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RESEARCH CONTRIBUTION

Centralizers in Mapping Class Groups and Decidability of Thurston Equivalence

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Abstract
We find a constructive bound for the word length of a generating set for the centralizer of an element of the Mapping Class Group. As a consequence, we show that it is algorithmically decidable whether two postcritically finite branched coverings of the sphere are Thurston equivalent.

Keyword Mapping class group, Thurston equivalence, Centralizer, Algorithmic decidability

1 Introduction

The study of Thurston maps is central to Complex Dynamics in one variable. A Thurston map is an orientation-preserving branched covering f of the 2-sphere whose branch points have finite orbits; it can be described in a combinatorial language, for example, by introducing a triangulation of S^2 whose set of vertices includes the critical orbits of f. Thurston defined a natural *combinatorial equivalence* relation, also known as *Thurston equivalence*. It is a fundamental question if, given combinatorial descriptions of two Thurston maps f and g, it is *algorithmically decidable* whether they are Thurston equivalent. This question is answered in the affirmative in the present paper.

Let us briefly outline the history of the problem and the strategy of the proof. The story began with the the result of Thurston [6] which described, in a topological language, which Thurston maps with hyperbolic orbifolds are combinatorially equivalent to rational maps of $\hat{\mathbb{C}}$. In the case when an equivalent rational map exists, it is essentially unique. Such a rational function should be seen as a *canonical geometrization* of the Thurston map f. Given two maps f and g which are thus geometrizable, it can be algorithmically decided whether they are equivalent by comparing the coefficients of the corresponding geometrizations (see [2]).

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According to Thurston's Theorem [6], a *Thurston obstruction* for the existence of a Thurston equivalent rational map is a finite collection of simple closed curves, f-invariant up to homotopy, with certain topological properties. Due to work of Pilgrim [14], an obstructed Thurston map could be thought of as a combination of several Thurston maps. There exists a *canonical Thurston obstruction*, which decomposes S^2 into two-spheres, connected by annuli. The two-spheres in this decomposition are permuted by the dynamics of f, and eventually fall into periodic cycles. First-return maps of such cycles are either Thurston maps or homeomorphisms, and form the *canonical decomposition* of f.

In [16], it has been shown that there exists an algorithm that finds a canonical decomposition of an obstructed Thurston map as well as a canonical geometrization of all cycles in that decomposition (see Theorem 7.1 for a precise statement). In the present article, we use this constructive canonical geometrization of a Thurston map to build an algorithm solving the Thurston equivalence problem.

A prototype of the algorithm has been already presented in [16] where decidability of Thurston equivalence has been shown for a subclass of Thurston maps that are only allowed to have maps with hyperbolic orbifolds in their canonical decompositions. This restriction significantly simplifies the problem as the group of self-equivalences of an unobstructed Thurston map with a hyperbolic orbifold is trivial (which follows from the fact that an equivalence between two unobstructed Thurston maps with hyperbolic orbifolds is unique [6]).

However, in the general case, there exist non-trivial self-equivalence groups for the maps forming the canonical decomposition. Our principal result is a complexity bound on a generating set of the group of self-equivalences in the case when a first return map on a component in the canonical decomposition of a Thurston map is a homeomorphism, that is, a bound on a generating set of centralizers of elements of the Mapping Class Group. We prove the following theorem, which is of an independent interest:

Theorem 1.1 *For every element ϕ of the Mapping Class Group, the centralizer of ϕ has a generating set where every element has a word length that is bounded by a uniform multiple of the word length of ϕ.*

Armed with this statement, we constructively characterize the generators of all of the self-equivalence groups involved in the canonical decomposition. This allows us to replace a countable search for a Thurston equivalence between two decomposed maps to solving a finite number of linear problems. In this way, we obtain:

Theorem 1.2 *There exists an algorithm which for any two Thurston maps f and g outputs an equivalence ϕ if f and g are equivalent, and outputs* ***maps are not equivalent*** *otherwise.*

We note that a different approach to the problem of algorithmically verifying Thurston equivalence has been studied in [3].

2 Background

2.1 Thurston Maps

All maps considered in the present article are assumed to be orientation-preserving. Let $f : S^2 \to S^2$ be a branched covering self-map of the two-dimensional topological sphere. We define the *postcritical set* P_f by

$$P_f := \bigcup_{n>0} f^{\circ n}(\Omega_f),$$

where Ω_f is the set of critical points of f. When the postcritical set P_f is finite, we say that f is *postcritically finite*.

A *(marked) Thurston map* is a pair (f, Q_f) where $f : S^2 \to S^2$ is a postcritically finite ramified covering of degree at least 2 and Q_f is a finite collection of marked points $Q_f \subset S^2$ which contains P_f and is f-invariant: $f(Q_f) \subset Q_f$. In particular, all elements of Q_f are pre-periodic for f.

Thurston equivalence. Two marked Thurston maps (f, Q_f) and (g, Q_g) are *Thurston (or combinatorially) equivalent* if there are homeomorphisms $\phi_0, \phi_1 : S^2 \to S^2$ such that

(1) the maps ϕ_0, ϕ_1 coincide on Q_f, send Q_f to Q_g and are isotopic rel Q_f;
(2) the diagram

$$\begin{array}{ccc} S^2 & \xrightarrow{\phi_1} & S^2 \\ \downarrow f & & \downarrow g \\ S^2 & \xrightarrow{\phi_0} & S^2 \end{array}$$

commutes.

We will call (ϕ_0, ϕ_1) an *equivalence pair*.

Let Q be a finite collection of points in S^2. Recall that a simple closed curve $\gamma \subset S^2 - Q$ is *essential* if it does not bound a disk, is *non-peripheral* if it does not bound a punctured disk.

Definition 2.1 A *multicurve* Γ on (S^2, Q) is a set of disjoint, nonhomotopic, essential, non-peripheral simple closed curves on $S^2 \backslash Q$. Let (f, Q_f) be a Thurston map, and set $Q = Q_f$. A multicurve Γ on $S \backslash Q$ is *f-stable* if for every curve $\gamma \in \Gamma$, each component α of $f^{-1}(\gamma)$ is either trivial (meaning inessential or peripheral) or homotopic rel Q to an element of Γ.

Definition 2.2 A *Levy cycle* is a multicurve

$$\Gamma = \{\gamma_0, \gamma_1, \ldots, \gamma_{n-1}\}$$

such that each γ_i has a nontrivial preimage γ_i', where the topological degree of f restricted to γ_i' is 1 and γ_i' is homotopic to $\gamma_{(i-1) \mod n}$ rel Q. A Levy cycle is *degenerate* if each γ_i has a preimage γ_i' as above such that γ_i' bounds a disk D_i and the restriction of f to D_i is a homeomorphism and $f(D_i)$ is homotopic to $D_{(i+1) \mod n}$ rel Q.

To any multicurve is associated its *Thurston linear transformation* $f_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$, best described by the following transition matrix:

$$M_{\gamma\delta} = \sum_\alpha \frac{1}{\deg(f : \alpha \to \delta)},$$

where the sum is taken over all the components α of $f^{-1}(\delta)$ which are isotopic rel Q to γ. Since this matrix has nonnegative entries, it has a leading eigenvalue $\lambda(\Gamma)$ that is real and nonnegative (by the Perron–Frobenius theorem).

The celebrated Thurston's theorem [6] is the following:

Thurston's theorem *Let $f : S^2 \to S^2$ be a marked Thurston map with hyperbolic orbifold. Then f is Thurston equivalent to rational function g with a finite set of marked pre-periodic orbits if and only if $\lambda(\Gamma) < 1$ for every f-stable multicurve Γ. The rational function g is unique up to conjugation by an automorphism* of $\mathbb{P}^1$.

In view of this, an f-stable multicurve Γ with $\lambda(\Gamma) \geq 1$ is called a *Thurston obstruction*.

In [16], the second and third authors obtained a similar statement for Thurston maps with parabolic orbifolds (here by a hyperbolic matrix we mean a matrix that does not have eigenvalues with absolute value 1):

Theorem 2.3 *Let f be a Thurston map with postcritical set P and marked set $Q \supset P$ such that the associated orbifold is parabolic and the associated matrix is hyperbolic. Then either f is equivalent to a quotient of an affine map or f admits a degenerate Levy cycle.*

Furthermore, in the former case the affine map is defined uniquely up to affine conjugacy.

3 Centralizers of Elements in the Mapping Class Group

Let ϕ be an element of the mapping class group $\mathrm{Mod}(S)$ of a surface S of finite type. Let

$$C(\phi) = \left\{ \psi \in \mathrm{Mod}(S) \,\middle|\, \psi\,\phi = \phi\,\psi \right\}$$

be the centralizer of ϕ in $\mathrm{Mod}(S)$. Fix a generating set $\mathcal{G}$ for $\mathrm{Mod}(S)$ and let $\|\cdot\|_\mathcal{G}$ denote the word length with respect to this generating set. For $M > 0$ define

$$C(\phi, M) = \left\{ \psi \in C(\phi) \,\middle|\, \|\psi\|_\mathcal{G} \leq M \right\}.$$

In this section, we prove the following version of Theorem 1.1:

Theorem 3.1 *There is a constant M_0, depending on S and the generating set $\mathcal{G}$, so that for every $\phi \in \mathrm{Mod}(S)$, $C\big(\phi, M_0\|\phi\|_{\mathcal{G}}\big)$ generates $C(\phi)$.*

A computational consequence of the above theorem is the following:

Corollary 3.2 *There is an algorithm which, given $\phi \in \mathrm{Mod}(S)$, outputs a set of generators of $C(\phi)$.*

3.1 Some Tools

Our main tool is the following theorem of J. Tao.

Theorem 3.3 [17] *For any fixed generating set $\mathcal{G}$ for $\mathrm{Mod}(S)$, there exists a constant K, such that if $\phi, \phi' \in \mathrm{Mod}(S)$ are conjugate, then there is a conjugating element η with*

$$\|\eta\|_{\mathcal{G}} \leq K \max(\|\phi\|_{\mathcal{G}} + \|\phi'\|_{\mathcal{G}}).$$

Let us introduce the following notations: $a \asymp_C b$, will mean $a < Cb + C$ and $b < Ca + C$, and $a \overset{*}{\prec} b$ will mean $a < Nb$ for some fixed N, and $a \overset{*}{\asymp} b$ will mean $a \overset{*}{\prec} b$ and $b \overset{*}{\prec} a$.

We will also need the Masur–Minsky distance formula [13]. For every subsurface $R \subset S$, they define a measure of complexity between two curve systems $d_R(\cdot, \cdot)$ called the *subsurface projection distance* (see [13] for more details).

Theorem 3.4 *For any generating set $\mathcal{G}$, any marking μ_0, and any threshold k that is sufficiently large, there is a uniform constant C so that, for any $\eta \in \mathrm{Mod}(S)$, we have*

$$\|\eta\|_{\mathcal{G}} \asymp_C \sum_{R \subset S} \Big[d_R\big(\mu_0, \eta(\mu_0)\big)\Big]_k.$$

Here the sum is over all subsurfaces R of S, and the function $[\cdot]_k$ is a truncation function with $[x]_k = x$ when $x \geq k$ and 0 otherwise.

3.2 Special Cases

Proposition 3.5 *Theorem 3.1 holds if ϕ is finite order.*

Proof There are finitely many conjugacy classes of finite order elements in $\mathrm{Mod}(S)$ (see, for example, [9, Theorem 7.13]). By Theorem 3.3, it is sufficient to show that, for each finite order element ϕ, $C(\phi)$ is finitely generated. Indeed, consider a set $\mathcal{F}$ of finite order elements by picking one representative from every conjugacy class. If each $C(\xi)$, $\xi \in \mathcal{F}$ is finitely generated, then there is a uniform upper-bound M_1 for the word length of all elements in any such generating set. If ϕ is conjugate to $\phi' \in \mathcal{F}$, there is a conjugating element η, where $\|\eta\|_{\mathcal{G}} \leq K \max(\|\phi\|_{\mathcal{G}} + \|\phi'\|_{\mathcal{G}})$. Then we can find a generating set for $C(\phi)$ by conjugating a generating set for $C(\phi')$. But $\|\phi'\|_{\mathcal{G}}$

is uniformly bounded ($\mathcal{F}$ is finite). Hence, there is M_0 where the word length of this generating set for $C(\phi)$ are bounded by $M_0\|\phi\|_{\mathcal{G}}$.

Now let ϕ be any finite order element. To see that $C(\phi)$ is finitely generated, let Σ be the orbifold quotient of S by ϕ and $\mathrm{Mod}^{\mathrm{o}}(\Sigma)$ be the orbifold mapping class group of Σ. Then $\mathrm{Mod}^{\mathrm{o}}(\Sigma)$ is a finite index subgroup of $\mathrm{Mod}(\Sigma)$ and hence (say, using Schreier's lemma) is finitely generated. There is a finite index sub-group of $\mathrm{Mod}^{\mathrm{o}}(\Sigma)$ that lifts to sub-group C_Σ of $\mathrm{Mod}(S)$ (see MacLachlan and Harvey [12, Theorem 10]) which is also finitely generated. Finally, $C(\phi)$ is a finite extension of C_Σ and hence is also finitely generated. □

Proposition 3.6 *Theorem 3.1 holds if ϕ is a pseudo-Anosov element.*

Proof By [11] $C(\phi)$ is a virtually cyclic where the degree of the extension is uniformly bounded, in particular $C(\phi)$ is finitely generated. In fact, if F_- and F_+ are the stable and unstable measured foliations associated with ϕ, then any $\psi \in C(\phi)$ preserves the pair (F_-, F_+) as a set.

To prove the statement, it is sufficient to show that, for any $\psi \in C(\phi)$ there is a power m so that $\|\psi\phi^m\|_{\mathcal{G}} \overset{*}{\prec} \|\phi\|_{\mathcal{G}}$. Indeed, this shows that $C(\phi)$ is generated by ϕ and elements in $C(\phi)$ whose word length is less than a multiple of $\|\phi\|_{\mathcal{G}}$.

We use Theorem 3.4 to find such a bound. First, we claim that there exists an integer m so that

$$d_S(\mu_0, \psi\phi^m(\mu_0)) \overset{*}{\prec} d_S(\mu_0, \phi(\mu_0)). \tag{1}$$

Let $\mathcal{B} = \mathcal{B}_\phi$ be the quasi-axis of ϕ in the curve graph of S, that is a geodesic in the curve graph that is preserved by a power $\phi^{m'}$ of ϕ (see [5]). Then $\mathcal{B}$ limits to $F_\pm$ in the boundary of the curve graph. And assuming $\mathcal{B}$ is tight, there are only finitely many such quasi-axes and ϕ permutes them (again see [5]). Hence, for some power m'', $\psi\phi^{m''}$ also preserves $\mathcal{B}$. Choose $m = m'' + p \cdot m'$ so that the translation of length $\psi\phi^m$ along $\mathcal{B}$ is less than or equal to that of $\phi^{m'}$. Both the distance from μ_0 to $\mathcal{B}$ and the translation distance of $\phi^{m'}$ along $\mathcal{B}$ are bounded by the word length of ϕ. Hence, the claims follows.

Choose such m' so that the translation length of $\phi^{m'}$ is large enough to ensure that the geodesic in the curve graph S connecting μ_0 to $\phi^{m'}(\mu_0)$ passes near $\mathcal{B}$ (the curve graph is Gromov hyperbolic). Then for every subsurface $R \subsetneq S$, if $d_R\big(\mu_0, \phi^{m'}(\mu_0)\big)$ is large, then either

$$d_R\big(\mu_0, F_+\big) \quad \text{or} \quad d_R\big(F_-, \phi^{m'}(\mu_0)\big)$$

is large. That is, there is a constant k so that

$$d_R\big(\mu_0, \phi^{m'}(\mu_0)\big) \geq \min\Big(d_R\big(\mu_0, F_+\big), d_R\big(F_-, \phi^{m'}(\mu_0)\big)\Big) - k.$$

Using $d_R\big(F_-, \phi^{m'}(\mu_0)\big) = d_R\big(F_-, \mu_0\big)$, we get

$$\Big[d_R\big(\mu_0, \phi^{m'}(\mu_0)\big)\Big]_{2k} \overset{*}{\succ} \Big[d_R\big(\mu_0, F_+\big)\Big]_k + \Big[d_R\big(F_-, \mu_0\big)\Big]_k.$$

Hence,

$$\|\phi\|_{\mathcal{G}} \stackrel{*}{\asymp} \|\phi^{m'}\|_{\mathcal{G}} \asymp_C \sum_{R\subset S} \Big[d_R\big(\mu_0, \eta(\mu_0)\big)\Big]_{2k}$$
$$\stackrel{*}{\succ} d_S(\mu_0, \phi^{m'}(\mu_0)) + \sum_{R\subsetneq S} \Big[d_R(\mu_0, F_-)\Big]_k + \sum_{R\subsetneq S} \Big[d_R(\mu_0, F_+)\Big]_k. \tag{2}$$

Now, let m be as in (1) and let $\xi = \psi\phi^m$. Further assume k is large enough so that $d_R(F_-, F_+) < k$. Then

$$\|\xi\|_{\mathcal{G}} \stackrel{*}{\asymp} d_S(\mu_0, \xi(\mu_0)) + \sum_{R\subsetneq S} \Big[d_R(\mu_0, \xi(\mu_0)\Big]_{2k}$$
$$\stackrel{*}{\prec} d_S(\mu_0, \xi(\mu_0)) + \sum_{R\subsetneq S} \Big[d_R(\mu_0, F_+)\Big]_k + \Big[d_R(\xi(\mu_0), F_+)\Big]_k.$$

If $\xi(F_+) = F_+$,

$$\sum_{R\subsetneq S} \Big[d_R(\xi(\mu_0), F_+)\Big]_k = \sum_{R\subsetneq S} \Big[d_R(\mu_0, F_+)\Big]_k,$$

and if $\xi(F_+) = F_-$,

$$\sum_{R\subsetneq S} \Big[d_R(\xi(\mu_0), F_+)\Big]_k = \sum_{R\subsetneq S} \Big[d_R(\mu_0, F_-)\Big]_k.$$

In either case, the last two terms in estimate above given for $\|\xi\|_{\mathcal{G}}$ are less than the lower bound (2) given for $\|\phi\|_{\mathcal{G}}$. We also know from (1) that the first term is bounded above by $d_S(\mu_0, \phi(\mu_0))$ which is also bounded by above by a multiple of $\|\phi\|_{\mathcal{G}}$. The theorem follows. □

3.3 The General Case

Recall from the Nielsen–Thurston classification of surface homeomorphisms [7,18] that there is a normal form for any homeomorphism ϕ of a surface S of finite type. That is,

(1) There is a multicurve Γ_ϕ that is preserved by ϕ, called the *canonical reducing system*, defined as follows: consider the set $\mathcal{A}_\phi$ consisting of all curves α so that $\phi^k(\alpha) = \alpha$ up to isotopy, for some $k > 0$ and let Γ_ϕ be the boundary of the subsurface of S that is filled with curves in $\mathcal{A}_\phi$. The curve system Γ_ϕ is empty if ϕ is pseudo-Anosov or has finite order.

(2) The components of $S - \Gamma_\phi$ are decomposed into *ϕ–orbits* $\{V_1, \ldots, V_\ell\}$ where

$$\phi(V_i) = V_{i+1} \quad \text{for} \quad i \in \mathbb{Z}/\ell\mathbb{Z}.$$

(3) For every every ϕ-orbit $\mathcal{V} = \{V_1, \ldots, V_\ell\}$ the first return map $\phi^\ell \colon V_1 \to V_1$ is either finite order or pseudo-Anosov.

It is convenient to fix a topological surface δ that is homeomorphic to every V_i. Choosing a homeomorphism $\delta \to V_i$, the map

$$\phi^\ell|_{V_i} \colon V_i \to V_i$$

defines a conjugacy class in $\text{Mod}(\delta)$ that is independent of i or the homeomorphism from δ to V_i. That is, it depends only on the ϕ-orbit $\mathcal{V}$. We denote this conjugacy class by $[\phi_\mathcal{V}]$. We say the ϕ-orbit $\mathcal{V}$ is *of type* δ *with the first return map* $[\phi_\mathcal{V}]$.

We start by modifying the generating set $\mathcal{G}$ and conjugating ϕ so that they are compatible with each other.

If we choose $\varphi \in [\phi]$ with $\|\varphi\|_\mathcal{G} \stackrel{*}{\prec} \|\phi\|_\mathcal{G}$ then, by Theorem 3.3, the conjugating element η satisfies $\|\eta\|_\mathcal{G} \stackrel{*}{\prec} \|\phi\|_\mathcal{G}$. But η conjugates a generating set for $C(\varphi)$ to a generating set for $C(\phi)$ which means it would be enough to prove the theorem for φ. Our goal is to find a representative of the conjugacy class $[\phi]$ of ϕ which has (as much as possible) a standard form.

There are finitely many topological types possible for subsurfaces of S. Let Δ be the set of surfaces that can be a subsurface of S. That is, for every subsurface V of S, there is a (unique) surface $\delta \in \Delta$ that is homeomorphic to V. We fix a generating set $\mathcal{G}_\delta$ for every surface $\delta \in \Delta$. In fact, we assume $\mathcal{G}_\delta$ consists of Dehn twists around a finite set of curves μ_δ. Curves in μ_δ fill the surface δ, that is every curve in δ intersects a curve in μ_δ.

Also, up to a homeomorphism, there are finitely many multicurves on a surface S. Let Λ be a fixed set consisting of a representative for every homeomorphism type of a multicurve in S. For any simple closed curve γ in S, let D_γ denote the Dehn twist around γ. For each $\Gamma \in \Lambda$ let μ_Γ be a set of curves on S with the following properties.

(1) $\Gamma \subset \mu_\Gamma$.
(2) The set $\mathcal{G}_\Gamma = \{D_\gamma \mid \gamma \in \mu_\Gamma\}$ generates $\text{Mod}(S)$.
(3) for every subsurface V that is a component of $S - \Gamma$ that is homeomorphic to δ, there is a homeomorphism $m_V \colon \delta \to V$ so that $m_V(\mu_\delta)$ is exactly the set of curves in μ_Γ that are contained in V. In particular, $m_V(\mathcal{G}_\delta) \subset \mathcal{G}_\Gamma$ generates $\text{Mod}(V)$.

Note that, μ_Γ fills S. For the rest of this article, we assume

$$\mathcal{G}_0 = \bigcup_{\Gamma \in \Lambda} \mathcal{G}_\Gamma \quad \text{and} \quad \mu_0 = \bigcup_{\Gamma \in \Lambda} \mu_\Gamma.$$

Note that $\|\cdot\|_\mathcal{G}$ differs from $\|\cdot\|_{\mathcal{G}_0}$ by a uniform multiplicative amount.

We are now ready to construct φ. Let $\Gamma \in \Lambda$ be the curve system in Λ that has the same homeomorphism type as Γ_ϕ. Conjugate ϕ to ϕ' by a homeomorphism that sends Γ_ϕ to Γ. Then, ϕ' partitions the components of $S - \Gamma$ to ϕ'-orbits similar to ϕ. We then further modify ϕ' to φ whose orbits are the same as the orbits of ϕ' so that, if $\mathcal{V}' = \{V'_1, \ldots, V'_\ell\}$ is a ϕ'-orbit of size ℓ associated with the ϕ-orbit $\mathcal{V}$, then

(1) for $i = 1, \ldots, \ell - 1$, we have

$$m_{V_{i+1}}^{-1}\,\varphi\, m_{V_1} \colon\ \delta \to \delta \qquad \text{is the identity map.}$$

(2) The map

$$m_{V_1}^{-1}\,\varphi\, m_{V_\ell} \colon\ \delta \to \delta$$

is the representative of $[\phi_{\mathcal{V}}]$ that has the shortest word length with respect to $\mathcal{G}_\delta$. We can make this canonical by choosing, ahead of time, a representative for every conjugacy class in $\mathrm{Mod}(\delta)$.

Proposition 3.7 *For φ constructed as above, we have*

$$\|\varphi\|_{\mathcal{G}_0} \overset{*}{\prec} \|\phi\|_{\mathcal{G}}.$$

Proof The proposition follows from the Masur–Minsky distance formula (3.4), which we apply to φ. Since $\varphi(\Gamma) = \Gamma$, for every R that intersects Γ, we have

$$d_R(\mu, \varphi(\mu)) \overset{+}{\asymp} d_R(\Gamma, \varphi(\Gamma)) = O(1).$$

Hence, choosing k large enough, these terms disappear from the distance formula.

Also, for every φ-orbit, $\mathcal{V}'$ associated to the ϕ-orbit $\mathcal{V}$, we have

$$\begin{aligned}
\left\|\varphi^\ell\big|_{\mathcal{V}}\right\|_{\mathcal{G}_\delta} &\overset{*}{\asymp} \sum_{\rho \subset \delta} \Big[d_\rho(\mu_\delta, \phi_{\mathcal{V}}(\mu_\delta))\Big]_k \\
&\overset{*}{\asymp} \min_{\mu \in M_\delta} \sum_{\rho \subset \delta} \Big[d_\rho(\mu, \phi_{\mathcal{V}}(\mu))\Big]_k \\
&= \min_{\mu \in M_{V_1}} \sum_{R \subset V_1} \Big[d_R(\mu, \phi^\ell(\mu))\Big]_k \\
&\overset{*}{\prec} \sum_{R \subset V_1} \Big[d_R(\mu_0, \phi^\ell(\mu_0))\Big]_k \\
&\overset{*}{\prec} \sum_{R \subset S} \Big[d_R(\mu_0, \phi^\ell(\mu_0))\Big]_k \overset{*}{\asymp} \|\phi^\ell\|_{\mathcal{G}_0} \le \ell \|\phi\|_{\mathcal{G}_0}.
\end{aligned}$$

Now, let μ_S be a marking for S associated with the generating set $\mathcal{G}_S$ and let μ_0 be the marking in δ that is the image of the projection of μ_S to V_1 under $m_{V_1}^{-1}$. That is, for every sub-surface R of δ, we have

$$d_{m_{V_1}(R)}\big(\mu_S, m_{V_1}(\mu_0)\big) = O(1).$$

We can now compare the word length of $\phi_{\mathcal{V}}$ with that of ϕ^{ℓ} which send V_1 to itself.

$$\begin{aligned}\|\phi_{\mathcal{V}}\|_{\mathcal{G}_\delta} &\prec \sum_{R\subset\delta}\Big[d_R\big(\mu_0, \phi_{\mathcal{V}}(\mu_0)\big)\Big]_k \\ &\overset{*}{\asymp} \sum_{R\subset V_1}\Big[d_R\big(\mu_S, \phi_{\mathcal{V}}(\mu_S)\big)\Big]_k \\ &\prec \sum_{R\subset S}\Big[d_R\big(\mu_S, \phi^{\ell}(\mu_S)\big)\Big]_k \asymp \|\phi^{\ell}\|_{\mathcal{G}_S},\end{aligned}$$

where the last inequality is the distance formula in the surface S. The lemma now follows since $\|\phi^{\ell}\|_{\mathcal{G}_S} \leq \ell\,\|\phi\|_{\mathcal{G}_S}$ and ℓ and all other related constant are independent of ϕ. □

Proposition 3.8 *There is a constant M_δ, depending only on the generating sets $\mathcal{G}_\delta$ and $\mathcal{G}$, so that for every $\phi \in \mathrm{Mod}(S)$ that has a ϕ orbit $\mathcal{V} = \{V_1, \ldots, V_\ell\}$ of type δ, with the first return map $\phi_{\mathcal{V}}$, we have*

$$\|\phi_{\mathcal{V}}\|_{\mathcal{G}_\delta} \leq M_\delta\,\|\phi\|_{\mathcal{G}}.$$

Proof If $\phi_{\mathcal{V}}$ is finite order, then the statement is clear since there are only finitely many conjugacy classes of finite order elements and $\phi_{\mathcal{V}}$ is one of the finitely many fixed representatives of these classes. Hence, $\|\phi_{\mathcal{V}}\|_{\mathcal{G}_\delta}$ is uniformly bounded. However, we give a general argument that works in both cases using the Masur–Minsky distance formula Theorem 3.4.

Note that the changing η by a conjugation is the same as changing the marking μ_δ. Since we have chosen $\phi_{\mathcal{V}}$ to be the representative $[\phi_{\mathcal{V}}]$ with the smallest word length, we have

$$\|\phi_{\mathcal{V}}\|_{\mathcal{G}_\delta} \prec \sum_{R\subset\delta}\Big[d_R\big(\mu, \phi_{\mathcal{V}}(\mu)\big)\Big]_k \qquad \text{for any marking}\mu.$$

Here, $\prec$ means less than up a uniform multiplicative and additive error.

Now, let μ_S be a marking for S associated to the generating set $\mathcal{G}_S$ and let μ_0 be the marking in δ that is the image of the projection of μ_S to V_1 under $m_{V_1}^{-1}$. That is, for every sub-surface R of δ, we have

$$d_{m_{V_1}(R)}\big(\mu_S, m_{V_1}(\mu_0)\big) = O(1).$$

We can now compare the word length of $\phi_{\mathcal{V}}$ with that of ϕ^{ℓ} which send V_1 to itself.

$$\begin{aligned}\|\phi_{\mathcal{V}}\|_{\mathcal{G}_\delta} &\prec \sum_{R\subset\delta}\Big[d_R\big(\mu_0, \phi_{\mathcal{V}}(\mu_0)\big)\Big]_k \\ &\overset{*}{\asymp} \sum_{R\subset V_1}\Big[d_R\big(\mu_S, \phi_{\mathcal{V}}(\mu_S)\big)\Big]_k \\ &\prec \sum_{R\subset S}\Big[d_R\big(\mu_S, \phi^{\ell}(\mu_S)\big)\Big]_k \asymp \|\phi^{\ell}\|_{\mathcal{G}_S},\end{aligned}$$

where the last inequality is the distance formula in the surface S. The lemma now follows since $\|\phi^\ell\|_{\mathcal{G}_S} \leq \ell \, \|\phi\|_{\mathcal{G}_S}$ and ℓ and all other related constant are independent of ϕ. $\square$

Now, consider an element $\psi \in C(\phi)$. First notice that if $\alpha \in \mathcal{A}_\phi$, then

$$\phi^k(\psi(\alpha)) = \psi(\phi^k(\alpha)) = \psi(\alpha),$$

which means $\psi(\mathcal{A}_\phi) = \mathcal{A}_\phi$. Hence, ψ also preserves the subsurface that is filled with the curves in $\mathcal{A}_\phi$. Therefore, $\psi(\Gamma_\phi) = \Gamma_\phi$ and ψ permutes the components of $S - \Gamma_\phi$.

Assume $\psi(V_1) = W_1$, where V_1 is in a ϕ-orbit $\mathcal{V}$ and W_1 is in a ϕ-orbit $\mathcal{W}$. We observe that $\mathcal{V}$ and $\mathcal{W}$ are of the same type δ. Also, for every i,

$$\psi(V_i) = \psi(\phi^i(V_1)) = \phi^i(\psi(V_1)) = \phi^i(W_1) = W_i.$$

Hence, the orbit $\mathcal{V}$ is mapped to the $\mathcal{W}$, which in particular implies $|\mathcal{V}| = |\mathcal{W}| = \ell$. We further have

$$\psi|_{V_1} \, \phi|_{V_1}^\ell \, \psi|_{V_1}^{-1} = \phi|_{W_1}^\ell,$$

where $\psi|_{V_1}$ is the restriction of ψ to V_1 (similarly, $\phi|_{V_1}$ and $\phi|_{W_1}$ are restrictions of ϕ to V_1 and W_1, respectively). That is, $[\phi_\mathcal{V}] = [\phi_\mathcal{W}]$ which implies $\phi_\mathcal{V} = \phi_\mathcal{W}$.

To sum up, ψ induces a permutation of ϕ-orbits; however, it can only send an orbit to another orbit if the orbits have the same size, same topological type and if the associated first return maps are the same. Also, ψ has to send adjacent components of $S - \Gamma_\phi$ to adjacent components. To keep track of this information, we consider the a decorated dual graph defined as follows. Let G be a graph whose vertices are components of $S - \Gamma_\phi$ and edges are pairs of adjacent components. We decorate a vertex $V \in G$ (which is a component of $S - \Gamma_\phi$) with the name $\mathcal{V}$ of the associated ϕ-orbit, the topological type δ and the first return map $\phi_\mathcal{V} \in \mathrm{Mod}(\delta)$. We say a map $f \colon G \to G$ is an automorphism of the decorated graph if

(1) f is a graph automorphism.
(2) There is a permutation σ of the ϕ-orbits so that, if

$$f(V, \mathcal{V}, \delta, \phi_\mathcal{V}) = (W, \mathcal{W}, \delta', \phi_\mathcal{W}),$$

then $\delta = \delta'$, $|\mathcal{V}| = |\mathcal{W}|$, $\mathcal{W} = \sigma(\mathcal{V})$ and $\phi_\mathcal{V} = \phi_\mathcal{W}$.

Note that, the set of automorphisms of the decorated graph G form a group which we denote by $\mathrm{Aut}(G)$. For a given $\psi \in C(\phi)$, we denoted the induced graph map by f_ψ and the induced permutation of ϕ-orbits by σ_ψ. We have a homomorphism

$$\pi_G \colon C(\phi) \to \mathrm{Aut}(G)$$

projecting ψ to the induced action f_ψ on the decorated graph G.

For each orbit $\mathcal{V}$, let

$$C_{\mathcal{V}} \subset \ker(\pi_G)$$

be the set of elements of $C(\phi)$ that fix every subsurface in $S - \Gamma_\phi$ whose restriction to any subsurface that is not in $\mathcal{V}$ is identity. Then the $\cup_{\mathcal{V}} C_{\mathcal{V}}$ generates $\ker(\pi_G)$ and the intersection $\cap_{\mathcal{V}} C_{\mathcal{V}}$ is the set of multi-twists around the curves Γ. Also, for $\psi \in C_{\mathcal{V}}$ where $\mathcal{V} = \{V_1, \ldots, V_\ell\}$,

$$\psi|_{V_{i+1}} = \phi^{-i} \psi|_{V_1} \phi^i.$$

That is, the restriction $\psi|_{V_1}$ determines the restriction of ϕ to every other subsurface in the ϕ-orbit $\mathcal{V}$. Therefore, considering the homeomorphism

$$\pi_{\mathcal{V}} \colon C_{\mathcal{V}} \to \mathrm{Mod}(\delta), \quad \psi \to m_{V_1}^{-1} \psi|_{V_1} m_{V_1},$$

we have that any $\psi \in C_{\mathcal{V}}$ is determined, up to possibly a multi-twist around Γ, by its projection $\pi_{\mathcal{V}}(\phi)$ to $\mathrm{Mod}(\delta)$ which lies in $C(\phi_{\mathcal{V}})$.

We have shown that every element of $C(\phi)$ is determined, up a multi-twist around Γ, by its projection to $\mathrm{Aut}(G)$ and to $C(\phi_{\mathcal{V}})$. We now examine which multi-twists around curves in Γ lie in $C(\phi)$. Consider the action of ϕ on Γ. It decomposes Γ into orbits $\overline{\gamma} = \{\gamma_1, \ldots, \gamma_j\}$ where $\phi(\gamma_i) = \gamma_{i+1}$ for $i \in \mathbb{Z}/j\mathbb{Z}$. We call such orbit $\overline{\gamma}$ an admissible multicurve if ϕ^j sends γ_1 to γ_1 preserving the orientation. For any admissible multicurve $\overline{\gamma}$, define

$$D_{\overline{\gamma}} = D_{\gamma_1} \ldots D_{\gamma_j},$$

where D_{γ_i} is a Dehn twist if γ_i is non-separating and a half-twist if γ_i is separating. That is, $D_{\overline{\gamma}}$ is the product of Dehn twists (or half-twists) around the curves in $\overline{\gamma}$. The set of multi-twists around the curve in Γ that commute with ϕ is generated by $\{D_{\overline{\gamma}}\}_{\overline{\gamma}}$. Note that this maybe an empty set. This is because, if an element $\psi \in C(\phi)$ twists around γ is also has to twist by the same amount around $\phi(\gamma)$. However, if ϕ^j sends γ to itself reserving the orientation, then ϕ^j conjugates D_γ to D_γ^{-1}. Hence, D_γ does not commute with ϕ and no Dehn twists around such γ is possible.

There is no homomorphism back from $\mathrm{Aut}(G)$ or $C(\phi_{\mathcal{V}})$ to $\mathrm{Mod}(S)$. But to find a generating set for $C(\phi)$ it is enough to choose a section. To summarize the above discussion, we have shown:

Summary

Let $\mathcal{G}_{\mathcal{V}}$ be a generating set for $C(\phi_{\mathcal{V}})$ and consider arbitrary sections

$$\pi_G^{-1} \colon \mathrm{Aut}(G) \rightsquigarrow C(\phi) \qquad \text{and} \qquad \pi_{\mathcal{V}}^{-1} \colon C(\phi_{\mathcal{V}}) \rightsquigarrow C_{\mathcal{V}}.$$

Then $C(\phi)$ is generated by the union of the following sets:

(1) The set $\{D_{\overline{\gamma}}, \gamma \in \Gamma\}$, where $\overline{\gamma} \subset \Gamma$ is an admissible multicurve.
(2) The image of $\mathrm{Aut}(G)$ under π_G^{-1}.
(3) The images of $\mathcal{G}_{\mathcal{V}}$ under maps $\pi_{\mathcal{V}}^{-1}$.

What remains is to bound the word length of the elements of this generating set. An element in (1) is a product of Dehn twists around a uniformly bounded number of curves and these Dehn twists are already in our generating set. For (2), we build the section to be as close to the identity as possible. Namely, for any $f \in \mathrm{Aut}(G)$ and induced permutation σ, let $\sigma(V_i) = W_j$ where V_i in the ϕ-orbit $\mathcal{V}$ and W_j in a ϕ-orbit $\mathcal{W}$. We define ψ to be the map that also sends V_i to W_j and so that $m_{W_j}^{-1}\psi m_{V_i}$ is the identity. Then ψ is clearly in $C(\phi)$. For (3), given an element $g \in \mathcal{G}_{\mathcal{V}}$ associated to a ϕ-orbit $\mathcal{V} = \{V_1, \ldots, V_\ell\}$ there is mapping class ψ, that acts on subsurfaces of $S - \mathcal{A}_\phi$ the same way as ϕ, its restriction to $\mathcal{V}$ is the same as ϕ and is the identity on every other orbit. Again, ψ clearly commutes with ϕ. The desired upper-bound for the word length of ψ follows from Proposition 3.8.

4 Self-equivalences of Thurston Maps

The results of the previous section imply the following theorem.

Theorem 4.1 *Let f be either a Thurston map with empty canonical obstruction or a homeomorphism. Then the group $C(f)$ of all self-equivalences of f is finitely generated. Moreover, there is an algorithm that finds a generating set for $C(f)$.*

Proof I. If f is an unobstructed Thurston map with hyperbolic orbifold then it is equivalent to a rational map (possibly with extra marking) and $C(f)$ is trivial (cf. [4,6]). The same argument applies in the case of a Thurston map with parabolic orbifold unless P_f contains exactly 4 points, and f is equivalent to a quotient of an affine map $Ax + b$ with hyperbolic associated matrix A (see [16]).

II. Let f be a Thurston map with parabolic orbifold such that $C(f)$ is non-trivial. If $Q_f = P_f$ contains exactly 4 points, then the pure mapping class group of (S^2, Q_f) is isomorphic to the modular group $\Lambda = \mathrm{PGL}(2, \mathbb{Z})/\mathrm{PGL}(2, \mathbb{Z}/2\mathbb{Z})$. In this case, $C(f)$ is the subgroup of Λ of all matrices that commute with A. It consists of the matrices which diagonalize simultaneously with A, and thus its generating set can be easily computed.

If Q_f has more than four points, let us denote $C(f, P_f)$ the group of self-equivalences of f with only the points in P_f marked. Clearly, $C(f)$ is isomorphic to a finite index subgroup of $C(f, P_f)$. Indeed, if a self-equivalence ϕ is homotopic to the identity in (S^2, Q_f) it will also be homotopic to the identity in (S^2, P_f). Therefore every self-equivalence can be represented by an affine homeomorphism. Some elements of $C(f, P_f)$, however, may have affine representatives that do not fix points in Q_f but instead send them to different pre-periodic orbits. Determining which subgroup of $C(f, P_f)$ fixes points in Q_f is a straightforward exercise in linear algebra.

III. If f is a homeomorphism then Corollary 3.2 can be applied. $\square$

5 Hurwitz Classification of Branched Covers

Let X and Y be two finite type Riemann surfaces. We recall that two finite degree branched covers ϕ and ψ of Y by X are *equivalent in the sense of Hurwitz* if there exist homeomorphisms $h_0 : Y \to Y$, $h_1 : X \to X$ such that

$$h_0 \circ \phi = \psi \circ h_1.$$

An equivalence class of branched covers is known as a *Hurwitz class*. Enumerating all Hurwitz classes with a given ramification data is a version of the *Hurwitz problem*. The classical paper of Hurwitz [10] gives an elegant and explicit solution of the problem for the case $X = \hat{\mathbb{C}}$.

We will need the following narrow consequence of Hurwitz's work (for a modern treatment, see [1]):

Theorem 5.1 *There exists an algorithm $\mathcal{A}$ which, given PL branched covers ϕ and ψ of PL spheres and a PL homeomorphism h_0 mapping the critical values of ϕ to those of ψ, does the following:*

(1) decides whether ϕ and ψ belong to the same Hurwitz class or not;
(2) if the answer to (1) is affirmative, decides whether there exists a homeomorphism h_1 such that $h_0 \circ \phi = \psi \circ h_1$.

6 Equivalence on Thick Parts

6.1 Canonical Obstructions and Thin–Thick Decompositions of Thurston Maps

Let f be a Thurston map, and $\Gamma = \{\gamma_j\}$ an f-stable multicurve. Consider a finite collection of disjoint closed annuli $A_{0,j}$ which are homotopic to the respective γ_j. For each $A_{0,j}$ consider only non-trivial preimages; these form a collection of annuli $A_{1,k}$, each of which is homotopic to one of the curves in Γ. Following Pilgrim, we say that the pair (f, Γ) is in a *standard form* (see Fig. 1) if there exists a collection of annuli $A_{0,j}$, which we call *decomposition annuli*, as above such that the following properties hold:

(a) for each curve γ_j the annuli $A_{1,k}$ in the same homotopy class are contained inside $A_{0,j}$;
(b) moreover, the two outermost annuli $A_{1,k}$ as above share their outer boundary curves with $A_{0,j}$.

A Thurston map with a multicurve in a standard form can be decomposed as follows. First, all annuli $A_{0,j}$ are removed, leaving a collection of spheres with holes, denoted $S_0(j)$. For each j, there exists a unique connected component $S_1(j)$ of $f^{-1}(\cup S_0(j))$ which has the property $\partial S_0(j) \subset \partial S_1(j)$. Any such component $S_1(j)$ is a sphere with holes, with boundary curves being of two types: boundaries of the removed annuli, or boundaries of trivial preimages of the removed annuli.

The holes in $S_0(j) \subset S^2$ can be filled as follows. Let χ be a boundary curve of a component D of $S^2 \setminus S_0(j)$. Let $k \in \mathcal{N}$ be the first iterate $f^k : \chi \to \chi$, if it exists. For

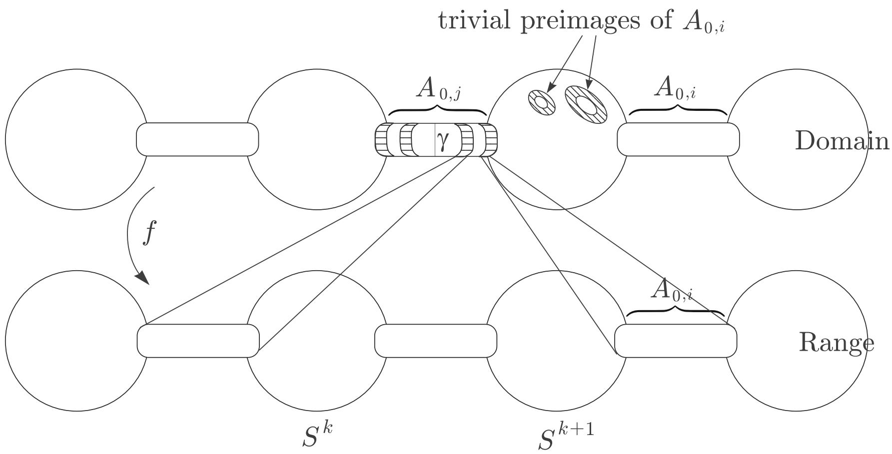

Fig. 1 Pilgrim's decomposition of a Thurston map

each $0 \leq i \leq k-1$ the curve $\chi_i \equiv f^i(\chi)$ bounds a component D_i of $S^2 \backslash S_0(m_i)$ for some m_i. Denote d_i the degree of $f : \chi_i \to \chi_{i+1}$. Select homeomorphisms

$$h_i : \bar{D}_i \to \bar{\mathcal{D}} \text{ so that } h_{i+1} \circ f \circ h_i^{-1}(z) = z^{d_i}.$$

Set $\tilde{f} \equiv f$ on $\cup S_0(j)$. Define new punctured spheres $\tilde{S}(j)$ by adjoining cups $h_i^{-1}(\bar{\mathcal{D}} \backslash \{0\})$ to $S_0(j)$. Extend the map $\tilde{f}$ to each D_i by setting

$$\tilde{f}(z) = h_{i+1}^{-1} \circ (h_i(z))^{d_i}.$$

We have thus replaced every hole with a cap with a single puncture. We call such a procedure *patching* a component.

By construction, the map

$$\tilde{f} : \cup \tilde{S}(j) \to \cup \tilde{S}(j),$$

contains a finite number of periodic cycles of punctured spheres. For every periodic sphere $\tilde{S}(j)$ denote by $\mathcal{F}$ the first return map $\tilde{f}^{k_j} : \tilde{S}(j) \to \tilde{S}(j)$. This is again a Thurston map or a homeomorphism. The collection of maps $\mathcal{F}$ and the combinatorial information required to glue the spheres $S_0(j)$ back together is what Pilgrim called a *decomposition* of f along Γ; we will denote it $\mathcal{S}_\Gamma$.

Pilgrim showed:

Theorem 6.1 *For every obstructed marked Thurston map f with an obstruction Γ there exists an equivalent map g such that (g, Γ) is in a standard form, and thus can be decomposed.*

Pilgrim [14] defined a *canonical decomposition* of a Thurston map f based on his definition of a canonical Thurston obstruction. His original definition was framed in the language of iteration on a Teichmüller space; we will give an equivalent definition discovered by the second author [15]:

Theorem 6.2 *Suppose f is an obstructed Thurston mapping. Then there exists a unique minimal (with respect to inclusion) obstruction Γ_f, which is the canonical obstruction in the sense of [14], with the following properties.*

- *If a first-return map $\mathcal{F}$ of a cycle of components in $\mathcal{S}_{\Gamma_f}$ is a $(2, 2, 2, 2)$-map, then every curve of every simple Thurston obstruction for $\mathcal{F}$ has two postcritical points of f in each complementary component and the two eigenvalues of $\hat{\mathcal{F}}_*$ are equal or non-integer.*
- *If the first-return map $\mathcal{F}$ of a cycle of components in $\mathcal{S}_{\Gamma_f}$ is not a $(2, 2, 2, 2)$-map nor a homeomorphism, then there exists no Thurston obstruction for $\mathcal{F}$.*

Definition 6.3 If Γ_f is the canonical obstruction, then the decomposition $\mathcal{S}_{\Gamma_f}$ is the *canonical decomposition* of f. In this case, we call the components of the complement of the decomposition annuli the *thick parts*, and the decomposition annuli themselves the *thin parts*.

From this point, only canonical decompositions of Thurston maps will be considered.

Definition 6.4 By *equivalence on thick parts* ϕ between f and g, we mean a homeomorphism defined on the union of patched thick parts of f onto the union of patched thick parts of g such that the following holds:

- Denote ϕ_W the restriction of ϕ to any patched thick component W. If X is a periodic patched thick component of $\tilde{f}$ then $Y = \phi(X)$ is periodic for $\tilde{g}$ with the same period. If $\mathcal{F}_X, \mathcal{G}_Y$ denote the first return maps of X and Y, respectively, then ϕ_X is an equivalence of $\mathcal{F}_X$ and $\mathcal{G}_Y$.
- Let X be a periodic patched thick component and let W be a preimage of X so that

$$\tilde{f}^n(W) = X.$$

Denote $Y = \phi_X(X)$ and $Z = \phi(W)$. Then $\tilde{g}^n(Z) = Y$ and ϕ_W is a lift of ϕ_X through actions of $\tilde{f}$ and $\tilde{g}$:

$$\phi_X \circ \tilde{f}^n = \tilde{g}^n \circ \phi_W.$$

6.2 Centralizer on Thick Parts

For each periodic patched thick component $X = \mathcal{F}(X)$ of the canonical decomposition of f denote $C_X(f) \subset \mathrm{PMCG}(X)$, the group of self-equivalences of the first return Thurston mapping $\mathcal{F}|_X : X \to X$.

Definition 6.5 We define the *centralizer on thick parts* $C_{\text{thick}}(f)$ of f to be the group of all self-equivalences of f on thick parts. By the previous definition, $C_{\text{thick}}(f)$ is isomorphic to the subgroup of the free abelian product

$$C_{\text{periodic}}(f) \equiv \prod_{\text{periodic components } X} C_X(f)$$

consisting of all elements ϕ such that for every thick patched component W with $\tilde{f}^n(W) = X$, one can define ϕ_W so that $\phi_X \circ \tilde{f}^n = \tilde{f}^n \circ \phi_W$ (that is, ϕ can be lifted via the action of $\tilde{f}$ to all strictly pre-periodic preimages of X).

Note that since all C_X are finitely generated (Theorem 4.1) and $C_{\text{thick}}(f)$ is a subgroup of finite index, $C_{\text{thick}}(f)$ is also finitely generated. Furthermore,

Lemma 6.6 *A generating set of $C_{thick}(f)$ can be computed explicitly.*

Proof By Theorem 4.1, for each periodic component X, a generating set A of $C_{\text{periodic}}(f)$ can be computed explicitly. Given the topological complexity of the covering maps $\tilde{f}^n : W \to X$ for all thick preimages of periodic components, it is straightforward to obtain an upper bound on the word length (in terms of the elements of A) of the generating set of $C_{\text{thick}}(f)$. By Theorem 5.1, we can verify algorithmically, which of the words, whose length is under this bound, correspond to elements of $C_{\text{thick}}(f)$. □

Consider two equivalences on thick parts ϕ and ψ between two Thurston maps f and g. Then $\phi^{-1} \circ \psi$ is a self-equivalence of f. This yields the following.

Lemma 6.7 *Let ϕ be an equivalence on thick parts between two Thurston maps f and g. Then any other equivalence can be written $\phi \circ l$, where $l \in C_{thick}(f)$.*

7 Algorithmic Geometrization of Thick Parts

The second and third authors proved the following [16, Theorem 6.1]:

Theorem 7.1 *(Canonical geometrization) There exists an algorithm which for any Thurston map f finds its canonical obstruction Γ_f.*

Furthermore, let $\mathcal{F}$ denote the collection of the first return maps of the canonical decomposition of f along Γ_f. Then the algorithm outputs the following information:

- *for every first return map with a hyperbolic orbifold, the unique (up to Möbius conjugacy) marked rational map equivalent to it;*
- *for every first return map of type $(2, 2, 2, 2)$ the unique (up to affine conjugacy) affine map of the form $z \mapsto Az + b$ where $A \in SL_2(\mathbb{Z})$ and $b \in \frac{1}{2}\mathbb{Z}^2$ with marked points which is equivalent to f after quotient by the orbifold group G;*
- *for every first return map which has a parabolic orbifold not of type $(2, 2, 2, 2)$ the unique (up to Möbius conjugacy) marked rational map map equivalent to it, which is a quotient of a complex affine map by the orbifold group.*

8 Extending Equivalence from Thick to Thin Parts

The following is standard (see, e.g. [8]):

Proposition 8.1 *For every Thurston obstruction* $\Gamma = \{\alpha_1, \ldots, \alpha_n\}$, *the Dehn twists* $T_{\alpha_j}, \ j = 1 \ldots n$ *generate a free Abelian subgroup of* $PMCG(S \backslash Q_f)$.

We write $\mathbb{Z}^\Gamma \simeq \mathbb{Z}^n$ to denote the subgroup generated by T_{α_j}.
We will need the following straightforward generalization of [16, Proposition 7.7]:

Proposition 8.2 *Let* f, g *be equivalent Thurston maps. Let the pair* (ϕ_1, ϕ_2) *realize the equivalence of the thick components of* f *and* g. *Extend* ϕ_1 *to a homeomorphism of the whole sphere* $S^2 \backslash Q_f$, *defining it on the thin parts in an arbitrary fashion. Then there exist* $m \in \mathbb{Z}^\Gamma$, $\psi \in C_{thick}$, *and an equivalence pair* (h_1, h_2) *for* f, g *such that* $h_1 = \phi_1 \circ \psi \circ m$.

Notice that if $h_1 \circ f = g \circ h_2$, where $h_1 = \phi_1 \circ m_1$ for some $m_1 \in \mathbb{Z}^\Gamma$, then h_2 is homotopic to $\phi_1 \circ m_2$ for some other $m_2 \in \mathbb{Z}^\Gamma$. If $m_1 = m_2$ then h_1 is homotopic to h_2 and these two homeomorphisms realize an equivalence between f and g. Since we cannot check whether this happens for all elements of $\mathbb{Z}^\Gamma$, we will require the following proposition [16, Proposition 7.8]:

Proposition 8.3 *There exists explicitly computable* $N \in \mathbb{N}$ *such that if* $n \in \mathbb{Z}^\Gamma$ *where all coordinates of* n *are divisible by* N, *then*

$$(\phi_1 \circ (m_1 + n)) \circ f = g \circ (\phi_2 \circ (m_2 + M_\Gamma n)),$$

whenever

$$(\phi_1 \circ m_1) \circ f = g \circ (\phi_2 \circ m_2).$$

9 Checking Thurston Equivalence

We are now ready to present an algorithm which checks whether two Thurston maps f and g are equivalent or not.
Algorithm

(1) Find the canonical obstructions $\Gamma_f = \{\alpha_1, \ldots, \alpha_n\}$ and $\Gamma_g = \{\beta_1, \ldots, \beta_n\}$ (Theorem 7.1).
(2) Check whether the cardinality of the canonical obstructions $\Gamma_f = \{\alpha_1, \ldots, \alpha_n\}$ and $\Gamma_g = \{\beta_1, \ldots, \beta_n\}$ is the same, and whether the corresponding Thurston matrices coincide. If not, output **maps are not equivalent** and halt.
(3) Denote the thin parts (decomposition annuli) of f and g by A_i and B_i respectively. Construct the first return maps $\mathcal{F}$ and $\mathcal{G}$ of the periodic patched thick parts for f and g and geometrize them (7.1). Are the geometrizations of $\mathcal{F}$ and $\mathcal{G}$ the same up to reordering of the components of the first return map? If not, output **maps are not equivalent** and halt.
(4) **for all** permutations $\sigma \in S_n$ **do**

(5) Is there a homeomorphism

$$h_\sigma : S^2 \backslash Q_f \to S^2 \backslash Q_g$$

sending $A_i \to B_{\sigma(i)}$? If not, **continue.**

(6) Is it true that for every periodic patched thick component X of f the geometrization of $\mathcal{F}|_X$ is the same as the geometrization of $\mathcal{G}_{h_\sigma(X)}$? If not, **continue.**

(7) For all thick components C_j^f check whether the Hurwitz classes of the patched coverings

$$\tilde{f} : \widetilde{C_j^f} \to \widetilde{f(C_j^f)} \text{ and } \tilde{g} : \widetilde{h_\sigma(C_j^f)} \to \widetilde{g(h_\sigma(C_j^f))}$$

are the same (5.1). If not, **continue.**

(8) Construct equivalence pairs (η_0^X, η_1^X) between first return maps $\mathcal{F}_X$ and $\mathcal{G}_{h_\sigma(X)}$ of periodic patched thick components corresponding by h_σ and the group $C_{\text{periodic}}(g)$ of self-equivalences of $\mathcal{G}$. If the maps of some pair are not equivalent, **continue.**

(9) Find an equivalence between first return maps $\mathcal{F}_X$ and $\mathcal{G}_{h_\sigma(X)}$ in the form $\phi_X = \psi \circ \eta_0^X$ with $\psi \in C_{\text{periodic}}(g)$ that can be lifted via branched covers $\tilde{f}$ and $\tilde{g}$ to every preimage of every thick component and preserves the set of marked points. Since C_{thick} is a finite index subgroup of $C_{\text{periodic}}(g)$, this is a finite check (for representatives of each coset), which can be carried out algorithmically by (Theorem 5.1 and Lemma 6.7). If not possible, **continue.**

(10) Lift the equivalences, to obtain a homeomorphism ϕ_1 defined on all thick parts.

(11) Compute $C_{\text{thick}}(g)$ (Lemma 6.7).

(12) Pick some initial homeomorphisms $a_i : A_i \to B_{\sigma(i)}$ so that the boundary values agree with ϕ_1. This defines ϕ_1 on the whole sphere.

(13) Find the set of vectors $m_1 \in \mathbb{Z}^\Gamma$ with coordinates between 0 and $N-1$, where N is as in Proposition 8.3 such that $h_1 = \phi_1 \circ m_1$ lifts through f and g so that

$$(\phi_1 \circ m_1) \circ f = g \circ h_2.$$

For all vectors m_1 in this set **do**

(14) By the discussion above $h_2 = m_2 \circ \phi_2$ with $m \in \mathbb{Z}^\Gamma$. Compute m_2.

(15) Find the finite index subgroup G_1 of $C_{\text{thick}}(f)$ of all elements ψ such that $\psi \circ h_1$ lifts through f and g (Lemma 6.6).

(16) For every $\psi \in G_1$ we have

$$\psi \circ h_1 \circ f = g \circ n_\psi \circ \psi \circ h_2$$

where $n_\psi \in \mathbb{Z}^\Gamma$. The map $\psi \mapsto n_\psi - m_2$ is a homomorphism.

(17) Similarly, find the finite index subgroup G_2 of $\mathbb{Z}^\Gamma$ of all elements k such that $k \circ h_1$ lifts through f and g. For every $k \in G_1$ we have

$$k \circ h_1 \circ f = g \circ n_k \circ h_2$$

where $n_k \in \mathbb{Z}^{\Gamma}$. The map $k \mapsto n_k - m_2$ is also a homomorphism (linear).

(18) Using generators of $C_{\text{thick}}(g)$ construct ψ and k such that $k + m_1 = n_k + n_\psi + m_2$. If $m_1 - m_2$ is not in the image of $n_k + n_\psi - k$, **continue.**

(19) Output **maps are equivalent** and $\psi \circ k \circ h_1$; halt.

(20) **end do**

(21) **end do**

(22) output **maps are not equivalent** and halt.

If the algorithm exits on step 17, then $\phi \circ h_0$ realizes the equivalence between f and g, by construction. Otherwise, no such equivalence exists, by Proposition 8.2, and thus the above algorithm satisfies the conditions of our main theorem.

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