RENORMALIZING AN INFINITE RATIONAL IET

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ABSTRACT. We study an interval exchange transformation of $[0,1]$ formed by cutting the interval at the points $\frac{1}{n}$ and reversing the order of the intervals. We find that the transformation is periodic away from a Cantor set of Hausdorff dimension zero. On the Cantor set, the dynamics are nearly conjugate to the 2–adic odometer.

INTRODUCTION

We study variations of the following interval exchange transformation: Consider the interval $[0,1)$ and cut it into subintervals of the form $[1-\frac{1}{k}, 1-\frac{1}{k+1})$ for integers $k \geq 1$. We are interested in the dynamical system $T_1 : [0,1) \to [0,1)$ that reverses the order of the intervals, see Figure 1.

To study this map T_1, we are also interested in similar maps T_N on particular subintervals $X_N \subset [0,1)$. For this, let N be a positive integer and let X_N denote the half-open interval $[0, \frac{1}{N})$. Now consider the dynamical system $T_N : X_N \to X_N$ where X_N is cut into half-open intervals of the form $[\frac{1}{N} - \frac{1}{k}, \frac{1}{N} - \frac{1}{k+1})$ for $k \geq N$. Reversing the order of these intervals can be described by applying a translation by $\frac{1}{k} + \frac{1}{k+1} - \frac{1}{N}$ to each such interval. More formally, the map $T_N : X_N \to X_N$ is defined by

$$T_N(x) = x - \frac{1}{N} + \frac{1}{k} + \frac{1}{k+1} \quad \text{where} \quad k = \left\lfloor \frac{1}{\frac{1}{N} - x} \right\rfloor.$$

Here $\lfloor \star \rfloor$ denotes the greatest integer less than or equal to $\star$. The map T_N is nearly a bijection: it is one-to-one and its image is the open interval $(0, \frac{1}{N})$.

Following notation that is standard in the theory of dynamical systems, we use $T_N^j(x)$ to indicate the point that is obtained by applying this map j times to the point $x \in X_N$. A point x is called *periodic* under T_N if there exists an integer $j > 0$ such that $T_N^j(x) = x$. We will show:

Theorem 1. *For each positive integer N, there is a Cantor set $\bar{\Lambda}_N \subset [0, \frac{1}{N}]$ of Hausdorff dimension zero such that x is periodic under T_N if and only if there exists an $\epsilon > 0$ such that $(x, x+\epsilon) \cap \bar{\Lambda}_N = \emptyset$. In particular, x is periodic if $x \notin \bar{\Lambda}_N$, so the vast majority of points are periodic under the map T_N.*

Let Λ_N denote the set of points which are *aperiodic* (not periodic) under T_N. The dynamics of the restriction of T_N to Λ_N turn out to be related to the 2–adic odometer which we now define.

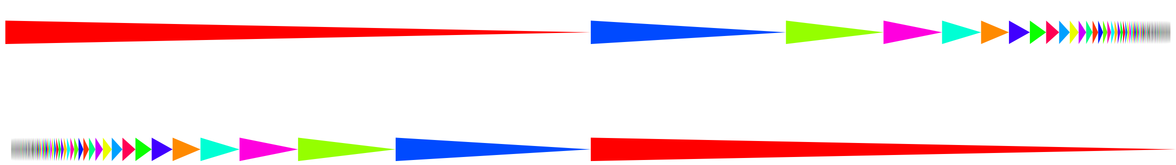

FIGURE 1. *Top:* The interval $[0,1)$ cut into intervals of the form $[1-\frac{1}{k}, 1-\frac{1}{k+1})$. *Bottom:* The images of these intervals under T_1.

Let $\mathcal{A}$ be the alphabet $\{0,1\}$ and $\mathbb{N} = \{0,1,2,\ldots\}$. The 2–adic integers are the set of formal sums

$$\sum_{k\in\mathbb{N}} \alpha_k 2^k \quad \text{with } \alpha_k \in \mathcal{A} \text{ for all } k. \tag{1}$$

We identify the 2–adic integers with the space $\mathcal{A}^{\mathbb{N}}$ consisting of all sequences $\alpha = (\alpha_0, \alpha_1, \ldots)$ with each $\alpha_k \in \mathcal{A}$. The 2–adic integers form an abelian group with the operation of addition allowing carrying of the form $1\cdot 2^k + 1\cdot 2^k = 1\cdot 2^{k+1}$. The *addition-by-one map* is given by adding $1\cdot 2^0$ to a 2–adic integer. In terms of sequences, the addition-by-one map is the map $f : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ defined by

$$f(\alpha)_k = \begin{cases} 0 & \text{if } k < j \\ 1 & \text{if } k = j \\ \alpha_k & \text{if } k > j \end{cases} \quad \text{where} \qquad j = \min\left(\{k :\ \alpha_k = 0\} \cup \{+\infty\}\right). \tag{2}$$

This map is also called the 2*–adic odometer.* It is a homeomorphism when we equip $\mathcal{A}$ with the discrete topology and $\mathcal{A}^{\mathbb{N}}$ with the product topology. It is well known that f is *minimal* (all orbits are dense) and *uniquely ergodic* (there is only one invariant Borel probability measure) [Pyt02, §1.6.2].

Let $\mathcal{N}$ be the set of all 2–adic integers $\alpha \in \mathcal{A}^{\mathbb{N}}$ which end in an infinite sequence of ones, i.e.,

$$\mathcal{N} = \{\alpha \in \mathcal{A}^{\mathbb{N}} : \quad \text{there exists a } K \text{ such that } \alpha_k = 1 \text{ for } k > K\}.$$

Another characterization of this set is as the set of 2–adic integers α such that there exists an $n > 0$ for which $f^n(\alpha) = \overline{0}$, where $\overline{0} \in \mathcal{A}^{\mathbb{N}}$ is the zero element defined by $\overline{0}_k = 0$ for all k.

We show that the restriction of T_N to the aperiodic set Λ_N mirrors the action of the 2–adic odometer:

Theorem 2. *For each positive integer N and $T = T_N$, there is a continuous bijection $h = h_N$ from $\mathcal{A}^{\mathbb{N}} \smallsetminus \mathcal{N}$ to the aperiodic set $\Lambda_N \subset X_N$ such that $T \circ h(\alpha) = h \circ f(\alpha)$ for all $\alpha \in \mathcal{A}^{\mathbb{N}}$.*

We give an explicit description of the aperiodic set Λ_N and an explicit description of the map h in § 4.

The *least period* of a periodic point $x \in X_N$ under T_N is the smallest $k > 0$ such that $T_N^k(x) = x$. An interesting question this work leaves open is (see also Remark 7):

Question 3. Which integers $p > 0$ appear as least periods of periodic points under T_N? For each such p what is the Lebesgue measure of the set of periodic points of least period p?

Connections to other work. Another infinite interval exchange transformation (IET) is given by the *Van der Corput map*:

$$S : [0,1) \to [0,1); \quad x \mapsto x - 1 + 2^k + 2^{k-1} \quad \text{where} \quad k = \lfloor \log_2(1-x) \rfloor. \tag{3}$$

This map is nearly conjugate to the 2–adic odometer; see discussions in [Pyt02, §5.2.3], [Sil08, §3.8] and [LT16, §2]. This map turns out to be semi-conjugate to the restriction of T_N to Λ_N as described in Theorem 2.

Polygon and polytope exchange transformations (PETs) are higher dimensional analogs of IETs. There are numerous examples in the literature of such maps admitting an open dense set of periodic points but with interesting dynamics on the complimentary sets. See for example [AH13], [Goe00], [Goe03], [Hoo13], [Sch14], [Yi18]. This sort of behaviour is impossible for IETs formed by permuting finitely many intervals [MT02, Theorem 6.6]. Part of the purpose of this article is to illustrate that this phenomenon arises in natural infinite IETs.

It is not the case that every infinite IET has a minimal component where the restriction of the map to this component is conjugate to an odometer. For example, there exists an infinite minimal IET of $[0, 1]$ with positive entropy such that all lengths are 2–adic rationals (see [DHV, §4]) but odometers have entropy zero.

1. Generalities

Interval exchanges. For us, an *interval exchange transformation* (IET) is a one-to-one piecewise translation $T : X \to X$ where $X \subset \mathbb{R}$ is a bounded interval. That is, we have a partition of X into countably many subintervals $X = \bigsqcup_{j\in J} I_j$ and a choice of translations $\tau_j \in \mathbb{R}$ for $j \in J$ such that the map

$$T : X \to X; \quad x \mapsto x + \tau_j \quad \text{when } x \in I_j$$

is injective.

We call T *rational* if each τ_j lies in $\mathbb{Q}$. The following is a classical observation:

Proposition 4. *If T is a rational IET and τ_j takes only finitely many values, then every orbit of T is periodic. More generally, if $T : X \to X$ is a rational IET and $x \in X$, then x has a periodic orbit unless*

$$\{\tau_j : \text{ there is an } n \geq 0 \text{ such that } T^n(x) \in I_j\} \quad \text{is infinite.}$$

Proof. Since each $\tau_j \in \mathbb{Q}$ and there are only finitely many translations τ_j, there is a $d \in \mathbb{Q}$ such that $\frac{\tau_j}{d} \in \mathbb{Z}$ for all j. Observe that T permutes the finitely many points in $(x + d\mathbb{Z}) \cap X$. □

When we were working on this project, we wondered how common it is to have a dense set of periodic points for a rational IET which is *infinite* in the sense that $\{\tau_j\}$ is infinite. Some experimental work of Anna Tao (undergraduate, CCNY) seems to suggest that this sort of periodicity is rare. However, we still wonder if there are natural classes of infinite rational IETs in which having a dense set of periodic points is typical.

At this point, there are a number of infinite rational IETs in the literature. Equation (3) gives an infinite rational IET without periodic orbits, and there are other examples corresponding to p–adic odometers and the Chacon middle third transformation [Dow05, §3] [LT16]. One way to get such a rational IET is from straight-line flows in directions of rational slope on an infinite-type translation surface all of whose saddle connections have holonomy in $\mathbb{Q}^2$. Symmetric surfaces of this form have been described in [Cha04], [Bow13] and [LT16].

Return maps. If $Y \subset X$ is an interval, the *first return time* of $y \in Y$ to Y is

$$r(y) = \min \left(\{n > 0 : T^n(y) \in Y\} \cup \{+\infty\}\right).$$

Assuming $r < +\infty$ on Y, we define the *first return map* $\hat{T} : Y \to Y$ to be the map

$$\hat{T} : Y \to Y; \quad y \mapsto T^{r(y)}(y).$$

If T is an IET in the sense above, then so is $\hat{T}$. Furthermore, $\hat{T}$ is rational whenever T is rational.

2. Basic return maps

Here we prove some basic results about the maps $T_N : X_N \to X_N$ defined in the introduction. First we fully describe the return map to $X_{N(N+1)}$.

Lemma 5. *For any N, the first return map of T_N to the interval $X_{N(N+1)} = [0, \frac{1}{N(N+1)})$ is given by $T_{N(N+1)}$. Furthermore, the return time is 2 on all of $X_{N(N+1)}$.*

Proof. For each $x \in X_{N(N+1)}$ we see that $k = \lfloor \frac{1}{1/N - x} \rfloor = N$ and thus $T_N(x) = x + \frac{1}{N+1}$. This shows $T_N(X_{N(N+1)}) = [\frac{1}{N+1}, \frac{1}{N})$ and in particular, no point has least period 1. We have that

$$\left[\frac{1}{N+1}, \frac{1}{N}\right) = \bigcup_{\ell \geq N(N+1)} \left[\frac{1}{N} - \frac{1}{\ell}, \frac{1}{N} - \frac{1}{\ell+1}\right).$$

Set $x' = T_N(x) \in [\frac{1}{N+1}, \frac{1}{N})$ and $k' = \lfloor \frac{1}{1/N - x'} \rfloor \geq N(N+1)$. We compute

$$T_N^2(x) = T_N(x') = x' - \frac{1}{N} + \frac{1}{k'} + \frac{1}{k'+1} = x - \frac{1}{N(N+1)} + \frac{1}{k'} + \frac{1}{k'+1}. \tag{4}$$

Now observing that

$$k' = \left\lfloor \frac{1}{\frac{1}{N} - x'} \right\rfloor = \left\lfloor \frac{1}{\frac{1}{N} - x - \frac{1}{N+1}} \right\rfloor = \left\lfloor \frac{1}{\frac{1}{N(N+1)} - x} \right\rfloor,$$

we see from (4) that $T_N^2(x)$ coincides with $T_{N(N+1)}(x)$. □

We get periodic points as a consequence:

Corollary 6. *For any N, every point in $[\frac{1}{N(N+1)}, \frac{1}{N+1})$ has a periodic orbit under T_N.*

Proof. Observe that $T_N\big([\frac{1}{N(N+1)}, \frac{1}{N+1})\big) = [\frac{1}{N(N+1)}, \frac{1}{N+1})$, because T_N reverses the order of intervals and we already know $T_N\big([0, \frac{1}{N(N+1)})\big) = [\frac{1}{N+1}, \frac{1}{N})$ and $T_N\big([\frac{1}{N+1}, \frac{1}{N})\big) = (0, \frac{1}{N(N+1)})$. Moreover, there are only finitely many distinct translations occurring on this interval, namely the translations associated to $(\frac{1}{k+1}, \frac{1}{k}]$ for values of k satisfying $N+1 \leq k < N(N+1)$. Proposition 4 then guarantees that every point in $[\frac{1}{N(N+1)}, \frac{1}{N+1})$ is periodic. □

Remark 7. *In the case $N = 2$, every point in the interval $[\frac{1}{6}, \frac{1}{3})$ has a periodic orbit under T_2 that has least period 10. In general, however, there may be points in $[\frac{1}{N(N+1)}, \frac{1}{N+1})$ that do not have the same least period. For example, for $N = 3$, points may have least period either 920 or 930 under T_3.*

To describe more examples for larger N, we define another family of IETs

$$R_{m,n} : \left[\frac{1}{m}, \frac{1}{n}\right) \to \left[\frac{1}{m}, \frac{1}{n}\right)$$

for all $m > n > 0$ by breaking this interval into subintervals of the form $[\frac{1}{k+1}, \frac{1}{k})$ for $m > k \geq n$ and reversing the order of the subintervals. Note that the restriction of T_N to the interval $[\frac{1}{N(N+1)}, \frac{1}{N+1})$ is $R_{N(N+1),N+1}$. In fact, there are many subintervals in X_1 of the form $[\frac{1}{m}, \frac{1}{n})$ that are preserved by a power of T_1 and where the first return map is $R_{m,n}$.

For example, the interval $[\frac{1}{42}, \frac{1}{7})$ is sent to itself by T_6. The restriction of T_6 to $[\frac{1}{42}, \frac{1}{7})$ coincides with $R_{42,7}$ and with the restriction of T_1^4 to this interval. Analyzing the periodic orbits in $[\frac{1}{42}, \frac{1}{7})$ with

[Del18], *we see that there are nine different least periods occurring under T_6, namely:*

$$\begin{array}{ccccc} 272, & 2002, & 105252, & 125986, & 9515623638834, \\ 70542359811724, & 35513020871128, & 13883349533760, & 43184371863572. & \end{array}$$

Furthermore, the interval $[\frac{1}{42}, \frac{1}{7})$ itself has subintervals of the same type that are preserved by some power of T_1. Namely, $[\frac{1}{15}, \frac{1}{10})$ is sent to itself under T_1^8 which coincides with T_6^2 and with $R_{15,10}$. Also each of the intervals $[\frac{1}{18}, \frac{1}{15})$, $[\frac{1}{24}, \frac{1}{18})$ and $[\frac{1}{42}, \frac{1}{24})$ are sent to themselves under T_1^{16} which coincides with T_6^4 and the first return map is of the form $R_{m,n}$. In case of the interval $[\frac{1}{42}, \frac{1}{24})$, there are again five different least periods occurring.

In all these cases, every possible least period has to be a divisor of the least common multiple of the denominators $n, n+1, \dots, m$. It would be interesting to classify for which pairs (m, n) every point in the interval $[\frac{1}{m}, \frac{1}{n})$ has the same least period under $R_{m,n}$.

3. Cantor sets

In this section we work through a general construction of a Cantor set. We will see later in the article that the set $\bar{\Lambda}_N$ arises as such a Cantor set.

The *free monoid* on the alphabet $\mathcal{A} = \{0, 1\}$ is the set $\mathcal{A}^*$ of all finite sequences in $\mathcal{A}$ equipped with the binary operation of concatenation. An element $w \in \mathcal{A}$ is called a *word* and has a *length* $|w| \in \mathbb{N}$ representing the number of elements strung together. We write $\mathcal{A}^k$ to denote the collection of all $w \in \mathcal{A}^*$ with length k. The unique element $\varepsilon \in \mathcal{A}^*$ with length zero is called the *empty word* and is the unique identity element of the monoid.

Every element $w \in \mathcal{A}^*$ can be written as

$$w = w_0 w_1 \dots w_{|w|-1} \quad \text{with each } w_i \in \mathcal{A}.$$

Concatenation is the operation defined by

$$xy = x_0 x_1 \dots x_{|x|-1} y_0 y_1 \dots y_{|y|-1}.$$

More formally, xy is defined to be the finite sequence z of length $|x| + |y|$ such that

$$z_i = \begin{cases} x_i & \text{if } i < |x|, \\ y_{i-|x|} & \text{if } |x| \leq i \leq |x| + |y|. \end{cases}$$

We use exponential notation for repeated concatenation so that w^k denotes the concatenation of k copies of w. For example 0^9 denotes the word w where $|w| = 9$ and $w_i = 0$ for $i = 0, \dots, 8$.

We now informally describe the Cantor sets that we are interested in. We use a variant of the standard construction of the Cantor ternary set in $\mathbb{R}$, where the Cantor set is obtained by removing the middle third interval of $[0, 1]$, then removing the middle third intervals of the remaining segments, and so on. Our Cantor set is similarly defined as the intersection $\bigcap_{k \geq 0} C_k$ and each C_k is a finite union of closed intervals. The sets C_k are defined inductively starting with a single interval $C_0 = [a_0, b_0]$ and the set C_{k+1} is formed by removing middle intervals of equal length from each of the intervals making up C_k. In contrary to the construction of the Cantor ternary set, the ratio of the lengths of intervals making up C_{k+1} to the lengths of intervals making up C_k is not necessarily the same for all k. We denote these ratios by numbers s_k.

We now give a more formal construction of our Cantor set. Fix an initial interval $[a_0, b_0]$ and a sequence $s = \{s_k\}_{k \in \mathbb{N}}$ of real numbers satisfying

$$0 < s_k \leq \frac{1}{2} \text{ for all } k \in \mathbb{N} \quad \text{and} \quad \limsup s_k < \frac{1}{2}. \tag{5}$$

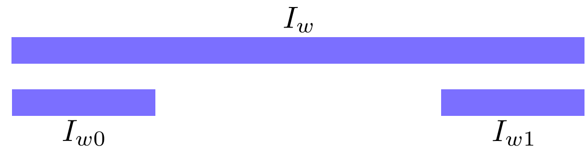

Figure 2. The intervals I_{w0} and I_{w1} produced from I_w when $s_{|w|} = \frac{1}{4}$.

We inductively define an interval I_w for each $w \in \mathcal{A}^*$. We define $I_\varepsilon = [a_0, b_0]$. Assuming I_w is defined to be $[a, b]$, we define

$$I_{w0} = [a, a + s_{|w|}(b-a)] \quad \text{and} \quad I_{w1} = [b - s_{|w|}(b-a), b]; \tag{6}$$

see Figure 2. Observe that if $s_{|w|} < \frac{1}{2}$ then $I_{w0} \cup I_{w1}$ is the interval I_w with the middle open interval removed whose length is $1 - 2s_{|w|}$ times the length of the whole interval. On the other hand, if $s_{|w|} = \frac{1}{2}$ the intervals I_{w0} and I_{w1} are formed by cutting I_w at the midpoint. In particular, the length of the interval I_w only depends on $|w|$ and the fixed sequence s. The length is given by $\ell_{|w|}$ where

$$\ell_0 = b_0 - a_0 \quad \text{and} \quad \ell_k = (b_0 - a_0) \prod_{j=0}^{k-1} s_j \quad \text{for } k \geq 1. \tag{7}$$

We define the Cantor set $\mathcal{C} = \mathcal{C}\big(s, [a_0, b_0]\big)$ by defining

$$C_k = \bigcup_{w \in \mathcal{A}^k} I_w \quad \text{and} \quad \mathcal{C} = \bigcap_{k \in \mathbb{N}} C_k.$$

It is a standard observation that as long as the sequence s satisfies the conditions in (5) that $\mathcal{C}$ is a Cantor set: it is compact, totally disconnected and perfect. The following is a standard result on the Hausdorff dimension of $\mathcal{C}$ (compare [Mat95, §4.10-11]).

Proposition 8. *If $\lim_{k\to\infty} s_k = 0$ then the Hausdorff dimension of $\mathcal{C}$ is zero.*

Proof. Recall that the *d–dimensional Hausdorff content* of $\mathcal{C}$ is

$$C_H^d(\mathcal{C}) = \inf \Big\{ \sum_i r_i^d : \text{ there is a covering of } \mathcal{C} \text{ by balls of radius } r_i > 0 \Big\}.$$

The *Hausdorff dimension* of $\mathcal{C}$ is $\inf\{d : \; C_H^d(\mathcal{C}) = 0\}$.

Fix $d > 0$. Now consider an integer $k > 0$ and consider that $\bigcup_{w \in \mathcal{A}^k} I_w$ contains $\mathcal{C}$. Each interval in the union has length ℓ_k and there are 2^k words in $\mathcal{A}^k$, so for this covering $\sum_i r_i^d$ yields $2^k(\ell_k/2)^d$. Observe from (7) that

$$\lim_{k\to\infty} 2^k \left(\tfrac{\ell_k}{2}\right)^d = \left(\tfrac{b_0 - a_0}{2}\right)^d \lim_{k\to\infty} \prod_{j=0}^{k-1} \left(2 s_j^d\right)$$

and since $s_j \to 0$, this limit is zero. This shows that the d–dimensional Hausdorff content is zero for any $d > 0$ and so the Hausdorff dimension is zero. □

We can now define the map h that was announced in Theorem 2 to give a continuous bijection from $\mathcal{A}^{\mathbb{N}} \smallsetminus \mathcal{N}$ to the aperiodic set of T_N. Recall that $\mathcal{A}^{\mathbb{N}}$ is the set of 2–adic integers, consisting of all sequences $\alpha = (\alpha_0, \alpha_1, \ldots)$ with each $\alpha_k \in \mathcal{A} = \{0,1\}$. Define the map $h : \mathcal{A}^{\mathbb{N}} \to \mathbb{R}$ depending on a sequence s as in (5) and on an interval $[a_0, b_0]$ by

$$h(\alpha) = a_0 + \sum_{k=0}^{\infty} \alpha_k(\ell_k - \ell_{k+1}). \tag{8}$$

We will see in Lemma 10 that the function h is closely related to our construction of the Cantor set $\mathcal{C}(s, [a_0, b_0])$. We will also see in § 4 that h can be used to give a 2–adic infinite address to every point in the aperiodic set of T_N, and that h describes a semi-conjugacy to the 2–adic odometer. But first we observe that h can be used to describe the endpoints of the intervals I_w used in the construction of the Cantor set $\mathcal{C}(s, [a_0, b_0])$.

Proposition 9. *For each $w \in \mathcal{A}^*$, we have $I_w = \left[h(w\overline{0}), h(w\overline{1})\right]$, where $w\overline{0}$ and $w\overline{1}$ denote the elements of $\mathcal{A}^{\mathbb{N}}$ whose first $|w|$ entries are given by w and whose remaining entries are all zeros or all ones respectively.*

Proof. Fix w and let $k = |w|$. Observe that the lengths of I_w and $\left[h(w\overline{0}), h(w\overline{1})\right]$ match since the length of I_w is ℓ_k and

$$h\left(w\overline{1}\right) - h\left(w\overline{0}\right) = \sum_{j=k}^{\infty}(\ell_j - \ell_{j+1}) = \ell_k$$

since $\lim_{j\to\infty} \ell_j = 0$. It follows that checking $I_w = \left[h(w\overline{0}), h(w\overline{1})\right]$ is equivalent to checking that the left endpoint of I_w is $h(w\overline{0})$ or checking that the right endpoint of I_w is $h(w\overline{1})$.

We proceed by induction on the length of the word w. Observe that $h(\overline{0}) = a_0$ and hence $I_\varepsilon = \left[h(\overline{0}), h(\overline{1})\right]$. Now suppose that $I_w = [a, b]$, $h(w\overline{0}) = a$ and $h(w\overline{1}) = b$. We have to check that $I_{w0} = \left[h(w0\overline{0}), h(w0\overline{1})\right]$ and $I_{w1} = \left[h(w1\overline{0}), h(w1\overline{1})\right]$. The statement for I_{w0} holds because the left endpoint of I_{w0} coincides with the left endpoint of I_w by definition in (6), and by hypothesis we have $a = h(w\overline{0}) = h(w0\overline{0})$. The statement for I_{w1} holds because the right endpoint of I_{w1} coincides with the right endpoint of I_w by (6), and by hypothesis we have $b = h(w\overline{1}) = h(w1\overline{1})$. □

Lemma 10. *The image $h(\mathcal{A}^{\mathbb{N}})$ is the Cantor set $\mathcal{C} = \mathcal{C}(s, [a_0, b_0])$. Furthermore, h is one-to-one at all $x \in \mathcal{C}$ except at those x of the form $x = h(w0\overline{1}) = h(w1\overline{0})$ for some $w \in \mathcal{A}^k$ with $s_k = \frac{1}{2}$. The latter case happens only finitely often and in this case, h is two-to-one at x.*

Proof. First we show that for any $\alpha \in \mathcal{A}^{\mathbb{N}}$ we have $h(\alpha) \in \mathcal{C}$. We must show $h(\alpha) \in C_k$ for every k. Fix a k and set $w = \alpha_0\alpha_1 \ldots \alpha_{k-1}$. Then observe that

$$h\left(w\overline{0}\right) \le h(\alpha) \le h\left(w\overline{1}\right)$$

which implies $h(\alpha) \in I_w \subset C_k$.

Now suppose $x \in \mathcal{C}$. We study the number of preimages of x under h. Observe that for each $k \ge 0$ there exists a $w \in \mathcal{A}^k$ such that $x \in I_w$. We break into two cases. First suppose that for each k there exists a unique $w \in \mathcal{A}^k$ such that $x \in I_w$. Denote each such word by w^k. Observe that w' is an initial word of w^k if and only if $I_{w'} \supset I_{w^k}$. It follows that for $j < k$, w^j is the initial subword of w^k of length j. Then we can unambiguously define $\alpha \in \mathcal{A}^{\mathbb{N}}$ by $\alpha_i = w_i^k$ for some $k > i$. Now observe that $h(\alpha) \in I_{w^k}$ for each k. Since the length of I_{w^k} tends to zero as $k \to \infty$, we see that $h(\alpha) = x$. Finally, suppose $\beta \in \mathcal{A}^{\mathbb{N}}$ is distinct from α. Then there is a k such that the initial word of length k of β differs from w^k. We see that $h(\beta) \in I_{\beta_0\ldots\beta_{k-1}}$ but x is not in this interval, so $h(\beta) \neq x$. Thus h is one-to-one at x.

If we are not in the first case, then there is a smallest k such that there are two words in $\mathcal{A}^k$ for which x lies in both the corresponding intervals. From the argument about initial words in the previous paragraph, we see that because k is smallest, the two words have the same initial words. That is, the two words must have the form $w0$ and $w1$. Thus we have $x \in I_{w0} \cap I_{w1}$. By (6) we see that $I_{w0} \cap I_{w1} \neq \emptyset$ if and only if $s_k = \frac{1}{2}$. And if this intersection is non-empty then the intersection just consists of the midpoint of I_w. In this case, x is the right endpoint of I_{w0} and the left endpoint of I_{w1}. So by Proposition 9 we see $x = h(w0\overline{1}) = h(w1\overline{0})$. Furthermore, it can be deduced by an inductive application of (6) that for any $j > 0$, we have $w' \in \mathcal{A}^{k+j}$ and $x \in I_{w'}$ if and only if $w' \in \{w01^j, w10^j\}$. Then if $\beta \in \mathcal{A}^{\mathbb{N}} \setminus \{w0\overline{1}, w1\overline{0}\}$, there is some initial word w' of β of length $k + j$ such that $w' \notin \{w01^j, w10^j\}$ and we have $h(\beta) \in I_{w'}$ but x is not in this interval, so $h(\beta) \neq x$. This shows that h is two-to-one at x. Furthermore, there are only finitely many $k > 0$ such that $s_k = \frac{1}{2}$ because of (5), so this case only appears finitely often. □

Recall from the introduction that $\mathcal{N} = \{w\overline{1} : w \in \mathcal{A}^*\} \subset \mathcal{A}^{\mathbb{N}}$. This is an important set for us, and we prove the following.

Proposition 11.

(1) *The restriction of h to $\mathcal{A}^{\mathbb{N}} \setminus \mathcal{N}$ is injective.*
(2) *The Cantor set $\mathcal{C}$ is the closure of $h(\mathcal{A}^{\mathbb{N}} \setminus \mathcal{N})$.*
(3) *The set $h(\mathcal{A}^{\mathbb{N}} \setminus \mathcal{N})$ is the set of all $x \in \mathcal{C}$ such that $(x, x + \epsilon) \cap \mathcal{C} \neq \emptyset$ for all $\epsilon > 0$.*

Proof. Statement (1) is a consequence of Lemma 10 since h is one-to-one at all points except that it is possible that $x = h(w0\overline{1}) = h(w1\overline{0})$. But we have $w0\overline{1} \in \mathcal{N}$.

Since $\mathcal{C} = h(\mathcal{A}^{\mathbb{N}})$ and $\mathcal{C}$ is closed by construction, to prove statement (2) we just need to find for each $\alpha \in \mathcal{N}$ a sequence $\alpha^k \in \mathcal{A}^{\mathbb{N}} \setminus \mathcal{N}$ such that $h(\alpha^k)$ converges to $h(\alpha)$. For each k, let $w^k = \alpha_0 \ldots \alpha_{k-1} \in \mathcal{A}^k$ and define $\alpha^k = w^k\overline{0}$. Then both $h(\alpha)$ and $h(\alpha^k)$ lie in I_{w^k} for each k and the length of I_{w^k} tends to zero so we see that $h(\alpha) = \lim h(\alpha^k)$ as desired.

Finally consider statement (3). First suppose that $\alpha \in \mathcal{A}^{\mathbb{N}} \setminus \mathcal{N}$. Then there exists a sequence $k_j \to \infty$ such that $\alpha_{k_j} = 0$. For $j \geq 0$, define $\beta^j \in \mathcal{A}$ so that the sequence agrees with α except that $\beta^j_{k_j} = 1$. Observe that by definition of h, we have $h(\beta^j) > h(\alpha)$ and $\lim h(\beta^j) = h(\alpha)$. This proves that $\big(h(\alpha), h(\alpha) + \epsilon\big)$ intersects $\mathcal{C} = h(\mathcal{A}^{\mathbb{N}})$ for all $\epsilon > 0$.

On the other hand, suppose that $x \in \mathcal{C} \setminus h(\mathcal{A}^{\mathbb{N}} \setminus \mathcal{N})$. We need to show that there exists an $\epsilon > 0$ such that $(x, x+\epsilon) \cap \mathcal{C} = \emptyset$. If $x = h(\overline{1})$ then this is clearly true since $h(\overline{1})$ is the right endpoint of I_ε by Lemma 13 and $\mathcal{C} \subset I_\varepsilon$. Otherwise there exists a $w \in \mathcal{A}^*$ such that $x = h(w0\overline{1})$. Furthermore, h is one-to-one at x since otherwise we would have $x = h(w1\overline{0})$ as well which would contradict that $x \notin h(\mathcal{A}^{\mathbb{N}} \setminus \mathcal{N})$. Setting $k = |w|$ we see therefore that $s_k < \frac{1}{2}$ by Lemma 10. Since $x = h(w0\overline{1})$ we see that x is the right endpoint of I_{w0}. Let $[a, b] = I_w$. Then we see in the notation of (6) that $x = a + s_k(b - a)$ and the removed interval $I_w \setminus (I_{w0} \cup I_{w_1})$ is $\big(x, x + (1 - 2s_k)(b - a)\big)$ which gives an interval of positive length not intersecting $\mathcal{C}$ as required. □

4. The conjugacy

Fix a positive integer N and extend it to a sequence inductively by defining

$$N_0 = N \quad \text{and} \quad N_{k+1} = N_k(1 + N_k) \text{ for all } k \geq 0.$$

By an inductive application of Lemma 5 we see:

Corollary 12. *For each k, the first return map of T_N to X_{N_k} is T_{N_k}.*

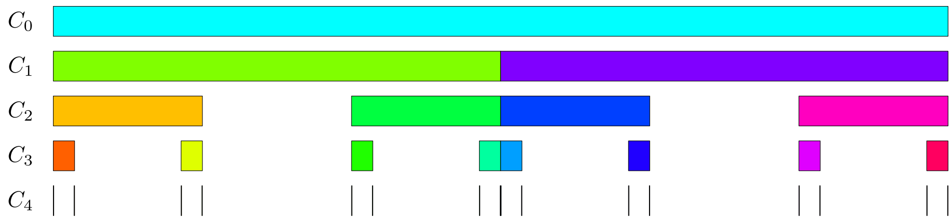

FIGURE 3. The construction of the Cantor set $\mathcal{C}$ when $N = 1$.

Set $[a_0, b_0] = [0, \frac{1}{N}]$ and define the sequence $s = \{s_k\}$ by $s_k = \frac{1}{1+N_k}$. With this data, we define the Cantor set $\mathcal{C} = \mathcal{C}(s, [a_0, b_0])$ and the map $h : \mathcal{A}^{\mathbb{N}} \to \mathbb{R}$ as in § 3. See Figure 3 for a sketch of $\mathcal{C}$ when $N = 1$. Observe that this choice of $[a_0, b_0]$ and of s and application of (7) yields $\ell_0 = \frac{1}{N} - 0 = \frac{1}{N_0}$ and inductively we have

$$\ell_k = \ell_{k-1} \cdot s_{k-1} = \frac{1}{N_{k-1}} \cdot \frac{1}{1 + N_{k-1}} = \frac{1}{N_k} \quad \text{for every } k.$$

We use this information to define the intervals I_w as before.

Recall the definition of the 2–adic odometer $f : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ in (2). We want to extend this addition-by-one map to $\mathcal{A}^* = \cup_{k \geq 0} \mathcal{A}^k$. At words of the form 1^k for some $k \geq 0$, we leave the map f undefined. We define $f : \mathcal{A}^k \smallsetminus \{1^k\} \to \mathcal{A}^k$ such that

$$\big(f(w)\big)_i = \begin{cases} 0 & \text{if } i < j \\ 1 & \text{if } i = j \\ w_i & \text{if } i > j \end{cases} \quad \text{where} \quad j = \min\{i : \ w_i = 0\}. \tag{9}$$

For this section, if I is a closed interval, we write $I^\star$ to denote I with its right endpoint removed.

The key to the results announced in the introduction is the following:

Lemma 13. *For any $w \in \mathcal{A}^k \smallsetminus \{1^k\}$, the restriction of T_N to $I_w^\star$ is a translation carrying $I_w^\star$ to $I_{f(w)}^\star$. If $w = 1^j 0$ for some $j \geq 0$ then this is a translation by $-\frac{1}{N_0} + \frac{1}{N_j} + \frac{1}{1+N_j}$.*

Proof. First we prove this for the special case when $w = 1^j 0$. By Proposition 9, the endpoints of I_w are

$$h\left(w\overline{0}\right) = \sum_{i=0}^{j-1} \left(\frac{1}{N_i} - \frac{1}{N_{i+1}}\right) = \frac{1}{N_0} - \frac{1}{N_j},$$

$$h\left(w\overline{1}\right) = \sum_{i=0}^{j-1} \left(\frac{1}{N_i} - \frac{1}{N_{i+1}}\right) + \sum_{i=j+1}^{\infty} \left(\frac{1}{N_i} - \frac{1}{N_{i+1}}\right) = \frac{1}{N_0} - \frac{1}{N_j} + \frac{1}{N_{j+1}} = \frac{1}{N_0} - \frac{1}{1 + N_j}.$$

Thus if $x \in I_w^\star$ then we see by definition of T_N that

$$T_N(x) = x - \frac{1}{N_0} + \frac{1}{N_j} + \frac{1}{1 + N_j}. \tag{10}$$

The word $f(w)$ is a string of j zeros followed by a one. Thus, we see that the endpoints of $I_{f(w)}$ are

$$h\left(f(w)\overline{0}\right) = \frac{1}{N_j} - \frac{1}{N_{j+1}} = \frac{1}{1 + N_j},$$

$$h\left(f(w)\overline{1}\right) = \sum_{i=j}^{\infty} \left(\frac{1}{N_i} - \frac{1}{N_{i+1}}\right) = \frac{1}{N_j}.$$

Observe that these new endpoints differ from the endpoints of I_w found earlier by a translation by $-\frac{1}{N_0}+\frac{1}{N_j}+\frac{1}{1+N_j}$ which is exactly how T_N acts. This proves the second statement of the lemma.

Now suppose that $w' \in \mathcal{A}^*$ is a word which has at least one zero. As in (9), we can then define $j = \min\{i :\ w'_i = 0\}$. Hence $w = w'_0 \dots w'_j$ is a word consisting of j ones followed by a zero, so the previous paragraph implies that T_N restricted to $I^\star_w$ is a translation by $-\frac{1}{N_0}+\frac{1}{N_j}+\frac{1}{1+N_j}$. Recall that $I^\star_{w'} \subset I^\star_w$ which implies that the restriction of T_N to $I^\star_{w'}$ also acts by the same translation. The intervals $I_{w'}$ and $I_{f(w')}$ have the same length and their left endpoints differ by

$$h\left(f(w')\overline{0}\right) - h\left(w'\overline{0}\right) = h\left(f(w)\overline{0}\right) - h\left(w\overline{0}\right) = -\frac{1}{N_0}+\frac{1}{N_j}+\frac{1}{1+N_j}$$

so that indeed $T_N(I^\star_{w'}) = I^\star_{f(w')}$. □

Theorem 14. *If $x \in X_N \setminus h(\mathcal{A}^{\mathbb{N}} \setminus \mathcal{N})$, then x is periodic under T_N.*

Proof. Let $x \in X_N \setminus h(\mathcal{A}^{\mathbb{N}} \setminus \mathcal{N})$. Then either x is not contained in the closed set $\mathcal{C}$ or we can apply statement (3) of Proposition 11. In both cases, there is an ϵ such that $(x, x+\epsilon) \cap \mathcal{C} = \emptyset$. Since $X_N = I^\star_\varepsilon$, the interval $(x, x+\epsilon)$ must lie in one of the gaps of the Cantor set, i.e., there is a $w \in \mathcal{A}^k$ such that

$$(x, x+\epsilon) \subset I_w \setminus (I_{w0} \cup I_{w1}).$$

It follows that $x \in I^\star_w \setminus (I^\star_{w0} \cup I^\star_{w1})$. Then $I^\star_{0^k}$ has the same length as $I^\star_w$ and so we have $I^\star_w = \tau + I^\star_{0^k}$ for some $\tau \in \mathbb{R}$ acting by translation. Set $x_0 = x - \tau \in I^\star_{0^k}$. Observe that there exists an $m \geq 0$ such that $f^m(0^k) = w$, where f is as in (9). By Lemma 13, we know that T_N^m restricted to $I^\star_{0^k}$ is a translation carrying this interval $I^\star_{0^k}$ to $I^\star_w$. Thus $T_N^m(x_0) = x$. It also follows that $T_N^m(I^\star_{0^{k+1}}) = I^\star_{w0}$ and $T_N^m(I^\star_{0^k1}) = I^\star_{w1}$ and in particular $x_0 \notin I^\star_{0^{k+1}} \cup I^\star_{0^k1}$.

Now observe that $I^\star_{0^k} = X_{N_k} = [0, 1/N_k)$ by Proposition 9, and by Corollary 12 the first return map of T_N to this interval is T_{N_k}. Since $I^\star_{0^{k+1}} = [0, 1/N_{k+1})$ and $I^\star_{0^k1} = [1/N_k - 1/N_{k+1}, 1/N_k)$, Corollary 6 tells us that x_0 is periodic under T_{N_k} and therefore also periodic under T_N. Since $x = T_N^m(x_0)$, x is also periodic. □

Theorem 15. *For any $\alpha \in \mathcal{A}^{\mathbb{N}} \setminus \mathcal{N}$, we have $T_N \circ h(\alpha) = h \circ f(\alpha)$. In particular, no point in $h(\mathcal{A}^{\mathbb{N}} \setminus \mathcal{N})$ has a periodic orbit.*

Proof. Fix $\alpha \in \mathcal{A}^{\mathbb{N}} \setminus \mathcal{N}$. Define $j = \min\left(\{k :\ \alpha_k = 0\} \cup \{+\infty\}\right)$ as in (2). Since $\alpha \notin \mathcal{N}$ we have $j < +\infty$. The initial word of α then has the form 1^j0 and the initial word of $f(\alpha)$ is 0^j1. The rest of the sequence $f(\alpha)$ agrees with α. Therefore we have

$$h \circ f(\alpha) - h(\alpha) = \left(\frac{1}{N_j} - \frac{1}{N_{j+1}}\right) - \sum_{i=0}^{j-1}\left(\frac{1}{N_i} - \frac{1}{N_{i+1}}\right) = -\frac{1}{N_0}+\frac{1}{N_j}+\frac{1}{1+N_j}. \tag{11}$$

Let $x = h(\alpha)$. Then $x \in I^\star_{1^j0}$ and T_N acts as a translation by $-\frac{1}{N_0}+\frac{1}{N_j}+\frac{1}{1+N_j}$ on $I^\star_{1^j0}$; see Lemma 13. Thus by equation (11) we see that $h \circ f(\alpha) = T_N \circ h(\alpha)$. Since f has no periodic orbits and h restricted to $\mathcal{A}^{\mathbb{N}} \setminus \mathcal{N}$ is injective, we see that T_N has no periodic orbits in $h(\mathcal{A}^{\mathbb{N}} \setminus \mathcal{N})$. □

We finish by proving the first two theorems of our article.

Proof of Theorems 1 and 2. Recall that Λ_N denoted the set of points in X_N with aperiodic orbits under T_N. Together Theorem 14 and Theorem 15 guarantee that $\Lambda_N = h(\mathcal{A}^{\mathbb{N}} \setminus \mathcal{N})$. Theorem 15 then directly implies Theorem 2. Statement (2) in Proposition 11 shows that the closure $\bar{\Lambda}_N$ is the Cantor set $\mathcal{C}$. Further by statement (3) of Proposition 11 we see that Λ_N has the form claimed in Theorem 1. The fact that Λ_N has Hausdorff dimension zero follows from Proposition 8. □

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