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Bounded combinatorics and the Lipschitz metric on Teichmüller space

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Abstract Considering the Teichmüller space of a surface equipped with Thurston's Lipschitz metric, we study geodesic segments whose endpoints have bounded combinatorics. We show that these geodesics are cobounded, and that the closest-point projection to these geodesics is strongly contracting. Consequently, these geodesics are stable. Our main tool is to show that one can get a good estimate for the Lipschitz distance by considering the length ratio of finitely many curves.

Keywords Teichmuller space · Teichmuller metric · Lipschitz metric · Fellow traveling · Stability

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1 Introduction

Let $\mathcal{T}(S)$ be the Teichmüller space of a surface S of finite type, that is, the space of marked hyperbolic (or conformal) structures on S. In [18], Thurston introduced an asymmetric metric d_L for $\mathcal{T}(S)$ which we refer to as the Lipschitz metric. For two marked hyperbolic structures x and y, $d_L(x, y)$ is defined to be the logarithm of the infimum of Lipschitz constants of any homeomorphism from x to y that is homotopic to the identity. The geometry of the Lipschitz

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metric is very rich, as Thurston shows in his paper. However, many aspects of it remain unexamined.

It is known that Teichmüller space equipped with the Teichmüller metric or the Lipschitz metric is not Gromov hyperbolic because the thin parts of $\mathcal{T}(S)$ have a product like structure (see [4,11]). However, certain Teichmüller geodesics exhibit behaviors that resemble that of geodesics in a hyperbolic space. Namely, the closest-point projection to these geodesics is strongly contracting [12]. In this paper, we investigate whether a similar phenomenon is also present in the Lipschitz metric.

We use tools that have been developed and successfully applied in the study of Teichmüller geodesics, namely the curve graphs of different subsurfaces of S. When x is in the thick part, the geometry of x can be coarsely encoded by its associated short marking μ_x, which is a finite collection of simple closed curves. Given $x, y \in \mathcal{T}(S)$, there are many results relating the combinatorics of markings μ_x and μ_y to the behavior of the Teichmüller geodesic connecting x and y. (See [4,15,16], or [17] for a review of some of these results in one paper.)

Contrasting with the Teichmüller metric, there is no unique geodesic in the Lipschitz metric from x to y. But one hopes that qualitative information about a Lipschitz geodesic can still be extracted from the end markings μ_x and μ_y. The first natural situation to consider is when μ_x and μ_y have bounded combinatorics. That is when, for every proper subsurface Y of S, the distance $d_Y(\mu_x, \mu_y)$ in the curve graph of Y between the projections of μ_x and μ_y to Y is uniformly bounded (see Definition 2.2). For the Teichmüller metric, this is in fact equivalent to the Teichmüller geodesic between x and y being cobounded (see [15,17]).

Our first result is that if μ_x and μ_y have bounded combinatorics, then every Lipschitz geodesic from x to y is cobounded. In fact, they are all well approximated by the unique Teichmüller geodesic connecting x and y.

Theorem A (Bounded combinatorics implies cobounded) *Assume, for $x, y \in \mathcal{T}(S)$ in the thick part of Teichmüller space, that $d_Y(\mu_x, \mu_y)$ is uniformly bounded for every proper subsurface $Y \subset S$. Then any geodesic $\mathcal{G}_L$ in the Lipschitz metric connecting x to y fellow travels the Teichmüller geodesic $\mathcal{G}_T$ with endpoints x and y. Consequently, $\mathcal{G}_L$ is cobounded.*

To restate Theorem A more succinctly is to say that $\mathcal{G}_T$, viewed as a set in the Lipschitz metric, is quasi-convex. A standard argument for showing a set is quasi-convex is to show that the closest-point projection to the set is strongly contracting. Indeed, this is how we prove Theorem A.

Theorem B (Lipschitz projection to Teichmüller geodesics) *Let $\mathcal{G}_T$ be a cobounded Teichmüller geodesic. Then the image of a Lipschitz ball disjoint from $\mathcal{G}_T$ under the closest-point projection to $\mathcal{G}_T$ (with respect to the Lipschitz metric) has uniformly bounded diameter. That is, the closest-point projection to $\mathcal{G}_T$ is strongly contracting.*

This is analogous to Minsky's theorem [12] that the closest-point projection in the Teichmüller metric to a cobounded Teichmüller geodesic is strongly contracting. Combining Theorem A and Theorem B, we obtain:

Theorem C (Strongly contracting for projections to Lipschitz geodesics) *Suppose $\mathcal{G}_L$ is a Lipschitz geodesic whose endpoints have bounded combinatorics. Then the closest-point projection to $\mathcal{G}_L$ is strongly contracting.*

Theorem C is a negative-curvature phenomenon. A natural consequence is stability of $\mathcal{G}_L$. In other words,

Corollary D (Stability of Lipschitz geodesics) *If $\mathcal{G}_L$ is a Lipschitz geodesic whose endpoints have bounded combinatorics, then any quasi-geodesic with the same endpoints as $\mathcal{G}_L$ fellow travels $\mathcal{G}_L$.*

It would be interesting to know whether the converse of Theorem A holds. In the Teichmüller metric, a geodesic stays in the thick part if and only if the endpoints have bounded combinatorics. However, this seems not to be the case for the Lipschitz metric. We investigate the behavior of a Lipschitz geodesic where the endpoints do not necessarily have bounded combinatorics in a subsequent paper.

Summary of the proofs. We use the detour through a Teichmüller geodesic for two reasons. First, because it is already established that $\mathcal{G}_T$ is cobounded if and only if the endpoints have bounded combinatorics. But also because the lengths of curves (both hyperbolic length and extremal length) along a cobounded Teichmüller geodesic are known to behave like a cosh function; the length of a curve α is minimal at the balanced point $\mathcal{G}_T(t_\alpha)$ and grows exponentially fast in both directions.

Our proof of Theorem B is to a large extent inspired by Minsky's proof in the Teichmüller setting. However, the following crucial length estimate used by Minsky has no analogue in our setting. Given a curve α and $x \in \mathcal{T}(S)$, let $\mathrm{Ext}_x(\alpha)$ and $\ell_x(\alpha)$ denote respectively the extremal length and the hyperbolic length of α in x. For every two curves α and β, Minsky showed that

$$\mathrm{Ext}_x(\alpha)\,\mathrm{Ext}_x(\beta) \geq \mathrm{i}(\alpha, \beta)^2, \tag{1}$$

where $\mathrm{i}(\alpha, \beta)$ is the geometric intersection number between α and β. While the Teichmüller distance is computed using extremal length ratios of curves (Eq. 2), the Lipschitz distance is computed using hyperbolic length ratios (Eq. 3). However, there is no analogue of Eq. (1) for hyperbolic length. For x in the thin part of Teichmüller space, the product $\ell_x(\alpha)\ell_x(\beta)$ can be arbitrarily close to zero, while $\mathrm{i}(\alpha, \beta)$ can be arbitrarily large.

Our approach to the proof of Theorem B is to give an effective description of the closest-point projection $\pi_{\mathcal{G}_T}(x)$ of a point $x \in \mathcal{T}(S)$ to a Teichmüller geodesic $\mathcal{G}_T$ (the closest-point projection is with respect to the Lipschitz metric). Let μ_x be a short marking on x. Then $\pi_{\mathcal{G}_T}(x)$ is near $\mathcal{G}_T(t_\alpha)$, where t_α is the balanced time of a curve $\alpha \in \mu_x$ (see Lemma 4.4). This follows from the cosh-like behavior of lengths along a Teichmüller geodesic and the following:

Theorem E (Candidate curves) *For $x, y \in \mathcal{T}(S)$, we have*

$$d_L(x, y) \stackrel{+}{\asymp} \max_{\alpha \in \mu_x} \log \frac{\ell_y(\alpha)}{\ell_x(\alpha)},$$

where $d_L(x, y)$ is the Lipschitz distance from x to y and $\stackrel{+}{\asymp}$ means equal up to an additive error depending only on the topology of S.

A special case of Theorem E where x and y are assumed to be in the thick part of $\mathcal{T}(S)$ was done in [3]. Thurston's formula (Eq. 3) for the Lipschitz distance implies that there is some curve α such that $\log \frac{\ell_y(\alpha)}{\ell_x(\alpha)}$ is a good estimate for $d_L(x, y)$. Theorem E implies that, to find such an α, one only needs to examine the finitely many curves that appear in μ_x. We will call such a curve α in μ_x a *candidate curve* from x to y.

The proof of Theorem E requires some way of estimating the hyperbolic length of a curve in terms of a marking on S. We derive two formulas for this purpose and their proofs take up a large part of the paper. The first formula allows us to estimate, up to a multiplicative error, the length of any curve γ by a linear sum of the lengths of the curves in a short marking, with coefficients coming from the intersection of γ with the curves in the marking (Proposition 3.1). The proof relies on the geometry of the thick-thin decomposition of a hyperbolic surface. The second formula uses a topological argument to show that, if the short marking is replaced by an arbitrary marking, then the same formula still provides an upper bound for the length of the curve (Proposition 3.2). Using these two propositions, we prove Theorem E and Theorem B in Sect. 4. These propositions also have analogues in extremal length, which we use to sketch an alternate proof of Minsky's theorem at the end of Sect. 4. We end the paper with a proof of Theorem A and Theorem C in Sect. 5.

Analogues with Weil-Petersson geodesics. As we have mentioned before, a Teichmüller geodesic is cobounded if and only if its endpoints have bounded combinatorics. In [2], Brock, Masur and Minsky showed a similar result for bi-infinite geodesics in the Weil-Petersson metric on Teichmüller space. As in our paper, the main tool is to show that some projection map is contracting. In their case, what they need (and what they show) is that the projection in the pants decomposition complex to any hierarchy path satisfying the *non-annular bounded combinatorics* property is *coarsely contracting* [2, Theorem 4.1].

Analogues with Outer space. Let $\mathcal{X}_n$ be the Outer Space, the space of marked metric graphs of rank n modulo homothety. The space $\mathcal{X}_n$ is naturally equipped with the Lipschitz metric, on which $\mathrm{Out}(\mathbb{F}_n) = \mathrm{Aut}(\mathbb{F}_n)/\mathrm{Inn}(\mathbb{F}_n)$ acts as isometries.

In [1], Algom-Kfir established a version of Theorem C for a family of geodesics in $\mathcal{X}_n$. It was shown that the closest-point projection to axes of fully irreducible elements of $\mathrm{Out}(\mathbb{F}_n)$ is strongly contracting. This result gives another parallel between fully irreducible elements of $\mathrm{Out}(\mathbb{F}_n)$ and pseudo-Anosov elements of the mapping class group of S. A generalization of this result for a larger class of paths (lines of minima) appears in [6].

An analogue of Theorem E exists for $\mathcal{X}_n$. By a result of White, to compute the Lipschitz distance from one graph to another, it suffices to consider the length ratios of a finite collection of loops. (See [1, Proposition 2.3] for a proof of this fact.)

2 Preliminaries

Teichmüller and Lipschitz metrics. Let S be a connected, oriented surface of finite type with $\chi(S) < 0$. The Teichmüller space $\mathcal{T}(S)$ of S is the space of marked conformal structures on S up to isotopy. Via uniformization, $\mathcal{T}(S)$ is also the space of marked (finite-area) hyperbolic metrics on S up to isotopy.

In this paper, we consider two metrics on $\mathcal{T}(S)$, the Teichmüller metric and Lipschitz metric. Given $x, y \in \mathcal{T}(S)$, the *Teichmüller distance* between them is defined to be

$$d_T(x, y) = \frac{1}{2} \inf_f \log K(f),$$

where $f\colon x \to y$ is a $K(f)$-quasi-conformal map preserving the marking. (See [5,7] for background information.) Introduced by Thurston in [18], the *Lipschitz distance from x to y* is defined to be

$$d_L(x, y) = \inf_f \log L(f),$$

where $f\colon x \to y$ is a $L(f)$-Lipschitz map preserving the marking. Unlike the Teichmüller metric, the Lipschitz metric is not symmetric, so the order of the two points matters when computing distance.

Both metrics can be described in terms of certain length ratios of curves. By a *curve* on S, we will always mean a free isotopy class of an essential simple closed curve. Essential means the curve is not homotopic to a point or a puncture of S. Given a curve α on S, the extremal length of α in $x \in \mathcal{T}(S)$ is

$$\mathrm{Ext}_x(\alpha) = \sup_\rho \frac{\ell_\rho(\alpha)^2}{\mathrm{Area}(\rho)},$$

where ρ is any metric in the conformal class of x, $\ell_\rho(\alpha)$ is the ρ-length of the shortest curve in the homotopy class of α, and $\mathrm{Area}(\rho)$ is the area of x equipped with the metric ρ. For the Teichmüller metric, Kerckhoff showed:

$$d_T(x, y) = \frac{1}{2} \log \sup_\alpha \frac{\mathrm{Ext}_y(\alpha)}{\mathrm{Ext}_x(\alpha)}, \tag{2}$$

where the sup is taken over all curves on S [8]. For the Lipschitz metric, Thurston showed:

$$d_L(x, y) = \log \sup_\alpha \frac{\ell_y(\alpha)}{\ell_x(\alpha)}, \tag{3}$$

where $\ell_x(\alpha)$ is the hyperbolic length of α in the unique hyperbolic metric in the conformal class of x [18].

A point $x \in \mathcal{T}(S)$ is called *ϵ-thick* (or *ϵ-thin*) if the length of the shortest curve on x is greater or equal to ϵ (or less than ϵ). In the thick part of $\mathcal{T}(S)$, it is known that the two metrics are the same up to an additive error.

Theorem 2.1 [3] *For every ϵ there exists a constant c such that whenever $x, y \in \mathcal{T}(S)$ are ϵ-thick,*

$$\left| d_T(x, y) - d_L(x, y) \right| \le c.$$

Curve graphs and subsurface projection. Given two curves α and β on S, we define their *intersection number* $\mathrm{i}(\alpha, \beta)$ to be the minimal number of intersections between any representatives of homotopy classes of α and β.

The *curve graph* $\mathcal{C}(S)$ of S is defined as follows: the vertices are curves on S and the edges are pairs of distinct curves that have minimal possible intersections. This minimum is 1 for the once-punctured torus, 2 for the four-holed sphere, and 0 for all other surfaces. Note that a pair of pants (three-holed sphere) does not have any essential curves. We equip $\mathcal{C}(S)$ with a metric by assigning length one to every edge.

We use a different definition for the curve graph $\mathcal{C}(A)$ of an annulus A (sphere with two boundary components). By an *arc* on A we always mean a homotopy class of a simple arc ω connecting the two boundary components of A where the homotopy is taken relative to the endpoints of ω. The intersection $\mathrm{i}(\omega, \omega')$ of two arcs is the minimal number of intersections between any representatives of homotopy classes of ω and ω'. The vertices of $\mathcal{C}(A)$ are arcs on A and the edges are pairs of arcs with zero intersection. We also equip $\mathcal{C}(A)$ with a metric as above.

From [14], we recall the definition of *subsurface projection*

$$\pi_Y : \mathcal{C}(S) \to \mathcal{P}\big(\mathcal{C}(Y)\big).$$

First suppose Y is not an annulus. Let $\alpha \in \mathcal{C}(S)$. If α is disjoint from Y, then $\pi_Y(\alpha) = \emptyset$ and if α is contained in Y, then $\pi_Y(\alpha) = \alpha$. In all other cases, the restriction of α to Y is a collection of arcs. Let ω be one such arc. The endpoints of ω lie on two (not necessarily distinct) boundary components β and β' of Y. Let $\mathcal{N}_\omega$ be a regular neighborhood in Y of $\omega \cup \beta \cup \beta'$. Then $\mathcal{N}_\omega$ always has a boundary component that is a non-trivial curve in Y. Let $\pi_Y(\alpha)$ be the union of all essential boundary curves of $\mathcal{N}_\omega$, where ω ranges over all arcs in the restriction of α with Y. The set $\pi_Y(\alpha)$ is non-empty with diameter at most two in $\mathcal{C}(S)$ [14].

Given an annular subsurface A of S with *core curve* γ, the Gromov compactification of the annular cover of S corresponding to $\gamma \in \pi_1(S)$ is well-defined and is independent of the choice of the hyperbolic metric on S. For any $\alpha \in \mathcal{C}(S)$, the projection $\pi_A(\alpha)$ is defined to be the set of lifts of α to A that are essential arcs. Note that a lift has well-defined endpoints in the Gromov boundary of A. The set $\pi_A(\alpha)$ has diameter at most two in $\mathcal{C}(A)$.

Short markings and bounded combinatorics. A pants curve system on S is a collection of mutually disjoint curves which cut S into pairs of pants. A *marking* μ on S is a pants curve system $\mathcal{P}$ with additionally a set of *transverse* curves $\mathcal{Q}$ satisfying the following properties. We require each curve $\alpha \in \mathcal{P}$ to have a unique transverse curve $\beta \in \mathcal{Q}$ that intersects α minimally (once or twice) and is disjoint from all other curves in $\mathcal{P}$. We will often say α and β are *dual* to each other, and write $\overline{\alpha} = \beta$ or $\overline{\beta} = \alpha$. This notion of a marking was introduced first by Masur and Minsky [14]; however their terminology is clean marking.

Given $x \in \mathcal{T}(S)$, a *short marking* μ_x on x is a marking where the pants curve system is constructed using the algorithm that picks the shortest curve on x, then the second shortest disjoint from the first, and so on. Once the pants curve system is complete, the transverse curves are then chosen to be as short as possible. Note that a short marking on x may not be unique, but all short markings on x form a bounded set in $\mathcal{C}(S)$. Thus, we will refer to μ_x as *the associated short marking* on x.

Let $x, y \in \mathcal{T}(S)$ and μ_x and μ_y be the associated short markings. For any $Y \subseteq S$, define

$$d_Y(\mu_x, \mu_y) = \operatorname{diam}_{\mathcal{C}(Y)} \big(\pi_Y(\mu_x), \pi_Y(\mu_y)\big),$$

where $\pi_Y(\mu_x)$ is the union of the projection of the curves of μ_x to Y.

Definition 2.2 Two points $x, y \in \mathcal{T}(S)$ are said to have *K-bounded combinatorics* if there exists a constant $K > 0$ such that for every proper subsurface $Y \subset S$,

$$d_Y(\mu_x, \mu_y) \leq K.$$

Cobounded geodesics. Given $x, y \in \mathcal{T}(S)$, we denote by $\mathcal{G}_T(x, y)$, or $\mathcal{G}_T$ when endpoints are not emphasized, the Teichmüller geodesic connecting x and y. We denote by $\mathcal{G}_L(x, y)$ (or $\mathcal{G}_L$) a Lipschitz geodesic *from x to y*. In either the Teichmüller or the Lipschitz metric, a geodesic is *ϵ-cobounded* if every point on the geodesic is ϵ-thick. Given x and y, there is a unique Teichmüller geodesic connecting them. On the other hand, Thurston proved the existence of a Lipschitz geodesic from x to y [18], but in general there may be more than one.

The following theorem is due to Rafi. The second direction also follows from the work of Minsky (see [10,13]).

Theorem 2.3 [15] *For every ϵ, $K > 0$, there exists a constant $\epsilon' > 0$ such that the following holds. If $x, y \in \mathcal{T}(S)$ are ϵ-thick and have K-bounded combinatorics, then the Teichmüller geodesic $\mathcal{G}_T$ with endpoints x and y is ϵ'-cobounded.*

Conversely, for every $\epsilon > 0$, there exists $K' > 0$ such that, if $\mathcal{G}_T$ is ϵ-cobounded (possibly an infinite or bi-infinite ray), then any two points on $\mathcal{G}_T$ have K'-bounded combinatorics.

For the rest of this paper, we will fix $\epsilon > 0$ to be less than the Margulis constant. Unless otherwise specified, by thick or thin, we will always mean ϵ-thick or ϵ-thin. We will also fix a constant K so that bounded combinatorics will mean K-bounded combinatorics. A cobounded geodesic will always mean ϵ'-cobounded with ϵ' as in Theorem 2.3. Once ϵ and K are fixed, the dependence of other constants on ϵ and K can be ignored; we can treat constants which depend on ϵ and K as if they depended only on the topology of S.

In this paper, we will try to understand a Lipschitz geodesic $\mathcal{G}_L$ whose endpoints have bounded combinatorics. Our main tool will be to compare the geometry of $\mathcal{G}_L$ with the geometry of the unique Teichmüller geodesic $\mathcal{G}_T$ connecting the same endpoints. We will use the fact that $\mathcal{G}_T$ is cobounded to show that the closest-point projection to $\mathcal{G}_T$ in the Lipschitz metric is contracting (Theorem 4.5). This will imply that $\mathcal{G}_L$ and $\mathcal{G}_T$ fellow travel, and hence $\mathcal{G}_L$ is also cobounded (for some ϵ'' depending only on S) (Theorem 5.1).

Thick-thin decomposition of a hyperbolic surface. Fix $0 < \epsilon_1 < \epsilon_0 < \epsilon$. For any $x \in \mathcal{T}(S)$, we recall the notion of (ϵ_0, ϵ_1) *thick-thin decomposition* of x (see [11]). Let $\mathcal{A}$ be the (possibly empty) set of curves in x whose hyperbolic lengths are less than ϵ_1. For each $\alpha \in \mathcal{A}$, let A_α be the regular neighborhood of the x-geodesic representative of α with boundary length ϵ_0. Note that, since ϵ_0 is less than the Margulis constant, the annuli are disjoint. Let $\mathcal{Y}$ be the set of components of $x \setminus (\bigcup_{\alpha\in\mathcal{A}} A_\alpha)$. We denote this decomposition of x by $(\mathcal{A}, \mathcal{Y})$.

Note that if $(\mathcal{A}, \mathcal{Y})$ is a thick-thin decomposition for x and μ_x is a short marking, then $\mathcal{A}$ always forms a subset of the pants curve system in μ_x.

Notations. Throughout this paper we will adopt the following notations. Below, $\mathfrak{a}$ and $\mathfrak{b}$ represent various quantities such as distances between two points or lengths of a curve, and C and D are constants that depend only on the topology of S.

(1) $\mathfrak{a} \overset{*}{\prec} \mathfrak{b}$ if $\mathfrak{a} \le C\mathfrak{b}$,
(2) $\mathfrak{a} \overset{+}{\prec} \mathfrak{b}$ if $\mathfrak{a} \le \mathfrak{b} + D$,
(3) $\mathfrak{a} \overset{*}{\asymp} \mathfrak{b}$ if $\mathfrak{a} \overset{*}{\prec} \mathfrak{b}$ and $\mathfrak{b} \overset{*}{\prec} \mathfrak{a}$.
(4) $\mathfrak{a} \overset{+}{\asymp} \mathfrak{b}$ if $\mathfrak{a} \overset{+}{\prec} \mathfrak{b}$ and $\mathfrak{b} \overset{+}{\prec} \mathfrak{a}$.

We will also often use the notation $\mathfrak{a} = O(1)$ to mean $\mathfrak{a} \overset{*}{\prec} 1$.

3 Hyperbolic length estimates via markings

In this section we give some estimates of the hyperbolic length of a simple closed curve in terms of the number of times the curve intersects a marking on a surface and the length of the marking itself. Up to a multiplicative error, our expression provides an accurate estimate when the marking is short, but yields only an upper bound for a general marking.

Short Marking.

Proposition 3.1 *Let $x \in \mathcal{T}(S)$ and μ_x be a short marking on x. Then for every curve γ,*

$$\ell_x(\gamma) \overset{*}{\asymp} \sum_{\alpha\in\mu_x} \mathrm{i}(\gamma, \alpha)\, \ell_x(\overline{\alpha}),$$

and

$$\mathrm{Ext}_x(\gamma) \overset{*}{\asymp} \sum_{\alpha\in\mu_x} \mathrm{i}(\gamma, \alpha)^2\, \mathrm{Ext}_x(\overline{\alpha}).$$

Proof We first prove the statement for the hyperbolic length of γ. Consider the (ϵ_0, ϵ_1)-decomposition $(\mathcal{A}, \mathcal{Y})$ for x. For each $Y \in \mathcal{Y}$, let μ_Y be the set of curves in μ_x that are contained entirely in Y. Note that if $\alpha \in \mu_Y$, then so is $\overline{\alpha}$. The set μ_Y fills the surface Y, that is, every curve in Y intersects some curve in μ_Y. For every curve γ in Y define

$$\mathrm{i}(\gamma, \mu_Y) = \sum_{\alpha \in \mu_Y} \mathrm{i}(\gamma, \alpha).$$

It is a consequence of [4, Corollary 3.2] and [10] that $\ell_x(\gamma)$ can be estimated using the following sum:

$$\ell_x(\gamma) \stackrel{*}{\asymp} \sum_{Y \in \mathcal{Y}} \mathrm{i}(\gamma, \mu_Y) + \sum_{\alpha \in \mathcal{A}} \mathrm{i}(\gamma, \alpha) \left[\log \frac{1}{\ell_x(\alpha)} + \ell_x(\alpha)\, \mathrm{twist}_\alpha(x, \gamma) \right]. \tag{4}$$

Here, $\mathrm{twist}_\alpha(x, \gamma) = d_A(\overline{\alpha}, \gamma)$ (see [11,17] for more details). We need to show

$$\ell_x(\gamma) \stackrel{*}{\asymp} \sum_{Y \in \mathcal{Y}} \sum_{\alpha \in \mu_Y} \mathrm{i}(\gamma, \alpha)\, \ell_x(\overline{\alpha}) + \sum_{\alpha \in \mathcal{A}} \left[\mathrm{i}(\gamma, \alpha)\, \ell_x(\overline{\alpha}) + \mathrm{i}(\gamma, \overline{\alpha})\, \ell_x(\alpha) \right] \tag{5}$$

which is just a rephrasing of the statement of the proposition for the hyperbolic length. We will show that the right hand sides of Eqs. (4) and (5) are comparable.

To start, note that for every $\alpha \in \mu_Y$, we have $\ell_x(\overline{\alpha}) \stackrel{*}{\asymp} 1$. Hence

$$\sum_{Y \in \mathcal{Y}} \sum_{\alpha \in \mu_Y} \mathrm{i}(\gamma, \alpha)\, \ell_x(\overline{\alpha}) \stackrel{*}{\asymp} \sum_{Y \in \mathcal{Y}} \mathrm{i}(\gamma, \mu_Y). \tag{6}$$

Now consider $\alpha \in \mathcal{A}$. By the collar lemma, the hyperbolic length of the dual curve $\overline{\alpha}$ is roughly the width of the collar around α. That is,

$$\ell_x(\overline{\alpha}) \stackrel{*}{\asymp} \log \frac{1}{\ell_x(\alpha)}.$$

Summing over $\alpha \in \mathcal{A}$ we obtain

$$\sum_{\alpha \in \mathcal{A}} \mathrm{i}(\gamma, \alpha)\, \ell_x(\overline{\alpha}) \stackrel{*}{\asymp} \sum_{\alpha \in \mathcal{A}} \mathrm{i}(\gamma, \alpha)\, \log \frac{1}{\ell_x(\alpha)}. \tag{7}$$

We now compare the last terms. Assume γ intersects some curve $\alpha \in \mathcal{A}$. From the discussion in [11, Sect. 3] we have

$$\mathrm{twist}_\alpha(x, \gamma) \stackrel{+}{\prec} \frac{\mathrm{i}(\gamma, \overline{\alpha})}{\mathrm{i}(\gamma, \alpha)}.$$

To make the error multiplicative, we add a large term to the right side:

$$\mathrm{twist}_\alpha(x, \gamma) \stackrel{*}{\prec} \frac{\ell_x(\overline{\alpha})}{\ell_x(\alpha)} + \frac{\mathrm{i}(\gamma, \overline{\alpha})}{\mathrm{i}(\gamma, \alpha)}.$$

Summing over $\alpha \in \mathcal{A}$ and multiplying by $\mathrm{i}(\gamma, \alpha)\, \ell_x(\alpha)$ we obtain

$$\sum_{\alpha \in \mathcal{A}} \mathrm{i}(\gamma, \alpha)\, \ell_x(\alpha)\, \mathrm{twist}_\alpha(x, \gamma) \stackrel{*}{\prec} \sum_{\alpha \in \mathcal{A}} \mathrm{i}(\gamma, \alpha)\, \ell_x(\overline{\alpha}) + \mathrm{i}(\gamma, \overline{\alpha})\, \ell_x(\alpha).$$

Thus the right hand side of (4) is bounded above by the right hand side of (5) up to a multiplicative error.

It remains to find an upper bound for $\mathrm{i}(\gamma, \overline{\alpha})\, \ell_x(\alpha)$, $\alpha \in \mathcal{A}$, using terms in the right hand side of Eq. (4). Since our inequalities are up to a multiplicative error, finding an upper bound for each such term provides an upper bound for the sum.

Consider the regular neighborhood A_α of α. If ϵ_0 is small enough, γ intersects α every time it enters A_α. The number of intersection points between γ and $\overline{\alpha}$ inside of A_α is bounded by $\mathrm{i}(\gamma,\alpha)\,\mathrm{twist}_\alpha(x,\gamma)$ and the number of intersection points outside of A_α is less than the number of intersection points between γ and $\mathcal{P}$, the set of pants curves in μ_x (every time γ intersects $\overline{\alpha}$ it either twists around α and intersects α or it will intersect some curve in $\mathcal{P}$ before intersecting $\overline{\alpha}$ again). That is,

$$\mathrm{i}(\gamma,\overline{\alpha}) \stackrel{*}{\prec} \mathrm{i}(\gamma,\alpha)\ \mathrm{twist}_\alpha(x,\gamma) + \mathrm{i}(\gamma,\mathcal{P}).$$

Since, for any $\beta \in \mathcal{P}$, $\ell_x(\alpha) \le \ell_x(\overline{\beta})$ we have

$$\mathrm{i}(\gamma,\overline{\alpha})\,\ell_x(\alpha) \stackrel{*}{\prec} \mathrm{i}(\gamma,\alpha)\ell_x(\alpha)\ \mathrm{twist}_\alpha(x,\gamma) + \sum_{\beta\in\mathcal{P}} \mathrm{i}(\gamma,\beta)\ell_x(\overline{\beta}).$$

Up to a multiplicative error, this is less than the right hand side of (4). Thus the right hand side of (5) is bounded above by the right hand side of (4) up to a multiplicative error. Therefore, the two quantities are equal. This completes the proof of the first statement.

To prove the statement for extremal length, we can follow the same path. We have the following estimate for the extremal length of a curve (this is Theorem 7 in [9] which follows essentially from [11]) analogous to Eq. (4):

$$\mathrm{Ext}_x(\gamma) \stackrel{*}{\asymp} \sum_{Y\in\mathcal{Y}} \mathrm{i}(\gamma,\mu_Y)^2 + \sum_{\alpha\in\mathcal{A}} \mathrm{i}(\gamma,\alpha)^2 \left[\frac{1}{\mathrm{Ext}_x(\alpha)} + \mathrm{Ext}_x(\alpha)\,\mathrm{twist}_\alpha(x,\gamma)^2\right]$$

Similar to Eq. (6), we have

$$\sum_{Y\in\mathcal{Y}} \mathrm{i}(\gamma,\mu_Y)^2 \stackrel{*}{\asymp} \sum_{\alpha\in\mu_Y} \mathrm{i}(\gamma,\alpha)^2\ \mathrm{Ext}_x(\overline{\alpha}).$$

For any $\alpha \in \mathcal{A}$, the version of the collar lemma for extremal length says:

$$\mathrm{Ext}_x(\overline{\alpha}) \stackrel{*}{\asymp} \frac{1}{\mathrm{Ext}_x(\alpha)}.$$

The rest of the proof is essentially identical. □

Upper bound from any marking. In the following, we use a surgery argument on curves to derive an upper bound for the hyperbolic length of a curve using an arbitrary marking. Although we do not need such a precise estimate, our argument produces a multiplicative error of 2.

Proposition 3.2 *Let $x \in \mathcal{T}(S)$ and μ be an arbitrary marking on S. Then for every curve γ,*

$$\ell_x(\gamma) \stackrel{*}{\prec} \sum_{\alpha\in\mu} \mathrm{i}(\gamma,\alpha)\,\ell_x(\overline{\alpha}) \tag{8}$$

The outline of the proof is as follows. Let $\mathcal{P}$ be the pants curve system in μ. We first perturb γ so that the restriction of γ to every pair of pants $P \in S \setminus \mathcal{P}$ is a union of *admissible* arcs. These are arcs for which the inequality (8) holds. Perturbing γ will only increase its length. Hence, if (8) holds for every arc, it holds for γ as well.

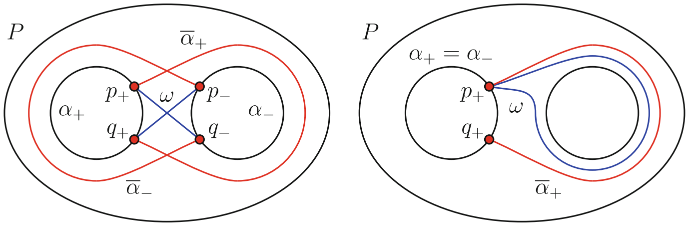

Fig. 1 There are 12 non-admissible arcs in P. For each pair of distinct boundary components of P, there are two non-admissible arcs as depicted in the left figure (both arcs are labeled ω). For each boundary component of P, there are two non-admissible arcs. The figure on the right depicts one such arc ω. The second one is obtained via a reflection across the x-axis

Admissible arcs. Let P be a pair of (embedded) pants in the pants decomposition associated with the marking μ. Equip P with the hyperbolic metric inherited from x. For every boundary curve $\alpha \in \partial P$, let $\overline{\alpha}$ be a simple geodesic arc in P with endpoints on α separating the other two boundary components of P, and let E be the set of endpoints of arcs $\overline{\alpha}$. Let ω be any simple geodesic arc whose endpoints are in E, and let $\mathrm{i}(\omega, \overline{\alpha})$ represent the number of intersection points in the interior of P. Assume that one endpoint of ω lies in α_- and the other lies in α_+. We say ω is *admissible* if

$$\ell_x(\omega) \overset{*}{\prec} \ell_x(\overline{\alpha}_+) + \ell_x(\overline{\alpha}_-) + \mathrm{i}(\omega, \overline{\alpha}_+)\, \ell_x(\alpha_+) + \mathrm{i}(\omega, \overline{\alpha}_-)\, \ell_x(\alpha_-).$$

As we shall see, most arcs are admissible.

Lemma 3.3 *Let ω be a simple geodesic arc with endpoints in E. Then ω is admissible unless it is one of the arcs depicted in Fig. 1. In particular, if $\mathrm{i}(\omega, \overline{\alpha}) > 0$ for some $\alpha \in \partial P$ then ω is admissible.*

Proof First suppose ω starts and ends on two different boundary components of P. Let ω_1 and ω_2 be the arcs depicted in Fig. 2. Then, up to homotopy, ω is a concatenation of either ω_1 or ω_2 with several copies of α_+, several copies of α_- and at most one copy of the arcs $[p_+, q_+]$ or $[p_-, q_-]$. The number of copies of α_+ needed is at most $\mathrm{i}(\omega, \overline{\alpha}_+)$ and the number of copies of α_- needed is at most $\mathrm{i}(\omega, \overline{\alpha}_-)$. The length of ω is less than the sum of these arcs.

Note that the lengths of ω_1 and ω_2 are both less than $\ell_x(\overline{\alpha}_+) + \ell_x(\overline{\alpha}_-)$. The lengths of copies of $\alpha_\pm$ needed is less than or equal to $\mathrm{i}(\omega, \overline{\alpha}_\pm)\, \ell_x(\alpha_\pm)$. If either $\mathrm{i}(\omega, \overline{\alpha}_+)$ or $\mathrm{i}(\omega, \overline{\alpha}_-)$ is non-zero then the quantity $\mathrm{i}(\omega, \overline{\alpha}_\pm)\, \ell_x(\alpha_\pm)$ is also an upper bound for the length of the segment $[p_\pm, q_\pm]$. Hence, if ω is not admissible, then it is disjoint from $\overline{\alpha}_\pm$ and it is not ω_1 or ω_2. The arcs depicted in the left side of Fig. 1 are the only possibilities.

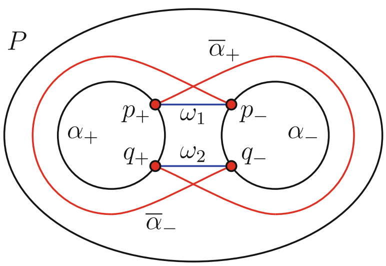

Fig. 2 The arcs ω_1 and ω_2

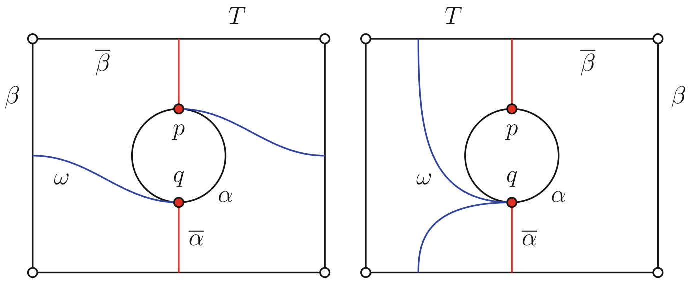

Fig. 3 There are 6 non-admissible arcs in T. On the left is a non-admissible arc ω whose endpoints are distinct. Another non-admissible arc of the same type can be obtained via a reflection across the x-axis. On the *right* is a non-admissible arc ω which starts and ends at the same point. The other 3 non-admissible arcs of this type can be obtained via reflections across the x-axis and the y-axis

A similar argument works when ω starts and ends on the same curve, that is, when $\alpha_+ = \alpha_-$. In this case, if ω is not admissible, then it must be disjoint from $\overline{\alpha}_+$ but not equal to it. There are only two such arcs, one with both endpoints at p_+ (see the right side of Fig. 1) and one with both endpoints at p_-. □

In the case that a pair of pants is not embedded in x (when one curve in x appears twice as a boundary of a pair of pants), the dual curve does not intersect the pants curves twice and the above arguments do not apply. In this case, the definition of an admissible arc has to be modified. Let T be a torus with one boundary component that is an image of a pair of pants associated to μ. Let α be the boundary curve of T and $\overline{\alpha}$ be a simple geodesic arc with endpoints on α. Also, let β be a simple closed curve in T that is disjoint from $\overline{\alpha}$, and let $\overline{\beta}$ be the dual curve to β: a simple closed geodesic that intersects each of β and $\overline{\alpha}$ exactly once. Let $E = \{p, q\}$ be the endpoints of $\overline{\alpha}$, and let ω be a simple geodesic arc with endpoints in E. We say ω is admissible if

$$\ell_x(\omega) \overset{*}{\prec} \ell_x(\overline{\alpha}) + \mathrm{i}(\omega, \overline{\alpha})\, \ell_x(\alpha) + \mathrm{i}(\omega, \beta)\, \ell_x(\overline{\beta}) + \mathrm{i}(\omega, \overline{\beta})\, \ell_x(\beta).$$

Lemma 3.4 *Let ω be a simple geodesic arc with endpoints in E. Then ω is admissible unless it is an arc of a type depicted in Fig. 3. In particular, if* $\mathrm{i}(\omega, \overline{\alpha}) > 0$ *then ω is admissible.*

Proof Up to homotopy, the arc ω is a concatenation of several copies of α, one-half of $\overline{\alpha}$, a simple closed curve δ, then again one-half of $\overline{\alpha}$ (could be the same half or the other half), and finally several copies of α. One may have to add the arc $[p, q]$ to the beginning or to the end to ensure the arc described above and ω have the same endpoints. First we claim

$$\ell_x(\delta) \leq \mathrm{i}(\omega, \beta)\, \ell_x(\overline{\beta}) + \mathrm{i}(\omega, \overline{\beta})\, \ell_x(\beta).$$

Consider the fundamental group of T with a base point at the intersection of β and $\overline{\beta}$. Then a curve homotopic to δ can be written as a product of copies of β and $\overline{\beta}$. The number of copies of β and $\overline{\beta}$ needed is exactly $\mathrm{i}(\omega, \overline{\beta})$ and $\mathrm{i}(\omega, \beta)$ respectively. This proves the claim.

The number of copies of α needed is bounded above by $\mathrm{i}(\omega, \overline{\alpha})$. If $\mathrm{i}(\omega, \overline{\alpha})$ is non-zero then the quantity $\mathrm{i}(\omega, \overline{\alpha})\, \ell_x(\alpha)$ is also an upper bound for the length of the segment $[p, q]$. Hence, ω is admissible if $\mathrm{i}(\omega, \overline{\alpha}) > 0$ or if the arc $[p, q]$ is not required to construct ω. Arcs of type depicted in Fig. 3 are the only exceptions. □

Proof of Proposition 3.2 If γ is a curve in μ then the statement of the proposition is clearly true. We can further assume that there is a pants curve $\alpha_0 \in \mu$ so that γ intersects both α_0 and $\overline{\alpha}_0$. Otherwise, γ has to pass only through pants in the form discussed in Lemma 3.4.

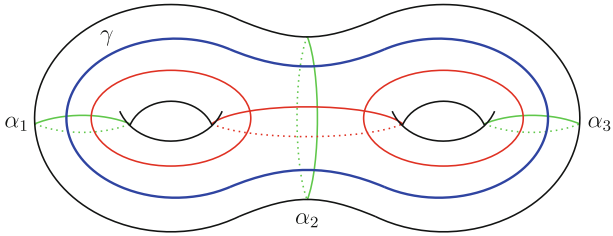

Fig. 4 The curve which goes around both holes of the surface is γ. The union of the other curves forms the marking μ. The curves α_1, α_2, and α_3 are the pants curves of μ. For each i, the transverse curve $\overline{\alpha}_i$ to α_i is the unlabeled curve which intersects only α_i. The curve γ does not intersect both α_i and $\overline{\alpha}_i$ for any $i = 1, 2, 3$

That means, S is a union of two one-holed tori. That is, S is a genus two surface and μ and γ are as depicted in Fig. 4. In this case, it is easy to produce a curve homotopic to γ as a concatenation of curves in μ and hence the proposition holds.

We claim γ can be homotoped to a curve γ' so that γ' is a union of admissible arcs and a sub-arc of α_0. The curve γ' has the same intersection pattern with the pants curves of μ and the intersection number of γ with every transverse curve is the same as the sum of the *interior intersection number* of γ' with these curves. The proposition then follows from Lemmas 3.3 and 3.4.

First perturb γ slightly so that it does not pass through any intersection point between α and $\overline{\alpha}$ for a pants curve $\alpha \in \mu$. We change γ by replacing the restriction of γ to a pair of pants P or a torus T to admissible arcs. Start with the pair of pants P_0 with the boundary curve α_0 and a sub-arc ω_0 of γ that starts from α_0 and ends in α_1 (α_1 may equal α_0). Replace ω_0 with an admissible arc ω_0' that has the same intersection pattern with the dual arcs in P_0. Let r_0 and r_1 be the endpoints of ω_0' in α_0 and α_1 respectively. Now let P_2 be the pair of pants (or once-punctured torus) with α_1 as a boundary component that is not P_0 and let ω_1 be the continuation of ω_0 in P_1. Again, replace ω_1 with an admissible arc ω_1', but make sure ω_1' starts at r_1. This is always possible by Lemmas 3.3 and 3.4; we can push the intersection point of ω_1 with α_2 either to the right or to the left and one of these two will result in an admissible arc. Continue in this fashion, replacing the arc ω_k which is a continuation of ω_{k-1} in the pair of pants (or once-punctured torus) P_k with an admissible arc making sure that the starting point r_k of ω_k' matches the endpoint of ω_{k-1}'. We can do this until we reach the starting point after K steps. Then $\alpha_K = \alpha_0$. We can ensure the arc ω_{K-1}' is admissible and it starts from r_{K-1}. But r_K may not equal r_0. In this case, we add a sub-arc ω' of α_0 to close up γ' to a curve homotopic to γ.

If we now add up the inequalities defining admissibility, we get that the sum of the lengths of arcs ω_i' is less than the right-hand side of the inequality (8). Also the term $\ell_x(\overline{\alpha}_0)$ appears in the right hand side of (8) and provides an upper bound for the length of ω'. That is, the right-hand side of Eq. (8) is an upper bound for the length of γ' and hence for $\ell_x(\gamma)$. This finishes the proof. □

Remark 3.5 If x is in the thick part of Teichmüller space, then Proposition 3.2 also holds for extremal length. This follows from the fact that in the thick part, hyperbolic length is coarsely equal to the square root of the extremal length (see Lemma 4.2).

4 Bounded projection to a Teichmüller geodesic

In this section, our main goal is to prove Theorem B of the introduction. The first step is to prove Theorem E, which allows us to estimate the Lipschitz distance from x to y by considering only how much a short marking on x is stretched. The special case of Theorem E when both x and y are in the thick part was proved in [3]. We restate Theorem E.

Theorem 4.1 (Candidate curves) *Let $x, y \in \mathcal{T}(S)$ and let μ_x be a short marking on x. Then*

$$d_L(x, y) \overset{+}{\asymp} \log \max_{\alpha \in \mu_x} \frac{\ell_y(\alpha)}{\ell_x(\alpha)}.$$

A curve $\alpha \in \mu_x$ satisfying $d_L(x, y) \overset{+}{\asymp} \log \frac{\ell_y(\alpha)}{\ell_x(\alpha)}$ is called a candidate curve from x to y.

Proof of Theorem 4.1 By Thurston's theorem (Eq. 3), there exists a curve γ such that $\log \frac{\ell_y(\gamma)}{\ell_x(\gamma)}$ is within a uniform additive error of $d_L(x, y)$. We invoke Propositions 3.1 and 3.2 to compute the hyperbolic length of γ on x and y, using the fact that μ_x is short on x but may not be short on y:

$$\ell_x(\gamma) \overset{*}{\asymp} \sum_{\alpha \in \mu_x} \mathrm{i}(\gamma, \alpha)\ell_x(\overline{\alpha}), \qquad \ell_y(\gamma) \overset{*}{\prec} \sum_{\alpha \in \mu_x} \mathrm{i}(\gamma, \alpha)\ell_y(\overline{\alpha}).$$

We have

$$\begin{aligned} e^{d_L(x,y)} \overset{*}{\asymp} \frac{\ell_y(\gamma)}{\ell_x(\gamma)} &\overset{*}{\prec} \frac{\sum_{\alpha \in \mu_x} \mathrm{i}(\gamma, \alpha)\, \ell_y(\overline{\alpha})}{\sum_{\alpha \in \mu_x} \mathrm{i}(\gamma, \alpha)\, \ell_x(\overline{\alpha})} \\ &\overset{*}{\prec} \max_{\alpha \in \mu_x} \frac{\ell_y(\alpha)}{\ell_x(\alpha)}. \end{aligned}$$

The opposite inequality directly follows from the definition of Lipschitz distance. $\square$

Given a closed set $\mathcal{K} \subset \mathcal{T}(S)$ and $x \in \mathcal{T}(S)$, define

$$d_L(x, \mathcal{K}) = \inf_{y \in \mathcal{K}} d_L(x, y).$$

The *closest-point projection* of $x \in \mathcal{T}(S)$ to $\mathcal{K}$ with respect to the Lipschitz metric is

$$\pi_{\mathcal{K}}(x) = \big\{ y \in \mathcal{K} \mid d_L(x, y) = d_L(x, \mathcal{K}) \big\}.$$

The projection is always nonempty, but it could contain more than one point. We can also project a set $B \subset \mathcal{T}(S)$ to $\mathcal{K}$: $\pi_{\mathcal{K}}(B) = \cup_{x \in B} \pi_{\mathcal{K}}(x)$.

We will use Theorem 4.1 to analyze the closest-point projection in the Lipschitz metric to a cobounded Teichmüller geodesic $\mathcal{G}_T$. Parametrizing $\mathcal{G}_T$ by arc length (in the Teichmüller metric), we denote points along $\mathcal{G}_T$ by $\mathcal{G}_T(t)$. Along $\mathcal{G}_T$, we have the following relationship between the hyperbolic length and the extremal length of a curve:

Lemma 4.2 [11] *For any x in the thick part of $\mathcal{T}(S)$ and any curve α,*

$$\ell_x(\alpha) \overset{*}{\asymp} \sqrt{\mathrm{Ext}_x(\alpha)}.$$

Furthermore, the length of α in either sense varies along $\mathcal{G}_T(t)$ coarsely like $\cosh(t)$ [17, Equation (2)]. Therefore, it makes sense to talk about a point $x_{t_\alpha} = \mathcal{G}_T(t_\alpha)$ on which the length of α is minimal, and away from x_{t_α} in either direction the length of α grows exponentially. If there are several minimal points, then we choose t_α arbitrarily among them. We call t_α the *balanced time* of α.

The first statement of the following lemma is a consequence of [12, Lemma 3.3]. The second statement follows immediately from the first one and Lemma 4.2.

Lemma 4.3 *There exist constants c_1, c_2, and D, depending only on S, so that for any curves α and β and any cobounded Teichmüller geodesic $\mathcal{G}_T$,*

$$|t_\alpha - t_\beta| \geq D \implies \mathrm{i}(\alpha, \beta)^2 \geq c_1\, e^{2|t_\alpha - t_\beta|}\, \mathrm{Ext}_{x_{t_\alpha}}(\alpha)\, \mathrm{Ext}_{x_{t_\beta}}(\beta)$$

and

$$|t_\alpha - t_\beta| \geq D \implies \mathrm{i}(\alpha, \beta) \geq c_2\, e^{|t_\alpha - t_\beta|}\, \ell_{x_{t_\alpha}}(\alpha)\, \ell_{x_{t_\beta}}(\beta).$$

Lemma 4.4 *Let $\mathcal{G}_T$ be a cobounded Teichmüller geodesic. Suppose $x \in \mathcal{T}(S)$ is a point not on $\mathcal{G}_T$ and $x_t \in \pi_{\mathcal{G}_T}(x)$. Then for any $\alpha \in \mu_x$, we have $|t - t_\alpha| = O(1)$.*

Proof Let $\beta \in \mu_x$ be a candidate curve from x to x_{t_α}. The curves α and β have bounded intersection number, so by Lemma 4.3, $|t_\alpha - t_\beta| = O(1)$ (note that since $\mathcal{G}_T$ is cobounded, the quantities $\ell_{x_{t_\alpha}}(\alpha)$ and $\ell_{x_{t_\beta}}(\beta)$ are bounded below). Away from t_β, the length of β grows exponentially. We have

$$e^{d_L(x, x_t)} \geq \frac{\ell_{x_t}(\beta)}{\ell_x(\beta)} \overset{*}{\succ} e^{\left(|t - t_\alpha| - |t_\alpha - t_\beta|\right)} \frac{\ell_{x_{t_\alpha}}(\beta)}{\ell_x(\beta)}.$$

Taking log on both sides yields

$$d_L(x, x_t) \overset{+}{\succ} |t - t_\alpha| - |t_\alpha - t_\beta| + d_L(x, x_{t_\alpha}).$$

Since x_t is the closest-point projection of x to $\mathcal{G}_T$, $d_L(x, x_t) \leq d_L(x, x_{t_\alpha})$. Together this implies $|t - t_\alpha| = O(1)$. □

By a *Lipschitz ball* of radius R centered at x, we will mean the set

$$B_L(x, R) = \{y \in \mathcal{T}(S) \mid d_L(x, y) \leq R\}.$$

The following is a precise formulation of Theorem B.

Theorem 4.5 (Lipschitz projection to Teichmüller geodesics) *There exists a constant b depending only on S such that, for any cobounded Teichmüller geodesic $\mathcal{G}_T$, any $x \in \mathcal{T}(S)$, and any constant $R < d_L(x, \mathcal{G}_T)$, we have*

$$\mathrm{diam}_L\left(\pi_{\mathcal{G}_T}\left(B_L(x, R)\right)\right) \leq b.$$

Proof Let $y \in B_L(x, R)$, and let μ_x and μ_y be the associated short markings on x and y respectively. Let $x_t \in \pi_{\mathcal{G}_T}(x)$. By Lemma 4.4, we can choose $\alpha \in \mu_x$ such that

$$d_L(x, \mathcal{G}_T) \overset{+}{\asymp} \log \frac{\ell_{x_t}(\alpha)}{\ell_x(\alpha)},$$

and Theorem 4.1 implies

$$\log \frac{\ell_{x_t}(\alpha)}{\ell_x(\alpha)} \overset{+}{\asymp} \log \frac{\ell_{x_{t_\alpha}}(\alpha)}{\ell_x(\alpha)},$$

where t_α is the balance time for α along $\mathcal{G}_T$. Hence

$$d_L(x, \mathcal{G}_T) \overset{+}{\asymp} \log \frac{\ell_{x_{t_\alpha}}(\alpha)}{\ell_x(\alpha)}.$$

Similarly, choose $\beta \in \mu_y$ so that

$$d_L(y, \mathcal{G}_T) \stackrel{+}{\asymp} \log \frac{\ell_{x_{t_\beta}}(\beta)}{\ell_y(\beta)}.$$

The theorem will hold if $|t_\alpha - t_\beta|$ is uniformly bounded.

Let D be the constant of Lemma 4.3. If $|t_\alpha - t_\beta| < D$, then we are done. So suppose $|t_\alpha - t_\beta| \geq D$, in which case

$$\mathrm{i}(\alpha, \beta) \stackrel{*}{\succ} e^{|t_\alpha - t_\beta|}\, \ell_{x_{t_\alpha}}(\alpha)\, \ell_{x_{t_\beta}}(\beta).$$

Since $\beta \in \mu_y$, by Proposition 3.1, $\ell_y(\alpha) \stackrel{*}{\succ} i(\alpha, \beta)\ell_y(\overline{\beta})$. Therefore,

$$\begin{aligned} e^{d_L(x,y)} \geq \frac{\ell_y(\alpha)}{\ell_x(\alpha)} &\stackrel{*}{\succ} \frac{\mathrm{i}(\alpha, \beta)\, \ell_y(\overline{\beta})}{\ell_x(\alpha)} \\ &\stackrel{*}{\succ} \frac{e^{|t_\alpha - t_\beta|}\, \ell_{x_{t_\alpha}}(\alpha)\, \ell_{x_{t_\beta}}(\beta)\, \ell_y(\overline{\beta})}{\ell_x(\alpha)}. \end{aligned}$$

Applying log to both sides above yields

$$d_L(x, y) \stackrel{+}{\succ} |t_\alpha - t_\beta| + d_L(x, \mathcal{G}_T) + \log\left(\ell_{x_{t_\beta}}(\beta)\, \ell_y(\overline{\beta})\right).$$

On the other hand, $d_L(x, y) \leq R < d_L(x, \mathcal{G}_T)$, so the proof will be complete if the product $\ell_{x_{t_\beta}}(\beta)\, \ell_y(\overline{\beta})$ is bounded from below. Since $\mathcal{G}_T$ is (ϵ')-cobounded, the length of every curve on x_{t_β} is bounded below, so we only need to consider the situation when $\ell_y(\overline{\beta})$ is small (say $\ell_y(\overline{\beta}) < \epsilon'$). In this case, since β and $\overline{\beta}$ intersect, β has to be long ($\ell_y(\beta) \stackrel{*}{\succ} \log \frac{1}{\epsilon'}$). But β is the candidate curve from y to a point in $\pi_{\mathcal{G}_T}(y)$ which we know is at most a bounded distance away from x_{t_β}. Thus,

$$\frac{\ell_{x_{t_\beta}}(\beta)}{\ell_y(\beta)} \stackrel{*}{\succ} \frac{\ell_{x_{t_\beta}}(\overline{\beta})}{\ell_y(\overline{\beta})}.$$

We conclude

$$\ell_{x_{t_\beta}}(\beta)\, \ell_y(\overline{\beta}) \stackrel{*}{\succ} \ell_{x_{t_\beta}}(\overline{\beta})\, \ell_y(\beta) \stackrel{*}{\succ} 1.$$

□

Projection in the Teichmüller metric. We now sketch a short proof that the closest-point projection with respect to the Teichmüller metric to a cobounded Teichmüller geodesic is strongly contracting. This was first established by Minsky in [12]. This part is independent from the rest of the paper.

Let $\Pi_{\mathcal{G}_T}$ be the closest-point projection to $\mathcal{G}_T$ with respect to the Teichmüller metric.

Theorem 4.6 [12] *For any cobounded Teichmüller geodesic $\mathcal{G}_T$ and for any Teichmüller ball B disjoint from $\mathcal{G}_T$*, $\mathrm{diam}_T\left(\Pi_{\mathcal{G}_T}(B)\right)$ *is uniformly bounded.*

Proof As discussed before, Proposition 3.2 holds for extremal length as long as x is in the thick part (see Remark 3.5). Therefore we have an analogue of Theorem 4.1: For any $x \in B$ and any $x_t \in \Pi_{\mathcal{G}_T}(x)$, there exists a candidate curve $\alpha \in \mu_x$ from x to x_t. The same argument for Lemma 4.4 will also show that x_t is a bounded distance from x_{t_α}. Replacing hyperbolic length by extremal length, we can carry out the same analysis as in Theorem 4.5 to finish the proof. □

5 Bounded projection to and stability of Lipschitz geodesics

In this section, we prove Theorem A and Theorem C of the introduction. Before we restate and prove the theorems, we first define what it means to fellow travel in the Lipschitz metric.

Let $\mathcal{G}_T(t)\colon\ [0,d] \to \mathcal{T}(S)$ and $\mathcal{G}_L(t)\colon\ [0,d] \to \mathcal{T}(S)$ be respectively a Teichmüller and a Lipschitz geodesic parametrized by arc length (in their respective metric). We will say $\mathcal{G}_L$ and $\mathcal{G}_T$ fellow travel in the Lipschitz metric if there exists a constant R depending only on S such that, for every $t \in [0,d]$,

$$\max\left\{d_L\big(\mathcal{G}_L(t), \mathcal{G}_T(t)\big), d_L\big(\mathcal{G}_T(t), \mathcal{G}_L(t)\big)\right\} \le R.$$

Theorem 5.1 (Lipschitz geodesic fellow travels Teichmüller geodesic)
Suppose $x, y \in \mathcal{T}(S)$ are thick and have bounded combinatorics. Then any Lipschitz geodesic $\mathcal{G}_L$ from x to y is cobounded. In fact, $\mathcal{G}_L$ fellow travels the Teichmüller geodesic with endpoints x and y. More precisely, let $d = d_L(x,y)$ and let $\mathcal{G}_T : \mathbb{R} \to \mathcal{T}(S)$ be the Teichmüller geodesic such that $\mathcal{G}_T(0) = x$ and passing through y. Then $\mathcal{G}_L\colon\ [0,d] \to \mathcal{T}(S)$ fellow travels $\mathcal{G}_T\colon\ [0,d] \to \mathcal{T}(S)$.

By previous result in Theorem 4.5, the Lipschitz closest-point projection to $\mathcal{G}_T$ is strongly contracting. This implies that, if one moves along $\mathcal{G}_L$, the rate of progress of the Lipschitz projection to $\mathcal{G}_T$ is inversely proportional to the distance between $\mathcal{G}_L$ and $\mathcal{G}_T$. (A segment of length R passing through a point z that has distance R from $\mathcal{G}_T$ projects to a subset of $\mathcal{G}_T$ with uniformly bounded size.) In order to apply a standard short-cut argument (see proof of Theorem 5.1), we need an additional fact about the asymmetry of d_L which is a corollary of [3, Proposition 4.1].

Lemma 5.2 *Let $x \in \mathcal{T}(S)$ be thick. Then there exists a constant C depending only on S such that for any $y \in \mathcal{T}(S)$*

$$d_L(x,y) \le C d_L(y,x).$$

Proof From [3, Proposition 4.1] we have (in [3] d_L is the symmetrized Lipschitz metric):

$$d_T(x,y) \overset{*}{\asymp} \max\{d_L(x,y), d_L(y,x)\}. \tag{9}$$

By Eq. (2), there is a curve α such that $d_T(y,x) \overset{*}{\asymp} \frac{1}{2}\log\frac{\mathrm{Ext}_y(\alpha)}{\mathrm{Ext}_x(\alpha)}$. Since x is thick, by Lemma 4.2, $\mathrm{Ext}_x(\alpha) \overset{*}{\asymp} \ell_x(\alpha)^2$. Since the extremal length is defined as a supremum over all metrics in a conformal class, we have $\mathrm{Ext}_y(\alpha) \overset{*}{\succ} \ell_y(\alpha)^2$. Hence,

$$d_L(y,x) \ge \log\frac{\ell_x(\alpha)}{\ell_y(\alpha)} \overset{*}{\succ} \frac{1}{2}\log\frac{\mathrm{Ext}_x(\alpha)}{\mathrm{Ext}_y(\alpha)} \overset{*}{\asymp} d_T(y,x).$$

Also by Eq. (9), $d_T(x,y) \overset{*}{\succ} d_L(x,y)$. The lemma follows from the symmetry of the Teichmüller metric. □

Proof of Theorem 5.1 By assumption, x and y have bounded combinatorics, thus $\mathcal{G}_T$ is cobounded by Theorem 2.3. We will first show that there exists R such that, for any $x \in \mathcal{G}_L$, there exists $x' \in \mathcal{G}_T$ with $d_L(x,x') \le R$. In view of Lemma 5.2 and our definition of a Lipschitz ball, this shows that $\mathcal{G}_L$ is contained in a CR Lipschitz neighborhood of $\mathcal{G}_T$.

For any $r > 0$, suppose a subinterval $[\overline{x}, \overline{y}] \subset \mathcal{G}_L$ is such that $d_L(\overline{x}, \mathcal{G}_T) = d_L(\overline{y}, \mathcal{G}_T) = r$, but $d_L(\overline{z}, \mathcal{G}_T) > r$ for all other points $\overline{z} \in [\overline{x}, \overline{y}]$. By cutting $[\overline{x}, \overline{y}]$ into segments of length at most r and projecting each piece to $\mathcal{G}_T$, we have

$$d_L\left(\pi_{\mathcal{G}_T}(\overline{x}), \pi_{\mathcal{G}_T}(\overline{y})\right) \leq \frac{b}{r}\, d_L(\overline{x}, \overline{y}) + b,$$

where b is the constant of Theorem 4.5. Now fix $r = 2b$. By the triangle inequality,

$$\begin{aligned} d_L(\overline{x}, \overline{y}) &\leq d_L\left(\overline{x}, \pi_{\mathcal{G}_T}(\overline{x})\right) + d_L\left(\pi_{\mathcal{G}_T}(\overline{x}), \pi_{\mathcal{G}_T}(\overline{y})\right) + d_L\left(\pi_{\mathcal{G}_T}(\overline{y}), \overline{y}\right) \\ &\leq r + \left(\frac{b}{r} d_L(\overline{x}, \overline{y}) + b\right) + C d_L\left(\overline{y}, \pi_{\mathcal{G}_T}(\overline{y})\right) \\ &\leq 2b + \left(\frac{1}{2} d_L(\overline{x}, \overline{y}) + b\right) + C2b. \end{aligned}$$

We obtain $d_L(\overline{x}, \overline{y}) \leq 6b + 4Cb$. Therefore, any $\overline{z} \in [\overline{x}, \overline{y}]$ is contained in an $R = 8b + 4Cb$ Lipschitz neighborhood of $\mathcal{G}_T$. By Lemma 5.2, we conclude that $\mathcal{G}_L$ is contained in a CR Lipschitz neighborhood of $\mathcal{G}_T$. In particular, $\mathcal{G}_L$ is cobounded (for some constant depending only on S).

Now parametrize $x_t = \mathcal{G}_L(t)$ and $y_t = \mathcal{G}_T(t)$ such that $x = \mathcal{G}_L(0) = \mathcal{G}_T(0)$. We have shown that for any $t \in [0, d]$, $d = d_L(x, y)$, there exists s such that $d_L\left(x_t, y_s\right) \leq R$. The proof will be complete if $s \stackrel{+}{\asymp} t$. We have:

$$s = d_T(x, y_s) \stackrel{+}{\asymp} d_L(x, y_s) \stackrel{+}{\asymp} d_L(x, x_t) = t.$$

Thus for every $t \in [0, d]$, we have $d_L(x_t, y_t) \stackrel{+}{\prec} 1$. The same thing is true for $d_L(y_t, x_t)$ since $\mathcal{G}_L$ is cobounded. □

We now show that the closest-point projection to $\mathcal{G}_L$ is also strongly contracting. As a corollary, $\mathcal{G}_L$ is stable. The precise formulations are below.

Theorem 5.3 (Bounded projection to Lipschitz geodesics) *Suppose $x, y \in \mathcal{T}(S)$ are thick and have bounded combinatorics. There exists a constant R such that whenever $\mathcal{G}_L$ is a Lipschitz geodesic from x to y and B is a Lipschitz ball with*

$$d_L(B, \mathcal{G}_L) = \min_{z \in B} d_L(z, \mathcal{G}_L) > R,$$

then the Lipschitz projection of B to $\mathcal{G}_L$ is uniformly bounded.

Proof Let $\mathcal{G}_T$ be the Teichmüller geodesic from x to y. Let R be the minimum constant such that $\mathcal{G}_L$ is contained in the R Lipschitz neighborhood of $\mathcal{G}_T$ (Theorem 5.1). With this R, any Lipschitz ball B satisfying the criterion of the theorem is disjoint from $\mathcal{G}_T$. Therefore, by Theorem 4.5, the projection of B to $\mathcal{G}_T$ has uniformly bounded diameter. To see that the projection of B to $\mathcal{G}_L$ also has uniformly bounded diameter, it suffices to show that, for any $z \in B$, the distance between $\pi_{\mathcal{G}_T} \circ \pi_{\mathcal{G}_L}(z)$ and $\pi_{\mathcal{G}_T}(z)$ is uniformly bounded.

We refer to Fig. 5 for this proof. By Lemma 4.4, $\pi_{\mathcal{G}_T}(z)$ is uniformly bounded from $x_{t_\alpha} = \mathcal{G}_T(t_\alpha)$, where $\alpha \in \mu_z$ is a candidate curve for the Lipschitz distance from z to $\mathcal{G}_T$, and t_α is the balanced time for α. Now let $w \in \pi_{\mathcal{G}_L}(z)$ and let $x_t \in \pi_{\mathcal{G}_T}(w)$. We will show $|t_\alpha - t|$ is uniformly bounded. Choose a point $w' \in \mathcal{G}_L$ so that $d_L(w', x_{t_\alpha})$ is minimal. In particular, $d_L(w', x_{t_\alpha}) \leq R$, and

$$d_L(z, w) \leq d_L(z, w') \leq d_L(z, x_{t_\alpha}) + CR, \tag{10}$$

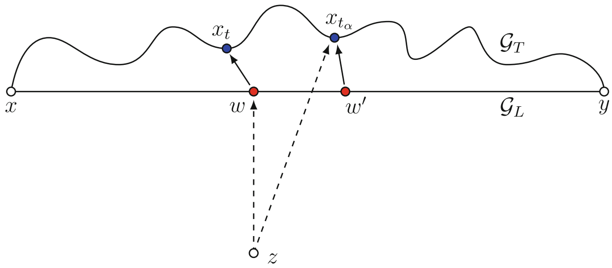

Fig. 5 Bounded projection to Lipschitz geodesics

where C is the constant of Lemma 5.2. On the other hand,

$$\begin{aligned} d_L(z, w) &\geq \log \frac{\ell_w(\alpha)}{\ell_z(\alpha)} \\ &= \log \frac{\ell_w(\alpha)}{\ell_{x_t}(\alpha)} + \log \frac{\ell_{x_t}(\alpha)}{\ell_{x_{t_\alpha}}(\alpha)} + \log \frac{\ell_{x_{t_\alpha}}(\alpha)}{\ell_z(\alpha)} \\ &\stackrel{+}{\succ} \log \frac{\ell_w(\alpha)}{\ell_{x_t}(\alpha)} + |t_\alpha - t| + d_L(z, x_{t_\alpha}). \end{aligned} \tag{11}$$

Since $x_t \in \pi_{\mathcal{G}_T}(w)$, $d_L(w, x_t) \leq R$. Hence,

$$\log \frac{\ell_w(\alpha)}{\ell_{x_t}(\alpha)} = -\log \frac{\ell_{x_t}(\alpha)}{\ell_w(\alpha)} \geq -d_L(w, x_t) \geq -R.$$

Putting this together with Eqs. (10) and (11) yields $|t_\alpha - t| \stackrel{+}{\prec} (C + 1)R$. □

Corollary 5.4 (Stability of Lipschitz geodesics) *Suppose $x, y \in \mathcal{T}(S)$ are thick and have bounded combinatorics. Then any Lipschitz quasi-geodesic from x to y (after reparametrization) fellow travels any Lipschitz geodesic from x to y.*

Proof The same argument in the proof of Theorem 5.1 can be applied here. Except now $\mathcal{G}_L$ will play the role of $\mathcal{G}_T$, and any Lipschitz quasi-geodesic from x to y will play the role of $\mathcal{G}_L$. □

We remark that, in general, a Lipschitz geodesic from x to y is not a Lipschitz geodesic from y to x, even after reparametrization. One does not even expect the Hausdorff distance between a geodesic from x to y and a geodesic from y to x to be bounded. (The Hausdorff distance is the smallest R such that each is contained in an R Lipschitz neighborhood of the other). However, the notion of bounded combinatorics is a symmetric notion, as it is defined using distances in curve graphs. Since Teichmüller geodesics are independent of the order of the endpoints, we can also deduce the following corollary.

Corollary 5.5 *Suppose $x, y \in \mathcal{T}(S)$ are thick and have bounded combinatorics. Then the Hausdorff distance between any Lipschitz geodesic from x to y and any Lipschitz geodesic from y to x is uniformly bounded.*

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