Research Statement – Artane Jérémie Siad

1 Introduction

My research interests are in number theory and, more specifically, in arithmetic statistics. In recent projects, I have focused on the distribution of class groups in families of number fields. Over the integers, every number can be uniquely expressed as a product of primes. For number fields in general (finite extensions of $\mathbb{Q}$ by roots of polynomials), however, this phenomenon can fail. This failure is measured by a finite abelian group called the *class group* of the field, whose size is its *class number*. The class number is 1 if and only if unique factorisation holds in the ring of integers. Statistically, we can think of the class group as a random variable valued in finite abelian groups. Understanding the distribution of this random variable over natural families of number fields counts as one of the deepest and most central questions in number theory. In addition to being interesting in its own right, class group statistics have applications to several problems. These include average rank and average size of Selmer groups of elliptic curves, the solubility of specific Diophantine equations (Thue equations coming from norm forms), and the existence of points on hyperelliptic curves [4, 5, 25, 7].

In 1801, Gauss introduced class groups in his *Disquisitiones Arithmeticae* [19], and posed what is recorded as the first question concerning their behaviour over families: are there infinitely many quadratic fields with class number 1? Despite this problem remaining open to this day, Gauss was nevertheless able to prove that there were infinitely many quadratic fields having odd class number. An application of my main theorem generalises this result of Gauss to fields of any even degree.

Theorem 1.1 (S., [27]). *Let $n \geq 4$ be an even integer. For each choice of signature (r_1, r_2), there are infinitely many fields of degree n and signature (r_1, r_2) that have odd class number.*

These types of results on the indivisibility of class numbers for infinitely many fields originate from Gauss, and include work of Davenport–Heilbronn for quadratic fields with class numbers indivisible by 3 [15], Bhargava for cubic fields with class number indivisible by 2 [3], Wiles for quadratic fields with class number indivisible by any prime $l \geq 5$ [29], and Ho–Shankar–Varma for fields of any odd degree with class number indivisible by 2 [20]. I also show that the statement of Theorem 1.1 holds even if we ask for the fields to satisfy certain prescribed sets of congruence conditions. These existence results are interesting because constructing even a single field satisfying a prescribed set of congruence conditions and with odd class number is usually very difficult. Theorem 1.1 comes from bounding the 2-torsion part of the class group of monogenic fields of even degree. This situation lies at the edge of current research on class group heuristics, to which we now turn.

Despite Gauss's original problem on whether there are infinitely many quadratic fields with class number 1 still being open, remarkable progress has been made in modern times towards providing a conjectural answer to the broad question of how class groups distribute over the family of all fields of a fixed degree and signature. In the 1980s, taking as a starting point work of Davenport–Heilbronn [15] and a great deal of computer-generated data, Cohen–Lenstra–Martinet–Malle used the philosophy that random objects should appear with frequency inversely proportional to the size of their automorphism groups to formulate precise conjectures describing the distribution of the p-power torsion part of the class group over the family of all fields a given degree and signature for "good" primes p [11, 13, 14, 12, 23]. A prime is called "good" if it is coprime to the fixed degree under consideration and "bad" otherwise. Despite remarkable progress on refining, understanding and streamlining these heuristics [34, 33, 1, 2, 22, 21], grey zones still exist where very little about the distribution of class groups is known (even numerically) and where the Cohen–Lenstra philosophy seems in need of revision.

The first grey zone concerns the distribution of class groups over more general families than the family of all number fields of a fixed degree and signature. Bhargava–Varma showed in [9, 10] that in the two cases where predictions arising from these heuristics have been verified, the averages remain the same when one imposes finitely many local conditions or even certain infinite sets of local conditions. Thus, the expectation is that the Cohen-Lenstra heuristics are unaffected by local conditions. As we shall see, this is not at all the case for global conditions. As such, determining the precise "law" which governs randomness in the class group of general families of fields is our first motivating question.

Problem 1.1. *How do class groups distribute over general families of number fields? Are there universality classes? If so, what determines membership to a universality class?*

The second grey zone for the Cohen–Lenstra heuristics concerns the situation at "bad" primes. There has been nothing proposed, so far, to describe the distribution of p-torsion in the class group for "bad" primes p. In particular, there is no prediction for the average number of the 2-torsion elements in the class group, narrow class group or oriented class group of number fields of even degree. The most serious reason for this gap seems to be the presence of *genus theory*. Indeed, it is unclear how the $1/\mathrm{Aut}(\cdot)$ philosophy of Cohen–Lenstra should mix with the "deterministic" input from genus theory.

Problem 1.2. *What is the distribution of the p^∞-torsion part of the class group for bad primes p? In particular, what role does genus theory play?*

The outline of my statement is as follows. In Section 2 and Section 3, I describe my thesis results and their proofs. These results form a contribution to our understanding of both of the problems above. In the first part of my thesis, I compute the average number of 2-torsion elements in the class group over a family defined by a global condition (monogenicity) and show that this average differs from that expected by the Cohen–Lenstra heuristics. This part touches on Problem 1.1, suggesting that monogenic fields belong to a universality class for which the Cohen–Lenstra heuristics need to be revised. In the second part of my thesis, I compute the average 2-torsion in a setting where genus theory and global conditions play a role. In particular, I find that genus theory is the only added complexity to consider for bad primes and quantify its contribution, which partially addresses Problem 1.2 for the family of monogenic fields. In Section 4, I describe work in progress to extend the geometry of numbers methods to count integral points in new situations. Lastly, in Section 5, I outline some of my future projects.

2 Results

In my thesis, I contributed to our understanding of Problems 1.1 and 1.2 by computing the average number of 2-torsion elements in the class group of monogenic fields of both even and odd degree. A number field is said to be monogenic if its ring of integers can be expressed as $\mathbb{Z}[\alpha]$ for some algebraic integer α. Monogenic fields cannot be specified by imposing local conditions. It was shown by Bhargava-Hanke-Shankar in [6] that the average number of non-trivial 2-torsion elements in the class group, $\mathrm{Avg}(\mathrm{Cl}_2, \mathcal{F})$, for $\mathcal{F}$ the family of monogenic cubic fields of a fixed signature was twice $\mathrm{Avg}(\mathrm{Cl}_2, \mathcal{F}')$ where $\mathcal{F}'$ is the family of all cubic fields of that signature. This is extremely surprising, as it suggests that the property of monogenicity, which is a global condition, tends to enlarge the class group! There is no reason to expect this a priori.

In the first part of my thesis, [26], I have shown that this phenomenon of monogenicity doubling the average number of non-trivial 2-torsion elements extends to number fields of any odd degree.

Theorem 2.1 (S. [26])**.** *Let $n \geq 3$ be an odd integer. Let $\mathfrak{R}$ be an acceptable family of monogenised fields of degree n and signature (r_1, r_2), ordered by naive height. The average number of 2-torsion elements in the class group of fields in $\mathfrak{R}$ satisfies the bound:*

$$\mathrm{Avg}(\mathrm{Cl}_2, \mathfrak{R}) \leq 1 + \frac{2}{2^{r_1+r_2-1}}$$

with equality conditional on a tail estimate. (Heuristically, average over all fields should be $1 + \frac{1}{2^{r_1+r_2-1}}$.)

Interestingly, a delicate interplay between the theory of integral quadratic forms and Tamagawa numbers of various special orthogonal groups in a mass formula accounts for the deviation of the monogenic average from that expected by Cohen–Lenstra–Martinet–Malle. This provides evidence that global conditions do affect the probabilistic assumptions of Cohen-Lenstra. It also opens up the possibility that one could refine the Cohen–Lenstra–Martinet–Malle style heuristic to handle global conditions by incorporating invariants associated with those conditions.

In the second part of my thesis, [27], I compute the average number of 2-torsion elements in the class group of monogenic fields of even degree at least 4. This situation is exciting because no heuristic is available to describe the distribution of the 2-torsion part over the full family. Our averages are, in fact, the first of their kind to be computed for a "bad" prime in degree at least 3.

Theorem 2.2 (S. [27])**.** *Let $n \geq 4$ be an even integer and $r_1 > 0$. Let $\mathfrak{R}$ be an acceptable family of monogenised fields (unramified at 2 and with local conditions at 2 given modulo 2) of degree n and signature (r_1, r_2), ordered by naive height. Let $r_p(\mathfrak{R})$ denote the density of fields of $\mathfrak{R}$ for which p ramifies as $(p) = \mathfrak{a}^2$ for some ideal $\mathfrak{a}$. The average number of 2-torsion elements in the class group of fields in $\mathfrak{R}$ satisfies the bound:*

$$\begin{aligned}\mathrm{Avg}(\mathrm{Cl}_2, \mathfrak{R}) \leq & \frac{1}{2} \prod_{p \equiv 1 \ mod\ 4} (1 + r_p(\mathfrak{R})) \left(\prod_{p \equiv 3 \ mod\ 4} (1 - r_p(\mathfrak{R})) + \prod_{p \equiv 3 \ mod\ 4} (1 + r_p(\mathfrak{R})) \right) \\ & + \frac{1 + 2\prod_{p \neq 2}(1 + r_p(\mathfrak{R}))}{2^{r_1 + r_2}}\end{aligned}$$

with equality conditional on tail estimate.

This result has several exciting new corollaries. Since it represents an averaged version of Gauss' genus theory, it allows us to conclude that there are infinitely many fields of any fixed even degree ≥ 4 and signature that have odd class number. This corollary is new (the analogous result in odd degree was proven in an amazing series of works by Bhargava [3], Bhargava–Varma [9], and Ho–Shankar–Varma [20]) and directly generalises Gauss' theorem that there are infinitely many quadratic fields with odd class number. Also, we obtain the average number of elements in the narrow class group. That average shows that there are infinitely many fields of any fixed even degree ≥ 4 and signature that have units of every signature.

Theorem 2.3. *Let $n \geq 4$ be an even integer. For each choice of signature (r_1, r_2) with $r_2 = 2$ if $n = 4$ and $r_2 > 0$ otherwise, there are infinitely many fields of degree n and signature (r_1, r_2) that have units of units of every signature.*

We can view this corollary as an existence statement for solutions to generalisations of the negative Pell equation.

3 Proof

We describe the proof in the even degree case. The strategy is to use Wood's parametrisation [31, 32] of 2-torsion ideal classes in rings associated to monic binary forms in terms of integral orbits for certain representations to reduce the question to an asymptotic counting problem in the geometry of numbers. The calculation reduces to counting integral orbits for a group acting on a variety. However, the exact choice of group (GL vs SL) and variety is important. A slight change in the choice of group and variety leads to a significant difference in the arithmetic information that is encoded. This feature is unique to the even degree case and arises because there are fields of even degree that have no unit of negative norm (in odd degree -1 always has norm -1).

When computing averages for 2-torsion in the class group, the relevant space is the set of $\mathrm{SL}_n^{\pm}(\mathbb{Z})$-orbits of pairs of integral symmetric matrices (A, B) with the constraint $\det(A) = (-1)^{\frac{n}{2}}$ *or* $\det(A) = -1 \cdot (-1)^{\frac{n}{2}}$. In order to apply the geometry of numbers, we borrow the idea of [6] and "linearise" the problem by noting that up to $\mathrm{SL}_n^{\pm}(\mathbb{Z})$, there are only finitely many equivalence classes of symmetric integral matrices of determinant $(-1)^{\frac{n}{2}}$ or $-1\cdot(-1)^{\frac{n}{2}}$. We denote this finite collection by $\mathscr{L}_{\mathbb{Z}}$. Counting $\mathrm{SL}_n^{\pm}(\mathbb{Z})$ orbits on the space of pairs (A,B) with the constraint $\det(A)=\pm1\cdot(-1)^{\frac{n}{2}}$ is thus reduced to counting $\mathrm{O}_{\mathrm{A}_0}(\mathbb{Z})$ orbits on the space of pairs (A_0,B). This count is then handled using geometry of numbers and we get the inequality of the main theorem conditional on a tail estimate. However, the arguments are complicated by the fact that we consider an infinite set of representations simultaneously and by the fact that the A_0 have maximal $\mathbb{Q}$ anisotropic subspaces of varying dimensions. The latter is a novel feature not present in literature.

The counts on the individual slices are then aggregated into the full count by summing over all the elements of $\mathscr{L}_{\mathbb{Z}}$. Calculating this sum is delicate because it relies on evaluating non-trivial masses for each of the A_0. The evaluation of these masses requires establishing equidistribution results at 2 and the Archimedean place. A feature unique to the even degree case appears when computing the local masses at the remaining places. There is extra local mass at the primes which are evenly ramified when $\det(A_0)=+1\cdot(-1)^{\frac{n}{2}}$ and less mass at primes congruent to 3 (mod 4) when $\det(A_0)=-1\cdot(-1)^{\frac{n}{2}}$! This behaviour encodes genus theory!

4 Work in progress

For many of the representations studied in arithmetic statistics, there is a wealth of virtually untapped information contained in the *cuspidal regions* of fundamental domains (see for example [8]). I am currently working on techniques to count in these *cuspidal regions* for the monogenic representations studied in my thesis in a joint project with Arul Shankar, Ashvin Swaminathan and Ila Varma. Our goal is to compute the average number of 2-torsion *ideals* (not ideal classes) of monogenic rings of any (even or odd) degree. We hope that the techniques developed in this project will generalise and allow us to discover the arithmetic information contained in most of the representations studied so far in arithmetic statistics.

Up to now, most results in arithmetic statistics over number fields have relied crucially on Davenport's Lemma to count integral points in fundamental domains lying in a vector space. In order to compute averages for families that are more general than the family of all fields, the available parametrisations force us to count integral points lying on varieties having potentially high co-dimension in the ambient space (if a linearisation trick is unavailable). In ongoing work with Iman Setayesh and Arul Shankar, we are adapting methods from ergodic theory and dynamics, particularly techniques of Eskin-McMullen [17] and of Eskin-Mozes-Shah [18], to count arithmetically interesting objects parametrised by integral points in fundamental domains lying on homogeneous varieties.

In addition to the projects above, I am currently working on two direct extensions to my thesis.

First, the results I have obtained have been for finite extensions of the base field $\mathbb{Q}$. It is natural to consider these averages for finite extensions of a global base field different from $\mathbb{Q}$. This presents some new difficulties but will potentially uncover some beautiful phenomena.

The second project is joint work with Ashvin Swaminathan. Swaminathan has discovered a new parametrisation of 2-torsion ideal classes for fields associated with binary forms with any fixed leading coefficient N in terms of the 2 torsion data for the "monicised" binary form. This allows him to apply my counting results to compute the 2-torsion in the class group of fields associated to binary forms with any fixed leading coefficient N. I am currently working with Swaminathan to determine the average 2-torsion in the class group of fields associated to binary forms whose the leading coefficient N is allowed to vary within ranges.

5 Future projects

5.1 Universality classes of class group distribtutions

A future project related to class groups would be to understand in a systematic way exactly how global conditions affect the distribution of class groups. The first phase of my research will be to work over function fields to develop and prove conjectures regarding universality classes of class group distributions and the key invariants of general families. Function fields are a natural place to start because they offer a natural definition of a general family of fields in terms of a family of curves over a base. Furthermore, the richer geometric structure of function fields means that in many cases, the function field analogues of notoriously hard problems in number theory become tractable. For example, the function field analogue of the Cohen-Lenstra heuristics for imaginary quadratic fields has been proven by Ellenberg-Venkatesh-Westerland in [16]. In my thesis, I found that the average number of 2-torsion elements in the class group of the special family consisting of monogenic fields differed from the value predicted by the heuristic of Cohen-Lenstra-Martinet-Malle. In my work, this deviation was explained by the interplay between a special symmetry group and an arithmetic input (the theory of quadratic forms and genus theory in even degree). This gives hope that a few arithmetic and geometric invariants will be able to detect universality classes for class group distributions.

These results would then be transferred to number fields via a large q-limit process. We do not, however, have a good notion of a thin family of number fields. Nonetheless, there is work of Sarnark, Shin and Templier, where they overcome this difficulty when "average class numbers of number fields" is replaced by "distribution of low-lying zeroes of L-functions" [24]. One type of canonical family appearing in their work corresponds to L-functions arising from families of automorphic representations on GL_n obtained by applying Langlands' principle of functoriality to certain collections of automorphic representations on a connected reductive algebraic group that can be isolated using the trace formula. This gives valuable ideas as to how to approach a refinement of Cohen-Lenstra for general families.

The third phase of my research will be to prove results towards these new conjectures for general families of number fields. Here, progress in the geometry of numbers will almost certainly be needed. The techniques developed in my ongoing work with Iman Setayesh and Arul Shankar will be a crucial first step to making progress towards computing average p-power torsion in the class group of general families of number fields.

5.2 The arithmetic statistics of degree 4 del Pezzo surfaces

A del Pezzo surface of degree four is a smooth surface in $\mathbb{P}^4$ given by the intersection of two quadrics. Let us denote the moduli scheme of these surfaces by $\mathcal{M} \subset \mathrm{Gr}_{\mathbb{Q}}(2, 15)$. Assuming a conjecture of Colliot-Thélène, it is known that a positive proportion of surfaces in $\mathcal{M}$ are globally soluble. Colliot-Thélène's conjecture states that the Hasse principle holds for some $X \in \mathcal{M}$ whenever a certain group known as the Brauer-Manin obstruction $(H^1(\mathbb{Q}, \mathrm{Pic}\,\overline{X}))$ is trivial. A variety is said to satisfy the Hasse principle if local solubility (the existence of points over all completions of $\mathbb{Q}$) implies global solubility. In [28], Browning combined work of Swinnerton-Dyer and recent results to show that the Brauer-Manin obstruction is generically empty over $\mathcal{M}$ and that a positive proportion of $\mathcal{M}$ is locally soluble, implying that positive proportion of $\mathcal{M}$ is globally soluble by Colliot-Thélène's conjecture. The problem is that this conjecture is wide open for the case of degree four del Pezzo surfaces, with the best results showing that it follows from other equally open conjectures, for instance, the finiteness of Ш for elliptic curves and Schinzel's hypothesis [30].

An interesting avenue of research would be to prove unconditionally that a positive proportion of curves in $\mathcal{M}$ are globally soluble (which would also be a proof of Colliot Thélène's conjecture for a positive proportion of degree 4 del Pezzo surfaces). I have been able to use Wood's parametrisation [32] to parametrise $\mathbb{Q}$-soluble del Pezzo surfaces in the moduli space $\mathcal{M}$ by finding a criterion on the δ parameters. The challenge is now to work over $\mathbb{Z}$ in order to apply the geometry of numbers to this new criterion. This is challenging because this criterion is of a type which has not been studied in arithmetic statistics so far.

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