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LENGTH OF A CURVE IS QUASI-CONVEX ALONG A TEICHMÜLLER GEODESIC

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Abstract

We show that for every simple closed curve α, the extremal length and the hyperbolic length of α are quasi-convex functions along any Teichmüller geodesic. As a corollary, we conclude that, in Teichmüller space equipped with the Teichmüller metric, balls are quasi-convex.

1. Introduction

In this paper we examine how the extremal length and the hyperbolic length of a measured lamination change along a Teichmüller geodesic. We prove that these lengths are quasi-convex functions of time. The convexity issues in Teichmüller space equipped with Teichmüller metric are hard to approach and are largely unresolved. For example, it is not known whether it is possible for the convex hull of three points in Teichmüller space to be the entire space. (This is an open question of Masur.)

Let S be a surface of finite topological type. Denote the Teichmüller space of S equipped with the Teichmüller metric by $\mathcal{T}(S)$. For a Riemann surface x and a measured lamination μ, we denote the extremal length of μ in x by $\operatorname{Ext}_x(\mu)$ and the hyperbolic length of μ in x by $\operatorname{Hyp}_x(\mu)$.

Theorem A. *There exists a constant* K*, such that for every measured lamination* μ*, any Teichmüller geodesic* $\mathcal{G}$*, and points* $x, y, z \in \mathcal{T}(S)$ *appearing in that order along* $\mathcal{G}$*, we have*

$$\operatorname{Ext}_y(\mu) \leq \mathsf{K} \max\big(\operatorname{Ext}_x(\mu), \operatorname{Ext}_z(\mu)\big),$$

and

$$\operatorname{Hyp}_y(\mu) \leq \mathsf{K} \max\big(\operatorname{Hyp}_x(\mu), \operatorname{Hyp}_z(\mu)\big).$$

In sec §7, we provide some examples showing that the quasi-convexity is the strongest statement one can hope for:

Theorem B. *The hyperbolic length and the extremal length of a curve are in general not convex functions of time along a Teichmüller geodesic.*

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This contrasts with the results of Kerckhoff [**Ker83**], Wolpert [**Wol87**], and Bestvina-Bromberg-Fujiwara-Souto [**BBFS09**]. They proved, respectively, that the hyperbolic length functions are convex along earthquake paths, Weil-Petersson geodesics, and certain shearing paths.

As a consequence of Theorem A, we show that balls in Teichmüller space are quasi-convex.

Theorem C. *There exists a constant* c *so that, for every* $x \in \mathcal{T}(S)$*, every radius,* r *and points* y *and* z *in the ball* $\mathcal{B}(x, \mathsf{r})$*, the geodesic segment* $[y, z]$ *connecting* y *to* z *is contained in* $\mathcal{B}(x, \mathsf{r} + \mathsf{c})$*.*

We also construct an example of a long geodesic that stays near the boundary of a ball, suggesting that balls in $\mathcal{T}(S)$ may not be convex.

A Teichmüller geodesic can be described very explicitly as a deformation of a flat structure on S, namely, by stretching the horizontal direction and contracting the vertical direction. Much is known about the behavior of a Teichmüller geodesic. Our proof consists of combining the length estimates given in [**Min96**, **Raf07b**, **CR07**] with the descriptions of the behavior of a Teichmüller geodesic developed in [**Raf05**, **Raf07a**, **CRS08**].

As a first step, for a curve γ and a quadratic differential q, we provide an estimate for the extremal length of γ in the underlying conformal structure of q (Theorem 8) by describing what are the contributions to the extremal length of γ from the restriction of γ to various pieces of the flat surface associated to q. These pieces are either *thick subsurfaces* or annuli with large moduli. We then introduce the notions of *essentially horizontal* and *essentially vertical* (Corollary 10 and Definition 12). Roughly speaking, a curve γ is essentially horizontal in q if the restriction of γ to some piece of q contributes a definite portion of the total extremal length of γ and if γ is *mostly horizontal* in that piece. We show that, while γ is essentially vertical, its extremal length is *essentially decreasing*, and while γ is essentially horizontal, its extremal length is *essentially increasing* (Theorem 15). This is because the flat length of the portion of γ that is mostly horizontal grows exponentially fast and becomes more and more horizontal. The difficulty with making this argument work is that the thick-thin decomposition of q changes as time goes by and the portion of γ that is horizontal and has a significant extremal length can spread onto several thick pieces. That is why we need to talk about the contribution to the extremal length of γ from every sub-arc of γ (Lemma 13). The theorem then follows from careful analysis of various possible situations. The proof for the hyperbolic length follows a similar path and is presented in §6.

Notation. The notation $A \stackrel{\cdot}{\asymp} B$ means that the ratio A/B is bounded both above and below by constants depending on the topology of S

only. When this is true, we say A is *comparable* with B, or A and B are comparable. The notation $A \stackrel{.}{\prec} B$ means that A/B is bounded above by a constant depending on the topology of S.

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2. Background

Hyperbolic metric. Let x be a Riemann surface or equivalently (using uniformization) a complete hyperbolic metric on S. By a *curve* on S we always mean a free homotopy class of non-trivial non-peripheral simple closed curve. Every curve γ has a unique geodesic representative in the hyperbolic metric x which we call the *x–geodesic representative* of γ. We denote the hyperbolic length of the x–geodesic representative of γ by $\operatorname{Hyp}_x(\gamma)$ and refer to it as the x–length γ.

For a small positive constant ϵ_1, the thick-thin decomposition of x is a pair $(\mathcal{A}, \mathcal{Y})$, where $\mathcal{A}$ is the set of curves in x that have hyperbolic length less than ϵ_1, and $\mathcal{Y}$ is the set of components of $S \setminus (\cup_{\alpha \in \mathcal{A}} \alpha)$. Note that, so far, we are only recording the topological information. One can make this into a geometric decomposition as follows: for each $\alpha \in \mathcal{A}$, consider the annulus that is a regular neighborhood of the x–geodesic representative of α and has boundary length of ϵ_0. For $\epsilon_0 > \epsilon_1 > 0$ small enough, these annuli are disjoint (the Margulis Lemma) and their complement is a union of subsurfaces with horocycle boundaries of length ϵ_0. For each $Y \in \mathcal{Y}$ we denote this representative of the homotopy class of Y by Y_x.

If μ is a set of curves, then $\operatorname{Hyp}_x(\mu)$ is the sum of the lengths of the x–geodesic representatives of the curves in μ. A short marking in Y_x is a set μ_Y of curves in Y so that $\operatorname{Hyp}_x(\mu_Y) = O(1)$ and μ_Y fills the surface Y (that is, every curve intersecting Y intersects some curve in μ_Y).

If γ is a curve and $Y \in \mathcal{Y}$, the *restriction* $\gamma|_{Y_x}$ of γ to Y_x is the union of arcs obtained by taking the intersection of the x–geodesic representative of γ with Y_x. Let $\gamma|_Y$ be the set of homotopy classes (rel ∂Y) of arcs in Y with end points on ∂Y. We think of $\gamma|_Y$ as a set of weighted arcs to keep track of multiplicity. Note that $\gamma|_Y$ has only topological information while $\gamma|_{Y_x}$ is a set of geodesic arcs. An alternate way of defining $\gamma|_Y$ is to consider the cover $\widetilde{Y} \to S$ corresponding to Y; that is, the cover where $\widetilde{Y}$ is homeomorphic to Y and such that $\pi_1(\widetilde{Y})$ projects to a subgroup of $\pi_1(S)$ that is conjugate to $\pi_1(Y)$. Use the hyperbolic metric to construct a boundary at infinity for $\widetilde{Y}$. Then $\gamma|_{\widetilde{Y}}$ is the homotopy class of arcs in $\widetilde{Y}$ that are lifts of γ and are not boundary parallel. Now the natural homeomorphism from $\widetilde{Y}$ to Y sends $\gamma|_{\widetilde{Y}}$ to $\gamma|_Y$.

By $\operatorname{Hyp}_x(\gamma|_Y)$, we mean the x–length of the shortest representatives of $\gamma|_Y$ in Y_x. It is known that (see, for example, [**CR07**])

$$\operatorname{Hyp}_x(\gamma|_Y) = \operatorname{Hyp}_x(\gamma|_{Y_x}) \stackrel{.}{\asymp} \mathrm{i}(\gamma, \mu_Y), \tag{1}$$

where $\mathrm{i}(\cdot,\cdot)$ is the geometric intersection number and $\mathrm{i}(\gamma, \mu_Y)$ is the sum of the geometric intersection numbers between γ and curves in μ_Y.

Euclidean metric. Let q be a quadratic differential on x. In a local coordinate z, q can be represented as $q(z)dz^2$ where $q(z)$ is holomorphic (when x has punctures, q is allowed to have poles of degree one at punctures). We call the metric $|q| = |q(z)|(dx^2 + dy^2)$ the flat structure of q. This is a locally flat metric with singularities at zeros of $q(z)$ (see [**Str84**] for an introduction to the geometry of q). The q–geodesic representative of a curve is not always unique; there may be a family of parallel copies of geodesics foliating a flat cylinder. For a curve α, we denote this flat cylinder of all q–geodesic representatives of α by F_α^q or F_α if q is fixed.

Consider again the thick-thin decomposition $(\mathcal{A}, \mathcal{Y})$ of x. (If q is a quadratic differential on x, we sometimes call this the thick-thin decomposition of q. Note that $(\mathcal{A}, \mathcal{Y})$ depends only on the underlying conformal structure.) For $Y \in \mathcal{Y}$, the homotopy class of Y has a representative with q–geodesic boundaries that is disjoint from the interior of the flat cylinders F_α, for every $\alpha \in \mathcal{A}$. We denote this subsurface by Y_q.

Example 1. The subsurface Y_q may be degenerate in several ways. It may have an empty interior (see [**Raf05**] for an example of this and more careful discussion). Even when Y_q has some interior, it is possible for the boundaries of Y_q to intersect each other (not transversally, however) or not to be embedded curves. To give the reader some intuition of possible difficulties, we construct an example of a subsurface Y and a curve β intersecting Y essentially so that the geodesic representative of β intersects Y_q at only one point.

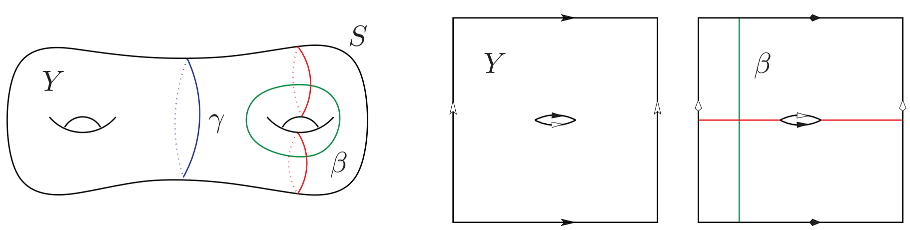

Figure 1. The arc β intersect, Y_q at one point.

Let q be the quadratic differential obtained by gluing two Euclidean squares along a slit at the middle and let Y be the subsurface homeomorphic to a twice punctured torus containing one of these squares (see Fig. 1). Let γ be the curve going around the slit and let β be a curve in the second torus that intersects the boundary components of Y once each and is disjoint from γ. The picture on the left represents these objects topologically and the picture on the right depicts Y_q and a geodesic representative for β in q. We can see that Y_q has some interior but the geodesic representatives of two boundaries of Y intersect each other. In fact, they intersect the geodesic representative of γ. The intersection of β with Y_q is just a single point.

Let $\operatorname{diam}_q(Y)$ denote the q–diameter of Y_q. We recall the following theorem relating the hyperbolic and the flat lengths of a curve in Y.

Theorem 2 ([**Raf07b**]). *For every curve γ in Y,*

$$\ell_q(\gamma) \stackrel{\cdot}{\asymp} \operatorname{Hyp}_x(\gamma)\, \operatorname{diam}_q(Y).$$

Since Y_q can be degenerate, one has to be more careful in defining $\ell_q(\gamma|_Y)$. Again we consider the cover $\widetilde{Y} \to S$ corresponding to Y and this time we equip $\widetilde{Y}$ with the locally Euclidean metric $\tilde{q}$ that is the pullback of q. The subsurface Y_q lifts isometrically to a subsurface $\widetilde{Y}_q$ in $\widetilde{Y}$. Consider the lift $\tilde{\gamma}$ of the q–geodesic representative of γ to $\widetilde{Y}$ and the restriction of $\tilde{\gamma}$ to Y_q. We define $\ell_q(\gamma|_Y)$ to be the $\tilde{q}$–length of this restriction. Note that $\ell_q(\gamma|_Y)$ may equal zero. (See the example at the end of [**Raf07b**].) However, a modified version of Equation (1) still holds true for $\ell_q(\gamma|_Y)$:

Proposition 3. *For every curve γ,*

$$\frac{\ell_q(\gamma|_Y)}{\operatorname{diam}_q(Y)} + \mathrm{i}(\gamma, \partial Y) \stackrel{\cdot}{\asymp} \mathrm{i}(\gamma, \mu_Y).$$

Proof. As above, consider the cover $\widetilde{Y} \to S$, the subsurface $\widetilde{Y}_q$ that is the isometric lift of Y_q, and the lift $\tilde{\gamma}$ of the q–geodesic representative of γ. For every curve $\alpha \in \mu_Y$, there is a lift of α that is a simple closed curve. To simplify notation, we denote this lift again by α and the collection of these curves by μ_Y. Let $d = \operatorname{diam}_q(Y)$, let Z be the d–neighborhood of $\widetilde{Y}_q$ in $\widetilde{Y}$, and let ω be an arc in Z constructed as follows. Choose an arc of $\gamma|_{\tilde{Y}_q}$ (which is potentially just a point) and at each end point p, extend this arc perpendicularly to $\partial\widetilde{Y}_q$ until it hits ∂Z at a point p' (see Fig. 2).

From the construction, we have

$$\ell_q(\omega) = \ell_q(\omega|_{\tilde{Y}_q}) + 2d.$$

Summing over all such arcs, we have:

$$\sum_{\omega} \ell_q(\omega) = \ell_q(\gamma|_Y) + d\, \mathrm{i}(\gamma, \partial Y).$$

Also,

$$\sum_{\omega} \mathrm{i}(\omega, \mu_Y) = \mathrm{i}(\gamma, \mu_Y).$$

Hence, to prove the lemma, we only need to show

$$d\, \mathrm{i}(\omega, \mu_Y) \stackrel{\cdot}{\asymp} \ell_q(\omega). \tag{2}$$

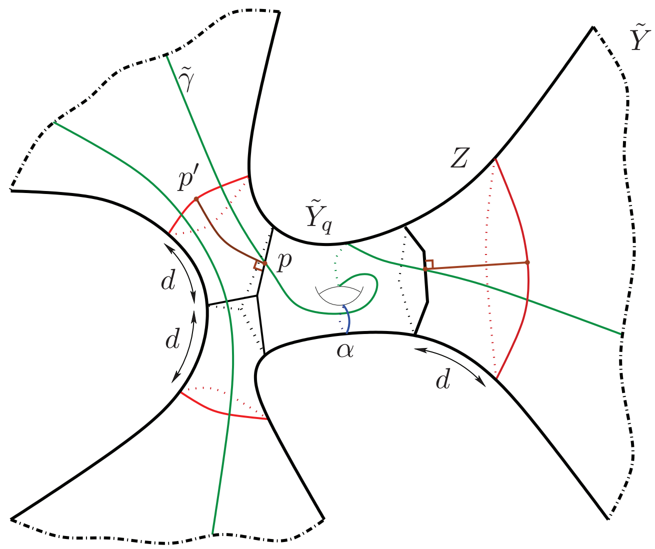

Figure 2. An arc in $\gamma|_{\tilde{Y}_q}$ can be extended to an arc with end points on Z.

The arguments needed here are fairly standard. In the interest of brevity, we point the reader to some references instead of repeating the arguments. Let α be a curve in μ_Y. By Theorem 2, the q–length of the shortest essential curve in Z (which has hyperbolic length comparable with 1) is comparable with d; hence the argument in the proof of [**Raf07b**, lemma 5] also implies

$$\ell_q(\alpha)\,\ell_q(\omega) \overset{\cdot}{\succ} d^2\,\mathrm{i}(\omega,\alpha).$$

Therefore, $l_q(\omega) \overset{\cdot}{\succ} d\,\mathrm{i}(\omega,\alpha)$. Summing over curves $\alpha \in \mu_Y$ (the number of which depends on the topology of S only), we have

$$\ell_q(\omega) \overset{\cdot}{\succ} d\,\mathrm{i}(\omega,\mu_Y).$$

It remains to show the other direction of Equation (2). Here, one needs to construct paths in Y_q (traveling along the geodesics in μ_Y) representing arcs in $\gamma|_Y$ whose lengths are of order $d\,\mathrm{i}(\gamma,\mu_Y)$. This is done in the proof of [**Raf07b**, theorem 6]. q.e.d.

Regular and expanding annuli. Let $(\mathcal{A},\mathcal{Y})$ be the thick-thin decomposition of q and let $\alpha \in \mathcal{A}$. Consider the q–geodesic representative of α and the family of regular neighborhoods of this geodesic in q. Denote the largest regular neighborhood that is still homeomorphic to an annulus by A_α. The annulus A_α contains the flat cylinder F_α in the middle and two *expanding* annuli on each end which we denote by E_α and G_α:

$$A_\alpha = E_\alpha \cup F_\alpha \cup G_\alpha.$$

We call E_α and G_α expanding because if one considers the foliation of these annuli by curves that are equidistant to the geodesic representative of α, the length of these curves increases as one moves away from the q–geodesic representative of α. This is in contrast with F_α, where all the equidistant curves have the same length. (See [**Min92**] for precise definition and discussion.) We denote the q–distance between the boundaries of A_α by $d_q(\alpha)$ and q–distance between the boundaries of E_α, F_α, and G_α by $e_q(\alpha)$, $f_q(\alpha)$, and $g_q(\alpha)$ respectively. When α and q are fixed, we simply use e, f, and g.

Lemma 4 ([**CRS08**])**.** *For $\alpha \in \mathcal{A}$,*

$$\frac{1}{\operatorname{Ext}_x(\alpha)} \stackrel{\cdot}{\asymp} \operatorname{Mod}_x(E_\alpha) + \operatorname{Mod}_x(F_\alpha) + \operatorname{Mod}_x(G_\alpha).$$

Furthermore,

$$\operatorname{Mod}_x(E_\alpha) \stackrel{\cdot}{\asymp} \log\left(\frac{e}{\ell_q(\alpha)}\right), \qquad \operatorname{Mod}_x(G_\alpha) \stackrel{\cdot}{\asymp} \log\left(\frac{g}{\ell_q(\alpha)}\right),$$

and

$$\operatorname{Mod}_x(F_\alpha) = \frac{f}{\ell_q(\alpha)}.$$

Let γ be a curve. The restriction $\gamma|_{A_\alpha}$ is the set of arcs obtained from restricting the q–geodesic representative of γ to A_α, and $\ell_q(\gamma|_{A_\alpha})$ is the sum of the q–lengths of these curves.

Lemma 5. *For the thick-thin decomposition $(\mathcal{A}, \mathcal{Y})$ of q, we have*

$$\ell_q(\gamma) \stackrel{\cdot}{\asymp} \sum_{Y \in \mathcal{Y}} \ell_q(\gamma|_Y) + \sum_{\alpha \in \mathcal{A}} \ell_q(\gamma|_{A_\alpha}).$$

Proof. The annuli A_α are not necessarily disjoint. But the cardinality of $\mathcal{A}$ is uniformly bounded and $\ell_q(\gamma) \geq \ell_q(\gamma|_{A_\alpha})$. Similarly, the number of elements in $\mathcal{Y}$ is uniformly bounded and $\ell_q(\gamma) \geq \ell_q(\gamma|_Y)$. Hence

$$\ell_q(\gamma) \stackrel{\cdot}{\succ} \sum_{Y \in \mathcal{Y}} \ell_q(\gamma|_Y) + \sum_{\alpha \in \mathcal{A}} \ell_q(\gamma|_{A_\alpha}). \tag{3}$$

To see the inequality in the other direction, we note that every segment in the q–geodesic representative of γ is either contained in some A_α, $\alpha \in \mathcal{A}$ or in some Y_q, $Y \in \mathcal{Y}$. q.e.d.

Teichmüller geodesics. Let $q = q(z)dz^2$ be a quadratic differential on x. It is more convenient to use the *natural parameter* $\zeta = \xi + i\eta$, which is defined away from its singularities as

$$\zeta(w) = \int_{z_0}^{w} \sqrt{q(z)}\, dz.$$

In these coordinates, we have $q = d\zeta^2$. The lines $\xi = const$ with transverse measure $|d\xi|$ define the *vertical* measured foliation, associated to q.

Similarly, the *horizontal* measured foliation is defined by $\eta = const$ and $|d\eta|$. The transverse measure of an arc α with respect to $|d\xi|$, denoted by $h_q(\alpha)$, is called the *horizontal length* of α. Similarly, the *vertical length* $v_q(\alpha)$ is the measure of α with respect to $|d\eta|$.

A Teichmüller geodesic can be described as follows. Given a Riemann surface x and a quadratic differential q on x, we can obtain a 1–parameter family of quadratic differentials q_t from q so that, for $t \in \mathbb{R}$, if $\zeta = \xi + i\eta$ are natural coordinates for q, then $\zeta_t = e^t \xi + ie^{-t}\eta$ are natural coordinates for q_t. Let x_t be the conformal structure associated to q_t. Then $\mathcal{G} : \mathbb{R} \to \mathcal{T}(S)$, which sends t to x_t, is a Teichmüller geodesic.

Let $\mathcal{G}\colon [a,b] \to \mathcal{T}(S)$ be a Teichmüller geodesic and q_a and q_b be the initial and terminal quadratic differentials. We use $\ell_a(\boldsymbol{\cdot})$ for the q_a–length of a curve; we use $\operatorname{Ext}_a(\boldsymbol{\cdot})$ for the extremal length of a curve in q_a. Similarly, we denote by $\operatorname{Mod}_a(\boldsymbol{\cdot})$ the modulus of an annulus in q_a. We denote the thick-thin decomposition of q_a by $(\mathcal{A}_a, \mathcal{Y}_b)$. We also write $e_a(\alpha), d_a(\alpha),\ f_a(\alpha)$, and $\ell_a(\alpha)$ in place of $e_{q_a}(\alpha), d_{q_a}(\alpha),\ f_{q_a}(\alpha)$, and $\ell_{q_a}(\alpha)$. When the curve α is fixed, we simplify notation even further and use e_a, d_a, f_a, and ℓ_a. Also, we denote the flat annulus and the expanding annuli corresponding to α in q_a by F^a_α, E^a_α, and G^a_α, or by F^a, E^a, and G^a when α is fixed. Similar notation applies to q_b. The following technical statement will be useful later.

Corollary 6. *Let α be a curve in the intersection of $\mathcal{A}_a$ and $\mathcal{A}_b$. Then*

$$\frac{\operatorname{Ext}_a(\alpha)}{\ell_a(\alpha)} \buildrel{\cdot}\over{\prec} e^{(b-a)} \frac{\operatorname{Ext}_b(\alpha)}{\ell_b(\alpha)},$$

Proof. The length of an arc along a Teichmüller geodesic changes at most exponentially fast. That is, e^{b-a} is an upper-bound for $\frac{e_b}{e_a}$, $\frac{f_b}{f_a}$, $\frac{g_b}{g_a}$, and $\frac{\ell_b}{\ell_a}$. Let $k = \frac{\ell_b}{\ell_a}$. Then

$$\frac{\ell_b \operatorname{Mod}_b(E^b)}{\ell_a \operatorname{Mod}_a(E^a)} \buildrel{\cdot}\over{\asymp} k \frac{\log \frac{e_b}{\ell_b}}{\log \frac{e_a}{\ell_a}} \le k \frac{\log\left(\frac{e^{b-a}}{k}\frac{e_a}{\ell_a}\right)}{\log \frac{e_a}{\ell_a}} \le k \frac{\frac{e^{b-a}}{k} \log \frac{e_a}{\ell_a}}{\log \frac{e_a}{\ell_a}} \le e^{b-a}.$$

By a similar argument,

$$\frac{\ell_b \operatorname{Mod}_b(G^b)}{\ell_a \operatorname{Mod}_a(G^a)} \buildrel{\cdot}\over{\prec} e^{b-a},$$

We also have

$$\frac{\ell_b \operatorname{Mod}_b(F^b)}{\ell_a \operatorname{Mod}_a(F^a)} = \frac{f_b}{f_a} \le e^{b-a}.$$

Then, by Lemma 4 and the estimates above,

$$\frac{\operatorname{Ext}_a}{\ell_a} \div \frac{\operatorname{Ext}_b}{\ell_b} \buildrel{\cdot}\over{\asymp} \frac{\ell_b\big(\operatorname{Mod}_b(E^b) + \operatorname{Mod}_b(F^b) + \operatorname{Mod}_b(G^b)\big)}{\ell_a\big(\operatorname{Mod}_a(E^a) + \operatorname{Mod}_a(F^a) + \operatorname{Mod}_a(G^a)\big)} \buildrel{\cdot}\over{\prec} e^{b-a},$$

which is the desired inequality. q.e.d.

Twisting. In this section we define several notions of twisting and discuss how they relate to each other. First, consider an annulus A with core curve α and let $\tilde{\beta}$ and $\tilde{\gamma}$ be homotopy classes of arcs connecting the boundaries of A (here, homotopy is relative to the end points of an arc). The relative twisting of $\tilde{\beta}$ and $\tilde{\gamma}$ around α, $\mathrm{tw}_\alpha(\tilde{\beta}, \tilde{\gamma})$, is defined to be the geometric intersection number between $\tilde{\beta}$ and $\tilde{\gamma}$. If α is a curve on a surface S, and β and γ are two transverse curves to α, we lift β and γ to the annular cover $\tilde{S}_\alpha$ of S corresponding to α. The curve β (resp., γ) has at least one lift $\tilde{\beta}$ (resp., $\tilde{\gamma}$) that connects the boundaries of $\tilde{S}_\alpha$. We define $\mathrm{tw}_\alpha(\beta, \gamma)$ to be $\mathrm{tw}_\alpha(\tilde{\beta}, \tilde{\gamma})$. This is well defined up to a small additive error ([**Min96**, §3]).

When the surface S is equipped with a metric, one can ask how many times the geodesic representative of γ twists around a curve α. However, this needs to be made precise. When x is a Riemann surface, we define $\mathrm{tw}_\alpha(x, \gamma)$ to be equal to $\mathrm{tw}_\alpha(\beta, \gamma)$ where β is the shortest hyperbolic geodesic in x intersecting α. For a quadratic differential q, the definition is slightly different. We first consider F_α and let β be an arc connecting the boundaries of F_α that is perpendicular to the boundaries. We then define $\mathrm{tw}_\alpha(q, \gamma)$ to be the geometric intersection number between β and any arc in $\gamma|_{F_\alpha}$. These two notions of twisting are related as follows:

Theorem 7 (Theorem 4.3 in [**Raf07a**])**.** *Let q be a quadratic differential in the conformal class of x, and let α and γ be two intersecting curves; then*

$$\big|\,\mathrm{tw}_\alpha(q, \gamma) - \mathrm{tw}_\alpha(x, \gamma)\big| \overset{\cdot}{\prec} \frac{1}{\mathrm{Ext}_x(\alpha)}.$$

3. An Estimate for the Extremal Length

In [**Min96**], Minsky has shown that the extremal length of a curve is comparable to the maximum of the contributions to the extremal length from the pieces of the thick-thin decomposition of the surface. Using this fact and some results in [**Raf07b**] and [**Raf07a**], we can state a similar result relating the flat length of a curve γ to its extremal length.

Theorem 8. *For a quadratic differential q on a Riemann surface x, the corresponding thick-thin decomposition $(\mathcal{A}, \mathcal{Y})$ and a curve γ on x, we have*

$$\begin{aligned}\mathrm{Ext}_x(\gamma) \overset{\cdot}{\asymp} &\sum_{Y \in \mathcal{Y}} \frac{\ell_q(\gamma|_Y)^2}{\mathrm{diam}_q(Y)^2} \\ &+ \sum_{\alpha \in \mathcal{A}} \left(\frac{1}{\mathrm{Ext}_x(\alpha)} + \mathrm{tw}_\alpha^2(q, \gamma)\,\mathrm{Ext}_x(\alpha) \right) \mathrm{i}(\alpha, \gamma)^2.\end{aligned}$$

Proof. First we recall [**Min96**, theorem 5.1] where Minsky states that the extremal length of a curve γ in x is the maximum of the contributions to the extremal length from each thick subsurface and from crossing each short curve. The contribution from each curve $\alpha \in \mathcal{A}$ is given by an expression [**Min96**, equation (4.3)] involving $\mathrm{i}(\alpha, \gamma)$, $\mathrm{tw}_\alpha(x, \gamma)$ and $\mathrm{Ext}_x(\alpha)$. For each subsurface $Y \in \mathcal{Y}$, the contribution to the extremal length from $\gamma|_Y$ is shown to be [**Min96**, theorem 4.3] the square of the hyperbolic length of γ restricted to a representative of Y with a horocycle boundary of a fixed length in x. This is known to be comparable to the square of the intersection number of γ with a short marking μ_Y for Y.

To be more precise, let μ_Y be a set of curves in Y that fill Y so that $\ell_x(\mu_Y) = O(1)$. Then Minsky's estimate can be written as

$$\begin{aligned} \mathrm{Ext}_x(\gamma) \stackrel{\cdot}{\asymp} & \sum_{Y \in \mathcal{Y}} \mathrm{i}(\gamma, \mu_Y)^2 + \\ & \sum_{\alpha \in \mathcal{A}} \left(\frac{1}{\mathrm{Ext}_x(\alpha)} + \mathrm{tw}_\alpha^2(x, \gamma)\, \mathrm{Ext}_x(\alpha) \right) \mathrm{i}(\alpha, \gamma)^2. \end{aligned} \tag{4}$$

From Theorem 7,

$$\big| \mathrm{tw}_\alpha(x, \gamma) - \mathrm{tw}_\alpha(q, \gamma) \big| = O\left(\frac{1}{\mathrm{Ext}_x(\alpha)} \right),$$

and hence,

$$1 + \mathrm{tw}_\alpha(x, \gamma)\, \mathrm{Ext}_x(\alpha) \stackrel{\cdot}{\asymp} 1 + \mathrm{tw}_\alpha(q, \gamma)\, \mathrm{Ext}_x(\alpha).$$

Squaring both sides, and using $(a+b)^2 \stackrel{\cdot}{\asymp} a^2 + b^2$, we get

$$1 + \mathrm{tw}_\alpha^2(x, \gamma)\, \mathrm{Ext}_x(\alpha)^2 \stackrel{\cdot}{\asymp} 1 + \mathrm{tw}_\alpha^2(q, \gamma)\, \mathrm{Ext}_x(\alpha)^2.$$

We now divide both sides by $\mathrm{Ext}_x(\alpha)$ to obtain

$$\left(\frac{1}{\mathrm{Ext}_x(\alpha)} + \mathrm{tw}_\alpha^2(x, \gamma)\, \mathrm{Ext}_x(\alpha) \right) \stackrel{\cdot}{\asymp} \left(\frac{1}{\mathrm{Ext}_x(\alpha)} + \mathrm{tw}_\alpha^2(q, \gamma)\, \mathrm{Ext}_x(\alpha) \right).$$

That is, the second sum in Minsky's estimate is comparable to the second sum in the statement of our proposition.

Now consider the inequality in Proposition 3 for every $Y \in \mathcal{Y}$. After taking the square and adding up, we get

$$\sum_{Y \in \mathcal{Y}} \frac{\ell_q(\gamma|_Y)^2}{\mathrm{diam}_q(Y)^2} + \sum_{\alpha \in \mathcal{A}} \mathrm{i}(\gamma, \alpha)^2 \stackrel{\cdot}{\asymp} \sum_{Y \in \mathcal{Y}} \mathrm{i}(\gamma, \mu_Y)^2,$$

But the term $\sum_{\alpha \in \mathcal{A}} \mathrm{i}(\gamma, \alpha)^2$ is insignificant compared with the term $\frac{\mathrm{i}(\gamma,\alpha)^2}{\mathrm{Ext}_x(\alpha)}$ appearing in the right side of Equation (4). Therefore, we can replace the term $\sum_{Y \in \mathcal{Y}} \mathrm{i}(\gamma, \mu_Y)^2$ in Equation (4) with $\sum_{Y \in \mathcal{Y}} \frac{\ell_q(\gamma|_Y)^2}{\mathrm{diam}_q(Y)^2}$ and obtain the desired inequality. q.e.d.

To simplify the situation, one can provide a lower bound for extremal length using the q–length of γ and the sizes of the subsurface Y_q, $Y \in \mathcal{Y}$, and A_α, $\alpha \in \mathcal{A}$.

Corollary 9. *For any curve γ, the contribution to the extremal length of γ from A_α, $\alpha \in \mathcal{A}$, is bounded below by $\frac{\ell_q(\gamma|_{A_\alpha})^2}{d_q(\alpha)^2}$. In other words,*

$$\operatorname{Ext}_x(\gamma) \stackrel{\cdot}{\succ} \sum_{Y \in \mathcal{Y}} \frac{\ell_q(\gamma|_Y)^2}{\operatorname{diam}_q(Y)^2} + \sum_{\alpha \in \mathcal{A}} \frac{\ell_q(\gamma|_{A_\alpha})^2}{d_q(\alpha)^2}.$$

Proof. Recall the notations E_α, F_α, G_α, e, f, and g from the background section. Denote the q–length of α by a. Every arc of $\gamma|_{A_\alpha}$ has to cross A_α and twist around α, $\operatorname{tw}_\alpha(q,\gamma)$–times. Hence, its length is less than $d_q(\alpha) + \operatorname{tw}_\alpha(q,\gamma)a$. Therefore,

$$\ell_q(\gamma|_{A_\alpha})^2 \stackrel{\cdot}{\prec} \operatorname{i}(\alpha,\gamma)^2 \big(d_q(\alpha)^2 + \operatorname{tw}_\alpha^2(q,\gamma)a^2\big).$$

Thus

$$\begin{aligned}\left(\frac{\ell_q(\gamma|_{A_\alpha})}{d_q(\alpha)\operatorname{i}(\alpha,\gamma)}\right)^2 &\stackrel{\cdot}{\asymp} \frac{d_q(\alpha)^2 + \operatorname{tw}_\alpha^2(q,\gamma)a^2}{d_q(\alpha)^2} \stackrel{\cdot}{\asymp} 1 + \frac{\operatorname{tw}_\alpha^2(q,\gamma)}{d_q(\alpha)^2/a^2} \\ &\stackrel{\cdot}{\prec} \frac{1}{\operatorname{Ext}_x(\alpha)} + \frac{\operatorname{tw}_\alpha^2(q,\gamma)}{\log\frac{e}{a} + \frac{f}{a} + \log\frac{g}{a}} \\ &\stackrel{\cdot}{\asymp} \frac{1}{\operatorname{Ext}_x(\alpha)} + \operatorname{tw}_\alpha^2(x,\gamma)\operatorname{Ext}_x(\alpha)\end{aligned}$$

We can now multiply both sides by $\operatorname{i}^2(\alpha,\gamma)$ and replace the common term in the second sum of the estimate in Theorem 8 with $\left(\frac{\ell_q(\gamma|_{A_\alpha})}{d_q(\alpha)}\right)^2$ to obtain the corollary.

The estimate here seems excessively generous, but there is a case where the two estimates are comparable. This happens when α is not very short, the twisting parameter is zero, and $\gamma|_{A_\alpha}$ is a set of $\operatorname{i}(\gamma,\alpha)$–many arcs of length comparable to one. q.e.d.

Essentially horizontal curves. The goal of this subsection is to define essentially horizontal curves. We will later show that if a curve is essentially horizontal at some point along a Teichmüller geodesic, its extremal length is coarsely non-decreasing from that point on. First we show that, for a quadratic differential q and a curve γ, there is a subsurface of q where the contribution to the extremal length of γ coming from this subsurface if a definite portion of the extremal length of γ in q. Then we will call γ essentially horizontal (see definition below); if restricted to at least one such subsurface, γ is mostly horizontal (the horizontal length of the restriction is larger than the vertical length of the restriction). The coarsely non-decreasing property of an essentially horizontal curve is a consequence of the fact that the flat length of a mostly horizontal curve grows exponentially fast along a Teichmüller geodesic.

Corollary 10. *Let $(\mathcal{A}, \mathcal{Y})$ be a thick-thin decomposition for q, and let γ be a curve that is not in $\mathcal{A}$. Then*

1) *For every $Y \in \mathcal{Y}$,*

$$\operatorname{Ext}_x(\gamma) \stackrel{\cdot}{\succ} \frac{\ell_q(\gamma|_Y)^2}{\operatorname{diam}_q(Y)^2}.$$

2) *For $\alpha \in \mathcal{A}$ and a flat annulus F_α whose core curve is α,*

$$\operatorname{Ext}_x(\gamma) \stackrel{\cdot}{\succ} \frac{\ell_q(\gamma|_{F_\alpha})^2 \operatorname{Ext}_x(\alpha)}{\ell_q(\alpha)^2}.$$

3) *For $\alpha \in \mathcal{A}$ and an expanding annulus E_α whose core curve is α,*

$$\operatorname{Ext}_x(\gamma) \stackrel{\cdot}{\succ} \mathrm{i}(\alpha, \gamma)^2 \operatorname{Mod}_x(E_\alpha).$$

Furthermore, at least one of these inequalities is an equality up to a multiplicative error.

Remark 11. In light of Theorem 8, we can think of the terms appearing in the right hand sides of the above inequalities as the contribution to the extremal length of γ coming from Y, F_α, and E_α respectively. Then Corollary 10 states that one of these subsurfaces contributes a definite portion of the extremal length of γ.

Proof of Corollary 10. Parts one and three follow immediately from Theorem 8. We prove part two. As before,

$$\ell_q(\gamma|_{F_\alpha})^2 \stackrel{\cdot}{\prec} (\operatorname{tw}_\alpha(q,\gamma)^2 \ell_q(\alpha)^2 + f_q(\alpha)^2)\, \mathrm{i}(\alpha,\gamma)^2. \tag{5}$$

Hence

$$\begin{aligned}\frac{\ell_q(\gamma|_{F_\alpha})^2 \operatorname{Ext}_x(\alpha)}{\ell_q(\alpha)^2} &\stackrel{\cdot}{\prec} \frac{\operatorname{tw}_\alpha(q,\gamma)^2 \ell_q(\alpha)^2 + f_q(\alpha)^2}{\ell_q(\alpha)^2} \operatorname{Ext}_x(\alpha)\, \mathrm{i}(\alpha,\gamma)^2 \\ &\stackrel{\cdot}{\prec} \operatorname{tw}_\alpha(q,\gamma)^2 \operatorname{Ext}_x(\alpha)\, \mathrm{i}(\alpha,\gamma)^2 \\ &\quad + \operatorname{Ext}_x(\alpha) \operatorname{Mod}_x(F_\alpha)^2\, \mathrm{i}(\alpha,\gamma)^2.\end{aligned}$$

But $\operatorname{Ext}_x(\alpha) \operatorname{Mod}_x(F_\alpha)^2 \leq \frac{1}{\operatorname{Ext}_x(\alpha)}$ and thus, by Theorem 8, the above expression is bounded above by a multiple of $\operatorname{Ext}_x(\gamma)$.

To see that one of the inequalities has to be an equality, we observe that the number of pieces in the thick-thin decomposition $(\mathcal{A}, \mathcal{Y})$ is uniformly bounded. Therefore, some term in Theorem 8 is comparable to $\operatorname{Ext}_x(\gamma)$. If this is a term in the first sum, then the inequality in part one is an equality. Assume for $\alpha \in \mathcal{A}$ that

$$\operatorname{Ext}_x(\gamma) \stackrel{\cdot}{\asymp} \frac{\mathrm{i}(\alpha,\gamma)^2}{\operatorname{Ext}_x(\alpha)}.$$

We either have $\mathrm{Ext}_x(E_\alpha) \overset{.}{\asymp} \mathrm{Ext}_x(\alpha)$ or $\mathrm{Ext}_x(F_\alpha) \overset{.}{\asymp} \mathrm{Ext}_x(\alpha)$. In the first case, the estimate in part three is comparable to $\mathrm{Ext}_x(\gamma)$. In the second case,

$$\mathrm{Ext}_x(\gamma) \overset{.}{\asymp} \frac{\mathrm{i}(\alpha,\gamma)^2}{\mathrm{Ext}_x(F_\alpha)} = \left(\frac{\mathrm{i}(\alpha,\gamma)^2 f_q(\alpha)^2}{\ell_q(\alpha)^2}\right)\left(\frac{\ell_q(\alpha)}{f_q(\alpha)}\right) \overset{.}{\prec} \frac{\ell_q(\gamma|_{F_\alpha})^2}{\ell_q(\alpha)^2}\,\mathrm{Ext}_x(\alpha),$$

which means the inequality in part two is an equality.

The only remaining case is when

$$\mathrm{Ext}_x(\gamma) \overset{.}{\asymp} \mathrm{tw}_\alpha(q,\gamma)^2\,\mathrm{Ext}_x(\alpha)\,\mathrm{i}(\alpha,\gamma)^2.$$

In this case, the estimate in part two is comparable to $\mathrm{Ext}_x(\gamma)$. This follows from $\ell_q(\gamma|_{F_\alpha}) \overset{.}{\succ} \mathrm{tw}_\alpha(q,\gamma)\ell_q(\alpha)\,\mathrm{i}(\alpha,\gamma)$. q.e.d.

Definition 12. We say that γ is *essentially horizontal*, if at least one of the following holds:

1) $\mathrm{Ext}_x(\gamma) \overset{.}{\asymp} \frac{\ell_q(\gamma|_Y)^2}{\mathrm{diam}_q(Y)^2}$ and $\gamma|_Y$ is mostly horizontal (i.e., its horizontal length is larger than its vertical length) for some $Y \in \mathcal{Y}$.
2) $\mathrm{Ext}_x(\gamma) \overset{.}{\asymp} \frac{\ell_q(\gamma|_{F_\alpha})^2\,\mathrm{Ext}_x(\alpha)}{\ell_q(\alpha)^2}$ and $\gamma|_{F_\alpha}$ is mostly horizontal for some flat annulus F_α whose core curve is $\alpha \in \mathcal{A}$.
3) $\mathrm{Ext}_x(\gamma) \overset{.}{\asymp} \mathrm{i}(\alpha,\gamma)^2\,\mathrm{Mod}_x(E_\alpha)$ for some expanding annulus E_α whose core curve is $\alpha \in \mathcal{A}$.

Extremal length of geodesic arcs. Consider the q–geodesic representative of a curve γ and let ω be an arc of this geodesic. We would like to estimate the contribution that ω makes to the extremal length of γ in q. Let $(\mathcal{A},\mathcal{Y})$ be the thick-thin decomposition of q. Let λ_ω be the maximum over $\mathrm{diam}_q(Y)$ for subsurfaces $Y \in \mathcal{Y}$ that ω intersects and over all $d_q(\alpha)$ for curves $\alpha \in \mathcal{A}$ that ω crosses. Let σ_ω be the q–length of the shortest curve β that ω intersects. We define

$$\mathrm{X}(\omega) = \frac{\ell_q(\omega)^2}{\lambda_\omega^2} + \log\frac{\lambda_\omega}{\sigma_\omega},$$

and claim that the contribution from ω to the extremal length of γ is at least $\mathrm{X}(\omega)$. This is stated in the following lemma:

Lemma 13. *Let Ω be a set of disjoint sub-arcs of γ. Then*

$$\mathrm{Ext}_q(\gamma) \overset{.}{\succ} |\Omega|^2 \min_{\omega\in\Omega} \mathrm{X}(\omega).$$

Proof. Let $(\mathcal{A}, \mathcal{Y})$ be the thick-thin decomposition of q. We have

$$\operatorname{Ext}_x(\gamma) \geq \sum_{Y \in \mathcal{Y}} \frac{\ell_q(\gamma|_Y)^2}{\operatorname{diam}_q(Y)^2} + \sum_{\alpha \in \mathcal{A}} \frac{\ell_q(\gamma|_{A_\alpha})^2}{d_q(\alpha)^2} \tag{6}$$

$$\overset{\cdot}{\succ} \left(\sum_{Y \in \mathcal{Y}} \frac{\ell_q(\gamma|_Y)}{\operatorname{diam}_q(Y)} + \sum_{\alpha \in \mathcal{A}} \frac{\ell_q(\gamma|_{A_\alpha})}{d_q(\alpha)} \right)^2 \tag{7}$$

$$\geq \left(\sum_{Y \in \mathcal{Y}} \sum_{\omega \in \Omega} \frac{\ell_q(\omega|_Y)}{\operatorname{diam}_q(Y)} + \sum_{\alpha \in \mathcal{A}} \sum_{\omega \in \Omega} \frac{\ell_q(\omega|_{A_\alpha})}{d_q(\alpha)} \right)^2 \tag{8}$$

$$\overset{\cdot}{\succ} \left(\sum_{\omega \in \Omega} \left(\sum_{Y \in \mathcal{Y}} \frac{\ell_q(\omega|_Y)}{\lambda_\omega} + \sum_{\alpha \in \mathcal{A}} \frac{\ell_q(\omega|_{A_\alpha})}{\lambda_\omega} \right) \right)^2 \geq \left(\sum_{\omega \in \Omega} \frac{\ell_q(\omega)}{\lambda_\omega} \right)^2. \tag{9}$$

Inequality (6) follows from Corollary 9. To obtain (7), we are using

$$\sum_{i=1}^{n} x_i^2 \geq \frac{1}{n} \Big(\sum_{i=1}^{n} x_i \Big)^2$$

and the fact that the number of components in $\mathcal{Y}$ and in $\mathcal{A}$ are uniformly bounded. Line (8) follows from the fact that arcs in Ω are disjoint subarcs of γ. To get (9), we then rearrange terms and use the fact that for all non-zero terms, $\operatorname{diam}_q(Y)$ and $d_q(\alpha)$ are less than λ_ω.

Now let $\alpha_1, \ldots, \alpha_k$ be the sequence of curves in $\mathcal{A}$ that ω intersects as it travels from the shortest curve β to the largest subsurface it intersects, which has size of at most λ_ω. Note that either $\alpha_1 = \beta$ or α_1 is the boundary of the thick subsurface containing β_ω. Either way, $\ell_q(\alpha_1) \overset{\cdot}{\prec} \sigma_\omega$. Also, $d_q(\alpha_i) \overset{\cdot}{\succ} \ell_q(\alpha_{i+1})$. This is because α_i and α_{i+1} are boundaries of some subsurface $Y \in \mathcal{Y}$. Finally, $d_q(\alpha_k) \overset{\cdot}{\succ} \lambda_\omega$. Therefore,

$$\sum_{i=1}^{k} \log \frac{d_q(\alpha_i)}{\ell_q(\alpha_i)} = \log \prod_{i=1}^{k} \frac{d_q(\alpha_i)}{\ell_q(\alpha_i)} \overset{\cdot}{\succ} \log \frac{d_q(\alpha_k)}{\ell_q(\alpha_1)} \overset{\cdot}{\succ} \log \frac{\lambda_\omega}{\sigma_\omega}.$$

Since $|\mathcal{A}| \overset{\cdot}{\asymp} 1$, we can conclude that, for each ω, there is curve α so that $\operatorname{i}(\alpha, \omega) \geq 1$ and $\log \dfrac{d_q(\alpha)}{\ell_q(\alpha)} \overset{\cdot}{\succ} \log \dfrac{\lambda_\omega}{\sigma_\omega}$. Using Theorem 8,

$$\begin{aligned}
\operatorname{Ext}_q(\gamma) &\stackrel{\cdot}{\succ} \sum_{\alpha\in\mathcal{A}} \frac{\mathrm{i}(\alpha,\gamma)^2}{\operatorname{Ext}_q(\alpha)} \stackrel{\cdot}{\succ} \left(\sum_{\alpha\in\mathcal{A}} \frac{\mathrm{i}(\alpha,\gamma)}{\sqrt{\operatorname{Ext}_q(\alpha)}}\right)^2 \\
&\stackrel{\cdot}{\succ} \left(\sum_{\omega\in\Omega}\sum_{\alpha\in\mathcal{A}} \mathrm{i}(\alpha,\omega)\sqrt{\log\frac{d_q(\alpha)}{\ell_q(\alpha)}}\right)^2 \\
&\stackrel{\cdot}{\succ} \left(\sum_{\omega\in\Omega} \sqrt{\log\frac{\lambda_\omega}{\sigma_\omega}}\right)^2.
\end{aligned}$$

Combining the above two inequalities, we get

$$\begin{aligned}
\operatorname{Ext}_q(\gamma) &\stackrel{\cdot}{\succ} \left(\sum_{\omega\in\Omega} \frac{\ell_q(\omega)}{\lambda_\omega}\right)^2 + \left(\sum_{\omega\in\Omega} \sqrt{\log\frac{\lambda_\omega}{\sigma_\omega}}\right)^2 \\
&\stackrel{\cdot}{\succ} \left(\sum_{\omega\in\Omega} \sqrt{X(\omega)}\right)^2 \geq |\Omega|^2 \min_\omega X(\omega).
\end{aligned}$$

q.e.d.

We also need the following technical lemma.

Lemma 14. *Let q_a and q_b be two points along a Teichmüller geodesic and let $(\mathcal{A}_a, \mathcal{Y}_a)$ and $(\mathcal{A}_b, \mathcal{Y}_b)$ be their thick-thin decompositions respectively. Let $Y \in \mathcal{Y}_a$:*

- *If $\beta \in \mathcal{A}_b$ intersects Y, then $d_b(\beta) \leq e^{(b-a)} \operatorname{diam}_a(Y)$.*
- *If $Z \in \mathcal{Y}_b$ intersects Y, then $\operatorname{diam}_b(Z) \leq e^{(b-a)} \operatorname{diam}_a(Y)$.*

Similarly, if $\alpha \in \mathcal{A}_a$,

- *If $\beta \in \mathcal{A}_b$ intersects α, then $d_b(\beta) \leq e^{(b-a)} \ell_a(\alpha)$.*
- *If $Z \in \mathcal{Y}_b$ intersects α, then $\operatorname{diam}_b(Z) \leq e^{(b-a)} \ell_a(\alpha)$.*

Proof. Let γ be the shortest curve system in q_a that fills Y. Then $l_b(\gamma) \stackrel{\cdot}{\prec} e^{(b-a)} \operatorname{diam}_a(Y)$. If Y intersects $\beta \in \mathcal{A}_b$, then some curve in γ has to intersect A_β essentially and we have

$$d_b(\alpha) \leq l_b(\gamma) \stackrel{\cdot}{\prec} e^{(b-a)} \operatorname{diam}_a(Y).$$

If Y intersects some subsurface $Z \in \mathcal{Y}_b$, then Z has an essential arc ω in Z whose q_b–length is less than the q_b–length of γ. Also, if Y intersects a boundary component δ of Z,

$$\ell_b(\gamma) \stackrel{\cdot}{\succ} d_b(\delta) \geq \ell_b(\delta).$$

By doing a surgery between ω and δ, one obtains an essential curve in Z whose q_b–length is less than a fixed multiple of γ. Hence

$$\operatorname{diam}_b(Z) \stackrel{\cdot}{\prec} l_b(\gamma) \stackrel{\cdot}{\prec} e^{(b-a)} \operatorname{diam}_a(Y),$$

which is what we claimed. The argument for $\alpha \in \mathcal{A}_a$ is similar. q.e.d.

4. The Main Theorem

Let $\mathcal{G}\colon \mathbb{R} \to \mathcal{T}(S)$ be a Teichmüller geodesic. We denote the Riemann surface $\mathcal{G}(t)$ by $\mathcal{G}_t$ and the corresponding quadratic differential in $\mathcal{G}_t$ by q_t. For a curve γ, denote the extremal length of γ on $\mathcal{G}_t$ by $\mathrm{Ext}_t(\gamma)$ and the thick-thin decomposition of q_t simply by $(\mathcal{A}_t, \mathcal{Y}_t)$.

Theorem 15. *There exists a constant* K*, such that for every measured foliation* μ*, any Teichmüller geodesic* $\mathcal{G}$*, and points* $x, y, z \in \mathcal{T}(S)$ *appearing in that order along* $\mathcal{G}$*, we have*

$$\mathrm{Ext}_y(\mu) \le \mathsf{K} \max\big(\mathrm{Ext}_x(\mu), \mathrm{Ext}_z(\mu)\big).$$

Proof. Let the times $a < b < c \in \mathbb{R}$ be such that $x = \mathcal{G}_a$, $y = \mathcal{G}_b$, and $z = \mathcal{G}_c$. Recall ([**Ker80**]) that the extremal length

$$\mathrm{Ext}\colon \mathcal{MF}(S) \times \mathcal{T}(S) \to \mathbb{R}$$

is a continuous function, and that the weighted simple closed curves are dense in $\mathcal{MF}(S)$. Since the limit of quasi-convex functions is itself quasi-convex, and a multiple of a quasi-convex function is also quasi-convex (with the same constant in both cases), it is sufficient to prove the theorem for simple closed curves only. That is, we can assume that every leaf of μ is homotopic to a curve γ and the transverse measure is one.

Suppose first that the extremal length $\mathrm{Ext}_b(\gamma)$ is small. Then it is comparable to the hyperbolic length of γ [**Mas85**]. In this case the result is already known. More precisely, if γ is in $\mathcal{A}_a$, $\mathcal{A}_b$, and $\mathcal{A}_c$, the statement for the hyperbolic length follows from [**Raf07a**, corollary 3.4] and [**Raf07a**, theorem 1.2]. Also, if $\gamma \in \mathcal{A}_b$ but either $\gamma \notin \mathcal{A}_a$ or $\gamma \notin \mathcal{A}_c$, then the statement is clearly true. Therefore, we can assume there is a lower bound on the length of γ at b, where the lower bound depends on the topology of x only.

Choose any element in the thick-thin decomposition of q_b with significant contribution to the extremal length of γ, i.e. where the corresponding inequality in Corollary 10 is an equality up to a multiplicative error. The restriction of γ to this subsurface is either mostly horizontal or mostly vertical. That is, γ is either essentially horizontal or essentially vertical. If γ is essentially horizontal, Proposition 16 implies $\mathrm{Ext}_b(\gamma) \stackrel{\cdot}{\prec} \mathrm{Ext}_c(\gamma)$ and we are done. Otherwise, γ is essentially vertical. In this case, we can reverse time, changing the role of the horizontal and vertical foliations, and using Proposition 16 to again conclude $\mathrm{Ext}_b(\gamma) \stackrel{\cdot}{\prec} \mathrm{Ext}_a(\gamma)$. This finishes the proof. q.e.d.

Proposition 16. *If* γ *is essentially horizontal for the quadratic differential* q_a*, then for every* $b > a$ *we have*

$$\mathrm{Ext}_b(\gamma) \succ \mathrm{Ext}_a(\gamma).$$

Proof. We argue in 3 cases according to which inequality in Corollary 10 is an equality up a multiplicative error.

Case 1. Assume there is a subsurface $Y \in \mathcal{Y}_a$ so that

$$\mathrm{Ext}_a(\gamma) \stackrel{.}{\asymp} \frac{\ell_a(\gamma|_Y)^2}{\mathrm{diam}_a(Y)^2}$$

such that $\gamma|_Y$ is mostly horizontal. We then have $\ell_b(\gamma|_Y) \stackrel{.}{\succ} e^{(b-a)}\ell_a(\gamma|_Y)$. Let $\mathcal{Z}$ be the set of subsurfaces in $\mathcal{Y}_b$ that intersect Y and let $\mathcal{B}$ be a set of annuli A_β, where $\beta \in \mathcal{A}_b$ and β intersects Y. Then Y is contained in the union of $\bigcup_{Z\in\mathcal{Z}} Z_b$ and $\bigcup_{\beta\in\mathcal{B}} A_b(\beta)$. Therefore,

$$\ell_b(\gamma|_Y) \le \sum_{Z\in\mathcal{Z}} \ell_b(\gamma|_Z) + \sum_{\beta\in\mathcal{B}} \ell_b(\gamma|_{A_b(\beta)}).$$

We also know from Lemma 14 that

$$\mathrm{diam}_b(Z) \le e^{(b-a)}\,\mathrm{diam}_a(Y) \qquad \text{and} \qquad d_b(\beta) \le e^{(b-a)}\,\mathrm{diam}_a(Y).$$

Therefore,

$$\begin{aligned}\mathrm{Ext}_b(\gamma) &\stackrel{.}{\succ} \sum_{Z\in\mathcal{Z}} \frac{\ell_b(\gamma|_Z)^2}{\mathrm{diam}_b(Z)^2} + \sum_{\beta\in\mathcal{B}} \frac{\ell_b(\gamma|_{A_b(\beta)})^2}{(d_b(\beta))^2}\\ &\stackrel{.}{\succ} \frac{\sum_{Z\in\mathcal{Z}} \ell_b(\gamma|_Z)^2 + \sum_{\beta\in\mathcal{B}} \ell_b(\gamma|_{A_b(\beta)})^2}{e^{2(b-a)}\,\mathrm{diam}_a(Y)^2}\\ &\stackrel{.}{\succ} \left(\frac{e^{(b-a)}\ell_a(\gamma|_Y)}{e^{(b-a)}\,\mathrm{diam}_a(Y)}\right)^2 \stackrel{.}{\asymp} \mathrm{Ext}_a(\gamma).\end{aligned}$$

Case 2. Assume that there is a curve $\alpha \in \mathcal{A}_a$ so that

$$\mathrm{Ext}_a(\gamma) \stackrel{.}{\asymp} \frac{\ell_a(\gamma|_{F_\alpha})^2\,\mathrm{Ext}_a(\alpha)}{\ell_a(\alpha)^2},$$

and $\gamma|_{F_\alpha}$ is mostly horizontal. Then $\ell_b(\gamma|_{F_\alpha}) \stackrel{.}{\succ} e^{(b-a)}\ell_a(\gamma|_{F_\alpha})$. If α is still short in q_b, i.e. if $\alpha \in \mathcal{A}_b$, then the proposition follows from Corollary 6.

Otherwise, let $\mathcal{Z}$ be the set of subsurfaces in $\mathcal{Y}_b$ that intersect α and let $\mathcal{B}$ be the set of curves in $\mathcal{A}_b$ that intersect α. Since F_α has geodesic boundaries, it is contained in the union of $\bigcup_{Z\in\mathcal{Z}} Z_b$ and $\bigcup_{\beta\in\mathcal{B}} A_b(\beta)$. The rest of the proof is exactly as in the previous case with the additional observation that $\mathrm{Ext}_b(\alpha) \ge \epsilon_1 \ge \mathrm{Ext}_a(\alpha)$.

Case 3. Assume there is an expanding annulus E with large modulus and the core curve α such that

$$\mathrm{Ext}_a(\gamma) \stackrel{.}{\asymp} i(\alpha,\gamma)^2\,\mathrm{Mod}_a(E)$$

and $\gamma|_E$ is mostly horizontal. Let Ω be the set of sub-arcs of γ that start and end in α and whose restriction to E is at least (1/3)–horizontal (that is, the ratio of the horizontal length to the vertical length is bigger than 1/3–times). We have

$$|\Omega| \geq (1/4)\, i(\alpha, \gamma).$$

Otherwise, (3/4) of arcs are 3-vertical, which implies that the total vertical length is larger than the total horizontal length. Recall that

$$\mathrm{Mod}_a(E) \stackrel{\cdot}{\asymp} \log(d_a(\alpha)/\ell_a(\alpha)).$$

For $\omega \in \Omega$ we have $\ell_a(\omega) \geq 2d_a(\alpha)$. Since the restriction of ω to E is mostly horizontal, we have

$$\ell_b(\omega) \succ e^{(b-a)} d_a(\alpha).$$

The arc ω intersects α, so $\sigma_\omega \leq \ell_b(\alpha) \leq e^{(b-a)}\ell_a(\alpha)$. Therefore,

$$X(\omega) \stackrel{\cdot}{\succ} \frac{e^{2(b-a)}(d_a(\alpha))^2}{\lambda_\omega^2} + \log \frac{\lambda_\omega}{e^{(b-a)}\ell_a(\alpha)}.$$

This expression is minimum when $\lambda_\omega = \sqrt{2}e^{(b-a)}d_a(\alpha)$. That is,

$$X(\omega) \stackrel{\cdot}{\succ} \log \frac{d_a(\alpha)}{\ell_a(\alpha)} \stackrel{\cdot}{\asymp} \mathrm{Mod}_a(E).$$

Using Lemma 13, we have

$$\mathrm{Ext}_b(\gamma) \succ |\Omega|^2 \min_{\omega\in\Omega} \mathrm{X}(\omega) \stackrel{\cdot}{\succ} i(\alpha,\gamma)^2\, \mathrm{Mod}_a(E) \stackrel{\cdot}{\asymp} \mathrm{Ext}_a(\gamma).$$

This finishes the proof in this case. q.e.d.

5. Quasi-Convexity of a Ball in Teichmüller Space

Consider a Riemann surface x. Let $\mathcal{B}(x, \mathsf{r})$ denote the ball of radius r in $\mathcal{T}(S)$ centered at x.

Theorem 17. *There exists a constant* c *such that, for every* $x \in \mathcal{T}(S)$*, every radius,* r *and point* y *and* z *in the ball* $\mathcal{B}(x, \mathsf{r})$*, the geodesic segment* $[y, z]$ *connecting* y *to* z *is contained in* $\mathcal{B}(x, \mathsf{r} + \mathsf{c})$*.*

Proof. Let u be a point on the segment $[y, z]$. It is sufficient to show that

$$d_{\mathcal{T}}(x, u) \leq \max\big(d_{\mathcal{T}}(x, y), d_{\mathcal{T}}(x, z)\big) + c.$$

There is a measured foliation μ such that

$$d_{\mathcal{T}}(x, u) = \frac{1}{2}\log\frac{\mathrm{Ext}_u(\gamma)}{\mathrm{Ext}_x(\gamma)}.$$

Also, from the quasi-convexity of extremal lengths (Theorem 15) we have

$$\mathrm{Ext}_u(\mu) \leq \mathsf{K} \max\big(\mathrm{Ext}_y(\mu), \mathrm{Ext}_z(\mu)\big)$$

Therefore,

$$d_{\mathcal{T}}(x,u) \le \frac{1}{2}\log\left(\frac{\mathsf{K}\max\big(\operatorname{Ext}_y(\gamma),\operatorname{Ext}_z(\gamma)\big)}{\operatorname{Ext}_x(\gamma)}\right)$$
$$\le \max\big(d_{\mathcal{T}}(x,y), d_{\mathcal{T}}(x,z)\big) + c.$$

q.e.d.

6. Quasi-Convexity of Hyperbolic Length

In this section, we prove the analogue of Theorem 15 for the hyperbolic length:

Theorem 18. *There exists a constant K', such that for every measured foliation μ, any Teichmüller geodesic $\mathcal{G}$, and times $a < b < c \in \mathbb{R}$, we have*

$$\operatorname{Hyp}_b(\mu) \le \mathsf{K}' \max\big(\operatorname{Hyp}_a(\mu), \operatorname{Hyp}_c(\mu)\big).$$

Proof. The argument is identical to the one for Theorem 15, with Corollary 20 and Proposition 23 being the key ingredients. They are stated and proved below. q.e.d.

Our main goal for the rest of this section is Proposition 23. To make the reading easier, we often take note of the similarities and skip some arguments when they are nearly identical to those for the extremal length case.

In place of Theorem 8, we have

Theorem 19. *For a quadratic differential q on a Riemann surface x, the corresponding thick-thin decomposition $(\mathcal{A},\mathcal{Y})$, and a curve γ on x, we have*

$$\operatorname{Hyp}_x(\gamma) \stackrel{*}{\asymp} \sum_{Y\in\mathcal{Y}} \frac{\ell_q(\gamma|_Y)}{\operatorname{diam}_q(Y)} + \sum_{\alpha\in\mathcal{A}}\left[\log\frac{1}{\operatorname{Ext}_x(\alpha)} + \operatorname{tw}_\alpha(q,\gamma)\operatorname{Ext}_x(\alpha)\right] i(\alpha,\gamma). \tag{10}$$

Proof. The hyperbolic length of a curve γ is, up to a universal multiplicative constant, the sum of the lengths of γ restricted to the pieces of the thick-thin decomposition of the surface. The hyperbolic length of $\gamma|_Y$ is comparable to the intersection number of γ with a short marking μ_Y of Y, which is, by Proposition 3, up to a multiplicative error,

$$\frac{\ell_q(\gamma|_Y)}{\operatorname{diam}_q(Y)} + \sum_{\alpha\in\partial Y} i(\gamma,\alpha).$$

The contribution from each curve $\alpha \in \mathcal{A}$ is (see, for example, [**CRS08**, corollary 3.2]),

$$\left[\log \frac{1}{\mathrm{Hyp}_x(\alpha)} + \mathrm{Hyp}_x(\alpha)\, \mathrm{tw}_\alpha(x,\gamma)\right] i(\alpha,\gamma).$$

Thus, we can write an estimate for the hyperbolic length of γ as

$$\mathrm{Hyp}_x(\gamma) \stackrel{\cdot}{\asymp} \sum_{Y \in \mathcal{Y}} \frac{\ell_q(\gamma|_Y)}{\mathrm{diam}_q(Y)} + \sum_{\alpha \in \mathcal{A}} \left[\log \frac{1}{\mathrm{Hyp}_x(\alpha)} + \mathrm{Hyp}_x(\alpha)\, \mathrm{tw}_\alpha(x,\gamma)\right] i(\alpha,\gamma).$$

Note that we are not adding 1 to the sum in the parentheses above since the sum is actually substantially greater.

To finish the proof, we need to replace $\mathrm{Hyp}_x(\alpha)$ with $\mathrm{Ext}_x(\alpha)$ and $\mathrm{tw}_\alpha(x,\gamma)$ with $\mathrm{tw}_\alpha(q,\gamma)$. Maskit has shown [**Mas85**] that, when $\mathrm{Hyp}_x(\alpha)$ is small,

$$\frac{\mathrm{Hyp}_x(\alpha)}{\mathrm{Ext}_x(\alpha)} \stackrel{\cdot}{\asymp} 1.$$

Hence, we can replace $\mathrm{Hyp}_x(\alpha)$ with $\mathrm{Ext}_x(\alpha)$. Further, it follows from Theorem 7 that

$$|\, \mathrm{tw}_\alpha(q,\gamma)\, \mathrm{Ext}_x(\alpha) - \mathrm{tw}_\alpha(x,\gamma)\, \mathrm{Ext}_x(\alpha)| = O(1).$$

Since $\log \frac{1}{\mathrm{Ext}_x(\alpha)}$ is at least 1 for $\alpha \in \mathcal{A}$, we have

$$\log \frac{1}{\mathrm{Ext}_x(\alpha)} + \mathrm{tw}_\alpha(q,\gamma)\, \mathrm{Ext}_x(\alpha) \stackrel{\cdot}{\asymp} \log \frac{1}{\mathrm{Ext}_x(\alpha)} + \mathrm{tw}_\alpha(x,\gamma)\, \mathrm{Ext}_x(\alpha),$$

which means that we can replace $\mathrm{tw}_\alpha(x,\gamma)$ with $\mathrm{tw}_\alpha(q,\gamma)$. q.e.d.

We almost immediately have:

Corollary 20. *Let $(\mathcal{A}, \mathcal{Y})$ be a thick-thin decomposition for q and let γ be a curve that is not in $\mathcal{A}$. Then*

1) *For every $Y \in \mathcal{Y}$,*

$$\mathrm{Hyp}_x(\gamma) \stackrel{\cdot}{\succ} \frac{\ell_q(\gamma|_Y)}{\mathrm{diam}_q(Y)}.$$

2) *For every $\alpha \in \mathcal{A}$ and the flat annulus F_α whose core curve is α,*

$$\mathrm{Hyp}_x(\gamma) \stackrel{\cdot}{\succ} \log \mathrm{Mod}_x(F_\alpha) i(\alpha,\gamma),$$

3) *For every $\alpha \in \mathcal{A}$,*

$$\mathrm{Hyp}_x(\gamma) \stackrel{\cdot}{\succ} \mathrm{tw}_\alpha(q,\gamma)\, \mathrm{Ext}_x(\alpha) i(\alpha,\gamma).$$

4) *For every $\alpha \in \mathcal{A}$ and an expanding annulus E_α whose core curve is α,*

$$\mathrm{Hyp}_x(\gamma) \stackrel{\cdot}{\succ} \log \mathrm{Mod}_x(E_\alpha) i(\alpha,\gamma).$$

Furthermore, at least one of these inequalities is an equality up to a multiplicative error.

Proof. Parts (1–4) follow immediately from Theorem 19 and the fact that the reciprocal of the extremal length of a curve α is bounded below by the modulus of any annulus homotopic to α. Further, since the number of pieces in the thick-thin decomposition $(\mathcal{A}, \mathcal{Y})$ is uniformly bounded, some term in Theorem 19 has to be comparable with $\mathrm{Hyp}_x(\gamma)$. The only non-trivial case is when that term is $\log \frac{1}{\mathrm{Ext}_x(\alpha)} i(\alpha, \gamma)$ for some $\alpha \in \mathcal{A}$. But by Lemma 4, either

$$\frac{1}{\mathrm{Ext}_x(\alpha)} \stackrel{\cdot}{\asymp} \mathrm{Mod}_x(F_\alpha),$$

or

$$\frac{1}{\mathrm{Ext}_x(\alpha)} \stackrel{\cdot}{\asymp} \mathrm{Mod}_x(E_\alpha),$$

and the lemma holds. q.e.d.

As in §3, we need a notion of *essentially horizontal* for hyperbolic length. We say that γ is *essentially horizontal* if at least one of the following holds

1) $\mathrm{Hyp}_x(\gamma) \stackrel{\cdot}{\asymp} \frac{\ell_q(\gamma|_Y)}{\mathrm{diam}_q(Y)}$ and $\gamma|_Y$ is mostly horizontal (i.e., its horizontal length is larger than its vertical length) for some $Y \in \mathcal{Y}$.

2) $\mathrm{Hyp}_x(\gamma) \stackrel{\cdot}{\asymp} \log \mathrm{Mod}_x(F_\alpha) i(\alpha, \gamma)$ and $\gamma|_{F_\alpha}$ is mostly horizontal for some flat annulus F_α whose core curve is $\alpha \in \mathcal{A}$.

3) $\mathrm{Hyp}_x(\gamma) \stackrel{\cdot}{\asymp} \mathrm{tw}_\alpha(q, \gamma)\, \mathrm{Ext}_x(\alpha) i(\alpha, \gamma)$ and $\gamma|_{F_\alpha}$ is mostly horizontal for some flat annulus F_α whose core curve is $\alpha \in \mathcal{A}$.

4) $\mathrm{Hyp}_x(\gamma) \stackrel{\cdot}{\asymp} \log \mathrm{Mod}_x(E_\alpha) i(\alpha, \gamma)$ for some expanding annulus E_α whose core curve is $\alpha \in \mathcal{A}$.

Further, Corollary 9 is replaced with

Corollary 21. *For any curve γ, the contribution to the hyperbolic length of γ from A_α, $\alpha \in \mathcal{A}$, is bounded below by $\frac{\ell_q(\gamma|_{A_\alpha})}{d_q(\alpha)}$. In other words,*

$$\mathrm{Hyp}_x(\gamma) \stackrel{\cdot}{\succ} \sum_{Y \in \mathcal{Y}} \frac{\ell_q(\gamma|_Y)}{\mathrm{diam}_q(Y)} + \sum_{\alpha \in \mathcal{A}} \frac{\ell_q(\gamma|_{A_\alpha})}{d_q(\alpha)}.$$

Proof. Identical to the proof of Corollary 9 after removing the squares and taking log when necessary. q.e.d.

Instead of the function $\mathrm{X}(\omega)$, to estimate the hyperbolic length of an arc, we define

$$\mathrm{H}(\omega) = \frac{\ell_q(\omega)}{\lambda_\omega} + \log \max \left\{ \log \frac{\lambda_\omega}{\sigma_\omega}, 1 \right\}.$$

In place of Lemma 13 we get

Lemma 22. *Let Ω be a set of disjoint sub-arcs of γ. Then*

$$\mathrm{Hyp}_x(\gamma) \succ |\Omega| \min_{\omega \in \Omega} \mathrm{H}(\omega).$$

Proof. Identical to the proof of Lemma 13 after removing the squares and taking log when necessary. q.e.d.

Finally, we have the analog of Proposition 16.

Proposition 23. *If γ is essentially horizontal for the quadratic differential q_a, then for every $b > a$ we have*

$$\mathrm{Hyp}_b(\gamma) \succ \mathrm{Hyp}_a(\gamma).$$

Proof. By the definition of essentially horizontal, there are four cases to consider. We deal with two of them, the flat annulus case and the twisting case, at once in Case 2.

Case 1. There is a thick subsurface Y where γ is mostly horizontal and such that

$$\mathrm{Hyp}_a(\gamma) \stackrel{\cdot}{\asymp} \frac{\ell_a(\gamma|_Y)}{\mathrm{diam}_a(Y)},$$

The proof is as in the extremal length case after removing the squares.

Case 2. There exists a curve $\alpha \in \mathcal{A}$ so that

$$\mathrm{Hyp}_a(\gamma) \stackrel{\cdot}{\asymp} \log \mathrm{Mod}_a(F_\alpha) i(\alpha, \gamma), \tag{11}$$

or

$$\mathrm{Hyp}_a(\gamma) \stackrel{\cdot}{\asymp} \mathrm{tw}_\alpha(a, \gamma)\, \mathrm{Ext}_a(\alpha) i(\alpha, \gamma), \tag{12}$$

and $\gamma|_{F_\alpha}$ is mostly horizontal. We argue in three sub-cases.

Case 2.1. Suppose first that α is no longer short at $t = b$ and either (11) or (12) holds. Let $\mathcal{Z}$ be the set of subsurfaces in $\mathcal{Y}_b$ that intersect α and let $\mathcal{B}$ be the set of curves in $\mathcal{A}_b$ that intersect α. Then, by Corollary 21 and Lemma 14,

$$\begin{aligned} \mathrm{Hyp}_b(\gamma) &\stackrel{\cdot}{\succ} \sum_{Z \in \mathcal{Z}} \frac{\ell_b(\gamma|_Z)}{\mathrm{diam}_b(Z)} + \sum_{\beta \in \mathcal{B}} \frac{\ell_b(\gamma|_{A_\beta})}{d_b(\beta)} \\ &\stackrel{\cdot}{\succ} \sum_{Z \in \mathcal{Z}} \frac{\ell_b(\gamma|_Z)}{e^{b-a}\ell_a(\alpha)} + \sum_{\beta \in \mathcal{B}} \frac{\ell_b(\gamma|_{A_\beta})}{e^{b-a}\ell_a(\alpha)}. \end{aligned}$$

But F_α is contained in $\left(\bigcup_{Z\in\mathcal{Z}} Z\right) \cup \left(\bigcup_{\beta\in\mathcal{B}} A_\beta\right)$:

$$\begin{aligned}
&\stackrel{\cdot}{\succ} \frac{\ell_b(\gamma|_{F_\alpha})}{e^{b-a}\ell_a(\alpha)} \stackrel{\cdot}{\succ} \frac{\ell_a(\gamma|_{F_\alpha})}{\ell_a(\alpha)} \\
&\stackrel{\cdot}{\succ} \max\left\{ \log \operatorname{Mod}_a(F_\alpha) i(\alpha,\gamma), \operatorname{tw}_\alpha(a,\gamma) \operatorname{Ext}_a(\alpha) i(\alpha,\gamma)\right\} \\
&\stackrel{\cdot}{\asymp} \operatorname{Hyp}_a(\gamma).
\end{aligned}$$

Case 2.2. Suppose now that $\alpha \in \mathcal{A}_b$ and that (11) holds. If α is mostly vertical at time a, the extremal length of α is decreasing exponentially fast for some interval $[a,c]$. That is, $\operatorname{Mod}_c(F_\alpha) \stackrel{\cdot}{\succ} \operatorname{Mod}_a(F_\alpha)$. It is sufficient to show that for $b \geq c$,

$$\operatorname{Hyp}_b(\gamma) \stackrel{\cdot}{\succ} \log \operatorname{Mod}_c(F_\alpha)\, \mathrm{i}(\alpha,\gamma).$$

Our plan is to argue that, while the modulus of F_α is decreasing, the hyperbolic length of γ is not decreasing by much because the curve is twisting very fast around α. We need to estimate the twisting of γ around α. Let ω be one of the arcs of $\gamma|_{F_\alpha}$. Note that ω is mostly horizontal at c (since it was at a) and its length is larger than $f_c(\alpha)$. Also, since α is mostly horizontal at c, $f_t(\alpha)$ is decreasing exponentially fast at $t = c$. Hence, after replacing c with a slightly larger constant, we can assume ω is significantly larger than $f_a(\alpha)$ and therefore, the number of times ω twists around α is approximately the length ratio of ω and α (see equation 15 and 16 in [**Raf07a**] and the related discussion for more details). That is, for $c \leq t \leq b$, $\operatorname{tw}_\alpha(q_t,\gamma)$ is essentially constant:

$$\operatorname{tw}_\alpha(q_t,\gamma) \stackrel{\cdot}{\asymp} \frac{\ell_t(\omega)}{\ell_t(\alpha)} \stackrel{\cdot}{\asymp} \frac{e^{(t-a)}\ell_a(\omega)}{e^{(t-a)}\ell_a(\alpha)} = \frac{\ell_a(\omega)}{\ell_a(\alpha)}.$$

Therefore,

$$\operatorname{Mod}_c(F_\alpha) = \frac{f_c(\alpha)}{\ell_c(\alpha)} \leq \frac{\ell_a(\omega)}{\ell_a(\alpha)} \stackrel{\cdot}{\asymp} \operatorname{tw}_\alpha(q_c,\gamma).$$

Keeping in mind that, for $k \geq 0$, the function $f(x) = -\log x + kx > \log k$, we have

$$\begin{aligned}
\operatorname{Hyp}_b(\gamma) &\stackrel{\cdot}{\succ} \left[\log \frac{1}{\operatorname{Ext}_b(\alpha)} + \operatorname{tw}_\alpha(b,\gamma)\operatorname{Ext}_b(\alpha)\right] i(\alpha,\gamma) \\
&\stackrel{\cdot}{\succ} \log\left(\operatorname{tw}_\alpha(q_b,\gamma)\right) i(\alpha,\gamma) \stackrel{\cdot}{\succ} \log \operatorname{Mod}_c(F_\alpha)\, i(\alpha,\gamma).
\end{aligned}$$

Case 2.3. Suppose that $\alpha \in \mathcal{A}_b$ and that (12) holds. Since γ crosses α, $\operatorname{Hyp}_a(\gamma)$ is greater than a large multiple of $\mathrm{i}(\alpha,\gamma)$. Hence

$$\operatorname{Hyp}_a(\gamma) \stackrel{\cdot}{\asymp} \operatorname{tw}_\alpha(a,\gamma)\operatorname{Ext}_a(\alpha) i(\alpha,\gamma)$$

implies that $\operatorname{tw}_\alpha(a,\gamma)$ is much larger than $\operatorname{Mod}_a(F_\alpha)$. That is, the angle between γ and α is small. Therefore, after perhaps replacing a with a slightly larger number, we can assume that α is mostly horizontal and that, for $a \le t \le b$,

$$\operatorname{tw}_\alpha(t,\gamma) \stackrel{.}{\asymp} \frac{\ell_t(\omega)}{\ell_t(\alpha)}, \tag{13}$$

Applying Theorem 19, Equation (13), Corollary 6, and Equation (12) in that order, we obtain:

$$\begin{aligned}
\operatorname{Hyp}_b(\gamma) &\stackrel{.}{\succ} [\operatorname{tw}_\alpha(b,\gamma)\operatorname{Ext}_b(\alpha)]\, i(\alpha,\gamma)\\
&\stackrel{.}{\asymp} \frac{\operatorname{Ext}_b(\alpha)}{\ell_b(\alpha)}\ell_b(\omega) i(\alpha,\gamma)\\
&\stackrel{.}{\succ} \frac{e^{b-a}}{e^{b-a}}\frac{\operatorname{Ext}_a(\alpha)}{\ell_a(\alpha)}\ell_a(\omega) i(\alpha,\gamma)\\
&\stackrel{.}{\succ} [\operatorname{tw}_\alpha(a,\gamma)\operatorname{Ext}_a(\alpha)]\, i(\alpha,\gamma).
\end{aligned}$$

Case 3. There is a curve $\alpha \in \mathcal{A}$ with expanding annulus E_α such that $\gamma|_{E_\alpha}$ is mostly horizontal with

$$\operatorname{Hyp}_a(\gamma) \stackrel{.}{\asymp} \log \operatorname{Mod}_a(E_\alpha) i(\alpha,\gamma),$$

Following the proof for the corresponding case for extremal length, we have

$$\begin{aligned}
\operatorname{H}(\omega) &\stackrel{.}{\succ} \frac{e^{b-a} d_a(\alpha)}{\lambda_\omega} + \log\max\left\{\log \frac{\lambda_\omega}{e^{b-a}\ell_a(\alpha)}, 1\right\}\\
&\stackrel{.}{\succ} \log\log \frac{d_a(\alpha)}{\ell_a(\alpha)} \stackrel{.}{\asymp} \log \operatorname{Mod}_a(E_\alpha).
\end{aligned}$$

One can verify the second inequality as follows. If $\frac{\lambda_\omega}{e^{b-a}} \le \sqrt{d_a(\alpha)\ell_a(\alpha)}$, then

$$\frac{e^{b-a} d_a(\alpha)}{\lambda_\omega} \ge \sqrt{\frac{d_a(\alpha)}{\ell_a(\alpha)}} \stackrel{.}{\succ} \log\log \frac{d_a(\alpha)}{\ell_a(\alpha)}.$$

Otherwise,

$$\log\log \frac{\lambda_\omega}{e^{b-a}\ell_a(\alpha)} \ge \log\log \sqrt{\frac{d_a(\alpha)}{\ell_a(\alpha)}} \stackrel{.}{\succ} \log\log \frac{d_a(\alpha)}{\ell_a(\alpha)},$$

and applying Lemma 22, we have

$$\operatorname{Hyp}_b(\gamma) \stackrel{.}{\succ} |\Omega| \min_{\omega\in\Omega} \operatorname{H}(\omega) \stackrel{.}{\succ} i(\alpha,\gamma)\log \operatorname{Mod}_a(E_\alpha) \stackrel{.}{\asymp} \operatorname{Hyp}_a(\gamma).$$

This finishes the proof. q.e.d.

7. Examples

This section contains two examples. In the first example we describe a Teichmüller geodesic and a curve whose length is not convex along this geodesic. The second example is of a very long geodesic that spends its entire length near the boundary of a round ball.

Example 24 (Extremal length and hyperbolic length are not convex)**.** To prove that the extremal and the hyperbolic lengths are not convex, we construct a quadratic differential and analyze these two lengths for a specific curve along the geodesic associated to this quadratic differential. We show that on some interval the average slope (in both cases) is some positive number and on some later interval the average slope is near zero. This shows that the two length functions are not convex. Note that scaling the weight of a curve by a factor k increases the hyperbolic and the extremal length of that curve by factors of k and k^2, respectively. Thus, after scaling, one can produce examples where the average slope is very large on some interval and near zero on some later interval.

Let $0 < a \ll 1$. Let T be a rectangular torus obtained from identifying the opposite sides of the rectangle $[0, a] \times [0, \frac{1}{a}]$. Also, let C be a euclidean cylinder obtained by identifying vertical sides of $[0, a] \times [0, a]$. Take two copies T_1 and T_2 of T, each cut along a horizontal segment of length $a/2$ (call it a slit), and join them by gluing C to the slits. This defines a quadratic differential q on a genus two surface x_0. The horizontal and the vertical trajectories of q are those obtained from the horizontal and the vertical foliation of $\mathbb{R}^2$ by lines parallel to the x–axis and y–axis respectively. We now consider the Teichmüller geodesic based at x_0 in the direction of q. Let α be a core curve of cylinder C. We will show that, for small enough a, $\mathrm{Ext}_{x_t}(\alpha)$ and $\mathrm{Hyp}_{x_t}(\alpha)$ are not convex along x_t.

Let ρ be the metric which coincides with the flat metric of q on C and on the two horizontal bands in T_i of width and height a with the slit in the middle, and is 0 otherwise. The shortest curve in the homotopy class of α has length a in this metric. Then we have

$$\mathrm{Ext}_{x_0}(\alpha) \geq \frac{a^2}{3a^2} = \frac{1}{3}. \tag{14}$$

Also, at time $t < 0$, we have $\mathrm{Mod}_{x_t}(C) = \frac{ae^{-t}}{ae^{t}} = e^{-2t}$ and, therefore,

$$\mathrm{Ext}_{x_t}(\alpha) \leq e^{2t}. \tag{15}$$

Hence we see that the extremal length of α grows exponentially on $(-\infty, 0)$. In particular, the average slope on the interval $\mathcal{J} = (-2, 0)$ is more than $\frac{1}{8}$. By proposition 1 and corollary 3 in [**Mas85**],

$$2e^{-\frac{1}{2}\operatorname{Hyp}_x(\alpha)} \leq \frac{\operatorname{Hyp}_x(\alpha)}{\operatorname{Ext}_x(\alpha)} \leq \pi, \tag{16}$$

and it is easy to see that the slope of $\operatorname{Hyp}_{x_t}(\alpha)$ on this interval is also greater than $\frac{1}{8}$.

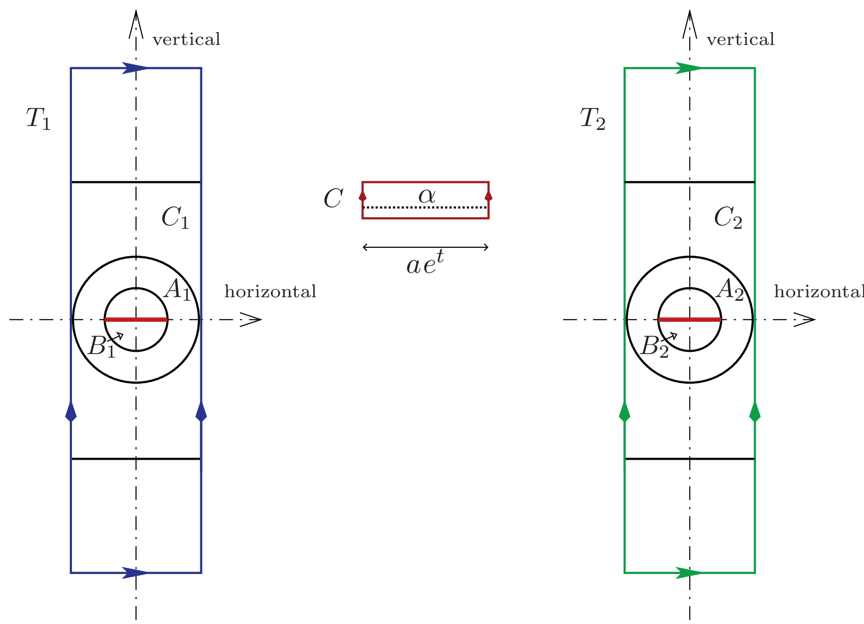

Figure 3. Metric ρ_t on x_t when $t > 0$.

Further along the ray, when $t > 0$, the modulus of C is decreasing exponentially. We estimate $\operatorname{Ext}_{x_t}(\alpha)$ for $t \in \mathcal{I} = (0, \frac{1}{2}\log\frac{1}{a^2})$. For the lower bound, consider the cylinder A which is the union of C and the maximal annuli in T_i whose boundary is a round circle centered at the middle of the slit. Then A contains two disjoint copies of annuli of inner radius $(ae^t/4)$ and outer radius $(ae^t/2)$ (the condition on t guarantees that these annuli do not touch the top or the bottom edges of T_1 and T_2). Both of these annuli have modulus of $\frac{1}{2\pi}\log 2$, and therefore $\operatorname{Mod}_{x_t}(A) \geq \frac{1}{\pi}\log 2$. Hence

$$\operatorname{Ext}_{x_t}(\alpha) \leq \frac{1}{\operatorname{Mod}_{x_t}(A)} \leq \frac{\pi}{\log 2}. \tag{17}$$

For the upper bound, we use the metric ρ_t defined as follows (see Fig. 3): Let A_i be the annuli in T_i centered at the midpoints of the corresponding slits with inner radius $\frac{ae^t+\delta}{4}$ and outer radius $\frac{ae^t-\delta}{2}$ for a very small δ. Let ρ_t be $\frac{1}{|z|}|dz|$ on A_i, and the flat metric $|dz|$ on C scaled so that the circumference is 2π. The complement of A_i and C consists of two annuli B_1, B_2 and two once-holed tori C_1 and C_2 with $B_i, C_i \in T_i$. On each of these components, we will define ρ_t so that the shortest representative of α has length at least 2π and the area is bounded above. More precisely, let $\rho_t = \frac{2\pi}{ae^t}|dz|$ on B_i. On C_i, let ρ_t be $\frac{2}{ae^t-\delta}|dz|$ if $|Im\, z| < \frac{1}{2}(\pi+1)(ae^t - \delta)$, and zero otherwise.

The area of C in this metric is $(2\pi ae^t)(2\pi ae^{-t}) = O(1)$. The pieces B_i and C_i have diameters of order $O(ae^t)$ in ρ_t and hence have area of order $O(1)$. The annuli A_i in this metric are isometric to flat cylinders of circumference 2π and width less than $\log 2$, which also has area one. Thus,

$$\mathrm{Area}_{\rho_t}(\mathcal{S}) = O(1).$$

Also, the ρ_t-length of any curve α' homotopic to α is $\ell_{\rho_t}(\alpha') \geq 2\pi$. Indeed, any curve contained in one of the annuli has ρ_t-length at least 2π. Morever, any sub-arc of α' with end points on a boundary component of an annulus can be homotoped relative to the end points to the boundary without increasing the length.

Since the area of ρ_t is uniformly bounded above (independent of a and t) and the length of α in ρ_t is larger than 2π, the extremal length

$$\mathrm{Ext}_{x_t}(\alpha) \geq \frac{\inf_{\alpha'\sim\alpha} \ell_{\rho_t}(\alpha')^2}{\mathrm{Area}_{\rho_t}(\mathcal{S})} \tag{18}$$

is bounded below on $\mathcal{I}$ by a constant independent of t and a. Combining this with (18), we see that, as $a \to 0$ (and hence the size of $\mathcal{I}$ goes to ∞), the average slope of $\mathrm{Ext}_{x_t}(\alpha)$ on $\mathcal{I}$ is near zero. In particular, the average slope on $\mathcal{I}$ can be made smaller than $\frac{1}{8}$, which implies that the function $\mathrm{Ext}_{x_t}(\alpha)$ is not convex. Combining (16) and the estimates of the extremal length above, we come to the same conclusion about $\mathrm{Hyp}_{x_t}(\alpha)$.

Example 25 (Geodesics near the boundary)**.** Here we describe how, for any $R > 0$, a geodesic segment of length comparable to R can stay near the boundary of a ball of radius R. This example suggests that metric balls in $\mathcal{T}(S)$ might not be convex.

REMARK 26. In [**MW95**], Masur and Wolf used a very similar example to show that the Teichmüller space is not Gromov hyperbolic.

Let x be a point in the thick part of $\mathcal{T}(S)$ and μ_x be the short marking of x. Pick any two disjoint curves α, β in μ_x. Let $y = \mathcal{D}^n_{(\alpha)}x$, and $z = \mathcal{D}^n_{(\alpha,\beta)}x$, where $\mathcal{D}_{(*)}$ is the Dehn twist around a multicurve $(*)$. The intersection numbers between the short markings of x, y, z satisfy

$$i(\mu_x, \mu_y) \stackrel{.}{\asymp} i(\mu_x, \mu_z) \stackrel{.}{\asymp} i(\mu_y, \mu_z) \stackrel{.}{\asymp} n.$$

Hence, by theorem 2.2 in [**CR07**], we have

$$d_{\mathcal{T}}(x,y) \stackrel{+}{\asymp} d_{\mathcal{T}}(x,z) \stackrel{+}{\asymp} d_{\mathcal{T}}(y,z) \stackrel{+}{\asymp} \log n.$$

That is, $[y,z]$ is a segment of length $\log n$ whose end points are near the boundary of the ball $\mathcal{B}(x, \log n)$. We will show, for $w \in [y,z]$, that $d_{\mathcal{T}}(x,w) \stackrel{+}{\asymp} \log n$, which means the entire geodesic $[y,z]$ stays near the boundary of the ball $\mathcal{B}(x, \log n)$. Let α' be a curve that intersects α, is disjoint from β, and $\operatorname{Ext}_y(\alpha') = O(1)$. Since α' intersects α,

$$\operatorname{Ext}_x(\alpha') \stackrel{\cdot}{\asymp} n^2,$$

and since α' is disjoint from β,

$$\operatorname{Ext}_z(\alpha') = O(1).$$

By Theorem 15,

$$\operatorname{Ext}_w(\alpha') \leq \mathsf{K} \max \left\{ \operatorname{Ext}_y(\alpha'), \operatorname{Ext}_z(\alpha') \right\} = O(1).$$

We now have

$$d_{\mathcal{T}(S)}(w,x) \geq \frac{1}{2} \log \frac{\operatorname{Ext}_x(\alpha')}{\operatorname{Ext}_w(\alpha')} \stackrel{+}{\asymp} \log n.$$

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