A Hitchhiker's Guide to the Affine Grassmannian

Anne Dranowski

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University of Toronto

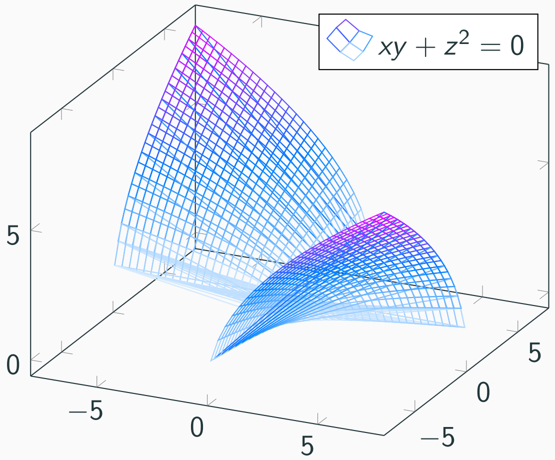


Figure 1: Stratum of Gr_{SL_2} (alias today's goal)

What is. . . an affine Grassmannian

Fix a complex reductive group G.

Write $O = \mathbb{C}[[t]]$ for power series (a PID) and K for Laurent series ($\mathrm{Frac}\, O$).

Definition A. $Gr = G(K)/G(O)$.

Fix a compact connected Lie group U.

Definition B. Write LU for maps $S^1 \to U$ and ΩU for those maps sending some fixed $z_0 \in S^1$ to $1 \in U$. $LU, \Omega U$ are groups under $f \cdot g(z) = f(z)g(z)$. Call them loop groups.

$\Omega U \cong LU/U$ by

1. constructing a map $LU \to \Omega U : f \mapsto f(z_0)^{-1} f$
2. considering the "loop rotation" action $w \cdot f(z) = f(wz)$ of S^1 on LU

Fix $n \in \mathbb{Z}$.

Let O be a PID and K its field of fractions.

Recall. A lattice L is a free O-module of the vector space K^n such that $K \otimes_O L \cong K^n$ as vector spaces.

Definition C1. $Gr = \{L : L \cong O^n\}$ (as O-modules).

This set carries a natural action of $GL_n K$. We recover **Definition A** by checking that the stabilizer of a given lattice is (isomorphic to) $GL_n O$.

Henceforth $G = GL_n K$, $O = \mathbb{C}[[t]]$ and $K = \mathbb{C}((t))$ and note the other-way-map

$$[g] \in G(K)/G(O) \mapsto O^n g^{-1}$$

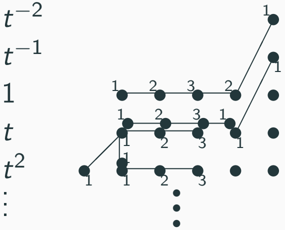
Fix $L \in Gr$.

Set

$$V_d(L) = \frac{t^{-d}L \cap O^n}{(tO)^n} \subseteq \frac{O^n}{(tO)^n} \cong \mathbb{C}^n$$

Lemma. $V_d(L)$ is increasing in d from 0 to $\mathbb{C}^n$.

$L \in Gr$ has a basis whose elements have some least power of t. Therefore multiplication by t^{-d} for $d > 0$ has the effect of pulling L over O^n.



Corollary. *Gr* can be written as a union of finite dimensional schemes.

Reason. *Gr* is a union of

$$Gr[a,b] = \{t^b O^n \subseteq L \subseteq t^a O^n\} \quad (a \leq b)$$

which can be identified with closed subschemes in some $Gr(k,(b-a)n)$ since $t^a O^n / t^b O^n \cong \mathbb{C}^{(b-a)n}$.

Thus e.g. the “Bruhat decomposition” -every invertible matrix M can be reduced to a unique permutation matrix $\widetilde{w}$ by upward row operations, rightward column operations and scaling columns- used to produce a basis of Schubert cycles for $H^\bullet$ of finite *Gr*s can be carefully generalized to affine *Gr*s.

For a finer decomposition consider the $\mathbb{C}^\times$ action on Gr

$$z \in \mathbb{C}^\times : L \mapsto zL = L$$

which scales t so that for $L = \mathrm{Span}_O(v_1, \ldots, v_n)$ for $v_i = \sum v^i_{jk} e_j t^k$

$$tv_i = \sum v^i_{jk} e_j (zt)^k$$

Taking $z \to 0$ has the effect of picking off least powers of basis elements, a tuple in $\mathbb{Z}^n$ which can be interpreted as a vertex of a moment polytope or as a coweight for G.

$$\begin{aligned}
&z \cdot (3e_1 t^{-1} + 7e_3 t^5 + e_6) \\
&= z^{-1}(3e_1 t^{-1} + 7e_3 z^6 t^5 + ze_6) \\
&= 3e_1 t^{-1} + 7e_3 z^6 t^5 + ze_6 \to 3e_1 t^{-1}
\end{aligned}$$

For $T \subset G$ a maximal torus, $X_*(T) = \mathrm{Hom}(\mathbb{C}^\times, T) \cong \mathbb{Z}^n$.

There is a map $X_*(T) \to Gr$ via the map $X_* \to G(K)$ defined by post-composing $\lambda : \mathbb{C}^\times \to T$ and $\operatorname{Spec} K \to \mathbb{C}^\times$ identifying $\lambda \in X_*$ and $t^\lambda \equiv \mathrm{diag}(t^{\lambda_1}, \ldots, t^{\lambda_n}) \in GL_n K$ or under the-other-way map $L_\lambda = \mathrm{Span}_O(e_i t^{\lambda_i} : 1 \leq i \leq n)$.

Related Facts.

- The fixed points of the T action on Gr are indexed by $X_*(T)$.
- The $z \to 0$ limits of the $\mathbb{C}^\times$ action on Gr are indexed by $X_*(T)$.
- The $G(O)$ orbits of Gr contain unique T-fixed points.
- Finally
$$Gr = \bigsqcup_{\lambda \in X_*} Gr^\lambda$$

Geometric Satake $:($

$$H^\bullet : IC_{Gr^\lambda} \in \mathcal{P}_G \mapsto V(\lambda) \in \underline{Rep}_G$$

Case $\lambda = \omega_k$ $:)$

$$H^\bullet(Gr(k,n)) \cong \bigwedge^k \mathbb{C}^n \quad \dim = \binom{n}{k}$$

Schubert varieties make up the basis on the left and k-element subsets of n index a basis on the right.

Emulating ω_k. Consider the linear map $t\cdot : K^n \to K^n$ sending $e_i t^j$ to $e_i t^{j+1}$ induced by multiplication by t on K.

Definition C2. $Gr^{>} = \{L \in Gr : t \cdot L \subset L\}$ sometimes called the *positive part of* Gr.

Fix $\lambda, \mu \in X_*(T)$ non-decreasing. Write Gr^{λ} for the set

$$\{L \in Gr^{>} : t\big|_{L/L_0} \text{ has jordan type } \lambda\}$$

and S^{μ} for the set

$$\{L \in Gr^{>} : \lim_{z \to 0} z \cdot L = L_{\mu}\}$$

where $L_\mu = \mathrm{Span}_{\mathcal{O}}(e_1 t^{\mu_1 - 1} \dots e_n t^{\mu_n - 1})$ and $z\cdot$ is our $\mathbb{C}^\times$ action from before.

Fact. The set $\overline{Gr^\lambda \cap S^\mu}$ has dimension equal $\dim V(\lambda)_\mu$ and its irreducible components, the so-called MV cycles, form a basis for $H^\bullet(\overline{Gr^\lambda})$ endowing it with a X_* grading, generalizing the case $\lambda = \omega_k$.

There is an action on $H^\bullet(\overline{Gr^\lambda})$ by multiplication by $c(\mathcal{L})$ where $\mathcal{L}$ denotes the det bundle on Gr and c Chern class.

Fact. This action is secretly an action of gl_n. It decomposes as

$$c_{\mu\nu} : H^\bullet(\overline{Gr^\lambda \cap S^\mu}) \to H^\bullet(\overline{Gr^\lambda \cap S^\nu})$$

with $c_{\mu\nu}$ nonzero only if $\nu = \mu + \alpha_i$ so that letting $E_i, F_i \in gl_n$ act by the appropriate components of $c(\mathcal{L}), c(\mathcal{L})^*$ defines $H^\bullet(\overline{Gr^\lambda})$ as an irrep of gl_n.

Definition D. Let $\lambda \geq \mu \in X_*$ viewed as partitions of N and consider the subset of gl_N defined by $\overline{\mathcal{O}_\lambda} \cap \mathcal{T}_\mu$ where $\mathcal{O}_\lambda = GL_N \cdot J_\lambda$ and by example $\mathcal{T}_{(3,2,2)}$ is elements of the form

$$\left[\begin{array}{ccc|cc|cc} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ * & * & * & * & * & * & * \\ \hline & & & 0 & 1 & 0 & \\ & * & * & * & * & * & * \\ \hline & & & & & 0 & 1 \\ & * & * & * & * & * & * \end{array}\right]$$

call it M_μ^λ.

Fact. The lattice POV supplies $M_\mu^\lambda \cong Gr_\mu^{\overline{\lambda}}$ with $L \in Gr_\mu^{\overline{\lambda}}$ being sent to the matrix of t

$$[t\big|_{L_0/L}]_B$$

in the basis

$$B = \{[e_1]\ldots[e_1 t^{\mu_1 - 1}], \ldots, [e_n]\ldots[e_n t^{\mu_n - 1}]\}$$

Examples

Fix $G = SL_2$, $\lambda = (2, 0)$, and $\mu = (1, 1)$.

In $Gr = G(K)/G(O)$ one defines

- $Gr_\mu = G_1[[t^{-1}]]t^\mu$ for $G_1 = \mathrm{Ker}(ev_\infty : Gr \mapsto G)$
- $Gr^\lambda = G(O)t^\lambda$
- $Gr_\mu^{\overline{\lambda}} = \overline{Gr^\lambda} \cap Gr_\mu$

Fact. $K^\times \cong \mathbb{Z} \times O^\times$ or $0 \neq g \in K$ can be written $t^n f$ for $f = f_0 + hot$ and $n \in \mathbb{Z}$.

Using this fact and the definitions, check that

$$\begin{aligned}
&\overline{G(O)t^{(2,0)}G(O)} \cap G_1[[t^{-1}]]t^{(1,1)}G(O) \\
&= \overline{\left\{ \begin{bmatrix} t+a & b \\ c & t+d \end{bmatrix} : \det = t^2 + (a+d)t + (ad-bc) = t^2 \right\}} \\
&\cong \overline{\{a+d=0, a^2+bc=0\}}
\end{aligned}$$

the 2-dimensional variety from slide 1.

On the other side $M_{(1,1)}^{(2,0)} = \overline{\mathcal{O}_{(2,0)}}$ and we check that

$$\begin{aligned}
&\overline{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}} \\
&= \overline{\left\{ \begin{bmatrix} -ac & a^2 \\ -c^2 & ac \end{bmatrix} \right\}} \\
&= \overline{\left\{ \begin{bmatrix} z & x \\ -y & z \end{bmatrix} : z^2 + xy = 0 \right\}}
\end{aligned}$$

What else is the affine Grassmannian

Definition E. Trivializable bundles definition.

4.2. **Global picture.** Let X be a curve, which in our case will always be $\mathbb{A}^1$. Let $\mathbb{A}^{(n)} = \mathbb{A}^1 \times \cdots \times \mathbb{A}^1 // \mathfrak{S}_n$ be the symmetric n-fold product of $\mathbb{A}^1$

Beilinson-Drinfeld Grassmannian [BD, MVi1, MVi2] is a (reduced) ind-scheme $\mathfrak{G}_{\mathbb{A}^{(n)}}$ whose $\mathbb{C}$-points are described as follows:

$$\mathfrak{G}_{\mathbb{A}^{(n)}}(\mathbb{C}) = \{(b, \mathcal{V}, t) \mid t : \mathcal{V}_{X-E} \to (X \times V)|_{X-E} \text{ is an isomorphism } \}, \tag{22}$$

where $b = (b_1, \ldots, b_n) \in \mathbb{A}^{(n)}$, $E = \{b_1, \ldots, b_n\} \subseteq \mathbb{A}^1$, $\mathcal{V}$ is a vector bundle of rank m, and t is the trivialization of $\mathcal{V}$ off E. The pairs $(\mathcal{V}, t)$ are considered up to an isomorphism.

Thank you for listening