ASYMPTOTIC DIMENSION OF BIG MAPPING CLASS GROUPS

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ABSTRACT. Even though big mapping class groups are not countably generated, certain big mapping class groups can be generated by a coarsely bounded set and have a well defined quasi-isometry type. We show that the big mapping class group of a stable surface of infinite type with a coarsely bounded generating set that contains an essential shift has infinite asymptotic dimension. This is in contrast with the mapping class groups of surfaces of finite type where the asymptotic dimension is always finite. We also give a topological characterization of essential shifts.

1. INTRODUCTION

In this paper, the surface Σ is an orientable, connected, second-countable 2–manifold without boundary. We further assume that Σ is stable, that is, every end of Σ has a stable neighborhood (see Definition 2.5). The mapping class group of Σ, denoted by $\mathrm{Map}(\Sigma)$, is the group of orientation preserving homeomorphisms of Σ up to isotopy. A surface Σ is said to be of finite type when $\pi_1(\Sigma)$ is finitely generated and is of infinite type otherwise. The mapping class groups of surfaces of infinite type are referred to as big mapping class groups.

When Σ is a surface of finite type, $\mathrm{Map}(\Sigma)$ is finitely generated. A finite generating set defines a word metric on $\mathrm{Map}(\Sigma)$ which is well-defined up to quasi-isometry independent of the particular finite generating set. The large scale geometry of the mapping class group, that is the geometry of the quasi-isometry class of such metrics, has been studied extensively [1, 5, 6, 7, 10, 15].

In contrast, big mapping class groups are not even countably generated. However, using the framework of Rosendal for coarse geometry of non locally compact groups [20], we can establish a notion large scale geometry for big mapping class groups when $\mathrm{Map}(\Sigma)$ has a coarsely bounded generating set. For a Polish topological group G, a subset $A \subset G$ is coarsely bounded, abbreviated CB, if every compatible left-invariant metric on G gives A finite diameter. We say G is locally CB if some neighborhood of the identity in G is CB, and we say G is CB generated if G has a generating set that is a union of a CB neighborhood of the identity and a finite set. Such a generating set defines a word metric on G that is well defined up to a quasi-isometry. Namely, word metrics on G associated to different CB generating sets are quasi-isometric to each other.

Mann-Rafi gave a classification of mapping classes of stable surfaces that are CB generated [13]. We are interested in the study of the coarse geometry of such big mapping class groups. In particular, we would like to know if big mapping class groups have finite asymptotic dimension.

Definition 1.1 (Asymptotic Dimension). Let X be a metric space. We say that $\mathrm{asdim}(X) \leq n$ if for every $R > 0$ there exists a covering of X by open sets $\{U_i\}_{i=1}^{\infty}$ such that $\sup_{i \in \mathbb{N}}\{\mathrm{diam}(U_i)\} < \infty$, and every ball of radius R in X intersects at most $n+1$ elements of the cover $\{U_i\}$. The asymptotic dimension is the least n for which $\mathrm{asdim}(X) \leq n$. We then write $\mathrm{asdim}(X) = n$. If no such n exists, then we say X has infinite asymptotic dimension.

The notion of asymptotic dimension was introduced by Gromov in [9] where he also proved that δ-hyperbolic groups have finite asymptotic dimension. Many other groups have also been shown to have finite asymptotic dimensions [3, 12, 16, 19]. The study of the asymptotic dimension in mapping

class groups started with the work of Bell-Fujiwara [3] who modified Gromov's argument to show that the curve complex has finite asymptotic dimension. Masur and Minsky showed that the curve complex is Gromov hyperbolic [14] and that the geometry of various curve complexes are closely linked with the coarse geometry of the mapping class group [15]. Using these facts, Bestvina, Bromberg and Fujiwara showed that, for surfaces of finite type, $\mathrm{Map}(\Sigma)$ has finite asymptotic dimension by embedding $\mathrm{Map}(\Sigma)$ in a finite product of *trees of curve complexes* [4].

In comparison with surfaces of finite type, there are new phenomena present in big mapping class groups. For example, there are homeomorphisms where parts of the surface are pushed to infinity (never recurring back). The simplest form of such a map is a shift map. Consider an infinite strip in $\mathbb{R}^2$ with $\mathbb{Z}$ acting on the strip by translations. Cut out an equivariant family of disks and attach identical surfaces Σ_i (possibly of infinite type) to the boundaries of these disks. Then $\mathbb{Z}$ still acts by translations on the strip, sending the surface Σ_i homeomorphically to the surface Σ_{i+1}. Embedding this in a larger strip σ, we can construct a homeomorphism h_σ of σ that acts as described above in the smaller strip, but the restriction of h_σ to the boundary of σ is the identity. Assume Σ contains a copy of σ where the ends of σ exit different ends of Σ. Then there is a homeomorphism of Σ (again called h_σ) that is as described in σ and is the identity map outside of σ. We call h_σ a *shift map*. We say a shift map $h_\sigma \in \mathrm{Map}(\Sigma)$ is *essential* if the group generated by h_σ, $\langle h_\sigma \rangle$, is not a coarsely bounded subgroup of $\mathrm{Map}(\Sigma)$.

Theorem 1.2 (Main Theorem)**.** *Assume Σ is stable and $\mathrm{Map}(\Sigma)$ is CB generated. If $\mathrm{Map}(\Sigma)$ contains an essential shift, then the asymptotic dimension of $\mathrm{Map}(\Sigma)$ is infinite.*

That is, the existence of an essential shift results in $\mathrm{Map}(\Sigma)$ having very non-trivial geometry. In fact, any subset of $\mathbb{Z}^n$ for any $n > 0$ can be embedded quasi-isometrically in $\mathrm{Map}(\Sigma)$ (see Theorem **??**).

We also provide the topological classification of essential shifts. To do this, it is more natural to work with a certain finite index subgroup of $\mathrm{Map}(\Sigma)$. Let E be the end space of Σ. Mann-Rafi [13] defined a partial order on E measuring the local complexity. Let $\mathrm{FMap}(\Sigma)$ be the subgroup of $\mathrm{Map}(\Sigma)$ that fixes the set of *isolated maximal points* in the end space E (see Section 2.4 for the definition). In the setting of CB (but infinite) generated groups, it is not always true that a finite index subgroup H of G is quasi-isometric to G. However, we show:

Theorem 1.3. *The group $\mathrm{FMap}(\Sigma)$ is quasi-isometric to $\mathrm{Map}(\Sigma)$.*

It turns out essential shifts are present when the end space is *two-sided* in a certain sense. Let $E(z)$ denote the $\mathrm{FMap}(\Sigma)$ orbit of z in E and $\mathrm{Accu}(z) \subset E$ be the accumulation set of $E(z)$. Also let $E^G \subset E$ be the set of non-planar ends of Σ (for $x \in E^G$, every neighborhood of x in Σ has non-zero genus).

Definition 1.4 (Two-sided)**.** We say $E(z)$ is *two-sided* if $E(z)$ is countable and $\mathrm{Accu}(z) = X \sqcup Y$ where X, Y are non-empty disjoint closed $\mathrm{FMap}(\Sigma)$–invariant subsets of E. We say E^G is *two-sided* if $E^G = X \sqcup Y$ where X, Y are non-empty disjoint closed $\mathrm{FMap}(\Sigma)$–invariant subsets of E. We say E is two-sided if either $E(z)$ is two-sided for some $z \in E$ or if E^G is two-sided.

Theorem 1.5 (Existence of essential shift)**.** $\mathrm{Map}(\Sigma)$ *contains an essential shift if and only if the end space E of Σ is two-sided.*

We can refine this theorem to give a characterization of exactly which shift maps are essential. For a shift map h_σ with support σ, let $x, y \in E$ be the ends of Σ associated to the ends of σ, that is, the ends σ exits towards. Let $E(\Sigma_i)$ be set of ends of Σ_i, where the Σ_i are the subsurfaces shifted by h_σ, and let $\mathcal{M}(\Sigma_i)$ be the set of maximal points in $E(\Sigma_i)$ (see Section 2.1).

Theorem 1.6 (Topological characterization of an essential shift)**.** *A shift map h_σ is essential if and only if either*

- Σ_i *has finite genus and* E^G *is two-sided giving a decomposition* $E^G = X \sqcup Y$ *where* $x \in X$ *and* $y \in Y$ *; or*
- *for some* $z \in \mathcal{M}(\Sigma_i)$, $E(z)$ *is two-sided giving a decomposition* $\mathrm{Accu}(z) = X \sqcup Y$ *where* $x \in X$ *and* $y \in Y$.

There are other topological properties of Σ that result in $\mathrm{Map}(\Sigma)$ having a non-trivial geometry. A subsurface R of Σ is called non-displaceable if for every $f \in \mathrm{Map}(\Sigma)$, $f(R)$ intersects R. In [11], it was shown that $\mathrm{Map}(\Sigma)$ acts non-trivially on an infinite diameter hyperbolic space if and only if Σ contains a non-displaceable subsurface. Using the classification of coarsely bounded mapping class groups given in [13], we show:

Theorem 1.7 (Two sources of non-trivial geometry)**.** *If* $\mathrm{Map}(\Sigma)$ *does not have an essential shift and* Σ *does not contain a non-displaceable subsurface then* $\mathrm{Map}(\Sigma)$ *is quasi-isometric to a point.*

That is, any surface that does have an essential shift map or a non-displaceable subsurface does not have interesting geometry. The finiteness of the asymptotic dimension of the mapping class groups in the remaining cases is still open.

Question 1.8. *Let* Σ *be a tame infinite type surface that contains a non-displaceable subsurface such that* $\mathrm{Map}(\Sigma)$ *has no essential shifts. Is* $\mathrm{asdim}(\mathrm{Map}(\Sigma))$ *always finite?*

Remark 1.9. The classification of CB generated big mapping class groups in [13] was carried out in a larger class of *tame* surfaces where only certain ends are assumed to have stable neighborhoods. Most of the arguments in this paper still work in the setting of tame surfaces as well, for example, an essential shift does always imply infinite asymptotic dimension. However, without the assumption that Σ is stable, E being two sided does not imply existence of a shift map. Hence the statement and some arguments are cleaner with this assumption. The class of stable surfaces, first used in [8], is a large and natural class of surfaces to work with and it includes all easily constructed infinite type surfaces.

Outline of the paper. We show $\mathrm{FMap}(\Sigma)$ has infinite asymptotic dimension by embedding the infinite dimensional cube, $Q^\infty \subset \{0,1\}^{\mathbb{N}}$, quasi-isometrically into $\mathrm{FMap}(\Sigma)$. We introduce the definition of asymptotic dimension in Section **??** and show that Q^∞ has infinite asymptotic dimension. In Section 2, we introduce the finite-index subgroup of the mapping class group $\mathrm{FMap}(\Sigma)$ and show that it is quasi-isometric to the mapping class group, $\mathrm{Map}(\Sigma)$.

In Section 3, we carry out our main arguments in the simple case of the *shark tank*, T, which is a cylinder with a $\mathbb{Z}$ action and a discrete set of punctures exiting both ends of the cylinder. We define a length function on $\mathrm{FMap}(T)$ which counts how many punctures from one side of the cylinder are mapped to the other side. We then use this length function to show that there is a quasi-isometric embedding from Q^∞ to $\mathrm{FMap}(T)$.

The proof in the general case is essentially the same, except one has to define appropriate length functions whenever E is two-sided. When $E(z)$ is two-sided for some $z \in E$, the length function counts the number of ends in $E(z)$ that are moved from one side to another. When E^G is two-sided, we rely on the action of $\mathrm{FMap}(\Sigma)$ on homology to construct a suitable length function (see Section 4). Using these length functions, we again show that there is a quasi-isometric embedding of Q^∞ to $\mathrm{FMap}(\Sigma)$ whenever E is two-sided which implies $\mathrm{FMap}(\Sigma)$ has infinite asymptotic dimension (see Section 5).

Theorems 1.5 and 1.6 are proven in Section 6 and Theorem 1.7 is proven in Section 7.

2. A FINITE INDEX SUBGROUP OF THE MAPPING CLASS GROUP

Recall from the introduction that Σ is an infinite-type, orientable, connected, second-countable 2–manifold without boundary with tame end space such that $\mathrm{Map}(\Sigma)$ has a CB generating set. In this section, we recall the definition of tame end space and describe the CB generating set for $\mathrm{Map}(\Sigma)$. We also introduce the finite-index subgroup $\mathrm{FMap}(\Sigma)$ of $\mathrm{Map}(\Sigma)$ and show that $\mathrm{FMap}(\Sigma)$ is quasi-isometric to $\mathrm{Map}(\Sigma)$. Then Fact **??** implies that these groups have the same asymptotic dimension.

2.1. **A partial order on the space of ends of Σ.** Each topological X has an *space of ends* which is defined as the inverse limit $\varprojlim_{K \subset \Sigma} \pi_0(X \smallsetminus K)$ as K ranges over the compact subsets of X. We denote the space of end of Σ by $E = E(\Sigma)$. For any subsurface $\Sigma' \subset \Sigma$, $E(\Sigma')$ is the space of end of Σ'. We always assume subsurfaces of Σ have compact boundary hence, to ensure $E(\Sigma') \subset E(\Sigma)$.

A point $x \in E$ is a *non-planar end* or is *accumulated by genus* if every neighborhood of x in Σ has non-zero genus. Otherwise, x is a *planar end* and it admits a neighborhood which can be embedded in the plane. We denote the subset of non-planar ends of Σ by E^G. Topologically, E is closed and totally disconnected and hence it is homeomorphic to a closed subset of the Cantor set. The space E^G is a closed subset of E.

Richards proved that orientable, boundary-less, infinite type surfaces are completely classified by their genus (possibly infinite), the space of ends E, and the subset of ends accumulated by genus E^G [18]. When we talk about homeomorphisms between subsets of E, we always assume they are *type preserving*. That is, we say $U \subset E$ is homeomorphic to $V \subset E$ if there is a homeomorphism $f\colon U \to V$ such that f sends $U \cap E^G$ homeomorphically to $V \cap E^G$. Every homeomorphism of Σ induces a (type preserving) homeomorphism on the space of ends and, by Richards classification, every homeomorphism of E is induced by some element of $\mathrm{Map}(\Sigma)$.

The following definition, given by Mann and Rafi, gives a ranking of the *local complexity of an end* providing a partial order on equivalence classes of ends. See [13, Section 4 and Section 6.3] for a detailed discussion.

Definition 2.1. Let $\preceq$ be the binary relation on E where $y \preceq x$ if, for every neighborhood U of x, there exists a neighborhood V of y and $f \in \mathrm{Mod}(\Sigma)$ so that $f(V) \subset U$. We say that *x and y are of the same type*, denoted $x \sim y$, if $x \preceq y$ and $y \preceq x$, and write $E(x)$ for the set $\{y \mid y \sim x\}$. One can easily verify that $\sim$ defines an equivalence relation.

From the definition of $\preceq$, we obtain a partial order on the set of equivalence classes under $\sim$. Indeed, the relation $\prec$, defined by $x \prec y$ if $x \preceq y$ and $x \nsim y$, gives a partial order on the set of equivalence classes under $\sim$.

Proposition 2.2 (Proposition 4.7 in [13]). *The partial order $\prec$ has maximal elements. Furthermore, for every maximal element x, the equivalence class $E(x)$ is either finite or a Cantor set.*

The above statement also applies to every clopen subset of E. Let $\mathcal{M} = \mathcal{M}(E)$ denote the set of maximal elements for $\prec$. For a clopen subset $A \subset E$, we denote the maximal ends in A by $\mathcal{M}(A)$.

2.2. $\mathrm{Map}(\Sigma)$ **is locally CB.** A *CB generating set* is the union of a CB neighborhood of the identity and a finite set.

Definition 2.3 (Locally CB). Let G be a topological group. A subset $H \subseteq G$ is *coarsely bounded*, abbreviated CB, in G if H has finite diameter with respect to every continuous left-invariant length

function on G. We say G is *locally CB* if there is a neighborhood of the identity that is a CB subset of G.

Mann and Rafi give a classification of surfaces Σ for which $\mathrm{Map}(\Sigma)$ is locally CB. In particular, they show that if $\mathrm{Map}(\Sigma)$ is locally CB, then Σ has the following structure [13, Proposition 5.4].

Proposition 2.4. *If* $\mathrm{Map}(\Sigma)$ *is locally CB, then there exists a subsurface of finite type* $L \subset \Sigma$ *giving a partition of the ends*

$$E = \bigsqcup_{A \in \mathcal{A}} A$$

where $|\mathcal{A}|$ *equals the number of boundary components of* L. *Each set* $A \in \mathcal{A}$ *is clopen and* $\mathcal{M}(A) \subset \mathcal{M}(E)$. *In fact, points in* $\mathcal{M}(A)$ *are all in the same equivalence class, and* $\mathcal{M}(A)$ *is either a singleton or a Cantor set.*

Furthermore, for every $A, B \in \mathcal{A}$, there is a clopen set $W_{A,B}$ such that if $E(x)$ intersects both A and B, it also intersects $W_{A,B}$ [13, Lemma 6.10].

2.3. **Stable surface.** In this paper, we assume the surface is stable.

Definition 2.5. We say a neighborhood U of a point $x \in E$ is *stable* if every neighborhood U' of x contains a smaller neighborhood U'' of x that is homeomorphic to U. The surface Σ is called *stable* if every point x in E has a stable neighborhood.

In fact, a stronger conclusion holds. Namely all clopen neighborhoods of x inside the stable neighborhood are homeomorphic.

Lemma 2.6 (Lemma 4.17 in [13]). *If* U *is a stable neighborhood of* $x \in E$, *then for any clopen neighborhood* $U' \subset U$ *of* x, U' *is homeomorphic to* U.

2.4. **The subgroup** $\mathrm{FMap}(\Sigma)$. As mentioned above, $\mathcal{M}(A)$ is either an isolated point in $\mathcal{M}(E)$ or a Cantor set. Let $\mathcal{A}_{iso}$ be the subset of $\mathcal{A}$ consisting of sets A where $\mathcal{M}(A)$ is isolated. Let x_A denote an element of $\mathcal{M}(A)$. The set A is always a stable neighborhood of x_A and if $x_B \in E(x_A)$ then A is homeomorphic to B. Any $f \in \mathrm{Map}(\Sigma)$ acts by a permutation on the set of isolated maximal ends,

$$M_{iso} = \{x_A \mid A \in \mathcal{A}_{iso}\}$$

which is a finite set. There exists a map

$$\sigma \colon \mathrm{Map}(\Sigma) \to \mathrm{Sym}(M_{iso})$$

induced by the action of $\mathrm{Map}(\Sigma)$ on M_{iso}. Since M_{iso} is finite, this implies that $\ker(\sigma)$ is a finite index subgroup of $\mathrm{Map}(\Sigma)$. We define $\mathrm{FMap}(\Sigma) = \ker(\sigma)$, that is, $\mathrm{FMap}(\Sigma)$ the subgroup of $\mathrm{Map}(\Sigma)$ that fixes the isolated maximal ends point-wise.

2.5. **Coarsely bounded generating sets.** In [13] Mann-Rafi gave a CB generating set for $\mathrm{FMap}(\Sigma)$ and $\mathrm{Map}(\Sigma)$. Let L be the surface of finite-type as above. The first collection of elements in our generating set are the elements fixing L

$$\nu_L = \big\{g \in \mathrm{Map}(\Sigma) \mid g|_L = id\big\}.$$

When $\mathrm{Map}(\Sigma)$ is CB generated, there is a finite set $F \subset \mathrm{FMap}(\Sigma)$ such that $\nu_L \cup F$ generates FMap (See [13, Section 6.4] for details). We say $\pi \in \mathrm{Sym}(M_{iso})$ is *type preserving* if for every $x_A \in M_{iso}$, $x_B = \pi(x_A) \in E(x_A)$. For every type preserving $\pi \in \mathrm{Sym}(M_{iso})$ we fix a homeomorphism p_π which permutes the sets in $\mathcal{A}$, sending A to B when $\pi(x_A) = x_B$, but fixes the finite type subsurface L set wise (that is the restriction of p_π to L is an element of $\mathrm{Map}(L)$). In fact, we can choose these so that $p_{\pi^{-1}} = p_\pi^{-1}$. Since there are only finitely many such permutations, the collection $P = \{p_\pi\}$ is finite. Then $\nu_L \cup F \cup P$ is a CB generating sets for $\mathrm{Map}(\Sigma)$.

2.6. FMap(Σ) **is quasi-isometric to** Map(Σ). Let S_{FMap} and S_{Map} be the generating sets for FMap(Σ) and Map(Σ) from the previous section. Let $\|\cdot\|_{\mathrm{FMap}}$ and $\|\cdot\|_{\mathrm{Map}}$ be the associated word lengths for FMap(Σ) and Map(Σ), respectively. Since FMap(Σ) and Map(Σ) are CB generated, the word lengths with respect to any other generating sets would be bi-Lipschitz equivalent to the word lengths associated to S_{FMap} and S_{Map}. Hence it is sufficient to consider these generating sets.

Also, recall that to show that a metric space (X, d_X) is quasi-isometric to its subspace (Y, d_Y) it is sufficient to check that the inclusion map $Y \to X$ is coarsely onto and a quasi-isometric embedding. That is there are constants $M, C > 0$ such that:

- for every $x \in X$ there is a $y \in Y$ such that $d_X(x, y) \leq C$.
- for all $y_1, y_2 \in Y$, we have

$$\frac{1}{M} d_Y(y_1, y_2) - C \leq d_X(y_1, y_2) \leq M \cdot d_Y(y_1, y_2) + C.$$

Theorem 2.7. *Let Σ be a surface such that* Map(Σ) *is CB generated. Then* (FMap, $\|\cdot\|_{\mathrm{FMap}}$) *and* (Map, $\|\cdot\|_{\mathrm{Map}}$) *are quasi-isometric.*

Proof. Let ν_L, F and P be as in Section 2.5 so that we have $S_{\mathrm{FMap}} = \nu_L \cup F$ and $S_{\mathrm{Map}} = \nu_L \cup F \cup P$. Notice that

$$\mathrm{Map}(\Sigma) = \bigcup_{p_\pi \in P} p_\pi \, \mathrm{FMap}(\Sigma).$$

Let H be the set of elements of FMap(Σ) that can be written in the form psp' where $s \in S_{\mathrm{Map}}$ and $p, p' \in P$. We begin by showing that

$$\max_{h \in H} \|h\|_{\mathrm{FMap}}$$

is finite. We consider the cases when s is in ν_L, or in $F \cup P$ separately.

For $s \in \nu_L$ and $p, p' \in P$, if $h = psp'$ is an element of FMap(Σ), we must have $p = p_\pi$ and $p' = p_{\pi^{-1}}$ since s acts trivially on M_{iso}. By the construction of P, this implies that $p' = p^{-1}$. Since, $s|_L = id$, we also have $h|_L = id$. That is, $h \in \nu_L$ and $\|h\|_{\mathrm{FMap}} = 1$.

Let H' be the subset of H consisting of elements of the type $h = psp'$, where $s \in F \cup P$. Then H' is a finite set. That is, letting

$$M = \max_{h \in H'} \|h\|_{\mathrm{FMap}},$$

we also have

$$M = \max_{h \in H} \|h\|_{\mathrm{FMap}} = \max\{M, 1\}.$$

Now, let $g \in \mathrm{FMap}(\Sigma)$. We can write $g = s_1 s_2 \ldots s_n$, where $s_i \in S_{\mathrm{Map}}$ and $n = \|g\|_{\mathrm{Map}}$. Let $g_i = s_i s_{i+1} \ldots s_n$ and let $p_i \in P$ be such that $g_i \in p_i \, \mathrm{FMap}(\Sigma)$, that is

$$g_i = s_i s_{i+1} \ldots s_n = p_i f_i$$

for some $f_i \in \mathrm{FMap}(\Sigma)$. Therefore,

$$p_i f_i = g_i = s_i \, g_{i+1} = s_i \, p_{i+1} \, f_{i+1},$$

which implies

$$f_i f_{i+1}^{-1} = p_i^{-1} s_i p_{i+1}.$$

Note that $p_i^{-1}, p_{i+1} \in P$, $s_i \in S_{\mathrm{Map}}$ and $f_i f_{i+1}^{-1} \in \mathrm{FMap}(\Sigma)$, hence $p_i^{-1} s_i p_{i+1} \in H$ and

$$\|p_i^{-1} \, s_i \, p_{i+1}\|_{\mathrm{FMap}} \leq M.$$

Now, by rewriting g as

$$g = (e \, s_1 \, p_2)(p_2^{-1} \, s_2 \, p_3)(p_3^{-1} \, s_3 \, p_4) \ldots (p_n \, s_n \, e)$$

we have that $p_i s_i p_{i+1} \in H$ for $1 \leq i \leq n$ (by setting $p_0 = p_{n+1} = e$ where e is the identity). Therefore,

$$\|g\|_{\text{FMap}} \leq M \cdot n = M \cdot \|g\|_{\text{Map}}.$$

Since $S_{\text{FMap}} \subset S_{\text{Map}}$, we always have $\|g\|_{\text{Map}} \leq \|g\|_{\text{FMap}}$. That is, the word lengths $\|\cdot\|_{\text{FMap}}$ and $\|\cdot\|_{\text{Map}}$ are bi-Lipschitz equivalent in $\text{FMap}(\Sigma)$. Also, for every $g \in \text{Map}(\Sigma)$ there is an $f \in \text{FMap}(\Sigma)$ and $p \in P$ such that $g = pf$, and thus $d_{\text{Map}}(f,g) \leq 1$. Hence $(\text{FMap}, \|\cdot\|_{\text{FMap}})$ and $(\text{Map}, \|\cdot\|_{\text{Map}})$ are quasi-isometric. $\qquad\square$

3. The Shark Tank

The shark tank, T, is a bi-infinite cylinder with a discrete countable set of punctures exiting in both ends. The end space $E = E(T)$ is homeomorphic to the following subset of $\mathbb{R}$:

$$\{0\} \cup \left\{\frac{1}{n}\right\}_{n=2}^{\infty} \cup \left\{\frac{n-1}{n}\right\}_{n=3}^{\infty} \cup \{1\}.$$

We call the ends of the cylinder which are accumulated by punctures the *limit ends*. These are also the maximal elements of E. The group $\text{FMap}(\Sigma)$ is the index two subgroup of $\text{Map}(T)$ which fixes the limit ends point wise. In this section, we prove that $\text{FMap}(T)$, and as a result $\text{Map}(T)$, have infinite asymptotic dimension.

The generating set given in [13] has a very simple form in the case of $\text{FMap}(T)$ which we now describe. Fix a curve β on T that separates the two limit ends of T. This decomposes the end space into two sets, which we denote by A and B. We denote the limit end in A by x_A and the limit end in B by x_B. In addition, we fix an ordering on the punctures in T (the non-limit ends) and label them with p_i, $i \in \mathbb{Z}$ such that $p_i \in A$ for $i \leq 0$ and $p_i \in B$ for $i > 0$.
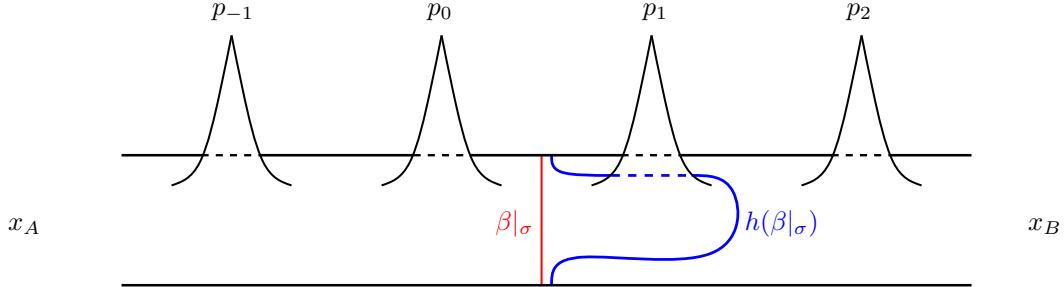

FIGURE 1. The map h is homotopic to the shift map h_σ whose support is the strip σ and sends p_i to p_{i+1}.

There is a homeomorphism $h\colon T \to T$ fixing x_A and x_B such that $h(p_i) = p_{i+1}$ and $f(\beta)$ is disjoint from β with β and $f(\beta)$ bounding a a surface with two boundary components and one puncture. We refer to h as *the shift map*. This is consistent with the definition of shift in the previous section; if we embed an strip σ in T containing all the punctures and limiting to x_A and x_B, the homeomorphism h_σ described in the previous section is homotopic to h since the complement of σ is a strip with no topology (see Figure 1).

Define

$$\nu_\beta = \big\{g \in \text{FMap}(\Sigma) \mid g(\beta) = \beta\big\}.$$

Then $\mathcal{S} = \nu_\beta \cup \{h_\sigma\}$ is the generating set for $\text{FMap}(T)$ given in [13, Section 6.4]. We now equip $\text{FMap}(T)$ with the word length associated to this generating set which we denote by $\|\cdot\|_{\text{FMap}}$.

Definition 3.1. For a group G, a function $\|\cdot\|\colon G \to \mathbb{R}_+$ is a called *length function* if it is continuous, if $\|\phi\| = \|\phi^{-1}\|$ for all $\phi \in \mathrm{FMap}(\Sigma)$ and if it satisfies the triangle inequality, namely, for $\phi, \psi \in G$ we have

$$\|\phi\,\psi\| \leq \|\phi\| + \|\psi\|.$$

We now define a length function on $\mathrm{FMap}(T)$, namely, for $\phi \in \mathrm{FMap}(\Sigma)$, define

$$(1) \qquad \|\phi\| = \big|\{p_i \in A \mid \phi(p_i) \in B\}\big| + \big|\{p_i \in B \mid \phi(p_i) \in A\}\big|.$$

Note that, since maps in $\mathrm{FMap}(T)$ fix the limit ends, they also fix a neighborhood of these ends and hence can move only a finitely many punctures from one side to another. That is, $\|\phi\|$ is a finite number.

Theorem 3.2. *The function $\|\cdot\|$ is a length function on* $\mathrm{FMap}(T)$. *Furthermore, for $\phi \in \mathrm{FMap}(T)$, we have*

$$(2) \qquad \|\phi\| \leq \|\phi\|_{\mathrm{FMap}}.$$

Proof. We start by checking the triangle inequality. Consider $\phi, \psi \in \mathrm{FMap}(T)$. For any $p_i \in A$ where $\psi\phi(p_i) \in B$, we either have

$$\Big(p_i \in A \quad \text{and} \quad \phi(p_i) \in B\Big) \qquad \text{or} \qquad \Big(\phi(p_i) \in A \quad \text{and} \quad \psi\phi(p_i) \in B\Big).$$

Therefore,

$$\{p_i \in A \mid \psi\phi(p_i) \in B\} \subset \{p_i \in A \mid \phi(p_i) \in B\} \cup \{p_i \in A \mid \phi(p_i) \in A \quad \text{and} \quad \psi\phi(p_i) \in B\}.$$

and hence (denoting $\phi(p_i)$ with q_i), we have

$$\big|\{p_i \in A \mid \psi\phi(p_i) \in B\}\big| \leq \big|\{p_i \in A \mid \phi(p_i) \in B\}\big| + \big|\{q_i \in A \mid \psi(q_i) \in B\}\big|.$$

Similarly,

$$\big|\{p_i \in B \mid \psi\phi(p_i) \in A\}\big| \leq \big|\{p_i \in B \mid \phi(p_i) \in A\}\big| + \big|\{q_i \in B \mid \psi(q_i) \in A\}\big|.$$

Therefore,

$$\|\phi\,\psi\| \leq \|\phi\| + \|\psi\|.$$

Now we check that $\|\cdot\|$ is continuous. Note that, for $\phi \in \nu_\beta$, we have $\|\phi\| = 0$. That is $\|\cdot\|$ is zero on some neighborhood of the identity. The triangle inequality above proves that $\|\cdot\|$ is continuous. Also, since ϕ is a homeomorphism,

$$\|\phi\| = \big|\{p_i \in A \mid \phi(p_i) \in B\}\big| + \big|\{p_i \in B \mid \phi(p_i) \in A\}\big|$$
$$= \big|\{q_i \in B \mid \phi^{-1}(q_i) \in A\}\big| + \big|\{q_i \in A \mid \phi^{-1}(q_i) \in A\}\big| = \|\phi^{-1}\|.$$

Hence, $\|\cdot\|$ is a length function.

To see the second assertion, let $s \in \mathcal{S}$ be an element of the generating set of $\mathrm{FMap}(T)$. If $s \in \nu_\beta$, then s fixes the base curve β. Therefore, $s(A) \subset A$ and $s(B) \subset B$, and therefore, $\|s\| = 0$. Alternatively, if $s = h_\sigma$ then p_0 is the only puncture in A that is mapped to B and no punctures from B are mapped to A. That is,

$$\|h\| = \big|\{p_i \in A \mid h(p_i) \in B\}\big| + \big|\{p_i \notin B \mid h(p_i) \in B\}\big| = 1 + 0 = 1.$$

Now, for $\phi \in \mathrm{FMap}(T)$, if $\|\phi\| = n$, then $\phi = s_1 \ldots s_n$ where $s_i \in \mathcal{S}$. By the triangle inequality,

$$\|\phi\| \leq \sum_{i=1}^{n} \|s_i\| \leq n = \|\phi\|_{\mathrm{FMap}}. \qquad \qquad \square$$

We now recall two useful facts about asymptotic dimension. The first fact states high dimensional space cannot be coarsely embedded in a small dimensional space.

Fact 3.3 (Theorem 5 of [2]). If $f\colon X \to Y$ is a quasi-isometric embedding between two metric spaces X and Y, then $\mathrm{asdim}(X) \le \mathrm{asdim}(Y)$.

The second fact is regarding the asymptotic dimension of $\mathbb{Z}^n$.

Fact 3.4 (Theorems 5 and 6 in [2]). Equip $\mathbb{Z}^n$ with the L^1 metric. That is, for $a = (a_1, .., a_n)$ and $b = (b_1, .., b_n)$ in $\mathbb{Z}^n$, define

$$d_{\mathbb{Z}^n}(a, b) = |a - b|_{\mathbb{Z}^n} = \sum_{i=1}^{n} |a_i - b_i|.$$

The asymptotic dimension of $\mathbb{Z}^n$ with respect to the L^1 metric is exactly n.

Due to these two facts, we now construct a map from $\mathbb{Z}^n$ to $\mathrm{FMap}(T)$ for all $n \in \mathbb{N}$ which will allow us to show that the asymptotic dimension of $\mathrm{Map}(T)$ is infinite. As the asymptotic dimension for $\mathbb{Z}^n$ is n for all $n \in \mathbb{N}$, by showing that the map we construct is a quasi-isometric embedding, we will be able to conclude that the asymptotic dimension of $\mathrm{FMap}(T)$ is greater than n for all $n \in \mathbb{N}$, which will show the asymptotic dimension is infinite.

Fix some $n \in \mathbb{N}$. We begin by defining a homeomorphism, associated to a given element $a \in \mathbb{Z}^n$. Notice that we are able to rearrange the puctures on the surface T to make n embedded infinite strips of punctures, $\sigma_1, \ldots, \sigma_n$, and to each strip we can define a puncture shift, $h_{\sigma_1}, \ldots, h_{\sigma_n}$. Indeed, with respect to our labelling, we create n strips of punctures by placing the punctures labelled $\ldots, 1 - 2n, 1 - n, 1, 1 + n, 1 + 2n, \ldots$ in the first strip σ_1, the punctures $\ldots, 2 - 2n, 2 - n, 2, 2 + n, 2 + 2n, \ldots$ in the second strip σ_2, so on and so forth until we have n strips, with the punctures labelled as $\ldots, i - 2n, i - n, i, i + n, i + 2n, \ldots$ in strip σ_i. Associated to each strip, the shift map h_{σ_i} has support in σ_i and sends puncture i to puncture $i + n$.

Now, for a given $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, we define the map Φ to be

$$\Phi\colon \mathbb{Z}^n \to \mathrm{FMap}(T)$$

(3)
$$a \mapsto \prod_{i=1}^{n} h_{\sigma_i}^{a_i}.$$

FIGURE 2. Consider the element $(4, -2, 5, -4) \in \mathbb{Z}^n$. This image illustrates the action of the map Φ on the base curve β.

For $a, b \in \mathbb{Z}^n$, we would like to compute $\|\Phi(b)^{-1}\Phi(a)\|$. For any $i > 0$ where $a_i \ge 0$, the map $\Phi(a)$ moves a_i punctures from A into B applying a puncture shift h_{σ_i} exactly a_i times. If $b_i \ge 0$, then $\Phi(b)^{-1}$ moves b_i punctures from B into A by applying a puncture shift $h_{\sigma_i}^{-1}$ exactly b_i times. If If $b_i < 0$, then $\Phi(b)^{-1}$ moves b_i punctures from A into B by applying a puncture shift h_{σ_i} exactly b_i times. Similarly, for any $i > 0$ where $a_i < 0$, the map $\Phi(a)$ moves a_i punctures from B into A applying a puncture shift $h_{\sigma_i}^{-1}$ exactly a_i times. If $b_i \ge 0$, then $\Phi(b)^{-1}$ moves b_i punctures from

B into A by applying a puncture shift $h_{\sigma_i}^{-1}$ exactly b_i times. If If $b_i < 0$, then $\Phi(b)^{-1}$ moves b_i punctures from A into B by applying a puncture shift h_{σ_i} exactly b_i times. This implies that

$$\|\Phi(b)^{-1}\Phi(a)(\beta)\| = |a - b|_{\mathbb{Z}^n}. \tag{4}$$

We now show that Φ is a quasi-isometric embedding from Q^∞ to $\mathrm{FMap}(T)$. Recall that the distance between two elements $f, g, \in \mathrm{FMap}(T)$ is $\|fg^{-1}\|_{\mathrm{FMap}}$, where $\|\cdot\|_{\mathrm{FMap}}$ is the word length with respect to the generating set $\mathcal{S}$. A quasi-isometric embedding is a map that preserve distances up to uniform additive and multiplicative errors.

Proposition 3.5. *For all $n \in \mathbb{N}$, for Φ as in Equation 3, and for $a, b \in \mathbb{Z}^n$, we have*

$$|a - b|_{\mathbb{Z}^n} \le \|\Phi(b)^{-1}\Phi(a)\|_{\mathrm{FMap}} \le |a - b|_{\mathbb{Z}^n} + 3.$$

That is, the map

$$\Phi \colon (\mathbb{Z}^n, |\cdot|_{\mathbb{Z}^n}) \to (\mathrm{FMap}(T), \|\cdot\|_{\mathrm{FMap}})$$

is a quasi-isometric embedding.

Proof. The left inequality follows from Equations (2) and (4). To prove the right inequality, let $a, b \in \mathbb{Z}^n$, we will find elements in $s_i \in \mathcal{S}$ such that $(\prod s_i) \circ \Phi(b)^{-1}\Phi(a)$ is the identity.

Let $\omega = \Phi(b)^{-1}\Phi(a)(\beta|_\sigma)$, and let $\omega_i = \omega|_{\sigma_i}$. We begin by rearranging the n strips of punctures back into a single strip of punctures. Recall for σ_i, the labeling of the punctures is $\dots, i - 2n, i - n, i, i + n, i + 2n, \dots$ inside of the strip. We rearrange the punctures so that we have a single strip with the labeling as indicated in Figure 1.

For each i such that $a_i - b_i \ge 0$, there are $a_i - b_i = k_{1_i}$ punctures between ω_i and $\beta|_\sigma$ in B. Let the total number of punctures between ω and $\beta|_\sigma$ in B be denoted by $k_1 = \sum_{i=1}^n k_{1_i}$. Using an element $s_1 \in \nu_\beta$ such that $h^{-k_1}(s_1(\omega_i)) \in A$ for all $1 \le i \le n$, we can rearrange the punctures in the single strip σ such that the k_1 punctures are in positions $p_1, p_2, \dots p_{k_1}$ in B. Then $h_\sigma^{-k_1}$ sends all these punctures to A. That is, $h_\sigma^{-k_1} s_1$ sends ω to an arc ω' that is completely to the left side of β.
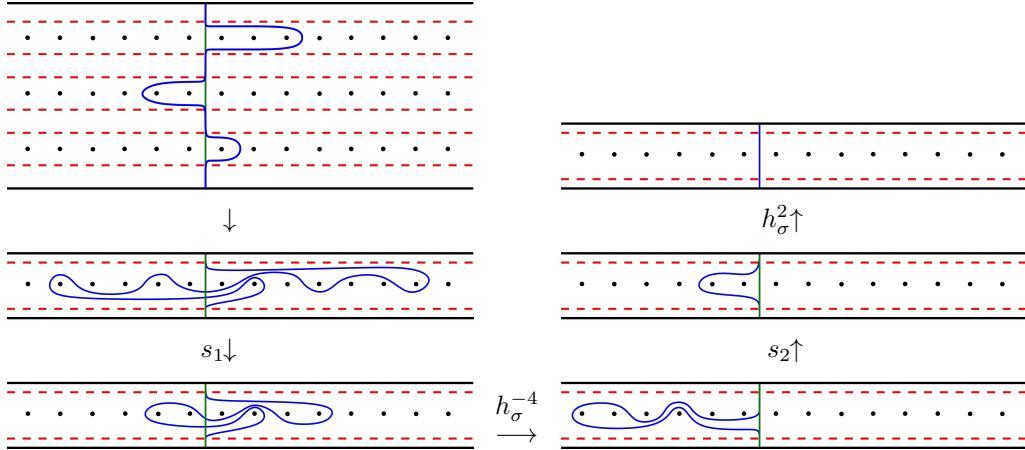

FIGURE 3. This image illustrates the elements of $\mathcal{S}$ required to build the inverse of $\Phi(3, -2, 1)$ where $(3, -2, 1) \in \mathbb{Z}^3$.

There are still k_2 punctures between ω' and $\beta|_\sigma$ in A since there are $|a_i - b_i|$ punctures between ω and $\beta|_\sigma$ in A whenever $a_i - b_i < 0$. We now find $s_2 \in \nu_\beta$ that lines up these punctures in the

positions $p_{-k_2+1}, \ldots, p_0$. Then $h_\sigma^{k_2}$ sends these punctures back to B. That is $h_\sigma^{k_2} s_2(\omega') = \beta|_\sigma$. Now the composition

$$\left(h_\sigma^{k_2} s_2 h_\sigma^{-k_1} s_1\right) \circ \Phi(b)^{-1} \Phi(a)$$

sends every puncture in A to a puncture in A and every puncture in B to a puncture in B. Hence, this map is equal to some element $s_3^{-1} \in \nu_\beta$. Therefore, we have shown that

$$\left(s_3 h_\sigma^{k_2} s_2 h_\sigma^{-k_1} s_1\right) \circ \Phi(b)^{-1} \Phi(a) = id.$$

Hence

$$\|\Phi(b)^{-1} \Phi(a)\|_{\text{FMap}} \leq k_1 + k_2 + 3,$$

where $k_1 + k_2 = |a - b|$. This completes the proof. $\qquad\qquad\square$

Theorem 3.6. *The group* $\text{FMap}(T)$ *has infinite asymptotic dimension.*

Proof. By Proposition, 3.5, for each $n \in \mathbb{N}$ the map $\Phi\colon \mathbb{Z}^n \to \text{FMap}(T)$ is a quasi-isometric embedding. Since quasi-isometric embeddings preserve asymptotic dimension by Fact 3.3, and the asymptotic dimension of $\mathbb{Z}^n$ is n by Fact 3.4, we have that for all n, the asymptotic dimension of $\text{FMap}(T)$ is at least n. This shows that the asymptotic dimension of $\text{FMap}(T)$ is infinite. $\qquad\square$

By Theorem 2.7, $(\text{FMap}(T), \|\cdot\|_\mathcal{S})$ and $(\text{Map}(T), \|\cdot\|_{S_{\text{Map}}})$ are quasi-isometric. Fact 3.3 tells us that asymptotic dimension is preserved under quasi-isometry. Therefore since $(\text{FMap}(T), \|\cdot\|_\mathcal{S})$ has infinite asymptotic dimension, so does $(\text{Map}(T), \|\cdot\|_{S_{\text{Map}}})$.

Theorem 3.7. *The group* $\text{Map}(T)$ *has infinite asymptotic dimension.*

4. THE LENGTH FUNCTIONS

As we saw in the last section, the length function $\|\cdot\|$, defined on $\text{FMap}(T)$, is how we provide a lower bound for the word length. In this section, we show that when the end space is two-sided, there is a similar length function on $\text{FMap}(\Sigma)$ that is bounded above by the word length. This implies that the word lengths of powers of an associated shift map grow linearly, and hence the shift map is essential.

4.1. **Two-sided end.** Assume, for $z \in E$, that $E(z)$ is two-sided. Recall from the introduction that this means $\text{Accu}(z) = X \sqcup Y$, where X, Y are non-empty disjoint closed $\text{FMap}(\Sigma)$–invariant subsets of E and where $\text{Accu}(z)$ is the accumulation set of $E(z)$.

Let the subsurface L be as in Proposition 2.4 giving the decomposition

$$E = \bigsqcup_{A \in \mathcal{A}} A.$$

As before, we fix a point $x_A \in \mathcal{M}(A) \subset \mathcal{M}(E)$.

Observe that, for $A \in \mathcal{A}$, either $X \cap A = \emptyset$ or $Y \cap A = \emptyset$. This is because, if $x \in X \cap A$, then $E(x) \subset X$ and $\text{Accu}(x) \subset X$. But x_A is an accumulation point of every type of point in A. That is, $x_A \in \text{Accu}(x) \subset X$. Similarly, if $y \in Y \cap A$, then $x_A \in \text{Accu}(y) \subset Y$. This contradicts the assumption that X and Y are disjoint. Hence, we can find a curve β in L so that $\Sigma \smallsetminus \beta$ consists of two subsurfaces Σ_+ and Σ_- such that $X \subset E(\Sigma_-)$ and $Y \subset E(\Sigma_+)$. Then there is a decomposition $\mathcal{A} = \mathcal{A}_+ \sqcup \mathcal{A}_-$ such that $E(\Sigma_+) = \cup_{A \in \mathcal{A}_+} A$ and $E(\Sigma_-) = \cup_{B \in \mathcal{A}_-} B$. We denote $E(\Sigma_+)$ by E_+ and $E(\Sigma_-)$ by E_-.

Theorem 4.1 (Length function associated to a two-sided end). *For $z \in E$, assume $E(z)$ is two-sided. Let β be the curve defined above so that*

$$\Sigma \smallsetminus \beta = \Sigma_- \cup \Sigma_+, \qquad X \subset E_- = E(\Sigma_-) \qquad \text{and} \qquad Y \subset E_+ = E(\Sigma_+).$$

Then, the function $\|\cdot\|\colon \text{FMap}(\Sigma) \to \mathbb{Z}$ *defined by*

$$(5) \qquad \|\phi\| = \left|\{p \in E(z) \mid p \in E_-, \quad \phi(p) \in E_+\}\right| + \left|\{p \in E(z) \mid p \in E_+, \quad \phi(p) \in E_-\}\right|$$

is a length function on $\mathrm{FMap}(\Sigma)$.

Proof. We first check that, for every $\phi \in \mathrm{FMap}(\Sigma)$, $\|\phi\|$ is finite. Suppose towards a contradiction that $\|\phi\|$ is infinite. By replacing ϕ with ϕ^{-1} if necessary, we may assume that ϕ maps infinitely many points from $E(z) \cap E_-$ into E_+. That is, there is a sequence $z_i \in E(z) \cap E_-$ such that $\phi(z_i) \in E_+$. After taking a sub-sequence, we can assume $z_i \to x \in X$. Since ϕ is continuous, and E_+ is closed, $\phi(x) \in E_+$. But this contradicts the fact that $X \subset E_-$ is $\mathrm{FMap}(\Sigma)$ invariant.

The proof of the triangle inequality and the fact that $\|\phi\| = \|\phi^{-1}\|$ are identical to the proof in Theorem 3.2. Also, the subgroup of $\mathrm{FMap}(\Sigma)$ that fixes L is a neighborhood of the identity where $\|\cdot\|$ is zero, hence $\|\cdot\|$ is continuous. □

Proceeding as in Section 3, we now show that the length function $\|\cdot\|$ is bounded above by a uniform multiple of the word length.

Theorem 4.2. *For $z \in E$, assume $E(z)$ is two-sided and let $\|\cdot\|$ be the associated length function. Let $\mathcal{S}$ be a CB generating set for $\mathrm{FMap}(\Sigma)$, and let $\|\cdot\|_{\mathcal{S}}$ denote the associated word length on $\mathrm{FMap}(\Sigma)$. Then there exists a constant $c > 0$ such that for every $\psi \in \mathrm{FMap}(\Sigma)$*

$$(6) \qquad\qquad \|\psi\| \leq c \cdot \|\psi\|_{\mathcal{S}}.$$

Proof. Since $\mathrm{FMap}(\Sigma)$ is CB generated, the word length associated to every two CB generating sets are Lipschitz equivalent. Hence, without loss of generality, we can assume $\mathrm{FMap}(\Sigma)$ is equipped with the generating set given in Section 2.5. That is, there is a finite set F such that $\mathcal{S} = \nu_L \cup F$. For $A \in \mathcal{A}$, denote the complementary component of $(\Sigma - L)$ associated to A by Σ_A. We further write

$$\nu_L = \cup_{A \in \mathcal{A}} \nu_A$$

where elements of ν_A have support in Σ_A.

If $s \in \nu_L$, then s fixes every $A \in \mathcal{A}$ set-wise. In particular, s fixes E_+ and E_- set-wise. Therefore, $\|s\| = 0$. Now define

$$c = \max_{s \in F} \|s\|.$$

Then, for $s \in \mathcal{S}$, $\|s\| \leq c$. For $\phi \in \mathrm{FMap}(\Sigma)$, if $\phi = s_1 \ldots s_n$, where $s_i \in S$ and $n = \|\phi\|_{\mathcal{S}}$, then

$$\|\phi\| \leq \sum_{i=1}^{n} \|s_i\| \leq c \cdot n.$$

That is, $\|\phi\| \leq c \cdot \|\phi\|_{\mathcal{S}}$. □

4.2. Two-sided E^G.

Assume E^G is two-sided. Recall from the introduction that this implies that $E^G = X \sqcup Y$, where X, Y are non-empty disjoint closed $\mathrm{FMap}(\Sigma)$–invariant subsets of E. As in the previous section, for every $A \in \mathcal{A}$, we have either $X \cap A = \emptyset$ or $Y \cap A = \emptyset$. Therefore, we can choose a curve β in L giving a decomposition $\Sigma = \Sigma_- \cup \Sigma_+$, where $X \subset E_- = E(\Sigma_-)$ and $Y \subset E_+ = E(\Sigma_+)$.

Heuristically, to follow the two-sided end case, for $\phi \in \mathrm{FMap}(\Sigma)$ we would like to count the genus of the subsurface of Σ_- that is moved by ϕ to Σ_+ plus the genus of the subsurface of Σ_+ that is moved by ϕ to Σ_-. But this is not the correct measurement. For example, consider a large genus subsurface Y that intersects both Σ_- and Σ_+ and let ϕ be a pseudo-Anosov homeomorphism with support in Y. Then no subsurface of Σ_- is moved to Σ_+. Instead, $\|\phi\|$ will be the genus of the subsurface Y.

We use the $\mathbb{Z}_2$–homology of the surface Σ, where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. We note that separating curves on infinite-type surfaces represent elements of homology which are not necessarily zero (see [17]). However, we quotient $H_1(\Sigma, \mathbb{Z}_2)$ by $H_1^{sep}(\Sigma, \mathbb{Z}_2)$, the subgroup of $H_1(\Sigma, \mathbb{Z}_2)$ generated by homology

classes that can be represented by simple separating closed curves on the surface. We denote this quotient by

$$\widehat{H} = \widehat{H}_1(\Sigma, \mathbb{Z}_2) = H_1(\Sigma, \mathbb{Z}_2)/H_1^{sep}(\Sigma, \mathbb{Z}_2).$$

Since the curve β represents a trivial class in $\widehat{H}$, the decomposition $\Sigma = \Sigma_+ \cup \Sigma_-$ gives a splitting $\widehat{H} = H_+ \oplus H_-$ where

$$H_+ = \widehat{H}_1(\Sigma_+, \mathbb{Z}/2) \qquad \text{and} \qquad H_- = \widehat{H}_1(\Sigma_-, \mathbb{Z}/2).$$

Note that $H_- \cap H_+ = \emptyset$ since Σ_- and Σ_+ are disjoint.

For $\phi \in \mathrm{FMap}(\Sigma)$, we denote the induced map on homology by $\phi^*\colon H_1(\Sigma, \mathbb{Z}/2) \to H_1(\Sigma, \mathbb{Z}/2)$. Since ϕ sends separating curves to separating curves, we also have an induced isomorphism $\phi^*\colon \widehat{H} \to \widehat{H}$.

Define $\|\cdot\|\colon \mathrm{FMap}(\Sigma) \to \mathbb{Z}_+$ by

$$(7) \qquad \|\phi\| = \mathrm{codim}\left(\widehat{H} : \left(H_+ \cap \phi^*(H_+)\right) \oplus \left(H_- \cap \phi^*(H_-)\right)\right),$$

where $\mathrm{codim}(V : W)$ is the co-dimension of a subspace W in a vector space V.

Theorem 4.3. *Assume E^G is two-sided. Then $\|\cdot\|\colon \mathrm{FMap}(\Sigma) \to \mathbb{N}$ is a length function.*

Proof. We start by showing that, for $\phi \in \mathrm{FMap}(\Sigma)$, $\|\phi\|$ is finite. Since ϕ fixes X set-wise, there is a small neighborhood of X in Σ_- that is contained in Σ_-. That is, there is a subsurface $\Sigma' \subset \Sigma_-$ such that $X \subset E(\Sigma')$ and $\phi(\Sigma') \subset \Sigma_-$. The subsurfaces $(\Sigma_- - \Sigma')$ and $(\Sigma_- - \phi(\Sigma'))$ both have finite genus because the fact that $E^G \cap E(\Sigma_-) = X$ implies that their end space is disjoint from E^G This means that

$$\mathrm{codim}(H_- : \widehat{H}_1(\Sigma', \mathbb{Z}_2)) < \infty \qquad \text{and} \qquad \mathrm{codim}(H_- : \widehat{H}_1(\phi(\Sigma'), \mathbb{Z}_2)) < \infty.$$

But

$$\widehat{H}_1(\Sigma', \mathbb{Z}_2) \cap \widehat{H}_1(\phi(\Sigma'), \mathbb{Z}_2) \subset H_- \cap \phi^*(H_-).$$

Therefore,

$$\mathrm{codim}(H_- : H_- \cap \phi^*(H_-)) \leq \mathrm{codim}(H_- : \widehat{H}_1(\Sigma', \mathbb{Z}_2)) + \mathrm{codim}(H_- : \widehat{H}_1(\phi(\Sigma'), \mathbb{Z}_2)) < \infty.$$

Similarly, $\mathrm{codim}(H_+ : H_+ \cap \phi^*(H_+)) < \infty$ and hence $\|\phi\| < \infty$.

We now check that $\|\cdot\|$ satisfies the triangle inequality. Consider $\phi, \psi \in \mathrm{FMap}(\Sigma)$. We have

$$\mathrm{codim}\left(H_\pm : H_\pm \cap \psi(\phi(H_\pm))\right) \leq \mathrm{codim}\left(H_\pm : H_\pm \cap \psi(H_\pm) \cap \psi(\phi(H_\pm))\right)$$
$$= \mathrm{codim}\left(H_\pm : H_\pm \cap \psi(H_\pm)\right) + \mathrm{codim}\left(H_\pm \cap \psi(H_\pm) : H_\pm \cap \psi(H_\pm) \cap \psi(\phi(H_\pm))\right)$$
$$\leq \mathrm{codim}\left(H_\pm : H_\pm \cap \psi(H_\pm)\right) + \mathrm{codim}\left(\psi(H_\pm) : \psi(H_\pm) \cap \psi(\phi(H_\pm))\right)$$
$$= \mathrm{codim}\left(H_\pm : H_\pm \cap \phi(H_\pm)\right) + \mathrm{codim}\left(H_\pm : H_\pm \cap \phi(H_\pm)\right).$$

Summing over $\pm$ we get

$$\|\psi \circ \phi\| \leq \|\psi\| + \|\phi\|.$$

The subgroup ν_L of $\mathrm{FMap}(\Sigma)$ is a neighborhood of the identity and for $\phi \in \nu_L$, ϕ preserves $\Sigma_\pm$. Hence, ϕ^* fixes $H_\pm$ and $\|\phi\| = 0$. This fact and the triangle inequality imply that $\|\cdot\|$ is continuous. Also, since

$$(\phi^*)^{-1}(H_\pm \cap \phi^*(H_\pm)) = (\phi^*)^{-1}(H_\pm) \cap (H_\pm)$$

we have $\|\phi\| = \|\phi^{-1}\|$. Therefore $\|\cdot\|$ is a length function. $\qquad\square$

We now show that the function $\|\cdot\|$ is bounded above by the word metric on $\mathrm{FMap}(\Sigma)$.

Theorem 4.4. *There exist a constant $c > 0$ such that*

$$(8) \qquad \|\phi\| \leq c \cdot \|\phi\|_{\mathcal{S}},$$

where $\|\cdot\|_{\mathcal{S}}$ denotes the word metric on $\mathrm{FMap}(\Lambda)$.

Proof. This proof follows the same reasoning as the proof of Theorem 4.2, except we use that E^G is 2-sided in place of $E(z)$ being two-sided. As before, we assume that $\mathrm{FMap}(\Sigma)$ is equipped with the generating set (ν_L, F) given in Section 2.5. If $s \in \nu_L$, then s fixes the curve β which is contained in L and hence it fixes $\Sigma_{\pm}$. In particular, s^* fixes $H_{\pm}$. Therefore, $\|\phi\| = 0$. Now, for

$$c = \max_{\phi \in F} \|\phi\|,$$

we have $\|\phi\| \leq c \cdot \|\phi\|_{\mathcal{S}}$. $\qquad \square$

5. Infinite asymptotic dimension

In this section, we prove that the mapping class group of an infinite type surface Σ with a two-sided end space has infinite asymptotic dimension. For any such surface, we construct an associated shift map and we use the length function from Section 4 to show that they are essential. We then follow the arguments in Section 3 to embed Q^{∞} in $\mathrm{FMap}(\Sigma)$.

Proposition 5.1. *If $E = E(\Sigma)$ is two-sided, then $\mathrm{FMap}(\Sigma)$ contains an essential shift.*

Proof. Assume first that E^G is two-sided. That is, $E^G = X \sqcup Y$ where X and Y are non-empty closed $\mathrm{FMap}(\Sigma)$ invariant sets. Let β be a curve in L separating X from Y and, as before, let Σ_- and Σ_+ be the components of $\Sigma - \beta$. Also, let $\|\cdot\|$ be the norm defined in Theorem 4.3.

Pick $x \in X$ and $y \in Y$. Since every neighborhood of x and y in Σ have non-zero genus, we can find a sequence of disjoint surfaces Σ_i, each homeomorphic to a genus one surface with one boundary component, such that $\Sigma_i \to x$ as $i \to -\infty$ and $\Sigma_i \to y$ as $i \to \infty$. We choose these so that $\Sigma_i \in \Sigma_-$ for $i \leq 0$ and $\Sigma_i \in \Sigma_+$ for $i > 0$.

We connect Σ_{i-1} to Σ_i by an arc ω_i so that the arcs ω_i are disjoint from each other and the other Σ_j, and so only ω_0 intersects β. Let σ be a regular neighborhood of the union of the Σ_i and ω_i. Then σ is a strip of infinite genus exiting towards x and y in each end and intersecting β in an arc $\beta|_{\sigma}$. Denote the components of $\sigma - \beta|_{\sigma}$ by σ_- and σ_+.

Let h_{σ} be the shift map with support in σ sending Σ_i to Σ_{i+1}. Choosing a basis for the homology of each Σ_i and extending it to the homology of Σ_- and Σ_+, we can decompose the homology of $\Sigma_{\pm}$ as follow:

$$H_- = \overline{H}_- \oplus \bigoplus_{i=-\infty}^{0} H_i \qquad \text{and} \qquad H_+ = \overline{H}_+ \oplus \bigoplus_{i=1}^{\infty} H_i$$

where $H_i = \widehat{H}(\Sigma_i, \mathbb{Z}_2)$ and $\overline{H}_{\pm}$ are the homology of $\Sigma_{\pm} - \sigma$. Then, h_{σ}^* fixes $\overline{H}_{\pm}$ and sends H_i to H_{i+1}. Therefore, for $n > 0$

$$(h_{\sigma}^*)^n(H_-) = H_- \oplus \bigoplus_{i=1}^{n} H_i. \qquad \text{and} \qquad (h_{\sigma}^*)^n(H_+) = \overline{H}_+ \oplus \bigoplus_{i=n+1}^{\infty} H_i$$

That is,

$$\|h_{\sigma}^n\| = \mathrm{codim}\left(\widehat{H} : \left(H_+ \cap (h_{\sigma}^*)^n(H_+)\right) \oplus \left(H_- \cap (h_{\sigma}^*)^n(H_-)\right)\right) = \dim\left(\bigoplus_{i=1}^{n} H_i\right) = 2n.$$

Theorem 4.4 implies that the diameter of the group $\langle h_{\sigma} \rangle$ is infinite. Hence, by definition, $\langle h_{\sigma} \rangle$ is not CB and hence h_{σ} is essential.

Now assume $E(z)$ is two-sided. That is, $\mathrm{Accu}(z) = X \sqcup Y$ where X and Y are non-empty closed $\mathrm{FMap}(\Sigma)$ invariant sets. The proof in this case is similar. Let β be a curve in L separating X from

Y, let Σ_- and Σ_+ be the components of $\Sigma - \beta$, and let $\|\cdot\|$ be the norm defined in Theorem 4.1. Pick $x \in X$ and $y \in Y$. Since every neighborhood of x and y in Σ has a point in $E(z)$, we can find a sequence of points $z_i \in E(z)$ where $z_i \to x$ as $i \to -\infty$ and $z_i \to y$ as $i \to \infty$, where $z_i \in E_-$ for $i \leq 0$ and $z_i \in \Sigma_+$ for $i > 0$. Choose disjoint stable neighborhoods E_i of z_i so that $\mathcal{M}(E_i) = z_i$; this is possible since $E(z)$ is countable. Let Σ_i be a subsurface of Σ with one boundary component and with $E(\Sigma_i) = E_i$. We can assume the Σ_i are disjoint from each other and from β. Then $\Sigma_i \to x$ as $i \to -\infty$, $\Sigma_i \to y$ as $i \to \infty$, $\Sigma_i \in \Sigma_-$ for $i < 0$, and $\Sigma_i \in \Sigma_+$ for $i \geq 0$.

As before, connect Σ_{i-1} to Σ_i by an arc ω_i so that the ω_i are disjoint from each other and the other Σ_j, and so only ω_0 intersects β. Let σ be a regular neighborhood of the union of Σ_i and ω_i. Then σ is a strip containing Σ_i exiting towards x and y in each end and intersecting β in an arc $\beta|_\sigma$. Then, h_σ acts as a shift on z_i with $h_\sigma^n(z_i) = z_{i+n}$. Therefore,

$$\|h_\sigma^n\| = n,$$

which implies $\langle h_\sigma \rangle$ has infinite diameter and h_σ is essential. $\qquad \square$

As in Section 3, we now construct a map from $\mathbb{Z}^n$ to $\mathrm{FMap}(\Sigma)$ for all $n \in \mathbb{N}$ which will allow us to show that the asymptotic dimension of $\mathrm{Map}(\Sigma)$ is infinite.

Recall that h_σ be the shift map with support in σ sending Σ_i to Σ_{i+1} Fix some $n \in \mathbb{N}$. We begin by defining a homeomorphism, associated to a given element $a \in \mathbb{Z}^n$. Notice that we are able to rearrange the subsurfaces on the strip σ to make n embedded infinite strips of subsurfaces, $\sigma_1, \ldots, \sigma_n$, and to each strip we can define a shift map, $h_{\sigma_1}, \ldots, h_{\sigma_n}$. Indeed, with respect to our labelling, we create n strips of subsurfaces by placing the subsurfaces labelled $\ldots, 1 - 2n, 1 - n, 1, 1 + n, 1 + 2n, \ldots$ in the first strip σ_1, the subsurfaces $\ldots, 2 - 2n, 2 - n, 2, 2 + n, 2 + 2n, \ldots$ in the second strip σ_2, so on and so forth until we have n strips, with subsurfaces labelled $\ldots, i - 2n, i - n, i, i + n, i + 2n, \ldots$ in strip σ_i. Associated to each strip, the shift map h_{σ_i} has support in σ_i and sends the subsurface labelled i to the subsurface labelled $i + n$.

Now, for a given $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, we define the map Φ to be

$$\Phi \colon \mathbb{Z}^n \to \mathrm{FMap}(T)$$

(9)
$$a \mapsto \prod_{i=1}^n h_{\sigma_i}^{a_i}.$$

For $a, b \in \mathbb{Z}^n$, we would like to compute $\|\Phi(b)^{-1}\Phi(a)\|$. For any $i > 0$ where $a_i \geq 0$, the map $\Phi(a)$ moves a_i subsurfaces from A into B by applying a shift h_{σ_i} exactly a_i times. If $b_i \geq 0$, then $\Phi(b)^{-1}$ moves b_i subsurfaces from B into A by applying a shift $h_{\sigma_i}^{-1}$ exactly b_i times. If If $b_i < 0$, then $\Phi(b)^{-1}$ moves b_i subsurfaces from A into B by applying a shift h_{σ_i} exactly b_i times. Similarly, for any $i > 0$ where $a_i < 0$, the map $\Phi(a)$ moves a_i subsurfaces from B into A applying a shift $h_{\sigma_i}^{-1}$ exactly a_i times. If $b_i \geq 0$, then $\Phi(b)^{-1}$ moves b_i subsurfaces from B into A by applying a shift $h_{\sigma_i}^{-1}$ exactly b_i times. If $b_i < 0$, then $\Phi(b)^{-1}$ moves b_i subsurfaces from A into B by applying a shift h_{σ_i} exactly b_i times. That is, the number of surfaces that are sent from $\Sigma_\pm$ to $\Sigma_\mp$ under $\Phi(b)^{-1}\Phi(a)$ is exactly equal to $|a - b|_{\mathbb{Z}^n}$.

When E^G is two sided, the surfaces Σ_i are genus one surfaces with one boundary component. Therefore, sending one of these surfaces from $\Sigma_\pm$ to $\Sigma_\mp$ contributes 2 to the norm defined in Equation (7) and we have

(10)
$$\|\Phi_w(b)^{-1}\Phi_w(a)(\beta)\| = 2|a - b|_{\mathbb{Z}^n}.$$

In the case where $E(z)$ is two-sided, Σ_i has one maximal end which is a point in $E(z)$. Therefore, sending one of these surfaces from $\Sigma_\pm$ to $\Sigma_\mp$ contributes 1 to the norm defined in Equation (5) and we have

(11)
$$\|\Phi_w(b)^{-1}\Phi_w(a)(\beta)\| = |a - b|_{\mathbb{Z}^n}.$$

We now show that Φ is a quasi-isometric embedding from Q^∞ to $\mathrm{FMap}(\Sigma)$.

Proposition 5.2. *Assume $E = E(\Sigma)$ is two-sided and let $\Phi\colon Q^\infty \to \mathrm{FMap}(\Sigma)$ be the map from Equation 9. Then there exists $C > 0$ such that, for every $a, b \in Q^\infty$ we have:*

$$\frac{1}{C} \cdot |a - b|_{\mathbb{Z}^n} \leq \|\Phi_w(a)\Phi_w(b)^{-1}\|_{\mathcal{S}} \leq C \cdot |a - b|_{\mathbb{Z}^n} + 3$$

and hence Φ is a quasi-isometric embedding from $(Q^\infty, |\cdot|_{\mathbb{Z}^n})$ to $(\mathrm{FMap}(\Sigma), \|\cdot\|_{\mathcal{S}})$.

Proof. To obtain inequality on the left, we assume C is larger than the constants in Theorems 4.2 and 4.4. We then combine Equation (10) with Theorem 4.4 in the case where E^G is two-sided and we combine Equation (11) with Theorem 4.2 in the case where $E(z)$ is two-sided.

Recall that $x, y \in E$ are the ends of Σ associated to σ. Let $A \in \mathcal{A}$ be the set containing x and $B \in \mathcal{A}$ be the set containing y. We now argue as in the proof of Theorem 3.5, but in place punctures, we use disjoint subsurfaces Σ_i. We find homeomorphisms $s_1, s_3 \in \nu_B$ and $s_2 \in \nu_A$ such that

$$(s_3 h_\sigma^{k_2} s_2 h_\sigma^{k_1} s_1) \circ \Phi_w(a)\Phi_w(b)^{-1} = id,$$

where k_1 is the number indices i where $a_i = 1$ and $b_i = 0$ and k_2 is the number of indices where $a_i = 0$ and $b_i = 1$. Further assuming that $C \geq \|h_\sigma\|_{\mathcal{S}}$, we have

$$\|\Phi_w(a)\Phi_w(b)^{-1}\|_{\mathcal{S}} \leq C \cdot (k_1 + k_2) + 3 = C \cdot |a - b|_{\mathbb{Z}^n} + 3$$

since the s_i are generators. This finishes the proof. $\square$

Theorem 1.2, restated below, now follows immediately.

Theorem 5.3. *Assume Σ has a two-sided end space. Then $\mathrm{FMap}(\Sigma)$ and $\mathrm{Map}(\Sigma)$ have infinite asymptotic dimension.*

Proof. By Proposition, 5.2, for each $n \in \mathbb{N}$ the map $\Phi\colon \mathbb{Z}^n \to \mathrm{FMap}(\Sigma)$ is a quasi-isometric embedding. Since quasi-isometric embeddings preserve asymptotic dimension by Fact 3.3, and the asymptotic dimension of $\mathbb{Z}^n$ is n by Fact 3.4, we have that for all n, the asymptotic dimension of $\mathrm{FMap}(\Sigma)$ is at least n. This shows that the asymptotic dimension of $\mathrm{FMap}(\Sigma)$ is infinite.

By Theorem 2.7, $\mathrm{FMap}(\Sigma)$ and $\mathrm{Map}(\Sigma)$ are quasi-isometric. Fact 3.3 tells us that asymptotic dimension is preserved under quasi-isometries, and hence $\mathrm{Map}(\Sigma)$ has infinite asymptotic dimension. $\square$

6. Equivalence of Algebraically and Topologically Essential

In this section we prove Theorems 1.5 and 1.6 from the introduction. Let h_σ be a shift map as described in the introduction. Recall that the support of h_σ is a strip σ, containing a collection of sub-surfaces Σ_i, exiting towards points $x, y \in E$. We make use of the following theorem of Rosendal giving an equivalent condition for the notion of coarsely bounded sets that is more suitable for our purposes.

Theorem 6.1 (Rosendal, Prop. 2.7 (5) in [20])**.** *Let A be a subset of a Polish group G. The following are equivalent*

- *A is coarsely bounded.*
- *For every neighborhood ν of the identity in G, there is a finite subset $\mathcal{F}$ and some $k \geq 1$ such that $A \subset (\mathcal{F}\nu)^k$.*

We prove several special cases of Theorem 1.5. We then combine them to give a proof of the general case.

Proposition 6.2. *Assume Σ_i is a surface of genus g with one boundary component. If h_σ is essential, then E^G is two-sided giving a decomposition $E^G = X \sqcup Y$ such that $x \in X$ and $y \in Y$. In particular, if E^G is not two-sided, then h_σ is not essential.*

Proof. Let $A, B \in \mathcal{A}$ be such that $x \in A$ and $y \in B$. There is a homeomorphic copy σ' of σ which is the concatenation of an infinite genus half-strip in Σ_A, a zero genus compact strip in L, and an infinite genus half-strip in Σ_B. Let g be a homeomorphism sending σ to σ'. For $\mathcal{F}$ and ν as in Theorem 6.1, let $\mathcal{F}' = g^{-1} \mathcal{F} g$ and $\nu' = g^{-1} \nu g$. Then

$$h_\sigma^n \in (\mathcal{F}\nu)^k \qquad \Longleftrightarrow \qquad h_{\sigma'}^n \in (\mathcal{F}'\nu')^k.$$

That is, $\langle h_\sigma \rangle$ is CB if and only if $\langle h_{\sigma'} \rangle$ is CB. In fact, in general, any set conjugate to a CB set is CB. Hence, it is enough to prove the Proposition for $h_{\sigma'}$.

Claim: Assume there is no decomposition $E^G = X \sqcup Y$ where $x \in X$, $y \in Y$ and X, Y are closed FMap(Σ)–invariant sets. Then, there is a sequence

$$A = A_0, A_1, \ldots, A_k = B, \qquad A_i \in \mathcal{A}$$

and a sequence of ends $z_i \in E^G$ and $z_i' \in E(z_i)$, $i = 0, \ldots, (k-1)$ such that $z_0 \in A_0 - \mathcal{M}(A_0)$, $z_{k-1}' \in A_k - \mathcal{M}(A_k)$ and, for $i = 1, \ldots, (k-1)$, $z_{i-1}', z_i \in A_i - \mathcal{M}(A_{i-1})$.

To see that this claim holds, let $\mathcal{A}'$ be the subset of $\mathcal{A}$ such that, for $C \in \mathcal{A}'$, there is a sequence as described above starting from A and ending in C. Let $\mathcal{B}' = \mathcal{A} - \mathcal{A}'$. Then, for every $A' \in \mathcal{A}'$, $B' \in \mathcal{B}'$ and $z \in E_{A'} \cap E^G$ we have $E(z) \cap E_{B'} = \emptyset$. Define

$$X = \bigcup_{A' \in \mathcal{A}'} E_{A'} \cap E^G \qquad \text{and} \qquad Y = \bigcup_{B' \in \mathcal{B}'} E_{B'} \cap E^G.$$

The sets X and Y are closed and FMap(Σ) invariant, and the definition of X implies that we have $x \in X$. The assumption implies that y cannot be in Y. Therefore $B \in \mathcal{A}'$ and the above sequence exists, which proves the claim.
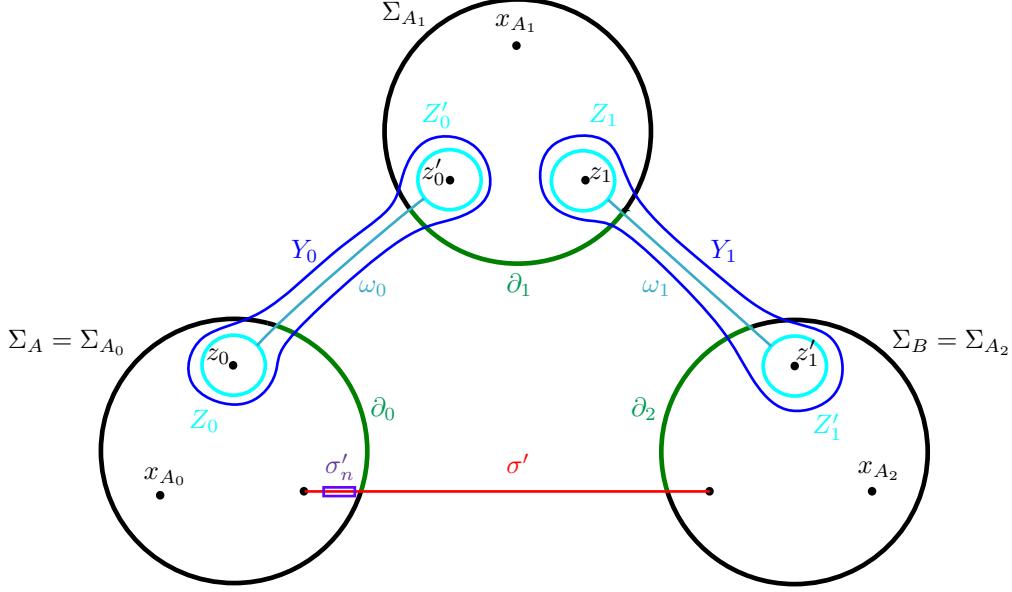

FIGURE 4. This is the picture of the our set up for $k = 2$. The arcs ω_i and ∂_i are chosen so that, the concatenation of $\partial_0, \omega_0|_L, \partial_1, \omega_1|_L$ is homotopic to the concatenation of $\sigma'|_L$ followed by ∂_2.

For $i = 0, \ldots, (k-1)$, fix disjoint sub-surfaces $Z_i \subset \Sigma_{A_i} - x_{A_i}$ and $Z_i' \subset \Sigma_{A_{i+1}} - x_{A_{i+1}}$ such that z_i is an end of Z_i, z_i' is an end of Z_i' and Z_i is homeomorphic to Z_i', for example, we can choose

Z_i so that $E(Z_i)$ is an stable neighborhood of z_i. Connect Z_i to Z_i' by an arc ω_i that is contained in $L \cup \Sigma_{A_i} \cup \Sigma_{A_{i+1}}$. Let Y_i be a regular neighborhood of $Z_i \cup Z_i' \cup \omega_i$. Further, we assume that subsurfaces Y_i are disjoint from each other and are disjoint from the strip σ' (see Figure 4). For $i = 1, \ldots (k-1)$, let ∂_i be a sub-arc of $\partial \Sigma_{A_i}$ connecting $\omega_{i-1} \cap \partial \Sigma_{A_i}$ to $\omega_i \cap \partial \Sigma_{A_i}$. Also, let $\omega_i|_L$ be the restriction of ω_i to L. We further assume that the concatenation of $\omega_i|_L$ and ∂_i, which is an arc connecting $\partial \Sigma_A$ to $\partial \Sigma_B$, is homotopic (relative $\partial \Sigma_A \cup \partial \Sigma_B$) to either arc in the the restriction of $\partial \sigma'$ to L, which we denote by $\partial \sigma'|_L$. Let ∂_0 be the sub-arc of $\partial \Sigma_A$ connecting a point in $\partial \sigma' \cap \partial \Sigma_A$ to the points $\omega_0 \cap \partial \Sigma_A$ and similarly let ∂_k be the sub-arc of $\partial \Sigma_B$ connecting a point in $\partial \sigma' \cap \partial \Sigma_B$ to the points $\omega_{k-1} \cap \partial \Sigma_B$. Again, these can be chosen such that the concatenation $(\partial_i \omega_i|_L)_{i=0}^{(k-1)}$ is homotopic to the concatenation of an arc in $\partial \sigma'|_L$ and ∂_k relative their end points.

Let g_i be a homeomorphism that sends Z_i to Z_i', whose support is in Y_i and that sends ω_i to itself in the reverse direction. Let $\mathcal{F} = \{g_i\}_{i=0}^{(k-1)}$.

We now show that, for every $n \in \mathbb{Z}$, $h_{\sigma'}^n \in (\nu_L \mathcal{F})^{k+1}$. Let σ_n' be a sub-strip of σ' of genus n so that the genus of what remains between σ_n' and ∂A is zero. Choose an element $f_0 \in \nu_{A_0}$ that sends σ_n' to a Z_0, which is possible since $z_0 \in E^G$ and therefore Z_0 has infinite genus. In fact, we can do this in a way such that $f_0(\sigma') \cap \Sigma_A$ starts the same as $\sigma' \cap \Sigma_A$ away from a small neighborhood of $\partial \Sigma_A$, then it follows ∂_0 (staying in a small neighborhood of $\partial \Sigma_A$) to the intersection point of $\partial \Sigma_A$ and $\omega_0 \cap \Sigma_A$, then follows $\omega_0 \cap \Sigma_A$ (staying in Y_0) to Z_0, then comes back the same way along $\omega_0 \cap \Sigma_A$ and ∂_0 and continues along σ' into B (see the left picture in Figure 5). Then, $g_0 \circ f_0(\sigma_n') \subset Z_0'$.
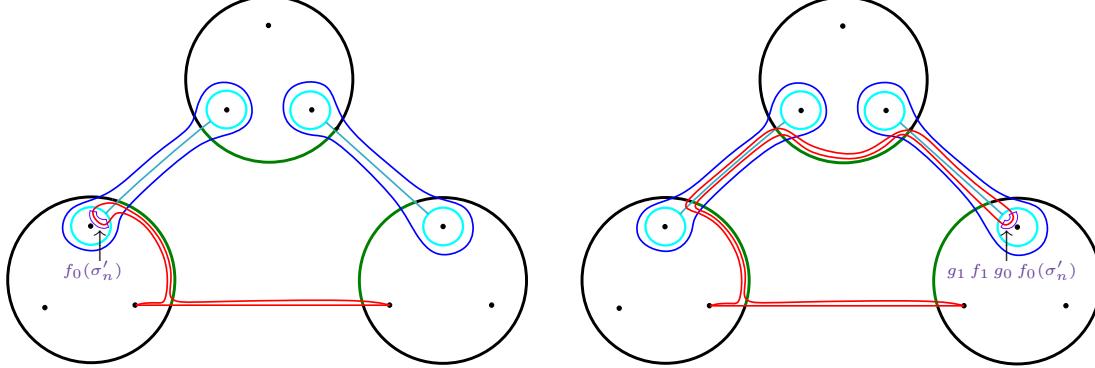

FIGURE 5. The picture on the left depicts the image of σ' and σ_n' under the map f_0. The picture on the right depicts the image of σ' and σ_n' under the map $\prod_{i=0}^{k-1} g_i \circ f_i(\sigma')$. The strip $\prod_{i=0}^{k-1} g_i \circ f_i(\sigma')$ can be homotoped to σ' inducing the shift map $h_{\sigma'}^n$.

We then find a $f_1 \in \nu_{A_1}$ that sends $g_0 \circ f_0(\sigma_n')$ to Z_1. We can do this in a way such that $f_1 \circ g_0 \circ f_0(\sigma')$ starts the same as σ' inside Σ_A (away from a small neighborhood of $\partial \Sigma_A$) then it follows ∂_0, then $\omega_0|_L$, then ∂_1 to the intersection point of $\partial \Sigma_{A_1}$ and ω_1, then it follows $\omega_1 \cap \Sigma_{A_1}$ (staying in Y_1) to Z_1, then comes back the same way along $\omega_1 \cap \Sigma_{A_1}$, ∂_1, $\omega_0|_L$ and ∂_0, and then follows σ' into B. We then apply g_1 and we have $g_1 \circ f_1 \circ g_0 \circ f_0(\sigma_n') \subset Z_1'$.

Following the same argument, we can find f_i, $i = 0, \ldots, (k-1)$ such that

$$\left(\prod_{i=0}^{k-1} g_i \circ f_i \right) (\sigma_n') \subset Z_{k-1}' \subset \Sigma_{A_k} = \Sigma_B.$$

The strip $\prod g_i \circ f_i(\sigma')$ starts the same as σ' in Σ_A, then follows the concatenation of the arcs $\omega_i|_L$ and ∂_i (which by assumption is homotopic to arcs $\partial \sigma'|_L$), then follows $\omega_k \cap \Sigma_B$ to Z_k, then back the same way to $\partial \Sigma_A \cap \sigma'$, and then continues the same as σ' (see the right picture in Figure 5).

The portion of $\prod_{i=0}^{k-1} g_i \circ f_i(\sigma'_n)$ traveling back from $\partial \Sigma_B$ to $\partial \Sigma_A$ has genus zero and can be homotoped into Σ_B. The portion going forward from $\partial \Sigma_A$ to $\partial \Sigma_B$ also has genus zero and can be homotoped to a neighborhood of the concatenation of $\partial \sigma'|_L$ and ∂_k. Hence, there is $f_k \in \nu_B$ such that $f_k \circ \prod g_i \circ f_i(\sigma')$ can be homotoped into σ' and hence $f_k \circ \prod g_i \circ f_i$ can be considered as a map with support in σ'. Since, σ'_n has moved from A to B, we in fact have $f_k \circ \prod g_i \circ f_i = h^n_{\sigma'}$. That is $h^n_{\sigma'} \in (\nu_L \mathcal{F})^{k+1}$. This finishes the proof. $\qquad \square$

Proposition 6.3. *Assume Σ_i is a surface with either zero or infinite genus and where $E(\Sigma_i)$ has a single maximal point z. If h_σ is essential, then $E(z)$ is two-sided giving the decomposition* $\mathrm{Accu}(x) = X \sqcup Y$ *where $x \in X$ and $y \in Y$.*

Proof. The proof is nearly identical to the proof of Proposition 6.2. We outline it here. Let x, y be the ends of Σ associated to the strip σ. Let $A, B \in \mathcal{A}$ be such that $x \in A$ and $y \in B$. First find $g \in \mathrm{FMap}(\Sigma)$ such that $\sigma' = g(\sigma)$ starts in Σ_A, continues in L, and then enters Σ_B. It is enough to to show that the group generated by $h_{\sigma'} = g h_\sigma g^{-1}$ is CB.

Note that $x, y \in \mathrm{Accu}(z)$. We can find a sequence

$$A = A_0, A_1, \ldots, A_k = B, \qquad A_i \in \mathcal{A}$$

and a sequence of ends $z_i \in \mathrm{Accu}(z)$ and $z'_i \in E(z_i)$, $i = 0, \ldots, (k-1)$ such that $z_0 \in A_0 - \mathcal{M}(A_0)$, $z'_{k-1} \in A_k - \mathcal{M}(A_k)$ and, for $i = 1, \ldots, (k-1)$, $z'_{i-1}, z_i \in A_i - \mathcal{M}(A_{i-1})$. A similar argument to the proof of Proposition 6.2 shows that if such a sequence does not exist, then $E(z)$ is two-sided which would be a contradiction.

We then construct sub-surfaces Z_i and Y_i and maps g_i as before, and set $\mathcal{F} = \{g_i\}_{i=0}^{k-1}$. Let σ'_n be a sub-strip of σ' that contains the n sub-surfaces Σ_i that are nearest to $\partial \Sigma_A$. As in the proof of Proposition 6.2, we can choose $f_i \in \nu_{A_i}$ such that $f_k \prod g_i \circ f_i$ sends σ' to the strip that is homotopic to σ' and moves σ_n from Σ_A to Σ_B. That is $h^n_{\sigma'} = f_k \prod g_i \circ f_i \in (\nu_L \mathcal{F})^{k+1}$. This finishes the proof. $\qquad \square$

Proposition 6.4. *Assume Σ_i is a surface with either zero or infinite genus, and where $\mathcal{M}(E(\Sigma_i))$ is a Cantor set. Then h_σ is not essential.*

Proof. As before, after conjugation, we can assume σ intersects only $\Sigma_A \cup L \cup \Sigma_B$, where $\Sigma_i \in \Sigma_A$ for $i \le 0$ and $\Sigma_i \in \Sigma_B$ for $i > 0$. Consider a simple closed curve in the strip σ separating Σ_i and Σ_{i+1} from the rest of the strip. Denote this subsurface by $\Sigma_{i,i+1}$. If the set of maximal ends in Σ_i is a Cantor set, the end space of Σ_i is homeomorphic to the end space of $\Sigma_{i,i+1}$ (see Figure 6).
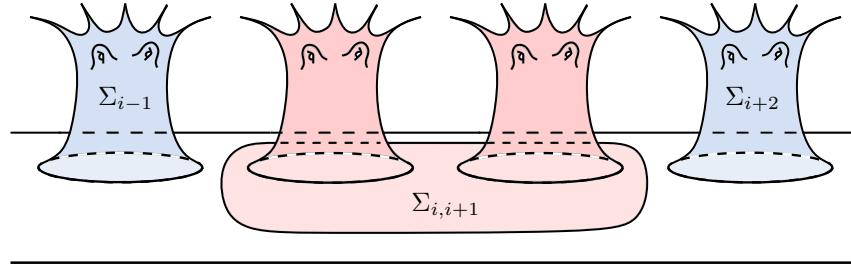

FIGURE 6. When $\mathcal{M}(E(\Sigma_i))$ is a cantor set the surface $\Sigma_{i,i+1}$ is homeomorphic to Σ_i.

Similarly, the subsurface $\Sigma_{i,i+n}$, which is a subsurface of σ with one boundary component containing the sub-surfaces $\Sigma_i, \Sigma_{i+1}, \ldots, \Sigma_{i+n}$, is also homeomorphic to Σ_i. Hence, for every n, there is a homeomorphism $f_n \in \nu_A$ with support in σ such that Σ_i is sent to Σ_{i+n} for $i < -n$, Σ_i is sent to itself for $i > 0$, and the surface $\Sigma_{-n,0}$ is sent to Σ_0. Similarly, there there is a homeomorphism $g_n \in \nu_A$ with support in σ such that Σ_i is sent to Σ_{i+n} for $i > 1$, Σ_i is sent

to itself for $i \leq 0$, and the surface Σ_1 is sent to $\Sigma_{1,n+1}$. Then $h_\sigma^n = g_n h_\sigma f_n$. That is, we can collect the surfaces $\Sigma_{-n} \ldots \Sigma_0$ to the surface Σ_0, shift σ once and the expand these surfaces to $\Sigma_1, \ldots, \Sigma_{n+1}$, which means we have written h_σ^n as a composition of 3 homeomorphisms. By letting $\mathcal{F} = \{h_\sigma\}$, we have that $\langle h_\sigma \rangle$ is contained in $(\nu_L \mathcal{F})^2$. Therefore, by Theorem 6.1, $\langle h_\sigma \rangle$ is a CB subset of $\mathrm{FMap}(\Sigma)$ and h_σ is not essential. $\qquad\square$

We are now in a position where we can prove Theorem 1.5, which states that $\mathrm{Map}(\Sigma)$ contains an essential shift if and only if the endspace of Σ is two-sided.

Proof of Theorem 1.5. One direction is already proven by Proposition 5.1. It remains to show that, if E is not two-sided then there is no essential shift map.

Assume E^G is not two-sided and there does not exist any $z \in E$ such that $E(z)$ is two-sided. Let h_σ be any shift map with support on a strip σ, containing the subsurfaces Σ_i, exiting towards $x, y \in E$. The end space $E(\Sigma_i)$ has finitely many maximal types each of which is either a Cantor set or a finite set. Therefore we can find finitely many sub-surfaces $\Sigma_i^j \subset \Sigma_i$, $j = 0, \ldots, \ell$, each with one boundary component such that:

(1) The surfaces Σ_i^j are disjoint. Furthermore, $\Sigma - \Sigma_i^j$ is a compact planar surface which means $E(\Sigma_i) = \sqcup_j E(\Sigma_i^j)$.

(2) For $j > 0$, Σ_i^j has zero genus or infinite genus. That is, if Σ_i has finite genus we include all the genus in the surface Σ_i^0 so every other subsurface of Σ_i contains no genus. If Σ_i already has zero genus or infinite genus, then there is no need to choose Σ_i^0. We can choose Σ_i^0 to be a disk or have j range from 1 to ℓ. Also, if $E(\Sigma_i^j) \cap E^G = \emptyset$ then we make sure Σ_i^j has genus zero.

(3) For every $j = 0, \ldots, \ell$, the surfaces Σ_i^j are homeomorphic for all $i \in \mathbb{Z}$. That is we decompose each Σ_i in the same way.

Now the strip σ can be decomposed to parallel strips σ^j each containing sub-surfaces Σ_i^j. Since h_{σ^j} have disjoint support, they commute and $h_\sigma = \prod h_{\sigma^j}$. That is, the group generated by h_{σ_j}, $j = 0, \ldots, \ell$, is an abelian group that contains the group generated by h_σ. Hence, if each $\langle h_{\sigma^j} \rangle$ is CB then $\langle h_{\sigma^0, \ldots, h_{\sigma^\ell}} \rangle$ is also CB and thus $\langle h_\sigma \rangle$ is CB.

Since E^G is not two-sided, by Proposition 6.2, h_{σ^0} is not essential and $\langle h_{\sigma^0} \rangle$ is CB. For $j = 1, \ldots, \ell$, if $\mathcal{M}(E(\Sigma_i^j))$ is a single point then $\langle h_{\sigma^0} \rangle$ is CB by Proposition 6.3. If $\mathcal{M}(E(\Sigma_i^j))$ is a Cantor set, then $\langle h_{\sigma^0} \rangle$ is CB by Proposition 6.4. This finishes the proof. $\qquad\square$

Finally, we prove Theorem 1.6 which tells us that a shift map is essential if and only if there is either a decomposition of the ends accumulated by genus, or there is a decomposition of the accumulation set of a maximal point of the surface.

Proof of Theorem 1.6. We begin by proving the forward direction. Consider the decomposition of the shift map $h_\sigma = \prod_j h_{\sigma^j}$ constructed in the proof of Theorem 1.5. As mentioned before, if h_σ is essential then some h_{σ_j} is essential. If h_{σ_0} is essential, then since Σ_i^0 contains finite genus, Proposition 6.2 implies the first bullet point in Theorem 1.6 holds. If $j > 0$ and $\mathcal{M}(E(\Sigma_i^j))$ is a single point, then by Proposition 6.3 the second bullet point in Theorem 1.6 holds. Finally, $\mathcal{M}(E(\Sigma_i^j))$ cannot be a Cantor set by Proposition 6.4, which proves the forward direction.

In the other direction, first suppose that E^G is two-sided giving a decomposition $E^G = X \sqcup Y$ where $x \in X$ and $y \in Y$, then the proof of Proposition 5.1 shows that h_{σ^0} is essential. In fact, there is a length function $\|\cdot\|$ such that, for $n > 0$, $\|h_{\sigma^0}^n\| = 2n$. Since, in this case, Σ_i^j have genus zero for $j > 0$, the shift maps h_{σ^j} act trivially on homology. Hence $\|h_\sigma^n\| = 2n$ and hence h_σ is essential.

Similarly, if there exists some $z \in \mathcal{M}(\Sigma_i)$ such that $E(z)$ is two-sided giving a decomposition $\mathrm{Accu}(z) = X \sqcup Y$ where $x \in X$ and $y \in Y$, then the proof of Proposition 5.1 shows that h_{σ^j} is

essential for some $j > 0$. In fact, there is a length function $\|\cdot\|$ such that, for $n > 0$, $\|h_{\sigma^0}^n\| = n$. It is possible that the Σ_i^j are homeomorphic for several different j. We can assume, after re-indexing, that $\Sigma_i^1, \ldots, \Sigma_i^{\ell'}$ all have same type of countable maximal end. Then $\|h_\sigma^n\| = \ell' \cdot n$ and hence h_σ is essential. $\square$

7. Two sources of non-trivial topology

In this section, we prove Theorem 1.7 which states that if Σ does not have a non-displaceable subsurface and $\mathrm{FMap}(\Sigma)$ does not have an essential shift then $\mathrm{Map}(\Sigma)$ and $\mathrm{FMap}(\Sigma)$ are CB. This implies that $\mathrm{Map}(\Sigma)$ and $\mathrm{FMap}(\Sigma)$ are quasi-isometric to a point and therefore do not have an interesting geometry. To prove this theorem, we make use of the notion of an avenue surface first introduced in [11]. Recall that we always assume that Σ is stable.

Definition 7.1. An *avenue surface* is a connected orientable surface Σ which does not contain any non-displaceable subsurface of finite type, and whose mapping class group $\mathrm{Map}(\Sigma)$ is CB-generated but not CB.

That is, the only possible examples for surfaces that have no non-displaceable subsurfaces, no essential shifts, and are not CB are the avenue surfaces. A topological description of avenue surfaces was given in [11].

Lemma 7.2 (Lemmas 4.5 and 4.6 in [11])**.** *Let Σ be an avenue surface. Then Σ has either 0 or infinite genus, and Σ has exactly two ends of maximal type, that is, $\mathcal{M}(E) = \{x_1, x_2\}$. Furthermore, for every $x \in E - \{x_1, x_2\}$, the set $E(x)$ accumulates to both x_1 and x_2.*

In order to state the classification of CB mapping class groups from [13], we must first define the notion of self-similarity for a space of ends, and the notion of telescoping for an infinite-type surface.

Definition 7.3. A space of ends (E, E^G) is said to be *self-similar* if for any decomposition $E = E_1 \sqcup E_2 \sqcup \ldots \sqcup E_n$ of E into pairwise disjoint clopen sets, there exists a clopen set D in some E_i such that $(D, D \cap E^G)$ is homeomorphic to (E, E^G).

A surface Σ is *telescoping* if there are ends $x_1, x_2 \in E$ and disjoint clopen neighborhoods V_i of x_i in Σ such that for all clopen neighborhoods $W_i \subset V_i$ of x_i, there exist homeomorphisms $f_i, h_i \in \mathrm{FMap}(\Sigma)$, with

$$f_i(W_i) \supset (\Sigma - V_{3-i}) \qquad h_i(W_i) = Vi, \qquad \text{and} \qquad h_i(V_{3-i}) = V_{3-i}.$$

A classification of CB mapping class groups was given in [13].

Theorem 7.4 (Theorem 1.7 in [13])**.** *The group $\mathrm{Map}(\Sigma)$ is CB if and only if Σ has infinite or zero genus and E is either self-similar or telescoping.*

We are now ready to prove Theorem 1.7 restated below.

Theorem 1.7. (Two sources of non-trivial geometry). *If $\mathrm{Map}(\Sigma)$ does not have an essential shift and Σ does not contain a non-displaceable subsurface then $\mathrm{Map}(\Sigma)$ is quasi-isometric to a point.*

Proof. By way of contradiction, suppose that Σ does not have a non-displaceable subsurface, $\mathrm{FMap}(\Sigma)$ does not contain an essential shift, and that $\mathrm{FMap}(\Sigma)$ is not CB. Then by assumption, Σ is an avenue surface. By Lemma 7.2, $\mathcal{M}(E) = \{x_1, x_2\}$ and for any other $x \in E - \{x_1, x_2\}$, $E(z)$ accumulates to both x_1 and x_2. Consider an end $z \in \mathcal{M}(E - \{x_1, x_2\})$, that is, an end that is maximal in $E - \{x_1, x_2\}$. If $E(z)$ is countable, then E is two-sided with $X = \{x_1\}$ and $Y = \{x_2\}$. This cannot happen since we are assuming there are no essential shifts (see Theorem 1.5). Therefore, $E(z)$ is a Cantor set.

Furthermore, either Σ has genus zero or it is infinite genus and $\{x_1, x_2\} \subset E^G$ (otherwise, $\mathrm{Map}(\Sigma)$ is not even locally CB, see [13, Theorem 1.4]). We now show that these assumptions imply that Σ is telescoping and, by Theorem 7.4, $\mathrm{Map}(\Sigma)$ is CB, which will prove our claim.

We check the definition of telescoping. For $i = 1, 2$, let V_i be disjoint stable neighborhoods of x_i such that $E - (V_1 \cup V_2)$ contains every maximal type in $E - \{x_1, x_2\}$. Let W_i be the given smaller neighborhoods of x_i. Since the V_i are stable neighborhoods, we have that V_i is homeomorphic to W_i. We claim that $E - (V_1 \cup V_2)$ is homeomorphic to $E - (V_1 \cup W_2)$. Since all maximal types in $E - \{x_1, x_2\}$ are present in $E - (V_1 \cup V_2)$, the sets $E - (V_1 \cup V_2)$ and $E - (V_1 \cup W_2)$ have the same maximal types. But all these types are Cantor sets. Therefore, for every $y \in V_2 - W_2$ there is a $z \in E - (V_1 \cup V_2)$ such that $E(y)$ accumulates to z. By [13, Lemma 4.18], there are small neighborhoods U_y of y and U_z of z such that $U_y \cup U_z$ is homeomorphic to U_z. The neighborhoods U_y give a covering of $V_2 - W_2$, hence there is a finite sub-covering. By making the neighborhoods smaller, we can assume that they are disjoint. Since each neighborhood U_y can be absorbed into U_z, the set $V_2 - W_2$ can be absorbed into $E - (V_1 \cup V_2)$ and hence, $E - (V_1 \cup V_2)$ is homeomorphic to $E - (V_1 \cup W_2)$.

Now consider surfaces Σ_{V_i} and Σ_{W_i} whose end points are V_i and W_i. We can choose these surfaces so that $\Sigma_{W_i} \subset \Sigma_{V_i}$ and so that Σ_{V_i} is disjoint from $\Sigma_{V_{3-i}}$. Also, we can assume $\Sigma_{V_i} - \Sigma_{W_i}$ and $\Sigma - (\Sigma_{W_1} \cup \Sigma_{W_2})$ all have genus zero or infinity. The end space of $\Sigma_{V_i} - \Sigma_{W_i}$ is homeomorphic to the end space of $\Sigma - (\Sigma_{W_1} \cup \Sigma_{W_2})$ and they have the same genus. Also Σ_{V_1} and Σ_{W_1} have homeomorphic end spaces and the genus. Hence, there is a map h_1 such that $h_1(\Sigma_{W_1}) = \Sigma_{V_1}$ and $h_1(\Sigma_{V_2}) = h_1(\Sigma_{V_2})$. The map h_2 in Definition 7.3 can also be similarly constructed.

The construction of the maps f_i follows similarly. As above, we can send Σ_{W_1} to $\Sigma - \Sigma_{V_2}$ while fixing some neighborhood W_2' of x_2 as long as W_2' is small enough so that $V_2 - W_2'$ intersect $E(z)$ for every $z \in \mathcal{M}(E - \{x_1, x_2\})$. This proves that Σ is telescoping and hence $\mathrm{Map}(\Sigma)$ is CB. Note that since $\mathrm{FMap}(\Sigma)$ and $\mathrm{Map}(\Sigma)$ are quasi-isometric this means $\mathrm{FMap}(\Sigma)$ is also CB which proves the theorem. $\square$

References

[1] J. Behrstock, B. Kleiner, Y. Minsky, and L. Mosher. Geometry and rigidity of mapping class groups. *Geometry & Topology*, 16(2):2012, 2012.

[2] Gregory Bell and Alexander Dranisnhikov. Asymptotic dimension in Będlewo. *Topology Proceedings*, 38:209–236, 2011.

[3] Gregory Bell and Koji Fujiwara. The asymptotic dimension of a curve graph is finite. *Journal of the London Mathematical Society*, 77(1):33–50, 2008.

[4] Mladen Bestvina, Kenneth Bromberg, and Koji Fujiwara. Constructing group actions on quasi-trees and applications to mapping class groups. *Publications mathématiques de l'IHÉS*, 122:1–64, 06 2010.

[5] Brian H. Bowditch. Coarse median spaces and groups. *Pacific Journal of Mathematics*, 261:53–93, 2013.

[6] Brian H. Bowditch. Large-scale rigidity properties of the mapping class groups. *Pacific Journal of Mathematics*, 293:1–73, 2018.

[7] A. Eskin, K. Rafi, and H. Masur. Large scale rank of Teichmüller space. *Duke Mathematical Journal*, 166(8):1517–1572, 2017.

[8] Federica Fanoni, Tyrone Ghaswala, and Alan McLeay. Homeomorphic subsurfaces and the omnipresent arcs, 2020. arXiv:2003.04750.

[9] Mikhail Gromov. Geometric group theory, vol. 2: Asymptotic invariants of infinite groups. In *London Mathematical Society Lecture Note Series, Volume 182*, pages vii + 295. Cambridge University Press, Cambridge, England, 1993.

[10] Ursula Hamenstädt. Geometry of the mapping class groups III: Quasi-isometric rigidity, 2007. arXiv:0512429.

[11] Camille Horbez, Yulan Qing, and Kasra Rafi. Big mapping class groups with hyperbolic actions: classification and applications, May 2020. arXiv:2005.00428.

[12] Lizhen Ji. Asymptotic dimension and the integral K-theoretic Novikov conjecture for arithmetic groups. *Journal of Differential Geometry*, 68(3):535–544, 2005.

[13] Katie Mann and Kasra Rafi. Large scale geometry of big mapping class groups. arXiv:1912.10914.

[14] H. Masur and Y. Minsky. Geometry of the complex of curves i: Hyperbolicity. *Inventiones mathematicae*, 138(1):103–149, 1999.

[15] H. Masur and Y. Minsky. Geometry of the complex of curves ii: Hierarchical structure. *Geometric and Functional Analysis*, 10(4):902–974, 2000.

[16] Denis Osin. Asymptotic dimension of relatively hyperbolic groups. *International Mathematics Research Notices*, 133(9):2143–2161, 2005.

[17] Priyam Patel and Nicholas Vlamis. Algebraic and topological properties of big mapping class groups. *Algebraic & Geometric Topology*, 18(7):4109–4142, 2018.

[18] Ian Richards. On the classification of noncompact surfaces. *Transactions of the American Mathematical Society*, 106:259–269, 1963.

[19] John Roe. Hyperbolic groups have finite asymptotic dimension. *Proceedings of the American Mathematical Society*, 133(9):2489–2490, 2005.

[20] Christian Rosendal. Coarse Geometry of Topological Groups. Book Manuscript. http://homepages.math.uic.edu/ rosendal/PapersWebsite/Coarse-Geometry-Book17.pdf.

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