QI-INVARIANT MODEL OF POISSON BOUNDARIES OF CAT(0) GROUPS

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Abstract. We study recurrent geodesic rays in a CAT(0) space. Let $\mathrm{thick}_\tau(T)$ be the amount of time $\tau[0,T]$ spends in the thick part. A parametrized unit speed geodesic τ in X will be called $(\gamma_0, L_0, M_0, \theta)$ *recurrent* if

$$E(\tau) = \lim_{T\to\infty} \frac{|t \in [0,T] : \tau(t) \in Thick(\gamma_0, L_0, M_0)|}{T} = \theta.$$

The geodesic τ will be called recurrent if for every γ_0, L_0, M_0 there is a $\theta > 0$ such that τ is $(\gamma_0, L_0, M_0, \theta)$ recurrent. We show that every recurrent geodesic ray is κ–Morse. Two applications are also described. The first one concerns the Poisson boundaries of CAT(0) groups and the second concerns the Patterson Sullivan measure on $\partial_\kappa X$. need a new one

1. Introduction

A major theme in recent research in metric geometry has been to find evidence of abundance of hyperbolic behavior in non-hyperbolic spaces. Well studied examples of such spaces include CAT(0) spaces with rank-1 geodesics: geodesics which do not bound a flat of infinite diameter. In a sense, these can be considered as geodesics in CAT(0) spaces exhibiting hyperbolic behavior. To any CAT(0) space can be associated the visual boundary consisting of asymptotic equivalence classes of geodesic rays. The limits of rank-1 geodesics in the visual boundary have been well studied, and shown to be generic in various senses of the word. However, neither the visual boundary nor its subset consisting of rank-1 directions is quasi-isometry invariant, which precludes it from being a topological invariant for groups acting properly and cocompactly on CAT(0) spaces. Thus, while genericity of rank-1 directions encapsulates abundance of hyperbolic behavior in rank-1 CAT(0) spaces it is difficult to translate in terms of properties of groups acting on such spaces. Qing and Rafi [QRT19], showed that a certain subset of the visual boundary consisting of limits of sublinearly Morse geodesics, when given a topology slightly different from the one induced from the visual boundary, called the sublinearly Morse boundary is in fact a quasi-isometry invariant.

In this paper, we show that this subset of the visual boundary is "generic" in several reasonable senses of the word. The visual boundary of a CAT(0) space X carries several natural families of measures corresponding to limits of different averaging procedures over orbits of a group acting on X properly and cocompactly. One are the so-called Patterson-Sullivan measures, studied in this context by Ricks [Ric17]. These are the weak limits of

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ball averages in the metric on X and are intimately related to the measure of maximal entropy on the unit tangent bundle of the geodesic flow on X. The other family consists of stationary measures associated to random walks coming from finitely supported measures μ on G: these are weak limit of measures on the orbit obtained by taking a large number of independent μ-distributed steps. We show that for either family of measures, almost every direction of the visual boundary is sublinearly Morse.

Theorem 1.1. *Let $G \curvearrowright X$ be a countable group of properly discontinuous, cocompact and isometric actions on a rank-1* CAT(0) *space X. Let ν be a measure on the visual boundary of X which is either the Patterson-Sullivan measure or the stationary measure for a finitely supported generating random walk on G. Then ν gives measure zero to the complement of sublinearly Morse directions.*

use intro-theorem notation?

As a corollary to the above statement about stationary measures we obtain:

Corollary 1.2. *Let μ be any measure on G whose finite support generates G as a semigroup. The sublinearly Morse boundary with either the subspace topology induced from the visual boundary or the Qing-Rafi topology is a topological model for the Poisson boundary of (G, μ).*

In order to prove Theorem 1.1, we prove that

Theorem A. *For any group G acting properly and cocompactly on a rank-1* CAT(0) *space X, and any finitely supported measure on G, the subliearly Morse boundary of G is a model for its Poisson boundary (G, μ) where μ is a finitely supported generating measure on G.*

We prove genericity of sublinearly Morse directions of the visual boundary by first proving that certain geodesics satisfying recurrent properties are sublinearly Morse. Namely, an infinite geodesic will be called strongly recurrent if it which spend a uniformly positive proportion of the time fellow travelling uniformly long contracting segments. We prove

Theorem 1.3. *A strongly recurrent infinite geodesic in any proper geodesic metric space is sublinearly Morse.*

We then use ergodic theoretic methods to prove

Theorem 1.4. *Let $G \curvearrowright X$ be a properly discontinuous and cocompact isometric action of a countable group on a rank-1* CAT(0) *space X. Let ν be a measure on the visual boundary of X which is either the Patterson-Sullivan measure or the stationary measure for a finitely supported symmetric generating random walk on G. Then ν gives measure zero to the complement of limit points of strongly reccurent geodesics.*

For the Patterson-Sullivan measure, Theorem 1.4 is a simple consequence of Birkhoff's ergodic theorem and the ergodicity of the geodesic flow in rank-1 CAT(0) spaces. For stationary measures coming from random walks, the genericity of strongly recurrent geodesics is derived from the double ergodicity of the Poisson boundary and follows the proof of a similar result for the Teichmüller geodesic flow proved by Baik-Gekhtman-Hamenstaedt [?].

Our arguments in proving do not use the CAT(0) property in an essential way and in fact are valid in any space with a reasonable geodesic flow. Indeed the results of [**?**] on genericity of strongly recurrent Teichmüller geodesics allow us to conclude:

Theorem 1.5. *Let S be a closed surface of genus at least 2 and $Mod(S)$ the associated mapping class group and $Teich(S)$ the associated Teichmüller space endowed. Let μ a probability measure on $Mod(S)$ whose finite support $Mod(S)$ generates $Mod(S)$. Then the sublinearly Morse boundary of $Teich(S)$ endowed with the Qing-Rafi cone topology is a topological model for the Poisson boundary of $(Mod(S), \mu)$.*

History. Kaimanovich [Kai00] proved that the Poisson boundary of hyperbolic groups are realized on their Gromov boundary. For CAT(0) groups, Karlson-Margulis [KM99] showed that random walk tracks geodesic rays sublinearly and thus the visual boundary realizes the Poisson boundary of CAT(0) spaces on which a CAT(0) group acts geometrically. However, visual boundaries are in general not QI-invariant and therefore not group-invariant, as shown by Croke-Kleiner [CK00]. Qing-Rafi-Tiozzo [QRT19] constructed κ–Morse boundaries for CAT(0) spaces that are QI-invariant and in the case of right-angled Artin groups, do realizes their Poisson boundaries. For mapping class groups, Kaimanovich-Masur showed that uniquely ergodic projective measured foliations with the corresponding harmonic measure can be identified with the Poisson boundary of random walks. Our proof of Theorem 1.4 follows a quantitative version of the Kaimanovich-Masur[KM96] result considered in the Teichmüller space setting by Baik-Gekhtman-Hamenstädt . need a citation. Also, discuss Choi

Organization of the paper. Section 2 recalls all necessary background for κ-Morse boundaries and establishes the geometric properties needed, i.e. a geodesic ray cannot fellow travel a contracting geodesic for a long time while keeping far away from it. Section 4.3 introduces the concept of recurrent geodesics and proves the main technical theorem that a a generic geodesic ray in a CAT(0) is strongly recurrent, where genericity can be defined with respect to a variety of natural measures on ∂X. Section 3 proves that strongly recurrency of a geodesic with respect to a contracting geodesic implies κ–Morse-ness, which leads to Theorem **??**. Section **??**

2. Sublinearly Morse quasi-geodesic rays in proper metric space

In geometric group theory, we are mainly interested in geometric properties of the associated spaces that are group-invariant. In the setting of finitely generated groups, group-invariance can be interpreted as *quasi-isometries* between metric spaces and objects, which we introduce now.

2.1. Quasi-isometries of groups and metric spaces.

Definition 2.1 (Quasi-isometric embedding)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. For constants $\mathsf{k} \geq 1$ and $\mathsf{K} \geq 0$, we say a map $\Phi\colon X \to Y$ is a (k, K)–*quasi-isometric* *embedding* if, for all $x_1, x_2 \in X$

$$\frac{1}{\mathsf{k}} d_X(x_1, x_2) - \mathsf{K} \le d_Y\big(\Phi(x_1), \Phi(x_2)\big) \le \mathsf{k}\, d_X(x_1, x_2) + \mathsf{K}.$$

If, in addition, every point in Y lies in the K–neighbourhood of the image of Φ, then f is called a (k, K)–quasi-isometry. When such a map exists, X and Y are said to be *quasi-isometric*.

A quasi-isometric embedding $\Phi^{-1}\colon Y \to X$ is called a *quasi-inverse* of Φ if for every $x \in X$, $d_X(x, \Phi^{-1}\Phi(x))$ is uniformly bounded above. In fact, after replacing k and K with larger constants, we assume that Φ^{-1} is also a (k, K)–quasi-isometric embedding,

$$\forall x \in X \quad d_X\big(x, \Phi^{-1}\Phi(x)\big) \le \mathsf{K} \qquad \text{and} \qquad \forall y \in Y \quad d_Y\big(y, \Phi\,\Phi^{-1}(x)\big) \le \mathsf{K}.$$

Geodesics and quasi-geodesic rays and segments. Fix a base point $\mathfrak{o} \in X$. A *geodesic ray* in X is an isometric embedding $\tau\colon [0,\infty) \to X$ such that $\tau(0) = \mathfrak{o}$. That is, by convention, a geodesic ray is always assumed to start from this fixed base-point. A *quasi-geodesic ray* is a continuous quasi-isometric embedding $\beta\colon [0,\infty) \to X$ such that $\beta(0) = \mathfrak{o}$. That is, there are constants $\mathsf{q} \ge 1$, $\mathsf{Q} > 0$ such that, for $s, t \in [0, \infty)$, we have

$$\frac{|s-t|}{\mathsf{q}} - \mathsf{Q} \le d_X\big(\beta(s), \beta(t)\big) \le \mathsf{q}\,|s-t| + \mathsf{Q}.$$

The additional assumption that quasi-geodesics are continuous is not necessary, but it is added for convenience and to make the exposition simpler. One can always adjust a quasi-isometric embedding slightly to make it continuous (see [BH09, Lemma III.1.11]).

Similar to above, a *geodesic segment* is an isometric embedding $\tau\colon [s,t] \to X$ and a *quasi-geodesic segment* is a continuous quasi-isometric embedding $\beta\colon [s,t] \to X$.

We often denote the images of a geodesic τ or a quasi-geodesic β as a subset of X again by τ and β respectively. That is, a point on β, is a point $x \in X$ such that $x = \beta(t_x)$ for some time t_x. We adopt the following notation for sub-segments of geodesics and quasi-geodesics. Suppose $\beta\colon [s,t] \to X$ is quasi-geodesic path and let $x, y \in X$ be two points on β, namely $x = \beta(t_x)$ and $y = \beta(t_y)$ for $t_x, t_y \in [s,t]$. Then $[x,y]_\beta$ is the subsegment of β starting from x and ending in y, that is $[x,y]_\beta = \beta|_{[t_x, t_y]}$. Also for points $x, y \in X$, we let $[x,y]$ represent a geodesic segment connecting x to y.

For a quasi-geodesic ray β and $r > 0$, we define $\beta|_r$ to be the quasi geodesic sub-segment of β that starts at $\mathfrak{o}$ and ends at the first point on β where the distance to $\mathfrak{o}$ is r.

Notations. We adopt the following notation for lines and segments in this paper. Suppose β is a specified path, then

$$[x,y]_\beta : \text{ the segment of } \beta \text{ from } x \in \beta \text{ to } y \in \beta.$$

In the special case where β is a geodesic, we suppress the subscript, i.e. we use $[x,y]$ denote geodesic segments between the two points. If β emanates from the base-point, then

$$\beta|_{\mathsf{r}} : \text{ the point on } \beta \text{ that is distance } \mathsf{r} \text{ from } \mathfrak{o}.$$

Contracting geodesics. Let Z be a closed subset of X and x be a point in X. By $d(x,Z)$ we mean the set-distance between x and Z, i.e.

$$d(x,Z) := \inf\{d(x,y) \mid y \in Z\}.$$

Let

$$\pi_Z(x) := \{y \mid d(x,y) = d(x,Z)\}$$

be the set of nearest-point projections from x to Z. Since X is a proper metric space, $\pi_Z(x)$ is non empty. We refer to $\pi_Z(x)$ as the *projection set* of x to Z. For a quasi-geodesic β and $x \in X$, we write x_β to denote *any* point in the projection set of x to β.

Definition 2.2. We say a closed subset $Z \subset X$ is N*–contracting* for a constant $N > 0$ if, for all pairs of points $x, y \in X$, we have

$$d(x,y) < d(x,Z) \implies d(x_Z, y_Z) \leq N.$$

Any such N is called a *contracting constant* for Z.

2.1.1. *Nearest-point projections in proper metric spaces.* Let Z be a closed subset of X and x be a point. By $d(x,Z)$ we mean the set-distance between x and Z, i.e.

$$d(x,Z) := \min\{d(x,y) | y \in Z\}.$$

Let

$$\pi_Z(x) := \{y | d(x,y) = d(x,Z)\}$$

be the set of nearest-point projections from x to Z. Since X is a proper metric space, projections of a point to a closed set always exist. We refer to the convex hull of $\pi_Z(x)$ as a *projection set*. We write x_β to denote *any* point in the projection set of x to β.

Lemma 2.3. [QRT22] *Consider a point $x \in X$ and a (q, Q)–quasi-geodesic segment β connecting a point $z \in X$ to a point $w \in X$. Let y be a point in x_β, and let γ be the concatenation of the geodesic segment $[x, y]$ and the quasi-geodesic segment $[y, z]_\beta \subset \beta$. Then $\gamma = [x, y] \cup [y, z]_\beta$ is a $(3\mathsf{q}, \mathsf{Q})$–quasi-geodesic.*

is this needed?

2.2. **κ-Morse and κ-contracting sets.** Now we introduce a large class of quasi-geodesic rays that are quasi-isometry invariant. Intuitively, these quasi-geodesics have a weak Morse property, i.e. their quasi-geodesics stay close asymptotically. To begin with, we fix a function that is sublinear in the following sense:

2.2.1. *Sublinear functions.* We fix a function

$$\kappa\colon [0, \infty) \to [1, \infty)$$

that is monotone increasing, concave and sublinear, that is

$$\lim_{t \to \infty} \frac{\kappa(t)}{t} = 0.$$

Note that using concavity, for any $a > 1$, we have

$$\kappa(at) \leq a\left(\frac{1}{a}\,\kappa(at) + \left(1 - \frac{1}{a}\right)\kappa(0)\right) \leq a\,\kappa(t). \tag{1}$$

Remark 2.4. The assumption that κ is increasing and concave makes certain arguments cleaner, otherwise they are not really needed. One can always replace any sublinear function κ, with another sublinear function $\overline{\kappa}$ so that $\kappa(t) \leq \overline{\kappa}(t) \leq \mathsf{C}\,\kappa(t)$ for some constant C and $\overline{\kappa}$ is monotone increasing and concave. For example, define

$$\overline{\kappa}(t) = \sup\left\{\lambda\kappa(u) + (1-\lambda)\kappa(v) \;\middle|\; 0 \leq \lambda \leq 1,\; u, v > 0,\; \text{and } \lambda u + (1-\lambda)v = t\right\}.$$

The requirement $\kappa(t) \geq 1$ is there to remove additive errors in the definition of κ–contracting geodesics.

Definition 2.5 (κ–neighborhood)**.** For a closed set Z and a constant n define the (κ, n)–neighbourhood of Z to be

$$\mathcal{N}_\kappa(Z, \mathsf{n}) = \left\{x \in X \;\middle|\; d_X(x, Z) \leq \mathsf{n} \cdot \kappa(x)\right\}.$$

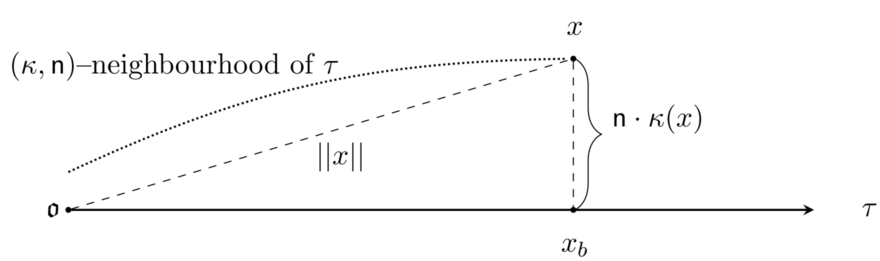

FIGURE 1. A κ-neighbourhood of a geodesic ray τ with multiplicative constant n.

In this paper, Z is either a geodesic or a quasi-geodesic. That is, we can write $\mathcal{N}_\kappa(\tau, \mathsf{n})$ to mean the (κ, n)–neighborhood of the image of the geodesic ray τ. Or, we can use phrases like "the quasi-geodesic β is κ–contracting" or "the geodesic τ is in a (κ, n)–neighbourhood of the geodesic c".

Definition 2.6. Let β and γ be two quasi-geodesic rays in X. If β is in some κ–neighbourhood of γ and γ is in some κ–neighbourhood of β, we say that β and γ κ*–fellow travel* each other. This defines an equivalence relation on the set of quasi-geodesic rays in X (to obtain transitivity, one needs to change n of the associated (κ, n)–neighbourhood). We refer to such an equivalence class as a κ*–equivalence class of quasi-geodesics*. We denote the κ–equivalence class that contains β by $[\beta]$ or we use the notation $\mathbf{b}$ for such an equivalence class when no quasi-geodesic in the class is given.

A metric space is called a unique geodesic space if any two points can be connected by a unique geodesic.

Lemma 2.7. [QRT19] *Let $b\colon [0,\infty) \to X$ be a geodesic ray in a unique geodesic space X. Then b is the unique geodesic ray in any (κ, n)–neighbourhood of b for any n. That is to say, distinct geodesic rays do not κ–fellow travel each other.*

κ–contracting and κ–Morse sets. First we recall the definition of κ–contracting and κ–Morse sets from [QRT22].

Definition 2.8 (weakly κ–Morse)**.** We say a closed subset Z of X is *weakly κ–Morse* if there is a function

$$\mathsf{m}_Z\colon \mathbb{R}^2_+ \to \mathbb{R}_+$$

so that if $\beta\colon [s,t] \to X$ is a (q, Q)–quasi-geodesic with end points on Z then

$$\beta[s,t] \subset \mathcal{N}_\kappa\big(Z, \mathsf{m}_Z(\mathsf{q}, \mathsf{Q})\big).$$

We refer to m_Z as the *Morse gauge* for Z. We always assume

$$\mathsf{m}_Z(\mathsf{q}, \mathsf{Q}) \geq \max(\mathsf{q}, \mathsf{Q}). \tag{2}$$

Definition 2.9 (Strongly κ–Morse)**.** We say a closed subset Z of X is *strongly κ–Morse* if there is a function $\mathsf{m}_Z\colon \mathbb{R}^2 \to \mathbb{R}$ such that, for every constants $\mathsf{r} > 0$, $\mathsf{n} > 0$ and every sublinear function κ', there is an $\mathsf{R} = \mathsf{R}(Z, \mathsf{r}, \mathsf{n}, \kappa') > 0$ where the following holds: Let $\eta\colon [0,\infty) \to X$ be a (q, Q)–quasi-geodesic ray so that $\mathsf{m}_Z(\mathsf{q}, \mathsf{Q})$ is small compared to r, let t_r be the first time $\|\eta(t_\mathsf{r})\| = \mathsf{r}$ and let t_R be the first time $\|\eta(t_\mathsf{R})\| = \mathsf{R}$. Then

$$d_X\big(\eta(t_\mathsf{R}), Z\big) \leq \mathsf{n} \cdot \kappa'(\mathsf{R}) \quad \Longrightarrow \quad \eta[0, t_\mathsf{r}] \subset \mathcal{N}_\kappa\big(Z, \mathsf{m}_Z(\mathsf{q}, \mathsf{Q})\big).$$

Definition 2.10 (κ–contracting)**.** Recall that, for $x \in X$, we have $\|x\| = d_X(\mathfrak{o}, x)$. For a closed subspace Z of X, we say Z is *κ–contracting* if there is a constant c_Z so that, for every $x, y \in X$

$$d_X(x,y) \leq d_X(x, Z) \quad \Longrightarrow \quad \mathrm{diam}_X\big(x_Z \cup y_Z\big) \leq \mathsf{c}_Z \cdot \kappa(\|x\|).$$

To simplify notation, we often drop $\|\cdot\|$. That is, for $x \in X$, we define

$$\kappa(x) := \kappa(\|x\|).$$

Now we show that one can surger quasi-geodesic segments so they all starts at the base-point:

Lemma 2.11. *Let τ be κ-Morse geodesic ray. Let $\beta\colon [a,b] \to X$ be a (q, Q)- quasi-geodesic segment with endpoints on τ. Them there exists $\beta'\colon [0, b']$ such β' is a $(9\mathsf{q}, \mathsf{Q})$–quasi-geodesic segment that starts at $\mathfrak{o}$ and ends at $\beta(b)$.*

Proof. Consider the nearest-point projection of $\mathfrak{o}$ to β and there is at least one point on β that realizes the distance, call it $p \in \beta$. Consider the concatenation $[\mathfrak{o}, p] \cap [p, \beta(b)]_\beta$. By Lemma 2.3 this is a $(3\mathsf{q}, \mathsf{Q})$–quasi-geodesic that emanates from $\mathfrak{o}$ and ends on β. □

Thus in this paper we sometimes the use following slightly different but equivalent charaterization of κ–Morse, which better suits the context:

Definition 2.12 (κ–Morse II)**.** We say a closed subset Z of X is *κ–Morse II* if there is a function

$$\mathsf{m}_Z \colon \mathbb{R}_+^2 \to \mathbb{R}_+$$

so that if $\beta \colon [0,t] \to X$ is a (q,Q)–quasi-geodesic starting at $\mathfrak{o}$ and with $\beta(t) \in Z$ then

$$\beta[0,t] \subset \mathcal{N}_\kappa\big(Z, \mathsf{m}_Z(\mathsf{q},\mathsf{Q})\big).$$

We refer to m_Z as the *Morse gauge* for Z. We always assume

$$\mathsf{m}_Z(\mathsf{q},\mathsf{Q}) \geq \max(\mathsf{q},\mathsf{Q}). \tag{3}$$

By the above definitions are equivalent. That is, we can check either condition and the others are implied.

There are strong relationships between these definitions, which we summarize in the following theorem. The equivalence between (1) and (3) is proven in Lemma 2.11, the equivalence between (1) and (2) is [QRT22, Theorem 3.10]. Lastly, the equivalence in CAT(0) space is established in [QRT19, Theorem A]. For definitions and properties of CAT(0) spaces, see Section 4.

Theorem 2.13 (Lemma 2.11, [QRT19, QRT22])**.** *Let X be a proper geodesic space, then the following conditions are equivalence for a quasi-geodesic ray τ.*

(1) τ is κ-Morse.
(2) τ is κ-weakly Morse.
(3) τ is κ-Morse II.

Furthermore, if X is a proper CAT(0) *space or the Teichmüller space of a finite type surface, then τ is κ-Morse if and only if it is κ-contracting.*

Lastly, a quasi-geodesic is called *sublinearly Morse* if it is κ-Morse for some sublinearly growing function κ. Two parametrized quasi-geodesics γ_1, γ_2 are said to be equivalent if their diverge sublinearly, i.e.

$$d(\gamma_1(t), \gamma_1(t))/t \to 0.$$

Let $\partial_\kappa X$ denote the set of equivalence classes of κ Morse quasi-geodesics and $\partial_{SL} X$ set of equivalence classes of κ sublinearly Morse quasi-geodesics. Qing, Rafi and Tiozzo introduced the metrizable *coarse cone topology* on the set $X \cup \partial_\kappa X$ which can be characterize as follows.

A sequence $x_n \in X$ converges to an equivalence class of a quasi-geodesic ζ in $\partial_\kappa X$ if $d(x_n, \zeta)/d(o, x_n) \to 0$.

It is shown in [QRT22] that $X \cup \partial_\kappa X$ with the coarse cone topology is a QI-invariant set.

3. Geodesics with enough Morse segments are sublinearly Morse

In this section, we introduce the notion of frequently contracting geodesics which are geodesics that contain sufficiently many (in a statistical sense) strongly contracting subsegments. We then give a criterion for a geodesic ray to be frequently contracting (Lemma 3.3).

We then prove that every frequently contracting geodesic is in fact sublinearly Morse (Corollary 3.7). In the next section, we use ergodic theory to show that in CAT(0) spaces and Teichmüller space, frequently contracting geodesics are generic in several reasonable senses of the word.

Definition 3.1. A unit speed parametrized geodesic ray $\tau\colon [0,\infty) \to X$ is (N,C)–*frequently contracting* for constant $N, C > 0$ if the following holds. For each $L > 0$ and $\theta \in (0,1)$ there is an $R_0 > 0$ such that for $R > R_0$ and $t > 0$ there is an interval of time $[s-L, s+L] \subset [t, t+\theta R]$ and an N–contracting geodesic γ such that,

$$u \in [s-L, s+L] \qquad \Longrightarrow \qquad d(\tau(u), \gamma) \leq C.$$

That is, every subsegment of τ of length θL contains a segment of length R that is C–close to an N–contracting geodesic γ. A bi-infinite geodesic τ is frequently contracting if the rays $t \to \tau(t)$ and $t \to \tau(-t)$ are both frequently contracting.

Definition 3.2. If for some t, $\tau(t-L, t+L)$ is C–close to some N–contracting geodesic γ, we say (in analogy with Teichmüller space) $\tau(t)$ is in the thick part of τ. Define

$$\text{thick}_\tau(T) = \Big|\Big\{t \in [0,T] : \tau(t-L, t+L) \text{ is } C\text{–close to some } N\text{–contracting geodesic } \gamma\Big\}\Big|.$$

That is, $\text{thick}_\tau(T)$ is the amount of time $\tau[0,T]$ spends in the thick part. We now give a sufficient condition for a geodesic ray to be frequently contracting.

Lemma 3.3. *Let $\tau\colon [0,\infty) \to X$ be a geodesic ray. Suppose there are constants $N, C > 0$ such that for each $L > 0$ there is a $m > 0$ where*

$$\lim_{T\to\infty} \frac{\text{thick}_\tau(T)}{T} = m.$$

Then τ is frequently contracting.

Proof. Suppose that τ is a geodesic ray satisfying the condition of the Lemma. Then

$$\lim_{s,t\to\infty} \frac{\text{thick}_\tau(t)/t}{\text{thick}_\tau(s)/s} = 1. \tag{4}$$

Now assume, by way of contradiction, that τ is not (N,C)–frequently contracting. Then there are constants $0 < \theta < 1$ and $L > 0$ and sequences $R_n \to \infty$ and $0 \leq t_n \leq (1-\theta)R_n$ such that $[t_n, t_n + \epsilon R_n]_\tau$ contains no segment of length $2L$ that is C–close to some N–contracting geodesic γ. That is,

$$\text{thick}_\tau(t_n) = \text{thick}_\tau(t_n + \theta R_n).$$

Therefore,

$$\frac{\text{thick}_\tau(t_n)/t_n}{\text{thick}_\tau(t_n+\theta R_n)/(t_n+\theta R_n)} = \frac{(t_n+\theta R_n)}{t_n} \geq \frac{(t_n+\theta t_n)}{t_n} = (1+\theta) > 1.$$

This contradicts Equation (4). The contradiction proves the desired result of this lemma. □

Our goal is to show that, if τ is frequently contracting, then the diameter of the projection of disjoint balls to τ is sublinearly small. It is in fact sufficient to show that the diameter of the projection of a disjoint balls is smaller than every linear function.

Proposition 3.4. *Let $\tau\colon [0,\infty) \to X$ be an (N, C)–frequently contracting geodesic. Then for every $\theta > 0$ there is $R_0 > 0$ such that for all $R \geq R_0$ the following holds. Assume*

$$d(x, y) \leq d(x, \tau) \qquad \textit{and} \qquad d(\mathfrak{o}, x) \leq R$$

Then

$$d(\pi_\tau(x), \pi_\tau(y)) \leq \theta R.$$

We recall several well known facts regarding the properties of contracting geodesics.

Lemma 3.5. *There are constants $C_1, D_1 > 0$ depending on N such that if γ is N–contracting and the geodesic segment $[x, y]$ is outside of the C_1–neighborhood of γ then the projection of $[x, y]$ to γ has diameter at most D_1.*

Lemma 3.6. *There is a constant $C_2 > 0$ depending only on N such that, for a N–contracting geodesic γ and for $x, y \in X$, if $d(\pi_\gamma(x), \pi_\gamma(y)) \geq D_2$, then the C_2–neighborhood of the geodesic segment $[x, y]$ contains the segment $[\pi_\gamma(x), \pi_\gamma(y)]_\gamma$.*

Proof of Proposition 3.4. Assume $\tau[s, t]$ is C close to some N–contracting geodesic γ with

$$d(\tau(s), \gamma(s')) \leq C \qquad \text{and} \qquad d(\tau(t), \gamma(t')) \leq C$$

for some times $s < t$ and $s' < t'$ where $L = (t - s)$ is large.

Claim. There is a D_2 (depending only on N and specified in Lemma 3.6)such that, for any $x \in X$, if $\pi_\tau(x) = \tau(u)$ for $u \leq s$ then $\pi_\gamma(x) = \gamma(u')$ for $u' \leq s' + D_2$.

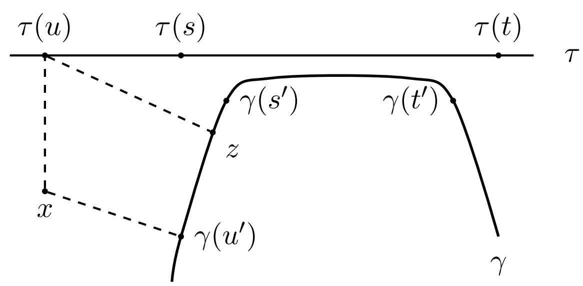

FIGURE 2. Claim.

This is because by Lemma 3.5, $z = \pi_\gamma(\pi_\tau(x))$ is near $\gamma(s')$. If $\pi_\gamma(x) = \gamma(u')$ where $(u' - s')$ is larger than D_2, then by Lemma 3.6, we have the a C_2–neighborhood of the geodesic $[x, \pi_\tau(x)]$ contains the sub-segment $\gamma[s', u']$. Choose w' such that $w' - s'$ is large and,

$$d(\gamma(w'), \gamma(w)) \leq C$$

for some w where $(w - s)$ is large. Therefore, $\gamma(w')$ and hence $\tau(w)$ are much closer to x than $\pi_\tau(x)$ which is a contradiction and thus the claim holds.

Now, if $d(\pi_\tau(x), \pi_\tau(y)) \geq \theta R$ for sufficiently large R, the segment $[\pi_\tau(x), \pi_\tau(y)]_\tau$ contains a subsegment $\tau[s,t]$ with $(t-s) \geq L$ that is C close to some γ. Then the projection of x, y to γ are $L - 2D_2$ apart. Which means the $(D_2 + C)$–neighborhood of the geodesic segment $[x, y]$ covers the segment $\tau[s,t]$. Hence $d(x,y) > d(x,\tau)$, contradicting the assumption. This finishes the proof of Proposition 3.4. □

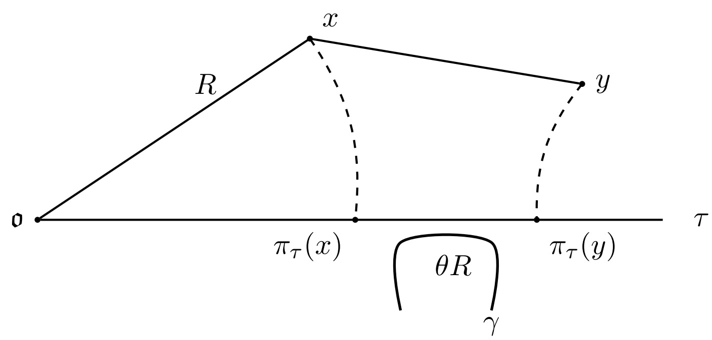

FIGURE 3. proof of Proposition 3.4

Corollary 3.7. *If τ is frequently contracting, then it is κ-contracting for some sublinear function κ. Hence it is also κ-Morse.*

Proof. Assume for contradiction that τ is not κ–contracting for any sublinear function κ. That is, there is a sequence of point $x^n, y^n \in X$ with $\|x^n\| \to \infty$, such that

$$d_X(x^n, y^n) \leq d_X(x^n, \tau).$$

However, we have

$$\limsup_{n\to\infty} \frac{\operatorname{diam}_X\left(x^n_\tau \cup y^n_\tau\right)}{\|x^n\|} \geq 3\theta > 0.$$

Taking a subsequence, we can in fact assume that, for every n,

$$\frac{\operatorname{diam}_X\left(x^n_\tau \cup y^n_\tau\right)}{\|x^n\|} \geq 2\theta. \tag{5}$$

Let R_0 be the constant associated to θ given by Proposition 3.4 and let n be such that $\|x_n\| \geq R_0$. Then for $R = \|x_n\|$, Proposition 3.4 implies that

$$\frac{\operatorname{diam}_X\left(x^n_\tau \cup y^n_\tau\right)}{\|x^n\|} \leq \theta,$$

which contradicts (5). The contradiction proves the corollary as desired. □

4. CAT(0) spaces, their boundaries and their isometries

A proper geodesic metric space is CAT(0) if it satisfies a certain metric analogue of nonpositive curvature. Roughly speaking, a space is CAT(0) if geodesic triangles in X are at least as thin as triangles in Euclidean space with the same side lengths. To be precise, for any given geodesic triangle $\triangle pqr$, consider the unique triangle $\triangle \overline{pqr}$ in the Euclidean plane with the same side lengths. For any pair of points x, y on edges $[p, q]$ and $[p, r]$ of the triangle $\triangle pqr$, if we choose points $\overline{x}$ and $\overline{y}$ on edges $[\overline{p}, \overline{q}]$ and $[\overline{p}, \overline{r}]$ of the triangle $\triangle \overline{pqr}$ so that $d_X(p, x) = d_{\mathbb{E}}(\overline{p}, \overline{x})$ and $d_X(p, y) = d_{\mathbb{E}}(\overline{p}, \overline{y})$ then,

$$d_X(x, y) \le d_{\mathbb{E}^2}(\overline{x}, \overline{y}).$$

For the remainder of the paper, we assume X is a proper CAT(0) space. A metric space X is *proper* if closed metric balls are compact. Here, we list some properties of proper CAT(0) spaces that are needed later (see [BH09]).

Lemma 4.1. *A proper* CAT(0) *space X has the following properties:*

i. It is uniquely geodesic, that is, for any two points x, y in X, there exists exactly one geodesic connecting them. Furthermore, X is contractible via geodesic retraction to a base point in the space.

ii. The nearest-point projection from a point x to a geodesic line b is a unique point denoted x_b. In fact, the closest-point projection map

$$\pi_b \colon X \to b$$

is Lipschitz.

iii. convexity: if $\beta \colon [0, 1] \to X$ is a quasi-geodesic segment with endpoint on the geodesic line $\gamma \colon [0, 1] \to X$, and $\beta(i) = \gamma(1), i = 0, 1$ then for every $0 \le s \le 1$, there exists t such that $\pi_\gamma(\beta(t)) = \gamma(s)$.

4.1. The visual boundary of CAT(0) spaces.

Definition 4.2 (visual boundary)**.** Let X be a CAT(0) space. The *visual boundary* of X, denoted ∂X, is the collection of equivalence classes of infinite geodesic rays, where τ and β are in the same equivalence class, if and only if there exists some $C \ge 0$ such that $d(\tau(t), \beta(t)) \le C$ for all $t \in [0, \infty)$. The equivalence class of τ in ∂X we denote $\tau(\infty)$.

Notice that by Proposition I. 8.2 in [BH09], for each τ representing an element of ∂X, and for each $x' \in X$, there is a unique geodesic ray τ' starting at x' with $\tau(\infty) = \tau'(\infty)$.

We describe the topology of the visual boundary by a neighbourhood basis: fix a base point $\mathfrak{o}$ and let τ be a geodesic ray starting at $\mathfrak{o}$. A neighborhood basis for τ is given by sets of the form:

$$\mathcal{U}_v\big(\tau(\infty), r, \epsilon) := \{\beta(\infty) \in \partial X \,|\, \beta(0) = \mathfrak{o} \text{ and } d(\tau(t), \beta(t)) < \epsilon \text{ for all } t < r\}.$$

In other words, two geodesic rays are close if they have geodesic representatives that start at the same point and stay close (are at most ϵ apart) for a long time (at least r). Notice that the above definition of the topology on ∂X references a base-point $\mathfrak{o}$. Nonetheless, Proposition I. 8.8 in [BH09] proves that the topology of the visual boundary is base-point invariant.

For each $o \in X$ and ζ in ∂X there is a unique unit speed parametrized geodesic ray $\tau_{o,\zeta}$ through o converging to ζ.

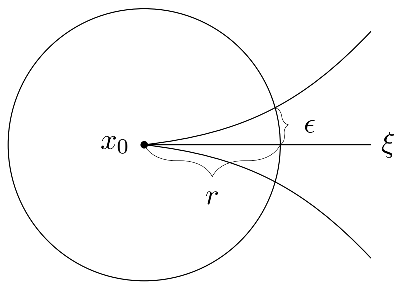

FIGURE 4. A basis for open sets

4.2. CAT(0) **spaces and their isometries.** A flat strip of a space X is a subset $F \subset X$ isometric to $\mathbb{R} \times I$ for some interval I with the Euclidean metric. It is a flat plane if $I = (-\infty, \infty)$ and a flat half plane if $I = [0, \infty)$. Call an infinite geodesic line in X *rank one* if its image does not bound a flat half-plane in X and *zero width* if it does not bound any flat strip. For any element $g \in Isom(X)$, the *translation length* of g is defined as

$$l(g) = \min\{d(x, gx) | x \in X\}.$$

Any geodesic contained in $\{p \in X : d(p, gp) = l(p)\}$ is called an *axis* of g. If a rank one geodesic γ is an axis of an isometry $g \in Isom(X)$, we call γ a rank one axis and g a rank one periodic isometry. A CAT(0) space is said to be rank one if it has a rank-1 geodesic. The space X is said to be geodesically complete if any geodesic segment can be extended to a bi-infinite geodesic. Any bi-infinite geodesic τ converges forwards and backwards to $\tau_+, \tau_- \in \partial X$ respectively then τ is the unique geodesic with these limit points in ∂X. Unless X is Gromov hyperbolic, not any two points of ∂X can be joined by a bi-infinite geodesic but the set of pairs that can be joined by a rank-1 geodesic form an open and dense of of $\partial X \times \partial X$.

The limit set $L(G) \subset \partial X$ of G is the unique minimal closed G-invariant subset of ∂X. An action by isometries $G \curvearrowright X$ is said to be rank-1 (and G is said to be a rank-1 group) if it contains two rank-1 isometries whose pairs of fixed points in ∂X are disjoint. A geodesically complete CAT(0) space admitting a rank-1 action $G \curvearrowright X$ always has a zero width geodesic with endpoints in $L(G)$([Ric17], Proposition 8.14). A cococompact action by isometries on a rank-1 CAT(0) space is always rank-1.

Let SX denote the *unit tangent bundle of* X, defined as the set of parametrized unit speed bi-infinite geodesics in X endowed with the compact-open topology.

Let $\pi_{fp}(v) = v(0)$ be the *footpoint projection* $\pi_{fp} : SX \to X$. Let

$$g_t : SX \to SX, \tau(s) \to \tau(s + t)$$

be the geodesic flow. Let $S_G X \subset SX$ denote the set of $v \in SX$ with $v_\pm \in L(G)$ For $x, y \in X$ and $\zeta, \alpha \in \partial X$ define the Busemann function $\beta_\zeta(x,y) = \lim_{z\to\zeta} d(x,z) - d(y,z)$ and the Gromov product $\rho_x(\zeta,\alpha) = \lim_{z\to\alpha, w\to\zeta}(d(z,x) + d(w,x) - d(z,w))/2$.

Given a basepoint $o \in X$ there is a map $H_o : SX \to \partial X \times \partial X \times \mathbb{R}$ given by

$$H_o(v) = (v_+, v_-, \beta_\gamma(o, v(0))).$$

The restriction of H_o to the set Z of zero width geodesics is one-to-one. Let $[SX]$ be the set of equivalence classes of SX under the equivalence relation given by v w if $H_o(v)$ $H_o(w)$. This equivalence relation does not depend on the basepoint o.

4.3. **Genericity of frequently contracting geodesics in** CAT(0) **spaces.** The visual boundary ∂X of a rank-1 CAT(0) space X carries several natural classes of measures, corresponding to different averaging constructions over orbits of a properly discontinuous group action $G \curvearrowright X$ One is the Patterson-Sullivan measure, studied in this context by Ricks [Ric17]. For geometric group actions, it can be described as the weak limit of distributions of G orbits for large spheres in the metric on X. The other family are stationary measure associated to random walks on G with finite increments: these are weak limits of pushforwards in X of convolution powers of finitely supported probability measures μ on G. In the next two sections we prove the following:

Theorem 4.3. *Let X be a* CAT(0) *space and let G act on X geometrically. Let ν be either of the following*

- *The Patterson-Sullivan measure on ∂X, if X is geodesically complete.*
- *The stationary measure coming from a finitely supported random walk on G.*

Then ν almost every point of ∂X is frequently contracting.

In fact, we will prove Theorem 4.3 for measures coming from a more general class of actions, which are not necessarily cocompact.

4.4. **Choosing a fixed sublinear function.** Fix this Theorem 4.3 implies that, with respect to either the Patterson-Sullivan or the stationary measure ν, ν a.e. $\zeta \in \partial X$ is sublinearly contracting. In this subsection, we show that there is a single sublinear function κ such that ν a.e. ζ for any $x \in X$ the geodesic ray $[x, \zeta)$ is $c_x(\zeta)\kappa$ contracting for some $c_x(\zeta) > 0$. Indeed, let $\Omega = \{\kappa_i\}$ be a countable collection of sublinear functions on $\mathbb{R}$ such that for any sublinear κ there is a $\kappa' \in \Omega$ and $C > 0$ with $\kappa \le C\kappa'$. Such a collection exists by the separability of the space of continued functions $X \to \mathbb{R}$. For each i let $A_i \subset \partial X$ be the collection ζ such that for any $x \in X$, $[x, \zeta)$ is $c_x(\zeta)\kappa_i$ contracting for some $c_x(\zeta) > 0$. We know $\nu(\cup A_i) = 1$. Moreover, each A_i is G-invariant. Thus, by ergodicity of $G \curvearrowright (X, \nu)$ for each i we have $\nu(A_i) \in \{0, 1\}$. Thus there is a single A_i with $\nu(A_i) = 1$. This means there is a single sublinear function κ and a full measure $A \subset \partial X$ such that for each $x \in X, \zeta \in A$ there is a $c_x(\zeta) > 0$ with $[x, \zeta)$ $c_x(\zeta)\kappa$ contracting, completing the proof.

5. Genericity with respect to Patterson-Sullivan measures

Let X be a proper CAT(0) space and $G < Isom(X)$ a rank-1 discrete group of isometries. Assume there is some zero width geodesic with endpoints in the limit set $L(G)$. This is always the case when X is geodesically complete.

5.1. **Patterson-Sullivan measures.** The quantity

$$\delta(G) = \lim \sup_{R \to \infty} R^{-1} \log |B_R o \cap Go|$$

is called the critical exponent of the action $G \curvearrowright X$. It is positive for any properly discontinuous faithful action of a nonamenable group, and hence for any rank-1 action on a proper CAT(0) space X (since loxodromics with disjoint endpoints generate a free group in $Isom(X)$. In general $\delta(G)$ may be infinite, but it is always finite when X admits some cocompact actions by a properly discontinuous group of isometries or when G is finitely generated.

The action $G \curvearrowright X$ is said to be *divergent* if the Poincare series $\sum_{g \in G} e^{-sd(go,o)}$ diverges at $s = \delta(G)$ and convergent otherwise. If the action $G \curvearrowright X$ is properly discontinuous and cocompact it is necessarily divergent ([Ric17], Theorem 3).

A $\delta(G)$-conformal density for $G \curvearrowright X$ is an absolutely continuous family of finite Borel measures $\nu_x, x \in X$ on $L(G)$ such that

$$d\nu_x / d\nu_y(\zeta) = \exp(\delta(G)\beta_\zeta(y, x))$$

and $g\nu_x = \nu_{g^{-1}x}$ for any $x, y \in X$ and $g \in G$. Any such family is determined by any one of the measures ν_o, $o \in X$ which we can normalize to be a probability measure. A $\delta(G)$ conformal density always exists when G is non-elementary and $\delta(G) < \infty$. When $G \curvearrowright X$ is divergent there is a unique conformal density for $G \curvearrowright X$(see [Lin17], Theorems 10.1 and 10.2 and the remark after Theorem 10.1); it (or any of the measures comprising it) is called the *Patterson-Sullivan measure*. When $G \curvearrowright X$ is cocompact, the Patterson-Sullivan measure can be interpreted as the unique weak limit of ball averages over G orbits in the metric on X. More precisely, for $x \in X$ we may consider the family of measures on X given by

$$\nu_{R,x} = |Gx \cap B_R x|^{-1} \sum_{g \in Gx \cap B_R x} D_{gx}$$

where D_x denotes the point mass at x. Considering $\nu_{R,x}$ as probability measures on the compact space $X \cup \partial X$, they converge (in the weak topology) as $R \to \infty$ to a scalar multiple of ν_x^{PS}. In the context of CAT(0) spaces, conformal densities and Patterson-Sullivan measures were introduced by Ricks [Ric17]. Conformal densities can be used to construct a G and geodesic flow invariant measure on SX as follows. When $G \curvearrowright X$ is divergent, there is up to scale a unique $\delta(G)$ conformal density $\{\nu_x\}_{x \in X}$. The measure $\nu_x \times \nu_x$ gives full measure to endpoints of zero width geodesics, and thus after taking the product with the arc-length normalized Lebesgue measure L can be considered a measure on SX. Using the conformal density property, we can find a G-invariant and geodesic flow invariant Radon measure $\tilde{m}$ on SX in the measure class of $\nu_x \times \nu_x \times dL$, see [Ric17] for details. This measure $\tilde{m}$ projects to a geodesic flow invariant measure m on SX/G; both m and $\tilde{m}$ are called the *Bowen-Margulis measure.* When the Bowen-Margulis measure on SX/G is finite (as is the case for instance when $G \curvearrowright X$ is cocompact) it is ergodic with respect to g_t ([Ric17], Theorem 3) and the Patterson-Sullivan measure is the weak limit of ball averages as in the cocompact case [Lin20]. We summarize the properties of the Patterson-Sullivan measure which we will use below.

Lemma 5.1. [Ric17, Theorem 3] *Suppose $G \curvearrowright X$ is a non-elementary divergent action with $\delta(G) < \infty$ on a rank-1* CAT(0) *space. Assume there is some zero width geodesic with endpoints in the limit set $L(G)$. Assume the Bowen-Margulis measure m on SX/G is finite. Then it is ergodic with respect to the geodesic flow and gives full weight to zero width geodesics. Furthermore, the Patterson-Sullivan measures ν_x on ∂X have full support on $L(G)$ and has no atoms, and the Bowen-Margulis measure $\tilde{m}$ has full support on $S_G X$.*

We now prove Theorem 4.3 for Patterson-Sullivan measures under the assumptions of Lemma 5.1. To do that we will show that for ν^{PS} a.e. $\zeta \in \partial X$ any geodesic ray converging to ζ satisfies the condition of Lemma 3.3. Since two geodesic rays converging to the same point of ∂X are asymptotic, and the property of being frequently contracting is invariant under asymptotic equivalence classes, it suffices to prove that with respect to the Bowen-Margulis measure $\tilde{m}$ on SX, almost every geodesic satisfies the condition of Lemma 3.3.

For any Borel $V \subset SX/G$ the Birkhoff ergodic theorem and the ergodicity of m with respect to the geodesic flow $g_t : t \in \mathbb{R}$ on SX/G implies for m almost every $v \in SX/G$

$$\lim_{T\to\infty} |\{t \in [0, T] : g_t v \in V\}|/T \to m(V).$$

Consequently, if W is any G-invariant Borel subset of SX and $W' \subset W$ a fundamental domain for the G action on W then for $\tilde{m}$ almost every $v \in SX$

$$\lim_{T\to\infty} |\{t \in [0, T] : g_t v \in W\}|/T \to m(W/G).$$

Note, the latter quantity is strictly positive when $\tilde{m}(W) > 0$.

Applying this for every L, N to the G-invariant Borel set $W_{L,N}$ consisting of $v \in SX$ such that $[\pi_{fp}(g_{-L}v), \pi_{fp}(g_L v)]$ c-fellow travels an N contracting geodesic we obtain for m a.e. $v \in SX$,

$$\lim_{T\to\infty} |\{t \in [0, T] : g_t v \in W_{L,N}\}|/T \to m(W_{L,N}/G).$$

Any v for which the left hand side of the above equation converges to a positive number defines a geodesic which satisfies the condition of Lemma 3.3. Thus, it suffices to show that there is an $N > 0$ such that for each $L > 0$ we have $\tilde{m}(W_{L,N}) > 0$.

To that end, let $N > 0$ be large enough so that there exists an N contracting axis $v_0 \in SX$ for a hyperbolic isometry $\gamma \in G$. Let $R, K > 0$. Let $W_{L,v_0} \subset SX$ be the set of v such that $d(\pi_{fp}(g_t v), v_0) < c$ for all $t \in (-L, L)\}$. Clearly $W_{L,v_0} \subset W_{L,N}$. Note W_{L,v_0} is an open subset of SX. Furthermore, it contains $v_0 \in S_G X$. Since $\tilde{m}$ has full support on $S_G X$ it follows that $\tilde{m}(W_{L,v_0}) > 0$ and thus $m(W_{L,N}) > 0$.

6. Stationary measures and random walks

The other family of measures on ∂X what does this mean? we are interested in are stationary measure associated to random walks coming from finitely supported measures μ on G which we now describe.

Let G be an infinite group. Let μ be a symmetric probability measure on G and let $\mu^{\mathbb{Z}}$ be the product measure on $G^{\mathbb{Z}}$.

Let $T : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$ be the following invertible transformation: T takes the two-sided sequence $(h_i)_{i\in\mathbb{Z}}$ to the sequence $(\omega_i)_{i\in\mathbb{Z}}$ with $\omega_0 = e$ and $g_n = g_{n-1}h_n$ for $n \neq 0$. Explicitly, this means

$$\omega_n = h_1 \cdots h_n \quad \text{ for } n > 0$$

and

$$\omega_n = h_0^{-1} h_{-1}^{-1} \cdots h_{-n+1}^{-1} \quad \text{ for } n < 0.$$

Similarly, let $\mu^{\mathbb{N}}$ be the product measure on $G^{\mathbb{N}}$. Let $T_+ : G^{\mathbb{N}} \to G^{\mathbb{N}}$ be the transformation that takes the one-sided infinite sequence $(h_i)_{i\in\mathbb{N}}$ to the sequence $(\omega_i)_{i\in\mathbb{N}}$ with $\omega_0 = e$ and $\omega_n = \omega_{n-1}h_n$ for $n \neq 0$. Explicitly, for $n > 0$ this means

$$\omega_n = h_1 \cdots h_n.$$

Let $\overline{P}$ be the pushforward measure $T_*\mu^{\mathbb{Z}}$ and P the pushforward measure $T_{+*}\mu^{\mathbb{N}}$.

The measure P describes the distribution of μ sample paths, i.e. of products of independent μ-distributed increments. Let $\hat{\mu}$ be the measure on G given by $\hat{\mu}(g) = \mu(g^{-1})$. Let $\hat{P}$ be the pushforward measure $T_{+*}\hat{\mu}^{\mathbb{N}}$. The measure space $(G^{\mathbb{Z}}, \overline{P})$ is naturally isomorphic to $(G^{\mathbb{N}}, P) \otimes (G^{\mathbb{N}}, \hat{P})$ via the map sending the bilateral path ω to the pair of unilateral paths $((\omega_n)_{n\in\mathbb{N}}, (\omega_{-n})_{n\in\mathbb{N}})$.

Let $\sigma : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$ be the left Bernoulli shift: $\sigma(\omega)_n = \omega_{n+1}$. By basic symbolic dynamics, σ is invertible, measure preserving and ergodic with respect to $\mu^{\mathbb{Z}}$. Therefore, when restricted to sequences with e at the $0th$ coordinate,

$$U = T \circ \sigma \circ T^{-1}$$

is invertible, measure preserving and ergodic with respect to $\overline{P}$. Note that for each $n \in \mathbb{Z}$,

$$(U\omega)_n = \omega_1^{-1}\omega_{n+1}$$

and more generally

$$(U^k\omega)_n = \omega_k^{-1}\omega_{n+k}.$$

Suppose G acts continuously on an infinite Hausdorff space B. A Borel probability measure ν on B is called (G, μ) stationary if

$$\nu(A) = \sum_{g\in G} \nu(g^{-1}A)\mu(g)$$

for all Borel $A \subset B$. Suppose now we have a bordification $Z = Y \cup B$ of a metric space X, such that for any basepoint $o \in Y$ and P almost every sample path $\omega = (\omega_n)_{n\in\mathbb{N}}$ the sequence $\omega_n o$ converges to a point $\omega_\infty \in B$ independent of the basepoint o. The probability measure on B given by $\nu(A) = P(\omega : \omega_\infty \in A)$ is then the unique (G, μ) stationary measure on B; moreover for P a.e. $\omega \in G^{\mathbb{N}}$ the pushforward measures $\omega_n^* \nu$ weakly converges to an atom concentrated at some $\omega_\infty \in B$. A space B with a stationary measure ν satisfying the last condition is called a (G, μ) boundary. A (G, μ) boundary (B, ν) is said to be a Poisson boundary of (G, μ) if it is maximal in the sense that for any other (G, μ) boundary (B', ν') there is a G equivariant measurable surjection $B \to B'$. The Poisson boundary is unique up to G-equivariant measurable isomorphism.

Lemma 6.1. *Suppose $G \curvearrowright B$ is a minimal action on a compact Hausdorff space such that every G orbit in B is infinite and ν a stationary measure on X. Then ν has no atoms and has full support on B.*

is there a citation for this? Karlsson and Margulis [KM99] showed that under mild conditions the visual boundary of a CAT(0) space provides a model for the Poisson boundary of a group acting on the space.

Theorem 6.2. [KM99] *Let X be a* CAT(0) *space, $o \in X$ and $G \curvearrowright X$ a nonamenable group acting on X by isometries with bounded exponential growth. Let μ be a probability measure on G whose finite support generates G as a semigroup. Then for $P = P^\mu$ almost every $\omega \in G^{\mathbb{N}}$, $\omega_n o$ converges to a point in ∂X. Moreover $(\partial X, \nu)$ is a model for the Poisson boundary of (G, μ) where ν is the unique μ stationary measure on ∂X.*

Lemma 6.3. [Kai00] *The action of any group G on the square of it's Poisson boundary with respect to the square of the stationary measure associated to a symmetric random walk preserves the measure class and is ergodic.*

If X is a rank-1 CAT(0) space, and $G \curvearrowright X$ a rank-1 group action then for any $c > 0$ the pairs of points of $L(G)$ which are the endpoints of a rank-1 which does not bound a flat strip of width $> c$ form a G-invariant open subset of $L(G)$. Consequently if ν is any nonatomic probability measure on $L(G) \subset \partial X$ with full support on $L(G)$ such that the G action on $\partial X \times \partial X$ preserves the measure class of $\nu \times \nu$ and is ergodic with respect to it, then there is a $c \geq 0$ such that $\nu \times \nu$ gives full weight to pairs of points bounding strips of width at least c.

We thus have:

Lemma 6.4. *Let X be a geodesically complete rank-1 CAT(0) space and $G \curvearrowright X$ a rank-1 group action. Let μ be a symmetric probability measure on G whose finite support generates G and ν the associated stationary measure on ∂X. Then $\nu \times \nu$ is ergodic with respect to the G action and for some $c > 0$ gives full weight to pairs of points defining geodesics of width at most c.*

By Kingman's ergodic theorem and the nonamenability of G there is an $l = l(\mu) > 0$ such that $l(\mu) = \lim_{n\to\infty} d(\omega_n o, o)/n$ for P-a.e. ω, called the drift of the random walk with respect to d. Karlsson-Margulis proved that for P almost every ω there is a parametrized unit speed geodesic $\tau \in SX$ such that $d(\tau(ln), \omega_n o)/n \to 0$.

Our goal is to show the following.

Theorem 6.5. *Let X, ν, μ be as in Lemma 6.4. Then for any basepoint $o \in X$ and ν almost every $\zeta \in \partial X$ the geodesic ray $\tau_{o,\zeta}$ through o converging to ζ is sublinearly Morse.*

We will prove this by showing that for ν a.e. ζ, $\tau_{o,\zeta}$ is frequently contracting. In fact, we will prove a stronger statement:

Proposition 6.6. *Let $g_0 \in G$ be a rank-1 periodic element and γ_0 its axis. Fix a basepoint $o \in X$. Then there is a $K > 0$ such that for ν a.e. ζ the geodesic ray $\tau_{o,\zeta}$ satisfies the following. For any $b > a > 0$ and $L > 0$ there is an $R_0 > 0$ such that for any $R > R_0$ there is a $g \in G$ such that the segment $\tau([aR, bR])$ contains a subsegment of $N_K(g\gamma_0)$ of length L.*

For each $\zeta_1, \zeta_2 \in \partial X$ and $p \in X$, let $\Psi(\zeta_1, \zeta_2) = E^{-1}(\zeta_1, \zeta_2)$ be the set of unit speed geodesics with endpoints $\zeta_1, \zeta_2 \in \partial X$. Let $\Psi(\zeta_1, \zeta_2, p)$ be the set of unit speed parametrizations of such geodesics γ such that $\gamma(0)$ is at minimal distance from p. We can make this choice in a G equivariant way, i.e. so that $g\Psi(\zeta_1, \zeta_2, p) = \Psi(g\zeta_1, g\zeta_2, gp)$. For a bilateral sample path ω converging to $\omega_-, \omega_+ \in \partial X$ write $\Psi(\omega, p)$ and $\Psi(\omega)$ instead of $\Psi(\omega_-, \omega_+, p)$ and $\Psi(\omega_+, \omega_-)$. for the image of the geodesic in X. Similarly for an unparametrized biinfinite geodesic γ and $p \in X$ we write γ_p for the unit speed parametrization with $\gamma_p(0)$ at minimum distance from p.

Proposition 6.6 will follow from the following bilateral statement.

Proposition 6.7. *Let $g_0 \in G$ be a rank-1 periodic element and γ_0 its axis. Fix a basepoint $o \in X$. Then there is a $K > 0$ such that for $\overline{P}$ a.e. biinfinite sample path ω any parametrization of any biinfinite geodesic $\gamma \in \Psi(\omega)$ satisfies the following. For any $\infty > b > a > -\infty$ and $L > 0$ there is an $R_0 > 0$ such that for any $R > R_0$ there is a $g \in G$ such that the segment $\gamma([aR, bR])$ contains a subsegment of $N_K(g\gamma_0)$ of length L.*

The remainder of this section is devoted to the proof of Proposition 6.7.

Let $\Omega(M, K, R)$ be the set of sample paths $\omega \in G^{\mathbb{Z}}$ such that, for all $\gamma \in \Psi(\omega)$, we have $d(o, \gamma) < R/10$ and $\gamma_{\omega,o}(t - M, t + M) \subset N_K(g\gamma_0)$ for some $g \in G$ and $t \in (-R/2 + M, R/2 - M)$.

Lemma 6.8. *There is a $K > 0$ such that for all $M > 0$ there is an function f with $\lim_{R\to\infty} f(R) = 0$ and $\overline{P}(\Omega(M, K, R)) > 1 - f(R)$.*

We first continue with the proof of Proposition 6.7 assuming Lemma 7.2 and will prove Lemma 7.2 afterwards.

Proof of Proposition 6.7 assuming Lemma 7.2. Assume without loss of generality that $a > 0$. Let $\Omega_0 \subset G^{\mathbb{Z}}$ denote the $\overline{P}$ full measure set of all ω such that $d(\omega_{\pm i}o, o)/i \to l$ and such that $\omega_{\pm i}o \to \zeta_\pm \in \partial X$ with (ζ_-, ζ_+) having width at most c. Note $\overline{P}(\Omega_0) = 1$. Consider $\omega \in \Omega_0$. Choose $R > 0$ large enough so that

$$1 - f(R) > (b - a)/(10a + 10b)$$

Note, $U^i\omega \in \Omega(M, K, R)$ if and only if for all $\gamma \in \Psi(\omega)$ $d(\omega_i o, \gamma) < R/10$ and $\gamma_{\omega_i o}(t - M, t + M) \subset N_K(g\gamma_0)$ for some $g \in G$ and $t \in (-R/2 + M, R/2 - M)$. This implies that for all $\gamma \in \Psi(\omega)$

$$\gamma_o(t_i - M, t_i + M) \subset N_K g_i \gamma_0$$

for some $g_i \in G$ and t_i with $|t_i - d(\omega_i o, \gamma_o(0))| < R/10$.

Let $s_i(\gamma) = d(\omega_i o, \gamma_o(0))$. Let

$$d = \sup\{d(o, go) \mid g \in \operatorname{supp}(\mu)\}.$$

Note since $d(\omega_i o, \omega_{i+1} o) \leq d$ for all i, for every $t > d(o, \gamma)$ there is some $i(t)$ with $|t - s_{i(t)}(\gamma)| < d$.

Hence, for large enough (depending on ω) n, for all $\gamma \in \Psi(\omega)$, if there is an i with

$$U^i \omega \in \Omega(M, K, R)$$

and

$$(2a + b)n/3 \leq s_i(\gamma) \leq (a + 2b)n/3$$

then $\gamma_o([an, bn])$ has a connected segment in $N_K(g\gamma_0)$ of length M for some $g \in G$. Moreover, $s_i(\gamma)/i \to l$ for all $\gamma \in \Psi(\omega)$. Thus, for large enough n, we have

$$(2a + b)n/3 \leq s_i(\gamma) \leq (a + 2b)n/3$$

for every i with

$$\frac{(3a + 2b)n}{5l} \leq i \leq \frac{(2a + 3b)n}{5l}.$$

Hence, unless $\gamma_o([an, bn])$ has a connected segment in $N_K(g\gamma_0)$ of length M for some $g \in G$, we have

$$U^i \omega \notin \Omega(M, K, R)$$

for any

$$\frac{(3a + 2b)n}{5L} \leq i \leq \frac{(2a + 3b)n}{5L}$$

If this holds for infinitely many n we have

$$\liminf_{N \to \infty} \frac{|\{i \in [0, N-1] \mid U^i \omega \in \Omega(M, K, R)\}|}{N} \leq 1 - \frac{b - a}{2a + 3b}.$$

On the other hand, by the Birkhoff ergodic theorem, for $\overline{P}$ ω we have:

$$\lim_{N \to \infty} \frac{|\{i \in [0, N-1] \mid U^i \omega \in \Omega(M, K, R)\}|}{N}$$
$$= \overline{P}(\Omega(M, K, R)) > 1 - (b - a)/(10a + 10b)$$

giving a contradiction. □

Finally, we prove Lemma 7.2.

Proof of Lemma 7.2. Clearly the $\overline{P}$ measure of $\omega \in G^{\mathbb{Z}}$ such that $d(o, [\omega_-, \omega_+]) < R/10$ converges to 1 with R. Thus it suffices to show that for each $M > 0$ the $\overline{P}$ measure of ω such that $\gamma_{\omega,o}(t - M, t + M) \subset N_K(g\gamma_0)$ for some $g \in G$ and $t \in (-R/2 + M, R/2 - M)$ converges to 1 as $R \to \infty$. Let $\Lambda(M, K)$ be the set of biinfinite sample paths ω such that $d(\gamma_o(t), \gamma_0(t)) < K$ for all $\gamma \in \Psi(\omega)$.

Claim 6.9. There is a $K > 0$ such that $\overline{P}(\Lambda(M, K)) > 0$ for all M.

Proof. Let K be large enough so that the periodic rank one geodesic γ_0 passes within $K/4$ of o and has width less than $K/4$. Let $\gamma_-, \gamma_+ \in \partial X$ be its limit points and parametrize γ_0 such that $\gamma_0(0)$ is at minimal distance to o. Let c' be the width of the flat strip bounded by γ_0 By [Bal95] for each $M > 0$ there are neighborhoods $A_\pm$ of $\gamma_\pm$ in ∂X such that any pair in $U_- \times U_+$ can be connected by a geodesic of width less than $K/2$ which passes within $K/2$ of $\gamma_0(t)$ for all $|t| < 4M$. Moreover, if $\zeta_n^- \to \gamma_-$ and $\zeta_n^+ \to \gamma_+$ and γ_n is a geodesic connecting $\zeta_n^\pm$ then γ_n converges locally uniformly to γ_0. Thus we have $d(\gamma(t), \gamma_0(t)) < K$ for $|t| < M$ for any geodesic γ, parametrized with $\gamma(0)$ at minimal distance to o, connecting pairs of points in $U^- \times U^+$. Let $\Lambda'(M, K)$ be the set of all sample paths ω with $\omega^\pm \in U^\pm$. By definition, $\Lambda'(M, K) \subset \Lambda(M, K)$. Since the $U^\pm$ are open neighborhoods of $\gamma_{0\pm} \in L(G)$ in ∂X and the harmonic measure ν has

full support on the limit set $L(G) \subset \partial X$, we have $\nu(U^\pm) > 0$ and hence $\overline{P}(\Lambda'(M, K)) > 0$ and thus $\overline{P}(\Lambda(M, K)) > 0$. □

Note, $U^i\omega \in \Lambda(M, K)$ if and only if $d(\gamma_o(t), \omega_i\gamma_0(t)) < K$ for all $\gamma \in \Psi(\omega)$. Note, $d(o, \omega_i o) \leq di$ and hence if

$$U^i\omega \in \Lambda(M, K)$$

for some i with

$$0 \leq i \leq \frac{R - M - 2K}{2d}$$

then for all $\gamma \in \Psi(\omega)$. $\gamma_o([-R, R])$ contains a length M segment in $N_K g\gamma_0$ for some $g \in G$. By the Birkhoff ergodic theorem, the $\overline{P}$ measure of sample paths ω such that $U^i\omega \notin \Lambda(M, K)$ for all i with

$$0 \leq i \leq \frac{R - M - 2K}{2d}$$

converges to 0 with R completing the proof. □

7. Genericity of sublinearly Morse geodesics in Teichmüller space

In this section we consider the Teichmüller space $T(S)$ of a closed genus $g \geq 2$ surface S with the Teichmüller metric $d = d_T$. There is a natural *Thurston compactification* of $T(S)$ by the space $\mathcal{PML}$ of all measured projective laminations on S. Any Teichmüller geodesic ray with a uniquely ergodic vertical foliation converges to its projective class in the Thurston compactification. Filling pairs of laminations are an open subset of $\mathcal{PML} \times \mathcal{PML}$, and any such pair determines a Teichmüller geodesic with corresponding vertical and horizontal measured foliations. Moreover, if a sequence of such pairs converges to the pseudo-Anosov pair (ϕ_-, ϕ_+), then the corresponding geodesics converge locally uniformly to $\gamma_{\phi_-,\phi_+,o}$. Moreover, any distinct pair of elements of $\mathcal{PML}$ is filling as long as at least one element of it is uniquely ergodic. We show that for several natural classes of measures on $\mathcal{PML}$ the limit points of sublinearly contracting geodesics are generic. The measures we consider are the Thurston measure on $\mathcal{PMF}$ (which can be considered as the analogue of the Lebesgue or Patterson-Sullivan measure on the boundary of a hyperbolic manifold) and stationary measures coming from finitely supported random walks on the mapping class group $MCG(S)$.

7.1. **Thurston measures on $\mathcal{PML}$.** Let ν be a normalized Thurston measure on $\mathcal{PML}$. The measure $\nu \times \nu$ gives full measure to uniquely ergodic foliations, and thus after taking the product with the arc-length (with respect to Teichnueller metric) normalized Lebesgue measure L can be considered a measure on the space Q^1 of unit area quadratic differentials, which can be seen as the (co)tangent bundle to Teichmüller space. We can find a G-invariant and Teichmüller geodesic flow invariant Radon measure $\tilde{m}$ on Q^1 in the measure class of $\nu \times \nu \times dL$, see [**?**] what's this paper? for details. This measure $\tilde{m}$ projects to a finite Teichmüller geodesic flow invariant and ergodic measure m on Q^1/G, called the Masur-Veech measure; both m and $\tilde{m}$ are called the *Bowen-Margulis measure*. The proof now proceeds as for the Patterson-Sullivan measure in the CAT(0) setting.

7.2. **Stationary measures on $\mathcal{PML}$.** A subgroup of $MCG(S)$ is called non-elementary if it contains two pseudo-Anosov elements with disjoint fixed point sets in $\mathcal{PML}$. A measure μ on $G = MCG(S)$ is said to be non-elementary if the semigroup generated by its support is a non-elementary subgroup.

Let μ be such a symmetric finitely supported non-elementary measure, $G < MCG(S)$ the subgroup generated by its support, and P, $\overline{P}$ the induced Markov measures on unilateral and bilateral sample paths respectively. Kaimanovich-Masur [KM96] proved that for P-almost every ω, and every $o \in \mathcal{T}(S)$, $\omega_n o$ converges to a uniquely ergodic point $\omega_\infty \in \mathcal{PML}$. In other words, there is a P-almost everywhere defined measurable map $\mathrm{bnd} : G^{\mathbb{N}} \to \mathcal{PML}$ sending ω to $\lim_{n\to\infty} \omega_n o \in \mathcal{PML}$. The measure on $\mathcal{PML}$ defined by

$$\nu = \mathrm{bnd}_* P = \lim_{n\to\infty} \mu^{*n}$$

is the unique μ stationary measure on $\mathcal{PML}$. In fact, $(\mathcal{PML}, \nu)$ is a model for the *Poisson boundary* of (G, μ). Moreover, ν gives full weight to uniquely ergodic foliations and has full support on the limit set $L(G) \subset \mathcal{PML}$ of the group $G < MCG(S)$ generated by the support of μ [KM96]. Let $l = \lim_{n\to\infty} d(\omega_n o, o)/n$ (for P a.e. ω) be the drift of the μ random walk. Tiozzo [Tio12] proved that P a.e. ω sublinearly tracks a geodesic τ in $T(S)$:

$$\lim_{n\to\infty} \frac{d(\tau(ln), \omega_n o)}{n} = 0$$

for any geodesic ray τ converging to $\omega_\infty \in \mathcal{PML}$.

We prove the following

Proposition 7.1. *Let $g_0 \in G$ be a pseudo-Anosov element and γ_0 its axis in $Teich(S)$. Fix a basepoint $o \in Teich(S)$. Then there is a $K > 0$ such that for $\overline{P}$ a.e. biinfinite sample path ω (any unit speed parametrization of) the biinfinite geodesic γ_ω satisfies the following. For any $\infty > b > a > -\infty$ and $L > 0$ there is an $R_0 > 0$ such that for any $R > R_0$ there is a $g \in G$ such that the segment $\gamma([aR, bR])$ contains a subsegment of $N_K(g\gamma_0)$ of length L.*

Let $\Omega(M, K, R)$ be the set of sample paths $\omega \in G^{\mathbb{Z}}$ such that $\omega_\pm \in PML$ are uniquely ergodic, $d(o, \gamma_\omega) < R/10$ and $\gamma_{\omega,o}(t - M, t + M) \subset N_K(g\gamma_0)$, for some $g \in G$ and $t \in (-R/2 + M, R/2 - M)$.

Lemma 7.2. *There is a $K > 0$ such that for all $M > 0$ there is an function f with $\lim_{R\to\infty} f(R) = 0$ and $\overline{P}(\Omega(M, K, R)) > 1 - f(R)$.*

We first continue with the proof of Proposition 6.7 assuming Lemma 7.2 and will prove Lemma 7.2 afterwards.

Proof of Proposition 6.7 assuming Lemma 7.2. Without loss of generality, let us assume that $a > 0$. Let $\Omega_0 \subset G^{\mathbb{Z}}$ denote the set of all ω such that

$$d(\omega_{\pm i}o, o)/i \to l$$

and such that $\omega_{\pm i}o \to \zeta_\pm \in \partial X$ which are distinct and uniquely ergodic. Note $\overline{P}(\Omega_0) = 1$. Consider $\omega \in \Omega_0$. Choose $R > 0$ large enough so that

$$1 - f(R) > (b - a)/(10a + 10b)$$

Note, $U^i\omega \in \Omega(M, K, R)$ if and only if for all $\gamma \in \Psi(\omega)$,

we have $d(\omega_i o, \gamma_\omega) < R/10$ and

$$\gamma_{\omega,\omega_i o}(t - M, t + M) \subset N_K(g\gamma_0) \text{ for some } g \in G$$

and we also have $t \in (-R/2 + M, R/2 - M)$. This implies that

$$\gamma_{\omega,o}(t_i - M, t_i + M) \subset N_K g_i \gamma_0$$

for some $g_i \in G$ and t_i with $|t_i - d(\omega_i o, \gamma_{\omega,o}(0))| < R/10$.

Let $s_i(\omega) = d(\omega_i o, \gamma_{\omega,o}(0))$. Let

$$d = \sup\{d(o, go) \mid g \in \text{supp}(\mu)\}.$$

Note since $d(\omega_i o, \omega_{i+1} o) \leq d$ for all i, for every $t > d(o, \gamma_\omega)$ there is some $i(t)$ with

$$|t - s_{i(t)}(\gamma)| < d.$$

Hence, for large enough (depending on ω) n, for all $\gamma \in \Psi(\omega)$, if there is an i with

$$U^i\omega \in \Omega(M, K, R)$$

and

$$(2a + b)n/3 \leq s_i(\gamma_\omega) \leq (a + 2b)n/3$$

then $\gamma_{\omega,o}([an, bn])$ has a connected segment in $N_K(g\gamma_0)$ of length M for some $g \in G$. Moreover, $s_i(\omega)/i \to l$. Thus, for large enough n, we have

$$(2a + b)n/3 \leq s_i(\gamma) \leq (a + 2b)n/3$$

for every i with

$$\frac{(3a + 2b)n}{5l} \leq i \leq \frac{(2a + 3b)n}{5l}.$$

Hence, unless $\gamma_{\omega,o}([an, bn])$ has a connected segment in $N_K(g\gamma_0)$ of length M for some $g \in G$, we have

$$U^i\omega \notin \Omega(M, K, R)$$

for any

$$\frac{(3a + 2b)n}{5L} \leq i \leq \frac{(2a + 3b)n}{5L}$$

If this holds for infinitely many n we have

$$\liminf_{N\to\infty} \frac{|\{i \in [0, N-1] \mid U^i\omega \in \Omega(M,K,R)\}|}{N} \le 1 - \frac{b-a}{2a+3b}.$$

On the other hand, by the Birkhoff ergodic theorem, for $\overline{P}$ ω we have:

$$\begin{aligned}&\lim_{N\to\infty} \frac{|\{i \in [0, N-1] \mid U^i\omega \in \Omega(M,K,R)\}|}{N}\\ &= \overline{P}(\Omega(M,K,R)) > 1 - (b-a)/(10a+10b)\end{aligned}$$

giving a contradiction. □

Finally, we prove Lemma 7.2.

Proof of Lemma 7.2. Clearly the $\overline{P}$ measure of $\omega \in G^{\mathbb{Z}}$ such that $d(o, [\omega_-, \omega_+]) < R/10$ converges to 1 with R. Thus it suffices to show that for each $M > 0$ the $\overline{P}$ measure of ω such that $\gamma_{\omega,o}(t-M, t+M) \subset N_K(g\gamma_0)$ for some $g \in G$ and $t \in (-R/2+M, R/2-M)$ converges to 1 as $R \to \infty$. Let $\Lambda(M,K)$ be the set of biinfinite sample paths ω such that $d(\gamma_{\omega,o}(t), \gamma_0(t)) < K$.

Claim 7.3. There is a $K > 0$ such that $\overline{P}(\Lambda(M,K)) > 0$ for all M.

Proof. Let K be large enough so that the pseudo-Anosov axis γ_0 passes within $K/2$ of o. Let $\gamma_-, \gamma_+ \in \partial X$ be its limit points and parametrize γ_0 such that $\gamma_0(0)$ is at minimal distance to o. There are neighborhoods $A_\pm$ of $\gamma_\pm$ in $\mathcal{PML}$ such that any pair in $U_- \times U_+$ determines a geodesic which passes within $K/2$ of $\gamma_0(t)$ for all $|t| < 4M$.

Moreover, if $\zeta_n^- \to \gamma_-$ and $\zeta_n^+ \to \gamma_+$ and γ_n is a geodesic connecting $\zeta_n^\pm$ then γ_n converges locally uniformly to γ_0. Thus we have $d(\gamma(t), \gamma_0(t)) < K$ for $|t| < M$ for any geodesic γ, parametrized with $\gamma(0)$ at minimal distance to o, connecting pairs of points in $U^- \times U^+$. Let $\Lambda'(M,K)$ be the set of all sample paths ω with $\omega^\pm \in U^\pm$. By definition, $\Lambda'(M,K) \subset \Lambda(M,K)$. Since the $U^\pm$ are open neighborhoods of $\gamma_{0\pm} \in L(G)$ in ∂X and the harmonic measure ν has full support on the limit set $L(G) \subset PML$, we have $\nu(U^\pm) > 0$ and hence

$$\overline{P}(\Lambda'(M,K)) > 0,$$

thus $\overline{P}(\Lambda(M,K)) > 0$. □

Note, $U^i\omega \in \Lambda(M,K)$ if and only if $d(\gamma_{\omega,\omega_i o}(t), \gamma_0(t)) < K$. Note, $d(o, \omega_i o) \le di$ is this a typo? and hence if

$$U^i\omega \in \Lambda(M,K)$$

for some i with

$$0 \le i \le \frac{R-M-2K}{2d}$$

then $\gamma_{\omega,o}([-R,R])$ contains a length M segment in $N_K g\gamma_0$ for some $g \in G$. By the Birkhoff ergodic theorem, the $\overline{P}$ measure of sample paths ω such that $U^i\omega \notin \Lambda(M,K)$ for all i with

$$0 \le i \le \frac{R-M-2K}{2d}$$

converges to 0 with R completing the proof. □

8. Identification of the Poisson Boundary

The following is proved in Theorem B of [QRT22].

Theorem 8.1. *Let G be a nonamenable group acting properly discontinuously by isometries on a proper geodesic metric space (X, d). Let μ be a finitely supported measure on G whose support generates G as a semigroup. Assume that μ almost every sample path ω sublinearly tracks some κ contracting geodesic τ, i.e. $d(\omega_n o, \tau)/n \to 0$ for some and equivalently every $o \in X$. Then the κ-contracting boundary with the unique μ stationary probability measure is a model for the Poisson boundary of (G, μ).*

It is known that when X is CAT(0) or Teichmüller space and the semigroup G generated by the support of μ is non-elementary, almost every μ sample path sublinearly tracks some X geodesic ray τ_ω ([Tio12] for Teichmüller space and [**?**] for CAT(0) spaces). Our results above show that there is a sublinear function κ such that for a.e. sample path ω, $\omega_n o$ converges to a $\zeta \in \partial X$ such that any geodesic $[o, \zeta)$ is κ sublinear. Thus τ_ω is almost surely κ-sublinear.

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