The Penrose inequality on perturbations of the Schwarzschild exterior

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Abstract

We prove a version the Penrose inequality for black hole space-times which are perturbations of the Schwarzschild exterior in a slab around a null hypersurface $\underline{\mathcal{N}}_0$. $\underline{\mathcal{N}}_0$ terminates at past null infinity $\mathcal{I}^-$ and $\mathcal{S}_0 := \partial\underline{\mathcal{N}}_0$ is chosen to be a marginally outer trapped sphere. We show that the area of $\mathcal{S}_0$ yields a lower bound for the Bondi energy of sections of past null infinity, thus also for the total ADM energy. Our argument is perturbative, and rests on suitably deforming the initial null hypersurface $\underline{\mathcal{N}}_0$ to one for which the natural "luminosity" foliation originally introduced by Hawking yields a monotonically increasing Hawking mass, *and* for which the leaves of this foliation become asymptotically round. It is to ensure the latter (essential) property that we perform the deformation of the initial null hypersurface $\underline{\mathcal{N}}_0$.

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1 Introduction

We prove a version of the Penrose inequality for metrics which are local perturbations of the Schwarzschild exterior, around a (past-directed) shear-free outgoing null surface $\underline{\mathcal{N}}_0$.

Our method of proof is an extension of the approach sometimes called the "null" Penrose inequality [18, 19], and in fact relies on ideas in the PhD thesis of Sauter under the direction of D. Christodoulou, [23]. Before stating our result, we recall in brief the original formulation of the Penrose inequality and the motivation behind it. This will make clear the motivation for proving this inequality for perturbations of the Schwarzschild solution.

Penrose proposed his celebrated inequality [20] as a *test* for what he called the "establishment view" on the evolution of dynamical black holes in the large. For a single black hole, the view was that the exterior region $(\mathcal{M}_{\text{ext}}, \mathbf{g})$ should evolve smoothly and eventually settle down, due to emission of matter and radiation into the black hole (through the future event horizon $\mathcal{H}^+$) and towards future null infinity $\mathcal{I}^+$. This "final state" would be non-radiating, and thus putatively stationary. In vacuum, such final states are believed to belong to the Kerr family of solutions, [14, 20].

While the mathematical verification of the above appears completely beyond the reach of current techniques (and indeed the above scenario rests on some fundamental conjectures of general relativity such as the weak cosmic censorship), it gives rise to a very rich family of problems. For example, the very last assertion is the celebrated "black hole uniqueness question", to which much work has been devoted. It has been resolved under the un-desirable assumption of real analyticity by combining the works of many authors–see [14, 9, 22, 11] and references in the last paper; more recently the author jointly with A. Ionescu and S. Klainerman has proven the result by replacing the assumption of real-analyticity by either that of closeness to the Kerr family of solutions, *or* that of small angular momentum on the horizon [1, 2, 3].

Given the magnitude of the challenge, Penrose proposed an inequality as a *test* of the above prediction: Indeed, if the above scenario were true, then the area of any section $\mathcal{S}$ of the future event horizon would have to satisfy:

$$\sqrt{\frac{\text{Area}[\mathcal{S}]}{16\pi}} \leq m_{\text{ADM}}. \tag{1.1}$$

where $m_{\rm ADM}$ stands for the ADM mass of an initial data set for $(\mathcal{M}_{\rm ext}, \mathbf{g})$. To see that the final state scenario implies this inequality, one needs to recall a few well-known facts: That this inequality holds for the Kerr space-times (in fact equality is achieved for the Schwarzschild solution), that the area of sections of the event horizon $\mathcal{H}^+$ is increasing towards the future [14], while the Bondi mass of sections of $\mathcal{I}^+$ is decreasing towards the future, while initially (at space-like infinity i^0) it agrees with the ADM mass.

Since the above formulation pre-supposes the entire event horizon $\mathcal{H}^+$, an alternative version of the above has been proposed, where (1.1) should hold for any $\mathcal{S}$ (inside the black hole) which is *marginally outer trapped sphere* (MOTS, from now on). Another way to then interpret the inequality (1.1) for such surfaces $\mathcal{S}$ is that all asymptoticaly flat vacuum space-times containing a MOTS of area 16π, the ADM mass must be bounded below by 1; moreover the bound should be achieved precisely by the Schwarzschild solution of mass 1. In short, the mass 1 Schwarzschild solution is the global minimizer of the ADM mass for asymptotically flat space-times containing a MOTS of area 16π.

Informally, our main result is a proof that the Schwarzschild solution of mass 1 is a *local* minimizer of the ADM energy, in the space of regular solutions close to Schwarzschild. In fact, our result requires only a portion of a space-time, which is a neighborhood of a past-directed outgoing null hypersurface $\underline{\mathcal{N}}_0$ which emanates from the MOTS $\mathcal{S}_0$ and extends up to past null infinity $\mathcal{I}^-$. It is in that portion of the space-time that we require our metric to be a perturbation of the Schwarzschild exterior.

Before proceeding to state the result precisely, we note the many celebrated results on the Penrose inequality in other settings. The Riemannian version of this inequality (corresponding to a time-symmetric initial data surface) was proven by Huisken-Ilmanen (for a connected MOTS which in fact is an outermost minimal surface on the initial data surface) [15] and Bray (for a MOTS with possibly many components) [8], and has led to many extensions to incorporate charge and angular momentum. See [19] for a review of these.

While the aforementioned results deal with an asymptoticaly flat (Riemannian) hypersurface in a space-time $(\mathcal{M}, \mathbf{g})$, our point of reference will be null hyper-surfaces in $(\mathcal{M}, \mathbf{g})$.

To state our assumptions clearly, recall the the Schwarzschild metric of mass m in Eddington-Finkelstein coordinates:

$$g_{\rm Schwarz} = -(1-\frac{2m}{r})d\underline{u}^2 + 2d\underline{u}dr + r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{1.2}$$

Here any hyper-surface $\{\underline{u} = \text{Const}, r \geq 2m\}$ is a shear-free (rotationally symmetric) past-directed outgoing null hypersurface, which terminates at past null infinity $\mathcal{I}^-$.[1] Consider the domain $\mathcal{D} := \{\underline{u} \in [\underline{u}_0, \underline{u}_0 + m), r \geq 2m\}$ in the Schwarzschild space-times. The portion $\{r = 2m\}$ of the boundary of this domain lies on the future event horizon $\mathcal{H}^+$ of the Schwarzschild exterior. In

[1] $\{\underline{u} = \text{Const}, r \geq 2m\}$ is a 3-dimensional smooth submanifold. We use the term "hypersurface" or "null surface" in this paper, with a slight abuse of language.

particular all the spheres $\{r = 2m, \underline{u} = \text{Const}\}$ are marginally outer trapped surfaces (the definition of this notion is recalled below).

Our main assumption on our space-time is that the metric g defined over the domain $\mathcal{D} := \{\underline{u} \in [\underline{u}_0, \underline{u}_0 + m), r \geq 2m, \theta \in [0, \pi), \phi \in [0, 2\pi)\}$ is a $\mathcal{C}^4$-perturbation of the Schwarzschild metric over the same domain, where we partly fix the gauge by requiring that the sphere $\mathcal{S}_0 := \{\underline{u} = \underline{u}_0, r = 2m\}$ is still marginally outer trapped and the hypersurface $\{r = 2m\}$ is still null. (We make the $\mathcal{C}^4$-closeness assumption precise below).

The inequality that we prove involves the notion of Bondi energy of a section of null infinity, compared to the area of a MOTS. We review these notions here:

Sections of Null infinity and the Bondi energy. We recall past null infinity $\mathcal{I}^-$ is an idealized boundary of the space-time, where past-directed null geodesics "end". In the asymptotically flat setting that we deal with here, the topology of $\mathcal{I}^-$ is $\mathbb{R} \times \mathbb{S}^2$. In the coordinates introduced above for the Schwarzschild exterior (and also in the perturbed Schwarzschild space-times we will consider), we can endow $\mathcal{I}^-$ with the coordinates $\underline{u}, \phi, \theta$.

A $\mathcal{C}^2$ *section* $\mathcal{S}_\infty := \{\underline{u} = f(\phi, \theta)\}$ of $\mathcal{I}^-$ can be seen as the boundary at infinity of an (incomplete) null surface $\underline{\mathcal{N}}$. We schematically write $\mathcal{S}_\infty = \partial_\infty \underline{\mathcal{N}}$. Now, consider any 1-parameter family of spheres $\mathcal{S}_t, t \geq 1$ foliating $\underline{\mathcal{N}}$ and converging (in a topological sense) towards $\mathcal{S}_\infty$. A (suitably regular) such foliation $\{\mathcal{S}_t\}$ is thought of as a reference frame relative to which certain natural quantities at $\mathcal{S}_\infty$ are defined. In particular, recalling the Hawking mass of any such sphere via (2.23) below, let:

Definition 1.1. *The Bondi energy of $\mathcal{S}_\infty := \partial_\infty \underline{\mathcal{N}}$ relative to the foliation (reference frame) $\mathcal{S}_t$ is defined to be :*

$$E_B := lim_{t\to\infty} m_{\text{Hawk}}[\mathcal{S}_t],$$

provided that:

$$\lim_{t\to\infty} \big(\frac{\text{Area}[\mathcal{S}_t]}{4\pi}\mathcal{K}[\mathcal{S}_t]\big) = 1.$$

(The limit here makes sense by pushing forward the metrics from $\mathcal{S}_t$ to $\mathcal{S}_\infty$, via a natural map that identifies the point on $\mathcal{S}_t$ with the one on $\mathcal{S}_\infty$ that lies on the same null generator of $\underline{\mathcal{N}}$).

We note that the *asymptotic roundness* required above is *necessary* for the limit of the Hawking masses to correspond to the Bondi Energy of $\mathcal{S}_\infty$ (relative to the foliation $\mathcal{S}_t, t \geq 1$). We also remark that the Bondi energy corresponds to the time-component of the Bondi-Sachs energy-momentum 4-vector $(E_B, \vec{P}_B)$ cf Chapter 9.9 in [21], relative to the reference frame given by $\mathcal{S}_t, t \geq 1$.

The Bondi *mass* is defined to be the Minkowski length of this vector:

$$m_B[\mathcal{S}_\infty] = \sqrt{(E_B)^2 - |\vec{P}_B|^2}.$$

This is in fact invariant under the changes of reference frame considered above. We refer to [21] Chapter 9.9 for the definition of these notions. We note that the theorem we prove here yields a lower bound for the Bondi energy, rather than the Bondi mass. In fact, we believe that the proof can be adapted to cover the case of the Bondi mass also,[2] since the two notions agree when the linear momentum vanishes (this is the center-of-mass reference frame).

The reason we do not pursue this here, is that the definition of linear momentum used in [21] uses spinors, and it is not immediately clear to the author how to translate this notion into the framework of Ricci coefficients used here.

Marginally outer trapped spheres:

Definition 1.2. *In an asymptoticaly flat space-time* $(\mathcal{M}, \mathbf{g})$*, a 2-sphere* $\mathcal{S} \subset \mathcal{M}$ *is called marginally outer trapped if, letting* $tr\chi^L[\mathcal{S}]$ *be its null expansion relative to a future-directed outgoing normal null vector field* L*, and also* $tr\chi^{\underline{L}}[\mathcal{S}]$ *its null expansion relative to the future-directed incoming null normal vector field* $\underline{L}$*, then for all points* $P \in \mathcal{S}$*:*

$$tr\chi^L[\mathcal{S}](P) = 0, tr\chi^{\underline{L}}[\mathcal{S}](P) < 0.$$

Our result is the following:

Theorem 1.3. *Consider any vacuum Einstein metric over* $\mathcal{D} := \{\underline{u} \in [\underline{u}_0, \underline{u}_0 + m), r \geq 2m, (\phi, \theta) \in \mathbb{S}^2\}$ *as above which is a perturbation of the Schwarzschild metric of mass* m *over* $\mathcal{D}$ *(measured to be* δ*-close, in a suitable norm that we introduce below), and such that the surface* $\mathcal{S}_0 := \{\underline{u} = \underline{u}_0, r = 2m\}$ *is marginally outer trapped.*

Then if $\delta \ll m$ *there exists a perturbation* $\mathcal{S}' \subset \mathcal{D}$ *of* $\mathcal{S}_0$*, which is also marginally outer trapped, and such that*

- $\text{Area}[\mathcal{S}'] \geq \text{Area}[\mathcal{S}]$.
- *The past-directed outgoing null surface* $\underline{\mathcal{N}}_{\mathcal{S}'}$ *emanating from* $\mathcal{S}'$ *is smooth, terminates at a cut* $\mathcal{S}^\infty \subset \mathcal{I}^-$*, and moreover the following Penrose inequality holds:*

$$E_{\text{B}}[\mathcal{S}^\infty] \geq \sqrt{\frac{\text{Area}[\mathcal{S}']}{16\pi}}, \tag{1.3}$$

  where $E_{\text{B}}[\mathcal{S}^\infty]$ *stands for the Bondi energy on the (asymptotically round) sphere* $\mathcal{S}^\infty$ *on* $\underline{\mathcal{N}}_{\mathcal{S}'}$*, associated with the luminosity foliation on* $\underline{\mathcal{N}}_{\mathcal{S}'}$*. (The latter foliation is recalled below)*

Furthermore, equality holds in the first inequality if and only if $\mathcal{S}' \subset \{r = 2m\}$*, and* $tr\chi^L = 0$ *on* $\{r = 2m\}$ *between* $\mathcal{S}, \mathcal{S}'$*. Equality holds in the second inequality if and only if* $\underline{\mathcal{N}}_{\mathcal{S}'}$ *is isometric (intrinsically and extrinsically) to a spherically symmetric outgoing null surface* $\{\underline{u} = \text{Const}\}$ *in a Schwarzschild space-time.*

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[2]We explain why this should be so in a remark in the next subsection.

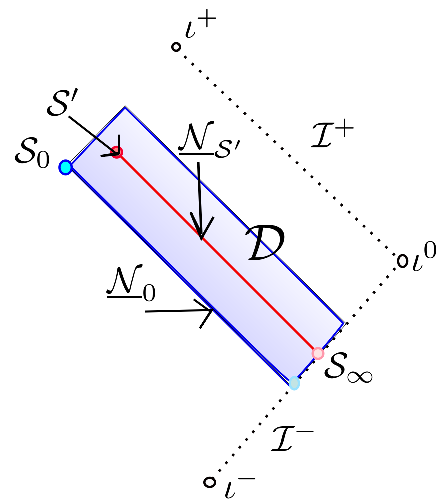

Figure 1: The Penrose diagram of the domain $\mathcal{D}$, the "old" MOTS $\mathcal{S}$ and the "new" MOTS $\mathcal{S}'$, and the smooth outgoing null surfaces $\underline{\mathcal{N}}_0, \underline{\mathcal{N}}_{\mathcal{S}'}$.

1.1 Outline of the paper.

Our proof rests on a perturbative argument. We mainly seek to exploit the evolution equation of the Hawking mass under a particular law of motion (introduced in [13]) on a *fixed*, *smooth*, outgoing null surface together with the possibility of perturbing the underlying null surface itself. We note that the possibility of varying the underlying null hypersurface as a possible approach method towards deriving the Penrose inequality was raised in Chapter 8 in [23].

The broad strategy is based on two observations:

- On the Schwarzschild space-time *and*, on small perturbations of the Schwarzschild space-time, the Hawking mass is *increasing* along "nearly" shear-free null hypersurfaces $\underline{\mathcal{N}}_0$ that emanate from a MOTS, when $\underline{\mathcal{N}}_0$ is foliated by a "luminosity parameter", originally introduced in [13], as we recall below. However, as observed in [23], the corresponding leaves of the foliation may *fail* to become asymptotically round.

- Under the closeness to Schwarschild assumption, we can perturb the underlying null hypersurface $\underline{\mathcal{N}}_0$ towards the future in order to induce a small conformal deformation of the metric "at infinity" associated with such a foliation. In fact (after renormalizing by the area), we can achieve *any* small conformal deformation. Moreover, all such new null hypersurfaces $\underline{\mathcal{N}}'$ emanate from MOTSs $\mathcal{S}'$ with $\mathrm{Area}[\mathcal{S}'] \geq \mathrm{Area}[\mathcal{S}]$.

The proof is then finished by invoking the implicit function theorem, since by the "closeness to Schwarzschild" assumption, the (renormalized) sphere at infinity

$\mathcal{S}_\infty$ of the original is "nearly round", thus it can be made exactly round by a small conformal deformation.

Remark. *As explained, our proof below in fact shows that* all *small, area-normalized, conformal transformations of the metric on the sphere at infinity* $\mathcal{S}_\infty$ *can be achieved by small perturbations of* $\underline{\mathcal{N}}_0$. *In particular, we believe that we can find a nearby surface* $\underline{\mathcal{N}}'$ *for which the* linear momentum *of the (perturbed) sphere at infinity vanishes, thus we obtain a lower bound for the Bondi mass, and not just the Bondi energy. However we do not pursue this here.*

Section Outline: In section 2 we state the assumptions of closeness to Schwarzschild precisely. We also recall the Ricci coefficients associated to a foliated null surface $\underline{\mathcal{N}}$, and (certain of) the null structure equations which link these to components of the ambient Weyl curvature. We also set up the framework for the analysis of perturbations of our given null surface $\underline{\mathcal{N}}_0$. In section 3 we study the "nearby" MOTSs to $\mathcal{S}_0$, when we deform $\mathcal{S}_0$ towards the future.

In section 4 we study the asymptotic behaviour of the expansion $\underline{\chi}$ of any smooth fixed null surface $\underline{\mathcal{N}}$, and moreover the variational behaviour of this relative to perturbations of $\underline{\mathcal{N}}$. We also recall the luminosity foliations and the monotonicity of the Hawking mass evolving along these foliations. We then show that the luminosity foliations asymptotically agree with *affine* foliations to a sufficiently fast rate, so that the Gauss curvatures and Bondi energies associated to the two foliations agree. This then enables us to *replace* the study of luminosity foliations by affine ones.

In particular in section 5 the first variation of the luminosity foliations is captured by "standard" Jacobi fields. We then note (relying on some calculations in [5]) that the effect of a variation of $\underline{\mathcal{N}}_0$ on the Gauss curvature at infinity is captured by a conformal transformation of the underlying metric over the sphere at infinity $\mathcal{S}_\infty$. In the latter half of that section we derive the solution of the relevant Jacobi fields. In conclusion, the first variation of the Gauss curvature is captured in the composition of two second-order operators, $\mathcal{F} \circ \mathcal{L}$, where $\mathcal{L}$ is a perturbation of the Laplace-Beltrami operator on the metric $\gamma_0 := \mathbf{g}|_{\mathcal{S}_0}$, and $\mathcal{F}$ is the operator $\Delta_{\gamma^\infty} + 2\mathcal{K}[\gamma^\infty]$, for γ^∞ being the metric at infinity associated with the luminosity foliation on $\underline{\mathcal{N}}_0$ and $\mathcal{K}[\gamma^\infty]$ its Gauss curvature. The proof is then completed in section 6 by an application of the implicit function theorem.

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2 Assumptions and Background.

We state the assumptions on the (local) closeness of our space-time to the Schwarzschild solution, as well as the norms in which this closeness is measured. The assumptions we make concern the metric in the domain $\mathcal{D}$, and the curvature components and their variational properties on a suitable family of smooth null hypersurfaces contained inside $\mathcal{D}$. (This is the family in which we perform the variations of $\underline{\mathcal{N}}_0$). As we note below, one expects that one does not really need to assume regularity on the whole domain $\mathcal{D}$. Rather, what we assume here should be derivable assuming data on the initial $\underline{\mathcal{N}}_0$, and a suitable portion $\underline{u} \in [\underline{u}_0, \underline{u}_0 + m)$ of $\mathcal{I}^-$ only. Finally, we note that the number of derivatives (of the various geometric quantities) can be dropped. Yet any such weakening of the assumptions is not in the scope of this paper, and would lengthen it substantially. The main contribution we wish to make here is to introduce the idea of deforming the null surfaces under consideration, as a possibly useful method for inequalities in general relativity, via an ODE analysis of the null structure equations.

2.1 Closeness to Schwarzschild and regularity assumptions.

In order to state the closeness assumption precisely from $(\mathcal{M}, \mathbf{g})$, we need to introduce the parameters that capture the closeness of our space-time to the Schwarzschild background.

The Scharzschild metric: Recall the form (1.2) of the Schwarzschild metric g_{Schwarz}. In particular note that r is both an affine parameter on each null surface $\underline{u} = \text{Const}$, and an area parameter, in that:

$$\text{Area}[\{r = B\} \bigcap \{\underline{u} = \text{Const}]\} = 4\pi B^2.$$

Consider the normalized null vector fields

$$\underline{L} = \partial_r, L = 2\partial_{\underline{u}} + (1 - \frac{2m}{r})\partial_r. \tag{2.1}$$

Then $g_{\text{Schwarz}}(L, \underline{L}) = 2$ and the Ricci coefficients of the Schwarzschild metric relative to this pair of null vectors are

$$tr\chi^L = (1 - \frac{2m}{r})\frac{2}{r}, tr\underline{\chi}^{\underline{L}} = \frac{2}{r}, \hat{\chi}^L = \hat{\underline{\chi}}^{\underline{L}} = 0, \zeta^L = 0. \tag{2.2}$$

In particular note that $tr\chi^L > 0$ on $\underline{\mathcal{N}}$ away from $\mathcal{H}^+ = \{r = 2m\}$, and moreover

$$\partial_r tr\chi|_{\mathcal{H}^+} = (2m)^{-1} > 0. \tag{2.3}$$

Coordinates for the perturbed metric: The metrics we will be considering will be perturbations of a Schwarzschild metric of mass $m > 0$ over a domain $\mathcal{D}$. We will be using a label u for the outgoing past-directed parameter instead of r; this is since it will no longer correspond with the area parameter r. We consider a metric g over $\mathcal{D} := \{\underline{u} \in [\underline{u}_0, \underline{u}_0 + m), u \geq 2m, \phi \in [0, 2\pi), \theta \in [0, \pi)\}$. We let $\mathcal{S}_0 := \{u = 2m, \underline{u} = \underline{u}_0\}$. (Below we will always be considering two coordinate systems to cover this sphere; all bounds will be assumed to hold with respect to either of the coordinate systems). The coordinates are normalized so that u is an affine parameter on each surface $\{\underline{u} = \text{Const}\}$, and the level set $\mathcal{N}[\mathcal{S}] := \{u = 2m\}$ is an outgoing null hypersurface, and moreover $\underline{u}$ is an affine parameter on $\mathcal{N}[\mathcal{S}_0]$. The coordinates ϕ, θ are fixed on the initial sphere $\mathcal{S}_0$ and are then extended to be constant on the generators of $\mathcal{N}[\mathcal{S}_0]$, and then *again* extended to be constant on each of the null generators of each $\{\underline{u} = \text{Const}\}$.

Note that this condition fixes the coordinates uniquely, up to normalizing $\partial_{\underline{u}}, \partial_u$ on the initial sphere, $\mathcal{S}_0$. We normalize so that letting $\underline{L} := \partial_u$ on that sphere $tr\underline{\chi}^{\underline{L}} = 2$ and $g(\underline{L}, L) = g(\underline{L}, \partial_{\underline{u}}) = 2$.

We will find it convenient to introduce a canonical frame on *any* smooth past-directed outgoing null surface $\underline{\mathcal{N}}$ emanating from a sphere $\mathcal{S}$; the frame is uniquely determined after we choose (two) coordinate systems that cover $\mathcal{S}$, and an affine parameter on $\underline{\mathcal{N}}$.

Definition 2.1. *Given coordinates ϕ, θ on $\mathcal{S}$, we extend them to be constant along the null generators of $\underline{\mathcal{N}}$. Given an affine parameter λ on $\underline{\mathcal{N}}$, with $\lambda = 1$ on $\mathcal{S} = \partial\underline{\mathcal{N}}$, we let $\Phi := \lambda^{-1}\partial_\phi, \Theta := \lambda^{-1}\partial_\theta$. We let $e^1 := \Phi, e^2 := \Theta$.*

We also consider a vector field L which is defined to be future-directed and null, and moreover satisfies:

$$g(e^1, L) = g(e^2, L) = g(L, \underline{L}) = 2. \tag{2.4}$$

We will be measuring the various natural tensor fields over a null surface $\underline{\mathcal{N}}$ with respect to the above frame. We introduce a measure of smoothness of such tensor fields:

Definition 2.2. *We say that a function f defined over any smooth infinite null surface $\underline{\mathcal{N}}$ with an affine parameter λ belongs to the class $O_2^\delta(\lambda^{-K})$ if in the coordinate system $\{\lambda, \phi^1 := \phi, \phi^2 := \theta\}$ on $\underline{\mathcal{N}}$ we have $|\lambda^K f| \le \delta, |\partial_{\phi^i}\lambda^K f| \le \delta$, $|\partial^2_{\phi^i\phi^j}\lambda^K f| \le \delta$. The classes O_1^δ, O^δ are defined analogously, by not requiring the last (respectively, the two last) estimates above to hold.*

We also considering any 1-parameter family of tensor fields $t_{ab\dots c}$ over the spheres $\mathcal{S}_\lambda$ that foliate $\underline{\mathcal{N}}_u$. (Thus the indices take values among e^1, e^2). We say that $t \in O_2^\delta(\lambda^{-K})$ if and only if any component $t(e_{i_1} \dots e_{i_d})$ in this frame belongs to $O_2^\delta(\lambda^{-K})$.

The parameter $\delta > 0$ will be our basic measure of closeness to the Schwarzschild metric. We **choose** $0 < \delta \ll m$.

We can now state our assumptions at the level of the metric, the connection coefficients and of the curvature tensor. Note that for the level sets $\underline{\mathcal{N}}_{\underline{u}}$ of the optical function $\underline{u}$ introduced above, we have chosen a *preferred* affine parameter, which we have denoted by u. The assumptions will be stated in terms of that u:

Assumptions on the metric: In the coordinate system above over $\mathcal{D}$, the metric $\mathbf{g}$ takes the following form, subject to the convention that components that are not specifically written out are zero:

$$g = (2+O^\delta(u^{-1}))d\underline{u}du - (1-\frac{2m}{u}+O^\delta(u^{-1}))d\underline{u}d\underline{u} + O^\delta(u^{-1})d\phi d\underline{u} + O^\delta(u^{-1})d\theta d\underline{u}$$
$$+ u^2\sum_{\phi,\theta} g_{\phi\theta}d\phi d\theta, \tag{2.5}$$

and the components $g_{\phi\phi}, g_{\phi\theta}, g_{\theta\theta}$ are assumed to satisfy in both coordinate systems:

$$|g_{\phi\phi}-(g_{\mathbb{S}^2})_{\phi\phi}| + |g_{\theta\theta}-(g_{\mathbb{S}^2})_{\theta\theta}| + |g_{\phi\theta}-(g_{\mathbb{S}^2})_{\phi\theta}| \le \delta. \tag{2.6}$$

where $g_{\mathbb{S}^2}$ stands for the standard round metric on the unit sphere $\mathbb{S}^2$, with respect to the chosen coordinates ϕ, θ.

Connection coefficients on $\mathcal{S}_0$: We assume that

$$tr\underline{\chi}^{\underline{L}}[\mathcal{S}_0] = m^{-1}, \sum_{i=0}^{2}|\partial^{(i)}\hat{\underline{\chi}}^{\underline{L}}| \le \delta, \sum_{i=0}^{2}|\partial^{(i)}\hat{\underline{\chi}}^{\underline{L}}| \le \delta, t\chi^L[\mathcal{S}_0] = 0, \sum_{i=0}^{2}|\partial^{(i)}\hat{\chi}^L| \le \delta$$
$$\sum_{i=0}^{2}|\partial^{(i)}\zeta^L| \le \delta. \tag{2.7}$$

Curvature bounds: Finally, the curvature components are assumed to decay towards $\mathcal{I}^-$ at rates that are consistent with those derived by Christodoulou and Klainerman in the stability of the Minkowski space-time, [10].[3] It is necessary, however, to strengthen that assuming that up to *two spherical* and one *transverse*[4] derivatives of our curvature components also satisfy suitable decay properties. (The latter can be seen as *strengthenings* of the decay derived in [10], which are however entirely consistent with those results).

To phrase this precisely, recall the independent components of the Weyl curvature in a suitable frame: Consider the level spheres $\mathcal{S}_u \subset \underline{\mathcal{N}}_{\underline{u}}$ of u on $\underline{\mathcal{N}}_{\underline{u}}$. Consider the frame Φ, Θ (Definition 2.1) on these level spheres, and let L be the future-directed null normal vector field to the spheres $\mathcal{S}_u$, normalized so that $g(L, \underline{L}) = 2$.

Then, letting the indices a, b below take values among the vectors Φ, Θ we require that for some $\delta > 0$ and *all* $\underline{u} \in [\underline{u}_0, \underline{u}_0 + m)$, $u \ge 2m$:

$$\alpha_{ab} := R_{LaLb} = O_2^\delta(\frac{1}{u}), \beta_a := R_{L\underline{L}La} = O_2^\delta(\frac{1}{u^2}), \rho := \frac{1}{4}R_{L\underline{L}L\underline{L}} = -\frac{2m}{u^3} + O_2^\delta(u^{-3})$$
$$\sigma := \frac{1}{4}R_{L12\underline{L}} = O_2^\delta(u^{-3}), \underline{\beta}_a := R_{\underline{L}L\underline{L}a} = O_2^\delta(u^{-3-\epsilon}), \underline{\alpha}_{ab} := R_{\underline{L}a\underline{L}b} = O_2^\delta(u^{-3-\epsilon}), \tag{2.8}$$

[3]Note that in [10] the bounds are derived towards $\mathcal{I}^+$.
[4]The transverse directions are L, $\underline{L}$

where the last two equations are assumed to hold for some $\epsilon > 0$.[5]

We also assume the same results for the L-derivatives and $\underline{L}$-derivatives of the Weyl curvature components, again for *all* $\underline{u} \in [\underline{u}_0, \underline{u}_0 + m)$, $u \geq 2M$:

$$\begin{aligned}&\nabla_L \alpha_{ab} = O_2^\delta(\frac{1}{u}), \nabla_L \beta_a = O_2^\delta(\frac{1}{u^2}), \nabla_L \rho = O_2^\delta(u^{-3})\\&\nabla_L \sigma = O_2^\delta(u^{-3}), \nabla_L \underline{\beta}_a = O_2^\delta(u^{-3-\epsilon}), \nabla_L \underline{\alpha}_{ab} = O_2^\delta(u^{-3-\epsilon}).\end{aligned} \tag{2.9}$$

$$\begin{aligned}&\nabla_{\underline{L}} \alpha_{ab} = O_2^\delta(\frac{1}{u^2}), \nabla_{\underline{L}} \beta_a = O_2^\delta(\frac{1}{u^3}), \nabla_{\underline{L}} \rho = O_2^m(u^{-4})\\&\nabla_{\underline{L}} \sigma = O_2^\delta(u^{-4}), \nabla_{\underline{L}} \underline{\beta}_a = O_2^\delta(u^{-4-\epsilon}), \nabla_{\underline{L}} \underline{\alpha}_{ab} = O_2^\delta(u^{-4-\epsilon}).\end{aligned} \tag{2.10}$$

Variations of null surfaces: As our theorem is proven by a perturbation argument, we now introduce the space of variations of our past-directed outgoing null surface $\underline{\mathcal{N}}_0$ which emanates from $\mathcal{S}_0$.

Considering any smooth positive function $\omega(\phi, \theta)$ over $\mathcal{S}_0$, we let:

$$\mathcal{S}_\omega := \{\underline{u} = \omega(\phi, \theta)\} \subset \mathcal{N}[\mathcal{S}_0]. \tag{2.11}$$

Definition 2.3. *Let $\underline{L}_\omega$ be the unique past-directed and outgoing null vector field normal to $\mathcal{S}_\omega$ and normalized so that $g(L, \underline{L}_\omega) = 2$. Let $\underline{\mathcal{N}}_\omega$ be the null surface emanating from $\underline{L}_\omega$. We extend $\underline{L}_\omega$ to an affine vector field: $\nabla_{\underline{L}_\omega} \underline{L}_\omega = 0$. Also let λ_ω be the corresponding affine parameter with $\lambda_\omega = 1$ on $\mathcal{S}_\omega$.*

The next assumption asserts that the decay assumptions for the curvature components and Ricci coefficients on all $\underline{\mathcal{N}}_{\underline{u}}$ *persist* under small deformations of $\underline{\mathcal{N}}_{\underline{u}}$, for all $\underline{u} \in [\underline{u}_0, \underline{u}_0 + m)$.

For any function $\omega \geq 0$ with $||\omega||_{\mathcal{C}^2} \leq 10^{-1} m$ we consider the components $\underline{\alpha}, \ldots, \alpha$ using the vector field $\underline{L}_\omega$, and its induced frame on $\underline{\mathcal{N}}_\omega$. We then assume that the differences of all the relevant curvature components and derivatives thereof between $\underline{\mathcal{N}}, \underline{\mathcal{N}}_\omega$ are bounded by $||\omega||_{\mathcal{C}^2}$. (Recall that by the unique construction of coordinates $\phi, \theta, \lambda_\omega$ on $\underline{\mathcal{N}}_\omega$ we have a natural map between $\underline{\mathcal{N}}, \underline{\mathcal{N}}_\omega$). Specifically:

Assumption 2.4. *For any $e^a, e^b, a, b \in \{1, 2\}$ and any $k, l \in \{1, 2\}$ below we assume:*

$$|\alpha^\omega_{ab} - \alpha_{ab}| \leq \frac{||\omega||_{\mathcal{C}^2}}{\lambda}, |\partial_{\phi^k} \alpha^\omega_{ab} - \partial_{\phi^k} \alpha_{ab}| \leq \frac{||\omega||_{\mathcal{C}^2}}{\lambda}, |\partial^2_{\phi^k \phi^l} \alpha^\omega_{ab} - \partial^2_{\phi^k \phi^l} \alpha_{ab}| \leq \frac{||\omega||_{\mathcal{C}^2}}{\lambda}; \tag{2.12}$$

we moreover assume the analogous bounds for the differences for the derivatives $\nabla_{\underline{L}_\omega} \alpha$ and $\nabla_{L_\omega} \alpha$, and any Ricci coefficient, curvature component, or rotational derivative thereof in (2.8), (2.9), (2.10) *evaluated against the frames e^i and identified via the coordinates constructed above.*

[5]Note that in [10] these bounds where derived for $\epsilon = \frac{1}{2}$. Any $\epsilon > 0$ is sufficient for our argument here.

Remark. *The above assumption can in fact be* derived *using* (2.8), (2.9), (2.10) *and by studying the geodesics emanating from $\underline{L}_\omega$ in the coordinate system constructed on $\mathcal{D}$. However to do this would be somewhat technical and is beyond the scope of this paper. So we prefer to state it as an assumption.*

Since our argument will be perturbational, we find it convenient to alternatively express $\omega = \tau \cdot e^v$, where τ will be a parameter of variation. Specifically:

Definition 2.5. *Consider any function $v \in \mathcal{C}^2[\mathcal{S}_0]$, $||v||_{\mathcal{C}^2(\mathcal{S}_0)} \leq 10^{-1}m$ and any number $\tau \in (0,1)$, and let:*

$$\mathcal{S}_{v,\tau} := \{\underline{u} = e^v \tau\} \subset \mathcal{N}[\mathcal{S}_0]. \tag{2.13}$$

We also let $\underline{L}_{v,\tau}$ be the unique past-directed and outgoing null vector field normal to $\mathcal{S}_{v,\tau}$ and normalized so that $g(L, \underline{L}_{v,\tau}) = 2$.

Let $\underline{\mathcal{N}}_{v,\tau}$ be the null surface emanating from $\underline{L}_{v,\tau}$. Also let $\lambda_{v,\tau}$ be the affine parameter on $\underline{\mathcal{N}}_{v,\tau}$ generated by $\underline{L}_{v,\tau}$, normalized so that $\lambda_{v,\tau} = 1$ on $\mathcal{S}_{v,\tau}$.

Next, we will be studying the perturbational properties of the parametrized null surfaces based on the geodesics that emanate from $\mathcal{S}_{v,\tau}$ in the direction of the null vector field $\underline{L}_{v,\tau}$.

A key in proving our theorem will be in equipping each of the null surfaces $\underline{\mathcal{N}}_{v,\tau}$ with a suitable foliation. As we will see, it is sufficient for our purposes to restrict attention to *affine* foliations of our surfaces $\underline{\mathcal{N}}_{v,\tau}$. The additional regularity assumption is then that the variation of the Gauss curvatures of the leaves of such affine foliations is captured to a sufficient degree by the Jacobi fields which encode the variations. To make this precise, we introduce the space of variations of affinely parameterized null surfaces.

For a given $v \in W^{4,p}(\mathcal{S}_0)$ (for any fixed $p > 2$ from now on), consider any 1-parameter family of smooth null surfaces $\underline{\mathcal{N}}_{v,\tau}$. Each of these surfaces can be identified with a family of affinelly parametrized null geodesics that rule them. In other words, consider any sphere $\mathcal{S}_{v,\tau} \subset \mathcal{N}[\mathcal{S}_0]$, and a smooth family of affinely parametrized null geodesics $\gamma_{v,\tau}(s, Q)$ for $Q \in \mathcal{S}_{v,\tau}$ (s is the affine parameter); then the null surface $\underline{\mathcal{N}}_{v,\tau}$, as long as it is smooth can be seen simply as the union of the null geodesics $\gamma_{v,\tau}(s, Q)$:

$$\underline{\mathcal{N}}_{v,\tau} = \bigcup_{s \in \mathbb{R}_+, Q \in \mathcal{S}_{v,\tau}} \gamma_{v,\tau}(s, Q). \tag{2.14}$$

Consider a function $f(\tau, \mathcal{S}_0)$ (i.e. $f(\tau, \phi, \theta)$ in coordinates). Note that the function $f\lambda_{v,\tau}$ is *still* an affine function on $\underline{\mathcal{N}}_{v,\tau}$.

Clearly, the variation of the null surfaces $\underline{\mathcal{N}}_{v,\tau}$ *and* the associated affine parameters $f \cdot \lambda_{v,\tau}$ is encoded in Jacobi fields along the 2-parameter family of geodesics $\gamma(\lambda, Q) \subset \underline{\mathcal{N}}$.

In particular, we consider the Jacobi fields over $\gamma_0(Q), Q \in \mathcal{S}_0$:

$$J_Q^{f,v}(\lambda) := \frac{d}{d\tau}|_{\tau=0} \gamma_{v,\tau}(f \cdot \lambda_{v,\tau}, Q). \tag{2.15}$$

Definition 2.6. *We let* $\mathcal{S}^{B,f}_{v,\tau} := \{\lambda^f_{v,\tau} = B\} \subset \underline{\mathcal{N}}_{v,\tau}$. *We let* $\mathcal{K}[\mathcal{S}^{B,f}_{v,\tau}]$ *be the Gauss curvature of that sphere.*[6] *We let* $\tilde{\mathcal{K}}[\mathcal{S}^{B,f}_{v,\tau}] := B^2\mathcal{K}[\mathcal{S}^{B,f}_{v,\tau}]$. *We call* $\tilde{\mathcal{K}}[\mathcal{S}^{B,f}_{v,\tau}]$ *the renormalized Gauss curvature of* $\mathcal{S}^{B,f}_{v,\tau}$.

The final regularity assumption on our space-time essentially states that the Jacobi fields (2.15) capture to a sufficient degree the variation of the Gauss curvatures of the *spheres at infinity* associated with the affinely parametrized $\underline{\mathcal{N}}_{v,\tau}$ above. To make this precise, define

Definition 2.7. *We say that a function* $F_{v,\tau}(\phi,\theta,B)$ *with* $v \in W^{4,p}(\mathcal{S}_0)$, $\tau \in [0,1)$ *and* $(\phi,\theta) \in \mathbb{S}^2, B \geq 1$ *lies in* $o(\tau)$ *if for the set* $\mathcal{B}(0,10^{-1}m) \subset W^{4,p}(\mathcal{S}_0)$,[7] *we have* $\tau^{-1}F_{v,\tau}(\phi,\theta,B) \to 0$ *as* $\tau \to 0$, *uniformly for all* $v \in \mathcal{B}(0,10^{-1}m)$, $(\phi,\theta) \in \mathbb{S}^2, B \geq 1$. *We also say that* $F_{v,\tau}(\phi,\theta,B)$ *lies in* $o_2(\tau)$ *if* F, $\partial_{\phi^i}F$ *and* $\partial^2_{\phi^i\phi^j}F$ *lie in* $o(\tau)$.

We say that $F_{v,\tau}(\phi,\theta,B) \in O(B^{-1})$ *if for all* $v \in \mathcal{B}(0,10^{-1}m) \in W^{4,p}(\mathcal{S}_0)$ *as above,* $\tau \in [0,1)$ *we have* $F(v,\tau,B) \leq CB^{-1}$ *for some fixed* $C > 0$. *We also say that* $F \in O_2(B^{-1})$ *if* F, $\partial_{\phi^i}F$ *and* $\partial^2_{\phi^i\phi^j}F$ *lie in* $O(B^{-1})$.

(Note that Definition 2.2 deals with functions that depend only on ϕ,θ,λ, while Definition 2.7 deals with functionals that also depend on $v \in W^{4,p}, \tau \in [0,1)$).

Our final regularity assumption is then as follows:

Assumption 2.8. *Let us consider the frame* $\underline{L},\Phi,\Theta,L$ *over* $\underline{\mathcal{N}}_0$ *as in Definition 2.1*

Consider a Jacobi field J *over* $\underline{\mathcal{N}}_0$ *as in* (2.15) *for* $v \in W^{4,p}, p > 2$.[8] *Express* J *with respect to the frame* $\underline{L},\Phi,\Theta,L$, *with components* $J^L, J^{\underline{L}}$ *and* J^Φ, J^Θ. *Assume that* $J^\Phi, J^\Theta = O_2(\lambda^{-1})$.

Then we assume that the (renormalized) Gauss curvature of the sphere $\{f\cdot\lambda_{v,\tau} = B\} \subset \underline{\mathcal{N}}_{v,\tau}$ *asymptotically agrees with that of the sphere obtained by flowing by* τ *in the direction of the Jacobi field starting from* $\underline{\mathcal{N}}_0$:

Let $\tilde{\mathcal{S}}_{v,\tau}$ *be the sphere* $\{f\lambda_{v,\tau} = B\} \subset \underline{\mathcal{N}}_{v,\tau}$ *and* $\mathcal{S}_{1-\text{flow}(\tau)}(B)$ *be the sphere that arises from* $\{\lambda_0 = B\} \subset \underline{\mathcal{N}}$ *by flowing along* $\underline{L}$ *by* $J^{\underline{L}}\tau$. *Finally let* L'_τ *be the null vector field that is normal to* $\mathcal{S}_{1-\text{flow}(\tau)}(B)$, *and normalized so that* $g(L'_\tau,\underline{L}) = 1$; *let* $\mathcal{S}_{2-\text{flow}(\tau)}(B)$ *be the sphere that arises from* $\mathcal{S}_{1-\text{flow}(\tau)}(B)$ *flowing along the* L'_τ *by* $J^L\cdot\tau$. *Then:*

$$B^2\mathcal{K}[\tilde{\mathcal{S}}_{v,\tau}] = B^2\mathcal{K}[\mathcal{S}_{2-\text{flow}(\tau)}(B)] + o(\tau) + O(B^{-1}). \tag{2.16}$$

We remark that the above property is entirely standard in a smooth metric for *finite* geodesic segments. (See the discussion on Jacobi fields in [16], for example). Thus the assumption here should be seen as a regularity assumption on the space-time metric near null infinity, in the rotational directions Φ,Θ and in the transverse direction L. One expects that this property of Jacobi fields can

[6] We think of the Gauss curvature as a function in the coordinates ϕ,θ.

[7] I.e. the ball of radius $10^{-1}m$ in the Banach space $W^{4,p}(\mathcal{S}_0)$.

[8] We suppress f,v,Q for simplicity.

be derived from the curvature fall-off assumptions we are making, since it also follows immediately when the space-time admits a sufficiently regular conformal compactification (by using the aforementioned result on finite geodesic segments as well as the conformal invariance of null geodesics). However proving this is beyond the scope of this paper, so we state this as an assumption.

2.2 The geometry of a null surface and the structure equations.

The analysis we will perform will require the use of the null structure equations linking the Ricci coefficients of a null surface with the ambient curvature components. We review these equations here. We will be using these equations both for future and past-directed outgoing surfaces $\mathcal{N}$ and $\underline{\mathcal{N}}$.

Let us first consider future-directed null outgoing surfaces $\mathcal{N}$, and let L be an affine vector field along $\mathcal{N}$. Let λ being a corresponding affine parameter and $\underline{L}$ be the null vector field with $\underline{L}$ normal to the level sets of λ and normalized so that $g(L, \underline{L}) = 2$.[9]

Given the Levi-Civita connection D of the space-time metric g, the Ricci coefficients on $\mathcal{N}$ for this parameter are the following:

- Define the *null second fundamental forms* $\chi, \underline{\chi}$ by

  $$\chi(X,Y) = g(D_X L, Y), \qquad \underline{\chi}(X,Y) = g(D_X \underline{L}, Y), \qquad X, Y.$$

  Since L and $\underline{L}$ are orthogonal to $\mathcal{S}$, both χ and $\underline{\chi}$ are symmetric. The trace and traceless parts of χ (with respect to $\not{\gamma}$),

  $$tr\chi = \not{\gamma}^{ab}\chi_{ab}, \qquad \hat{\chi} = \chi - \frac{1}{2}(tr\chi)\not{\gamma},$$

  are often called the *expansion* and *shear* of $\mathcal{N}$, respectively. The same trace-traceless decomposition can also be done for $\underline{\chi}$.

- Define the *torsion* ζ by

  $$\zeta(X) = \frac{1}{2} g(D_X L, \underline{L}).$$

The Ricci coefficients on a given sphere $\mathcal{S} \subset \mathcal{N}$ depend on the choice of the null pair $\underline{L}, L$. When we wish to highlight this dependence below, we will write $\chi^L, \underline{\chi}^{\underline{L}}, \zeta^{\underline{L}}$.

The Ricci and curvature coefficients are related to each other via a family of geometric differential equations, known as the *null structure equations* which we now review. For details and derivations, see, for example, [10, 17].

[9] Note that this is the reverse condition compared to the usual one, where both $\underline{L}, L$ are future-directed. This is also manifested in some of the null structure equations below.

Structure equations: We use the connection $\nabla\!\!\!/_{\underline{L}}$ which acts on smooth 1-parameter families of vector fields over the level sets of an affine parameter λ as follows:

$$\nabla\!\!\!/_{\underline{L}} X(\phi,\theta,\lambda_0) := \mathrm{proj}_{\{\lambda=\lambda_0\}} \nabla_{\underline{L}} X. \tag{2.17}$$

In other words, $\nabla\!\!\!/_{\underline{L}} X$ is merely the projection of $\nabla_{\underline{L}} X$ onto the level sphere of the affine parameter λ. The definition extends to tensor fields in the obvious way. We analogously define a connection $\nabla\!\!\!/_L$ on foliated future-directed outgoing null surfaces $\mathcal{N}$.

Then, the following structure equations hold on $\mathcal{N}$.

$$\begin{aligned} \nabla\!\!\!/_L \chi_{ab} &= -\gamma\!\!\!/^{cd}\chi_{ac}\chi_{bd} - \alpha_{ab}, \\ \nabla\!\!\!/_L \zeta_a &= 2\gamma\!\!\!/^{bc}\chi_{ab}\zeta_c - \beta_a, \\ \nabla\!\!\!/_L \underline{\chi}_{ab} &= -(\nabla\!\!\!/_a\zeta_b + \nabla\!\!\!/_b\zeta_a) + \frac{1}{2}\gamma\!\!\!/^{cd}(\chi_{ac}\underline{\chi}_{bd} + \chi_{bc}\underline{\chi}_{ad}) - 2\zeta_a\zeta_b - \rho\gamma\!\!\!/_{ab}. \end{aligned} \tag{2.18}$$

In particular the last equation implies:

$$\nabla\!\!\!/_L \mathrm{tr}\underline{\chi} = \frac{1}{2}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} - 2div\zeta - 2|\zeta|^2 - 2\rho + \hat{\chi}\hat{\underline{\chi}}. \tag{2.19}$$

An analogous system of Ricci coefficients and structure equations hold for the past-directed and outgoing null surfaces $\underline{\mathcal{N}}$. Now $\underline{L}$ will be an affine vector field on $\underline{\mathcal{N}}$ and λ will be the corresponding affine parameter. In this case, we let L be the null vector field that is normal to the level sets of λ and normalized so that $g(\underline{L},L)=2$.

The Ricci coefficients $\underline{\chi},\chi$ are then defined as above; we define ζ in this context to be $\zeta(X)=\frac{1}{2}g(D_X\underline{L},L)$. We then have the following evolution equations on $\underline{\mathcal{N}}$:

$$\begin{aligned} \nabla\!\!\!/_{\underline{L}} \underline{\chi}_{ab} &= -\gamma\!\!\!/^{cd}\underline{\chi}_{ac}\underline{\chi}_{bd} - \underline{\alpha}_{ab}, \\ \nabla\!\!\!/_{\underline{L}} \zeta_a &= 2\gamma\!\!\!/^{bc}\underline{\chi}_{ab}\zeta_c - \underline{\beta}_a, \\ \nabla\!\!\!/_{\underline{L}} \chi_{ab} &= -(\nabla\!\!\!/_a\zeta_b + \nabla\!\!\!/_b\zeta_a) - \frac{1}{2}\gamma\!\!\!/^{cd}(\chi_{ac}\underline{\chi}_{bd} + \chi_{bc}\underline{\chi}_{ad}) - 2\zeta_a\zeta_b - \rho\gamma\!\!\!/_{ab}. \end{aligned} \tag{2.20}$$

The last equation implies:

$$\nabla\!\!\!/_{\underline{L}} \mathrm{tr}\chi = -\frac{1}{2}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} - 2div\zeta - 2|\zeta|^2 - 2\rho - \hat{\chi}\hat{\underline{\chi}} \tag{2.21}$$

Hawking mass: For any space-like 2-sphere $\mathcal{S}\subset\mathcal{M}$, we consider any pair of normal vector fields to $\mathcal{S}$, $\underline{L},L$, with $\underline{L}$ and $-L$ being past-directed. Let $\chi(\mathcal{S}),\underline{\chi}(\mathcal{S})$ be the two second fundamental forms of $\mathcal{S}$ relative to these vector fields. Let $tr\chi, tr\underline{\chi}$ be the traces of these. We also let

$$r[\mathcal{S}] := \sqrt{\frac{\mathrm{Area}[\mathcal{S}]}{4\pi}} \tag{2.22}$$

Then the Hawking mass of $\mathcal{S}$ is defined via:

$$m_{\text{Hawk}}(\mathcal{S}) := \frac{r}{2}[\mathcal{S}](1 - \frac{1}{16\pi}\int_{\mathcal{S}} tr\chi tr\underline{\chi}). \tag{2.23}$$

Recall also the mass aspect function μ:

$$\mu = \mathcal{K} - \frac{1}{4}tr\chi tr\underline{\chi} - div\zeta. \tag{2.24}$$

In view of the Gauss-Bonnet theorem, we readily derive that:

$$\int_{\mathcal{S}} \mu dV_{\mathcal{S}} = \frac{8\pi}{r} m_{\text{Hawk}}(\mathcal{S}). \tag{2.25}$$

2.3 Transformation laws of Ricci coefficients, and perturbations of null surfaces.

Recall that $\mathcal{N}[\mathcal{S}_0]$ is equipped with an affine parameter $\underline{u}$ normalized so that $\underline{u} = \underline{u}_0$ on $\mathcal{S}_0$ and $L(\underline{u}) = 1$. Our proof will require calculating $\underline{\chi}^{\underline{L}_{v,\tau}}[\mathcal{S}_{v,\tau}]$ up to an error $o_2(\tau)$.

For this we will use certain transformation formulas for the Ricci coefficients of affinely parameterized null surfaces under changes of the affine foliation; we refer the reader to [5] for the details and derivations of these.

In order to reduce matters to that setting, we consider a new affine parameter $\underline{u}'$ on $\mathcal{N}[\mathcal{S}_0]$ defined via:

$$\underline{u}' - 1 := e^{-v}(\underline{u} - 1). \tag{2.26}$$

(Thus $\{\underline{u} = 1\} = \{\underline{u}' = 1\} = \mathcal{S}_0$, and $\mathcal{S}_{v,\tau} = \{\underline{u}' = \tau\}$).[10] We now invoke formula (2.11) in [5] for $\mathcal{S}_{v,\tau}$ which calculates the Ricci coefficients $\underline{\chi}', \zeta', \chi'$ *defined relative to the vector field* $e^v\underline{L}_{v,\tau}$, in terms of the Ricci coefficients $\underline{\chi}, \zeta, \chi$ defined relative to the original vector fields $\underline{L}$. While the primed $'$ and un-primed tensor fields live over different tangent spaces (the level sets of two different affine parameters), there is a natural identification between these level sets subject to which the formulas below make sense; essentially we compare the evaluations of these tensor fields against their respective coordinate vector fields. We refer the reader to [5] regarding this (technical) point.

Thus in our setting we calculate, on the spheres $\mathcal{S}_{v,\tau}$:[11]

$$\begin{aligned}\underline{\chi}'_{ab} = \underline{\chi}_{ab} &- 2(\underline{u}-1)\nabla\!\!\!/_{ab}v - 2(\underline{u}-1)(\nabla\!\!\!/_a v\cdot\zeta_b + \nabla\!\!\!/_b v\cdot\zeta_a) \\ &- (\underline{u}-1)^2 \gamma\!\!\!/^{cd}\nabla\!\!\!/_c v(\nabla\!\!\!/_d v\cdot\chi_{ab} - 2\nabla\!\!\!/_a v\cdot\chi_{bd} - 2\nabla\!\!\!/_b v\cdot\chi_{ad})\end{aligned} \tag{2.27}$$

[10] Note that by construction $\underline{L}_{v,\tau}$ is normal to the level sets of the affine parameter on $\mathcal{N}[\mathcal{S}_0]$.

[11] The differences in some signs and the absence of e^{-v} compared to (2.11) in [5] are due to the different orientations of $\underline{L}$ and $\underline{L}_{v,\tau}$, and their different scalings by e^v relative to L, L' in (2.11) in [5].

$$- 2(\underline{u} - 1)\nabla\!\!\!\!/_a v \nabla\!\!\!\!/_b v.$$

Replacing $\underline{u} - 1$ as in (2.26), (recall $\underline{u}' - 1 = \tau$) we find:

$$\underline{\chi}'_{ab} = \underline{\chi}_{ab} - 2\tau\nabla\!\!\!\!/_{ab} e^v - 2\tau(\nabla\!\!\!\!/_a e^v \cdot \zeta_b + \nabla\!\!\!\!/_b e^v \cdot \zeta_a) \\ - (s-1)^2 e^{2v} \gamma\!\!\!/^{cd} \nabla\!\!\!\!/_c v(\nabla\!\!\!\!/_d v \cdot \chi_{ab} - 2\nabla\!\!\!\!/_a v \cdot \chi_{bd} - 2\nabla\!\!\!\!/_b v \cdot \chi_{ad}). \tag{2.28}$$

Then, using (2.19) we find:

$$tr\underline{\chi}'(\mathcal{S}_{v,\tau}) = tr\underline{\chi}(\mathcal{S}_{v,\tau}) - 2\tau\Delta_{\mathcal{S}_0} e^v - 4\tau(\nabla\!\!\!\!/^a e^v \zeta_a) \\ + \tau^2 e^v \gamma^{cd} \nabla\!\!\!\!/_c v \nabla\!\!\!\!/_d v \cdot tr\chi - 4\tau^2 \nabla\!\!\!\!/_a v \nabla\!\!\!\!/_d v \cdot \chi^{ad} \tag{2.29}$$

$$= tr\underline{\chi}(\mathcal{S}_0) + e^v \tau \nabla_L tr\underline{\chi}(\mathcal{S}_0) - 2\tau\Delta_{\mathcal{S}_0} e^v - 4\tau(\zeta^a \nabla_a e^v) + o(\tau), \tag{2.30}$$

where $\Delta_{\mathcal{S}_0}$ is the Laplace-Beltrami operator for the restriction $\gamma\!\!\!/$ of the space-time metric g onto $\mathcal{S}_0$. Thus in particular, we derive that on $\mathcal{S}_{v,\tau}$:[12]

$$tr\underline{\chi}'[\mathcal{S}_{v,\tau}] = tr\underline{\chi}[\mathcal{S}_0] + \tau\{[\frac{1}{2} tr\chi[\mathcal{S}_0] tr\underline{\chi}[\mathcal{S}_0] - 2\rho[\mathcal{S}_0] + \hat{\chi}[\mathcal{S}_0] \cdot \hat{\underline{\chi}}[\mathcal{S}_0] \\ - 2div\zeta[\mathcal{S}_0] - 2|\zeta[\mathcal{S}_0]|^2] e^v - 2\Delta_{\mathcal{S}_0} e^v - 4(\zeta[\mathcal{S}_0]^a \nabla_a e^v)\} + o(\tau). \tag{2.31}$$

For future reference, we also recall some facts from [5] on the transformation law of the second fundamental forms $\underline{\chi}$ and χ on the level sets of suitable affine parameters λ on $\underline{\mathcal{N}}$. In particular, given a $\mathcal{C}^2$ function ω over $\mathcal{S} = \partial\underline{\mathcal{N}}$, we consider the new affine parameter λ' defined via:

$$\lambda' - 1 = e^\omega(\lambda - 1). \tag{2.32}$$

(Recall that $\lambda = 1$ on $\mathcal{S}$), we let $\underline{L}'$ the associated null vector field, and L' the null vector field that is normal to the level sets of λ', normalized so that $g(\underline{L}', L') = 2$. We let $\underline{\chi}', \chi'$ be the null second fundamental forms corresponding to $\underline{L}', L'$. Then (subject to the identification of coordinates described in section 2 in [5]), $\underline{\chi}', \chi'$ evaluated at any point on $\underline{\mathcal{N}}$ equal:

$$\underline{\chi}'_{ab} = e^\omega \underline{\chi}_{ab} \tag{2.33}$$

$$\zeta'_a = \zeta_a + (\lambda - 1)\gamma\!\!\!/^{bc} \nabla\!\!\!\!/_b v \cdot \chi_{ac} - \nabla\!\!\!\!/_a v, \tag{2.34}$$

$$\chi'_{ab} = e^{-\omega}\{\chi_{ab} - 2(\lambda-1)\nabla\!\!\!\!/_{ab}\omega - 2(\lambda-1)(\nabla\!\!\!\!/_a\omega \cdot \zeta_b + \nabla\!\!\!\!/_b\omega \cdot \zeta_a) \\ - (\lambda-1)^2 \gamma\!\!\!/^{cd} \nabla\!\!\!\!/_c\omega(\nabla\!\!\!\!/_d\omega \cdot \underline{\chi}_{ab} - 2\nabla\!\!\!\!/_a\omega \cdot \underline{\chi}_{bd} - 2\nabla\!\!\!\!/_b\omega \cdot \underline{\chi}_{ad}) \\ - 2(\lambda-1)\nabla\!\!\!\!/_a\omega\nabla\!\!\!\!/_b\omega\}. \tag{2.35}$$

An application of these formulas will be towards constructing new MOTSs, off of the original MOTS $\mathcal{S}_0$.

[12] The terms in the last line are all evaluated on the initial sphere $\mathcal{S}_0$. The difference between the Ricci coefficients on $\mathcal{S}_0$ and $\mathcal{S}_{v,\tau}$ is of order $o(\tau)$, by the evolution equations in the previous subsection.

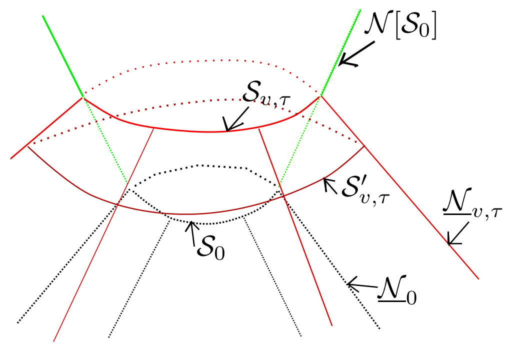

Figure 2: The “old” null surface $\underline{\mathcal{N}}_0$, and the “new” $\underline{\mathcal{N}}_{v,\tau}$ emanating from $\mathcal{S}'_{v,\tau}$.

3 New MOTS off of $\mathcal{S}_0$.

Our aim here is to capture the space of marginally outer trapped 2-spheres nearby the original sphere $\mathcal{S}_0$, but to its future, (with respect to the direction L). Recall $\mathcal{S}_{v,\tau}, \underline{L}_{v,\tau}$ from Definition 2.3 (where $e^v \cdot \tau = \omega$).

We let $\underline{\chi}_{v,\tau}$ be the null second fundamental form on $\underline{\mathcal{N}}_{v,\tau}$ corresponding to $\underline{L}_{v,\tau}$. We then claim:

Lemma 3.1. *Given any $v \in \mathcal{C}^2[\mathcal{S}_0]$ with $||v||_{\mathcal{C}^2} \leq (10)^{-1}m$, then for all $\tau \in [0,1)$ there exists a function $F(v,\tau) \geq 0$ for which $\mathcal{S}'_{v,\tau} := \{\lambda_{v,\tau} = F(v,\tau)\} \subset \underline{\mathcal{N}}_{v,\tau}$ is marginally outer trapped (see Figure 2). Furthermore, we claim that:*

$$
\begin{aligned}
&tr\underline{\chi}[\mathcal{S}'_{v,\tau}] = 2 + \tau\{[-2\rho[\mathcal{S}_0] + \hat{\chi}_{ab}[\mathcal{S}_0]\cdot\hat{\underline{\chi}}^{ab}[\mathcal{S}_0] \\
&-2div\zeta[\mathcal{S}_0] - 2|\zeta[\mathcal{S}_0]|^2]e^v - 2\Delta_{\mathcal{S}_0}e^v - 4(\zeta^a[\mathcal{S}_0]\nabla_a e^v) \\
&+ e^v|\hat{\chi}|^2[\mathcal{S}_0][-2\rho[\mathcal{S}_0] - 2div\zeta[\mathcal{S}_0] - 2|\zeta[\mathcal{S}_0]|^2 + \hat{\chi}[\mathcal{S}_0]\hat{\underline{\chi}}[\mathcal{S}_0]]^{-1}\big(-2-|\hat{\underline{\chi}}|^2[\mathcal{S}_0]\big)\} + o_2(\tau)
\end{aligned}
\tag{3.1}
$$

For future reference, we let:

$$
\begin{aligned}
&\overline{\mathcal{L}}(e^v) := \tau^{-1}[tr\underline{\chi}_{v,\tau} - 2] \\
&= [-2\rho[\mathcal{S}_0] + \hat{\chi}_{ab}[\mathcal{S}_0]\cdot\hat{\underline{\chi}}^{ab}[\mathcal{S}_0] \\
&-2div\zeta[\mathcal{S}_0] - 2|\zeta[\mathcal{S}_0]|^2]e^v - 2\Delta_{\mathcal{S}_0}e^v - 4(\zeta^a[\mathcal{S}]\nabla_a e^v) \\
&+ e^v|\hat{\chi}|^2[\mathcal{S}_0][-2\rho[\mathcal{S}_0] - 2div\zeta[\mathcal{S}_0] - 2|\zeta[\mathcal{S}_0]|^2 + \hat{\chi}[\mathcal{S}_0]\hat{\underline{\chi}}[\mathcal{S}_0]]^{-1}\big(-2-|\hat{\underline{\chi}}|^2[\mathcal{S}_0]\big).
\end{aligned}
\tag{3.2}
$$

Note that the operator $\overline{\mathcal{L}}$ can also be expressed in the form:

$$
\overline{\mathcal{L}}(e^v) = [\Delta_{\mathcal{S}_0} + O_2^\delta(1)\partial_i + (\frac{1}{2m^2} + O_2^\delta(1))]e^v. \tag{3.3}
$$

(The use of the symbol $O_2^\delta(1)$ is an abuse of notation, since the functions in question do not depend on λ–here it merely means that the functions involved as well as their first and second rotational derivatives are bounded by δ).

We remark also that the area of $\mathcal{S}'_{v,\tau}$ is not lesser than $\mathcal{S}_0$:

Lemma 3.2. *With $\mathcal{S}'_{v,\tau}$ as above (see Figure 2),*

$$\text{Area}[\mathcal{S}'_{v,\tau}] \geq \text{Area}[\mathcal{S}_0].$$

Furthermore we have equality in the above if and only if $tr\chi[\tilde{\mathcal{S}}] = 0$ for each sphere $\tilde{\mathcal{S}} \subset \mathcal{N}[\mathcal{S}_0]$ contained between $\mathcal{S}_0$ and $\mathcal{S}_{v,\tau}$ on $\mathcal{N}[\mathcal{S}_0]$, and moreover $F(v,\tau) = 0$.

Proof of Lemma 3.1: We start by invoking formula (2.29) to find that

$$\begin{aligned} &tr\underline{\chi}[\mathcal{S}_{v,\tau}] = tr\underline{\chi}[\mathcal{S}_0] + \tau\{[-2\rho[\mathcal{S}] + \hat{\chi}_{ab}[\mathcal{S}_0]\cdot\underline{\hat{\chi}}^{ab}[\mathcal{S}_0] \\ &- 2div\zeta[\mathcal{S}_0] - 2|\zeta[\mathcal{S}_0]|^2]e^v - 2\Delta_{\mathcal{S}_0}e^v - 4(\zeta^a[\mathcal{S}_0]\nabla_a e^v)\} + o_2(\tau) \end{aligned} \tag{3.4}$$

On the other hand, letting χ stand for the second fundamental form on $\mathcal{N}[\mathcal{S}_0]$, the first formula in (2.18) tells us that

$$tr\chi[\mathcal{S}_{v,\tau}] = -\tau e^v|\hat{\chi}|^2[\mathcal{S}_0] + o_2(\tau) \tag{3.5}$$

Now, using the mean-value theorem, we find that there exists a function $F(v,\tau)$ so that $\mathcal{S}'_{v,\tau}$ (as in defined in the theorem statement) is marginally outer trapped, and moreover:

$$F(v,\tau) = -\frac{tr\chi^L[\mathcal{S}_{v,\tau}]}{\nabla_{\underline{L}_{v,\tau}}tr\chi^L[\mathcal{S}_{v,\tau}]} + o_2(\tau) = \frac{\tau e^v|\hat{\chi}|^2[\mathcal{S}_0]}{\nabla_{\underline{L}_{v,\tau}}tr\chi[\mathcal{S}_{v,\tau}]} + o_2(\tau). \tag{3.6}$$

To show (3.1), we first recall (2.21).

$$\begin{aligned} &\nabla_{\underline{L}_{v,\tau}}tr\chi[\mathcal{S}_{v,\tau}] = -\frac{1}{2}tr\chi[\mathcal{S}_{v,\tau}]tr\underline{\chi}[\mathcal{S}_{v,\tau}] - 2\rho[\mathcal{S}_{v,\tau}] - 2div\zeta[\mathcal{S}_{v,\tau}] - 2|\zeta[\mathcal{S}_{v,\tau}]|^2 \\ &- \hat{\chi}[\mathcal{S}_{v,\tau}]\underline{\hat{\chi}}[\mathcal{S}_{v,\tau}] \end{aligned} \tag{3.7}$$

Note in particular that the assumed closeness to the Schwarzschild space-time implies that $\nabla_{\underline{L}_{v,\tau}}tr\chi[\mathcal{S}_{v,\tau}]$ is bounded below by $4^{-1}m^{-2}$, for $\tau \in [0,1)$. Using (2.21) we derive that:

$$F(v,\tau) = \tau e^v|\hat{\chi}|^2[\mathcal{S}_0]\{-2\rho[\mathcal{S}_0] - 2div\zeta[\mathcal{S}_0] - 2|\zeta[\mathcal{S}_0]|^2 - \hat{\chi}[\mathcal{S}_0]\underline{\hat{\chi}}[\mathcal{S}_0]\}^{-1} + o_2(\tau). \tag{3.8}$$

Therefore (3.1) follows by invoking the first (traced) formula in (2.20) to obtain:

$$
\begin{aligned}
&tr\underline{\chi}[\mathcal{S}'_{v,\tau}] = tr\underline{\chi}[\mathcal{S}_{v,\tau}] + F(v,\tau)[-\frac{1}{2}(tr\underline{\chi})^2[\mathcal{S}_{v,\tau}] - |\hat{\underline{\chi}}|^2[\mathcal{S}_{v,\tau}]] + o_2(\tau)\\
&= tr\underline{\chi}[\mathcal{S}_{v,\tau}] + F(v,\tau)[-\frac{1}{2}2^2 - |\hat{\underline{\chi}}|^2[\mathcal{S}_0]\big) + o_2(\tau)\\
&= tr\underline{\chi}[\mathcal{S}_0] + \tau\{[-2\rho[\mathcal{S}_0] - \hat{\chi}_{ab}[\mathcal{S}_0]\cdot\hat{\underline{\chi}}^{ab}[\mathcal{S}_0]\\
&- 2div\zeta[\mathcal{S}_0] - 2|\zeta[\mathcal{S}_0]|^2]e^v - 2\Delta_{\mathcal{S}_0}e^v - 4(\zeta^a[\mathcal{S}_0]\nabla_a e^v)\}\\
&- \tau e^v|\hat{\chi}|^2[\mathcal{S}_0]\{-2\rho[\mathcal{S}_0] - 2div\zeta[\mathcal{S}_0] - 2|\zeta[\mathcal{S}_0]|^2 - \hat{\chi}[\mathcal{S}_0]\hat{\underline{\chi}}[\mathcal{S}_0]\}^{-1}\big(2 + |\hat{\underline{\chi}}|^2[\mathcal{S}_0]\big) + o_2(\tau)
\end{aligned}
\tag{3.9}
$$

This proves (3.1).□

We remark that formula (3.9) can be used to define an appropriate *affine* parameter on $\underline{\mathcal{N}}_{v,\tau}$: Indeed, the above shows that there exists a function $A(v,\tau)$

$$
\begin{aligned}
&A(v,\tau) = -\tau\{[-2\rho[\mathcal{S}_0] - \hat{\chi}_{ab}[\mathcal{S}_0]\cdot\hat{\underline{\chi}}^{ab}[\mathcal{S}_0]\\
&- 2div\zeta[\mathcal{S}] - 2|\zeta[\mathcal{S}_0]|^2]e^v - 2\Delta_{\mathcal{S}_0}e^v - 4(\zeta^a[\mathcal{S}_0]\nabla_a e^v)\}\\
&- \tau e^v|\hat{\chi}|^2[\mathcal{S}_0]\{2\rho[\mathcal{S}_0] - 2div\zeta[\mathcal{S}_0] - 2|\zeta[\mathcal{S}_0]|^2 - \hat{\chi}[\mathcal{S}_0]\hat{\underline{\chi}}[\mathcal{S}_0]\}^{-1}\big(2 + |\hat{\underline{\chi}}|^2[\mathcal{S}_0]\big) + o_2(\tau)
\end{aligned}
\tag{3.10}
$$

so that if we let

$$\underline{L}^\sharp_{v,\tau,\nu} := (1 + A(v,\tau))\underline{L}_{v,\tau}, \tag{3.11}$$

and denote by $tr\underline{\chi}^\sharp_{v,\tau}$ the corresponding null second fundamental form of $\underline{\mathcal{N}}_{v,\tau}$ then:

$$tr\underline{\chi}^\sharp_{v,\tau}[\mathcal{S}'_{v,\tau}] = 2. \tag{3.12}$$

Note for future reference that $A(v,\tau)$ can be expressed in the form:

$$A(v,\tau) = \tau\{2\Delta_{\mathcal{S}_0} + O^\delta_2(1)\cdot\nabla - [\frac{1}{2m^2} + O^\delta_2(1)]\}e^v + o_2(\tau), \tag{3.13}$$

and in particular we can re-express (3.11) as:

$$\underline{L}^\sharp_{v,\tau,\nu} = e^{\tau\{2\Delta_{\mathcal{S}_0} + O^\delta_2(1)\cdot\nabla - [\frac{1}{2m^2} + O^\delta_2(1)]\}e^v + o_2(\tau)}. \tag{3.14}$$

Proof of Lemma 3.2: It suffices to show a localized version of our claim. Consider a natural map between $\mathcal{S}_0$ and $\mathcal{S}'_{v,\tau}$ which identifies the points on null generators of $\mathcal{N}[\mathcal{S}_0]$, $\underline{\mathcal{N}}_{v,\tau}$ that intersect on $\mathcal{S}_{v,\tau}$ (see the East and West "poles" on $\mathcal{S}_{v,\tau}$ in Figure 2. We can thus identify area elements on these spheres; thus for a given triplet of points $P_1, P_2 \in \mathcal{N}[\mathcal{S}_0]$ and $P_2, P_3 \in \underline{\mathcal{N}}_{v,\tau}$ that lie on the same null generators of $\mathcal{N}[\mathcal{S}_0]$, $\underline{\mathcal{N}}_{v,\tau}$ respectively, we think of $dV_{\mathcal{S}_{v,\tau}}(P_1)(dV_{\mathcal{S}_0}(P_2))^{-1}$, $dV_{\mathcal{S}_{v,\tau}}(P_3)(dV_{\mathcal{S}'_{v,\tau}}(P_2))^{-1}$ as a numbers. Letting x stand for a point in the null segment (P_1P_2) we then derive:

$$log[dV_{\mathcal{S}_{v,\tau}}(P_2)(dV_{\mathcal{S}_0}(P_1))^{-1}] = \int_0^{\tau e^v} tr\chi^L(x)dx \tag{3.15}$$

In particular since (by the Raychaudhuri equation) $tr\chi^L$ is a non-increasing, non-positive function along (P_1P_2), we derive:

$$|log[dV_{\mathcal{S}_{v,\tau}}(P_1)(dV_{\mathcal{S}_0}(P_2))^{-1}]| \le \tau e^v |tr\chi^L(P_2)| \tag{3.16}$$

We now use $tr\underline{\chi}^{\underline{L}_{v,\tau}}$ on $\underline{\mathcal{N}}_{v,\tau}$ to derive (letting $x \in (P_2P_3)$):

$$\begin{aligned} & log[dV_{dV_{\mathcal{S}'_{v,\tau}}(P_3)(\mathcal{S}_{v,\tau}}(P_2))^{-1}] = \int_0^{F_{P_2}(v,\tau)} tr\underline{\chi}^{\underline{L}_{v,\tau}}(x)dx \\ & \ge \frac{1}{2}F_{P_2}(v,\tau)tr\underline{\chi}^{\underline{L}_{v,\tau}}(P_2) \ge \frac{tr\chi^L(\mathcal{S}_{v,\tau})}{4m^2}, \end{aligned} \tag{3.17}$$

using (3.8) and (2.7) in the second inequality.

Thus, adding (3.16) and (3.17) we derive (for τ small enough) the inequality:

$$dV_{\mathcal{S}'_{v,\tau}}(P_3)(dV_{\mathcal{S}_0}(P_1))^{-1} \ge 1. \tag{3.18}$$

Moreover, clearly we have equality if and only if $tr\chi^L(P_2) = 0$. In this case, the Raychaudhuri equation implies that $tr\chi^L(P) = 0$ for all $P \in (P_1P_2)$; moreover clearly $F_{P_2}(v,\tau) = 0$.

Thus, we have derived that:

$$dV_{\mathcal{S}'_{v,\tau}} \ge dV_{\mathcal{S}_0} \tag{3.19}$$

with equality if and only if $tr\chi^L = 0$ in $\mathcal{N}[\mathcal{S}_0]$ between $\mathcal{S}_0$ and $\mathcal{S}_{v,\tau}$ and $F_{v,\tau} = 0$ on $\mathcal{S}_{v,\tau}$. $\Box$

We note for future reference that by a similar argument (integrating the structure equations (2.18), (2.20) in the L and $\underline{L}$-directions respectively, and using (2.7)) shows that for all v, τ as in Lemma 3.1 we have

$$|\partial^{(i)}\zeta|_{\mathcal{C}^0(\mathcal{S}'_{v,\tau})} \le \delta, |\partial^{(i)}\chi|_{\mathcal{C}^0(\mathcal{S}'_{v,\tau})} \le \delta \tag{3.20}$$

for all $i \le 2$.

The main method towards proving our result will be by expoliting a monotonicity property enjoyed by the Hawking mass on null hypersurfaces that are perturbations of the shear-free null hypersurfaces in Schwarzschild. We review this in the next subsection.

4 Monotonicity of Hawking mass on smooth null surfaces.

In this section we will review certain well-known monotonicity properties of the Hawking mass which go back to [13] (see also [23], whose notation we largely follow). We begin by understanding the asymptotic behaviour of the relevant Ricci coefficients for all hypersurfaces $\underline{\mathcal{N}}_{v,\tau}$ that we consider; we then proceed to study the behaviour of these coefficients under perturbations of the underlying hypersurfaces.

4.1 The asymptotics of the expansion of null surfaces relative to an affine vector field.

Understanding the asymptotic behaviour of the expansion of a smooth past-directed outgoing $\underline{\mathcal{N}}$ will be necessary for the construction and study of our luminosity foliation.

We introduce a convention: When we write a tensor (defined over level spheres in $\underline{\mathcal{N}}$) with lower case indices a, b we think of it as an abstract tensor field. When we use upper-case indices, we will be referring to its components evaluated against the vector fields $e^A, A = 1, 2$ defined in Definition 2.1.

Lemma 4.1. *Consider any infinite smooth past-directed outgoing null surface $\underline{\mathcal{N}}$. Assume that λ is an affine parameter on $\underline{\mathcal{N}}$ and let $\underline{L}^\lambda$ be the corresponding affine vector field. Assume that $\underline{\alpha}_{ab}(\lambda)$ defined relative to $\underline{L}^\lambda$ satisfies the fall-off condition*

$$\underline{\alpha}_{AB}(\lambda) \in O_2^\delta(\lambda^{-3-\epsilon}).$$

Assume also that $|tr\underline{\chi}(x) - 2|$ and $|\hat{\underline{\chi}}|(x)$ are sufficiently small on $\mathcal{S} = \partial\underline{\mathcal{N}}$. Then there exist continuous and bounded functions (tensors) $a(\phi, \theta, \lambda), h_{ab}(\phi, \theta, \lambda) \in O^\delta(1)$ which converge to continuous limits $a(\phi, \theta), b_{ab}(\phi, \theta)$ over $\mathcal{S}_\infty := \partial_\infty\underline{\mathcal{N}}$ as $\lambda \to \infty$ so that:

$$tr\underline{\chi}^\lambda = \frac{2}{\lambda} + \frac{a(x,\lambda)}{\lambda^2}, \hat{\underline{\chi}}_{ab} = \frac{h_{ab}(x,\lambda)}{\lambda^2}. \tag{4.1}$$

Furthermore the first and second spherical derivatives ∂_{ϕ^i}, $\partial^2_{\phi^i\phi^j}$ of the functions $a(x, \lambda), h_{ab}(x, \lambda)$ remain in $O^\delta(1)$ and have continuous limits as $\lambda \to \infty$.

We also note for future reference that Lemma 4.1 can also be applied to *families* of infinite null surfaces: Consider any 1-parameter family of infinite smooth outgoing null surfaces $\underline{\mathcal{N}}_\tau$, $\tau \in [0, 1]$, let λ is an affine parameter on each of these, and assume that the corresponding Weyl curvature component $\underline{\alpha}_\tau$ satisfies the same fall-off condition

$$|\underline{\alpha}_\tau| \in O_2^\delta(\lambda^{-3-\epsilon}).$$

Then the bounds (4.1) hold for $tr\underline{\chi}^\lambda_\tau$, $(\hat{\underline{\chi}}_{ab})_\tau$.

We also note a consequence of Lemma 4.1: If we let $\nabla\!\!\!\!/_A$ be the (intrinsic) connection on the level spheres $\mathcal{S}_\lambda$ of the affine parameter on $\underline{\mathcal{N}}$, then for $A, B \in \{1, 2\}$:

$$\nabla\!\!\!\!/_A e^B \in O_2(\lambda^{-1}). \tag{4.2}$$

We wish to study the first variation of the quantities above in τ, for the surfaces $\underline{\mathcal{N}}_{v,\tau}$ that we consider. To do this, we need the following definition:

Definition 4.2. *We say that a 1-parameter family of smooth outgoing past-directed infinite null surfaces $\underline{\mathcal{N}}_\tau$ as above is of class $\mathcal{R}(\delta)$ if there exists a natural map: $\Phi_\tau : \underline{\mathcal{N}} \to \underline{\mathcal{N}}_\tau$ with $\Phi_\tau(\lambda, \phi, \theta) = (\lambda, \phi_\tau, \theta_\tau)$ so that:*

- *the componets of $\underline{\alpha}_\tau$ measured relative to the frame $\lambda^{-1}(\Phi_\tau)_*\partial_\phi, \lambda^{-1}(\Phi_\tau)_*\partial_\theta$ are differentiable in τ and obey the bound*
$$|\partial_\tau(\underline{\alpha}_{AB})_\tau| \in O_2^\delta(\lambda^{-3-\epsilon}).$$
- *The metric components $(\gamma\!\!\!/_{ab})_\tau(\lambda)$ measured relative to the same frame satisfy: $\partial_\tau(\gamma_\tau)_{AB}(\lambda) = O(1)$.*
- *$tr\underline{\chi}_\tau = 2$ at $\mathcal{S}_\tau := \partial\underline{\mathcal{N}}_\tau$ and $|(\hat{\underline{\chi}}_{AB})_\tau|_{\mathcal{S}_\tau} \in O_2^\delta(\lambda^{-2})$.*

Lemma 4.3. *Consider a 1-parameter family of infinite null surfaces $\underline{\mathcal{N}}_{v,\tau}$, of class $\mathcal{R}(\delta)$ as described above. Assume that there exist functions $\mathcal{F}(v), \mathcal{G}(v) \in \mathcal{C}^2(\phi,\theta)$ (both depending only on ϕ, θ, and the function $v(\phi,\theta)$) and tensors $f^{ab} \in O_2^1(1)$,$\overline{f}_{ab} \in O_2^\delta(\lambda^{-3-\epsilon})$ depending on ϕ, θ, λ so that the first variation of the metric and curvature components is given by:*

$$\dot{\gamma\!\!\!/}_v^{AB} = f^{AB}(\lambda)\mathcal{F}(v), (\dot{\underline{\alpha}}_{AB})_v = \overline{f}_{AB}(\lambda)\mathcal{G}(v). \tag{4.3}$$

Then the function $\dot{a}_v := \partial_\tau|_{\tau=0}a_{v,\tau}$, and the components of the tensor field $(\dot{h}_{ab})_v := \partial_\tau|_{\tau=0}(h_{ab})_{v,\tau}$ can be expressed in the form

$$\dot{a}_v(\lambda) = b_1(\lambda)\mathcal{F}[v] + b_2(\lambda)\mathcal{G}[v], (\dot{h}_{AB})_v(\lambda) = b_3(\lambda)\mathcal{F}[v] + b_4(\lambda)\mathcal{G}[v], \tag{4.4}$$

where $b_i \in O_2^\delta(1)$, $i = 1, 2, 3, 4$.

Proof of Lemma 4.1: We refer to the evolution equation (2.20) for $\underline{\chi}$. This is a tensorial equation, while we are interested in the components of the tensors $\underline{\chi}$. To do this, we first recall the (tensorial) evolution equations on $\mathrm{tr}\underline{\chi}$ and $\hat{\underline{\chi}}$ on a null surface, see the first equation in (2.20):

$$\nabla\!\!\!/_\lambda \mathrm{tr}\underline{\chi} = -\frac{1}{2}(\mathrm{tr}\underline{\chi})^2 - |\hat{\underline{\chi}}|^2, \nabla\!\!\!/_\lambda \hat{\underline{\chi}}_{ab} = -\mathrm{tr}\underline{\chi}\hat{\underline{\chi}}_{ab} - \underline{\alpha}_{ab}. \tag{4.5}$$

As discussed above, we let $(\hat{\underline{\chi}}_{AB})$ be the evaluation $\hat{\underline{\chi}}(e_A, e_B)$; accordingly we let $(\underline{\alpha}_{AB})$. (The e_A, e_B are among Φ, Θ). Using the fact that $\underline{L}$ commutes with the two vector fields ∂_ϕ and ∂_θ, we derive the formulas:

$$\begin{aligned}&\nabla_{\underline{L}}\underline{L} = 0, g(\nabla_{\underline{L}}e_A, e_B) = -\frac{1}{\lambda}\gamma\!\!\!/_{AB} + \underline{\chi}_{AB}, g(\nabla_{\underline{L}}e_A, \underline{L}) = -g(\nabla_{\underline{L}}\underline{L}, e_A) = 0,\\ &g(\nabla_{\underline{L}}e_A, L) = \zeta_A.\end{aligned} \tag{4.6}$$

Also

$$\begin{aligned}&g(\nabla_{\underline{L}}L, L) = 0, g(\nabla_{\underline{L}}L, \underline{L}) = -g(\nabla_{\underline{L}}\underline{L}, L) = 0,\\ &g(\nabla_{\underline{L}}L, e_A) = -g(\nabla_{\underline{L}}e_A, L) = -g(\nabla_{e^A}\underline{L}, L) = -2\zeta_A.\end{aligned} \tag{4.7}$$

Furthermore, recall that $\underline{\chi}_{AB}$ is a symmetric $(0,2)$ tensor field over the level sets of λ. We let $(\underline{\chi}^\sharp)^B_A$ be the corresponding $(1,1)$-tensor defined via the relation:

$$(\underline{\chi}^\sharp)^B_A e_B = \underline{\chi}(e_A, e_B).$$

Therefore:

$$\nabla_{\underline{L}}\underline{L} = 0, \nabla_{\underline{L}} e_A = -\frac{1}{\lambda} e_A + (\underline{\chi}^\sharp_A)^B e_B + \zeta_A \underline{L}, \nabla_{\underline{L}} L = \zeta^A e_A. \tag{4.8}$$

Then by the definition (2.17) of $\not\nabla_{\underline{L}}$ and from (4.8) we have:

$$\not\nabla_{\underline{L}} e_A = -\frac{1}{\lambda} e_A + (\underline{\chi}^\sharp)^B_A e_B$$

We also recall the definitions of χ_{AB}, and note that:

$$g(\nabla_A L, L) = 0, g(\nabla_A L, \underline{L}) = -g(\nabla_A \underline{L}, L) = -2\zeta_A.$$

In view of this we find:

$$\nabla_A L = \chi^B_A e_B - \zeta_A L. \tag{4.9}$$

We derive

$$\begin{aligned} \underline{L}\not\gamma_{AB} &= \underline{L} g(e_A, e_B) = g(\nabla_{\underline{L}} e_A, e_B) + g(e_A, \nabla_{\underline{L}} e_B) = -\frac{2}{\lambda}\not\gamma_{AB} + 2\underline{\chi}(e_A, e_B) \\ &= -\frac{2}{\lambda}\not\gamma_{AB} + \not\gamma_{AB}\mathrm{tr}\underline{\chi} + 2\hat{\underline{\chi}}_{AB} = \frac{a}{\lambda^2}\not\gamma_{AB} + 2\lambda^{-2} h_{AB}. \end{aligned} \tag{4.10}$$

Then, using the second formula in (4.5) and the definition of covariant differentiation, we find, for e_A, e_B among Φ, Θ

$$\begin{aligned} \not\nabla_\lambda(\hat{\underline{\chi}}_{AB}) &= (\not\nabla_\lambda \hat{\underline{\chi}})(e_A, e_B) + \hat{\underline{\chi}}(\not\nabla_\lambda e_A, e_B) + \hat{\underline{\chi}}(\not\nabla_\lambda e_B, e_A) \\ &= -\frac{2}{\lambda}\hat{\underline{\chi}}(e_A, e_B) - \mathrm{tr}\underline{\chi}\hat{\underline{\chi}}(e_A, e_B) - \underline{\alpha}_{AB} + \underline{\chi}_{AC}\hat{\underline{\chi}}^C_B + \underline{\chi}^C_B\hat{\underline{\chi}}_{CA} \\ &= -\frac{2}{\lambda}\hat{\underline{\chi}}(e_A, e_B) + 2\hat{\underline{\chi}}_{AC}\hat{\underline{\chi}}^C_B - \underline{\alpha}_{AB} \end{aligned}$$

Now, let $a := \lambda^2(tr\underline{\chi}^\lambda - \frac{2}{\lambda}), h_{ab} := \lambda^2 \hat{\underline{\chi}}_{ab}$.
Then using the (4.5) we derive the following equations for a, h:

$$\frac{d}{d\lambda} a = -\frac{1}{\lambda^2} a^2 - \frac{h_{AB} h_{CD} \not\gamma^{AC} \not\gamma^{BD}}{\lambda^2}. \tag{4.11}$$

$$\frac{d}{d\lambda}(h_{AB}) + \frac{2}{\lambda^2} h_{AD} \cdot h_{CB} \not\gamma^{CD} = -\lambda^2(\underline{\alpha}_{AB}). \tag{4.12}$$

Now, combining the above two equations with (4.10), the claimed asymptotic behavior along with the bound[13]

$$\not\gamma_{AB}[\mathcal{S}_\lambda] = \not\gamma_{AB}[\mathcal{S}_0] + O^\delta(1)$$

follow by a simple bootstrap argument, using

$$\lambda^2 \underline{\alpha}_{AB} \in O^\delta(\lambda^{-1-\epsilon}).$$

To derive the claim on the spherical derivatives of a, h, we take first one derivative ∂_{ϕ^i} of (4.11), (4.12), (4.10). We then obtain a system of three linear first order ODEs:

$$\begin{aligned}
&\frac{d}{d\lambda}(\partial_{\phi^i} a) = -\frac{2}{\lambda^2}(\partial_{\phi^i} a)a - \frac{2}{\lambda^2}\sum_{A,B,C,D=1}^{2}[(\partial_{\phi^i} h_{AB})h_{CD}\not\gamma^{AC}\not\gamma^{BD} \\
&+ (\partial_{\phi^i}\not\gamma^{AC})h_{AB}h_{CD}\not\gamma^{BD}] \\
&= O^\delta(\lambda^{-2})(\partial_{\phi^i} a) + \sum_{C,D=1}^{2} O^\delta(\lambda^{-2})(\partial_{\phi^i} h_{CD}) + \sum_{A,C=1}^{2} O^\delta(\lambda^{-2})(\partial_{\phi^i}\not\gamma^{AC}),
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
&\frac{d}{d\lambda}(\partial_{\phi^i} h_{AB}) = -\frac{4}{\lambda^2}(\partial_{\phi^i} h_{AC})h_{BD}\not\gamma^{CD} - \frac{2}{\lambda^2}(\partial_{\phi^i}\not\gamma^{CD})h_{AC}h_{BD} + \lambda^2(\partial_{\phi^i}\underline{\alpha}_{AB}) \\
&\sum_{C,D=1}^{2} O^\delta(\lambda^{-2})(\partial_{\phi^i} h_{CD}) + \sum_{C,D=1}^{2} O^\delta(\lambda^{-2})(\partial_{\phi^i} h_{CD}) - \lambda^2(\partial_{\phi^i}\underline{\alpha}_{AB}),
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
&\frac{d}{d\lambda}(\partial_{\phi^i}\not\gamma_{AB}) = \frac{\partial_{\phi^i} a}{\lambda^2}\not\gamma_{AB} + \frac{a}{\lambda^2}\partial_{\phi^i}\not\gamma_{AB} + 2\lambda^{-2}\partial_{\phi^i}\not\gamma_{AB} \\
&O(\lambda^{-2})\partial_{\phi^i} a + O(\lambda^{-2})\partial_{\phi^i}\not\gamma_{AB}.
\end{aligned} \tag{4.15}$$

The $O(\lambda^{-2})$ terms in the last lines of the above equations follow from the bounds we have already derived in the previous step.

Our claim on the first derivatives thus follows from standard formulas for this first order system of linear equations. Then replacing this in (4.11), we derive the claim for $\partial_{\phi^i} a$. The claim for the second derivatives follows by taking a further rotational derivative of the above equations and repeating the same argument. □

For future reference, we note that the above imply the bounds

$$\not\gamma_{AB}[\mathcal{S}_\lambda] = \not\gamma_{AB}[\mathcal{S}_0] + O_2^\delta(1), \tag{4.16}$$

which also capture up to two of the rotational derivatives of $\not\gamma_{AB}$.

[13]This follows from the definition of $\underline{\chi}$ and the asymptotics in (4.1).

Proof Lemma 4.3: We again consider the evolution equations (4.11), (4.12), (4.10) for $a_{v,\tau}$ and the evaluation of $h_{v,\tau}$ against frame e^1, e^2. (So now a, h_{AB} depend on the parameters v, τ).

We then consider the $\partial_\tau|_{\tau=0}$-derivative of this system. (Recall that $\partial_\tau|_{\tau=0}$ stands for a Jacobi field J–see (2.15)). Since $\lambda_{v,\tau}$ and ∂_τ commute by construction, we find:

$$
\begin{aligned}
&\frac{d}{d\lambda}(\dot{a}_v) = -\frac{2}{\lambda^2}(\dot{a}_v)a - \frac{2}{\lambda^2}\sum_{A,B,C,D=1}^{2}[(\dot{h}_{AB})_v h_{CD}{\gamma\mkern-9mu/}^{AC}{\gamma\mkern-9mu/}^{BD} + (\dot{\gamma\mkern-9mu/}^{AC})h_{AB}h_{CD}{\gamma\mkern-9mu/}^{BD}_{v,\tau}] \\
&= O_2^\delta(\lambda^{-2})(\dot{a}_v) + O_2^\delta(\lambda^{-2})(\dot{h}_{ij})_v + \sum_{A,C=1}^{2} O_2^\delta(\lambda^{-2})(\dot{\gamma\mkern-9mu/}_v^{AC})
\end{aligned}
\tag{4.17}
$$

$$
\begin{aligned}
&\frac{d}{d\lambda}(\dot{h}_{AB})_v = -\frac{4}{\lambda^2}(\dot{h}_{AC})_v h_{BD}{\gamma\mkern-9mu/}^{CD} - \frac{2}{\lambda^2}(\dot{\gamma\mkern-9mu/}^{CD})h_{AC}h_{BD} - \lambda^2(\dot{\underline{\alpha}}_{AB})_v \\
&= \sum_{C,D=1}^{2} O_2^\delta(\lambda^{-2})(\dot{h}_{CD})_v + \sum_{C,D=1}^{2} O_2^\delta(\lambda^{-2})(\dot{h}_{CD})_v - \lambda^2(\dot{\underline{\alpha}}_{AB})_v
\end{aligned}
\tag{4.18}
$$

$$
\begin{aligned}
&\frac{d}{d\lambda}(\dot{\gamma\mkern-9mu/}_{AB}) = \frac{\dot{a}}{\lambda^2}{\gamma\mkern-9mu/}_{AB} + \frac{a}{\lambda^2}\dot{\gamma\mkern-9mu/}_{AB} + 2\lambda^{-2}\dot{\gamma\mkern-9mu/}_{AB} \\
&O_2^\delta(\lambda^{-2})\dot{a} + O_2^\delta(\lambda^{-2})\dot{\gamma\mkern-9mu/}_{AB}.
\end{aligned}
\tag{4.19}
$$

Thus, recalling that $\dot{a}_v = 0$ at $\lambda = 1$, our result again follows by the above first-order system of linear ODEs, by integration of these first order equations. This completes the proof of our Lemma.□

We note that (4.1) together with the formula[14]

$$
{\gamma\mkern-9mu/}_{ab}(\{\lambda = r_1(\phi,\theta)\}) - {\gamma\mkern-9mu/}_{ab}(\{\lambda = r_2(\phi,\theta)\}) = 2\int_{\{\lambda=r_1(\phi,\theta)\}}^{\{\lambda=r_2(\phi,\theta)\}} \underline{\chi}^{\underline{L}}_{ab}(\lambda), \tag{4.20}
$$

where λ is any chosen affine parameter on $\underline{\mathcal{N}}_{v,\tau}$ and $\underline{L}$ the associated affine vector field, implies the existence of the limit (where a, b are evaluated against the coordinate vector field $\partial_{\phi^1}, \partial_{\phi^2}$)

$$
(\gamma^{\infty,\lambda}_{v,\tau})_{ab} := lim_{\lambda_{v,\tau}\to\infty}\lambda_{v,\tau}^{-2}{\gamma\mkern-9mu/}_{ab}[\mathcal{S}_{\lambda_{v,\tau}}] \tag{4.21}
$$

on each $\underline{\mathcal{N}}_{v,\tau}$.

Using (4.21), applied to any affine function on each $\underline{\mathcal{N}}_{v,\tau}$ we naturally associate a "metric at infinity" (for the chosen affine function). We also check that in view of the regularity in the angular directions of $\underline{\chi}$ in (4.1), the limit of

[14]The indices a, b below take values among the coordinate vector fields $\partial_{\phi^1}, \partial_{\phi^2}$.

the Gauss curvatures of $\lambda^{-2}\gamma\!\!\!/_{ab}[\mathcal{S}_{\lambda_{v,\tau}}]$. This limit in fact agrees with the Gauss curvature of the limiting metric $\gamma^{\infty,\lambda}_{v,\tau}$. As we will see below, on any fixed $\underline{\mathcal{N}}_{v,\tau}$, any smooth change of the affine parameter $\lambda_{v,\tau}$ induces a conformal change on the metric at infinity, defined (for any affine parameter) via (4.21).

We also note a few useful facts about the asymptotics of the Ricci coefficients ζ, χ on the affinely parametrized null surfaces $\underline{\mathcal{N}}_{v,\tau}$:

Lemma 4.4. *Given any surface $\underline{\mathcal{N}}_{v,\tau}$ in our space of perturbations, we claim that (letting $L_{v,\tau}$ be the null normal to the level sets of $\lambda_{v,\tau}$ normalized so that $g(L_{v,\tau}, \underline{L}_{v,\tau}) = 2$):*

$$\zeta^{L_{v,\tau}}_A = O^\delta_1(\lambda^{-2}_{v,\tau}), \chi^{L_{v,\tau}}_{AB} \in O^\delta_1(\lambda^{-1}_{v,\tau}). \tag{4.22}$$

and moreover:

$$tr\chi^{L_{v,\tau}}[\mathcal{S}_\lambda] \geq \frac{(\lambda_{v,\tau} - 1)}{4m^2\lambda^2_{v,\tau}}. \tag{4.23}$$

Proof: Recall from (3.20) that

$$\chi^{L_{v,\tau}}[\mathcal{S}'_{v,\tau}] = 0, \sum_{i=0}^{2} |\partial^{(i)}\zeta^{L_{v,\tau}}_A[\mathcal{S}'_{v,\tau}]| \leq \delta, \sum_{i=0}^{2} |\partial^{(i)}\underline{\chi}^{L_{v,\tau}}_{AB}| \leq \delta$$

The claim on ζ follows from the second equation in (2.20) (a linear ODE, given the bounds we have on $\underline{\chi}$) by multiplying by λ^2 and evaluating against e_A, $A = 1, 2$. To derive the claim on the rotational derivatives of ζ_a, we just differentiate the evolution equations by ∂_{ϕ^i} and invoke the solution of first order ODEs, along with the derived and assumed bounds on the angular derivatives of $\underline{\chi}$ and $\underline{\beta}$. Once the claim has been derived for ζ_a, we refer to the third equation in (2.20) and repeat the same argument for χ_{ab}. This proves (4.22).

To derive (4.23) we invoke (4.1) and multiply the equation (2.21) by λ, and derive an equation:

$$\frac{d}{d\lambda}[\lambda tr\chi] + \frac{atr\chi}{\lambda} = -2\lambda div\zeta - 2\lambda|\zeta|^2 - 2\lambda\rho + \lambda\hat{\chi}\hat{\underline{\chi}}. \tag{4.24}$$

In the RHS of the above, all terms can be seen as perturbations of the main term $-2\lambda\rho$, in view of the bounds we have already derived. Thus our result follows by the bounds on ρ in (2.8) and treating the above as a first order ODE in $\lambda \mathrm{tr}\chi$, using the smallness of δ compared to m. $\Box$

For future reference we note two key facts: The first is that by integration of the evolution equation (4.24), we derive that $\lambda tr\chi$ has a limit over $\mathcal{S}^\infty_{v,\tau}$, which is a $\mathcal{C}^2$ function. We denote this limit by $tr\chi^{\lambda,\infty}$ to stress the dependence on the choice of affine function λ. Note that by Lemmas 4.1, 4.4, and formula (4.48), for any of the hypersurfaces $\underline{\mathcal{N}} = \underline{\mathcal{N}}_{v,\tau}$ this limit in fact agrees with the limit of the (renormalized) Gauss curvatures of the level spheres $\mathcal{S}_\lambda$:

$$lim_{\lambda\to\infty}\lambda tr\chi^\lambda = 2lim_{\lambda\to\infty}\lambda^2\mathcal{K}[\mathcal{S}_\lambda]. \tag{4.25}$$

In this connection, we make a note on the transformation law of $tr\chi$ on a given null surface $\underline{\mathcal{N}}$ which satisfies the conclusion of Lemmas 4.1 and 4.4. We will be particularly interested in a function ω_τ equal to the *exponent* in (3.14). We let $\widetilde{tr\chi}^{\lambda,\infty}$ to be the limit of $\lambda tr\chi^\lambda[\mathcal{S}_\lambda]$ as $\lambda \to \infty$. We also let $\widetilde{tr\chi}^{\lambda'_\tau}$ to be the limit of $\lambda'_\tau tr\chi^\lambda[\mathcal{S}_{\lambda'_\tau}]$ as $\lambda'_\tau \to \infty$

Using that choice of ω, the definition (4.21) and the asymptotics for $\chi, \underline{\chi}, \zeta$, we find that (2.35) implies that letting λ'_τ be the new affine parameter defined via (2.32) for $\omega := \omega_\tau$ we have:

$$\widetilde{tr\chi}^{\lambda'_\tau} = \widetilde{tr\chi}^{\lambda} + 2[\widetilde{tr\chi}^{\lambda} + \Delta_{\gamma^\infty,\lambda}]\omega_\tau + o(\tau). \tag{4.26}$$

4.2 Monotonicity of Hawking Mass.

In this subsection $\underline{\mathcal{N}}$ will stand for any infinite smooth past-directed outgoing null surface which satisfies the assumptions (and thus the conclusions) of Lemma 4.1. In particular recall that all the null surfaces $\underline{\mathcal{N}}_{v,\tau}$ considered in Lemma 3.1 satisfy these assumptions.

We recall a fact essentially due to Hawking, [13]:

Definition 4.5. *Consider any foliation of $\underline{\mathcal{N}}$ by a smooth family of 2-spheres $\mathcal{S}_s$, $s \in [1, +\infty)$. We consider the (unique) null geodesic generator $\underline{L}$ which is tangent to $\underline{\mathcal{N}}$ and defined via:*

$$\underline{L}s = 1. \tag{4.27}$$

We call s a luminosity parameter if:

$$tr\underline{\chi}^{\underline{L}}[\mathcal{S}_s] = \frac{2}{s}, \tag{4.28}$$

We call the family $\mathcal{S}_s \subset \underline{\mathcal{N}}, s \in [1, \infty)$ a luminosity foliation of $\underline{\mathcal{N}}$.

In particular $tr\underline{\chi}^{\underline{L}}$ (defined relative to $\underline{L}$) is constant on each sphere $\mathcal{S}_s$.

A key property of luminosity foliations of $\underline{\mathcal{N}}$ is that the Hawking mass is monotone for such a foliation, when $\underline{\mathcal{N}}$ is (extrinsically and intrinsically) close to the shear-free null surfaces in the Schwarzschild exteriors:

Lemma 4.6. *We let L be the conjugate null vector field to $\underline{L}$ on $\underline{\mathcal{N}}$ for the spheres $\mathcal{S}_s$ (i. e. $g(L, \underline{L}) = 2$, $L \perp \mathcal{S}_s, s \geq 1$) and also let $\underline{\chi}^{\underline{L}}, \chi^L, \zeta^{\underline{L}}$ be the null expansions and torsion of the spheres $\mathcal{S}_s$ defined relative to $\underline{L}, L$.*

For the luminosity foliation $\mathcal{S}_s$ of $\underline{\mathcal{N}}$ (satisfying (4.28)) we have:

$$\frac{d}{ds} m_{\mathrm{Hawk}}[\mathcal{S}_s] = \frac{r[\mathcal{S}_s]}{32\pi} \int_{\mathcal{S}_s} tr\chi^L |\hat{\underline{\chi}}^{\underline{L}}|^2 + tr\underline{\chi}^{\underline{L}} |\zeta^{\underline{L}}|^2 dV_s. \tag{4.29}$$

In particular when $tr\chi^L[\mathcal{S}_s] \geq 0$ and $tr\underline{\chi}^{\underline{L}}[\mathcal{S}_s] \geq 0$ for all $s \geq 1$ (as will be the case for all $\underline{\mathcal{N}}_{v,\tau}$ that we consider here in view of (4.23)), $m_{\mathrm{Hawk}}[\mathcal{S}_s]$ is non-decreasing in s, in the outward direction.

Proof: The proof of this follows [23], which elaborates the argument in [13]. We define a function κ over $\underline{\mathcal{N}}$ defined via:

$$\nabla_{\underline{L}}\underline{L} = \kappa\underline{L}. \tag{4.30}$$

We then recall the evolution equations on each $\mathcal{S}_s$:[15]

$$\begin{aligned}
&\nabla_{\underline{L}} tr\underline{\chi}^{\underline{L}} = -\frac{1}{2}(tr\underline{\chi})^2 - \frac{1}{2}|\hat{\chi}|^2 + \kappa tr\underline{\chi}, \\
&\frac{d}{ds} dV_{\mathcal{S}_s} = tr\underline{\chi}^n dV_{\mathcal{S}_s} = \frac{2}{s} dV_{\mathcal{S}_s}, \\
&\frac{d}{ds} tr\chi^L = -tr\chi tr\underline{\chi} + 2\mathcal{K} - 2|\zeta|^2 - 2div\zeta - \kappa tr\chi.
\end{aligned} \tag{4.31}$$

Now, let us study the evolution of the Hawking mass of such a foliation. Recall the mass aspect function (2.24) and its relation (2.25) with the Hawking mass of each $\mathcal{S}_s$. Now, using these and the evolution equations, along with the fact that $tr\underline{\chi}[\mathcal{S}_s] = \frac{2}{s}$, we derive:

$$\begin{aligned}
&\frac{d}{ds} m_{\mathrm{Hawk}}[\mathcal{S}_s] = \frac{d}{ds}\Big(\frac{r[\mathcal{S}_s]}{8\pi}\int_{\mathcal{S}_s} \mu dV_s\Big) \\
&= \frac{1}{2}\overline{tr\underline{\chi}} m_{\mathrm{Hawk}}[\mathcal{S}_s] - \frac{r[\mathcal{S}_s]}{16\pi}\int_{\mathcal{S}_s} tr\underline{\chi}\mu dV_s + \frac{r[\mathcal{S}_s]}{32\pi}\int_{\mathcal{S}_s} tr\chi|\hat{\underline{\chi}}|^2 + tr\underline{\chi}|\zeta|^2 dV_s.
\end{aligned} \tag{4.32}$$

Now, first writing $\mu = \overline{\mu} + (\mu - \overline{\mu})$ and then recalling that $tr\underline{\chi} = \overline{tr\underline{\chi}}$, we finally obtain (4.29). $\Box$

As we will see in the next two subsections, all the hypersurfaces $\underline{\mathcal{N}}_{v,\tau}$ that we consider here admit a luminosity foliation, and moreover the Hawking mass is monotone increasing along such a foliation. It is in proving this latter property that the assumption of closeness to the Schwarzschild solution is employed in an essential way.

4.3 Construction of constant luminosity foliations, and their asymptotic behaviour.

We now show how to construct a luminosity foliation on the surfaces $\underline{\mathcal{N}}_{v,\tau}$, using their affine parameters $\lambda_{v,\tau}$ as a point of reference. The construction here essentially follows [23], whose notation we also adopt.

Recall that the affine vector field $\underline{L}_{v,\tau}$ is normalized so that: $tr\underline{\chi}^{\underline{L}_{v,\tau}}[\mathcal{S}'_{v,\tau}] = 2$ on $\mathcal{S}'_{v,\tau}$. $\lambda_{v,\tau}$ is the corresponding affine parameter, with $\lambda|_{\mathcal{S}'_{v,\tau}} = 1$. We refer to $\lambda_{v,\tau}$ as the "background affine parameter" on $\underline{\mathcal{N}}_{v,\tau}$.

To stress that this construction can be performed separately on each $\underline{\mathcal{N}}_{v,\tau}$, we use subscripts v, τ on all relevant quantities below, except for the luminosity parameters. These we still write as s instead of $s_{v,\tau}$ for notational convenience.

[15] We write $tr\underline{\chi}, tr\chi, \zeta$ for short.

(Note that each $s_{v,\tau}$ lives over $\underline{\mathcal{N}}_{v,\tau}$). In the subsequent sections where we study the variation of $s_{v,\tau}$ under changes in τ, we will use $s_{v,\tau}$.

The function s that we are then seeking (on each null geodesic on $\underline{\mathcal{N}}_{v,\tau}$), can be encoded in a function $w_{v,\tau}(s,x)$ defined via the relation:

$$\lambda_{v,\tau} = w_{v,\tau}(s,x). \tag{4.33}$$

The requirement (4.28) can then be re-expressed as:

$$\frac{\partial}{\partial_s} w_{v,\tau}(s,x) = \frac{2}{s \cdot tr\underline{\chi}^{\lambda_{v,\tau}}(w_{v.\tau}(s,x),x)} \tag{4.34}$$

With the initial condition $w_{v,\tau}(1,x) = 1$.

In order to study the local and global existence of a solution to the above, it is useful to recall Lemma 4.1 on the the asymptotic behaviour of $tr\underline{\chi}^{\lambda_{v,\tau}}(\lambda,x)$ and its derivatives with respect to x. In particular we recall that

$$tr\underline{\chi}^{\lambda_{v,\tau}}(x) = \frac{2}{\lambda_{v,\tau}} + \frac{a_{v,\tau}(x)}{\lambda^2_{v,\tau}}, \tag{4.35}$$

where $a_{v,\tau} \in O^\delta_2(1)$, and thus for some $0 < c < C$

$$c \leq \lambda tr\underline{\chi}^{\lambda_{v,\tau}}(x) \leq C, \tag{4.36}$$

for all $(x,\lambda) \in \underline{\mathcal{N}}$. Equation (4.34) has a local solution in s. The equation (4.34) implies that $w_{v,\tau}(s,x)$ is increasing in s; the bounds (4.36) imply that the solution exists for all $s \geq 1$ (for each v,τ).

We next claim:

Lemma 4.7. *Let $\underline{\mathcal{N}}_{v,\tau}$ be as in Definition 2.5 and let $\lambda_{v,\tau}$ be the background affine parameter, and s the luminosity parameter constructed above. We claim that*

$$(\lambda_{v,\tau} - 1)^{-1}[s\lambda^{-1}_{v,\tau} - 1] \in O^\delta_2(\lambda^{-1}_{v,\tau}), \tag{4.37}$$

and moreover that there exists a $\mathcal{C}^2$ function $\varphi_{v,\tau}(\phi,\theta)$ so that:

$$s\lambda^{-1}_{v,\tau} = e^{\varphi_{v,\tau}(\phi,\theta)} + O^\delta_2(\lambda^{-1}_{v,\tau}). \tag{4.38}$$

Postponing the proof of the above for a moment, we define:

Definition 4.8. *Consider the new (affine) vector field $\underline{\tilde{L}}_{v,\tau} := e^{\varphi_{v,\tau}}\underline{L}_{v,\tau}$ and let $\tilde{\lambda}_{v,\tau}$ be its corresponding affine parameter.*

Observe that (4.38) implies:

$$s_{v,\tau}\tilde{\lambda}^{-1}_{v,\tau} = 1 + O^\delta_2(\lambda^{-1}_{v,\tau}). \tag{4.39}$$

In particular, the spheres $\mathcal{S}(\{s_{v,\tau} = B\} \subset \underline{\mathcal{N}}_{v,\tau})$ agree asymptotically (to leading order) with the spheres $\mathcal{S}(\{\tilde{\lambda}_{v,\tau} = B\}) \subset \underline{\mathcal{N}}_{v,\tau}$.

It will be necessary to calculate the *first variation* $\dot{\varphi}_v$ of the functions $\varphi_{v,\tau}$ in τ, which is defined via:

$$\dot{\varphi}_v := \frac{d}{d\tau}|_{\tau=0}\varphi_{v,\tau}. \tag{4.40}$$

We claim:

Lemma 4.9. *Consider a smooth 1-parameter family of null surfaces $\underline{\mathcal{N}}_{v,\tau}$ (emanating from spheres $\mathcal{S}_{v,\tau}$) to which the assumption of Lemma 4.3 applies. Then there exist functions $f^1(\lambda) \in O_2^1(1), f^2(\lambda) \in O_2^\delta(\lambda^{-2})$ so that*

$$\dot{\varphi}_v(\phi,\theta) = lim_{s\to\infty} f^1(s)\int_1^s f^2(t)\dot{a}_v(t)dt. \tag{4.41}$$

Proof of Lemmas 4.7, 4.9: We prove both Lemmas together. Recall the parameters $w_{v,\tau}(\lambda)$, $a_{v,\tau}(\lambda)$ on $\underline{\mathcal{N}}_{v,\tau}$ as in (4.33), (4.35).

We now derive the asymptotic behaviour of the solution $w_{v,\tau}$ of (4.34), for given v,τ: Given (4.35) our equation becomes:

$$\partial_s w_{v,\tau} = \frac{w_{v,\tau}}{s(1+\frac{a_{v,\tau}(w_{v,\tau})}{2w_{v,\tau}})}. \tag{4.42}$$

Letting $\tilde{w}_{v,\tau} := \frac{w_{v,\tau}}{s}$ we transform the above into a new equation on $\tilde{w}_{v,\tau}$:

$$\partial_s \tilde{w}_{v,\tau} = -\frac{a_{v,\tau}(\tilde{w}_{v,\tau}s)\tilde{w}_{v,\tau}}{s[2s\tilde{w}_{v,\tau} + a_{v,\tau}(\tilde{w}_{v,\tau}s)]}. \tag{4.43}$$

In view of the smallness of $a_{v,\tau}$, and since $\tilde{\lambda}_{v,\tau} = 1$ at $s = 1$, a simple bootstrap argument reveals that $\tilde{w}_{v,\tau}$ stays δ-close to 1 for all $s \geq 1$. Then just integrating the above equation shows that $\tilde{w}_{v,\tau}$ converges to a limit as $s \to \infty$.

$$\tilde{w}_{v,\tau}(s) \to \tilde{w}_{v,\tau}(\infty), \tag{4.44}$$

with $|\tilde{w}_{v,\tau}(\infty) - 1| = O(\delta)$ and $|\tilde{w}_{v,\tau}(s) - \tilde{w}_{v,\tau}(\infty)| \leq Cs^{-1}$. In particular, we can define a continuous function $\varphi_{v,\tau}(s,\phi,\theta)$ via:

$$\tilde{w}_{v,\tau}(s,\phi,\theta) = e^{\varphi_{v,\tau}(s,\phi,\theta)}. \tag{4.45}$$

The above combined show (4.37) and (4.38),[16] except for the angular regularity. We let $\varphi_{v,\tau}(\phi,\theta) := lim_{s\to\infty}\varphi_{v,\tau}(\phi,\theta)$. To obtain the extra regularity in the angular directions ϕ^1, ϕ^2, we just take $\partial_\phi, \partial_\theta$ derivatives of (4.43), up to two times. Then the claim on the extra regularity follows immediately from the resulting (linear) ODE, using the bounds we have derived in Lemma 4.1 on the angular derivatives of $a_{v,\tau}$. This proves Lemma 4.7.

The proof of Lemma 4.9 follows by just differentiating in τ equation (4.42) to derive (letting $\dot{a}_v(\lambda)$ stand for $\partial_\tau|_{\tau=0}a_{v,\tau}(\lambda)$ and $a'_{v,\tau}(\cdot)$ stands for the regular derivative in λ of $a_{v,\tau}(\lambda)$).

[16] For the former merely recall that $\lambda = s$ on this initial sphere $\mathcal{S}_{v,\tau}$.

$$\partial_s \dot{\tilde{w}}_v = -\frac{\dot{a}_v(\tilde{w}_0 s) + a'_{v,0}(\tilde{w}_0 s)\dot{\tilde{w}}_v s}{2s^2(1+\frac{a_0(w)}{2\tilde{w}_0 s})} + \frac{a_0(\tilde{w}s)}{2s^2(1+\frac{a_0(w)}{2\tilde{w}_0 s})^2}(\dot{a}_v(\tilde{w}_0 s) + a'_0(\tilde{w}_0 s)\dot{\tilde{w}}_v s). \tag{4.46}$$

(Here $w = \tilde{w}s$ lives over the original surface $\underline{\mathcal{N}}$). Thus, seeing the above as a first-order linear ODE in $\dot{\tilde{w}}_v(s)$ we derive:

$$\dot{\tilde{w}}_v(s) = O_2^\delta(1)\int_1^s O_2^\delta(t^{-2})\dot{a}_v(w_0(t))dt. \tag{4.47}$$

By the definition (4.45) of $\varphi_{v,\tau}(s,\phi,\theta)$ and passing to the limit $s \to \infty$, our claim follows. $\Box$

We have thus derived that any luminosity foliation on any $\underline{\mathcal{N}}_{v,\tau}$ is asymptotically equivalent to an *affine* foliation of the same null hypersurface $\underline{\mathcal{N}}_{v,\tau}$. The relation between the luminosity parameter s and the new affine parameter is given by (4.39).

As we have seen in the introduction, the main issue in capturing the Bondi energy at a section of $\mathcal{I}^-$ is to approximate that section by spheres that become asymptotically round. With that in mind, we introduce a definition:

Definition 4.10. *Given any $\underline{\mathcal{N}}_{v,\tau}$ and either an affine parameter λ or the luminosity parameter s, we let:*

$$\begin{aligned} \mathcal{K}^{\infty,\lambda}_{v,\tau}(\phi,\theta) &:= lim_{B\to\infty} B^2\mathcal{K}[\mathcal{S}(\{\lambda = B\})](\phi,\theta), \\ \mathcal{K}^{\infty,s}_{v,\tau}(\phi,\theta) &:= lim_{B\to\infty} B^2\mathcal{K}[\mathcal{S}(\{s = B\})](\phi,\theta). \end{aligned} \tag{4.48}$$

Regarding the first limit, recall the discussion after (4.21) on the existence of a limit of the (renormalized) Gauss curvatures for an *affine* foliation. Regarding the second limit, note that we have not yet derived its existence at this point.

We next claim that the (renormalized) limits of the Gauss curvatures and the Hawking masses of the two foliations by $\tilde{\lambda}_{v,\tau}$ and s as in Definition 4.8. agree. This will enable us to replace luminosity foliations with suitable *affine* foliations on each of the null surfaces $\underline{\mathcal{N}}_{v,\tau}$ that we are considering.

Lemma 4.11. *In the notation above we claim that on each $\underline{\mathcal{N}}_{v,\tau}$:*

$$\mathcal{K}^{\infty,s}_{v,\tau}(\phi,\theta) = \mathcal{K}^{\infty,\tilde{\lambda}_{v,\tau}}_{v,\tau}(\phi,\theta) \tag{4.49}$$

and

$$lim_{B\to\infty} m_{\text{Hawk}}[\mathcal{S}(\{s = B\})] = lim_{B\to\infty} m_{\text{Hawk}}[\mathcal{S}(\{\tilde{\lambda}_{v,\tau} = B\})] = 0. \tag{4.50}$$

Proof: To derive the first formula, we note that by the definition of $\underline{\chi}^{L}_{ab}(\lambda)$, for *any* affine vector field $\underline{L}$ with corresponding affine parameter λ:

$$\not\gamma_{ab}(\{\lambda = r_1(\phi,\theta)\}) - \not\gamma_{ab}(\{\lambda = r_2(\phi,\theta)\}) = 2\int_{\{\lambda=r_1(\phi,\theta)\}}^{\{\lambda=r_2(\phi,\theta)\}} \underline{\chi}^{\underline{L}}_{ab}(\lambda), \tag{4.51}$$

where the indices a, b are assigned values from among the vector fields $\partial_\phi, \partial_\theta$.

Then, we choose the affine parameter $\tilde{\lambda}_{v,\tau}$, and choose $r_1(\phi,\theta) = B$. We also choose $r_2(\phi,\theta)$ to be the function so that:

$$\{\tilde{\lambda}_{v,\tau} = r_2(\phi,\theta)\} = \{s_{v,\tau} = B\}.$$

Then invoking the asymptotics (4.1) of $\underline{\chi}$, equations (4.51) and (4.39), along with expression of the Gauss curvature in terms of second coordinate derivatives of the metric of the spheres, we derive (4.49).

To derive (4.50) we recall a formula for the Hawking mass:[17]

$$m_{\text{Hawk}}[\mathcal{S}] = r[\mathcal{S}]\int_{\mathcal{S}} -\rho - \frac{1}{2}\hat{\chi}\cdot\underline{\hat{\chi}} - div\zeta dV_{\mathcal{S}}. \tag{4.52}$$

To derive this, we have used (2.24), (2.25) and:

$$\mathcal{K}[\mathcal{S}] = -\rho - \frac{1}{2}\hat{\chi}[\mathcal{S}]\underline{\hat{\chi}}[\mathcal{S}] + \frac{1}{4}tr\chi[\mathcal{S}]tr\underline{\chi}[\mathcal{S}] \tag{4.53}$$

Thus, to prove (4.50), the main challenge is for any fixed large $B > 0$ to compare $tr\chi, \hat{\chi}$ on the spheres $\{s = B\}$ and $\{\tilde{\lambda} = B\}$. We will be using the formulas derived in section 2 of [4], for the distortion function:

$$e^{\psi_B} = \tilde{\lambda}(\{s = B\})B^{-1}. \tag{4.54}$$

(ψ_B is defined to be constant on the null generators of $\underline{\mathcal{N}}_{v,\tau}$). By (4.39) we derive that:

$$\psi_B = O_2^\delta(B^{-1}) \tag{4.55}$$

Using (4.51) we now compare the metric elements of the two spheres $\{s = B\}$ and $\{\tilde{\lambda} = B\}$, via the natural map that identifies points on the same null generator of $\underline{\mathcal{N}}$:

$$\gamma_{\{\tilde{\lambda}=B\}} = (1 + O(B^{-1}))\gamma_{\{s=B\}}. \tag{4.56}$$

A consequence of this is a comparison of the area elements and the areas of $\{s = B\}$ and $\{\tilde{\lambda} = B\}$:

$$\text{Area}[\{s = B\}] = (1 + O(B^{-1}))\text{Area}[\{\tilde{\lambda} = B\}]. \tag{4.57}$$

On the other hand, the transformation laws of section 2 in [4] imply that:

$$B^2\rho(\{s = B\}), B^2\rho(\{\tilde{\lambda} = B\})\} = O(B^{-1}). \tag{4.58}$$

[17]The $\chi, \underline{\chi}$ (along with their traces and trace-less parts) appearing below are defined relative to *any* pair of null vectors $L, \underline{L}$ which are normal to $\mathcal{S}$ and normalized so that $g(L, \underline{L}) = 2$.

Also, the transformation laws of subsection 2.3 in the present paper imply

$$B^2\{tr\chi[\mathcal{S}(\{s=B\})]tr\underline{\chi}[\mathcal{S}(\{s=B\})]-tr\chi[\mathcal{S}(\{\tilde{\lambda}=B\})]tr\underline{\chi}[\mathcal{S}(\{\tilde{\lambda}=B\})]\}=O(B^{-1}). \tag{4.59}$$

$$B^3\{\hat{\chi}[\mathcal{S}(\{s=B\})]\hat{\underline{\chi}}[\mathcal{S}(\{s=B\})]-\hat{\chi}[\mathcal{S}(\{\tilde{\lambda}=B\})]\hat{\underline{\chi}}[\mathcal{S}(\{\tilde{\lambda}=B\})]\}=O(B^{-1}). \tag{4.60}$$

These equations, combined with (4.52) prove our claim. □

Remark. *Note that the transformation laws* (2.33)*,* (2.34)*,* (2.35) *invoked above show that since the Ricci coefficients ζ, χ associated with the affine parameter $\tilde{\lambda}_{v,\tau}$ satisfy the bounds* (4.22) *of Lemma 4.4, then the same bounds are satisfied by the Ricci coefficient associated with the luminosity parameter s.*

We next prove that the luminosity foliations on all the hypersurfaces $\underline{\mathcal{N}}_{v,\tau}$ have the desired monotonicity of the Hawking mass:

Monotonicity of the Hawking mass for luminosity foliations:

Lemma 4.12. *Consider any hypersurface $\underline{\mathcal{N}}_{v,\tau}$ as in Definition 2.5, and let $\{\mathcal{S}_s\}_{s\geq 1}$ be its luminosity foliation. Then $tr\chi^{L_s}[\mathcal{S}_s]>0$ for all $s>1$. (Here L_s is the future-directed outgoing null normal to $\mathcal{S}_s$). In particular, in view of* (4.29)*, $m_{\text{Hawk}}[\mathcal{S}_s]$ is an increasing function in s.*

Proof: We show this in two steps. Firstly, observe that the level sets of the original affine parameter $\lambda_{v,\tau}$ satisfy $\mathrm{tr}\chi^{L_{v,\tau}}[\mathcal{S}_{\lambda_{v,\tau}}]>0$. This follows from (4.23) which yields a positive lower bound for $\mathrm{tr}\chi^{L_{v,\tau}}[\mathcal{S}_{\lambda_{v,\tau}}]$. Secondly we use this lower bound to derive $tr\chi^{L_s}[\mathcal{S}_s]>0$. This second claim in fact follows straightforwardly from (4.23), coupled with (4.37) (which encodes how the spheres $\mathcal{S}_{\{s=B\}}$ are small perturbations of the spheres $\mathcal{S}_{\{\lambda_{v,\tau}=B\}}$), and the transformation law (2.27). □

5 Variations of the null surfaces and their luminosity foliations.

We next seek to capture how a variation $\underline{\mathcal{N}}_{v,\tau}$ of the original $\underline{\mathcal{N}}_0$ induces a variation on the Gauss curvature of the metric at infinity of $\underline{\mathcal{N}}_{v,\tau}$ associated with the luminosity foliations on these surfaces. (See the first equation in (4.48)).

5.1 Varying null surfaces and luminosity foliations: The effect on the Gauss curvature at infinity.

We consider the family $\underline{\mathcal{N}}_{v,\tau}$ of smooth null surfaces in Definition 2.5. We also consider the associated functions $w_{v,\tau}(\phi,\theta,s)$ and let $s_{v,\tau}$ to be the luminosity parameters on $\underline{\mathcal{N}}_{v,\tau}$. (We write out $s_{v,\tau}$ instead of just s, to stress that we are studying the variation of $\underline{\mathcal{N}}_{v,\tau}$ and the parameters $s_{v,\tau}$ defined over them).

Our goal in this section is to calculate the first variation of the renormalized Gauss curvatures $\mathcal{K}^{\infty,s}_{v,\tau}$, around $\mathcal{K}^{\infty,s}_0$ (see (4.48)) which corresponds to the initial null hypersurface $\underline{\mathcal{N}}_0$. In other words we seek to capture:

$$\frac{d}{d\tau}|_{\tau=0} lim_{B\to\infty} B^2\mathcal{K}[\{s_{v,\tau}=B\}\subset \underline{\mathcal{N}}_{v,\tau}]. \tag{5.1}$$

Remark. *We note for future reference that the precise same calculation can be applied also to capture the first variation of Gauss curvatures around* any *null surface $\underline{\mathcal{N}}_\omega$ with $||\omega||_{W^{4,p}(\mathcal{S}_0)}\le 10^{-1}m$. This follows readily in view of the assumed bounds on the curvature on the surfaces $\underline{\mathcal{N}}_\omega$. In particular all the formulas we derive remain true, by just replacing the Ricci coefficients and curvature components on $\underline{\mathcal{N}}_0$ by those on $\underline{\mathcal{N}}_\omega$.*

Definition 5.1. *On each $\underline{\mathcal{N}}_{v,\tau}$ we let $s_{v,\tau}$ be the luminosity parameter. We let $\mathcal{S}_{v,\tau}[B]$ be the level set $\{s_{v,\tau}=B\}\subset\underline{\mathcal{N}}_{v,\tau}$ and $\gamma_{v,\tau}[B]$ the induced metric on this sphere. We then let:*

$$\gamma^\infty_{v,\tau} := lim_{B\to\infty} B^{-2}\gamma_{v,\tau}[B] \tag{5.2}$$

The limit is understood in the sense of components relative to the coordinate vector fields $\partial_\phi,\partial_\theta$. For $\tau=0$ we just denote the corresponding limit metric by γ^∞.

(Note that these limits exist, by combining (4.39) with (4.21), to derive that the corresponding limit exists for the level sets of the *affine* parameter $\tilde{\lambda}_{v,\tau}$). Note further (as mentioned above) that the same equations imply:

$$\mathcal{K}[\gamma^\infty_{v,\tau}] = lim_{B\to\infty} B^2\mathcal{K}[\{s_{v,\tau}=B\}]. \tag{5.3}$$

To capture the variation of the Gauss curvatures, we proceed in two steps: The variation of the null surfaces $\underline{\mathcal{N}}_{v,\tau}$, foliated by the backgound *affine* parameters $\lambda_{v,\tau}$ is encoded in the Jacobi fields (2.15) along the 2-parameter family of geodesics $\gamma_Q\subset\underline{\mathcal{N}}_{v,\tau}$. Recall the *new* affine parameters $\tilde{\lambda}_{v,\tau}$ defined in Definition 4.8, which asymptote to the luminosity foliations of the hypersurfaces $\underline{\mathcal{N}}_{v,\tau}$. We then define the family of *modified* Jacobi fields $\tilde{J}$ that correspond to these affine parameters:

$$\tilde{J}_{v,Q}(B) := \frac{d}{d\tau}|_{\tau=0}\{\tilde{\lambda}^Q_{v,\tau}=B\} \tag{5.4}$$

We consider $\tilde{J}_v$ expressed in the frame $\underline{L}, e^1, e^2, L.$ as in Definition 2.1:

Definition 5.2. *We denote the components of the Jacobi fields $\tilde{J}_v$ expressed with respect to the above frame by $J^{\underline{L}}_v, J^A_v, A=1,2$ and J^L_v. We will think of these components with respect to the background affine parameter λ on $\underline{\mathcal{N}}$. The* prime $\prime$ *will stand for the derivative with respect to λ. (In particular $(\tilde{J}^{\underline{L}}_v)' := \frac{d}{d\lambda}\tilde{J}^{\underline{L}}_v(\lambda)$.*

We let:

$$(\tilde{J}^{\underline{L}}_{v})'_{\infty} := lim_{\lambda\to\infty}(\tilde{J}^{\underline{L}}_{v})'(\lambda).$$

(The existence of this limit will be derived below, for every $v \in W^{4,p}(\mathcal{S}_0)$).

We claim that:

Proposition 5.3. *With the identification of coordinates described above:*

$$\mathcal{K}[\gamma^{\infty}_{v,\tau}] - \mathcal{K}[\gamma^{\infty}] = 2\tau[\Delta_{\gamma^{\infty}} + 2\mathcal{K}(\gamma^{\infty})]((\tilde{J}^{\underline{L}}_{v})'_{\infty}) + o(\tau). \tag{5.5}$$

In fact, using the definition 4.8 of $\tilde{\lambda}_{v,\tau}$ along with (4.40), we find readily that:

$$(\tilde{J}^{\underline{L}}_{v})'(\lambda) = (J^{\underline{L}}_{v})'(\lambda) + \dot{\varphi}_v(\lambda). \tag{5.6}$$

The evaluation of the two terms in the RHS of the above will be performed in the next section. For now, we prove the Proposition above:

Proof of Proposition (5.3)*:* The key insight behind the proof is that the variation of the spheres under study can be decomposed into one *tangential* to the original null surface $\underline{\mathcal{N}}_0$ and one *transverse* to it. We find that the transverse component of the variation only contributes an error term to the variation of the (renormalized) Gauss curvature. On the other hand, the tangential variation induces a (linearized) *conformal* change of the underlying metric, since it corresponds (up to error terms) to a first variation of affine foliations. In a different guise, this latter fact was also used in [5] (albeit on a single, un-perturbed null hypersurface); as noted there, the intuition behind this goes back to the ambient metric construction of Fefferman and Graham [12].

Returning to the proof, observe that it suffices to show that:

$$B^2\mathcal{K}[\{s_{v,\tau} = B\}] - B^2\mathcal{K}[\{s_0 = B\}] = 2\tau B^2[\Delta_{\gamma[B]} + 2\mathcal{K}(\gamma[B])][(\tilde{J}^{\underline{L}}_{v})'] + o(\tau) + O(B^{-1}). \tag{5.7}$$

Invoking the limit $B^{-2}\gamma[B] \to \gamma^{\infty}$ (the convergence being in $\mathcal{C}^2$, as noted in the proof of Lemma 4.11) we note that:

$$B^2[\Delta_{\gamma[B]} - \mathcal{K}(\gamma[B])](\tilde{J}^{\underline{L}}_{v})' \to [\Delta_{\gamma^{\infty}} - \mathcal{K}(\gamma^{\infty})](\tilde{J}^{\underline{L}}_{v})',$$

with the convergence being in $L^p(\mathcal{S})$. Thus it suffices to show (5.7) to derive (5.5).

To capture the difference in the LHS of (5.7), we will proceed in six steps, suitably approximating the spheres

$$\{s_{v,\tau} = B\} \subset \underline{\mathcal{N}}_{v,\tau}, \{s_0 = B\} \subset \underline{\mathcal{N}}_0.$$

Recall that $\underline{L}_{v,\tau}$ is the affine vector field on $\underline{\mathcal{N}}_{v,\tau}$ normalized so that $tr\underline{\chi}^{\underline{L}_{v,\tau}}[\mathcal{S}'_{v,\tau}] = 2$. Recall that $\lambda_{v,\tau}$ is the corresponding affine parameter. We also recall that $\tilde{\lambda}_{v,\tau}$ is the affine parameter on $\underline{\mathcal{N}}_{v,\tau}$ which asymptotes to the luminosity parameter $s_{v,\tau}$; see (4.39).

Definition 5.4. *We let $\underline{L}^\flat_{v,\tau}$ be a new affine vector field on $\underline{\mathcal{N}}_0$ defined via:*

$$\underline{L}^\flat_{v,\tau} := (1 + \tau(\tilde{J}^{\underline{L}})'_\infty)\underline{L},$$

and we let $\lambda^\flat_{v,\tau}$ be the corresponding affine function over $\underline{\mathcal{N}}_0$,[18] *normalized such that $\lambda^\flat_{v,\tau} = 1$ on $\mathcal{S}_0$ and $\underline{L}^\flat_{v,\tau}(\lambda^\flat_{v,\tau}) = 1$.*

Let $L'_{v,\tau}$ be a null vector field, normal to the level sets of $\lambda^\flat_{v,\tau}$ on $\underline{\mathcal{N}}_0$, with $g(L'_{v,\tau}, \underline{L}^\flat_{v,\tau}) = 2$.

We then let:

Definition 5.5. *1. $\mathcal{S}^1_0(B) := \{s_0 = B\} \subset \underline{\mathcal{N}}_0$.*

2. $\mathcal{S}^2_0(B) := \{\tilde{\lambda}_0 = B\} \subset \underline{\mathcal{N}}_0$.

3. $\mathcal{S}^3_{v,\tau}(B) := \{\lambda^\flat_{v,\tau} = B\} \subset \underline{\mathcal{N}}_0$.

4. $\mathcal{S}^4_{v,\tau}(B)$ is the sphere obtained from $\mathcal{S}^3_{v,\tau}(B)$ by flowing along the geodesics emanating from the vector field $L'_{v,\tau}$ by $J^{L'_{v,\tau}}\tau$ in the corresponding affine parameter.

5. $\mathcal{S}^5_{v,\tau}(B) := \{\tilde{\lambda}_{v,\tau} = B\} \subset \underline{\mathcal{N}}_{v,\tau}$.

6. $\mathcal{S}^6_{v,\tau}(B) := \{s_{v,\tau} = B\} \subset \underline{\mathcal{N}}_{v,\tau}$.

Our aim is to show that

$$B^2\mathcal{K}[\mathcal{S}^1_0(B)] - B^2\mathcal{K}[\mathcal{S}^6_{v,\tau}(B)], \tag{5.8}$$

equals the RHS of (5.5) (up to errors of the form $O(B^{-1})$, plus $o(\tau)$). (The difference in (5.8) is taken in the coordinates ϕ, θ defined over both spheres $\mathcal{S}^1_0(B), \mathcal{S}^6_{v,\tau}(B)$).

To do this, we seek to "move" from the first to the sixth sphere by successively moving along the intermediate spheres defined above.

Invoking Lemma 4.11, we find that:

$$|B^2\mathcal{K}[\mathcal{S}^1_0(B)] - B^2\mathcal{K}[\mathcal{S}^2_0(B)]| = O(B^{-1}), |B^2\mathcal{K}[\mathcal{S}^5_{v,\tau}(B)] - B^2\mathcal{K}[\mathcal{S}^6_{v,\tau}(B)]| = O(B^{-1}). \tag{5.9}$$

Next, our Assumption 2.8 on the space-time implies that:

$$|B^2\mathcal{K}[\mathcal{S}^5_{v,\tau}(B)] - B^2\mathcal{K}[\mathcal{S}^4_{v,\tau}(B)]| = o(\tau) + O(B^{-1}). \tag{5.10}$$

The above combined then show that:

[18] Recall that $\underline{\mathcal{N}}_0$ is the original null hypersurface. Thus these are 1-parameter families of affine vector fields over the *original* hypersurface.

$$B^2\mathcal{K}[\mathcal{S}^6_{v,\tau}(B)] - B^2\mathcal{K}[\mathcal{S}^1_{v,0}(B)] = B^2\mathcal{K}[\mathcal{S}^3_{v,\tau}(B)] - B^2\mathcal{K}[\mathcal{S}^2_{v,0}(B)] + B^2\mathcal{K}[\mathcal{S}^4_{v,\tau}(B)] - B^2\mathcal{K}[\mathcal{S}^3_{v,\tau}(B)] + O(B^{-1}) + o(\tau). \tag{5.11}$$

Thus it suffices to estimate the two differences of the RHS of the above, up to errors of the form $O(B^{-1})$ and $o(\tau)$.

We recall that the functions $tr\chi^{L'}$, $tr\chi^{L'_{v,\tau}}$ can be thought of as a scalar-valued functions over the null surface $\underline{\mathcal{N}}_0$; we can then consider their restriction to any sphere $\mathcal{S}^3_{v,\tau}$. This yields a scalar-valued functions over that sphere.

With this convention, recalling (4.26) we derive that:

$$B \cdot tr\chi^{L'_{v,\tau}}[\mathcal{S}^3_{v,\tau}(B)] = B(1 + 2\tau(\tilde{J}\tfrac{L}{v})'_\infty)tr\chi^{L'}[\mathcal{S}^2_0(B)] + 2\tau B^2\Delta_{\gamma_{\mathcal{S}^2_0(B)}}(\tilde{J}\tfrac{L}{v})'_\infty + o(\tau) + O(B^{-1}). \tag{5.12}$$

Now, we recall that γ^∞ is the metric induced "at infinity" on $\underline{\mathcal{N}}_0$ by the affine parameter $\tilde{\lambda}_0$. We apply formulas (4.53), (4.1) and recall the decay of the component ρ in (5.18) to the affine parameter $\tilde{\lambda}_0$ to derive that:

$$Btr\chi^{L'}[\mathcal{S}^2_0(B)] = 2\mathcal{K}[\gamma^\infty] + O(B^{-1}). \tag{5.13}$$

Combining the above two equations, we derive that:

$$B^2\mathcal{K}[\mathcal{S}^3_{v,\tau}(B)] = B^2\mathcal{K}[\mathcal{S}^2_0(B)] + 2\tau[\Delta_{\gamma^\infty} + 2\mathcal{K}[\gamma^\infty]](\tilde{J}\tfrac{L}{v})' + O(B^{-1}) + o(\tau). \tag{5.14}$$

Now we proceed to show

$$B^2\mathcal{K}[\mathcal{S}^4_{v,\tau}(B)] - B^2\mathcal{K}[\mathcal{S}^3_{v,\tau}(B)] = O(B^{-1}) + o(\tau) : \tag{5.15}$$

Let $\mathcal{H}_{v,\tau}$ be the (small, incomplete) future-directed null surface emanating from $\mathcal{S}^3_{v,\tau}$ in the direction of $L'_{v,\tau}$. Let $L'_{v,\tau}$ be the corresponding affine vector field on this surface, and $\underline{u}_{v,\tau}$ the corresponding affine parameter, with $\underline{u}_{v,\tau} = 0$ on $\mathcal{S}^3_{v,\tau}$. Let us denote by $\underline{L}^\sharp_{v,\tau}$ the transverse null vector field along $\mathcal{H}_{v,\tau}$ which is normal to the level sets of $\underline{u}_{v,\tau}$ so that $g(L'_{v,\tau}, \underline{L}^\sharp_{v,\tau}) = 2$. We then observe that by definition:

$$\mathcal{S}^4_{v,\tau} := \{\underline{u}_{v,\tau} = \tau\} \subset \mathcal{H}_{v,\tau}. \tag{5.16}$$

This, implies that

$$\begin{aligned} &B^2\chi^{L'_{v,\tau}}_{ab}[\mathcal{S}^4_{v,\tau}(B)](\underline{\chi}^{\underline{L}^\sharp_{v,\tau}})^{ab}[\mathcal{S}^4_{v,\tau}(B)] - B^2\chi^{L'_{v,\tau}}_{ab}[\mathcal{S}^3_{v,\tau}(B)](\underline{\chi}^{\underline{L}^\sharp_{v,\tau}})^{ab}[\mathcal{S}^3_{v,\tau}(B)] \\ &= B^2\nabla\!\!\!\!/_{L'_{v,\tau}}\{\chi^{L'_{v,\tau}}_{ab}[\mathcal{S}^3_{v,\tau}(B)](\underline{\chi}^{\underline{L}^\sharp_{v,\tau}})^{ab}[\mathcal{S}^3_{v,\tau}(B)]\} + o(\tau) \\ &= O(B^{-1}) + o(\tau). \end{aligned} \tag{5.17}$$

The last equation follows from the first and last formulas in (2.18) and the bounds in Lemma 4.4.

Furthermore, by our decay assumptions (2.8) on the curvature components (which are assumed on all $\underline{\mathcal{N}}_{v,\tau}$)

$$B^2\rho[\mathcal{S}^4_{v,\tau}(B)], B^2\rho[\mathcal{S}^3_{v,\tau}(B)] = O(B^{-1}). \tag{5.18}$$

Consequently, invoking equation (4.53), we derive (5.15).

This completes the proof of Proposition 5.3. □

Thus the main point that remains is to express $(\tilde{J}^{\underline{L}}_v)'_\infty$ in terms of the function e^v. We take this up in the next subsection.

5.2 Variations of the null surfaces and Jacobi fields.

The goal for the remainder of this section is to prove:

Proposition 5.6. *There exist* fixed *functions (i. e. independent of v)* $f^i \in \mathcal{C}^2(\mathcal{S}_0), i = 0,1,2,3$ *with* $|f^i|_{\mathcal{C}^2(\mathcal{S}_0)} \le \delta$ *so that:*

$$(\tilde{J}^{\underline{L}}_v)'_\infty = (1+f^0)\{-2\Delta_{\mathcal{S}_0}e^v + f^1\partial_1 e^v + f^2\partial_2 e^v + f^3 e^v\}. \tag{5.19}$$

This will be proven by combining Propositions 5.7 and 5.8 below, see formula (5.6).

We proceed to show:

Proposition 5.7. *We claim that there exist fixed functions (i. e. functions independent of v)* $k^1 \in O^\delta_2(\lambda^{-1}), k^2 \in O^\delta_2(\lambda^{-1}), \tilde{f}^{\underline{L}} \in O^\delta_2(\lambda),\ \tilde{f}^1, \tilde{f}^2 \in O^\delta_2(\lambda)$,[19] *so that:*

$$\begin{aligned} &J^L = e^v, J^1 = k^1 e^v, J^2 = k_2 e^v,\\ &(J^{\underline{L}})' = -2\Delta_{\mathcal{S}_0}e^v + \sum_{C=1}^{2}\tilde{f}^C e_C[e^v] + \tilde{f}^{\underline{L}}e^v. \end{aligned} \tag{5.20}$$

Moreover, the limit $lim_{\lambda\to\infty}\lambda^{-1}J^{\underline{L}}(\lambda)$ *exists, and is a* δ*-small (in* $\mathcal{C}^2$*) perturbation of the operator* $\Delta_{\mathcal{S}_0}[e^v]$.

Proof: By the construction above, (3.8) and the definition (2.15) we readily find that for each point $Q \in \mathcal{S}_0$ (with $\lambda^Q(Q) = 1$):

$$\begin{aligned} &J^L(1) = e^v, J^1(1), J^2(1) = 0,\\ &J^{\underline{L}}(1) = e^v|\hat{\chi}|^2[\mathcal{S}_0]\{-2\rho[\mathcal{S}_0] - 2div\zeta[\mathcal{S}_0] - 2|\zeta[\mathcal{S}_0]|^2 + \hat{\chi}[\mathcal{S}_0]\underline{\hat{\chi}}[\mathcal{S}_0]\}^{-1} \end{aligned} \tag{5.21}$$

[19] Recall that the vector fields e_1, e_2 below correspond to $\lambda^{-1}\partial_{\phi^1}, \lambda^{-1}\partial_{\phi^2}$.

Moreover, the requirement that J corresponds to a variation by *null* geodesics implies that $g(\underline{L}, \nabla_{\underline{L}} J) = 0$; this in turn forces $(J^L)'(1) = 0$. We also have by construction $(J^A)'(1) = 0$, for $A = 1, 2$. Lastly, we have derived in (3.10) that

$$(J^{\underline{L}})'(1) = -2\Delta_{\mathcal{S}_0} e^v + O_2^\delta(1) \cdot \nabla e^v - (2\rho[\mathcal{S}_0] + O_2^\delta(1))e^v. \tag{5.22}$$

Jacobi equations. We study the Jacobi equation relative to the frame $\underline{L}, e_1, e_2, L$. We recall the Jacobi equation:

$$(\nabla_{\underline{L}\underline{L}} J)^B = -R_{\underline{L}A\underline{L}}{}^B J^A. \tag{5.23}$$

In order to solve for the Jacobi field, we first need to express the LHS of the above in terms of $\nabla_{\underline{L}}$ derivatives of the various components, relative to the frame $\underline{L}, L, e^1, e^2$. Recalling the evolution equations (4.8), (4.5), (4.1) we calculate:[20]

$$\nabla_{\underline{L}\underline{L}}J = \nabla_{\underline{L}\underline{L}}(J^{\underline{L}}\underline{L} + J^L L + \sum_{A=1}^{2} J^A e_A)$$

$$= (J^{\underline{L}})''\underline{L} + (J^L)''L + \sum_{A=1}^{2}(J^A)''e_A + 2[(J^L)'\nabla_{\underline{L}}L + \sum_{A=1}^{2}(J^A)'\nabla_{\underline{L}}e_A]$$

$$+ J^L\nabla_{\underline{L}}(\nabla_{\underline{L}}L) + \sum_{A=1}^{2} J^A\nabla_{\underline{L}}(\nabla_{\underline{L}}e_A)$$

$$= (J^{\underline{L}})''\underline{L} + (J^L)''L + \sum_{A=1}^{2}(J^A)''e_A + 2(J^L)'\zeta^A e_A$$

$$+ 2\sum_{A=1}^{2}(J^A)'(-\frac{1}{\lambda}e_A + (\underline{\chi}^\sharp_A)^B e_B + \zeta_A\underline{L})$$

$$+ J^L\nabla_{\underline{L}}[\zeta^A e_A] + \sum_{A=1}^{2} J^A\nabla_{\underline{L}}(-\frac{1}{\lambda}e_A + (\underline{\chi}^\sharp_A)^B e_B + \zeta_A\underline{L})$$

$$= (J^{\underline{L}})''\underline{L} + (J^L)''L + \sum_{A=1}^{2}(J^A)''e_A + 2(J^L)'\zeta^A e_A$$

$$+ 2\sum_{A=1}^{2}(J^A)'(-\frac{1}{\lambda}e_A + (\underline{\chi}^\sharp_A)^B e_B + \zeta_A\underline{L})$$

$$+ J^L[\nabla_{\underline{L}}\zeta^A e_A - \frac{1}{\lambda}\zeta^A e_A + \zeta^A(\underline{\chi}^\sharp)^B_A e_B + \zeta^A\zeta_A\underline{L}] + \sum_{A=1}^{2} J^A((\nabla_{\underline{L}}(\chi^\sharp_A)^B e_B + \zeta'_A\underline{L})$$

$$+ \sum_{A=1}^{2} J^A[\frac{2}{\lambda^2}e_A - \frac{2}{\lambda}(\underline{\chi}^\sharp_A)^B e_B - \lambda^{-1}\zeta_A\underline{L} + (\underline{\chi}^\sharp_A)^B(\underline{\chi}^\sharp_B)^C e_C + (\underline{\chi}^\sharp_A)^B\zeta_B\underline{L}]$$

$$= (J^{\underline{L}})''\underline{L} + (J^L)''L + \sum_{A=1}^{2}(J^A)''e_A + (J^L)'\zeta^A e_A$$

$$+ 2\sum_{A=1}^{2}(J^A)'((\frac{2a\not\gamma^B_A + h^B_A}{2\lambda^2})e_B + \zeta_A\underline{L})$$

$$+ J^L[(\zeta^A)'e_A - \frac{1}{\lambda}\zeta^A e_A + \zeta^A(\underline{\chi}^B)_A e_B + \zeta^A\zeta_A\underline{L}]$$

$$+ \sum_{A=1}^{2} J^A((\frac{(a\not\gamma_{AB})'}{2\lambda^2} + \frac{(h^B_A)'}{\lambda^2})e_B + (\zeta_A)'\underline{L})$$

$$+ \sum_{A=1}^{2} J^A[-\frac{2}{\lambda}(\frac{a}{2\lambda^2}e_A + \frac{h^B_A}{\lambda^2}e_B) + \frac{a^2}{4\lambda^4}e_A + \frac{h^B_A h^C_B}{\lambda^4}e_C + (\underline{\chi}^\sharp_A)^B\zeta_B\underline{L}] - \frac{\zeta_A}{\lambda}\underline{L} \tag{5.24}$$

[20]Recall that $'$ stands for the regular $\frac{d}{d\lambda}$-derivative of scalar-valued quantities.

In short, using the equations (4.22) and (4.1), we derive the equation:

$$\begin{aligned}
\nabla_{\underline{L}\underline{L}}J &= (J^{\underline{L}})''\underline{L} + (J^L)''L + \sum_{A=1}^{2}(J^A)''e_A + (J^L)'\sum_{A=1}^{2}O_2^\delta(\lambda^{-2})e_A \\
&+ \sum_{A,B=1}^{2}O_2^\delta(\lambda^{-2})(J^A)'e_B + \sum_{A=1}^{2}O_2^\delta(\lambda^{-2})(J^A)'\underline{L} \\
&+ \sum_{A=1}^{2}J^L O_2^\delta(\lambda^{-3})e_A + \sum_{A,B=1}^{2}O_2^\delta(\lambda^{-3})J^A e_B + \sum_{A=1}^{2}J^A O_2^\delta(\lambda^{-3})\underline{L}
\end{aligned} \tag{5.25}$$

Now, setting $B = L$ in (5.23) we derive:

$$(\nabla_{\underline{L}\underline{L}}J)^L = R_{a\underline{L}\underline{L}}{}^L J^a = R_{a\underline{L}\underline{L}\underline{L}}J^a = 0. \tag{5.26}$$

Refer to equation (5.25). Observe that the coefficient of L is precisely $(J^L)''$. Thus we derive:

$$(J^L)''(\lambda) = 0$$

Therefore, using the initial conditions $J^L(1) = e^v$, $(J^L)'(1) = 0$ at $\lambda = 1$ we find:

$$J^L = e^v \tag{5.27}$$

We now consider the B-components of the Jacobi equation, with $B = 1, 2$. Recalling that $J^L = e^v$ we derive:

$$\begin{aligned}
&(J^B)'' + \sum_{A=1}^{2}[O_2^\delta(\lambda^{-2})(J^A)' + O_2^\delta(\lambda^{-3})J^A] + O_2^\delta(\lambda^{-3})e^v \\
&= -R^B_{\underline{L}S\underline{L}}J^S = R_{\underline{L}L\underline{L}b}J^L + R^B_{\underline{L}A\underline{L}}J^A = R^B_{\underline{L}L\underline{L}}e^v + R^B_{\underline{L}A\underline{L}}J^A \\
&= \underline{\beta}^B e^v + \underline{\alpha}^B_A J^A.
\end{aligned} \tag{5.28}$$

Moreover, since (5.28) is a linear ODE with trivial initial data, (since $J^B(1) = (J^B)'(1) = 0$ for $B = 1, 2$ as noted above), we derive that the solution J^1, J^2 depend *linearly* on the parameter e^v. In particular there exist two functions $\vartheta^1(\lambda), \vartheta^2(\lambda)$ so that:

$$J^B(\lambda) = \vartheta^B(\lambda)e^v. \tag{5.29}$$

Then, using the explicit form of the equation above to express it as a system of first order ODEs, we readily find that $\vartheta_b(\lambda) \in O_2^\delta(\lambda^{-1})$.

Finally considering the component $B = \underline{L}$ in the Jacobi equation, along with (5.25), (5.29), $J^L = e^v$, we find:

$$\begin{aligned}
&(J^{\underline{L}})'' + O_2^\delta(\lambda^{-3})e^v \\
&= -\sum_{S=1}^{2}R_{\underline{L}S\underline{L}}{}^{\underline{L}}J^S - R_{\underline{L}L\underline{L}L}J^L = -\sum_{S=1}^{2}R_{\underline{L}S\underline{L}}{}^{\underline{L}}J^S - 4\rho e^v
\end{aligned} \tag{5.30}$$

Using (2.8), along with the fact that $u = 2m\lambda$ by construction, we derive:

$$(J^{\underline{L}})''(\lambda) = [\frac{8m}{u^3} + O_2^\delta(\lambda^{-3})]e^v = [\frac{1}{m^2\lambda^3} + O_2^\delta(\lambda^{-3})]e^v. \quad (5.31)$$

Integrating the above in λ we derive:

$$(J^{\underline{L}})'(\lambda) = (J^{\underline{L}})'(1) + \frac{1}{m^2}\int_1^\lambda t^{-3}dt + O_2^\delta(\lambda^{-2}). \quad (5.32)$$

Therefore using the initial conditions (5.22), we find that:

$$lim_{\lambda\to\infty}(J^{\underline{L}})'(\lambda) = -2\Delta_{\mathcal{S}_0}e^v + \sum_{A=1}^{2} O_2^\delta(1)\partial_{\phi^A}e^v + O_2^\delta(1)e^v. \quad (5.33)$$

This precisely yields the last equation in (5.20). □

5.3 Jacobi fields and the first variation of Weyl curvature.

Recall (4.40). Our aim here is to show that:

Proposition 5.8. *There exist functions $n^1(\phi,\theta), n^2(\phi,\theta), n^3(\phi,\theta) \in O_2^\delta(\mathcal{S}_\infty)$ so that:*[21]

$$\dot{\varphi}_v = n^1\cdot[\Delta_{\mathcal{S}_0}e^v + n^2\sum_{i=1}^{2}\nabla_i e^v] + n^3 e^v.$$

Proof of Proposition 5.8: We can now use our explicit evaluation of the Jacobi field J (corresponding to the variation ∂_τ in the notation of (5.4)) obtained in the previous subsection. In particular, our first aim is to determine the dependence of the quantities

$$\partial_\tau|_{\tau=0}\not\gamma^{AB} := J(\not\gamma^{AB}), \partial_\tau|_{\tau=0}\underline{\alpha}_{AB} := J(\underline{\alpha}_{AB})$$

on the function v. We will show that they depend as in (4.3), and we in fact determine the parameters in the RHSs of that equation. Then, Lemma 4.3 yields an explicit formula for $\dot{a}_v(\lambda)$.

We first calculate the first term above. We start by observing that:

$$J\not\gamma^{AB} = -\sum_{C,D=1}^{2} J(\not\gamma_{CD})\not\gamma^{AC}\not\gamma^{BD}.$$

And then, recalling that $J\lambda = 0$ and $e^A\lambda = 0$ by construction, we calculate:

[21] Recall that O_2^δ is a slight abuse of notation and refers to functions defined over $\mathcal{S}_0$ (or $\mathcal{S}_\infty$, via the identification of points on the same null generators on $\underline{\mathcal{N}}_0$). It requires the function and to rotational derivatives ∂_{ϕ^i} to be bounded by δ.

$$J\not{\gamma}_{AB} = Jg(e_A, e_B) = g(\nabla_J e_A, e_B) + g(e_A, \nabla_J e_B) = \lambda^{-1} g(\nabla_J \partial_{\phi^A}, e_B)$$
$$+ \lambda^{-1} g(e^A, \nabla_J \partial_{\phi^B}) = \lambda^{-1} g(\nabla_{\phi^A} J, e_B) + \lambda^{-1} g(e^A, \nabla_{\phi^B} J).$$

To pursue this calculation, we express the Jacobi field J in terms of the frame $\underline{L}, L, e_1, e_2$:

$$J = J^{\underline{L}} \underline{L} + J^L L + \sum_{C=1}^{2} J^C e_C.$$

Thus:

$$J\not{\gamma}_{AB} = \lambda^{-1} \sum_{C=1}^{2} (\nabla_{\phi^A} J^C) g(e_C, e_B) + \lambda^{-1} \sum_{C=1}^{2} (\nabla_{\phi^B} J^C) g(e_A, e_C) + \underline{\chi}_{AB} J^{\underline{L}} + \chi_{AB} J^L$$
$$+ \sum_{C=1}^{2} J^C [g(\nabla_A e_C, e_B) + g(e_A, \nabla_B e_C)]. \tag{5.34}$$

In view of (4.2), and the formulas for the other components of J that we found above, in conjunction with (4.2), we derive that there exist functions $f^{AB}(\lambda) \in O_2^1(1)$, $r \in O_2^\delta(\lambda^{-1})$ so that:

$$\dot{\not{\gamma}}^{AB} = J\not{\gamma}_{AB} = f^{AB}(\lambda)[\mathcal{W}[e^v] + re^v], \tag{5.35}$$

where $\mathcal{W}[e^v]$ stands for the RHS of (5.33). Note that the above shows that the first assumption of Lemma 4.3 is fulfilled.

Next, we calculate the variation of the curvature component $\underline{\alpha}_{AB}$. We claim:

Lemma 5.9. *There exist functions $f^1_{AB}(\lambda), f^2_{AB}(\lambda) \in O_2^\delta(\lambda^{-3-\epsilon})$, $y^C_{AB}(\lambda) \in O_2^\delta(\lambda^{-3-\epsilon})$ (independent of v) so that for each $A, B \in \{1, 2\}$:*

$$\dot{\underline{\alpha}}_{AB} = f^1_{AB}\{\Delta_{\mathcal{S}_0} e^v - \rho e^v\} + f^2_{AB} e^v + y^C_{AB} e_C(e^v). \tag{5.36}$$

Observe that the above implies that the second assumption of Lemma 4.3 is fulfilled. Thus, combining the above Lemma with (5.35), and then invoking Lemmas 4.3 and 4.9, Proposition 5.8 follows immediately. Thus, matters are reduced to showing Lemma 5.9.

Proof: We calculate for each $A, B \in \{1, 2\}$.

$$\dot{\underline{\alpha}}_{AB} := \partial_\tau|_{\tau=0}[\underline{\alpha}(e_A, e_B)] = J[\underline{\alpha}(e_A, e_B)] = J[R(\underline{L}, e_A, e_B, \underline{L})] = (\nabla_J R)(\underline{L}, e_A, e_B, \underline{L})$$
$$+ R(\nabla_J \underline{L}, e_A, e_B, \underline{L}) + R(\underline{L}, \nabla_J e_A, e_B, \underline{L}) + R(\underline{L}, e_A, \nabla_J e_B, \underline{L}) + R(\underline{L}, e_A, e_B, \nabla_J \underline{L}). \tag{5.37}$$

Observe that by our decay assumptions (2.8), (2.9), (2.10) on the derivatives of the Weyl curvature components, as well as the bounds (5.27), (5.29), (5.33) that we have derived, we find:

$$
\begin{aligned}
&(\nabla_J R)(\underline{L}, e_A, e_B, \underline{L}) = J^{\underline{L}} \nabla_{\underline{L}} \underline{\alpha}_{AB} + J^L \nabla_L \underline{\alpha}_{AB} + J^C \nabla_C \underline{\alpha}_{AB} \\
&= O_2^\delta(\lambda^{-3-\epsilon})\{\Delta_{\mathcal{S}} e^v - 2\zeta^A \nabla_A e^v + [\frac{1}{4} tr\chi tr\underline{\chi} - \rho + \frac{1}{2}\hat{\chi} \cdot \hat{\underline{\chi}} - div\zeta - |\zeta|^2][\mathcal{S}_0] e^v\} \\
&+ O_2^\delta(\lambda^{-3-\epsilon}) e^v
\end{aligned}
\tag{5.38}
$$

In order to calculate the remaining terms in the RHS of (5.37), it suffices to calculate the second and third terms in the RHS; the fourth and fifth follow in the same way. We first consider the first term. By construction J and $\underline{L}$ commute. Thus:

$$
\begin{aligned}
&R(\nabla_J \underline{L}, e_A, e_B, \underline{L}) = R(\nabla_{\underline{L}} J, e_A, e_B, \underline{L}) = (J^{\underline{L}})' R(\underline{L}, e_A, e_B, \underline{L}) \\
&+ \sum_{C=1}^{2} (J^C)' R(e_C, e_A, e_B, \underline{L}) \\
&+ \sum_{C=1}^{2} J^C R(\nabla_{\underline{L}} e_C, e_A, e_B, \underline{L}) + J^L R(\nabla_{\underline{L}} L, e_A, e_B, \underline{L}) \\
&- \sum_{C=1}^{2} [\frac{J^C}{\lambda} R(e_C, e_A, e_B, \underline{L}) + J^C \underline{\chi}_C^D R(e_D, e_A, e_B, \underline{L}) + \zeta^C R(\underline{L}, e_A, e_B, \underline{L})
\end{aligned}
\tag{5.39}
$$

(We have used the fact that $(J^L)' = 0$). Thus, using (2.8), (2.9), (2.10), we derive that in the notation of Lemma 5.9:

$$
R(\nabla_J \underline{L}, e_A, e_B, \underline{L}) = (y_{AB}^C e_C) e^v + f_{AB}^2 e^v,
\tag{5.40}
$$

in the notation of Lemma 5.9. To evaluate the second term, note that since $J\lambda = 0$, $[J, \partial_\phi] = [J, \partial_\theta] = 0$ then J and $e^A, A = 1, 2$ also commute. Thus, invoking (4.9), and using the fact that $R(\underline{L}, \underline{L}, e^B, \underline{L}) = 0$ we find:

$$
\begin{aligned}
&R(\underline{L}, \nabla_J e_A, e_B, \underline{L}) = R(\underline{L}, \nabla_{e_A} J, e_B, \underline{L}) = \sum_{C=1}^{2} (e_A J^C) R(\underline{L}, e_C, e_B, \underline{L}) \\
&+ (e_A J^L) R(\underline{L}, L, e_B, \underline{L}) \\
&+ J^{\underline{L}} R(\underline{L}, \nabla_{e_A} \underline{L}, e_B, \underline{L}) + J^L R(\underline{L}, \nabla_{e_A} L, e_B, \underline{L}) + \sum_{C=1}^{2} J^C R(\underline{L}, \nabla_{e_A} e_C, e_B, \underline{L}).
\end{aligned}
\tag{5.41}
$$

Thus, using (5.37), the decay assumptions on the curvature coefficients (2.8), (2.9), (2.10), along with the bounds (5.27), (5.29), (5.33) we have obtained on the components of J (in the previous subsection), we derive:

$$R(\underline{L}, \nabla_J e_A, e_B, \underline{L}) = (y^C_{AB} e_C) e^v + f^2_{AB} e^v; \tag{5.42}$$

thus combining (5.38), (5.40), (5.42) above, Lemma 5.9 follows. This concludes the proof of Proposition 5.8. □

6 Finding the desired null hypersurface, via the implicit function theorem.

We now recall Propositions 5.3 and 5.6. Denote the RHS of (5.19) by $\mathcal{L}[e^v]$. We consider $\mathcal{L}$ acting on the space $W^{4,p}(\mathcal{S}_0)$, for a fixed $p > 2$.

We first carefully define the map to which we will apply the implicit function theorem. Recall that $\tau \cdot e^v = \omega$, and denote the sphere $\mathcal{S}_{v,\tau} \subset \mathcal{N}[\mathcal{S}_0]$ by $\mathcal{S}_\omega$. We have let $\underline{\mathcal{N}}_\omega := \underline{\mathcal{N}}_{v,\tau}$, and $\mathcal{S}'_\omega := \mathcal{S}'_{v,\tau} \subset \underline{\mathcal{N}}_\omega$. On each $\underline{\mathcal{N}}_\omega$, we recall the luminosity parameter s_ω with $\{s_\omega = 1\} = \mathcal{S}'_\omega$.

We define a natural map $\Psi : \underline{\mathcal{N}}_\omega \to \mathcal{S}_0$ which maps any point $Q \in \underline{\mathcal{N}}_\omega$ to the point $\Psi(Q) \in \mathcal{S}_0$ so that the null generator of $\mathcal{N}[\mathcal{S}_0]$ though $\Psi(Q)$ and the null generator of $\underline{\mathcal{N}}_\omega$ through Q intersect (on $\mathcal{S}_\omega$)–see Figure 2 where now $\mathcal{S}_{v,\tau} = \mathcal{S}_\omega$. For conceptual convenience, let us consider the "boundary at infinity" $\partial_\infty \underline{\mathcal{N}}_\omega$ of $\underline{\mathcal{N}}_\omega$, which inherits coordinates ϕ, θ from $\mathcal{S}_0$ by the map Ψ. We let $\mathcal{S}^\infty_\omega := \partial_\infty \underline{\mathcal{N}}_\omega$.

We then consider the operator:

$$\Phi^B : W^{4,p}(\mathcal{S}_0) \to L^p(\{s = B \subset \underline{\mathcal{N}}_\omega\})$$

defined via:

$$\Phi^B(\omega) := B^2 \mathcal{K}(\{s_\omega = B\} \subset \underline{\mathcal{N}}_\omega) \tag{6.1}$$

We also define:

$$\Phi^\infty : W^{4,p}(\mathcal{S}_0) \to L^p(\mathcal{S}^\infty_\omega), \Phi^\infty(\omega) := lim_{B \to \infty} \Phi^B(\omega) \tag{6.2}$$

We now claim that:

Proposition 6.1. *Choose any $p > 2$; the map $\Phi^\infty[\omega] : W^{4,p}(\mathcal{S}_0) \to L^p(\mathcal{S}^\infty_\omega)$, is well-defined and $\mathcal{C}^1$ for all $\omega \in \mathcal{B}(0, 10^{-1}M^{-1}) \subset W^{4,p}(\mathcal{S}_0)$, where $M(m) > 1$ is a precise constant that will appear in the proof.*[22] *Furthermore, letting*

$$\dot{\Phi}^{\omega,\infty} : W^{4,p}(\mathcal{S}_\omega) \to L^p(\mathcal{S}^\infty_\omega)$$

be the linearization around any $\omega \in \mathcal{B}(0, 10^{-1}M^{-1}) \subset W^{4,p}(\mathcal{S}_0)$, we claim that for any $\omega, \omega' \in \mathcal{B}(0, 10^{-1}M^{-1})$, $\omega, \omega' \geq 0$ we have:[23]

$$||\dot{\Phi}^{\omega,\infty} - \dot{\Phi}^{\omega',\infty}||_{W^{4,p}(\mathcal{S}_\omega) \to L^p(\mathcal{S}^\infty)} \leq M ||\omega - \omega'||_{W^{4,p}(\mathcal{S}_0)}. \tag{6.3}$$

[22] $\mathcal{B}(0, 10^{-1}M^{-1})$ stands for the ball of radius $10^{-1}M^{-1}$ in $W^{4,p}(\mathcal{S}_0)$.

[23] Note that the operators $\dot{\Phi}^{\omega,\infty}, \dot{\Phi}^{\omega',\infty}$ take values over different spheres, ($\mathcal{S}^\infty_\omega$ and $\mathcal{S}^\infty_{\omega'}$ respectively). Nonetheless we may compare and subtract the operators below via the natural map from these spheres to $\mathcal{S}_0$ as described above.

Moreover, for $0 < \delta << m$ small enough[24] *we claim that there exists a constant $C > 0$ with $|C-1|$ being small, and a function $\omega \in \mathcal{B}(0, 10^{-1}M^{-1}) \subset W^{4,p}(\mathcal{S}_0)$ so that:*

$$\Phi^\infty(\omega) = C. \tag{6.4}$$

Note that the second part of (6.4) implies Theorem 1.3 above. As we will note in the proof, there are in fact many such ω's–in fact a 3-dimensional space of such functions.

Proof of Proposition 6.1: We will be showing this by an application of the implicit function theorem. Returning to expressing $\omega = \tau v$, we have derived, combining Lemma 3.2 with Propositions 5.3 5.6, and the conventions on $\mathcal{L}[e^v]$ at the beginning of this section, that:

$$\frac{d}{d\tau}|_{\tau=0}\Phi^\infty(e^v\tau) := \dot{\Phi}^\infty[e^v] = (\Delta_{\gamma^\infty} + 2\mathcal{K}[\gamma^\infty]) \circ \mathcal{L}[e^v]. \tag{6.5}$$

In fact, as noted in Remark 5.1 the same formula holds for the variation around any sphere $\mathcal{S}'_\omega$, for all $\omega(\mathcal{S}^2) \geq 0$ with $||\omega||_{\mathcal{C}^2} \leq 10^{-1}m$. We denote the corresponding linearization as follows:

$$\frac{d}{d\tau}|_{\tau=0}\Phi^{\omega,\infty}(e^v\tau) := \dot{\Phi}^{\omega,\infty}[e^v] = (\Delta_{\gamma^\infty_\omega} + 2\mathcal{K}[\gamma^\infty_\omega]) \circ \mathcal{L}_\omega[e^v]. \tag{6.6}$$

Observe that the Sobolev embedding theorem $W^{4,p}(\mathcal{S}^2) \subset \mathcal{C}^3(\mathcal{S}^2)$ then implies that if we choose M to be the norm of this embedding times m, then $\Phi^\infty[\omega]$ is a well-defined map for all $\omega \geq 0$, $\omega \in \mathcal{B}(0, 10^{-1}M^{-1})$. Also (6.3) follows by keeping track of the transformation laws for all the geometric quantities that appear as coefficients in the RHS of (6.5), along with the embedding $W^{4,p}(\mathcal{S}^2) \subset \mathcal{C}^3(\mathcal{S}^2)$. Thus, the control of the modulus of continuity in (6.3) follows by keeping track of the transformation laws of these quantities under changes of ω. We omit the details on these points, as they are fairly standard.

To prove (6.4), we need to understand the mapping properties of $\dot{\Phi}^{\omega,\infty}$ for all $\omega \in \mathcal{B}(0, 10^{-1}M^{-1})$:

Observe that for all $\omega \in \mathcal{C}^2(\mathcal{S}_0)$ $||\omega||_{\mathcal{C}^2(\mathcal{S}_0)} \leq 10^{-1}m$ the operator $\mathcal{L}_\omega$ is a (non-self-adjoint) perturbation of the Laplacian $\Delta_{\mathbb{S}^2}$ on the round 2-sphere:

$$\mathcal{L}_\omega = \Delta_{\mathbb{S}^2} + \epsilon^{ij}\partial_{ij} + \beta^i\partial_i + \eta.$$

where the coefficients $\epsilon^{ij}, \beta^i, \eta$ (all of which depend on ω) are small and bounded in the $\mathcal{C}^3$ norm. This (by the continuous dependence of the spectrum of second order elliptic operators on the operator coefficients) implies that there exist co-dimension-1 subspaces $\mathcal{A}_\omega \subset W^{4,p}(\mathcal{S}_0)$, $\mathcal{T}_\omega \subset W^{2,p}(\mathcal{S}_0)$ so that the restriction $\mathcal{L}^r_\omega$ of $\mathcal{L}$ from $\mathcal{A}_\omega$ into $\mathcal{T}_\omega$ is one-to-one and onto, and moreover $\mathcal{L}_\omega$ is coercive,

$$||\mathcal{L}^r_\omega(\psi)||_{W^{2,p}} \geq 10^{-1}m^{-1}||\psi||_{\mathcal{A}_\omega}.$$

[24] Recall that $\delta > 0$ captures the closeness of the underlying space-time and $\underline{\mathcal{N}}$ to the ambient Schwarzschild space-time, around a shear-free null surface. In particular recall that $||\mathcal{K}^\infty(\mathcal{N}_0) - 1||_{W^{2,p}}$ is bounded by δ.

Thus in particular the inverse $(\mathcal{L}^r_\omega)^{-1}:\mathcal{T}\to\mathcal{A}_\omega$ is bounded:

$$||(\mathcal{L}^r_\omega)^{-1}\phi||_{W^{4,p}}\le 10m||\phi||_{W^{2,p}},\forall\phi\in\mathcal{T}_\omega.$$

Note that $\mathcal{A}_\omega$, $\mathcal{T}_\omega$ are annihilated by elements ψ_ω, ϕ_ω in the dual space which are δ-close (in the suitable norms) to the constant function 1 (this is because of the closeness of the operator $\mathcal{L}$ to the round Laplacian $\Delta_{\mathbb{S}^2}$):

$$\psi\in\mathcal{A}_\omega \iff \{<\psi,\psi_\omega>=0 \text{ for some fixed } \psi_\omega\in(W^{4,p})^*, ||1-\psi_\omega||_{(W^{4,p})^*}\le\delta\}. \tag{6.7}$$

$$\psi\in\mathcal{T}_\omega \iff \{<\psi,\phi_\omega>=0 \text{ for some fixed } \phi_\omega\in(W^{2,p})^*, ||1-\phi_\omega||_{(W^{2,p})^*}\le\delta\}. \tag{6.8}$$

Now, observe that (6.6) implies that the effect on the Gauss curvature of $\mathcal{S}^\infty_\omega$ under perturbations of $\underline{\mathcal{N}}_\omega$ (with a first variation of ω by e^v) agrees with that induced by *conformally* varying the metric γ^∞_ω, with $\mathcal{L}_\omega[e^v]$ being the first variation of the conformal factor. In particular, by integrating over these variations, we derive that $\Phi^\infty(\omega)$ (i.e. the Gauss curvature on $\mathcal{S}^\infty_\omega:=\partial_\infty\underline{\mathcal{N}}_\omega$) agrees with the Gauss curvature of a metric

$$e^{2\Sigma(\omega)}\gamma^\infty,$$

where $\Sigma(\omega)$ (for $\omega=e^v$) is defined via:

$$\Sigma(\omega)=\int_0^1\mathcal{L}_{\tau\omega}[\omega]d\tau. \tag{6.9}$$

In particular, the metric at infinity γ^∞_ω on $\underline{\mathcal{N}}_\omega$ arises from the metric at infinity γ^∞ on $\underline{\mathcal{N}}_0$ by multiplying against a conformal factor $e^{2\Sigma(\omega)}$, with $\Sigma(\omega)$ defined via (6.9). Furthermore, note that $\Sigma(\omega)$ is in $W^{2,p}(\mathcal{S}_0)$ (with $W^{2,p}$ norm bounded by $10^{-1}mM$). Thus, it suffices to show that there exists an $\omega\in W^{4,p}(\mathcal{S}_0)$ so that:

$$\mathcal{K}[e^{\Sigma(\omega)}\gamma^\infty]=C, \tag{6.10}$$

for some constant $C\sim 1$. This will follow from the implicit function theorem. First, note that all the operators $\mathcal{L}_{\tau\omega}$ are small perturbations of $\Delta_{\mathbb{S}^2}$, as noted above. Next, to make the implicit function theorem applicable, we slightly generalize this by considering (for any $\mathcal{C}^3$-small function α) the operator

$$\Sigma_\alpha[\omega]:=\int_0^1\mathcal{L}_{\tau\alpha}[\omega]d\tau,$$

which is *still* a small (non-self-adjoint) perturbation of the Laplacian; we are interested in the case where $\alpha=\omega$. Then, the implicit function theorem implies the following two facts: First, together with the bound on the modulus of continuity of $\dot{\Phi}^{\omega,\infty}$ for all $\omega\in\mathcal{B}(0,10^{-1}M^{-1})$ implies that the image $\Sigma(\mathcal{B}(0,10^{-1}M^{-1}))$ is a codimension-1 smooth (open) submanifold in $W^{2,p}(\mathcal{S}_0)$, with radius bounded below by $100^{-1}M^{-1}$.

Secondly, using (6.8), it follows that $\tilde{\Sigma} := \Sigma(\mathcal{B}(1, 10^{-1}M^{-1}))$ intersects the space of dilations (in $W^{2,p}$) transversely: Given any $P \in \tilde{\Sigma}$, considering the segment $\text{seg}(P) := t \cdot P$, $t \in [1 - 10\delta, 1 + 10\delta]$. Then the segment $\text{seg}(P)$ intersects $\tilde{\Sigma}$ only at P. Moreover, the intersection is "almost normal", in the sense that $\phi_P(P) \sim 1$, for ϕ being the element in the dual space in (6.8) that annihilates the tangent space of Σ at P.

Now, given the solution to the uniformization problem for the metric γ^∞, note that there exists a 4-parameter space of functions $U_{\lambda,q}$ for which

$$\mathcal{K}[U_{\lambda,q}\gamma^\infty] = \lambda.$$

Here $\lambda \in \mathbb{R}_+$ and $q \in SO(3,1)$ (the conformal group of $\mathbb{S}^2$), normalized so that $U_{\lambda,q} = \lambda^{-1}U_{1,q}$ (since the scaling of a conformal factor corresponds to a scaling of the resulting Gauss curvature). Thus the space $U_{\lambda,q}$, $\lambda \in \mathbb{R}_+, q \in SO(3,1)$ *intersects* $\tilde{\Sigma}$ *transversely* along a smooth 3-dimensional submanifold. Choosing any element of this intersection provides a conformal factor $e^{\Sigma(\omega)}$ for (6.10) to hold.

The three dimensions of freedom thus essentially correspond to the three dimensions of the conformal group modulo isometries, $SO(3,1)/SO(3)$. They corresspond to the fact that we can capture all constant-curvature (up to the constant, which is not fixed here) metrics conformal to γ^∞, nearby γ^∞. This completes our proof. □

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