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Hyperbolicity in Teichmüller space

KASRA RAFI

We give an inductive description of a Teichmüller geodesic, that is, we show that there is a sense in which a Teichmüller geodesic is assembled from Teichmüller geodesics in smaller subsurfaces. We then apply this description to answer various questions about the geometry of Teichmüller space, obtaining several applications: (1) We show that Teichmüller geodesics do not backtrack in any subsurface. (2) We show that a Teichmüller geodesic segment whose endpoints are in the thick part has the fellow traveling property and that this fails when the endpoints are not necessarily in the thick part. (3) We prove a thin-triangle property for Teichmüller geodesics. Namely, we show that if an edge of a Teichmüller geodesic triangle passes through the thick part, then it is close to one of the other edges.

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1 Introduction

Two points in Teichmüller space determine a unique Teichmüller geodesic that connects them. This old result of Teichmüller [19] looks somewhat surprising now, given the current understanding of the coarse geometry of Teichmüller space; the thin part of Teichmüller space has a product structure equipped with the sup metric that resembles a space with positive curvature (Minsky [14]). This implies that there are many *nearly geodesic* paths connecting two points but only one geodesic. Our goal is to understand the behavior of this unique geodesic and describe how the given data, two endpoints x, y in the Teichmüller space of a surface S, translate to concrete information about the geodesic segment $[x, y]$ connecting them. (In general, one can replace each of x or y with a projectivized measured foliation.) Much is known about this relationship; one can describe which curves are short along a Teichmüller geodesic and what the geometry of the surface in the complement of the short curves looks like. (See Rafi [15; 16; 17], Choi, Rafi and Series [3].) The first part of this paper is devoted to organizing and improving some of these results in a way that is more accessible and so that the theorems are stated in their full natural generality, not just tailored for a specific purpose. The culmination of these results provides a complete (coarse) description of a Teichmüller geodesic. We summarize some of this information in the statements

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below. In the interest of readability, the theorems in the introduction are stated without quantifiers. The precise version of each statement appears where the theorem is proved in the article. Let S be a surface of finite type and $\mathcal{T}(S)$ be the Teichmüller space of S.

Theorem A (See Theorem 5.3) *Let $\mathcal{G}\colon \mathbb{R} \to \mathcal{T}(S)$ be a Teichmüller geodesic. For every proper subsurface $Y \subsetneq S$, there is an interval of times I_Y (possibly empty) called the* active interval *for Y. During this interval, the subsurface Y is* isolated *in the surface and the restriction of $\mathcal{G}$ to Y behaves like a geodesic in $\mathcal{T}(Y)$. Outside of I_Y, the projection of $\mathcal{G}$ to the curve complex of Y moves by at most a bounded amount.*

See Theorem 5.3 for the precise version of this theorem. In fact, we know much more. We can determine for which subsurfaces Y the interval I_Y is nonempty, and in what order these intervals appear along $\mathbb{R}$. Applying the theorem inductively, we can describe the restriction of the geodesic to Y during I_Y (Section 5).

In the rest of the paper we consider some of the implications of the above theorem and we examine to what extent Teichmüller geodesics behave like geodesics in a hyperbolic space. It is known that the Teichmüller space is not hyperbolic; Masur showed that Teichmüller space is not δ–hyperbolic (Masur and Wolf [13]). However, there is a strong analogy between the geometry of Teichmüller space and that of a hyperbolic space. For example, the isometries of Teichmüller space are either hyperbolic, elliptic or parabolic (Thurston [21], Bers [1]) and the geodesic flow is exponentially mixing (Masur [9], Veech [22]). There is also a sense in which Teichmüller space is hyperbolic relative to its thin parts; Masur and Minsky [10] showed that electrified Teichmüller space is δ–hyperbolic.

Each application of Theorem A presented in this paper examines how the Teichmüller space equipped with the Teichmüller metric is similar to or different from a relatively hyperbolic space. Apart from their individual utility, these results also showcase how one can apply Theorem A to answer geometric problems in Teichmüller space.

As the first application, we show that Teichmüller geodesics do not *backtrack*. This is a generalization of a theorem of Masur and Minsky [10] stating that the shadow of a Teichmüller geodesic to the curve complex is an unparametrized quasigeodesic.

Theorem B (See Theorem 6.1) *The projection of a Teichmüller geodesic to the complex of curves of any subsurface Y of S is an unparametrized quasigeodesic in the curve complex of Y.*

This produces a sequence of markings, analogous to a resolution of a hierarchy [10], which is obtained directly from a Teichmüller geodesic. See Theorem 6.5 for an exact statement.

As the second application, we examine the fellow traveling properties of Teichmüller geodesics.

Theorem C (See Theorem 7.1) *Consider a Teichmüller geodesic segment* $[x, y]$ *with endpoints* x *and* y *in the thick part. Any other geodesic segment that starts near* x *and ends near* y *fellow travels* $[x, y]$.

We also provide contrasting examples to the above result.

Theorem D (See Theorem 7.3) *When the endpoints of a geodesic segment are allowed to be in the thin part, the above theorem does not hold.*

As our third application, we prove that geodesic triangles are slim while they pass through the thick part of Teichmüller space, suggesting similarities between Teichmüller space and relatively hyperbolic groups.

Theorem E (See Theorem 8.1) *For a geodesic triangle* $\triangle(x, y, z)$ *in Teichmüller space, if a large segment of* $[x, y]$ *is in the thick part, then it is close either to* $[x, z]$ *or to* $[y, z]$.

Organization of the paper In Section 2, we make the notion of coarsely describing a point in Teichmüller space precise. This means to record enough information so that one can estimate the length of any curve on the surface and the distance between two points in Teichmüller space. It turns out that it is sufficient to keep track of which curves are short as well as the length and the twisting parameter of the short curves.

A Teichmüller geodesic is the image of a quadratic differential under the Teichmüller geodesic flow. In Section 3 we discuss how one can translate the information given by the flat structure of a quadratic differential to obtain the combinatorial information needed to describe a point in $\mathcal{T}(S)$.

The precise statement for the description of a Teichmüller geodesic and some related statements are given in Section 5. Theorem B is proven in Section 6, Theorems C and D are proven in Section 7, and Theorem E is proven in Section 8.

Notation The notation $A \stackrel{*}{\asymp} B$ means that the ratio A/B is bounded both above and below by constants depending on the topology of S only. When this is true we say that A is *comparable* with B or that A and B are comparable. The notation $A \stackrel{*}{\prec} B$ means that A/B is bounded above by a constant depending on the topology of S. Similarly, $A \stackrel{+}{\asymp} B$ means $|A - B|$ is uniformly bounded and $A \stackrel{+}{\prec} B$ means $(B - A)$ is uniformly bounded above, in both cases by a constant that depends only on the topology of S.

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2 Combinatorial description of a point in Teichmüller space

In this section, we discuss the notion of a marking which provides a coarse combinatorial description of a point in Teichmüller space (see Definition 2.2). Given such a description of a point x in Teichmüller space we are able to estimate the extremal length of any curve at x (Theorem 3.1). Also, given the description of two points $x, y \in \mathcal{T}(S)$, we are able to estimate the Teichmüller distance between them (Theorem 2.4). We first establish terminology and the definitions of some basic concepts.

2.1 Teichmüller metric

Let S be a compact surface of hyperbolic type possibly with boundary. The Teichmüller space $\mathcal{T}(S)$ is the space of all conformal structures on S up to isotopy. In this paper, we consider only the Teichmüller metric on $\mathcal{T}(S)$. For two points $x, y \in \mathcal{T}(S)$ the Teichmüller distance between them is defined to be

$$d_{\mathcal{T}}(x,y) = \tfrac{1}{2}\log \max_f K_f,$$

where $f\colon x \to y$ ranges over all quasiconformal maps from x to y in the correct isotopy class and K_f is the quasiconformal constant of the map f. (See Gardiner and Lakic [4] and Hubbard [5] for background information.) A geodesic in this metric is called a Teichmüller geodesic.

Arcs and curves By a *curve* in S we mean a free isotopy class of an essential simple closed curve and by an *arc* in S we mean a proper isotopy class of an essential simple arc. In both cases, *essential* means that the given curve or arc is neither isotopic to a point nor can be isotoped to ∂S. The definition of an arc is slightly different when S is an annulus. In this case, an *arc* is an isotopy class of a simple arc connecting the two boundaries of S, relative to the endpoints of the arc. We use $\mathrm{i}(\alpha,\beta)$ to denote the geometric intersection number between arcs or curves α and β and we refer to it simply as the intersection number.

Define the arc and curve graph $\mathcal{AC}(S)$ of S as follows: the vertices are essential arcs and curves in S and the edges are pairs of vertices that have representatives with disjoint interiors. Giving the edges length one turns $\mathcal{AC}(S)$ into a connected metric space. The following is contained in Masur and Minsky [10; 11] and Klarreich [7].

Theorem 2.1 *The graph $\mathcal{AC}(S)$ is locally infinite, has infinite diameter and is Gromov hyperbolic. Furthermore, its boundary at infinity can be identified with $\mathcal{EL}(S)$, the space of ending laminations of S.*

Recall that $\mathcal{EL}(S)$ is the space of *irrational* laminations in $\mathcal{PML}(S)$ (the space of projectivized measured laminations) after forgetting the measure. An irrational lamination is one that has nonzero intersection number with every curve.

Measuring the twist It is often desirable to measure the number of times a curve γ twists around a curve α. This requires us to choose a notion of *zero twisting*. The key example is the case where S is an annulus with a core curve α. Then $\mathcal{AC}(S)$ is quasi-isometric to $\mathbb{Z}$. Choose an arc $\tau \in \mathcal{AC}(S)$ to serve as the origin. Then the *twist* of $\gamma \in \mathcal{AC}(S)$ about α is

$$\operatorname{twist}_\alpha(\gamma, \tau) = \mathrm{i}(\gamma, \tau)$$

relative to choice of origin τ.

In general, if α is a curve in S let S^α be the corresponding annular cover. A notion of zero twisting around α is given by a choice of arc $\tau \in \mathcal{AC}(S^\alpha)$. Then, for every $\gamma \in \mathcal{AC}(S)$ intersecting α essentially, we define

$$\operatorname{twist}_\alpha(\gamma, \tau) = \mathrm{i}(\tilde{\gamma}, \tau),$$

where $\tilde{\gamma}$ is any essential lift of γ to S^α. Since there may be several choices for $\tilde{\gamma}$, this notion of twisting is well defined up to an additive error of at most one.

A geometric structure on S often naturally defines a notion of zero twisting. For example, for a given point $x \in \mathcal{T}(S)$ and a curve α, we can define twisting around α in x as follows: lift x to the conformal structure x^α on S^α. Consider the hyperbolic metric associated to x^α and choose τ in x^α to be any hyperbolic geodesic perpendicular to α. Now, for every curve γ intersecting α nontrivially, define

$$\operatorname{twist}_\alpha(\gamma, x) = \operatorname{twist}_\alpha(\gamma, \tau) = \mathrm{i}(\tilde{\gamma}, \tau).$$

Similarly, for a quadratic differential q on S we can define $\operatorname{twist}_\alpha(\gamma, q)$; lift q to a singular Euclidean metric q^α and choose τ to be any Euclidean arc perpendicular to α. (See Section 3 for the definition of the Euclidean metric associated to q.)

Similarly, any foliation, arc or curve λ intersecting α essentially defines a notion of zero twisting. Since the intersection is essential the lift λ^α of λ to S^α contains an essential arc which we may use as τ. Anytime two geometric objects define notions of zero twisting, we can talk about the relative twisting between them. For example, for two quadratic differentials q_1 and q_2 and a curve α, let τ_1 be the arc in q_1^α that is perpendicular to α and τ_2 be the arc in q_2^α that is perpendicular to α. Considering both these arcs in S^α, it makes sense to talk about their geometric intersection number. We define

$$\mathrm{twist}_\alpha(q_1, q_2) = \mathrm{i}(\tau_1, \tau_2).$$

The expression $\mathrm{twist}_\alpha(x_1, x_2)$ for Riemann surfaces x_1 and x_2 is defined similarly.

Marking Our definition of *marking* differs slightly from that of [11] and contains more information.

Definition 2.2 A marking on S is a triple $\mu = (\mathcal{P}, \{l_\alpha\}_{\alpha \in \mathcal{P}}, \{\tau_\alpha\}_{\alpha \in \mathcal{P}})$, where:

- $\mathcal{P}$ is a pants decomposition of S.
- For $\alpha \in \mathcal{P}$, l_α is a positive real number which we think of as the length of α.
- For $\alpha \in \mathcal{P}$, τ_α is an arc in the annular cover S^α of S associated to α, establishing a notion of zero twisting around α.

For a curve α in S and $x \in \mathcal{T}(S)$, we define the extremal length of α in x to be

$$\mathrm{Ext}_x(\alpha) = \sup_{\sigma \in [x]} \frac{\ell_\sigma^2(\alpha)}{\mathrm{area}(\sigma)}.$$

Here, σ ranges over all metrics in the conformal class x and $\ell_\sigma(\alpha)$ is the infimum of the σ–length of all representatives of the homotopy class of the curve α. Using the extremal length, we define a map from $\mathcal{T}(S)$ to the space of markings as follows: For any $x \in \mathcal{T}(S)$, let $\mathcal{P}_x$ be the pants decomposition with the shortest extremal length in x obtained using the greedy algorithm. For $\alpha \in \mathcal{P}_x$, let $l_\alpha = \mathrm{Ext}_x(\alpha)$. As in the discussion of zero twist above, let τ_α be any geodesic in S^α that is perpendicular to α in x^α. We call this the *short marking at* x and denote it by μ_x.

As mentioned before, we can compute the extremal length of any curve in x from the information contained in μ_x up to a multiplicative error. The next theorem follows from [14] (see also [8, Theorem 8]):

Theorem 2.3 *For every curve γ, we have*

$$\mathrm{Ext}_x(\gamma) \stackrel{*}{\asymp} \sum_{\alpha \in \mathcal{P}} \left(\frac{1}{l_\alpha} + l_\alpha \cdot \mathrm{twist}_\alpha(\gamma, \tau_\alpha)^2 \right) \mathrm{i}(\alpha, \gamma)^2.$$

Subsurface projection To compute the distance between two points $x, y \in \mathcal{T}(S)$ we need to introduce the concept of subsurface projection. We call a collection of vertices in $\mathcal{AC}(S)$ having disjoint representatives a *multicurve*. For every proper subsurface $Y \subset S$ and any multicurve α in $\mathcal{AC}(S)$ we can project α to Y to obtain a multicurve in $\mathcal{AC}(Y)$ as follows: Let S^Y be the cover of S corresponding to $\pi_1(Y) < \pi_1(S)$ and identify the Gromov compactification of S^Y with Y. (To define the Gromov compactification, one needs first to pick a metric on S. However, the resulting compactification is independent of the metric. Since S admits a hyperbolic metric, every essential curve in S lifts to an arc which has well-defined endpoints in the Gromov boundary of S^Y.) Then for $\alpha \in \mathcal{AC}(S)$, the projection $\alpha|_Y$ is defined to be the set of lifts of α to S^Y that are essential curves or arcs. Note that $\alpha|_Y$ is a set of diameter one in $\mathcal{AC}(Y)$ since all the lifts have disjoint interiors.

For markings μ and ν, define

$$d_Y(\mu, \nu) = \mathrm{diam}_{\mathcal{AC}(Y)}(\mathcal{P}|_Y \cup \mathcal{R}|_Y),$$

where $\mathcal{P}$ and $\mathcal{R}$ are the pants decompositions for μ and ν respectively.

Distance formula In what comes below, the function $[a]_C$ is equal to a if $a \geq C$ and it is zero otherwise. Also, we modify the $\log(a)$ function to be one for $a \leq e$. We can now state the distance formula:

Theorem 2.4 [16, Theorem 6.1] *There is a constant $C > 0$ such that the following holds. For $x, y \in \mathcal{T}(S)$ let $\mu_x = (\mathcal{P}, \{l_\alpha\}, \{\tau_\alpha\})$ and $\mu_y = (\mathcal{R}, \{k_\beta\}, \{\sigma_\beta\})$ be the associated short markings. Then*

$$(1)\quad d_{\mathcal{T}}(x, y) \asymp \sum_Y \big[d_Y(\mu_x, \mu_y)\big]_C + \sum_{\gamma \notin \mathcal{P}\cup\mathcal{R}} \big[\log d_\gamma(\mu_x, \mu_y)\big]_C + \sum_{\alpha \in \mathcal{P}\smallsetminus\mathcal{R}} \log \frac{1}{l_\alpha} + \sum_{\beta \in \mathcal{R}\smallsetminus\mathcal{P}} \log \frac{1}{k_\beta} + \sum_{\gamma \in \mathcal{P}\cap\mathcal{R}} d_{\mathbb{H}}\big((1/l_\gamma, \mathrm{twist}_\gamma(x, y)), (1/k_\gamma, 0)\big).$$

Here, $d_{\mathbb{H}}$ is the distance in the hyperbolic plane.

Remark 2.5 In the theorem above, C can be taken to be as large as needed. However, increasing C will increase the constants hidden inside $\asymp$. Let $\mathfrak{L}$ be the left-hand side of Equation (1) and $\mathfrak{R}$ be the right-hand side. Then, a stronger version of this theorem can be stated as follows: There is $C_0 > 0$, depending only on the topology of S, and for every $C \geq C_0$ there are constants A and B such that

$$\frac{\mathfrak{L}}{A} - B \leq \mathfrak{R} \leq A\,\mathfrak{L} + B.$$

As a corollary, we have the following criterion for showing two points in Teichmüller space are a bounded distance apart. Let $\epsilon_0 > \epsilon_1 > 0$, let $\mathcal{A}_x$ be a set of curves in x that have extremal length less than ϵ_0 and assume that every other curve in x has a length larger than ϵ_1. Let ϵ_0', ϵ_1' and $\mathcal{A}_y$ be similarly defined for y.

Corollary 2.6 *Assume that the following hold for $x, y \in \mathcal{T}(S)$:*

(1) $\mathcal{A}_x = \mathcal{A}_y$.

(2) *For any subsurface Y that is not an annulus with core curve in $\mathcal{A}_x$, we have $d_Y(\mu_x, \mu_y) = O(1)$.*

(3) *For $\alpha \in \mathcal{A}_x$, $\ell_x(\alpha) \stackrel{*}{\asymp} \ell_y(\alpha)$.*

(4) *For $\alpha \in \mathcal{A}_x$, $\operatorname{twist}_\alpha(x, y) = O\big(1/\operatorname{Ext}_x(\alpha)\big)$.*

Then $d_{\mathcal{T}}(x, y) = O(1)$.

Proof Condition (2) implies that the first two terms in Equation (1) are zero. Since $\mathcal{A}_x = \mathcal{A}_y$, curves in $\mathcal{P} \smallsetminus \mathcal{R}$ and $\mathcal{R} \smallsetminus \mathcal{P}$ have lengths that are bounded below. Hence the third and the forth terms of Equation (1) are uniformly bounded. The conditions on the lengths and twisting of curves in $\mathcal{A}_x$ imply that the last term is uniformly bounded; for points $p, q \in \mathbb{H}$, $p = (p_1, p_2)$, $q = (q_1, q_2)$, if

$$(p_1 - q_1) \stackrel{*}{\asymp} p_2 \stackrel{*}{\asymp} q_2 \quad \text{then} \quad d_{\mathbb{H}}(p, q) = O(1). \qquad \square$$

3 Geometry of quadratic differentials

A geodesic in Teichmüller space is the image of a quadratic differential under the Teichmüller geodesic flow. Quadratic differentials are naturally equipped with a singular Euclidean structure. We, however, often need to compute the extremal length of a curve. In this section, we review how the extremal length of a curve can be computed from the information provided by the flat structure and how the flat length and the twisting information around a curve change along a Teichmüller geodesic.

Quadratic differentials Let $\mathcal{T}(S)$ be the Teichmüller space of S and $\mathcal{Q}(S)$ be the space of unit-area quadratic differentials on S. Recall that a quadratic differential q on a Riemann surface x can locally be represented as

$$q = q(z)\,dz^2,$$

where $q(z)$ is a meromorphic function on x with all poles having a degree of at most one. All poles are required to occur at the punctures. In fact, away from zeros and poles, there is a change of coordinates such that $q = dz^2$. Here $|q|$ locally defines a Euclidean metric on x and the expressions $\Im(\sqrt{q}) = 0$ and $\Re(\sqrt{q}) = 0$ define the horizontal and the vertical directions. Vertical trajectories foliate the surface except at the zeros and the poles. This foliation equipped with the transverse measure $|dx|$ is called the vertical foliation and is denoted by λ_-. The horizontal foliation is similarly defined and is denoted by λ_+.

A neighborhood of a zero of order k has the structure of the Euclidean cone with total angle $(k+2)\pi$ and a neighborhood of a degree-one pole has the structure of the Euclidean cone with total angle π. In fact, this locally Euclidean structure and this choice of the vertical foliation completely determine q. We refer to this metric as the q–metric on S.

Size of a subsurface For every curve α, the geodesic representatives of α in the q–metric form a (possibly degenerate) flat cylinder $F_q(\alpha)$. For any proper subsurface $Y \subset S$, let $\mathsf{Y} = \mathsf{Y}_q$ be the representative of the homotopy class of Y that has q–geodesic boundaries and that is disjoint from the interior of $F_q(\alpha)$ for every curve $\alpha \subset \partial Y$. When the subsurface is an annulus with core curve α we think of $\mathsf{F} = F_q(\alpha)$ as its representative with geodesic boundary. Define $\mathrm{size}_q(Y)$ to be the q–length of the shortest essential curve in Y and for a curve α let $\mathrm{size}_q(\mathsf{F})$ be the q–distance between the boundary components of F. When Y is a pair of pants, $\mathrm{size}_q(Y)$ is defined to be the diameter of Y.

An estimate for lengths of curves For every curve α in S, denote the extremal length of α in $x \in \mathcal{T}(S)$ by $\mathrm{Ext}_x(\alpha)$. For constants $\epsilon_0 > \epsilon_1 > 0$, the (ϵ_0, ϵ_1)–thick-thin decomposition of x is the pair $(\mathcal{A}, \mathcal{Y})$, where $\mathcal{A}$ is the set of curves α in x such that $\mathrm{Ext}_x(\alpha) \le \epsilon_0$ and $\mathcal{Y}$ is the set of homotopy classes of the components of x cut along $\mathcal{A}$. We further assume that the extremal length of any essential curve γ that is disjoint from $\mathcal{A}$ is larger than ϵ_1.

Consider the quadratic differential (x, q) and the thick-thin decomposition $(\mathcal{A}, \mathcal{Y})$ of x. Let $\alpha \in \mathcal{A}$ be the common boundary of subsurfaces Y and Z in $\mathcal{Y}$.

Let α^* be the geodesic representative of α in the boundary of Y and let $\mathsf{E} = E_q(\alpha, Y)$ be the largest regular neighborhood of α^* in the direction of Y that is still an embedded annulus. We call this annulus the expanding annulus with core curve α in the direction of Y. Define $M_q(\alpha, Y)$ to be $\mathrm{Mod}_x(\mathsf{E})$, where $\mathrm{Mod}_x(\cdot)$ is the modulus of an annulus in x. Recall from [15, Lemma 3.6] that

$$\mathrm{Mod}_x(\mathsf{E}) \stackrel{*}{\asymp} \log \frac{\mathrm{size}_q(Y)}{\ell_q(\alpha)} \quad \text{and} \quad \mathrm{Mod}_x(\mathsf{F}) = \frac{\mathrm{size}_q(\mathsf{F})}{\ell_q(\alpha)}.$$

Let $\mathsf{G} = E_q(\alpha, Z)$ and $M_q(\alpha, Z)$ be defined similarly.

The following statement relates the information about the flat length of curves to their extremal length. For a more general statement see [8, Lemma 3 and Theorem 7].

Theorem 3.1 *Let (x, q) be a quadratic differential and let $(\mathcal{A}, \mathcal{Y})$ be the thick-thin decomposition of x. Then:*

(1) *For $Y \in \mathcal{Y}$ and a curve γ in Y*

$$\mathrm{Ext}_x(\gamma) \stackrel{*}{\asymp} \frac{\ell_q(\gamma)^2}{\mathrm{size}(Y)^2}.$$

(2) *For $\alpha \in \mathcal{A}$ that is the common boundary of $Y, Z \in \mathcal{Y}$,*

$$\frac{1}{\mathrm{Ext}_x(\alpha)} \stackrel{*}{\asymp} \log \frac{\mathrm{size}_q(Y)}{\ell_q(\alpha)} + \frac{\mathrm{size}_q(F_q(\alpha))}{\ell_q(\alpha)} + \log \frac{\mathrm{size}_q(Z)}{\ell_q(\alpha)}$$
$$\stackrel{*}{\asymp} \mathrm{Mod}_x(\mathsf{E}) + \mathrm{Mod}_x(\mathsf{F}) + \mathrm{Mod}_x(\mathsf{G}).$$

Length and twisting along a Teichmüller geodesic A matrix $A \in \mathrm{SL}(2, \mathbb{R})$ acts on any $q \in \mathcal{Q}(S)$ locally by affine transformations. The total angle at a point does not change under this transformation. Thus the resulting singular Euclidean structure defines a quadratic differential that we denote by Aq. The Teichmüller geodesic flow, $g_t\colon \mathcal{Q} \to \mathcal{Q}$, is the action by the diagonal subgroup of $\mathrm{SL}(2, \mathbb{R})$:

$$g_t(q) = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} q.$$

The Teichmüller geodesic described by q is then a map

$$\mathcal{G}\colon \mathbb{R} \to \mathcal{Q}, \quad \mathcal{G}(t) = (x_t, q_t),$$

where $q_t = g_t(q)$ and x_t is the underlying Riemann surface for q_t.

The flat length of a curve along a Teichmüller geodesic is well behaved. Let the horizontal length $h_t(\alpha)$ of α in q be the transverse measure of α with respect to the vertical foliation of q_t and the vertical length $v_t(\alpha)$ of α be the transverse measure with respect to the horizontal foliation of q_t. We have (see the discussion on [15, page 186])

$$\ell_{q_t}(\alpha) \stackrel{*}{\asymp} h_t(\alpha) + v_t(\alpha).$$

Since the vertical length decreases exponentially fast and the horizontal length increases exponentially fast, for every curve α there are constants L_α and t_α such that

$$\ell_{q_t}(\alpha) \stackrel{*}{\asymp} L_\alpha \cosh(t - t_\alpha). \tag{2}$$

We call the time t_α the balanced time for α and the length L_α the minimum flat length for α.

We define the twisting parameter of a curve along a Teichmüller geodesic to be the relative twisting of q_t with respect to the vertical foliation. That is, for any curve α and time t, let τ_t be the arc in q_t^α, the annular cover of q_t, that is perpendicular to α and let λ_- be the vertical foliation of q_t (which is topologically the same foliation for every value of t). Define

$$\operatorname{twist}_t(\alpha) = \operatorname{twist}_\alpha(\tau_t, \lambda_-).$$

This is an increasing function that ranges from a minimum of zero to a maximum of $T_\alpha = d_\alpha(\lambda_-, \lambda_+)$. That is, τ_t looks like λ_- at the beginning and like λ_+ in the end. In fact, from [16, Equation 16] we have the explicit formula

$$\operatorname{twist}_t(\alpha) \stackrel{+}{\asymp} \frac{2\, T_\alpha\, e^{2(t-t_\alpha)}}{\cosh^2(t-t_\alpha)}. \tag{3}$$

Also, [3, Proposition 5.8] gives the following estimate on the modulus of $\mathsf{F}_t = F_{q_t}(\alpha)$:

$$\operatorname{Mod}_{q_t}(\mathsf{F}_t) \stackrel{*}{\asymp} \frac{T_\alpha}{\cosh^2(t-t_\alpha)}. \tag{4}$$

That is, the modulus of F_t is maximum when α is balanced and goes to zero as t goes to $\pm\infty$. The maximum modulus of F_t is determined purely by the topological information T_α, which is the relative twisting of λ_- and λ_+ around α. The size of F_t at q_t is equal to its modulus times the flat length of α at q_t. Hence,

$$\operatorname{size}_{q_t}(\mathsf{F}_t) = \frac{T_\alpha L_\alpha}{\cosh(t-t_\alpha)}. \tag{5}$$

4 Projection of a quadratic differential to a subsurface

In this section, we introduce the notion of an isolated surface in a quadratic differential. Let (x,q) be a quadratic differential, $Y \subset S$ be a proper subsurface and Y be the representative of Y with q–geodesic boundaries. Note that, when Y is nondegenerate, it is itself a Riemann surface that inherits its conformal structure from x. In this case, for a curve γ in Y, we use the expression $\operatorname{Ext}_{\mathsf{Y}}(\gamma)$ to denote the extremal length of γ in the Riemann surface Y. The following lemma is a consequence of [14, Lemma 4.2].

Lemma 4.1 (Minsky) *There exists a constant m_0 depending only on the topological type of S such that, for every subsurface Y with negative Euler characteristic the following holds. If $M_q(\alpha, Y) \geq m_0$ for every boundary component α of Y then for any essential curve γ in Y*

$$\operatorname{Ext}_{\mathsf{Y}}(\gamma) \stackrel{*}{\asymp} \operatorname{Ext}_x(\gamma).$$

Fixing m_0 as above, we say Y is *isolated* in q if, for every boundary component α of Y, $M_q(\alpha, Y) \geq m_0$. The large expanding annuli in the boundaries of Y isolate it in the sense that one does not need any information about the rest of the surface to compute extremal lengths of curves in Y. As we shall see, when Y is isolated, the restrictions of the hyperbolic metric of x to Y and the quadratic differential q to Y are at most a bounded distance apart in the Teichmüller space of Y.

For $x \in \mathcal{T}(S)$ and $Y \subset S$ we define the Fenchel–Nielsen projection of x to Y, a complete hyperbolic metric $x|_Y$ on Y, as follows: Extend the boundary curves of Y to a pants decomposition $\mathcal{P}$ of S. Then the Fenchel–Nielsen coordinates of $\mathcal{P}|_Y$ define a point $x|_Y$ of $\mathcal{T}(Y)$ (see [14] for a detailed discussion).

Now, we construct a projection map from q to $q|_Y$ by considering the representative with geodesic boundary Y and capping off the boundaries with punctured disks. It turns out that the underlying conformal structures of $q|_Y$ and $x|_Y$ are not very different, but the quadratic differential restriction commutes with the action of $\mathrm{SL}(2,\mathbb{R})$. When Y is not isolated in q, the capping-off process is not geometrically meaningful (or sometimes not possible). Hence, the process is restricted to the appropriate subset of $\mathcal{Q}$.

Theorem 4.2 *Let Y be a subsurface of S that is not an annulus and let $\mathcal{Q}_Y(S)$ be the set of quadratic differentials q such that Y is isolated in q. There is a map $\pi_Y\colon \mathcal{Q}_Y(S) \to \mathcal{Q}(Y)$, with $\pi_Y(q) = q|_Y$, such that*

$$d_{\mathcal{T}(Y)}(q|_Y, x|_Y) = O(1). \tag{6}$$

Furthermore, if, for $A \in \mathrm{SL}(2,\mathbb{R})$, both q and Aq are in $\mathcal{Q}_Y(S)$ then

$$d_{\mathcal{T}(Y)}\big((Aq)|_Y, A(q|_Y)\big) = O(1). \tag{7}$$

Proof We first define the map π_Y. Let (x,q) be a quadratic differential with Y isolated in q. Let Y be the representative of Y with q–geodesic boundaries. Our plan, nearly identical to that of [17], is to fill all components of $\partial \mathsf{Y}$ with locally flat once-punctured disks.

Fix $\alpha \subset \partial Y$ and recall that $\mathsf{E} = E_q(\alpha, Y)$ is an embedded annulus and α^* is a boundary of E. Let $a_1, \dots, a_n$ be the points on α^* which have angle $\theta_i > \pi$ in E. Note that this set is nonempty: if it were empty then E would meet the interior of the flat cylinder $F(\alpha)$; a contradiction. Let E' be the double cover of E and let α' be the preimage of α^*. Let q' be the lift of $q|_{\mathsf{E}}$ to E'. Along α' we attach a locally flat disk D' with a well-defined notion of a vertical direction as follows.

Label the lifts of a_i to E' by b_i and c_i. We will fill α' by symmetrically adding $2(n-1)$ Euclidean triangles to obtain a flat disk D' such that the total angle at each b_i and c_i is a multiple of π and is at least 2π.

We start by attaching a Euclidean triangle to vertices b_1, b_2, b_3, which we denote by $\triangle(b_1, b_2, b_3)$ (see Figure 1). We choose the angle $\angle b_2$ at the vertex b_2 so that $\theta_2 + \angle b_2$ is a multiple of π. Assuming $0 \leq \angle b_2 < \pi$, there is a unique such triangle. Attach an isometric triangle to c_1, c_2, c_3. Now consider the points b_1, b_3, b_4. Again, there exists a Euclidean triangle with one edge equal to the newly introduced segment $[b_1, b_3]$, another edge equal to the segment $[b_3, b_4]$ and an angle at b_3 that makes the total angle at b_3, including the contribution from the triangle $\triangle(b_1, b_2, b_3)$, a multiple of π. Attach this triangle to the vertices b_1, b_3, b_4 and an identical triangle to the vertices c_1, c_3, c_4. Continue in this fashion until finally adding triangles $\triangle(b_1, b_n, c_1)$ and $\triangle(c_1, c_n, b_1)$. Due to the symmetry, the two edges connecting b_1 and c_1 have equal length, and we can glue them together. We call the union of the added triangles D'. Notice that the involution on E' extends to D'. Let $\mathsf{D} = \mathsf{D}(\alpha)$ be the quotient of D', and note that D is a punctured disk attached to α^* in the boundary of E.

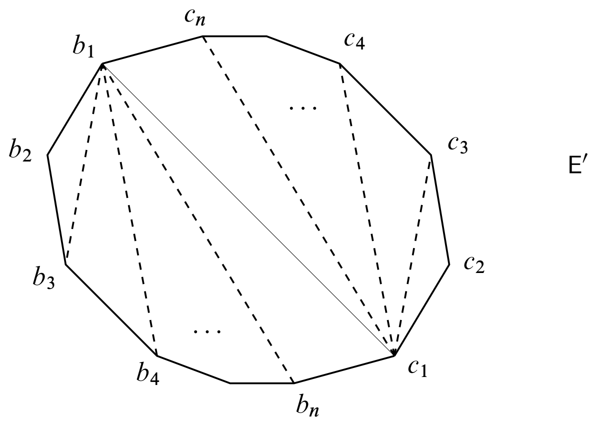

Figure 1: The filling of the annulus E' by the disk D'

For $i \neq 1$, the total angle at b_i and at c_i is a multiple of π and is larger than $\theta_i > \pi$; therefore, it is at least 2π. We have added $2(n-1)$ triangles. Hence, the sum of the total angles of all vertices is $2\sum_i \theta_i + 2(n-1)\pi$, which is a multiple of 2π. Therefore, the sum of the angles at b_1 and c_1 is also a multiple of 2π. But they are equal to each other, and each one is larger than π. This implies that they are both at least 2π. It follows that the quadratic differential q' extends over D' symmetrically with quotient an extension of q to D.

Thus, attaching the disk $\mathsf{D}(\alpha)$ to every boundary component α^* in $\partial\mathsf{Y}$ gives a point $q|_Y \in \mathcal{Q}(Y)$. This completes the construction of the map π_Y.

We now show that the distance in $\mathcal{T}(Y)$ between $q|_Y$ and $x|_Y$ is uniformly bounded. For this, we examine the extremal lengths of curves in two conformal structures. Since Y is isolated in q, the boundaries of Y are short in x. This implies, using [14], that for any essential curve γ in Y, the extremal lengths of γ in x and in $x|_Y$ are comparable:

$$\operatorname{Ext}_{x|_Y}(\gamma) \stackrel{*}{\asymp} \operatorname{Ext}_x(\gamma) \tag{8}$$

(see the proof of Theorem 6.1 in [14, page 283, line 19]). We need to show that the extremal lengths of γ in q and in $q|_Y$ are comparable as well. We obtain this after applying Lemma 4.1 twice. Once considering Y as a subset of q and once as a subset of $q|_Y$, Lemma 4.1 implies that

$$\operatorname{Ext}_x(\gamma) \stackrel{*}{\asymp} \operatorname{Ext}_{\mathsf{Y}}(\gamma) \stackrel{*}{\asymp} \operatorname{Ext}_{q|_Y}(\gamma).$$

Since the extremal lengths of curves are comparable, the distance in $\mathcal{T}(Y)$ between $x|_Y$ and $q|_Y$ is uniformly bounded above [6, Theorem 4].

We note that defining the map π_Y involved a choice of labeling of the points $\{a_i\}$. However, the argument above will work for any labeling. In fact, for any labeling of points in a boundary component of Y in q, one can use the corresponding labeling $A(\mathsf{Y})$ in (Aq) so that $A(q|_Y) = (Aq)|_Y$. Since all the different labelings result in points that are close in $\mathcal{T}(Y)$ to $x|_Y$, Equation (7) holds independently of the choices made. This finishes the proof. □

5 Projection of a Teichmüller geodesic to a subsurface

As mentioned before, a quadratic differential q defines a Teichmüller geodesic $\mathcal{G}\colon \mathbb{R} \to \mathcal{Q}(S)$ by taking

$$\mathcal{G}(t) = (x_t, q_t), \quad q_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} q,$$

where x_t is the underlying Riemann surface for q_t. Let λ_+ and λ_- be the horizontal and the vertical foliations of q_t.

Recall that a point $x \in \mathcal{T}(S)$ has an associated shortest marking μ_x. We similarly define, for any $(x, q) \in \mathcal{Q}(S)$, a shortest marking μ_q. The marking μ_q has the same pants decomposition and the same set of lengths $\{l_\alpha\}$ as μ_x. However, we use the flat metric of q to define the transversals τ_α, as follows. Recall that q^α is the annular cover of q with respect to α. Define τ_α to be any arc connecting the boundaries of q^α that is perpendicular to the geodesic representative of the core. That is, the transversal is the quadratic differential perpendicular instead of the hyperbolic perpendicular.

In what follows, we often replace q_t subscripts simply with t. For example, $\ell_t(\alpha)$ is short for $\ell_{q_t}(\alpha)$, while μ_t is short for μ_{q_t} and $M_t(\alpha, Y)$ is short for $M_{q_t}(\alpha, Y)$. We let t_α be the time when α is *balanced* along $\mathcal{G}$ (see Equation (2)). We need the following two statements. First we have a lemma that is contained in the proof of [16, Theorem 3.1].

Lemma 5.1 *There is a uniform constant $c \geq 0$ such that*

$$M_s(\alpha, Y) \leq M_t(\alpha, Y) + c$$

for all $s \leq t \leq t_\alpha$ and for all $t_\alpha \leq t \leq s$. □

Second we have a theorem that follows from the proof of [15, Theorem 5.5].

Theorem 5.2 *There are constants M_0 and C such that, if $M_t(\alpha, Y) \leq M_0 + c$ for some boundary component α, then either*

$$d_Y(\mu_t, \lambda_-) \leq C \quad \text{or} \quad d_Y(\mu_t, \lambda_+) \leq C.$$

We now define I_Y, the *active interval* for Y. Choose a large enough M_0 (we need $M_0 > m_0$ as in Lemma 4.1 and we need M_0 to satisfy Theorem 5.2). Define the interval $I_{\alpha,Y} \subset \mathbb{R}$ to be empty when $M_{t_\alpha}(\alpha, Y) < M_0$ and otherwise to be the largest interval containing t_α so that $M_t(\alpha, Y) \geq M_0$ for all $t \in I_{\alpha,Y}$. Define

$$I_Y = \bigcap_{\alpha \subset \partial Y} I_{\alpha,Y}.$$

Note that, by Lemma 5.1, for any t outside of I_Y, there is a boundary component α such that $M_t(\alpha, Y) \leq M_0 + c$.

Theorem 5.3 *Let $\mathcal{G}\colon \mathbb{R} \to \mathcal{Q}(S)$ be a Teichmüller geodesic with $\mathcal{G}(t) = (x_t, q_t)$. Let Y be a subsurface with the active interval I_Y. Then there exists a geodesic $\mathcal{F}\colon I_Y \to \mathcal{Q}(Y)$ with $\mathcal{F}(t) = (y_t, p_t)$, such that:*

- *If $[a,b] \cap I_Y = \varnothing$ then*
$$d_Y(\mu_a, \mu_b) = O(1).$$
- *For $t \in I_Y$,*
$$d_{\mathcal{T}(Y)}\big(x_t|_Y, y_t\big) = O(1).$$

In fact, we may take $p_t = q_t|_Y$.

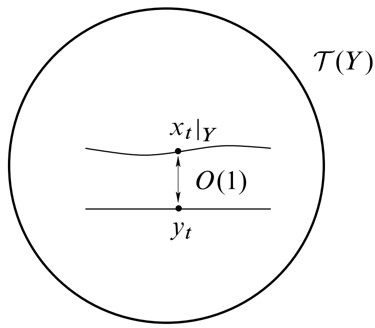

Figure 2: The projection of $\mathcal{G}$ to $\mathcal{T}(Y)$ fellow-travels the geodesic $\mathcal{F}$.

Proof For every $t \in [a,b]$, there exists a boundary component α such that $M_t(\alpha, Y) \le M_0 + c$. By Theorem 5.2

$$d_Y(\mu_t, \lambda_-) \le C \quad \text{or} \quad d_Y(\mu_t, \lambda_+) \le C.$$

Let $J_- \subset [a,b]$ be the set of times where the former holds and $J_+ \subset [a,b]$ be the set of times where the latter holds. If J_- or J_+ is empty, we are done by the triangle inequality. Otherwise, we note that these intervals are closed and have to intersect. This implies that $d_Y(\lambda_-, \lambda_+) \le 2C$. Again we are done after applying the triangle inequality; the bound on $d_Y(\mu_a, \mu_b)$ is at most $4C$. This proves the first conclusion of Theorem 5.3.

To obtain the second conclusion, we construct the candidate geodesic arc $\mathcal{F}$ in $\mathcal{T}(Y)$. Let $I_Y = [c,d]$. As suggested in the statement of the theorem, let $p_c = q_c|_Y$ and let $\mathcal{F} = (y_t, p_t)$ be the geodesic segment in $\mathcal{Q}(Y)$, $c \le t \le d$, defined by

$$p_t = \begin{bmatrix} e^{t-c} & 0 \\ 0 & e^{-t+c} \end{bmatrix} p_c.$$

In fact, if we make consistent choices in the construction of $q_t|_Y$ for different values of t, we have $p_t = q_t|_Y$. Now Equation (7) in Theorem 4.2 implies

$$d_{\mathcal{T}(Y)}(x_t|_Y, y_t) = O(1).$$

This finishes the proof. □

For a Teichmüller geodesic segment whose endpoints are in the thick part of the Teichmüller space, we can look at the short markings at the endpoints of the segment instead of the horizontal and the vertical foliations, to determine which subsurfaces are isolated along the geodesic segment. That is, the end invariants can be taken to be the short markings instead of the horizontal and the vertical foliations.

Corollary 5.4 *Let $\mathcal{G}\colon \mathbb{R} \to \mathcal{T}(S)$ be a Teichmüller geodesic. Suppose $a < b$ are times such that $\mathcal{G}(a)$ and $\mathcal{G}(b)$ are in the thick part. Then, for every subsurface Y, we have*

- *either $I_Y \subset [a,b]$,*

  $$\mathrm{i}(\lambda_-|_Y, \mu_b|_Y) = O(1) \quad \text{and} \quad \mathrm{i}(\lambda_+|_Y, \mu_a|_Y) = O(1),$$

  in particular

  $$d_Y(\lambda_-, \lambda_+) \stackrel{+}{\asymp} d_Y(\mu_a, \mu_b);$$

- *or $I_Y \cap [a,b] = \varnothing$ and*

  $$d_Y(\mu_a, \mu_b) = O(1).$$

Proof Since the endpoints lie in the thick part of Teichmüller space, the times a and b are not in any interval I_Y. That is, I_Y is either contained in $[a,b]$ or it is disjoint from it. If $I_Y = [c,d]$ then all markings μ_t, $t \in [-\infty, c]$, project to a bounded set in $\mathcal{AC}(Y)$. In fact, from [15, Theorem 5.5] we know that $\mathrm{i}(\mu_t|_Y, \lambda_+|_Y) = O(1)$. Therefore, $d_Y(\lambda_+, \mu_a) = O(1)$. Similarly, for $t \in [d, \infty]$, $\mathrm{i}(\mu_t|_Y, \lambda_-|_Y) = O(1)$ and $d_Y(\lambda_-, \mu_b) = O(1)$. The corollary follows immediately. □

Order of appearance of intervals I_Y By examining the subsurface projections one can determine which curves α are short along a Teichmüller geodesic $\mathcal{G}$. The following is the restatement of results in [15] in a way that is more suitable for our purposes. Let $\mathcal{G}$ be a Teichmüller geodesic with horizontal and vertical foliations $\lambda_\pm$ and, for a curve α, let $\mathcal{Z}(\alpha, D)$ be the set of subsurfaces Z that are disjoint from α and have $d_Z(\lambda_+, \lambda_-) \geq D$.

Theorem 5.5 *A curve α is short at some point along $\mathcal{G}$ if and only if α is the boundary of a subsurface Y so that Y is filled with subsurfaces with large projections. That is, there are constants ϵ, D_0 and D_1 such that:*

- *If $\mathrm{Ext}_t(\alpha) \le \epsilon$ then α is a boundary component of some subsurface Y, where Y is filled by subsurfaces in $\mathcal{Z}(\alpha, D_0)$.*
- *Suppose that α is a boundary component of Y and that Y is filled by elements of $\mathcal{Z}(\alpha, D_1)$. Then there is a time $t \in \mathbb{R}$ when $\mathrm{Ext}_t(\alpha) \le \epsilon$.*

Proof This is a restatement of [15, Theorem 1.1] after the following: two curves or arcs in $\mathcal{AC}(Y)$ have large intersection number if and only if their projections to some subsurface Z of Y are large. (This assertion is well known and follows from [2, Corollary D].) We have just translated the condition about intersection numbers to a condition about subsurface projections. □

One consequence of the above theorem is that the order in which the intervals I_Y appear in $\mathbb{R}$ is essentially determined by any geodesic g in $\mathcal{AC}(S)$ connecting λ_- to λ_+.

Proposition 5.6 *The boundary curves of any isolated surface are in a 2–neighborhood of a geodesic g in the curve complex. The order of appearance of intervals of isolations in $\mathbb{R}$ is coarsely determined by the order in which the vertices ∂Y appear along g.*

The proof uses both the description of a Teichmüller geodesic as well as some hyperbolicity result for the curve complex $\mathcal{AC}(S)$. Namely, we use Masur and Minsky's bounded geodesic image theorem:

Theorem 5.7 [11, Theorem 3.1] *If Y is an essential subsurface of S and g is a geodesic in $\mathcal{AC}(S)$ all of whose vertices intersect Y nontrivially, then the projected image of g in $\mathcal{AC}(Y)$ has uniformly bounded diameter.*

Proof of Proposition 5.6 By Theorem 5.5, a boundary curve α of any isolated subsurface Y is disjoint from some subsurface Z where the projection distance $d_Z(\lambda_+, \lambda_-)$ is large. By Theorem 5.7, the geodesic g has to miss Z as well. Hence α has a distance of at most 2 from g.

Write $g = g_- \cup g_0 \cup g_+$, where Z intersects every curve in g_- and g_+ and where g_0 has length 10 and ∂Z is disjoint from a curve at the middle of g_0. From Theorem 5.7 we have that the projection of g_- to $\mathcal{AC}(Z)$ is in a bounded neighborhood of $\lambda_-|_Z$ and the projection of g_+ to $\mathcal{AC}(Z)$ is in a bounded neighborhood of $\lambda_+|_Z$. Let Y' be another isolated surface. We claim that if the boundary of Y' is close to a point in g_-, then the interval $I_{Y'}$ appears after the interval I_Y.

Let $I_Y = [a,b]$ and let $t \in I_{Y'}$. Then $t \notin [a,b]$ because Y and Y' intersect (the distance between their boundaries is larger than 1) and their boundaries can not be short simultaneously. Note that $\partial Y'$ is part of the short marking μ_t. By Corollary 5.4, if $t < a$ then $\mathrm{i}(\mu_t|_Y, \lambda_+|_Y) = O(1)$. Hence,

$$\mathrm{i}(\mu_t|_Z, \lambda_+|_Z) = O(1) \quad \text{and} \quad d_Z(\mu_t, \lambda_+) = O(1).$$

But this is a contradiction because $\partial Y'$ is close to a point in g_- which projects to a point in $\mathcal{AC}(Z)$ near $\lambda_-|_Z$. Therefore, $t > b$. □

Remark 5.8 Note that, using Corollary 5.4, we can restate the above statements for Teichmüller geodesic segments $\mathcal{G}\colon [a,b] \to \mathcal{T}(S)$ where $\mathcal{G}(a)$ and $\mathcal{G}(b)$ are in the thick part. All statements hold after replacing λ_- and λ_+ with μ_a and μ_b respectively.

6 No backtracking

As before, let $\mathcal{G}$ be a Teichmüller geodesic with $\mathcal{G}(t) = (x_t, q_t)$ and let μ_t be the short marking associated to q_t. In this section we examine the projection of markings μ_t to the curve complex of a subsurface.

Theorem 6.1 *For every subsurface Y of S, the shadow of $\mathcal{G}$ in $\mathcal{AC}(Y)$ is an unparametrized quasigeodesic. That is, for $r \le s \le t \in \mathbb{R}$,*

$$d_Y(\mu_r, \mu_s) + d_Y(\mu_s, \mu_t) \stackrel{+}{\prec} d_Y(\mu_r, \mu_t).$$

Remark 6.2 We observe that the projection of μ_t to $\mathcal{AC}(Y)$ is a coarsely continuous path. That is, there is a constant B such that for every $t \in \mathbb{R}$ there is a δ where

$$\mathrm{i}(\mu_t, \mu_{t+\delta}) \le B \quad \text{and hence} \quad d_Y(\mu_t, \mu_{t+\delta}) = O(1).$$

To see this, note that since lengths change continuously, x_t and $x_{t+\delta}$ have the same thick-thin decompositions and the intersection between moderate length curves in x_t and $x_{t+\delta}$ is bounded. Also, twisting along the short curves changes coarsely continuously (see Equation (3)).

Remark 6.3 The reverse triangle inequality for a path (as given in the statement of the theorem) is a stronger condition than being an unparametrized quasigeodesic. However, in Gromov hyperbolic spaces such as $\mathcal{AC}(Y)$ the two conditions are equivalent. (See [11, Section 7] and [12, Section 2.1] for relevant discussions.)

Remark 6.4 This contrasts with the way geodesics behave in the Lipschitz metric on $\mathcal{T}(S)$, studied by Thurston in [20], where the projection of a geodesic to a subsurface can backtrack arbitrarily far. (Examples can easily be produced using Thurston's construction of minimal stretch maps [20] and the results in [2].)

Proof of Theorem 6.1 If $Y = S$, the above is a theorem of Masur and Minsky [11, Theorem 3.3], that is, we already know that the shadow of $\mathcal{G}$ to $\mathcal{AC}(S)$ is an unparametrized quasigeodesic. Let Y be a proper subsurface and consider the active interval $I_Y = [c, d]$. If Y is not an annulus, by the first part of Theorem 5.3, the shadows of $\mathcal{G}(-\infty, c]$ and $\mathcal{G}[d, \infty)$ have bounded diameter in $\mathcal{AC}(Y)$ and by the second part of Theorem 5.3 and again using [11, Theorem 3.3], the shadow of $\mathcal{G}[c, d]$ is an unparametrized quasigeodesic in $\mathcal{AC}(Y)$. It remains to check the case of an annulus. But in this case $\mathcal{AC}(Y)$ is quasi-isometric to Z and we need only to show that the twisting around the core of Y is increasing up to an additive error. This follows from Equation (3). □

One can naturally define a map from $\mathcal{T}(S)$ to the mapping class group of S. Namely, let $\mathrm{Map}(S)$ be the mapping class group of S equipped with the word metric associated to some generating set. Fix a marking μ_0. Then for any other marking μ, there is a mapping class ϕ such that the geometric intersection number between μ and $\phi(\mu_0)$ is uniformly bounded. In fact, the number of such mapping classes is finite and they form a uniformly bounded set in $\mathrm{Map}(S)$. We denote one such mapping class by ϕ_μ. Now, define the map

$$\Upsilon\colon \mathcal{T}(S) \to \mathrm{Map}(S) \quad \text{by} \quad \Upsilon(x) = \phi_{\mu_x}.$$

As a consequence of Theorem 6.1, we can prove a similar theorem for the shadow of a Teichmüller geodesic to the mapping class group but with additive and multiplicative errors.

Theorem 6.5 *Let $\mathcal{G}\colon R \to \mathcal{T}(S)$ be a Teichmüller geodesic. Then for $r \le s \le t \in \mathbb{R}$, we have*

$$d_{\mathrm{Map}}\big(\Upsilon(r), \Upsilon(s)\big) + d_{\mathrm{Map}}\big(\Upsilon(s), \Upsilon(t)\big) \succ d_{\mathrm{Map}}\big(\Upsilon(r), \Upsilon(t)\big).$$

Proof This follows from Masur–Minsky distance formula for the mapping class group: there is a constant C such that for any two mapping classes ϕ and ψ, we have

$$d_{\mathrm{Map}}(\phi, \psi) \asymp \sum_Y \big[d_Y\big(\phi(\mu_0), \psi(\mu_0)\big)\big]_C. \tag{9}$$

In fact C can be chosen arbitrarily large which makes the constants in $\asymp$ also larger but still uniform. Let K_1 be such that, for any $x \in \mathcal{T}(S)$ and any subsurface Y,

$$d_Y\big(\Upsilon(x)(\mu_0), \mu_x\big) \le K,$$

let K_2 be the additive error in Theorem 6.1 and let $C' > 2C + 6K_1 + K_2$. Also, let $\phi_r = \Upsilon(\mathcal{G}(r))$, $\phi_s = \Upsilon(\mathcal{G}(s))$ and $\phi_t = \Upsilon(\mathcal{G}(t))$. For a subsurface Y, we have

$$\begin{aligned} d_Y\big(\phi_r(\mu_0), \phi_t(\mu_0)\big) \ge C' = 2C + 6K_1 + K_2 & \\ \implies d_Y(\mu_r, \mu_t) \ge 2C + 4K_1 + K_2 & \\ \implies d_Y(\mu_r, \mu_s) \ge C + 2K_1 \quad \text{or} \quad d_Y(\mu_s, \mu_t) \ge C + 2K_1 & \\ \implies d_Y\big(\phi_r(\mu_0), \phi_s(\mu_0)\big) \ge C \quad \text{or} \quad d_Y\big(\phi_s(\mu_0), \phi_t(\mu_0)\big) \ge C. & \end{aligned}$$

Hence

$$\big[d_Y\big(\phi_r(\mu_0), \phi_t(\mu_0)\big)\big]_{C'} \overset{*}{\succ} \big[d_Y\big(\phi_r(\mu_0), \phi_t(\mu_0)\big)\big]_C + \big[d_Y\big(\phi_r(\mu_0), \phi_t(\mu_0)\big)\big]_C.$$

This is because, by the above computation, if the left-hand side is nonzero, both terms on the right can not disappear. And the largest one is at least half of the left-hand side, up to an additive error which can be absorbed in the multiplicative error. The theorem now follows after summing over all subsurfaces and applying Equation (9) twice; once with the threshold C and once with the threshold C'. □

7 Fellow traveling

Theorem 7.1 *There is a constant $D > 0$ such that, for points $x, \bar{x}, y$ and $\bar{y}$ in the thick part of $\mathcal{T}(S)$ where*

$$d_{\mathcal{T}}(x, \bar{x}) \le 1 \quad \text{and} \quad d_{\mathcal{T}}(y, \bar{y}) \le 1,$$

the geodesic segments $[x, y]$ and $[\bar{x}, \bar{y}]$ D–fellow travel in a parametrized fashion.

Remark 7.2 The proof also works when either x or y is replaced with a measured foliation in $\mathcal{PML}(S)$ and $\mathcal{G}$ and $\overline{\mathcal{G}}$ are infinite rays.

Proof After adjusting x and y along the geodesic extension through $[x, y]$ by a bounded amount, we may assume that $d_{\mathcal{T}}(x, y) = d_{\mathcal{T}}(\bar{x}, \bar{y})$. Let

$$\mathcal{G}\colon [0, l] \to \mathcal{T}(S) \quad \text{and} \quad \overline{\mathcal{G}}\colon [0, l] \to \mathcal{T}(S)$$

be Teichmüller geodesics connecting x to y and $\bar{x}$ to $\bar{y}$ respectively; $\mathcal{G}(t) = (x_t, q_t)$ and $\overline{\mathcal{G}}(t) = (\bar{x}_t, \bar{q}_t)$.

We first show that, for any curve α, $\ell_{q_t}(\alpha) \stackrel{*}{\asymp} \ell_{\bar{q}_t}(\alpha)$.

Since x and $\bar{x}$ are both in the thick part, for every curve α we have (part (1) of Theorem 3.1)

$$\mathrm{Ext}_x(\alpha) \stackrel{*}{\asymp} l_{q_0}(\alpha)^2 \quad \text{and} \quad \mathrm{Ext}_{\bar{x}}(\alpha) \stackrel{*}{\asymp} l_{\bar{q}_0}(\alpha)^2.$$

But $d_{\mathcal{T}}(x, \bar{x}) = 1$. Therefore,

$$\mathrm{Ext}_x(\alpha) \stackrel{*}{\asymp} \mathrm{Ext}_{\bar{x}}(\alpha) \quad \Longrightarrow \quad l_{q_0}(\alpha) \stackrel{*}{\asymp} l_{\bar{q}_0}(\alpha).$$

The same argument works to show that $l_{q_l}(\alpha) \stackrel{*}{\asymp} l_{\bar{q}_l}(\alpha)$. The flat length of a curve is essentially determined by two parameters. From Equation (2) we have $\ell_{q_t}(\alpha) \stackrel{*}{\asymp} L_\alpha \cosh(t - t_\alpha)$ and $\ell_{\bar{q}_t}(\alpha) \stackrel{*}{\asymp} \bar{L}_\alpha \cosh(t - \bar{t}_\alpha)$. Since the flat lengths of α are comparable at the beginning and the end they are always comparable. That is, $L_\alpha \stackrel{*}{\asymp} \bar{L}_\alpha$ and $t_\alpha \stackrel{+}{\asymp} \bar{t}_\alpha$.

We use Corollary 2.6 to prove $d_{\mathcal{T}}(x_t, \bar{x}_t) = O(1)$ by checking the four conditions.

Condition (1) We need to show that q_t and $\bar{q}_t$ have the same thick-thin decompositions. Fix an ϵ and let $(\mathcal{A}, \mathcal{Y})$ be the (ϵ, ϵ)–thick-thin decomposition of x_t. Let $\alpha \in \mathcal{A}$ and let E, F and G be as in Theorem 3.1. Since α is short, one of E, F or G must have a large modulus. That is, for every curve β intersecting α, we have

$$\frac{\ell_{q_t}(\beta)}{\ell_{q_t}(\alpha)} \stackrel{*}{\succ} \frac{1}{\epsilon}.$$

(In fact it may be larger than $e^{1/\epsilon}$.) Since the flat lengths in q_t and $\bar{q}_t$ are comparable, we also have

$$\frac{\ell_{\bar{q}_t}(\beta)}{\ell_{\bar{q}_t}(\alpha)} \stackrel{*}{\succ} \frac{1}{\epsilon}.$$

We show the extremal length of α is small in $\bar{x}_t$. If not, α would pass through some thick piece of $\bar{x}_t$ and it would intersect some curve β with $\mathrm{Ext}_{\bar{x}_t}(\beta) \stackrel{*}{\prec} 1$. That is, $\mathrm{Ext}_{\bar{x}_t}(\beta) \stackrel{*}{\prec} \mathrm{Ext}_{\bar{x}_t}(\alpha)$. Part (1) of Theorem 3.1 implies $\ell_{\bar{q}_t}(\beta) \stackrel{*}{\prec} \ell_{\bar{q}_t}(\alpha)$ which is a contradiction. That is, there is an ϵ_0 such that if $\mathrm{Ext}_{x_t}(\alpha) \le \epsilon$ then $\mathrm{Ext}_{\bar{x}_t}(\alpha) \le \epsilon_0$.

Arguing in the other direction, we can find ϵ_1 such that if $\mathrm{Ext}_{\bar{x}_t}(\alpha) \le \epsilon_1$ then $\mathrm{Ext}_{x_t}(\alpha) \le \epsilon$. That is, every curve not in $\mathcal{A}$ is ϵ_1–thick in $\bar{x}_t$. This proves that $(\mathcal{A}, \mathcal{Y})$ is an (ϵ_0, ϵ_1)–thick-thin decomposition for $\bar{x}_t$.

Condition (2) The size of a surface $Y \in \mathcal{Y}$ is the flat length of the shortest essential curve in Y. Hence, we have $\mathrm{size}_{q_t}(Y) \stackrel{*}{\asymp} \mathrm{size}_{\bar{q}_t}(Y)$. Now, Theorem 3.1 implies that for every curve γ in Y, if $\mathrm{Ext}_{x_t}(\gamma) \stackrel{*}{\asymp} 1$ then $\mathrm{Ext}_{\bar{x}_t}(\gamma) \stackrel{*}{\asymp} 1$ as well. But two curves of length one have bounded intersection numbers. Hence, they have bounded projection to every subsurface Z. This means $d_Z(\mu, \bar{\mu}) = O(1)$.

Condition (3) For each $\alpha \in \mathcal{A}$, as we saw before, $L_\alpha \stackrel{*}{\asymp} \overline{L}_\alpha$ and $t_\alpha \stackrel{+}{\asymp} \overline{t}_\alpha$. We now show that $T_\alpha \stackrel{+}{\asymp} \overline{T}_\alpha$. Since the endpoints of $\mathcal{G}$ and $\overline{\mathcal{G}}$ are close, we have

$$d_\alpha(\mu_0, \overline{\mu}_0) = O(1) \quad \text{and} \quad d_\alpha(\mu_l, \overline{\mu}_l) = O(1).$$

Also, from Corollary 5.4 we have

$$\begin{aligned} d_\alpha(\mu_0, \lambda_-) &= O(1), & d_\alpha(\mu_l, \lambda_+) &= O(1), \\ d_\alpha(\overline{\mu}_0, \overline{\lambda}_-) &= O(1), & d_\alpha(\overline{\mu}_l, \overline{\mu}_+) &= O(1). \end{aligned}$$

Hence, using the triangle inequality,

$$T_\alpha = d_\alpha(\lambda_-, \lambda_+) \stackrel{+}{\asymp} d_\alpha(\mu_0, \mu_l) \stackrel{+}{\asymp} d_\alpha(\overline{\mu}_0, \overline{\mu}_l) \stackrel{+}{\asymp} d_\alpha(\overline{\lambda}_-, \overline{\lambda}_+) = \overline{T}_\alpha.$$

Now Equation (4) implies that

$$\operatorname{Mod}_{x_t}(\mathsf{F}_t) \stackrel{*}{\asymp} \operatorname{Mod}_{\overline{x}_t}(\overline{\mathsf{F}}_t). \tag{10}$$

Also, as seen above, the size of all subsurfaces is comparable in q_t and $\overline{q}_t$. Therefore, by Theorem 3.1 $\operatorname{Ext}_{x_t}(\alpha) \stackrel{*}{\asymp} \operatorname{Ext}_{\overline{x}_t}(\alpha)$.

Condition (4) We show that $\operatorname{twist}_\alpha(q_t, \overline{q}_t) \operatorname{Ext}_{x_t}(\alpha) \stackrel{*}{\asymp} 1$. Since $d_\alpha(\lambda_-, \overline{\lambda}_-) = O(1)$,

$$\operatorname{twist}_\alpha(q_t, \overline{q}_t) \stackrel{+}{\asymp} \left| \operatorname{twist}_\alpha(q_t, \lambda_-) - \operatorname{twist}_\alpha(\overline{q}_t, \overline{\lambda}_-) \right|. \tag{11}$$

Denote $\operatorname{twist}_\alpha(q_t, \overline{q}_t)$ (as before) by $\operatorname{twist}_t(\alpha)$ and denote $\operatorname{twist}_\alpha(\overline{q}_t, \overline{\lambda}_-)$ by $\overline{\operatorname{twist}}_t(\alpha)$. We use Equation (3) and that $|t_\alpha - \overline{t}_\alpha| = O(1)$ and $|T_\alpha - \overline{T}_\alpha| = O(1)$ to estimate the right-hand side of Equation (11).

If $t \stackrel{+}{\prec} t_\alpha$ (and hence $t \stackrel{+}{\prec} \overline{t}_\alpha$), then

$$\operatorname{twist}_t(\alpha) \stackrel{*}{\prec} \frac{T_\alpha}{\cosh^2(t - t_\alpha)} \quad \text{and} \quad \overline{\operatorname{twist}}_t(\alpha) \stackrel{*}{\prec} \frac{T_\alpha}{\cosh^2(t - t_\alpha)}.$$

But $\operatorname{Ext}_t(\alpha) \stackrel{*}{\prec} \frac{1}{\operatorname{Mod}(\mathsf{F}_t)}$. Thus using Equation (4) we get

$$\left| \operatorname{twist}_t(\alpha) - \overline{\operatorname{twist}}_t(\alpha) \right| \operatorname{Ext}_t(\alpha) \stackrel{*}{\prec} \frac{T_\alpha}{\cosh^2(t - t_\alpha)} \frac{\cosh^2(t - t_\alpha)}{T_\alpha} \stackrel{*}{\prec} 1.$$

If $t \stackrel{+}{\succ} t_\alpha$, then

$$T_\alpha - \operatorname{twist}_t(\alpha) \stackrel{*}{\prec} \frac{T_\alpha}{\cosh^2(t - t_\alpha)} \quad \text{and} \quad T_\alpha - \overline{\operatorname{twist}}_t(\alpha) \stackrel{*}{\prec} \frac{T_\alpha}{\cosh^2(t - t_\alpha)}.$$

Hence, as before,

$$\left|\operatorname{twist}_t(\alpha) - \overline{\operatorname{twist}}_t(\alpha)\right| \operatorname{Ext}_t(\alpha) \overset{*}{\prec} \frac{\left|(T_\alpha - \operatorname{twist}_t(\alpha)) - (T_\alpha - \overline{\operatorname{twist}}_t(\alpha))\right|}{\operatorname{Mod}_t(\alpha)}$$
$$\overset{*}{\prec} \frac{T_\alpha}{\cosh^2(t-t_\alpha)} \frac{\cosh^2(t-t_\alpha)}{T_\alpha} \overset{*}{\prec} 1.$$

That is, the last condition in Corollary 2.6 holds and $d_{\mathcal{T}}(q_t, \bar{q}_t) = O(1)$. This finishes the proof. □

We now construct the counterexample.

Theorem 7.3 *For every constant* $\mathsf{d} > 0$*, there are points* $x, y, \bar{x}$ *and* $\bar{y}$ *in* $\mathcal{T}(S)$ *such that*

$$d_{\mathcal{T}}(x, \bar{x}) = O(1) \quad \textit{and} \quad d_{\mathcal{T}}(y, \bar{y}) = O(1),$$

and

$$d_{\mathcal{T}}\big([x,y], [\bar{x}, \bar{y}]\big) \overset{*}{\succ} \mathsf{d}.$$

Proof For a given d, we construct quadratic differentials q_0 and $\bar{q}_0$ with the following properties: Let q_t be the image of q_0 under the Teichmüller geodesic flow and let x_t be the underlying conformal structures of q_t. Let $\bar{q}_t$ and $\bar{x}_t$ be defined similarly. We will show that

$$d_{\mathcal{T}}(x_0, \bar{x}_0) = O(1), \quad d_{\mathcal{T}}(x_{2\mathsf{d}}, \bar{x}_{2\mathsf{d}}) = O(1),$$

and

$$d_{\mathcal{T}}(x_{\mathsf{d}}, \bar{x}_{\mathsf{d}}) \overset{*}{\succ} \mathsf{d}.$$

This is sufficient to show that $d_{\mathcal{T}}(x_{\mathsf{d}}, \bar{x}_t) \overset{*}{\succ} \mathsf{d}$ for any $t \in [0, 2\mathsf{d}]$. To see this note that for any $0 \le t < \mathsf{d}$, we have

$$d_{\mathcal{T}}(x_{\mathsf{d}}, \bar{x}_t) + d_{\mathcal{T}}(\bar{x}_t, \bar{x}_0) \overset{+}{\succ} \mathsf{d} \quad \text{and} \quad d_{\mathcal{T}}(x_{\mathsf{d}}, \bar{x}_t) + d_{\mathcal{T}}(\bar{x}_t, \bar{x}_{\mathsf{d}}) \overset{+}{\succ} d_{\mathcal{T}}(x_{\mathsf{d}}, \bar{x}_{\mathsf{d}}).$$

Summing up both sides, we get

$$2 d_{\mathcal{T}}(x_{\mathsf{d}}, \bar{x}_t) + d_{\mathcal{T}}(\bar{x}_0, \bar{x}_{\mathsf{d}}) \overset{+}{\succ} \mathsf{d} + d_{\mathcal{T}}(x_{\mathsf{d}}, \bar{x}_{\mathsf{d}}).$$

Hence,

$$2 d_{\mathcal{T}}(x_{\mathsf{d}}, \bar{x}_t) \overset{+}{\succ} d_{\mathcal{T}}(x_{\mathsf{d}}, \bar{x}_d).$$

A similar argument works for $\mathsf{d} < t \le 2\mathsf{d}$.

Let S be a surface of genus 2, γ be a separating curve in S and Y and Z be the components of $S \smallsetminus \gamma$. Consider a pseudo-Anosov map ϕ on a torus and choose a flat torus T on the axis of ϕ so that the vertical direction in T matches the unstable foliation of ϕ. Cut open a slit in T of size $\epsilon = c\, e^{-\mathsf{d}/2}$ and of angle $\pi/4$ (the constant $0 < c < 1$ is to be specified below). Fix a homeomorphism from Y to this slit torus and call this marked flat surface T_0. Define

$$T_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} T_0.$$

Note that T_t is still a marked surface. The length of the slit is minimum at $t = 0$ and grows exponentially as $t \to \pm\infty$. For $-\mathsf{d}/2 \le t \le \mathsf{d}/2$, the length of the slit is smaller than c but the length of the shortest essential curve in T_t in this interval is comparable with 1. Hence, for c small enough, $M_t(\gamma, Y) \ge m_0$ (see Section 4) and T_t looks like an isolated subsurface.

Now choose $\delta \ll \epsilon$ (specified below) and let q_0 be the quadratic differential defined by gluing T to $\delta T_{-\mathsf{d}/2}$. What we mean by this is that we first scale down $T_{-\mathsf{d}/2}$ by a factor δ. Then we cut open a slit in T of the same size and angle as the slit in $\delta T_{-\mathsf{d}/2}$ and then glue these two flat tori along this slit. Fixing a homeomorphism from Z to T slit open, we obtain a marking for q_0 that is well defined up to twisting around γ. Let $\mathcal{G}\colon [0, 2\mathsf{d}] \to \mathcal{T}(S)$ be the Teichmüller geodesic segment defined by q_0.

Construct $\bar{q}_0$ in a similar fashion by gluing T to $\delta T_{-3\mathsf{d}/2}$. Now choose the marking map from S to $\bar{q}_0$ so that q_0 and $\bar{q}_0$ have bounded relative twisting around γ. Let $\bar{\mathcal{G}}\colon [0, 2\mathsf{d}] \to \mathcal{T}(S)$ be the Teichmüller geodesic segment defined by $\bar{q}_0$.

Recall that, for $-\mathsf{d}/2 \le t \le \mathsf{d}/2$, the subsurface δT_t is isolated (scaling by δ does not change the value of $M_t(\alpha, Y)$) and by Theorem 4.2 the projection of T_t to the Teichmüller space of Y fellow-travels a Teichmüller geodesic. However, for $t > \mathsf{d}/2$ and $t < -\mathsf{d}/2$, the projection to the curve complex of Y changes by at most a bounded amount. That is, the active interval for Y along $\mathcal{G}$, I_Y, is $[0, \mathsf{d}]$ and that along $\bar{\mathcal{G}}$, $\bar{I}_Y$, is $[\mathsf{d}, 2\mathsf{d}]$. In particular,

$$d_Y(q_0, \bar{q}_0) = O(1) \quad \text{and} \quad d_Y(q_{2\mathsf{d}}, \bar{q}_{2\mathsf{d}}) = O(1).$$

Also, since no curve in Y or Z is ever short (the vertical and the horizontal foliation in Y and Z are cobounded), the twisting parameters around any curves inside Y or Z are uniformly bounded. Projections of q_0 and $\bar{q}_0$ to Z are identical and γ is short in both q_0 and $\bar{q}_0$. Therefore, to show $d_{\mathcal{T}}(x_0, \bar{x}_0) = O(1)$, it remains to show (Corollary 2.6) that the extremal lengths of γ in x_0 and $\bar{x}_0$ are comparable. We have (Theorem 3.1)

$$\operatorname{Ext}_{x_0}(\gamma) \stackrel{+}{\asymp} \log \frac{1}{\delta} \quad \text{and} \quad \operatorname{Ext}_{\bar{x}_0}(\gamma) \stackrel{+}{\asymp} \log \frac{1}{e^{\mathsf{d}}\delta} = \log \frac{1}{\delta} - \mathsf{d}.$$

But these quantities are comparable for δ small enough. A similar argument shows that $d_{\mathcal{T}}(x_{2\mathsf{d}}, \bar{x}_{2\mathsf{d}}) = O(1)$. Since Y is isolated in q_t for $0 \le t \le \mathsf{d}$ the shadow to the $\mathcal{AC}(Y)$ is an unparametrized quasigeodesic. In fact, since no curve is short in Y in that interval, the shadow is a parametrized quasigeodesic [18, Lemma 4.4]. That is,

$$d_{\mathcal{T}(Y)}(x_0, x_{\mathsf{d}}) \stackrel{*}{\asymp} \mathsf{d}.$$

But the active interval for Y along the geodesic $\overline{\mathcal{G}}$ is $[\mathsf{d}, 2\mathsf{d}]$. Therefore,

$$d_Y(\bar{x}_0, \bar{x}_{\mathsf{d}}) = O(1).$$

As before, we have $d_Y(x_0, \bar{x}_0) = O(1)$. Hence

$$d_Y(q_{\mathsf{d}}, \bar{q}_{\mathsf{d}}) \stackrel{*}{\asymp} \mathsf{d}.$$

Now, by Theorem 2.4, we have

$$d_{\mathcal{T}}(x_{\mathsf{d}}, \bar{x}_{\mathsf{d}}) \stackrel{*}{\succ} d_Y(x_{\mathsf{d}}, \bar{x}_{\mathsf{d}}) \stackrel{*}{\asymp} \mathsf{d}.$$

This finishes the proof. □

8 Thin triangles

Let x, y and z be three points in $\mathcal{T}(S)$ and let $\mathcal{G}\colon [a,b] \to \mathcal{T}(S)$ be the Teichmüller geodesic connecting x to y. In this section we prove Theorem E from the introduction.

Theorem 8.1 *For every ϵ, there are constants C and D such that the following holds. Let $[c,d]$ be a subinterval of $[a,b]$ with $(d-c) > C$ such that for every $t \in [c,d]$, $\mathcal{G}(t)$ is in the ϵ–thick part of $\mathcal{T}(S)$. Then, there is a $w \in [\mathcal{G}(c), \mathcal{G}(d)]$, where*

$$\min\big(d_{\mathcal{T}}\big(w, [x,z]\big), d_{\mathcal{T}}\big(w, [y,z]\big)\big) \le D.$$

Proof Consider the shadow map from $\mathcal{T}(S)$ to the curve complex $\mathcal{AC}(S)$ sending a point x to its short marking μ_x. The geodesic triangle $\triangle(\mu_x, \mu_y, \mu_z)$ in the arc and curve complex $\mathcal{AC}(S)$ is δ–slim. Since the shadow of $[x,y]$ is a quasigeodesic (Theorem 6.1) for any $w \in [x,y]$, μ_w is δ–close to the geodesic $[\mu_x, \mu_y]$ in $\mathcal{AC}(S)$. That is, for every $w \in [x,y]$, there is a Riemann surface u in either $[x,z]$ or $[y,z]$ so that $d_S(\mu_w, \mu_u) \le 3\delta$.

The projection of $[\mathcal{G}(c), \mathcal{G}(d)]$ to $\mathcal{AC}(S)$ is in fact a parametrized quasigeodesic [18, Lemma 4.4]. Hence, by making C large, we can assume that the shadow of $[\mathcal{G}(c), \mathcal{G}(d)]$ is as long as we like. Thus, we can choose $w \in [\mathcal{G}(c), \mathcal{G}(d)]$ so that μ_w is far from either the shadow of $[x,z]$ or the shadow of $[y,z]$. To summarize, without loss of generality, we can assume that there is a $w \in [\mathcal{G}(c), \mathcal{G}(d)]$ and a $u \in [x, z]$ such that $d_S(\mu_w, \mu_u) = O(1)$ and that neither μ_u nor μ_w is in the (10δ)–neighborhood of the geodesic $[\mu_y, \mu_z]$.

We claim that u is in the thick part of Teichmüller space. Using Theorem 5.5 it is enough to show, for every subsurface Y whose boundaries are close to μ_u in $\mathcal{AC}(S)$, that $d_Y(\mu_x, \mu_z) = O(1)$. Since μ_u is far away from $[\mu_y, \mu_z]$, Theorem 5.7 implies that $d_Y(\mu_y, \mu_z) = O(1)$. To prove the claim, we need to show that

$$d_Y(\mu_x, \mu_y) = O(1). \tag{12}$$

We prove (12) by contradiction. Assume $d_Y(\mu_x, \mu_y)$ is large. By Theorem 5.5, ∂Y is short at some point $v \in [x, y]$. But the shadow of $[x, y]$ is a quasigeodesic and the shadow of $[\mathcal{G}(c), \mathcal{G}(d)]$ is a parametrized quasigeodesic. Hence, by choosing C large enough, we can conclude that, for any such subsurface, $d_S(\partial Y, \mu_w) \stackrel{+}{\succ} d_S(\mu_v, \mu_w)$ is large. This contradicts the fact that

$$d_S(\partial Y, \mu_u) = O(1) \quad \text{and} \quad d_S(\mu_u, \mu_w) = O(1).$$

Hence, (12) holds and thus u is in the thick part of Teichmüller space.

We now claim, for any subsurface $Y \subset S$, that

$$d_Y(\mu_u, \mu_w) = O(1).$$

This is because any such subsurface Y should appear near the curve complex geodesic connecting μ_u and μ_w and hence ∂Y has a bounded distance from μ_w in $\mathcal{AC}(S)$. As before, assuming $d_Y(\mu_x, \mu_y)$ is large will result in a contradiction. Thus, $d_Y(\mu_x, \mu_y) = O(1)$. Since μ_u is far from the geodesic $[\mu_y, \mu_z]$, the bounded projection theorem implies that $d_Y(\mu_y, \mu_z) = O(1)$ and by the triangle inequality, $d_Y(\mu_x, \mu_z) = O(1)$. On the other hand, by Theorem 6.1

$$d_Y(\mu_x, \mu_y) = O(1) \quad \Longrightarrow \quad d_Y(\mu_x, \mu_w) = O(1),$$
$$d_Y(\mu_x, \mu_z) = O(1) \quad \Longrightarrow \quad d_Y(\mu_x, \mu_u) = O(1).$$

The triangle inequality implies $d_Y(\mu_w, \mu_u) = O(1)$. This proves the claim.

We have w and u are both in the thick part and that all subsurface projections between μ_u and μ_w are uniformly bounded. Corollary 2.6 implies that $d_{\mathcal{T}}(u, w) = O(1)$. □

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Department of Mathematics, University of Toronto
Room 6290, 40 George Street, Toronto ON M5S 2E4, Canada

rafi@math.toronto.edu

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