ξ^{m+1} is required to verify the Dirichlet boundary condition

$$\xi^{m+1}(x,t) = 0 \qquad\qquad \{x = 0, B\} \times [0, T]. \tag{4.4}$$

Under the assumptions of Theorem 3.2, we show inductively that for sufficiently small $T > 0$ (uniform in m), the sequence also satisfies the energy estimate

$$\mathcal{E}(\eta^m, \xi^m; T) \leq 2\mathcal{E}_0 \qquad\qquad m = 0, 1, \dots \tag{4.5}$$

We prove this in Section 5.

The main task that we undertake here is to prove Theorem 3.2 by showing that the sequence $(\eta^m, \xi^m)_{m\in\mathbb{N}}$ is actually Cauchy in the energy spaces we have introduced.

Proposition 4.1. *Let*

$$d\eta^{m+1} = \eta^{m+1} - \eta^m, \quad d\xi^{m+1} = \xi^{m+1} - \xi^m \qquad m = 0, 1, \ldots, \tag{4.6}$$

where η^m, ξ^m are the functions constructed above. Then under the assumptions in Theorem 3.2 on $\alpha, \sigma, \mathcal{E}_0, T$ the following contraction estimate holds:

$$\mathcal{E}(d\eta^{m+1}, d\xi^{m+1}; T) \le \frac{1}{2}\mathcal{E}(d\eta^m, d\xi^m; T) \qquad m = 1, 2, \ldots, \tag{4.7}$$

The previous proposition readily implies Theorem 3.2; the iterates (η^m, ξ^m) converge to a solution of the system (3.14) satisfying the assertions of the theorem.

Proof. It is carried out in §4.2. □

Some standard pointwise estimates adapted to our weighted norms are needed to proceed.

Lemma 4.1. *Given functions η^m, ξ^m, $m \in \mathbb{N}$, in the spaces (4.1), the following pointwise bounds are valid:*

$$\|\frac{\eta^m}{\ell^k}\|^2_{L^\infty(x)} \le C(k+1)\mathcal{E}_0 \qquad \|\frac{\xi^m}{\ell^k}\|^2_{L^\infty(x)} \le C(k+1)\mathcal{E}_0, \tag{4.8}$$

$$\|\frac{\xi^m_s}{\ell^k}\|^2_{L^\infty(x)} \le C\sqrt{\mathcal{E}_0}\big(\|\frac{\xi^m_{ss}}{\ell^{\alpha-1}}\|_{L^2} + k\|\frac{\xi^m_s}{\ell^\alpha}\|_{L^2}\big), \quad \int_0^t \|\frac{\xi^m_s}{\ell^k}\|^2_{L^\infty} d\tau \le C(k+1)\sqrt{T}\mathcal{E}_0, \tag{4.9}$$

for all $k = 0, \ldots \alpha - 1$, $\alpha \ge 1$, $t \in [0, T]$. If in addition $\mathcal{E}_0$ is small enough, the following estimates also hold:

$$\sup_{x \in (0,B)} \big(|\eta^m| + |\xi^m|\big) < \frac{1}{2} \qquad \inf_{x \in (0,B)} (\xi^m + 1)^{-2n} \ge \frac{1}{2}, \tag{4.10}$$

We note that (4.10) is the first main reason we consider small $\mathcal{E}_0$, which in particular guarantees the parabolicity of the second equation of (4.2).

Proof. We treat the estimate of $|\frac{\xi^m}{\ell^k}|$. The rest follow easily from the same argument. By the fundamental theorem of calculus we have

$$\Big|\frac{\xi^m\big((s(x,t),t\big)^2}{\ell^{2k}} - \frac{\xi^m\big(s(0,t),t\big)^2}{\ell^{2k}}\Big| \overset{(4.4)}{=} \Big|\int_{s(0,t)}^{s(x,t)} 2\frac{\xi^m}{\ell^k}\Big(\frac{\xi^m_s}{\ell^k} - k\frac{\xi^m}{\ell^{k+1}}\ell_s\Big)ds\Big|$$

$$\le 2\|\frac{\xi^m}{\ell^{k+\frac12}}\|_{L^2}\big(\|\frac{\xi^m_s}{\ell^{k-\frac12}}\|^2_{L^2} + Ck\|\frac{\xi^m}{\ell^{k+\frac12}}\|^2_{L^2}\big) \overset{(4.5)}{\le} C(k+1)\mathcal{E}_0 \qquad (\ell_s = O(1)\ (3.27))$$

In the case of $|\frac{\eta^m}{\ell^k}|$, instead of $x = 0$, we choose a reference point $x \in [0, +\infty]$ realizing its infimum, which is controlled by the L^2 norm and argue similarly as before. The estimate (4.10) follows from (4.8) for $k = 0$, provided the initial weighted energy is small enough.

As for (4.9), the second part obviously follows from the first by integrating in time and applying C-S, along with the energy estimate (4.5). An easy derivation of the first part is obtained by noticing that there exists a reference point $x_0 := x_0(t)$ for which $\xi^m_s(x_0, t) = 0$. Indeed, this is implied by the vanishing of $\xi^m(x, t)$ at the endpoints $x = 0, B$ (4.4). The above argument applies directly. □

To write our system for $d\eta^{m+1}, d\xi^{m+1}$ concisely, we introduce generic notation

$$B, D$$

to denote rational functions in η^m, ξ^m, $m = 0, 1, \ldots$, satisfying the following conditions:

- The denomerators of B, D have non-zero constant terms.
- The constant term in the numerator of B is non-zero, whereas the one in the numerator of D vanishes.

The next lemma is an immediate consequence of the pointwise estimates (4.8) and the energy estimate (4.5).

Lemma 4.2. *If B, D are functions as above and $\mathcal{E}_0$ is sufficiently small, then the following estimates hold:*

$$\|B(s,t)\|_{L^\infty(x)} < C \qquad\qquad \|\frac{D}{\ell^k}\|^2_{L^\infty(x)} \leq C\mathcal{E}_0, \tag{4.11}$$

where $k = 0, \ldots, \alpha - 1$ and

$$\|\frac{B_s}{\ell^{\alpha-1}}\|^2_{L^2} + \|\frac{D_s}{\ell^{\alpha-1}}\|^2_{L^2} \leq C\mathcal{E}_0, \tag{4.12}$$

for $0 \leq t \leq T$ and C a positive constant depending on the coefficients of the rational functions B, D.

Consider now the two systems (4.2) corresponding to the steps $m+1$ and m. We derive a new system for $d\eta^{m+1}, d\xi^{m+1}$ (4.6) by subtracting these two systems. Doing so, it is straightforward to check that we arrive at the following system:

$$\begin{aligned} d\eta_t^{m+1} &= \frac{\psi_s^2}{\psi^2} B d\xi^{m+1} + \frac{\psi_s}{\psi} B d\xi_s^{m+1} + 2n(n-1)\frac{\psi_s^2}{\psi^2} d\eta^{m+1} + dF_1^m \\ d\xi_t^{m+1} &= (\frac{\psi_{ss}}{\psi} + (n-1)\frac{\psi_s^2}{\psi^2})(B d\eta^{m+1} + B d\xi^{m+1}) + \frac{\psi_s}{\psi} B d\xi_s^{m+1} + \frac{\psi_s}{\psi} B d\eta_s^{m+1} \\ &\quad + \frac{d\xi_{ss}^{m+1}}{(\eta^m + 1)(\xi^m + 1)^{2n}} + dF_2^m, \end{aligned} \tag{4.13}$$

where

$$\begin{aligned} dF_1^m :=& \frac{\psi_s^2}{\psi^2} D d\xi^m + \frac{A}{s} B(d\xi^m + d\eta^m) + \frac{\psi_s}{\psi} B \xi_s^m d\xi^m + B d\xi_s^m(\xi_s^m + \xi_s^{m-1}) \\ &+ |\xi_s^{m-1}|^2 d\xi^m B \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} dF_2^m :=& (\frac{\psi_{ss}}{\psi} + (n-1)\frac{\psi_s^2}{\psi^2})(D d\xi^m + D d\eta^m) + \frac{A}{s} B d\xi^m + \frac{\psi_s}{\psi} B \xi_s^m (d\eta^m + d\xi^m) \\ &+ \xi_{ss}^m B(d\eta^m + d\xi^m) + B d\xi_s^m(\xi_s^m + \xi_s^{m-1}) + |\xi_s^{m-1}|^2 B(d\eta^m + d\xi^m) \\ &+ \frac{\psi_s}{\psi}\eta_s^m B(d\eta^m + d\xi^m) + B(\xi_s^m d\eta_s^m + \eta_s^{m-1} d\xi_s^m) + \eta_s^{m-1}\xi_s^{m-1} B(d\eta^m + d\xi^m) \end{aligned} \tag{4.15}$$

We note that the terms $d\xi_s^m(\xi_s^m + \xi_s^{m-1})$ and $\xi_{ss}^m B(d\eta^m + d\xi^m)$ are of the most problematic and an additional reason we need to consider small initial energy $\mathcal{E}_0$ in order to close the contraction mapping argument in H^1_α.

Similarly to Lemma 4.1, we have the following L^∞ estimates for the differences.

Lemma 4.3. *For every $m \in \mathbb{N}$ and $t \in [0,T]$ the following estimates hold:*

$$\|\frac{d\xi^m}{\ell^k}\|^2_{L^\infty(s)} \leq C(k+1)\|d\xi^m\|^2_{H^1_{k+1}} \qquad \|\frac{d\eta^m}{\ell^k}\|^2_{L^\infty(s)} \leq C(k+1)\|d\eta^m\|^2_{H^1_{k+1}} \tag{4.16}$$

and

$$\|\frac{d\xi^m_s}{\ell^k}\|^2_{L^\infty(s)} \leq C\|\frac{d\xi^m_s}{\ell^{\alpha-1}}\|_{L^2}\big(\|\frac{d\xi^m_{ss}}{\ell^{\alpha-1}}\|_{L^2} + k\|\frac{d\xi^m_s}{\ell^\alpha}\|_{L^2}\big), \tag{4.17}$$

$k = 0, \dots, \alpha - 1$.

4.2 Proof of Proposition 4.1: the contraction estimate (4.7)

In this subsection we show that the desired contraction estimate (4.7) follows from the next proposition, whose proof in turn we divide in three parts occupying the subsequent subsections §4.3, §4.4, §4.5.

Proposition 4.2. *The following estimates are valid in the time interval $[0,T]$. First, for $d\eta^{m+1}, d\xi^{m+1}$ in L^2_α we have*

$$\begin{aligned}
&\frac{1}{2}\big(\|d\eta^{m+1}\|^2_{L^2_\alpha[t]} + \|d\xi^{m+1}\|^2_{L^2_\alpha[t]}\big) + \alpha\sigma\int_0^t \big(\|\frac{d\eta^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)} + \|\frac{d\xi^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}\big)d\tau \\
&\leq C\int_0^t \big(\alpha^2 + \|\xi^m_s\|^2_{L^2}\big)\big(\|d\eta^{m+1}\|^2_{L^2_\alpha[\tau]} + \|d\xi^{m+1}\|^2_{L^2_\alpha[\tau]}\big)d\tau \\
&\quad + C(\alpha^2+\sigma)\int_0^t \big(\|\frac{d\eta^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)} + \|\frac{d\xi^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}\big)d\tau \\
&\quad + C\big[\mathcal{E}_0\sigma^2 T + (\mathcal{E}_0+1)\sigma\sqrt{T} + \sqrt{T}\mathcal{E}_0^2 + \mathcal{E}_0\big]\mathcal{E}(d\eta^m, d\xi^m; T)
\end{aligned} \tag{4.18}$$

and second for the first derivatives $d\eta^{m+1}_s, d\xi^{m+1}_s$ in $L^2_{\alpha-1}$

$$\begin{aligned}
&\frac{1}{2}\big(\|d\eta^{m+1}_s\|^2_{L^2_{\alpha-1}[t]} + \|d\xi^{m+1}_s\|^2_{L^2_{\alpha-1}[t]}\big) \\
&\quad + \alpha\sigma\int_0^t \big(\|\frac{d\eta^{m+1}_s}{\ell^\alpha}\|^2_{L^2(0,\delta)} + \|\frac{d\xi^{m+1}_s}{\ell^\alpha}\|^2_{L^2(0,\delta)}\big)d\tau + \frac{1}{6}\int_0^t \|d\xi^{m+1}_{ss}\|^2_{L^2_{\alpha-1}[\tau]}d\tau \\
&\leq C\int_0^t \big(\|\xi^m_s\|^2_{L^\infty} + \|\xi^{m-1}_s\|^2_{L^\infty} + \alpha^2\big)\big(\|d\eta^{m+1}_s\|^2_{L^2_{\alpha-1}[\tau]} + \|d\xi^{m+1}_s\|^2_{L^2_{\alpha-1}[\tau]}\big)d\tau \\
&\quad + C\int_0^t \big(\|\frac{\xi^m_{ss}}{\ell^{\alpha-1}}\|_{L^2} + \|\frac{\xi^{m-1}_{ss}}{\ell^{\alpha-1}}\|_{L^2}\big)\|d\xi^m_s\|_{L^\infty}\|d\eta^{m+1}_s\|_{L^2_{\alpha-1}[\tau]}d\tau \\
&\quad + C(\alpha^2+\sigma)\int_0^t \big(\|\frac{d\eta^{m+1}_s}{\ell^\alpha}\|^2_{L^2(0,\delta)} + \|\frac{d\xi^{m+1}_s}{\ell^\alpha}\|^2_{L^2(0,\delta)}\big)d\tau \\
&\quad + C\int_0^t \big(\|d\eta^{m+1}\|^2_{L^2_\alpha} + \|d\xi^{m+1}\|^2_{L^2_\alpha}\big)d\tau + C\sigma^2\int_0^t \big(\|\frac{d\eta^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)} + \|\frac{d\xi^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}\big)d\tau \\
&\quad + C\big[\mathcal{E}_0\sigma^2 T + (\mathcal{E}_0+1)\sigma\sqrt{T} + \mathcal{E}_0^2\sqrt{T} + \mathcal{E}_0\big]\mathcal{E}(d\eta^m, d\xi^m; T)
\end{aligned} \tag{4.19}$$

It is precisely at this point that the significance of the weights we introduced becomes apparent. We wish to close the above energy estimates by applying the standard Gronwall lemma. Unfortunately, this is not possible due to the terms in the RHSs of the estimates in the preceding proposition having larger exponents in the weights (by one) than the ones in the norms differentiated in the LHS, e.g., line three in (4.18). We call these terms 'critical'. Estimating the extra weight of the critical terms in $L^\infty(x)$ would not close either, as noted in Remark 3.21. We have to keep it in the norms. Thus, the only way to close the estimates is by absorbing these terms into the corresponding critical terms in the LHSs which work in our favor, e.g., (4.18) line one. That is where the role of the parameters α, σ comes into play:

Clearly, we may choose these parameters appropriately large such that the critical terms in the estimate (4.18), line three, are absorbed in the LHS. However, we notice that the critical terms in the estimate (4.19), lines five and six, cannot be directly absorbed by the corresponding ones in the estimates (4.18), (4.19), lines one and two respectively, since $C(\alpha^2+\sigma^2)$ dominates $\alpha\sigma$ (C is large in our setting); see coefficients α^2, σ^2 in the RHSs of (4.18) line three and (4.19) line six respectively. In order, to bypass this issue it is crucial that we can close the estimates of $d\eta^{m+1}, d\xi^{m+1}$, before moving on to estimate their derivatives. Since we are able to do that, we can then absorb the critical term in (4.19), line five, by choosing $\alpha\sigma > C(\alpha^2+\sigma)$ and use afterwards the already derived estimate of the zeroth order terms to estimate the critical terms in (4.19) line six, instead of absorbing them anywhere. This way we can close the estimates for the first order terms $d\eta_s^{m+1}, d\xi_s^{m+1}$ in $L^2_{\alpha-1}$ and obtain the desired contraction estimate (4.7) for small $T, \mathcal{E}_0$.

We will use below in the proof the following simple modified version of Gronwall's inequality.

Lemma 4.4. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function which satisfies:*

$$\frac{1}{2}f^2(t) \leq \frac{1}{2}f_0^2 + \int_a^t \Psi(\tau)f(\tau)d\tau, \quad t \in [a,b],$$

where $f_0 \in \mathbb{R}$ and Ψ nonnegative continuous in $[a,b]$. Then the estimate

$$\frac{1}{2}|f(t)| \leq \frac{1}{2}|f_0| + \int_a^t \Psi(\tau)d\tau, \quad t \in [a,b]$$

holds.

Proposition 4.2 implies the contraction (4.7): Choosing α, σ appropriately large such that

$$\alpha\sigma > C(\alpha^2+\sigma)+1,$$

the critical terms on the RHS of (4.18), line three, are be absorbed in the LHS. Hence, we may employ the standard (integral form of) Gronwall's inequality, applying the estimate (4.9), to close the estimate of the zeroth order terms $d\eta^{m+1}, d\xi^{m+1}$:

$$\begin{aligned}&\sup_{[0,T]}\big(\|d\eta^{m+1}\|^2_{L^2_\alpha} + \|d\xi^{m+1}\|^2_{L^2_\alpha}\big) + \int_0^T \big(\|\frac{d\eta_s^{m+1}}{\ell^\alpha}\|^2_{L^2(0,\delta)} + \|\frac{d\xi_s^{m+1}}{\ell^\alpha}\|^2_{L^2(0,\delta)}\big)d\tau \\ &\leq C\exp\{C(\alpha^2 T + \mathcal{E}_0\sqrt{T})\}\big[\mathcal{E}_0\sigma^2 T + (\mathcal{E}_0+1)\sigma\sqrt{T} + \sqrt{T}\mathcal{E}_0^2 + \mathcal{E}_0\big]\mathcal{E}(d\eta^m, d\xi^m; T)\end{aligned} \tag{4.20}$$

We proceed to the estimate of the first derivatives (4.19). For the same choice of α, σ as above (uniform C), we absorb the critical terms in the RHS, line five, involving the first order terms $d\eta_s^{m+1}, d\xi_s^{m+1}$. Also, utilizing the preceding estimate (4.20) we estimate the zeroth order terms on the RHS of (4.19), line six; including the critical terms with a bad sign coefficient of magnitude σ^2. Thus, we have

$$
\begin{aligned}
&\frac{1}{2}\big(\|d\eta_s^{m+1}\|^2_{L^2_{\alpha-1}[t]} + \|d\xi_s^{m+1}\|^2_{L^2_{\alpha-1}[t]}\big) \\
&\quad + \int_0^t \big(\|\frac{d\eta_s^{m+1}}{\ell^\alpha}\|^2_{L^2(0,\delta)} + \|\frac{d\xi_s^{m+1}}{\ell^\alpha}\|^2_{L^2(0,\delta)}\big)d\tau + \frac{1}{6}\int_0^t \|d\xi_{ss}^{m+1}\|^2_{L^2_{\alpha-1}[\tau]}d\tau \\
&\leq C\int_0^t \big(\|\xi_s^m\|^2_{L^\infty} + \|\xi_s^{m-1}\|^2_{L^\infty} + \alpha^2\big)\big(\|d\eta_s^{m+1}\|^2_{L^2_{\alpha-1}[\tau]} + \|d\xi_s^{m+1}\|^2_{L^2_{\alpha-1}[\tau]}\big)d\tau \\
&\quad + C\int_0^t \big(\|\frac{\xi_{ss}^m}{\ell^{\alpha-1}}\|_{L^2} + \|\frac{\xi_{ss}^{m-1}}{\ell^{\alpha-1}}\|_{L^2}\big)\|d\xi_s^m\|_{L^\infty}\|d\eta_s^{m+1}\|_{L^2_{\alpha-1}[\tau]}d\tau \qquad (4.21)\\
&\quad + C\big(\sigma^2 e^{C(\alpha^2T+\mathcal{E}_0\sqrt{T})}+1\big)\big[\mathcal{E}_0\sigma^2T + (\mathcal{E}_0+1)\sigma\sqrt{T} + \mathcal{E}_0^2\sqrt{T} + \mathcal{E}_0\big]\mathcal{E}(d\eta^m, d\xi^m;T)
\end{aligned}
$$

Employing Lemma 4.4 for

$$
\begin{aligned}
f^2(t) &= \|d\eta_s^{m+1}\|^2_{L^2_{\alpha-1}[t]} + \|d\xi_s^{m+1}\|^2_{L^2_{\alpha-1}[t]} \\
\frac{1}{2}f_0^2 &= \text{the last term in (4.21)} \\
\Psi &= C\big(\|\xi_s^m\|^2_{L^\infty} + \|\xi_s^{m-1}\|^2_{L^\infty} + \alpha^2\big)\big(\|d\eta_s^{m+1}\|^2_{L^2_{\alpha-1}[\tau]} + \|d\xi_s^{m+1}\|^2_{L^2_{\alpha-1}[\tau]}\big)^{\frac{1}{2}} \\
&\quad + C\big(\|\frac{\xi_{ss}^m}{\ell^{\alpha-1}}\|_{L^2} + \|\frac{\xi_{ss}^{m-1}}{\ell^{\alpha-1}}\|_{L^2}\big)\|d\xi_s^m\|_{L^\infty}
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\sup_{t\in[0,T]}\big(\|d\eta_s^{m+1}\|^2_{L^2_{\alpha-1}[t]} + \|d\xi_s^{m+1}\|^2_{L^2_{\alpha-1}[t]}\big) \leq \int_0^T \Psi(\tau)d\tau \qquad (4.22)\\
&+ C\big(\sigma^2 e^{C(\alpha^2T+\mathcal{E}_0\sqrt{T})}+1\big)\big[\mathcal{E}_0\sigma^2T + (\mathcal{E}_0+1)\sigma\sqrt{T} + \mathcal{E}_0^2\sqrt{T} + \mathcal{E}_0\big]\mathcal{E}(d\eta^m, d\xi^m;T).
\end{aligned}
$$

Finally, applying C-S and (4.5),(4.17) we estimate

$$
\begin{aligned}
\int_0^T \Psi d\tau &\leq C(\alpha^2T + \mathcal{E}_0\sqrt{T})\sup_{t\in[0,T]}\big(\|d\eta_s^{m+1}\|^2_{L^2_{\alpha-1}[t]} + \|d\xi_s^{m+1}\|^2_{L^2_{\alpha-1}[t]}\big)^{\frac{1}{2}} \qquad (4.23)\\
&\quad + C\sqrt{\mathcal{E}_0}\cdot\mathcal{E}(d\eta^m, d\xi^m;T)
\end{aligned}
$$

Hence, for $T > 0$ small we absorb the first term in (4.23) to the LHS of (4.22) and close the estimates of $d\eta_s^{m+1}, d\xi_s^{m+1}$.

From the above estimates we deduce the contraction estimate (4.7), provided $T, \mathcal{E}_0$ are sufficiently small. □

4.3 Proof of Proposistion 4.2 I: Estimates for the non-linear terms

We establish some estimates for the functions dF_1^m, dF_2^m (4.14),(4.15) that we will use in proving the estimates in Proposition 4.2.

Proposition 4.3. *For any function $u \in L^2(0,T;L^2_\alpha)$ and $t \in [0,T]$ the following estimates hold:*

$$
\int_0^t \|dF_1^m\|^2_{L^2_{\alpha-1}}d\tau \leq C\Big(\mathcal{E}_0\sigma^2T + (\mathcal{E}_0+1)\sigma\sqrt{T} + \sqrt{T}\mathcal{E}_0^2\Big)\mathcal{E}(d\eta^m, d\xi^m;T), \qquad (4.24)
$$

$$\int_0^t \|dF_2^m\|^2_{L^2_{\alpha-1}}d\tau \le C\Big(\mathcal{E}_0\sigma^2 T + (\mathcal{E}_0+1)\sigma\sqrt{T} + \sqrt{T}\mathcal{E}_0^2 + \mathcal{E}_0\Big)\mathcal{E}(d\eta^m, d\xi^m; T) \tag{4.25}$$

and

$$\begin{aligned}
&\int_0^t\int \frac{u\cdot\partial_s(dF_1^m)}{\ell^{2\alpha}}dsd\tau \le \frac{C}{\varepsilon}\sigma\int_0^t \|\frac{u}{\ell^\alpha}\|^2_{L^2(0,\delta)}d\tau + \frac{C}{\varepsilon}\int_0^t \|\frac{u}{\ell^{\alpha-1}}\|^2_{L^2}d\tau\\
&+\frac{C}{\varepsilon}\int_0^t \big(\|\xi_s^m\|^2_{L^\infty} + \|\xi_s^{m-1}\|^2_{L^\infty}+1\big)\|\frac{u}{\ell^{\alpha-1}}\|^2_{L^2}d\tau \qquad (4.26)\\
&+C\int_0^t \big(\|\frac{\xi^m_{ss}}{\ell^{\alpha-1}}\|_{L^2} + \|\frac{\xi^{m-1}_{ss}}{\ell^{\alpha-1}}\|_{L^2}\big)\|d\xi^m_s\|_{L^\infty}\|\frac{u}{\ell^{\alpha-1}}\|_{L^2}d\tau \qquad (0<\varepsilon<1)\\
&+C\Big(\mathcal{E}_0\sigma^2T + \mathcal{E}_0\sigma\sqrt{T} + \varepsilon(\mathcal{E}_0+1) + (\mathcal{E}_0+1)^2\sqrt{T}\Big)\mathcal{E}(d\eta^m, d\xi^m; T)
\end{aligned}$$

We remark that the only part that 'does not belong' in the above estimates, is the last summand in (4.25) from which we do not gain any smallness in T. This term comes from estimating $\xi^m_{ss}B(d\eta^m + d\xi^m)$ in dF_2^m (4.15) below.

Proof. Recall the leading behavior of the coefficients (3.15), (3.22). Plugging (4.14) in the norm below we estimate:

$$\begin{aligned}
\|dF_1^m\|^2_{L^2_{\alpha-1}} &\le \|\frac{\psi_s^2}{\psi^2}\frac{Dd\xi^m}{\ell^{\alpha-1}}\|^2_{L^2} + \|\frac{A}{s}\frac{B(d\xi^m + d\eta^m)}{\ell^{\alpha-1}}\|^2_{L^2} + \|\frac{\psi_s}{\psi}B\xi^m_s\frac{d\xi^m}{\ell^{\alpha-1}}\|^2_{L^2} \qquad (4.27)\\
&\quad + \|B\frac{d\xi^m_s}{\ell^{\alpha-1}}(\xi^m_s + \xi^{m-1}_s)\|^2_{L^2} + \||\xi_s^{m-1}|^2B\frac{d\xi^m}{\ell^{\alpha-1}}\|^2_{L^2}\\
&\le C\mathcal{E}_0\|\frac{d\xi^m}{s^2\ell^{\alpha-2}}\|^2_{L^2} \qquad \text{(using the estimate (4.11) for the fraction } \tfrac{D}{\ell})\\
&\quad + CA^2(t)\big(\|\frac{d\xi^m}{s\ell^{\alpha-1}}\|^2_{L^2} + \|\frac{d\eta^m}{s\ell^{\alpha-1}}\|^2_{L^2}\big) + C\|\xi^m_s\|^2_{L^\infty}\|\frac{d\xi^m}{s\ell^{\alpha-1}}\|^2_{L^2}\\
&\quad + C\big(\|\xi^m_s\|^2_{L^\infty} + \|\xi^{m-1}_s\|^2_{L^\infty}\big)\|\frac{d\xi^m_s}{\ell^{\alpha-1}}\|^2_{L^2} + C\|\xi^m_s\|^2_{L^\infty}\|d\xi^m\|^2_{L^\infty}\|\frac{\xi_s^{m-1}}{\ell^{\alpha-1}}\|^2_{L^2}
\end{aligned}$$

Employing the comparison estimate $\ell^2/s^2 \le C\sigma$ (3.28) for the first three terms in the RHS of the second inequality above and the L^∞ estimate of $d\xi^m$ (4.16) for the last term we obtain

$$\begin{aligned}
\|dF_1^m\|^2_{L^2_{\alpha-1}} \le C\big(&\mathcal{E}_0\sigma^2 + A^2(t)\sigma + \|\xi^m_s\|^2_{L^\infty}\sigma + \|\xi^m_s\|^2_{L^\infty} + \|\xi^{m-1}_s\|^2_{L^\infty} \qquad (4.28)\\
&+ \|\xi^{m-1}_s\|^2_{L^\infty}\mathcal{E}_0\big)\big(\|d\xi^m\|^2_{H^1_\alpha} + \|d\eta^m\|^2_{L^2_\alpha}\big)
\end{aligned}$$

After integrating in time and applying (3.23),(4.9) we arrive at (4.24).

Similarly, for the case of dF_2^m plugging in (4.15) we derive:

$$\begin{aligned}
&\|dF_2^m\|^2_{L^2_{\alpha-1}} \qquad (4.29)\\
&\le \|(\frac{\psi_{ss}}{\psi} + (n-1)\frac{\psi_s^2}{\psi^2})\frac{D(d\xi^m + d\eta^m)}{\ell^{\alpha-1}}\|^2_{L^2} + \|\frac{A}{s}B\frac{d\xi^m}{\ell^{\alpha-1}}\|^2_{L^2} + \|\frac{\psi_s}{\psi}B\xi^m_s\frac{d\eta^m + d\xi^m}{\ell^{\alpha-1}}\|^2_{L^2}\\
&\quad + \|\xi^m_{ss}B\frac{d\eta^m + d\xi^m}{\ell^{\alpha-1}}\|^2_{L^2} + \|B\frac{d\xi^m_s}{\ell^{\alpha-1}}(\xi^m_s + \xi^{m-1}_s)\|^2_{L^2} + \||\xi^{m-1}_s|^2B\frac{d\eta^m + d\xi^m}{\ell^{\alpha-1}}\|^2_{L^2}\\
&\quad + \|\frac{\psi_s}{\psi}\eta^m_sB\frac{d\eta^m + d\xi^m}{\ell^{\alpha-1}}\|^2_{L^2} + \|B(\xi^m_s\frac{d\eta^m_s}{\ell^{\alpha-1}} + \eta^{m-1}_s\frac{d\xi^m_s}{\ell^{\alpha-1}})\|^2_{L^2} + \|\eta^{m-1}_s\xi^{m-1}_sB\frac{d\eta^m + d\xi^m}{\ell^{\alpha-1}}\|^2_{L^2}\\
&\le C\mathcal{E}_0\big(\|\frac{d\xi^m}{s^2\ell^{\alpha-2}}\|^2_{L^2} + \|\frac{d\eta^m}{s^2\ell^{\alpha-2}}\|^2_{L^2}\big) \qquad \text{(applying (4.11) for } \tfrac{D}{\ell} \text{ and } B)
\end{aligned}$$

$$+ CA^2(t)\|\frac{d\xi^m}{s\ell^{\alpha-1}}\|^2_{L^2} + C\|\xi^m_s\|^2_{L^\infty}\big(\|\frac{d\eta^m}{s\ell^{\alpha-1}}\|^2_{L^2} + \|\frac{d\xi^m}{s\ell^{\alpha-1}}\|^2_{L^2}\big)$$
$$+ C\big(\|d\eta^m\|^2_{L^\infty} + \|d\xi^m\|^2_{L^\infty}\big)\|\frac{\xi^m_{ss}}{\ell^{\alpha-1}}\|^2_{L^2} + C\big(\|\xi^m_s\|^2_{L^\infty} + \|\xi^{m-1}_s\|^2_{L^\infty}\big)\|\frac{d\xi^m_s}{\ell^{\alpha-1}}\|^2_{L^2}$$
$$+ C\|\xi^{m-1}_s\|^2_{L^\infty}\big(\|d\eta^m\|^2_{L^\infty} + \|d\xi^m\|^2_{L^\infty}\big)\|\frac{\xi^{m-1}_s}{\ell^{\alpha-1}}\|^2_{L^2}$$
$$+ C\big(\|\frac{d\eta^m}{s}\|^2_{L^\infty} + \|\frac{d\xi^m}{s}\|^2_{L^\infty}\big)\|\frac{\eta^m_s}{\ell^{\alpha-1}}\|^2_{L^2} + C\|\xi^m_s\|^2_{L^\infty}\|\frac{d\eta^m_s}{\ell^{\alpha-1}}\|^2_{L^2}$$
$$+ C\|d\xi^m_s\|^2_{L^\infty}\|\frac{\eta^{m-1}_s}{\ell^{\alpha-1}}\|^2_{L^2} + C\|\xi^{m-1}_s\|^2_{L^\infty}\big(\|d\eta^m\|^2_{L^\infty} + \|d\xi^m\|^2_{L^\infty}\big)\|\frac{\eta^{m-1}_s}{\ell^{\alpha-1}}\|^2_{L^2}$$

We employ once more the comparison estimate (3.28), the energy estimate of the iterates (4.5) and the L^∞ estimates for $d\eta^m, d\xi^m, d\xi^m_s$ to get

$$\begin{aligned}
&\|dF^m_2\|^2_{L^2_{\alpha-1}} \qquad (4.30)\\
&\le C\big(\mathcal{E}_0\sigma^2 + A^2(t)\sigma + \|\xi^m_s\|^2_{L^\infty}\sigma + \sigma\mathcal{E}_0\big)\big(\|d\xi^m\|^2_{H^1_\alpha} + \|d\eta^m\|^2_{H^1_\alpha}\big)\\
&\quad + C\big(\|\xi^m_s\|^2_{L^\infty} + \|\xi^{m-1}_s\|^2_{L^\infty} + \|\xi^{m-1}_s\|^2_{L^\infty}\mathcal{E}_0\big)\big(\|d\xi^m\|^2_{H^1_\alpha} + \|d\eta^m\|^2_{H^1_\alpha}\big)\\
&\quad + C\|\frac{\xi^m_{ss}}{\ell^{\alpha-1}}\|^2_{L^2}\big(\|d\xi^m\|^2_{H^1_\alpha} + \|d\eta^m\|^2_{H^1_\alpha}\big) + \mathcal{E}_0\|\frac{d\xi^m_s}{\ell^{\alpha-1}}\|_{L^2}\|\frac{d\xi^m_{ss}}{\ell^{\alpha-1}}\|_{L^2}
\end{aligned}$$

Integrating from $0 \le \tau \le t$, applying C-S to the last term above and utilizing (4.9) we achieve the estimate (4.25).[5]

We proceed to the relevant estimates of $\partial_s(dF^m_1)$. This time, to be comprehensive, we plug in each term in the RHS of (4.14) at the time and estimate it separately. Recall again the singular orders of the coefficients (3.15), (3.22) and the ones of their spatial derivatives (3.17). Applying C-S to each arising term we have:

$$\begin{aligned}
&\int \frac{u}{\ell^{2\alpha-2}}\partial_s\Big[\frac{\psi_s^2}{\psi^2}Dd\xi^m\Big]ds \qquad (4.31)\\
&= \int \frac{u}{\ell^{2\alpha-2}}\Big[\partial_s(\frac{\psi_s^2}{\psi^2})Dd\xi^m + \frac{\psi_s^2}{\psi^2}D_s d\xi^m + \frac{\psi_s^2}{\psi^2}Dd\xi^m_s\Big]ds\\
&\le \|\frac{u}{s\ell^{\alpha-1}}\|^2_{L^2} + C\mathcal{E}_0\|\frac{d\xi^m}{s^2\ell^{\alpha-2}}\|^2_{L^2}\\
&\qquad\qquad \text{(by C-S and the pointwise estimate of } D \text{ (4.11), } k=1)\\
&\quad + \|\frac{u}{s\ell^{\alpha-1}}\|^2_{L^2} + C\|\frac{d\xi^m}{s}\|^2_{L^\infty}\|\frac{D_s}{\ell^{\alpha-1}}\|^2_{L^2} + \|\frac{u}{s\ell^{\alpha-1}}\|^2_{L^2} + C\mathcal{E}_0\|\frac{d\xi^m_s}{s\ell^{\alpha-2}}\|^2_{L^2}\\
&\le C\sigma\|\frac{u}{\ell^\alpha}\|^2_{L^2(0,\delta)} + C\|\frac{u}{\ell^{\alpha-1}}\|^2_{L^2(\delta,+\infty)} \qquad \text{(recall def. (3.26); estimate } \ell^2/s^2 \text{ (3.28))}\\
&\quad + C\mathcal{E}_0\sigma^2\|\frac{d\xi^m}{\ell^\alpha}\|^2_{L^2} + C\mathcal{E}_0\sigma\|\frac{d\xi^m_s}{\ell^{\alpha-1}}\|^2_{L^2} \quad \text{(by (4.12) for } D_s \text{ and } d\xi^m\text{-}L^\infty \text{ estimate (4.16))}
\end{aligned}$$

Similarly, utilizing the estimates on B (4.11), (4.12) we obtain

$$\begin{aligned}
&\int \frac{u}{\ell^{2\alpha-2}}\partial_s\Big[\frac{A}{s}B(d\xi^m + d\eta^m)\Big]ds \qquad (4.32)\\
&= \int \frac{u}{\ell^{2\alpha-2}}\Big[\partial_s(\frac{A}{s})B(d\xi^m + d\eta^m) + \frac{A}{s}B_s(d\xi^m + d\eta^m) + \frac{A}{s}B(d\xi^m_s + d\eta^m_s)\Big]ds\\
&\le \|\frac{u}{s\ell^{\alpha-1}}\|^2_{L^2} + CA^2(t)\big(\|\frac{d\xi^m}{s\ell^{\alpha-1}}\|^2_{L^2} + \|\frac{d\eta^m}{s\ell^{\alpha-1}}\|^2_{L^2}\big)
\end{aligned}$$

[5] The second last term in the RHS of (4.30) is the first problematic term that forces us to assume further smallness of the initial energy $\mathcal{E}_0$.

$$\|\frac{u}{s\ell^{\alpha-1}}\|_{L^2}^2+CA^2(t)\big(\|d\xi^m\|_{L^\infty}^2+\|d\eta^m\|_{L^\infty}^2\big)\|\frac{B_s}{\ell^{\alpha-1}}\|_{L^2}^2$$
$$+\|\frac{u}{s\ell^{\alpha-1}}\|_{L^2}^2+CA^2(t)\big(\|\frac{d\xi_s^m}{\ell^{\alpha-1}}\|_{L^2}^2+\|\frac{d\eta_s^m}{\ell^{\alpha-1}}\|_{L^2}^2\big)$$
$$\leq C\sigma\|\frac{u}{\ell^\alpha}\|_{L^2(0,\delta)}^2+C\|\frac{u}{\ell^{\alpha-1}}\|_{L^2(\delta,+\infty)}^2+CA^2(t)(\mathcal{E}_0+\sigma+1)\big(\|d\xi^m\|_{H_\alpha^1}^2+\|d\eta^m\|_{H_\alpha^1}^2\big)$$

and

$$\int\frac{u}{\ell^{2\alpha-2}}\partial_s\Big[\frac{\psi_s}{\psi}B\xi_s^m d\xi^m\Big]ds \tag{4.33}$$
$$=\int\frac{u}{\ell^{2\alpha-2}}\Big[\partial_s(\frac{\psi_s}{\psi})B\xi_s^m d\xi^m+\frac{\psi_s}{\psi}B_s\xi_s^m d\xi^m+\frac{\psi_s}{\psi}B\xi_{ss}^m d\xi^m+\frac{\psi_s}{\psi}B\xi_s^m d\xi_s^m\Big]ds$$
$$\leq\|\frac{u}{s\ell^{\alpha-1}}\|_{L^2}^2+C\|\xi_s^m\|_{L^\infty}^2\|\frac{d\xi^m}{s\ell^{\alpha-1}}\|_{L^2}^2+C\|\xi_s^m\|_{L^\infty}^2\|d\xi^m\|_{L^\infty}^2\|\frac{B_s}{\ell^{\alpha-1}}\|_{L^2}^2$$
$$+\frac{C}{\varepsilon}\|\frac{u}{s\ell^{\alpha-1}}\|_{L^2}^2+\varepsilon\|d\xi^m\|_{L^\infty}^2\|\frac{\xi_{ss}^m}{\ell^{\alpha-1}}\|_{L^2}^2+\|\frac{u}{s\ell^{\alpha-1}}\|_{L^2}^2+C\|\xi_s^m\|_{L^\infty}^2\|\frac{d\xi_s^m}{\ell^{\alpha-1}}\|_{L^2}^2$$
$$\leq\frac{C}{\varepsilon}\sigma\|\frac{u}{\ell^\alpha}\|_{L^2(0,\delta)}^2+\frac{C}{\varepsilon}\|\frac{u}{\ell^{\alpha-1}}\|_{L^2(\delta,+\infty)}^2+C\sigma\|\xi_s^m\|_{L^\infty}^2\|\frac{d\xi^m}{\ell^\alpha}\|_{L^2}^2 \qquad \text{(using (3.28))}$$
$$+\big(\varepsilon\|\frac{\xi_{ss}^m}{\ell^{\alpha-1}}\|_{L^2}^2+C\mathcal{E}_0\|\xi_s^m\|_{L^\infty}^2+C\|\xi_s^m\|_{L^\infty}^2\big)\|d\xi^m\|_{H_\alpha^1}^2$$

The last term to be estimated is a bit more involved. We follow the same plan employing the estimates on B (4.11), (4.12) and the L^∞ estimates of $d\xi^m, d\xi_s^m$ (4.16), (4.17).

$$\int\frac{u}{\ell^{2\alpha-2}}\partial_s\Big[Bd\xi_s^m(\xi_s^m+\xi_s^{m-1})+|\xi_s^{m-1}|^2d\xi^m B\Big]ds \tag{4.34}$$
$$=\int\frac{u}{\ell^{2\alpha-2}}\Big[B_sd\xi_s^m(\xi_s^m+\xi_s^{m-1})+Bd\xi_{ss}^m(\xi_s^m+\xi_s^{m-1})+Bd\xi_s^m(\xi_{ss}^m+\xi_{ss}^{m-1})$$
$$+2\xi_s^{m-1}\xi_{ss}^{m-1}d\xi^mB+|\xi_s^{m-1}|^2d\xi_s^mB+|\xi_s^{m-1}|^2d\xi^mB_s\Big]ds$$
$$\leq C\big(\|\xi_s^m\|_{L^\infty}^2+\|\xi_s^{m-1}\|_{L^\infty}^2\big)\|\frac{u}{\ell^{\alpha-1}}\|_{L^2}^2+\|d\xi_s^m\|_{L^\infty}^2\|\frac{B_s}{\ell^{\alpha-1}}\|_{L^2}^2$$
$$+\varepsilon\|\frac{d\xi_{ss}^m}{\ell^{\alpha-1}}\|_{L^2}^2+\frac{C}{\varepsilon}\big(\|\xi_s^m\|_{L^\infty}^2+\|\xi_s^{m-1}\|_{L^\infty}^2\big)\|\frac{u}{\ell^{\alpha-1}}\|_{L^2}^2$$
$$+C\|\frac{u}{\ell^{\alpha-1}}\|_{L^2}\|d\xi_s^m\|_{L^\infty}\big(\|\frac{\xi_{ss}^m}{\ell^{\alpha-1}}\|_{L^2}+\|\frac{\xi_{ss}^{m-1}}{\ell^{\alpha-1}}\|_{L^2}\big)$$
$$+\frac{C}{\varepsilon}\|\xi_s^{m-1}\|_{L^\infty}^2\|\frac{u}{\ell^{\alpha-1}}\|_{L^2}^2+\varepsilon\|d\xi^m\|_{L^\infty}\|\frac{\xi_{ss}^{m-1}}{\ell^{\alpha-1}}\|_{L^2}^2$$
$$+\|\xi_s^{m-1}\|_{L^\infty}^2\|\frac{u}{\ell^{\alpha-1}}\|_{L^2}^2+C\|\xi_s^{m-1}\|_{L^\infty}^2\|\frac{d\xi_s^m}{\ell^{\alpha-1}}\|_{L^2}^2+C\|\xi_s^{m-1}\|_{L^\infty}^2\|d\xi^m\|_{L^\infty}^2\|\frac{B_s}{\ell^{\alpha-1}}\|_{L^2}^2$$
$$\leq\frac{C}{\varepsilon}\big(\|\xi_s^m\|_{L^\infty}^2+\|\xi_s^{m-1}\|_{L^\infty}^2\big)\|\frac{u}{\ell^{\alpha-1}}\|_{L^2}^2+C\|\frac{u}{\ell^{\alpha-1}}\|_{L^2}\|d\xi_s^m\|_{L^\infty}\big(\|\frac{\xi_{ss}^m}{\ell^{\alpha-1}}\|_{L^2}+\|\frac{\xi_{ss}^{m-1}}{\ell^{\alpha-1}}\|_{L^2}\big)$$
$$+C\big[(\mathcal{E}_0+1)\|\xi_s^{m-1}\|_{L^\infty}^2+\varepsilon\|\frac{\xi_{ss}^m}{\ell^{\alpha-1}}\|_{L^2}^2\big]\|d\xi^m\|_{H_\alpha^1}^2+\varepsilon\|\frac{d\xi_{ss}^m}{\ell^{\alpha-1}}\|_{L^2}^2+\mathcal{E}_0\|\frac{d\xi_s^m}{\ell^{\alpha-1}}\|_{L^2}\|\frac{d\xi_{ss}^m}{\ell^{\alpha-1}}\|_{L^2}$$

We remark here that the control of the term $Bd\xi_s^m(\xi_{ss}^m+\xi_{ss}^{m-1})$ in the above estimate, which results to the second term on the RHS of the last inequality, is one of the most delicate that we have to perform[6]; essentially due to the fact that our energies depend

[6]In fact, if this term in the equation had been slightly more nonlinear, the overall scheme would break down.

on just one derivative in η. This term also forces us to consider small initial energy $\mathcal{E}_0$ to close the estimates; cf. the last term in the estimate (4.23).

Combining (4.31)-(4.34) we obtain

$$\begin{aligned}
&\int \frac{u\cdot\partial_s(dF_1^m)}{\ell^{2\alpha-2}}ds \qquad (4.35)\\
&\le \frac{C}{\varepsilon}\sigma\|\frac{u}{\ell^\alpha}\|^2_{L^2(0,\delta)}+\frac{C}{\varepsilon}\big(\|\xi_s^m\|^2_{L^\infty}+\|\xi_s^{m-1}\|^2_{L^\infty}+1\big)\|\frac{u}{\ell^{\alpha-1}}\|^2_{L^2(\delta,+\infty)}\\
&\quad+C\|\frac{u}{\ell^{\alpha-1}}\|_{L^2}\|d\xi_s^m\|_{L^\infty}\big(\|\frac{\xi_{ss}^m}{\ell^{\alpha-1}}\|_{L^2}+\|\frac{\xi_{ss}^{m-1}}{\ell^{\alpha-1}}\|_{L^2}\big)\\
&\quad+C\Big[\mathcal{E}_0\sigma^2+(\mathcal{E}_0+\sigma+1)A^2(t)+\sigma\|\xi_s^m\|^2_{L^\infty}+\varepsilon\|\frac{\xi_{ss}^m}{\ell^{\alpha-1}}\|^2_{L^2}\\
&\quad+(\mathcal{E}_0+1)\big(\|\xi_s^m\|^2_{L^\infty}+\|\xi_s^{m-1}\|^2_{L^\infty}\big)\Big]\big(\|d\xi^m\|^2_{H^1_\alpha}+\|d\eta^m\|^2_{H^1_\alpha}\big)+\varepsilon\|\frac{d\xi_{ss}^m}{\ell^{\alpha-1}}\|^2_{L^2}\\
&\quad+\mathcal{E}_0\|\frac{d\xi_s^m}{\ell^{\alpha-1}}\|_{L^2}\|\frac{d\xi_{ss}^m}{\ell^{\alpha-1}}\|_{L^2}
\end{aligned}$$

Thus, integrating on $[0,t]$ and employing once more the estimates (3.23),(4.9) we conclude the desired estimate (4.26). This completes the proof Proposition 4.3. □

4.4 Proof of Proposition 4.2 II: L^2_α estimates of $d\eta^{m+1}, d\xi^{m+1}$

We prove (4.18). Let us commence with the L^2_α estimates of $d\eta^{m+1}$. Taking the time derivative of the L^2_α norm of $d\eta^{m+1}$ and using (3.27), (3.25) we derive

$$\begin{aligned}
\frac12\partial_t\|d\eta^{m+1}\|^2_{L^2_\alpha}&=\int\frac{d\eta^{m+1}d\eta_t^{m+1}}{\ell^{2\alpha}}ds-\alpha\int\frac{|d\eta^{m+1}|^2}{\ell^{2\alpha+1}}\partial_t\ell ds+\frac12\int\frac{|d\eta^{m+1}|^2}{\ell^{2\alpha}}\partial_t ds\\
&\le\int\frac{d\eta^{m+1}d\eta_t^{m+1}}{\ell^{2\alpha}}ds-\alpha\sigma\|\frac{d\eta^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}+C\alpha\|\frac{d\eta^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)} \qquad (4.36)\\
&\quad+C\alpha\|\frac{d\eta^{m+1}}{\ell^\alpha}\|^2_{L^2(\delta,+\infty)}+C\|\frac{d\eta^{m+1}}{s\ell^\alpha}\|^2_{L^2}
\end{aligned}$$

As usual, we estimate the last term employing (3.28)

$$\|\frac{d\eta^{m+1}}{s\ell^\alpha}\|^2_{L^2}\le C\sigma\|\frac{d\eta^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}+C\|\frac{d\eta^{m+1}}{\ell^\alpha}\|^2_{L^2(\delta,+\infty)} \qquad (4.37)$$

Recall (3.15), (3.22) and the pointwise bound of B (4.11) to derive

$$\begin{aligned}
&\int\frac{d\eta^{m+1}d\eta_t^{m+1}}{\ell^{2\alpha}}ds \qquad \text{(plugging in the RHS of (4.13))}\\
&=\int\frac{d\eta^{m+1}}{\ell^{2\alpha}}\Big[\frac{\psi_s^2}{\psi^2}Bd\xi^{m+1}+\frac{\psi_s}{\psi}Bd\xi_s^{m+1}+2n(n-1)\frac{\psi_s^2}{\psi^2}d\eta^{m+1}+dF_1^m\Big]ds \qquad (4.38)\\
&\le\|\frac{d\eta^{m+1}}{s\ell^\alpha}\|^2_{L^2}+C\|\frac{d\xi^{m+1}}{s\ell^\alpha}\|^2_{L^2}+\varepsilon\|\frac{d\xi_s^{m+1}}{\ell^\alpha}\|^2_{L^2}+\frac{C}{\varepsilon}\|\frac{d\eta^{m+1}}{s\ell^\alpha}\|^2_{L^2}\\
&\quad+C\|\frac{d\eta^{m+1}}{s\ell^\alpha}\|^2_{L^2}+\|\frac{d\eta^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2}+\|dF_1^m\|^2_{L^2_{\alpha-1}}\\
&\le\frac{C}{\varepsilon}\sigma\|\frac{d\eta^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}+\frac{C}{\varepsilon}\|\frac{d\eta^{m+1}}{\ell^\alpha}\|^2_{L^2(\delta,+\infty)} \qquad \text{(employing (3.28), } 0<\varepsilon<1)\\
&\quad+C\sigma\|\frac{d\xi^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}+C\|\frac{d\xi^{m+1}}{\ell^\alpha}\|^2_{L^2(\delta,+\infty)}+\varepsilon\|\frac{d\xi_s^{m+1}}{\ell^\alpha}\|^2_{L^2}+\|dF_1^m\|^2_{L^2_{\alpha-1}}
\end{aligned}$$

We proceed to the case of $d\xi^{m+1}$ slightly differently. We control the L^2_α norm of the term $(\eta^m+1)^{\frac12}d\xi^{m+1}$ instead. Of course, it is evident from (4.10) that it is the same thing as estimating $d\xi^{m+1}$. We should note that it is not needed to go through this procedure if $\mathcal{E}_0$ is small enough, but we wish to provide a more general plan. Similarly to (4.36), keeping in mind the pointwise estimate on η^m (4.10), we deduce

$$\begin{aligned}\frac12\partial_t\|(\eta^m+1)^{\frac12}d\xi^{m+1}\|^2_{L^2_\alpha}\le&\int\frac{(\eta^m+1)d\xi^{m+1}d\xi^{m+1}_t}{\ell^{2\alpha}}ds-\frac12\alpha\sigma\|\frac{d\xi^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}\\&+C\alpha\|\frac{d\xi^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}+C\alpha\|\frac{d\xi^{m+1}}{\ell^{\alpha}}\|^2_{L^2(\delta,+\infty)}\\&+C\|\frac{d\xi^{m+1}}{s\ell^{\alpha}}\|^2_{L^2}+\|\eta^m_t\|_{L^\infty}\|\frac{d\xi^{m+1}}{\ell^{\alpha}}\|^2_{L^2}\end{aligned}\tag{4.39}$$

The second last term is controlled via (3.28), as in (4.37). We estimate the last term from the equation satisfied by η^m_t, analogous of the first equation in (4.2), using the pointwise estimate on the iterates (4.8) and the comparison estimate (3.28), replacing the singular orders of the coefficients (3.15), (3.22) with the weights ℓ^k, $k=1,2$.

$$\begin{aligned}&\|\eta^m_t\|_{L^\infty}\|\frac{d\xi^{m+1}}{\ell^\alpha}\|^2_{L^2}\\&\le C\big(\sqrt{\mathcal{E}_0}\sigma+\sqrt{\sigma}\|\xi^m_s\|_{L^\infty}+A(t)\sqrt{\sigma\mathcal{E}_0}+\|\xi^{m-1}_s\|^2_{L^\infty}\big)\|\frac{d\xi^{m+1}}{\ell^\alpha}\|^2_{L^2}\end{aligned}\tag{4.40}$$

Moving on to the main term, plugging in the RHS of (4.13), we have

$$\begin{aligned}&\int\frac{(\eta_m+1)d\xi^{m+1}d\xi^{m+1}_t}{\ell^{2\alpha}}ds\\&=\int\frac{(\eta_m+1)d\xi^{m+1}}{\ell^{2\alpha}}\Big[(\frac{\psi_{ss}}{\psi}+(n-1)\frac{\psi_s^2}{\psi^2})(Bd\eta^{m+1}+Bd\xi^{m+1})+\frac{\psi_s}{\psi}Bd\xi^{m+1}_s\\&\quad+\frac{d\xi^{m+1}_{ss}}{(\eta^m+1)(\xi^m+1)^{2n}}+\frac{\psi_s}{\psi}Bd\eta^{m+1}_s+dF_2^m\Big]ds\\&\le C\|\frac{d\xi^{m+1}}{s\ell^\alpha}\|^2_{L^2}+C\|\frac{d\eta^{m+1}}{s\ell^\alpha}\|^2_{L^2}+\varepsilon\|\frac{d\xi^{m+1}_s}{\ell^\alpha}\|^2_{L^2}+\frac{C}{\varepsilon}\|\frac{d\xi^{m+1}}{s\ell^\alpha}\|^2_{L^2}\qquad\text{(by (4.11) for } B)\\&\quad+\int\frac{d\xi^{m+1}d\xi^{m+1}_{ss}}{\ell^{2\alpha}(\xi^m+1)^{2n}}ds+\int\frac{\psi_s}{\psi}\frac{Bd\xi^{m+1}d\eta^{m+1}_s}{\ell^{2\alpha}}ds+\int\frac{(\eta_m+1)d\xi^{m+1}dF_2^m}{\ell^{2\alpha}}ds\\&\le\frac{C}{\varepsilon}\sigma\|\frac{d\xi^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}+\frac{C}{\varepsilon}\|\frac{d\xi^{m+1}}{\ell^\alpha}\|^2_{L^2(\delta,+\infty)}+\varepsilon\|\frac{d\xi^{m+1}_s}{\ell^\alpha}\|^2_{L^2}\qquad\text{(using (3.28))}\\&\quad+C\sigma\|\frac{d\eta^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}+C\|\frac{d\eta^{m+1}}{\ell^\alpha}\|^2_{L^2(\delta,+\infty)}+\|dF_2^m\|^2_{L^2_{\alpha-1}}\\&\quad+\int\frac{d\xi^{m+1}d\xi^{m+1}_{ss}}{\ell^{2\alpha}(\xi^m+1)^{2n}}ds+\int\frac{\psi_s}{\psi}\frac{Bd\xi^{m+1}d\eta^{m+1}_s}{\ell^{2\alpha}}ds\end{aligned}\tag{4.41}$$

We treat the last two terms separately integrating by parts. At this point the role of the Dirichlet boundary condition (4.4) comes into play.

$$\begin{aligned}&\int\frac{d\xi^{m+1}d\xi^{m+1}_{ss}}{\ell^{2\alpha}(\xi^m+1)^{2n}}ds\\&=-\int\frac{|d\xi^{m+1}_s|^2}{\ell^{2\alpha}(\xi^m+1)^{2n}}ds+\int\frac{2n\xi^m_sd\xi^{m+1}d\xi^{m+1}_s}{\ell^{2\alpha}(\xi^m+1)^{2n+1}}ds+2\alpha\int\frac{d\xi^{m+1}d\xi^{m+1}_s}{\ell^{2\alpha+1}(\xi^m+1)^{2n}}\partial_s\ell ds\end{aligned}\tag{4.42}$$

$$\leq -\frac{1}{2}\|\frac{d\xi_s^{m+1}}{\ell^\alpha}\|_{L^2}^2 + \varepsilon\|\frac{d\xi_s^{m+1}}{\ell^\alpha}\|_{L^2}^2 + \frac{C}{\varepsilon}\|\xi_s^m\|_{L^\infty}^2\|\frac{d\xi^{m+1}}{\ell^\alpha}\|_{L^2}^2 \qquad \text{(see (4.10))}$$
$$+ \varepsilon\|\frac{d\xi_s^{m+1}}{\ell^\alpha}\|_{L^2}^2 + \frac{C\alpha^2}{\varepsilon}\|\frac{d\xi^{m+1}}{\ell^{\alpha+1}}\|_{L^2}^2 \qquad (\ell_s = O(1)\ (3.27))$$
$$\leq (2\varepsilon - \frac{1}{2})\|\frac{d\xi_s^{m+1}}{\ell^\alpha}\|_{L^2}^2 + \frac{C}{\varepsilon}\|\xi_s^m\|_{L^\infty}^2\|\frac{d\xi^{m+1}}{\ell^\alpha}\|_{L^2}^2 + \frac{C\alpha^2}{\varepsilon}\|\frac{d\xi^{m+1}}{\ell^{\alpha+1}}\|_{L^2(0,\delta)}^2 + \frac{C\alpha^2}{\varepsilon}\|\frac{d\xi^{m+1}}{\ell^\alpha}\|_{L^2}^2$$

Similarly, by (3.15), (4.11), (4.12) we obtain [7]

$$\int \frac{\psi_s}{\psi}\frac{Bd\xi^{m+1}d\eta_s^{m+1}}{\ell^{2\alpha}}ds \tag{4.43}$$
$$= -\int \partial_s(\frac{\psi_s}{\psi})\frac{Bd\xi^{m+1}d\eta^{m+1}}{\ell^{2\alpha}}ds - \int \frac{\psi_s}{\psi}\frac{B_s d\xi^{m+1}d\eta^{m+1}}{\ell^{2\alpha}}ds - \int \frac{\psi_s}{\psi}\frac{Bd\xi_s^{m+1}d\eta^{m+1}}{\ell^{2\alpha}}ds$$
$$+ 2\alpha\int \frac{\psi_s}{\psi}\frac{Bd\xi^{m+1}d\eta^{m+1}}{\ell^{2\alpha+1}}\partial_s\ell ds$$
$$\leq \|\frac{d\eta^{m+1}}{s\ell^\alpha}\|_{L^2}^2 + C\|\frac{d\xi^{m+1}}{s\ell^\alpha}\|_{L^2}^2 + \frac{C}{\varepsilon}\|\frac{d\eta^{m+1}}{s\ell^\alpha}\|_{L^2}^2 + \varepsilon\|\frac{d\xi^{m+1}}{\ell}\|_{L^\infty}^2\|\frac{B_s}{\ell^{\alpha-1}}\|_{L^2}^2$$
$$+ \varepsilon\|\frac{d\xi_s^{m+1}}{\ell^\alpha}\|_{L^2}^2 + \frac{C}{\varepsilon}\|\frac{d\eta^{m+1}}{s\ell^\alpha}\|_{L^2}^2 + \|\frac{d\eta^{m+1}}{s\ell^\alpha}\|_{L^2}^2 + \alpha^2\|\frac{d\xi^{m+1}}{\ell^{\alpha+1}}\|_{L^2}^2$$
$$\leq \frac{C}{\varepsilon}\sigma\|\frac{d\eta^{m+1}}{\ell^{\alpha+1}}\|_{L^2(0,\delta)}^2 + \frac{C}{\varepsilon}\|\frac{d\eta^{m+1}}{\ell^\alpha}\|_{L^2(\delta,+\infty)}^2 \qquad \text{(employing (3.28), } 0<\varepsilon<1)$$
$$+ (C\sigma + C\varepsilon\mathcal{E}_0 + \alpha^2)\|\frac{d\xi^{m+1}}{\ell^{\alpha+1}}\|_{L^2(0,\delta)}^2 \qquad \text{(by the } L^\infty \text{ estimate (4.16) of } d\xi^m/\ell)$$
$$+ \varepsilon(1 + C\mathcal{E}_0)\|\frac{d\xi_s^{m+1}}{\ell^\alpha}\|_{L^2}^2 + C\alpha^2\|\frac{d\xi^{m+1}}{\ell^\alpha}\|_{L^2}^2$$

Putting the above estimates (4.36)-(4.43) all together we conclude that

$$\frac{1}{2}\partial_t\big(\|d\eta^{m+1}\|_{L^2_\alpha}^2 + \|(\eta^m+1)^{\frac{1}{2}}d\xi^{m+1}\|_{L^2_\alpha}^2\big) + \frac{1}{2}\alpha\sigma\big(\|\frac{d\eta^{m+1}}{\ell^{\alpha+1}}\|_{L^2(0,\delta)}^2 + \|\frac{d\xi^{m+1}}{\ell^{\alpha+1}}\|_{L^2(0,\delta)}^2\big)$$
$$\leq (4\varepsilon + C\varepsilon\mathcal{E}_0 - \frac{1}{2})\|\frac{d\xi_s^{m+1}}{\ell^\alpha}\|_{L^2}^2 + \frac{C}{\varepsilon}\big(\alpha^2 + \|\xi_s^m\|_{L^2}^2\big)\big(\|d\eta^{m+1}\|_{L^2_\alpha}^2 + \|d\xi^{m+1}\|_{L^2_\alpha}^2\big) \tag{4.44}$$
$$+ C(\frac{\alpha^2}{\varepsilon} + \frac{\sigma}{\varepsilon} + \varepsilon\mathcal{E}_0)\big(\|\frac{d\eta^{m+1}}{\ell^{\alpha+1}}\|_{L^2(0,\delta)}^2 + \|\frac{d\xi^{m+1}}{\ell^{\alpha+1}}\|_{L^2(0,\delta)}^2\big) + \|dF_1^m\|_{L^2_{\alpha-1}}^2 + \|dF_2^m\|_{L^2_{\alpha-1}}^2$$

Choosing ε small enough, the first term in the RHS of the preceding estimate has a negative sign and hence it can be dropped. Integrating on $[0,t]$ and taking into account the integrated estimates of dF_1^m, dF_2^m in Proposition 4.3, we obtain the desired estimate (4.18) in Proposition 4.2.

4.5 Proof of Proposition 4.2 III: $L^2_{\alpha-1}$ estimates of $d\eta_s^{m+1}, d\xi_s^{m+1}$

In this subsection we prove (4.19). Recall the bounds on the derivatives of the weight

[7] The possibility to control this next term using an integration by parts to offload the derivative from $d\eta^{m+1}$ is essential in order to close our estimates for the L^2_α norms of $d\xi^{m+1}, d\eta^{m+1}$, without recourse to the higher derivatives.

ℓ (3.27), the volume form ds (3.25) and the commutator $[\partial_s, \partial_t]$ (3.24) to obtain

$$\begin{aligned}
\frac{1}{2}\partial_t\|d\eta_s^{m+1}\|^2_{L^2_{\alpha-1}} &= \int \frac{d\eta_s^{m+1}\partial_t d\eta_s^{m+1}}{\ell^{2\alpha-2}}ds - (\alpha-1)\int \frac{|d\eta_s^{m+1}|^2}{\ell^{2\alpha-1}}\partial_t \ell ds + \frac{1}{2}\int \frac{|d\eta_s^{m+1}|^2}{\ell^{2\alpha-2}}\partial_t ds \\
&\leq \int \frac{d\eta_s^{m+1}\partial_s d\eta_t^{m+1}}{\ell^{2\alpha-2}}ds - (\alpha-1)\sigma\|\frac{d\eta_s^{m+1}}{\ell^\alpha}\|^2_{L^2(0,\delta)} \\
&\quad + C(\alpha-1)\|\frac{d\eta_s^{m+1}}{\ell^\alpha}\|^2_{L^2} + C\|\frac{d\eta_s^{m+1}}{s\ell^{\alpha-1}}\|^2_{L^2}
\end{aligned} \tag{4.45}$$

As usual, from (3.28)

$$\|\frac{d\eta_s^{m+1}}{s\ell^{\alpha-1}}\|^2_{L^2} \leq C\sigma\|\frac{d\eta_s^{m+1}}{\ell^\alpha}\|^2_{L^2(0,\delta)} + C\|\frac{d\eta_s^{m+1}}{\ell^{\alpha-1}}\|^2_{L^2(\delta,+\infty)},$$

In order to estimate the first term in the RHS of the inequality (4.45) we plug in $d\eta_t^{m+1}$ from the first equation of (4.13) and treat each generated term separately. For all three of the subsequent bounds we apply C-S at each term, using the estimates on the coefficients (3.15) and the relevant function B (4.11), (4.12):

$$\begin{aligned}
&\int \frac{d\eta_s^{m+1}}{\ell^{2\alpha-2}}\partial_s\left[\frac{\psi_s^2}{\psi^2}Bd\xi^{m+1}\right]ds \qquad (4.46)\\
&= \int \frac{d\eta_s^{m+1}}{\ell^{2\alpha-2}}\left[\partial_s(\frac{\psi_s^2}{\psi^2})Bd\xi^{m+1} + \frac{\psi_s^2}{\psi^2}B_s d\xi^{m+1} + \frac{\psi_s^2}{\psi^2}Bd\xi_s^{m+1}\right]ds \\
&\leq \|\frac{d\eta_s^{m+1}}{s\ell^{\alpha-1}}\|^2_{L^2} + C\|\frac{d\xi^{m+1}}{s^2\ell^{\alpha-1}}\|^2_{L^2} + C\|\frac{d\xi^{m+1}}{s}\|^2_{L^\infty}\|\frac{B_s}{\ell^{\alpha-1}}\|^2_{L^2} + C\|\frac{d\xi_s^{m+1}}{s\ell^{\alpha-1}}\|^2_{L^2} \\
&\leq C\sigma\|\frac{d\eta_s^{m+1}}{\ell^\alpha}\|^2_{L^2(0,\delta)} + C\|\frac{d\eta_s^{m+1}}{\ell^{\alpha-1}}\|^2_{L^2(\delta,+\infty)} + C\sigma^2\|\frac{d\xi^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)} \quad \text{(employing (3.28))} \\
&\quad + C\|\frac{d\xi^{m+1}}{\ell^\alpha}\|^2_{L^2(\delta,+\infty)} + C\sigma\|\frac{d\xi_s^{m+1}}{\ell^\alpha}\|^2_{L^2(0,\delta)} \quad \text{(by the } L^\infty \text{ estimate (4.16) on } d\xi^{m+1}) \\
&\quad + C\|\frac{d\xi_s^{m+1}}{\ell^{\alpha-1}}\|^2_{L^2(\delta,+\infty)}
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\int \frac{d\eta_s^{m+1}}{\ell^{2\alpha-2}}\partial_s\left[\frac{\psi_s}{\psi}Bd\xi_s^{m+1}\right]ds \qquad (4.47)\\
&= \int \frac{d\eta_s^{m+1}}{\ell^{2\alpha-2}}\left[\partial_s(\frac{\psi_s}{\psi})Bd\xi_s^{m+1} + \frac{\psi_s}{\psi}B_s d\xi_s^{m+1} + \frac{\psi_s}{\psi}Bd\xi_{ss}^{m+1}\right]ds \\
&\leq \|\frac{d\eta_s^{m+1}}{s\ell^{\alpha-1}}\|^2_{L^2} + C\|\frac{d\xi_s^{m+1}}{s\ell^{\alpha-1}}\|^2_{L^2} + \frac{C}{\varepsilon}\|\frac{d\eta_s^{m+1}}{s\ell^{\alpha-1}}\|^2_{L^2} + \varepsilon\|d\xi_s^{m+1}\|^2_{L^\infty}\|\frac{B_s}{\ell^{\alpha-1}}\|^2_{L^2} \\
&\quad + \varepsilon\|\frac{d\xi_{ss}^{m+1}}{\ell^{\alpha-1}}\|^2_{L^2} \\
&\leq \frac{C}{\varepsilon}\sigma\|\frac{d\eta_s^{m+1}}{\ell^\alpha}\|^2_{L^2(0,\delta)} + \frac{C}{\varepsilon}\|\frac{d\eta_s^{m+1}}{\ell^{\alpha-1}}\|^2_{L^2(\delta,+\infty)} \qquad \text{(by (3.28), } 0<\varepsilon<1) \\
&\quad + C\sigma\|\frac{d\xi_s^{m+1}}{\ell^\alpha}\|^2_{L^2(0,\delta)} + C(1+\varepsilon\mathcal{E}_0)\|\frac{d\xi_s^{m+1}}{\ell^{\alpha-1}}\|^2_{L^2} \\
&\quad + \varepsilon(1+C\mathcal{E}_0)\|\frac{d\xi_{ss}^{m+1}}{\ell^{\alpha-1}}\|^2_{L^2} \qquad \text{(by the } L^\infty \text{ estimate (4.17) on } d\xi_s^{m+1},\ k=0, \text{ and C-S)}
\end{aligned}$$

and

$$\int \frac{d\eta_s^{m+1}}{\ell^{2\alpha-2}}\partial_s\Big[2n(n-1)\frac{\psi_s^2}{\psi^2}d\eta^{m+1}\Big]ds \tag{4.48}$$

$$= \int \frac{d\eta_s^{m+1}}{\ell^{2\alpha-2}}\Big[2n(n-1)\partial_s(\frac{\psi_s^2}{\psi^2})d\eta^{m+1} + 2n(n-1)\frac{\psi_s^2}{\psi^2}d\eta_s^{m+1}\Big]ds$$

$$\le C\|\frac{d\eta_s^{m+1}}{s\ell^{\alpha-1}}\|_{L^2}^2 + C\|\frac{d\eta^{m+1}}{s^2\ell^{\alpha-1}}\|_{L^2}^2$$

$$\le C\sigma\|\frac{d\eta_s^{m+1}}{\ell^{\alpha}}\|_{L^2(0,\delta)}^2 + C\|\frac{d\eta_s^{m+1}}{\ell^{\alpha-1}}\|_{L^2(\delta,+\infty)}^2 + C\sigma^2\|\frac{d\eta^{m+1}}{\ell^{\alpha+1}}\|_{L^2(0,\delta)}^2 \qquad \text{(by (3.28))}$$

$$+ C\|\frac{d\eta^{m+1}}{\ell^{\alpha}}\|_{L^2(\delta,+\infty)}^2$$

We proceed to the case of $d\xi_s^{m+1}$. Similarly to (4.45), using in addition the boundary condition (4.4) upon integrating by parts we have

$$\frac{1}{2}\partial_t\|d\xi_s^{m+1}\|_{L^2_{\alpha-1}}^2 \le \int \frac{d\xi_s^{m+1}\partial_s d\xi_t^{m+1}}{\ell^{2\alpha-2}}ds - (\alpha-1)\sigma\|\frac{d\xi_s^{m+1}}{\ell^{\alpha}}\|_{L^2(0,\delta)}^2$$

$$+ C(\alpha-1)\|\frac{d\xi_s^{m+1}}{\ell^{\alpha}}\|_{L^2}^2 + C\|\frac{d\xi_s^{m+1}}{s\ell^{\alpha-1}}\|_{L^2}^2$$

$$= -\int \frac{d\xi_{ss}^{m+1}d\xi_t^{m+1}}{\ell^{2\alpha-2}}ds + (2\alpha-2)\int \frac{d\xi_s^{m+1}d\xi_t^{m+1}}{\ell^{2\alpha-1}}\ell_s ds \tag{4.49}$$

$$- (\alpha-1)\sigma\|\frac{d\xi_s^{m+1}}{\ell^{\alpha}}\|_{L^2(0,\delta)}^2 + C(\alpha-1)\|\frac{d\xi_s^{m+1}}{\ell^{\alpha}}\|_{L^2}^2 + C\|\frac{d\xi_s^{m+1}}{s\ell^{\alpha-1}}\|_{L^2}^2$$

There are two main terms we must estimate here. In both estimates we plug in $d\xi_t^{m+1}$ from (4.13), distributing the singularities in the coefficients (3.15) by applying C-S and the usual pointwise estimates. We start first with the term

$$(2\alpha-2)\int \frac{d\xi_s^{m+1}d\xi_t^{m+1}}{\ell^{2\alpha-1}}\ell_s ds \tag{4.50}$$

$$= (2\alpha-2)\int \frac{d\xi_s^{m+1}}{\ell^{2\alpha-1}}\ell_s\Big[(\frac{\psi_{ss}}{\psi} + (n-1)\frac{\psi_s^2}{\psi^2})(Bd\eta^{m+1} + Bd\xi^{m+1}) + \frac{\psi_s}{\psi}Bd\xi_s^{m+1}$$

$$+ |B|d\xi_{ss}^{m+1} + \frac{\psi_s}{\psi}Bd\eta_s^{m+1} + dF_2^m\Big]ds$$

$$\le \alpha^2\|\frac{d\xi_s^{m+1}}{\ell^{\alpha}}\|_{L^2}^2 + C\|\frac{d\eta^{m+1}}{s^2\ell^{\alpha-1}}\|_{L^2}^2 + C\|\frac{d\xi^{m+1}}{s^2\ell^{\alpha}}\|_{L^2}^2 + C\|\frac{d\xi_s^{m+1}}{s\ell^{\alpha-1}}\|_{L^2}^2$$

$$+ \varepsilon\|\frac{d\xi_{ss}^{m+1}}{\ell^{\alpha-1}}\|_{L^2}^2 + \frac{C}{\varepsilon}\alpha^2\|\frac{d\xi_s^{m+1}}{\ell^{\alpha}}\|_{L^2}^2 + \alpha^2\|\frac{d\xi_s^{m+1}}{\ell^{\alpha}}\|_{L^2}^2 + C\|\frac{d\eta_s^{m+1}}{s\ell^{\alpha-1}}\|_{L^2}^2$$

$$+ (2\alpha-2)\int \frac{d\xi_s^{m+1}dF_1^m}{\ell^{2\alpha-1}}\ell_s ds$$

$$\le \frac{C}{\varepsilon}\alpha^2\|\frac{d\xi_s^{m+1}}{\ell^{\alpha}}\|_{L^2(0,\delta)}^2 + \frac{C}{\varepsilon}\alpha^2\|\frac{d\xi_s^{m+1}}{\ell^{\alpha-1}}\|_{L^2(\delta,+\infty)}^2 + C\sigma^2\|\frac{d\eta^{m+1}}{\ell^{\alpha+1}}\|_{L^2(0,\delta)}^2 \qquad \text{(by (3.28))}$$

$$+ C\|\frac{d\eta^{m+1}}{\ell^{\alpha}}\|_{L^2(\delta,+\infty)}^2 + C\sigma^2\|\frac{d\xi^{m+1}}{\ell^{\alpha+1}}\|_{L^2(0,\delta)}^2 + C\|\frac{d\xi^{m+1}}{\ell^{\alpha}}\|_{L^2(\delta,+\infty)}^2$$

$$+ C\sigma\|\frac{d\xi_s^{m+1}}{\ell^{\alpha}}\|_{L^2(0,\delta)}^2 + C\|\frac{d\xi_s^{m+1}}{\ell^{\alpha-1}}\|_{L^2(\delta,+\infty)}^2 + C\sigma\|\frac{d\eta_s^{m+1}}{\ell^{\alpha}}\|_{L^2(0,\delta)}^2$$

$$+ C\|\frac{d\eta_s^{m+1}}{\ell^{\alpha-1}}\|_{L^2(\delta,+\infty)}^2 + \varepsilon\|\frac{d\xi_{ss}^{m+1}}{\ell^{\alpha-1}}\|_{L^2}^2 + \|dF_2^m\|_{L^2_{\alpha-1}}^2 \qquad (\ell_s = O(1)\ (3.27))$$

and analogously for

$$
\begin{aligned}
&-\int \frac{d\xi_{ss}^{m+1} d\xi_t^{m+1}}{\ell^{2\alpha-2}} ds \qquad (4.51)\\
=&-\int \frac{d\xi_{ss}^{m+1}}{\ell^{2\alpha-2}} \Big[\big(\frac{\psi_{ss}}{\psi} + (n-1)\frac{\psi_s^2}{\psi^2}\big)(Bd\eta^{m+1} + Bd\xi^{m+1}) + \frac{\psi_s}{\psi} Bd\xi_s^{m+1} \\
&+ \frac{d\xi_{ss}^{m+1}}{(\eta^m+1)(\xi^m+1)^{2n}} + \frac{\psi_s}{\psi} Bd\eta_s^{m+1} + dF_2^m \Big] ds \\
\leq\, & \varepsilon \| \frac{d\xi_{ss}^{m+1}}{\ell^{\alpha-1}} \|_{L^2}^2 + \frac{C}{\varepsilon} \| \frac{d\xi^{m+1}}{s^2 \ell^{\alpha-1}} \|_{L^2}^2 + \frac{C}{\varepsilon} \| \frac{d\eta^{m+1}}{s^2 \ell^{\alpha-1}} \|_{L^2}^2 + \frac{C}{\varepsilon} \| \frac{d\xi_s^{m+1}}{s\ell^{\alpha-1}} \|_{L^2}^2 \\
&- \int \frac{|d\xi_{ss}^{m+1}|^2}{\ell^{2\alpha-2}(\eta^m+1)(\xi^m+1)^{2n}} ds + \varepsilon \| \frac{d\xi_{ss}^{m+1}}{\ell^{\alpha-1}} \|_{L^2}^2 + \frac{C}{\varepsilon} \| \frac{d\eta_s^{m+1}}{s\ell^{\alpha-1}} \|_{L^2}^2 + \frac{C}{\varepsilon} \| dF_2^m \|_{L^2_{\alpha-1}}^2 \\
\leq\, & 2\varepsilon \| \frac{d\xi_{ss}^{m+1}}{\ell^{\alpha-1}} \|_{L^2}^2 + \frac{C}{\varepsilon}\sigma^2 \| \frac{d\xi^{m+1}}{\ell^{\alpha+1}} \|_{L^2(0,\delta)}^2 + \frac{C}{\varepsilon} \| \frac{d\xi^{m+1}}{\ell^{\alpha}} \|_{L^2(\delta,+\infty)}^2 \qquad \text{(by (3.28))} \\
&+ \frac{C}{\varepsilon}\sigma^2 \| \frac{d\eta^{m+1}}{\ell^{\alpha+1}} \|_{L^2(0,\delta)}^2 + \frac{C}{\varepsilon} \| \frac{d\eta^{m+1}}{\ell^{\alpha}} \|_{L^2(\delta,+\infty)}^2 + \frac{C}{\varepsilon}\sigma \| \frac{d\xi_s^{m+1}}{\ell^{\alpha}} \|_{L^2(0,\delta)}^2 \\
&+ \frac{C}{\varepsilon} \| \frac{d\xi_s^{m+1}}{\ell^{\alpha-1}} \|_{L^2(\delta,+\infty)}^2 + \frac{C}{\varepsilon}\sigma \| \frac{d\eta_s^{m+1}}{\ell^{\alpha}} \|_{L^2(0,\delta)}^2 + \frac{C}{\varepsilon} \| \frac{d\eta_s^{m+1}}{\ell^{\alpha-1}} \|_{L^2(\delta,+\infty)}^2 \\
&- \frac{1}{3} \| \frac{d\xi_{ss}^{m+1}}{\ell^{\alpha-1}} \|_{L^2}^2 + \frac{C}{\varepsilon} \| dF_2^m \|_{L^2_{\alpha-1}}^2 \qquad \text{(from the poinwtise estimate (4.10))}
\end{aligned}
$$

Summary: Combining (4.45)-(4.51) we deduce

$$
\begin{aligned}
&\frac{1}{2}\partial_t \big(\| d\eta_s^{m+1} \|_{L^2_{\alpha-1}}^2 + \| d\xi_s^{m+1} \|_{L^2_{\alpha-1}}^2 \big) + (\alpha-1)\sigma \big(\| \frac{d\eta_s^{m+1}}{\ell^\alpha} \|_{L^2(0,\delta)}^2 + \| \frac{d\xi_s^{m+1}}{\ell^\alpha} \|_{L^2(0,\delta)}^2 \big) \\
\leq\, & C\big(\frac{\alpha^2}{\varepsilon} + \varepsilon \mathcal{E}_0 \big) \big(\| d\eta_s^{m+1} \|_{L^2_{\alpha-1}}^2 + \| d\xi_s^{m+1} \|_{L^2_{\alpha-1}}^2 \big) + \int \frac{d\eta_s^{m+1} \partial_s (dF_1^m)}{\ell^{2\alpha-2}} ds \qquad (4.52) \\
&+ \frac{C}{\varepsilon}(\alpha^2+\sigma) \big(\| \frac{d\eta_s^{m+1}}{\ell^\alpha} \|_{L^2(0,\delta)}^2 + \| \frac{d\xi_s^{m+1}}{\ell^\alpha} \|_{L^2(0,\delta)}^2 \big) + (4\varepsilon + C\varepsilon\mathcal{E}_0 - \frac{1}{3}) \| \frac{d\xi_{ss}^{m+1}}{\ell^{\alpha-1}} \|_{L^2}^2 \\
&+ \frac{C}{\varepsilon} \big(\| d\eta^{m+1} \|_{L^2_\alpha}^2 + \| d\xi^{m+1} \|_{L^2_\alpha}^2 \big) + \frac{C}{\varepsilon}\sigma^2 \big(\| \frac{d\eta^{m+1}}{\ell^{\alpha+1}} \|_{L^2(0,\delta)}^2 + \| \frac{d\xi^{m+1}}{\ell^{\alpha+1}} \|_{L^2(0,\delta)}^2 \big) \\
&+ \frac{C}{\varepsilon} \| dF_2^m \|_{L^2_{\alpha-1}}^2
\end{aligned}
$$

Let $\varepsilon > 0$ be small such that

$$
4\varepsilon + C\varepsilon\mathcal{E}_0 < \frac{1}{6}.
$$

Integrating on $[0,t]$ and invoking the integrated estimates (4.25), (4.26) for $u = d\eta_s^{m+1}$, we finally arrive at the estimate (4.19) in Proposition 4.2.

5 The Linear step in the iteration

In the beginning of §4.1 we took for granted that at each step $m+1$, $m = 0, 1, \ldots$, the linear system (4.2) possessed a solution with prescribed regularity and energy bounds. We prove these assertions here.

Definition 5.1. *For this section, we will let f, g, F_1, F_2 stand for generic functions in the spaces*

$$
f, g \in L^\infty(0,T;H^1) \qquad F_1 \in L^2(0,T;H^1), \quad F_2 \in L^2(0,T;L^2) \qquad (5.1)
$$

satisfying the bounds

$$\frac{1}{2} + \|f\|_{L^\infty_x} < \|f\|_{L^\infty_x} + g(x,t) < C \qquad \|g_s\|^2_{L^\infty(0,T;L^2)} < \varepsilon, \tag{5.2}$$

for appropriate positive constants c, C, ε small, and

$$\int_0^T \|\frac{F_i}{\ell^{\alpha-1}}\|^2_{L^2} dt < +\infty, \qquad i = 1, 2 \tag{5.3}$$

$$\int \frac{u \cdot \partial_s F_1}{\ell^{2\alpha-2}} ds \le C\sigma \|\frac{u}{\ell^\alpha}\|^2_{L^2(0,\delta)} + G_1(t)\|\frac{u}{\ell^{\alpha-1}}\|^2_{L^2} + G_2(t),$$

for a.e. $0 \le t \le T$ and the general function $u \in L^2(0,T;L^2_\alpha)$, where $G_1(t), G_2(t)$ are positive $[0,T]$-integrable functions.

Motivated by (4.2), we consider the following linear system:

$$\begin{aligned}
\eta_t &= \frac{\psi_s^2}{\psi^2} f\xi + \frac{\psi_s}{\psi} f\xi_s + 2n(n-1)\frac{\psi_s^2}{\psi^2}\eta + F_1 \\
\xi_t &= (\frac{\psi_{ss}}{\psi} + (n-1)\frac{\psi_s^2}{\psi^2}) f \cdot (\eta + \xi) + \frac{\psi_s}{\psi} f\xi_s + g\xi_{ss} + \frac{\psi_s}{\psi} f\eta_s + F_2 \\
\eta\Big|_{t=0} &= \eta_0 \qquad \xi\Big|_{t=0} = \xi_0, \qquad \xi = 0, \text{ on } \{x = 0, B\} \times [0,T]
\end{aligned} \tag{5.4}$$

We prove:

Theorem 5.1. *There exist α, σ sufficiently large such that (5.4) has a unique solution up to time $T > 0$ in the spaces*

$$\begin{aligned}
&\eta \in L^\infty(0,T;H^1_\alpha) \cap L^2(0,T;H^1_{\alpha+1}) \qquad \xi \in L^\infty(0,T;H^1_\alpha) \cap L^2(0,T;H^2_{\alpha+1}) \\
&\eta_t \in L^\infty(0,T;L^2_{\alpha-2}) \cap L^2(0,T;H^1_{\alpha-1}) \qquad \xi_t \in L^2(0,T;L^2_{\alpha-1})
\end{aligned} \tag{5.5}$$

Further, the solution satisfies the energy estimate

$$\mathcal{E}(\eta, \xi; T) \le \widetilde{C}\Big[\mathcal{E}_0 + \sum \int_0^T \|\frac{F_i}{\ell^{\alpha-1}}\|^2_{L^2} dt + \int_0^T G_2(t)dt\Big] =: \widetilde{C}C_0(T), \tag{5.6}$$

for some positive constant $\widetilde{C}$.

It is easy to see that the linear system (4.2) is of the type (5.4), if the energy $\mathcal{E}(\eta^m, \xi^m; T)$ is small enough. Taking the latter as an induction hypothesis, Theorem 5.1 then implies the existence of η^{m+1}, ξ^{m+1} satisfying the same assertions, provided $T, \mathcal{E}_0 > 0$ are sufficiently small (uniformly in m).

5.1 Plan of the proof of Theorem 5.1

We perform a new iteration for (5.4), first solving the first equation (ODE) for η (using a previously-soved-for $\tilde{\xi}$)[8] and then plugging η into the second (and main) equation of (5.4) to solve for the new ξ. Let

$$\tilde{\xi} \in L^\infty(0,T;H^1_\alpha) \cap L^2(0,T;H^2_{\alpha+1}) \tag{5.7}$$

[8]This way we avoid some additional problems having to do with the fact that the level of regularity of η is lower, by one derivative, than the one we have for ξ.

be a function satisfying

$$\|\tilde{\xi}\|^2_{L^\infty(0,T;H^1_\alpha)} + \|\tilde{\xi}\|^2_{L^2(0,T;H^2_{\alpha+1})} \le \widetilde{C} C_0(T), \tag{5.8}$$

with improved bounds for

$$\int_0^T \|\frac{\tilde{\xi}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)} dt \le \frac{\widetilde{C}}{\sigma^2} C_0(T) \qquad \int_0^T \|\frac{\tilde{\xi}_s}{\ell^\alpha}\|^2_{L^2(0,\delta)} dt \le \frac{\widetilde{C}}{\sigma} C_0(T); \tag{5.9}$$

$\widetilde{C}$ is some positive constant to be determined later. We consider the system

$$\begin{aligned}
\eta_t &= \frac{\psi_s^2}{\psi^2} f\tilde{\xi} + \frac{\psi_s}{\psi} f\tilde{\xi}_s + 2n(n-1)\frac{\psi_s^2}{\psi^2}\eta + F_1 \\
\xi_t &= (\frac{\psi_{ss}}{\psi} + (n-1)\frac{\psi_s^2}{\psi^2}) f \cdot (\eta + \xi) + \frac{f}{\psi^2}\xi + \frac{\psi_s}{\psi} f\xi_s + g\xi_{ss} + \frac{\psi_s}{\psi} f\eta_s + F_2 \\
\eta\Big|_{t=0} &= \eta_0 \qquad \xi\Big|_{t=0} = \xi_0, \qquad\qquad \xi = 0, \text{ on } \{x = 0, B\} \times [0, T]
\end{aligned} \tag{5.10}$$

Claim: For suitably large α, σ the preceding system has a unique solution

$$\begin{aligned}
\eta &\in L^\infty(0,T;H^1_\alpha) \cap L^2(0,T;H^1_{\alpha+1}) \qquad \xi \in L^\infty(0,T;H^1_\alpha) \cap L^2(0,T;H^2_{\alpha+1}) \\
\eta_t &\in L^\infty(0,T;L^2_{\alpha-2}) \cap L^2(0,T;H^1_{\alpha-1}) \qquad\qquad\qquad \xi_t \in L^2(0,T;L^2_{\alpha-1}),
\end{aligned} \tag{5.11}$$

which satisfies the energy estimates

$$\mathcal{E}(\eta, \xi; T) \le \widetilde{C} C_0(T) \tag{5.12}$$

and

$$\int_0^T \|\frac{\xi}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)} dt \le \frac{\widetilde{C}}{\sigma^2} C_0(T) \qquad \int_0^T \|\frac{\xi_s}{\ell^\alpha}\|^2_{L^2(0,\delta)} dt \le \frac{\widetilde{C}}{\sigma} C_0(T). \tag{5.13}$$

Observe that if we can prove this, a standard iteration argument (passing to a subsequence, weak limits etc.) yields a solution η, ξ of the original linear problem (5.4) in the same space (5.11) and satisfying the same estimates as above. This reduces the proof of Theorem 5.1 to proving our claim above.

5.2 A priori estimates for η

The function η given by the (ODE) first equation of (5.10) satisfies the following energy estimates for $\alpha, \sigma, \widetilde{C}$ large, $T > 0$ small:

$$\|\eta\|^2_{L^\infty(0,T;H^1_\alpha)} + \|\eta\|^2_{L^\infty(0,T;H^1_{\alpha+1})} \le \frac{\widetilde{C}}{10} C_0(T) \tag{5.14}$$

and

$$\int_0^T \|\frac{\eta}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)} d\tau \le \frac{C}{\alpha}\frac{\widetilde{C}}{\sigma^2} C_0(T) \qquad \int_0^T \|\frac{\eta_s}{\ell^\alpha}\|^2_{L^2(0,\delta)} d\tau \le \frac{C}{\alpha}\frac{\widetilde{C}}{\sigma} C_0(T). \tag{5.15}$$

Sketch of the argument . The relevant derivations are the same (and in fact a lot less involved) with the ones in the non-linear case §4 (see Proposition 4.2). There is a slight difference in the very last argument before closing the estimates, which we present separately here. For example, following §4.4, we derive

$$\begin{aligned}
&\frac{1}{2}\partial_t\|\eta\|^2_{L^2_\alpha}+\alpha\sigma\|\frac{\eta}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)} \\
&\le C(\sigma+\alpha)\|\frac{\eta}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}+C\alpha\|\eta\|^2_{L^2_\alpha}+C\sigma\|\frac{\tilde{\xi}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)} \\
&\quad +C\|\frac{\tilde{\xi}}{\ell^{\alpha}}\|^2_{L^2}+C\|\frac{\tilde{\xi}_s}{\ell^{\alpha}}\|^2_{L^2}+\|\frac{F_1}{\ell^{\alpha}}\|^2_{L^2}
\end{aligned} \tag{5.16}$$

Choosing α,σ such that

$$\frac{1}{2}\alpha\sigma>C(\sigma+\alpha)$$

and integrating in time and utilizing (5.8), (5.9) we have

$$\begin{aligned}
&\frac{1}{2}\|\eta\|^2_{L^2_\alpha[t]}+\frac{\alpha\sigma}{2}\int_0^t\|\frac{\eta}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}d\tau \\
&\le\frac{1}{2}\|\eta_0\|^2_{L^2_\alpha}+C\alpha\int_0^t\|\eta\|^2_{L^2_\alpha[\tau]}d\tau+C(\frac{1}{\sigma}+T)\widetilde{C}C_0(T)+\int_0^T\|\frac{F_1}{\ell^\alpha}\|^2_{L^2}d\tau
\end{aligned} \tag{5.17}$$

The part of (5.14), (5.15) involving the zeroth order terms follows from (5.17) by Gronwall's inequality. $\square$

5.3 The weak solution ξ: A Galerkin-type argument

Now that we have solved the first equation of (5.10) for η and obtained the required energy estimates, we plug it into the second equation of the system (5.10) and solve for ξ via a modified Galerkin method. We initially seek a weak solution

$$\xi\in L^\infty(0,T;L^2_\alpha)\cap L^2(0,T;H^1_{\alpha+1,0})\qquad \ell^2\xi_t\in L^2(0,T;H^{-1}_{\alpha+1}) \tag{5.18}$$

satisfying

$$\begin{aligned}
\int_0^T\big(\xi_t,v\big)_{L^2_\alpha}dt=\int_0^T\Big[&\big((\frac{\psi_{ss}}{\psi}+(n-1)\frac{\psi_s^2}{\psi^2})f\cdot(\eta+\xi),v\big)_{L^2_\alpha}+\big(\frac{\psi_s}{\psi}f\xi_s,v\big)_{L^2_\alpha} \\
&-\big(g_s\xi_s,v\big)_{L^2_\alpha}-\big(g\xi_s,v_s\big)_{L^2_\alpha}+2\alpha\big(g\xi_s,v\frac{\ell_s}{\ell}\big)_{L^2_\alpha} \\
&+\big(\frac{\psi_s}{\psi}f\eta_s,v\big)_{L^2_\alpha}+\big(F_2,v\big)_{L^2_\alpha}\Big]dt,\qquad \xi\Big|_{t=0}=\xi_0
\end{aligned} \tag{5.19}$$

for all

$$v\in L^\infty(0,T;H^1_{\alpha,0}(s))\cap L^2(0,T;H^1_{\alpha+1}(s)), \tag{5.20}$$

where by $\big(\cdot,\cdot\big)_{L^2_\alpha}$ we denote the inner product in L^2_α

$$\big(v_1,v_2\big)_{L^2_\alpha}:=\int\frac{v_1v_2}{\ell^{2\alpha}}ds. \tag{5.21}$$

and by $H^1_{\alpha,0}$ the closure of compactly supported functions in $H^1_\alpha(0,B)$; $H^{-1}_{\alpha+1}$ being the dual of $H^1_{\alpha+1,0}$. In view of the regularity (5.18), ξ is actually continuous in time and hence the initial condition in (5.19) makes sense.

Let $\{u_k(x)\}_{k=1}^\infty$ be an orthonormal basis of $L^2(0,B)$, which is also a basis of $H^1_0\big((0,B)\big)$; consisting of smooth, bounded functions. Then for each $t\in[0,T]$ (abusing slightly the notation of the endpoints of integration)

$$w_k(s,t) := \ell^\alpha u_k(s) \qquad\qquad k=1,2,\dots \tag{5.22}$$

is an orthonormal basis of L^2_α and a basis of $H^1_{\alpha,0}$. We note that

$$\begin{aligned}\int_0^T\int_0^B \frac{1}{\ell^2}dsdt &\overset{(3.28)}{\leq} C\int_0^T\int_0^B\frac{1}{s^2}ds \leq C\int_0^T \frac{1}{s(0,t)}ds \\ &\overset{(3.18)}{\leq} C\int_0^T\frac{1}{\sqrt{t}}dt \leq C\sqrt{T} < +\infty,\end{aligned} \tag{5.23}$$

from which it follows that the set

$$\mathrm{span}\big\{d_k(t)w_k(s,t)\;\big|\; t\in[0,T],\; k=1,2\dots\big\}, \tag{5.24}$$

$d_k(t)$ smooth, is also dense in $L^2(0,T;H^1_{\alpha+1,0}(s))$. Similarly to (5.23), by definition (5.22) and (3.27), we verify the asymptotics

$$\begin{aligned}&\int\frac{w_{k_1}w_{k_2}}{s^2\ell^{2\alpha}}ds = O(\frac{1}{\sqrt{t}}) &&\int\frac{\partial_s w_{k_1}w_{k_2}}{s\ell^{2\alpha}}ds = O(\frac{1}{\sqrt{t}})\\ &\int\frac{\partial_s w_{k_1}\partial_s w_{k_2}}{\ell^{2\alpha}}ds = O(\frac{1}{\sqrt{t}}) &&\int\frac{\partial_t w_{k_1}w_{k_2}}{\ell^{2\alpha}}ds = O(\frac{1}{\sqrt{t}}),\end{aligned} \tag{5.25}$$

without assuming of course any uniformity in the RHSs with respect to the indices $k_1,k_2\in\{1,2,\dots\}$.

Given $\nu\in\{1,2,\dots\}$, we construct Galerkin approximations of the solution of (5.19), which lie in the span of the first ν basis elements:

$$\xi^\nu := \sum_{k=1}^\nu a_k(t)w_k \qquad\qquad a_k(0) := \int\frac{\xi_0 w_k(x,0)}{x^{2\alpha}}dx \tag{5.26}$$

solving

$$\begin{aligned}\big(\xi^\nu_t, w_k\big)_{L^2_\alpha} = {}&\big((\frac{\psi_{ss}}{\psi}+(n-1)\frac{\psi_s^2}{\psi^2})f\cdot(\eta+\xi^\nu), w_k\big)_{L^2_\alpha} + \big(\frac{\psi_s}{\psi}f\xi^\nu_s, w_k\big)_{L^2_\alpha}\\ &-\big(g_s\xi^\nu_s, w_k\big)_{L^2_\alpha} - \big(g\xi^\nu_s,\partial_s w_k\big)_{L^2_\alpha} + 2\alpha\big(g\xi^\nu_s, w_k\frac{\ell_s}{\ell}\big)_{L^2_\alpha}\\ &+\big(\frac{\psi_s}{\psi}f\eta_s, w_k\big)_{L^2_\alpha} + \big(F_2, w_k\big)_{L^2_\alpha},\end{aligned} \tag{5.27}$$

for $t\in[0,T]$ and every $k=1,\dots,\nu$.

Proposition 5.1 (Galerkin approximations)**.** *For each $\nu=1,2,\dots$ there exists a unique function ξ^ν of the form (5.26) satisfying (5.27).*

Proof. Employing (5.25) we see that

$$\big(\xi_t^\nu, w_k\big)_{L^2_\alpha} = a_k'(t) + \sum_{j=1}^{\nu} a_j(t) O(\frac{1}{\sqrt{t}})$$

and also utilizing (3.15), (5.2)

$$\big((\frac{\psi_{ss}}{\psi} + (n-1)\frac{\psi_s^2}{\psi^2}) f \cdot \xi^\nu, w_k\big)_{L^2_\alpha} + \big(\frac{\psi_s}{\psi} f \xi_s^\nu, w_k\big)_{L^2_\alpha} = \sum_{j=1}^{\nu} a_j(t) O(\frac{1}{\sqrt{t}}).$$

Further, by our assumption on g (5.1) and (5.25) it holds

$$\begin{aligned} -\big(g_s \xi_s^\nu, w_k\big)_{L^2_\alpha} - \big(g \xi_s^\nu, \partial_s w_k\big)_{L^2_\alpha} + 2\alpha\big(g \xi_s^\nu, w_k \frac{\ell_s}{\ell}\big)_{L^2_\alpha} \qquad & (\ell_s = O(1) \ (3.27)) \\ = \sum_{j=1}^{\nu} a_k(t) O(1) + \sum_{j=1}^{\nu} a_k(t) O(\frac{1}{\sqrt{t}}), & \end{aligned}$$

Lastly, setting

$$\begin{aligned} d_k(t) &:= \big((\frac{\psi_{ss}}{\psi} + (n-1)\frac{\psi_s^2}{\psi^2}) f \cdot \eta, w_k\big)_{L^2_\alpha} + \big(\frac{\psi_s}{\psi} f \eta_s, w_k\big)_{L^2_\alpha} + \big(F_2, w_k\big)_{L^2_\alpha} \\ &\leq C \| \frac{\eta}{s\ell^\alpha} \|_{L^2}^2 + \int \frac{1}{s^2} ds + C \| \frac{\eta_s}{\ell^\alpha} \|_{L^2}^2 + \int \frac{1}{s^2} ds + C \| \frac{F_2}{\ell^{\alpha-1}} \|_{L^2}^2 + \int \frac{1}{\ell^2} ds \end{aligned}$$

we observe that (5.27) reduces to a linear first order ODE system of the form

$$a_k'(t) = \sum_{j=1}^{\nu} a_k(t) O(\frac{1}{\sqrt{t}}) + \sum_{j=1}^{\nu} a_k(t) O(1) + d_k(t) \qquad\qquad k = 1, \ldots, \nu$$

having coefficients which are singular at $t = 0$, but luckily they are all integrable on $[0, T]$. This implies local existence and uniqueness of the system and hence of ξ^ν at each step $\nu \in \{1, 2, \ldots\}$. $\square$

Proposition 5.2 (Energy estimates)**.** *For $\alpha, \sigma, \widetilde{C}$ appropriately large and $T > 0$ small the following estimates hold:*

$$\|\xi^\nu\|^2_{L^\infty(0,T;L^2_\alpha(s))} + \|\xi_s^\nu\|^2_{L^2(0,T;H^1_{\alpha+1,0})} \leq \frac{\widetilde{C}}{10} C_0(T), \tag{5.28}$$

$$\int_0^T \| \frac{\xi^\nu}{\ell^{\alpha+1}} \|^2_{L^2(0,\delta)} dt \leq \frac{C}{\alpha} \frac{\widetilde{C}}{\sigma^2} C_0(T) \tag{5.29}$$

and

$$\Big(\int_0^T \big(\xi_t^\nu, v\big)_{L^2_\alpha} dt \Big)^2 \leq \frac{\widetilde{C}}{10} C_0(T) \int_0^T \|v\|^2_{H^1_{\alpha+1},0} dt, \tag{5.30}$$

for every $\nu = 1, 2, \ldots$, $v = \sum_{k=1}^{\nu} d_k(t) w_k$.

Proof. Multiplying the equation (5.27) with $a_k(t)$ and summing over $k = 1, \dots, \nu$, we can then follow the argument in §5.2 to prove (5.28),(5.29). Next, we readily compute using the equation (5.27):

$$\begin{aligned}(\xi_t^\nu, v)_{L^2_\alpha} \leq C\big(\|\frac{v}{\ell^{\alpha+1}}\|_{L^2} + \|\frac{v_s}{\ell^\alpha}\|_{L^2}\big)\Big[\|\frac{\eta}{s^2\ell^{\alpha-1}}\|_{L^2} + \|\frac{\xi^\nu}{s^2\ell^{\alpha-1}}\|_{L^2} + \|\frac{\xi_s^\nu}{s\ell^{\alpha-1}}\|_{L^2}\\ + \alpha^2\|\frac{\xi_s^\nu}{\ell^\alpha}\|_{L^2} + \|\frac{\eta_s}{s\ell^{\alpha-1}}\|_{L^2} + \|\frac{F_2}{\ell^{\alpha-1}}\|_{L^2}\Big]\end{aligned}$$

Employing the comparison (3.28) and (5.14), (5.15) along with the already derived (5.28), (5.29) we arrive at (5.30). $\square$

The estimates in Proposition 5.2 suffice to pass to a subsequence (applying a diagonal argument due to (5.30)), yielding in the limit a weak solution ξ (5.18),(5.19) verifying the energy bounds

$$\|\xi\|^2_{L^\infty(0,T;L^2_\alpha(s))} + \int_0^T \|\frac{\xi_s}{\ell^\alpha}\|^2_{L^2} dt \leq \frac{\widetilde{C}}{10} C_0(T) \tag{5.31}$$

and

$$\int_0^T \|\frac{\xi}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)} dt \leq \frac{C}{\alpha}\frac{\widetilde{C}}{\sigma^2} C_0(T). \tag{5.32}$$

Uniqueness follows by the linearity of (5.19), since the difference of any two weak solutions satisfies the corresponding estimates with zero initial data and zero inhomogeneous terms.

5.4 Improved regularity and energy estimates for ξ

We now show that ξ is in fact a strong solution of (5.10). Let $0 < t_0 < T$ be a fixed positive time. Looking at the second equation of (5.10) for $t \in [t_0, T]$, we observe that the coefficients involving ψ and its derivatives are smooth and bounded, while $f, g \in L^\infty(0,T;H^1)$ (5.1). Moreover, from §5.2 we have $\eta \in L^\infty(0,T;H^1)$ and by assumption $F_i \in L^2(0,T;L^2)$, $i = 1, 2$. Hence, by standard theory of parabolic equations the weak solution ξ (5.18) of (5.10) that we established in §5.3, having "initial data" $\xi(s, t_0) \in H^1$ (for a.e. $0 < t_0 < T$), attains interior regularity

$$\xi \in L^\infty(t_0,T;H^1_0) \cap L^2(t_0,T;H^2) \qquad\qquad \xi_t \in L^2(t_0,T;L^2)$$

Since $t_0 \in (0,T)$ is arbitrary, we can improve the regularity of the preceding solution

$$\xi \in L^\infty(0,T;H^1_{\alpha,0}) \cap L^2(0,T;H^2_{\alpha+1}) \qquad\qquad \xi_t \in L^2(0,T;L^2_{\alpha-1}) \tag{5.33}$$

by straightforwardly using the second equation in (5.10) to derive the desired energy estimates for the higher order terms. Recall that for fixed $t > 0$, the weight ℓ^2 is bounded above and below (Definition 3.1). Thus, it makes sense to (time) differentiate the $L^2_{\alpha-1}$ of ξ and plug in directly the equation (5.10) to obtain (as in the non-linear case for $d\xi_s^{m+1}$ §4.5):

$$\frac{1}{2}\frac{d}{dt}\|\xi_s\|^2_{L^2_{\alpha-1}} + \alpha\sigma\|\frac{\xi_s}{\ell^\alpha}\|^2_{L^2(0,\delta)} + \frac{1}{4}\|\frac{\xi_{ss}}{\ell^{\alpha-1}}\|^2_{L^2} \tag{5.34}$$
$$\leq C(\alpha^2+\sigma)\|\frac{\xi_s}{\ell^\alpha}\|^2_{L^2(0,\delta)} + C\alpha^2\|\xi_s\|^2_{L^2_{\alpha-1}}$$
$$+ C\sigma^2\big(\|\frac{\eta}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)} + \|\frac{\xi}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}\big) + C\sigma\big(\|\frac{\eta_s}{\ell^\alpha}\|^2_{L^2(0,\delta)} + \|\frac{\xi_s}{\ell^\alpha}\|^2_{L^2(0,\delta)}\big)$$
$$+ C\big(\|\eta\|^2_{L^2_\alpha} + \|\xi\|^2_{L^2_\alpha} + \|\eta_s\|^2_{L^2_{\alpha-1}}\big) + C\|\frac{F_2}{\ell^{\alpha-1}}\|^2_{L^2}$$

Let α,σ large such that $\frac{1}{2}\alpha\sigma > C(\alpha^2+\sigma)$. Invoking (5.8), (5.9), (5.14), (5.15), (5.31), (5.32) upon integrating on $[0,T]$ we deduce

$$\frac{1}{2}\|\xi_s\|^2_{L^2_{\alpha-1}[t]} + \frac{1}{2}(\alpha-1)\sigma\int_0^t\|\frac{\xi_s}{\ell^\alpha}\|^2_{L^2}d\tau + \frac{1}{4}\int_0^t\|\frac{\xi_{ss}}{\ell^{\alpha-1}}\|^2_{L^2}d\tau \tag{5.35}$$
$$\leq \frac{1}{2}\|\partial_x\xi_0\|^2_{L^2_{\alpha-1}} + C\alpha^2\int_0^t\|\xi_s\|^2_{L^2_{\alpha-1}[\tau]}d\tau + C(\frac{1}{\alpha}+T)\widetilde{C}C_0(T) + C\int_0^T\|\frac{F_2}{\ell^{\alpha-1}}\|^2_{L^2}d\tau$$

Employing Gronwall's inequality, $t\in[0,T]$, we finally conclude ($T>0$ small, α large)

$$\|\xi_s\|^2_{L^\infty(0,T;L^2_{\alpha-1}(s))} + \int_0^T\|\frac{\xi_{ss}}{\ell^{\alpha-1}}\|^2_{L^2}d\tau \leq \frac{\widetilde{C}}{10}C_0(T) \tag{5.36}$$

and

$$\int_0^T\|\frac{\xi_s}{\ell^\alpha}\|^2_{L^2(0,\delta)}d\tau \leq \frac{\widetilde{C}}{\sigma}C_0(T) \tag{5.37}$$

This completes the proof of the *claim* in the outline of the plan §5.1 and consequently of Theorem 5.1 and the realization of the linear step in the iteration of the non-linear PDE (4.2).

A Analysis of the singular Ricci solitons

Generally, for metrics of the form (2.1) [6, §1.3.2] the Ricci tensor is given by

$$Ric(g) = -n\frac{\psi_{xx}}{\psi}dx^2 + (n-1-\psi\psi_{xx}-(n-1)(\psi_x)^2)g_{\mathbb{S}^n} \tag{A.1}$$

and the Hessian of a radial function ϕ by

$$\nabla\nabla\phi = \phi_{xx}dx^2 + \psi\psi_x\phi_x g_{\mathbb{S}^n}, \tag{A.2}$$

where $\dot{} = \frac{d}{dx}$. Therefore, equation (2.2) reduces to a coupled ODE system of the form

$$\begin{cases} n\psi_{xx} - \psi\phi_{xx} = \lambda\psi \\ \psi\psi_{xx} + (n-1)\psi_x^2 - (n-1) - \psi\psi_x\phi_x = \lambda\psi^2. \end{cases} \tag{A.3}$$

Following [6, Chapter 1, §5.2], we introduce the transformation

$$W = \frac{1}{-\phi_x + n\frac{\dot\psi}{\psi}}, \qquad X = \sqrt{n}W\frac{\dot\psi}{\psi}, \qquad Y = \frac{\sqrt{n(n-1)}W}{\psi}, \tag{A.4}$$

along with a new independent variable y defined via

$$dy = \frac{dx}{W}. \tag{A.5}$$

For the above set of variables, the ODE system (A.3) becomes

$$\left(\, ' = \frac{d}{dy}\right) \begin{cases} W' = W(X^2 - \lambda W^2) \\ X' = X^3 - X + \frac{Y^2}{\sqrt{n}} + \lambda(\sqrt{n} - X)W^2 \\ Y' = Y(X^2 - \frac{X}{\sqrt{n}} - \lambda W^2) \end{cases} \tag{A.6}$$

We readily check (see also [6, §1.5.2]) that the equilibrium points of the above system are

$$(0,0,0) \qquad\qquad (0,\pm 1,0) \qquad\qquad (0,\frac{1}{\sqrt{n}},\pm\sqrt{1-\frac{1}{n}}).$$

and also $(\pm\frac{1}{\sqrt{\lambda n}},\frac{1}{\sqrt{n}},0)$, when $\lambda > 0$.

In the present article we are concerned with the trajectories emanating from the equilibrium point $(0,1,0)$, for all $\lambda \in \mathbb{R}$ (in our primary analysis). The linearization of (A.6) at $(0,1,0)$ takes the diagonal form

$$\begin{pmatrix} W \\ X-1 \\ Y \end{pmatrix}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1-\frac{1}{\sqrt{n}} \end{pmatrix} \begin{pmatrix} W \\ X-1 \\ Y \end{pmatrix} \tag{A.7}$$

Note that for $n > 1$, all eigenvalues (diagonal entries) are positive, which implies that $(0,1,0)$ is a source of the system. Whence, if a trajectory of (A.6) is initially ($y = 0$) close to $(0,1,0)$, i.e.,

$$|(W(0), X(0) - 1, Y(0))| < \varepsilon,$$

for $\varepsilon > 0$ sufficiently small (indicated by the RHS of (A.6)), then standard ODE theory (e.g., see [9]) yields the estimate

$$|(W(y), X(y) - 1, Y(y))| \leq \sqrt{3}\varepsilon e^{\mu y} \qquad\qquad y \leq 0, \tag{A.8}$$

for some $0 < \mu < 1 - \frac{1}{\sqrt{n}}$ (least eigenvalue).[9] We will show that these trajectories correspond to an essential singularity of the original metric (2.1) at $x = 0$.

A.1 Asymptotics at $x = 0$

We will be considering solutions of the system (A.6), with $(W(0), X(0), Y(0))$ sufficiently close to the equilibrium point $(0,1,0)$ and with $Y(0), W(0) > 0$. (The reflection-symmetric trajectories over $\{Y = 0\}$ and $\{W = 0\}$ are easily seen to correspond to the same metric, while the trajectories with $Y(0) = 0$ do not to correspond to Riemannian metrics.)

We proceed to derive the asymptotic behavior, as $y \to -\infty$, of the variables W, X, Y. Changing back to x, using (A.5), we determine the desired asymptotic behavior of the unknown functions in the original system (A.3), as $x \to 0^+$. The final estimates will confirm that $x = 0$ is actually a singular point of the metric g, where in fact the curvature blows up.

[9]The latter estimate improves as the initial conditions approach the equilibrium point $(0,1,0)$; in other words one can pick μ closer to the eigenvalue $1 - \frac{1}{\sqrt{n}}$ by taking ε sufficiently small.

Proposition A.1. *The above initial conditions for the system (A.6) furnish trajectories (W, X, Y), $y \in (-\infty, 0]$, which correspond to solutions (ψ, ϕ_x) of the system (A.3) defined locally for $x \in (0, \delta)$, $\delta > 0$, verifying the asymptotics:*

$$\begin{aligned}
W = x + O(x^{2\mu+1}), \qquad X = 1 + O(x^{\mu}), \qquad Y = \frac{\sqrt{n(n-1)}}{a} x^{1-\frac{1}{\sqrt{n}}} + O(x^{2\mu+1-\frac{1}{\sqrt{n}}}),\\
\psi = a x^{\frac{1}{\sqrt{n}}} + O(x^{\frac{2\mu+1}{\sqrt{n}}}),\ a > 0 \qquad \frac{\psi_x}{\psi} = \frac{1}{\sqrt{n}}\frac{1}{x} + O(x^{\mu-1}), \qquad\qquad (A.9)\\
\phi_x = \frac{\sqrt{n}-1}{x} + O(x^{\mu-1}),\ \ \frac{\psi_{xx}}{\psi} = -\frac{\sqrt{n}-1}{n}\frac{1}{x^2} + O(x^{\mu-2}),\ \ \phi_{xx} = -\frac{\sqrt{n}-1}{x^2} + O(x^{\mu-2})
\end{aligned}$$

Proof. Let $X(y) = 1 + g(y)$. Plugging into the equation of W' in (A.6) we obtain

$$W(y) = W(0)\exp\big\{y + \int_0^y W(z)g(z)(2+g(z))dz - \lambda\int_0^y W^3(z)dz\big\},$$

where according to (A.8), for $y \leq 0$,

$$\Big|\int_0^y W(z)g(z)(2+g(z))dz\Big| \leq 3\varepsilon^2(2+\sqrt{3}\varepsilon)\frac{1-e^{2\mu y}}{2\mu}$$

and

$$\Big|-\lambda\int_0^y W^3(z)dz\Big| \leq |\lambda|\cdot 3\sqrt{3}\varepsilon^3\frac{1-e^{3\mu y}}{3\mu}.$$

Thus,

$$\begin{aligned}
W(y) =& C_1 e^y + W(0)e^y\Big[\exp\big\{\int_0^y W(z)g(z)(2+g(z))dz - \lambda\int_0^y W^3(z)dz\big\}\\
&- \exp\big\{-\int_{-\infty}^0 W(z)g(z)(2+g(z))dz + \lambda\int_{-\infty}^0 W^3(z)dz\big\}\Big],
\end{aligned}$$

where $C_1 = W(0)\exp\big\{-\int_{-\infty}^0 W(z)g(z)(2+g(z))dz + \lambda\int_{-\infty}^0 W^3(z)dz\big\} > 0$. Using (A.8) again, we readily estimate the second term as above

$$W(y) = C_1 e^y + O(e^{(2\mu+1)y}) \qquad\qquad y \leq 0.$$

Similarly, from the equation of Y' (A.6) we obtain

$$Y(y) = C_2 e^{(1-\frac{1}{\sqrt{n}})y} + O(e^{(2\mu+1-\frac{1}{\sqrt{n}})y}) \qquad\qquad y \leq 0.$$

for an appropriate positive ($Y(0) > 0$) constant C_2. As for X, directly from (A.8) we have the bound

$$X = 1 + g(y) = 1 + O(e^{\mu y}) \qquad\qquad y \leq 0,$$

which we can retrieve from the equation of X' by integrating on $(-\infty, y)$ and using (A.8), along with the previously derived estimates for W, Y.

Recall the transformation (A.4) to derive asymptotics for the variables in (A.3): ($y \leq -M$, $M > 0$ large)

$$\begin{aligned}
\psi =& \frac{\sqrt{n(n-1)}W}{Y} = \frac{\sqrt{n(n-1)}(C_1 e^y + O(e^{(2\mu+1)y}))}{C_2 e^{(1-\frac{1}{\sqrt{n}})y} + O(e^{(2\mu+1-\frac{1}{\sqrt{n}})y})} = \frac{\sqrt{n(n-1)}C_1}{C_2} e^{\frac{1}{\sqrt{n}}y} + O(e^{(2\mu+\frac{1}{\sqrt{n}})y})\\
\frac{\psi_x}{\psi} =& \frac{X}{\sqrt{n}W} = \frac{1 + O(e^{\mu y})}{\sqrt{n}(C_1 e^y + O(e^{(2\mu+1)y}))} = \frac{1}{\sqrt{n}C_1}e^{-y} + O(e^{(\mu-1)y})\\
\phi_x =& n\frac{\psi_x}{\psi} - \frac{1}{W} = \frac{\sqrt{n}}{C_1}e^{-y} + nO(e^{(\mu-1)y}) - \frac{1}{C_1 e^y + O(e^{(2\mu+1)y})} = \frac{\sqrt{n}-1}{C_1}e^{-y} + O(e^{(\mu-1)y}).
\end{aligned}$$

Also, going back to the second equation of (A.3) and dividing both sides by ψ^2 yields

$$
\begin{aligned}
\frac{\psi_{xx}}{\psi} &= -(n-1)\frac{\psi_x^2}{\psi^2} + \frac{n-1}{\psi^2} + \frac{\psi_x}{\psi}\phi_x + \lambda \\
&= -(n-1)\big[\frac{1}{\sqrt{n}C_1}e^{-y} + O(e^{(\mu-1)y})\big]^2 + \frac{n-1}{\big[\frac{\sqrt{n(n-1)}C_1}{C_2}e^{\frac{1}{\sqrt{n}}y} + O(e^{(2\mu+\frac{1}{\sqrt{n}})y})\big]^2} \\
&\quad + \big[\frac{1}{\sqrt{n}C_1}e^{-y} + O(e^{(\mu-1)y})\big]\big[\frac{\sqrt{n}-1}{C_1}e^{-y} + O(e^{(\mu-1)y})\big] + \lambda \\
&= -\frac{\sqrt{n}-1}{n}\frac{e^{-2y}}{C_1^2} + O(e^{(\mu-2)y}).
\end{aligned}
$$

Furthermore, the first equation of (A.3) gives

$$
\phi_{xx} = n\frac{\psi_{xx}}{\psi} - \lambda = -(\sqrt{n}-1)\frac{e^{-2y}}{C_1^2} + nO(e^{(\mu-2)y}) + \lambda = -(\sqrt{n}-1)\frac{e^{-2y}}{C_1^2} + O(e^{(\mu-2)y}).
$$

Having derived asymptotics, as $y \to -\infty$, for all the unknown functions appearing in the problem, we would like to derive corresponding asymptotics in the independent variable x that we started with. For that we recall (A.5) and normalize so that $x \to 0^+$ as $y \to -\infty$ to deduce

$$
x = \int W dy = \int C_1 e^y + O(e^{(2\mu+1)y}) dy = C_1 e^y + O(e^{(2\mu+1)y}) \qquad (y \leq 0),
$$

Hence, it follows

$$
C_1 e^y = x + O(x^{2\mu+1}),
$$

for $y \leq -M$, $M > 0$ large. Going back to each of the above estimates, we confirm the rest of the asymptotics in Proposition A.1 for $a = \frac{\sqrt{n(n-1)}C_1^{1-\frac{1}{\sqrt{n}}}}{C_2} > 0$. □

Remark A.1. *One could also consider the trajectories which emanate from the other equilibrium* $(0,-1,0)$ *of (A.6) (also a source). These can be seen to correspond to solitons with profile*

$$
\psi(x) \sim x^{-\frac{1}{\sqrt{n}}} \qquad \phi_x(x) = -\frac{1+\sqrt{n}}{x}, \qquad \text{as } x \to 0^+.
$$

They are in fact defined for all dimensions $n+1 \geq 2$, *and in the steady case* ($\lambda = 0$), *dim* $n+1 = 2$, *can be explicitly written out as:*

$$
\psi(x) = \frac{1}{x} \qquad \phi_x(x) = -\frac{2}{x}, \qquad x \in (0, +\infty).
$$

Notice that these metrics are also singular at $x = 0$, *but their evolution under the Ricci flow (through diffeomorphisms) is almost the opposite from the metrics we obtain near the equilibrium at* $(0,1,0)$*; see §2.1. In particular, they remain singular for all time. However, these solitons are beyond the scope of this paper.*

A.2 The steady singular solitons; asymptotics at $x = +\infty$

In the steady case, $\lambda = 0$, we can push the domain of the solutions considered in Proposition A.1 all the way up to $+\infty$. A very useful tool in the analysis of the trajectories of (A.6) is the Lyapunov function [6, §1.4.3]

$$L = X^2 + Y^2, \qquad\qquad (L-1)' = X^2(L-1), \tag{A.10}$$

which implies that the unit disk is a stable region of the critical point $(0,0)$. Further, it follows from (A.10) that the equation of W' in (A.6) is actually redundant, reducing the system to

$$\begin{cases} X' = X^3 - X + \frac{Y^2}{\sqrt{n}} \\ Y' = Y(X^2 - \frac{X}{\sqrt{n}}) \end{cases} \tag{A.11}$$

We remark that the unique trajectory emanating from the equilibrium point $(\frac{1}{\sqrt{n}}, \sqrt{1-\frac{1}{n}})$ and converging (as $y \to +\infty$) to the origin $(0,0)$ corresponds to the well-known Bryant soliton (see [6]).

The source considered in (A.7) corresponds to the point $(1,0)$. Thus, if we consider solutions of (A.6) with initial point $(X(0), Y(0))$ satisfying $X^2(0) + Y^2(0) < 1$, $Y(0) > 0$ and lying close enough to $(1,0)$, we easily conclude that the trajectory $(X(y), Y(y))$ approaches the origin $(0,0)$, as $y \to +\infty$ (at an exponential rate). Whence it exists for all $y \in (-\infty, +\infty)$. In fact, these trajectories emanating from $(1,0)$ translate back to Ricci soliton metrics of the form (2.1), which exist (and are smooth) for all $x \in (0, +\infty)$ and have the leading behavior described in Proposition A.1 at $x = 0$.

One can easily see that the set of all such trajectories fills up the domain in the unit disc bounded by the Bryant soliton trajectory $\big($which emanates from $(\frac{1}{\sqrt{n}}, \sqrt{1-\frac{1}{n}})\big)$ and the positive X-axis. The asymptotics of these trajectories at $+\infty$ are easily seen to matching those of the one corresponding of the Bryant soliton. This has to do with the Lyapunov function (A.10) and the uniform convergence of the trajectories at the origin $(0,0)$, as $y \to +\infty$. Following [6, Chapter 1, §4] we arrive at the next proposition.

Proposition A.2. *The soliton metrics corresponding to the (X,Y)-orbits above are complete towards $x = +\infty$ and satisfy the asymptotics*

$$cx^{\frac{1}{2}} \le \psi \le Cx^{\frac{1}{2}} \qquad cx^{-\frac{1}{2}} \le \dot{\psi} \le Cx^{-\frac{1}{2}} \qquad -Cx^{-\frac{3}{2}} \le \ddot{\psi} \le -cx^{-\frac{3}{2}}, \tag{A.12}$$

for $x > M$ large, retrieving from (A.3) the asymptotics of the derivatives of ϕ

$$-C < \phi_x < -c \qquad\qquad -Cx^{-2} \le \phi_{xx} \le -cx^{-2}. \tag{A.13}$$

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