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COMPARISON BETWEEN TEICHMÜLLER AND LIPSCHITZ METRICS

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Abstract

We study the Lipschitz metric on a Teichmüller space (defined by Thurston) and compare it with the Teichmüller metric. We show that in the thin part of the Teichmüller space the Lipschitz metric is approximated up to a bounded additive distortion by the sup-metric on a product of lower-dimensional spaces (similar to the Teichmüller metric as shown by Minsky). In the thick part, we show that the two metrics are equal up to a bounded additive error. However, these metrics are not comparable in general; we construct a sequence of pairs of points in the Teichmüller space, with distances that approach zero in the Lipschitz metric while they approach infinity in the Teichmüller metric.

1. *Introduction*

The Teichmüller distance between two points σ and τ in Teichmüller space $\mathcal{T}(S)$ is defined in terms of the minimal quasiconformal constant $K(\sigma,\tau)$ between σ and τ. Thurston [**12**] introduced an analogous metric on $\mathcal{T}(S)$ by considering the least possible value of the global Lipschitz constant $\Lambda(\sigma,\tau)$ from σ to τ. On the one hand, Kerckhoff [**3**] showed that $K(\sigma,\tau)$ can be formulated in terms of the ratio of extremal lengths of simple closed curves

$$K(\sigma,\tau) = \sup_\alpha \frac{\mathrm{Ext}_\tau(\alpha)}{\mathrm{Ext}_\sigma(\alpha)} \tag{1}$$

and on the other, it was shown by Thurston [**12**] that the minimal Lipschitz constant $\Lambda(\sigma,\tau)$ is given by the ratio of lengths in the hyperbolic metric

$$\Lambda(\sigma,\tau) = \sup_\alpha \frac{\ell_\tau(\alpha)}{\ell_\sigma(\alpha)}. \tag{2}$$

A comparison of $K(\sigma,\tau)$ and the ratio of lengths in equation (2) was first given by Wolpert [**13**], who proved that for any K-quasiconformal map f from σ to τ and for any simple closed curve α,

$$\frac{\ell_\tau(f(\alpha))}{\ell_\sigma(\alpha)} \leqslant K.$$

This implies, in particular, that

$$\Lambda(\sigma,\tau) \leqslant K(\sigma,\tau). \tag{3}$$

In this paper, we compare the Teichmüller and Lipschitz metrics by comparing the two ratios in equations (1) and (2). Our method is to analyse the ratio of hyperbolic lengths in much the same way as the ratio of extremal lengths was analysed by Minsky [**7**] to show that certain regions in the thin part of the Teichmüller space have product structures. However, since $K(\sigma,\tau)$ is symmetric and $\Lambda(\sigma,\tau)$ is not [**12**], it is necessary to choose some symmetric version of Λ to make the comparison more meaningful. Thus, we take

$$L(\sigma,\tau) = \max\{\Lambda(\sigma,\tau), \Lambda(\tau,\sigma)\}$$

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and define the Teichmüller and Lipschitz metrics, respectively, as

$$d_{\mathcal{T}}(\sigma,\tau) = \tfrac{1}{2}\log K(\sigma,\tau),$$
$$d_{\mathrm{L}}(\sigma,\tau) = \log L(\sigma,\tau).$$

Note that the factor 1/2 has been left out in the Lipschitz metric. This is because we can compare the two metrics up to an additive error on the thick part of the Teichmüller space, as we shall shortly see.

Although $\Lambda(\sigma,\tau)$ is not symmetric, it is easy to check that it satisfies the following ordered triangle inequality:

$$\log(\Lambda(\rho,\tau)) \leqslant \log(\Lambda(\rho,\sigma)) + \log(\Lambda(\sigma,\tau))$$

and further satisfies the property that $\log(\Lambda(\sigma,\tau)) = 0$ if and only if $\sigma = \tau$. Thus $d_{\mathrm{L}}(\sigma,\tau)$ defines a genuine metric in that it is symmetric, takes the value zero if and only if $\sigma=\tau$, and satisfies the triangle inequality. In [**11**], it was shown that on the Teichmüller space of the torus, the Teichmüller metric and a similarly defined Lipschitz metric are, in fact, equal (see also [**1**]). In contrast, we show that for a hyperbolic surface S, the two metrics are not comparable.

THEOREM A. *There are sequences $\sigma_n, \tau_n \in \mathcal{T}(S)$ such that, as $n \to \infty$,*

$$d_{\mathrm{L}}(\sigma_n,\tau_n) \to 0, \qquad d_{\mathcal{T}}(\sigma_n,\tau_n) \to \infty.$$

We have recently been made aware that the fact that the two metrics are not metrically equivalent was first shown by Li [**4**].

As is often the case, however, no incongruities occur on the thick part of the Teichmüller space, and the two metrics are quasi-isometric to one another. In fact, they are equal up to a bounded additive error. This is a consequence of the following theorem, proved in Section 2.

THEOREM B. *For $\rho \in \mathcal{T}(S)$, let μ_ρ be a short marking for ρ. For every $\epsilon > 0$, there is a constant c depending on the surface S and on ϵ such that, for any σ,τ in the ϵ-thick part of $\mathcal{T}(S)$, the following quantities differ from one another by at most c:*

(1) $d_{\mathcal{T}}(\sigma,\tau)$;
(2) $d_{\mathrm{L}}(\sigma,\tau)$;
(3) $\log \max_{\alpha\in\mu_\sigma}(\ell_\tau(\alpha)/\ell_\sigma(\alpha))$;
(4) $\log \max_{\alpha\in\mu_\tau}(\ell_\sigma(\alpha)/\ell_\tau(\alpha))$.

In particular, in order to estimate the Teichmüller distance between two points in the thick part, one need only compare the lengths of a finite number of curves (that is, those in the short marking) with respect to the two metrics.

To compare the metrics on the thin part of the Teichmüller space, we prove in Section 3 an analog of Minsky's product region theorem [**7**]. Let Γ be a collection of k disjoint, homotopically distinct, simple closed curves on S and let $\mathrm{Thin}_\epsilon(S,\Gamma)$ be the set of $\sigma \in \mathcal{T}(S)$ such that $\ell_\sigma(\gamma) \leqslant \epsilon$ for all $\gamma \in \Gamma$. Let $\mathcal{T}_\Gamma = \mathcal{T}(S\setminus\Gamma) \times U_1 \times \ldots \times U_k$, where $S \setminus \Gamma$ is the analytically finite surface obtained from S by pinching all the curves in Γ and where U_i is the subset $\{(x,y) : y \geqslant 1/\epsilon\}$ of the upper half-plane. The Fenchel–Nielsen coordinates on $\mathcal{T}(S)$ give rise to a natural homeomorphism $\Pi : \mathrm{Thin}_\epsilon(S,\Gamma) \to \mathcal{T}_\Gamma$. Then Minsky's product region theorem states the following.

THEOREM 1.1 (Minsky [**7**]). *Let $d_{\mathcal{T}_\Gamma}$ be the sup metric*

$$d_{\mathcal{T}_\Gamma} = \sup\left\{ d_{\mathcal{T}(S\setminus\Gamma)}, \tfrac{1}{2}d_{\mathbb{H}_1}, \ldots, \tfrac{1}{2}d_{\mathbb{H}_k} \right\}$$

on $\mathcal{T}_\Gamma$, where $d_{\mathcal{T}(S\setminus\Gamma)}$ is the Teichmüller metric on $\mathcal{T}(S\setminus\Gamma)$ and $d_{\mathbb{H}_i}$ the restriction of the hyperbolic metric on the upper half-plane to U_i. Then, for ϵ sufficiently small, there is a *constant c depending on ϵ such that for any $\sigma, \tau \in \mathrm{Thin}_\epsilon(S,\Gamma)$*

$$|d_{\mathcal{T}}(\sigma,\tau) - d_{\mathcal{T}_\Gamma}(\Pi(\sigma),\Pi(\tau))| < c.$$

In the analog for the Lipschitz metric, we define the sup-metric

$$d_{L_\Gamma} = \sup\{d_{L(S\setminus\Gamma)}, d_{L(\gamma_1)}, \ldots, d_{L(\gamma_k)}\}$$

on $\mathcal{T}_\Gamma$, where $d_{L(S\setminus\Gamma)}$ is the Lipschitz metric on $\mathcal{T}(S \setminus \Gamma)$ and $d_{L(\gamma_i)}$ is a modification of the hyperbolic metric on U_i (see Section 3 for details).

THEOREM C. *For ϵ sufficiently small, there is a constant c depending on ϵ such that for any $\sigma, \tau \in \mathrm{Thin}_\epsilon(S,\Gamma)$,*

$$|d_{\mathrm{L}}(\sigma,\tau) - d_{L_\Gamma}(\Pi(\sigma),\Pi(\tau))| < c.$$

A more precise statement is given in Theorem 3.5. Our proof is parallel to Minsky's, but requires only elementary hyperbolic geometry, since we need not deal with extremal lengths.

As a consequence of Theorem B, one can deduce the following purely combinatorial result. For a subsurface Z, let $d_Z(\mu_1,\mu_2)$ be the distance between the projections of μ_1 and μ_2 to Z, measured in the arc complex of Z (see [**5**, **9**] for details); see below for notation.

COROLLARY D. *There is a constant k such that for any markings μ_1 and μ_2 on S,*

$$\log i(\mu_1,\mu_2) \asymp \sum_Y [d_Y(\mu_1,\mu_2)]_k + \sum_A \log [d_A(\mu_1,\mu_2)]_k\,, \tag{4}$$

where Y ranges over all subsurfaces of S that are not annuli, A ranges over all annuli, and where $[x]_k = 0$ if $x < k$ and $[x]_k = x$ if $x \geqslant k$.

Masur and Minsky [**5**] provide an estimate, similar to the right-hand side of (4), for the number of elementary moves needed to change μ_1 to μ_2. Using their result and examining how the intersection number between the two markings changes as a result of applying a sequence of elementary moves to one of them, one can show that the right-hand side of (4) is an upper bound for $\log i(\mu_1,\mu_2)$ (there is no clear combinatorial argument for proving the inequality in the other direction). In this context, Corollary D states that along an efficient path in the marking space, the intersection number increases at the fastest possible rate.

1.1. *Notation*

Often, we shall compare two functions f and g on $\mathcal{T}(S)$ and use the notation $f \prec g$ and $f \asymp g$ to mean, respectively, that there are positive constants k and c such that $f \leqslant kg + c$ and such that $g/k - c \leqslant f \leqslant kg + c$. We also use $f \overset{*}{\prec} g$, $f \overset{+}{\asymp} g$ to mean, respectively, that there is only a multiplicative constant, or only an additive constant, involved. In particular, $f \overset{*}{\asymp} 1$ means that the function f is bounded both above and below by positive constants. The constants k and c usually depend on the topological type of S, which will not be subsequently mentioned. Other dependencies will be explicitly noted.

2. *The thick part*

Let S be a surface of finite topological type. Given $\epsilon > 0$, the ϵ-thick part of the Teichmüller space is the set of $\sigma \in \mathcal{T}(S)$ such that the infimum of the injectivity radius measured in σ, taken over all points in S, is greater than ϵ. When we simply say 'the thick part', we mean that it is the ϵ-thick part for some ϵ which has already been chosen.

A *marking* on S is a collection of homotopically distinct, simple closed curves in S obtained by first choosing a pants curves system, that is, a collection of mutually disjoint curves that cut S into pairs of pants (where a hole may be a puncture of S), and then by choosing an additional collection of curves that together with the pants system cuts the surface into disks and punctured disks. To make the choice of a marking less arbitrary, additional conditions on the choice of curves are often specified.

For $\sigma \in \mathcal{T}(S)$, we define a *short marking* μ_σ as follows. First choose a pants system by taking the shortest curve in S, then the next shortest curve disjoint from the first, and so on until a complete pants system $\underline{\alpha}$ is formed. Throughout this paper, when we say the 'length of a curve', we always mean the length of its geodesic representative. Next, choose a 'dual' curve δ_α for each $\alpha \in \underline{\alpha}$ that is disjoint from $\underline{\alpha} \setminus \alpha$, and that is the shortest among all such curves. There may be a finite number of possible short markings for σ.

A lemma of Bers states that there is a uniform constant N such that every $\sigma \in \mathcal{T}(S)$ has a pants curves system $\underline{\alpha}$ with the property that $\ell_\sigma(\alpha) < N$ for all $\alpha \in \underline{\alpha}$. Hence, if σ is in the ϵ-thick part of $\mathcal{T}(S)$ so that all the curves in a short marking μ have length bounded below as well, then the lengths of the dual curves are bounded above, and so $\ell_\sigma(\mu) = \sum_{\alpha \in \mu} \ell_\sigma(\alpha)$ is bounded above by some quantity depending only on ϵ. Conversely, given a marking μ and a number $B > 0$, the set of metrics $\sigma \in \mathcal{T}(S)$ such that $\ell_\sigma(\mu) \leqslant B$ has a bounded diameter in $\mathcal{T}(S)$, where the bound depends only on B (see, for example, [**6**]). Thus there is a coarse correspondence between the thick part of the Teichmüller space and the set of markings. This idea is implicit in the theorems that follow.

2.1. *Proof of Theorem* B

First we need the following lemma. Let $g : \mathbb{R} \to \mathcal{T}(S)$ be the Teichmüller geodesic that passes through σ and τ and let q_t be the family of quadratic differentials representing g. We assume that all quadratic differential metrics have been normalized to have area 1.

LEMMA 2.1. *Let μ be a marking on S that has the same number of curves as any short marking* (*that is, $6g(S) - 6 + 2p$, where $g(S)$ is the genus of S and p is the number of punctures*)*. Then there exist ℓ_0 and t_0 such that*

$$\ell_{q_t}(\mu) \stackrel{*}{\asymp} \ell_0 \, e^{|t-t_0|}.$$

Proof. Recall that a quadratic differential q_t defines a pair of measured foliations on the surface S, called the horizontal and the vertical foliations. For every curve α the horizontal length $h_t(\alpha)$ of α is the intersection number of α with the vertical foliation, and the vertical length $v_t(\alpha)$ of α is the intersection number of α with the horizontal foliation. Then we have (see, for example, [**8**])

$$\ell_{q_t}(\alpha) \stackrel{*}{\asymp} h_t(\alpha) + v_t(\alpha).$$

Let t_α be the time when α is balanced, that is, the time when the horizontal length and the vertical length of α are equal. Let $\ell_\alpha = \ell_{q_{t_\alpha}}(\alpha)$. Along a Teichmüller geodesic, the horizontal length of α increases and the vertical length of α decreases exponentially fast. Therefore

$$\ell_{q_t}(\alpha) \stackrel{*}{\asymp} \ell_\alpha \cosh(t - t_\alpha).$$

Thus, for every marking μ,

$$\ell_{q_t}(\mu) = \sum_{\alpha \in \mu} \ell_{q_t}(\alpha) \stackrel{*}{\asymp} \sum_{\alpha \in \mu} \ell_\alpha \cosh(t - t_\alpha). \tag{5}$$

Denote the right-hand side of (5) by $f(t)$. Let t_0 be the time when $f(t)$ is minimised and let $\ell_0 = f(t_0)$. Since

$$\cosh(t - t_\alpha) \leqslant \cosh(t_0 - t_\alpha)\, e^{|t-t_0|},$$

we have

$$\sum_{\alpha\in\mu} \ell_\alpha \cosh(t - t_\alpha) \leqslant \sum_{\alpha\in\mu} \ell_\alpha \cosh(t_0 - t_\alpha)\, e^{|t-t_0|} = \ell_0\, e^{|t-t_0|}. \tag{6}$$

To prove the inequality in the other direction, we observe that the derivative of $f(t)$ with respect to t at $t = t_0$ is $\sum_\alpha \ell_\alpha \sinh(t_0 - t_\alpha) = 0$, which implies that

$$\sum_{\alpha\in\mu} \ell_\alpha\, e^{t_0 - t_\alpha} = \sum_{\alpha\in\mu} \ell_\alpha\, e^{t_\alpha - t_0} = \ell_0.$$

If n is the number of curves in μ, then the above equation implies that there exist $\beta, \gamma \in \mu$ such that

$$\ell_\beta\, e^{t_0 - t_\beta} \geqslant \frac{\ell_0}{n} \quad \text{and} \quad \ell_\gamma\, e^{t_\gamma - t_0} \geqslant \frac{\ell_0}{n}.$$

Thus we have

$$\begin{aligned} f(t) = \sum_{\alpha\in\mu} \ell_\alpha \cosh(t - t_\alpha) &\geqslant \ell_\beta \cosh(t - t_\beta) + \ell_\gamma \cosh(t - t_\gamma) \\ &\geqslant \frac{1}{2}\Big[\ell_\beta\, e^{t-t_0}\, e^{t_0 - t_\beta} + \ell_\gamma\, e^{t_\gamma - t_0}\, e^{t_0 - t}\Big] \\ &\geqslant \frac{\ell_0}{2n}\, e^{|t-t_0|}. \end{aligned} \tag{7}$$

Equations (6) and (7) show that $f(t) \stackrel{*}{\asymp} \ell_0\, e^{|t-t_0|}$. This along with Equation (5) prove the lemma. □

Proof of Theorem B. We show that the first three quantities are comparable, and the proof for the remaining term is obtained by reversing the orientation of g. Suppose that for $a < b$, we have $g(a) = \sigma$, $g(b) = \tau$ so that $d_{\mathcal{T}}(\sigma, \tau) = b - a$. Since the moduli space of the thick part is compact, we know that the hyperbolic lengths of curves in σ and τ are proportional to their quadratic differential lengths in q_a and q_b, respectively (see [**10**] for a more general discussion). Therefore, there are multiplicative constants depending only on ϵ such that for any simple closed curve α,

$$\frac{\ell_\tau(\alpha)}{\ell_\sigma(\alpha)} \stackrel{*}{\asymp} \frac{\ell_{q_b}(\alpha)}{\ell_{q_a}(\alpha)}. \tag{8}$$

Moreover, since

$$\ell_{q_b}(\alpha) \stackrel{*}{\asymp} \ell_\alpha \cosh(b - t_\alpha) \leqslant e^{b-a} \ell_\alpha \cosh(a - t_\alpha) \stackrel{*}{\asymp} e^{b-a} \ell_{q_a}(\alpha),$$

it follows from equation (8) that

$$d_{\mathrm{L}}(\sigma, \tau) \stackrel{+}{\prec} d_{\mathcal{T}}(\sigma, \tau).$$

Therefore, since we clearly have

$$\log \max_{\alpha\in\mu_\sigma} \frac{\ell_\tau(\alpha)}{\ell_\sigma(\alpha)} \leqslant d_{\mathrm{L}}(\sigma, \tau),$$

it remains to be shown that there is a curve $\alpha \in \mu_\sigma$ such that

$$b - a \stackrel{+}{\prec} \log \frac{\ell_{q_b}(\alpha)}{\ell_{q_a}(\alpha)}.$$

Let $\ell_{q_t}(\mu_\sigma) \stackrel{*}{\asymp} \ell_0\, e^{|t-t_0|}$ as in Lemma 2.1. Then

$$\ell_0\, e^{|b-a|-|a-t_0|} \stackrel{*}{\prec} \ell_{q_b}(\mu_\sigma) \stackrel{*}{\prec} \ell_0\, e^{|b-a|+|a-t_0|}. \tag{9}$$

First we show that $|a-t_0|$ is bounded above. Since σ is in the thick part, the q_a-length and the σ-length of μ_σ are comparable to one another. Moreover, since μ_σ is a short marking in σ, its σ-length is bounded both above and below. Therefore, we have

$$\ell_0\, e^{|a-t_0|} \stackrel{*}{\asymp} \ell_{q_a}(\mu_\sigma) \stackrel{*}{\asymp} \ell_\sigma(\mu_\sigma) \stackrel{*}{\asymp} 1. \tag{10}$$

Furthermore, we can see that ℓ_0 is bounded below as follows. A marking divides the surface into disks and punctured disks. For any quadratic differential q, the q-area of a disk or a punctured disk is less than the square of its perimeter. Therefore, we have for all t that

$$1 = \text{area}_{q_t}(S) \stackrel{*}{\prec} \sum_{\alpha\in\mu_\sigma} \ell_{q_t}(\alpha)^2. \tag{11}$$

By applying the above equation to $t = t_0$, we obtain $\ell_0 \stackrel{*}{\succ} 1$. It then follows from equation (10) that $|a-t_0| \stackrel{*}{\prec} 1$, as desired. Thus, it follows from equation (9) that

$$\ell_{q_b}(\mu_\sigma) \stackrel{*}{\asymp} e^{b-a}. \tag{12}$$

As we saw in equation (10), the q_a-lengths of curves in μ_σ are bounded above and below. Combining this with equation (12) and the fact that the q_b-length of μ_σ is the sum of the q_b-lengths of its curves, we see that there is a curve $\alpha \in \mu_\sigma$ such that

$$\ell_{q_b}(\alpha) \stackrel{*}{\asymp} \ell_{q_b}(\mu_\sigma) \stackrel{*}{\asymp} e^{b-a} \stackrel{*}{\asymp} \ell_{q_a}(\alpha)e^{b-a},$$

which is what we wanted. □

THEOREM 2.2. *Let σ and τ be points in the ϵ-thick part of the Teichmüller space and let μ_σ and μ_τ be their short markings, respectively. Then there is an additive constant depending only on ϵ such that*

$$d_{\mathcal{T}}(\sigma,\tau) \stackrel{+}{\asymp} \log\, i(\mu_\sigma,\mu_\tau),$$

where $i(\mu_\sigma,\mu_\tau)$ is the total number of intersections between the curves in μ_σ and the curves in μ_τ.

Proof. The τ-length of a curve is proportional to its intersection number with μ_τ (see, for example, [**6**, Lemma 4.7]). Therefore,

$$i(\mu_\sigma,\mu_\tau) \stackrel{*}{\asymp} \sum_{\alpha\in\mu_\sigma} \ell_\tau(\alpha) \stackrel{*}{\asymp} \max_{\alpha\in\mu_\sigma} \ell_\tau(\alpha). \tag{13}$$

Since σ is in the thick part of $\mathcal{T}(S)$, we have $\ell_\sigma(\alpha) \stackrel{*}{\asymp} 1$ for every curve $\alpha \in \mu_\sigma$. Thus, it follows from Theorem B that

$$\log \max_{\alpha\in\mu_\sigma} \ell_\tau(\alpha) \stackrel{+}{\asymp} \log \max_{\alpha\in\mu_\sigma} \frac{\ell_\tau(\alpha)}{\ell_\sigma(\alpha)} \stackrel{+}{\asymp} d_{\mathcal{T}}(\sigma,\tau). \tag{14}$$

The theorem follows from equations (13) and (14). □

REMARK 2.3. The above theorem implies that the logarithm of the intersection number is almost a distance function on the marking space. In particular, it satisfies a quasi-triangle inequality. That is, for markings μ_1, μ_2, and μ_3 we have

$$\log i(\mu_1,\mu_3) \stackrel{+}{\prec} \log i(\mu_1,\mu_2) + \log i(\mu_2,\mu_3).$$

This 'distance function' is similar, but not comparable, to the distance defined on the space of markings in [**5**].

Proof of Corollary D. For markings μ_1 and μ_2, one can find the points σ_1 and σ_2 in the thick part of the Teichmüller space such that μ_1 and μ_2 are short markings in σ_1 and σ_2, respectively. In [**9**], a combinatorial formula is given for the Teichmüller distance between any two points in the thick part of the Teichmüller space. It states that $d_{\mathcal{T}}(\sigma_1, \sigma_2)$ is comparable to the right-hand side of equation (4). Also, Theorem 2.2 states that $\log i(\mu_1, \mu_2) \stackrel{+}{\asymp} d_{\mathcal{T}}(\sigma_1, \sigma_2)$. These two results together prove the corollary. □

3. *Product regions in the Lipschitz metric*

In this section, we prove the analog of Minsky's product region theorem for the Lipschitz metric.

3.1. *An* (ϵ_0, ϵ_1)*-decomposition*

First, we need to recall the notion of an (ϵ_0, ϵ_1)-decomposition defined in [**7**]. Let $0 < \epsilon_1 < \epsilon_0$ be two numbers less than the Margulis constant $c_0 = 0.2629\ldots$; see [**14**]. Let σ be a hyperbolic metric on S and suppose $\gamma_1, \ldots, \gamma_k$ are geodesics with lengths $\ell_\sigma(\gamma_i) \leqslant \epsilon_1$. Let $A_1, \ldots, A_k$ be the collection of annular neighborhoods of $\gamma_1, \ldots, \gamma_k$, respectively, such that the boundary components of A_i each have length ϵ_0. A component Q of $S \setminus \bigcup A_i$ is called a *hyperbolic* component and the entire collection $\mathcal{P}$ of hyperbolic components and annular components is called an (ϵ_0, ϵ_1)-*decomposition*. We assume that ϵ_0 and ϵ_1 are chosen so that any simple geodesic that intersects an annular component A is either the core of A or is made up of arcs that run from one boundary component of A to another. We remark that in [**7**], what we have described is called a *partial* (ϵ_0, ϵ_1)-decomposition. There, the term (ϵ_0, ϵ_1)-decomposition is reserved for the case where $\{\gamma_1, \ldots, \gamma_k\}$ is the full set of curves, the lengths of which satisfy $\ell_\sigma(\gamma_i) \leqslant \epsilon_1$.

In the course of arguments to follow, we shall further require that $\epsilon_0/\epsilon_1 > 2$ so that certain desired estimates hold (see, for example, Lemma 3.6). We therefore assume that ϵ_0 and ϵ_1 have been chosen once and for all to satisfy all the conditions stated above and henceforth use the notation $f \stackrel{*}{\asymp} g$, $f \stackrel{+}{\asymp} g$, and so on, to mean that the multiplicative or additive constants that appear depend only on this choice of ϵ_0 and ϵ_1 (and on the topological type of S).

3.2. *Decomposing the length of a curve*

Consider the intersection of a simple closed curve ζ with the components of an (ϵ_0, ϵ_1)-decomposition. For a hyperbolic component Q, let $\mathcal{C}(Q, \partial Q)$ denote the homotopy classes of simple closed curves in Q and of essential arcs in Q with endpoints on ∂Q, under homotopies that keep any endpoints of arcs on ∂Q. Define the orthogonal projection ζ_Q of ζ to be the geodesic representative of $\zeta \cap Q$ in $\mathcal{C}(Q, \partial Q)$ that has the shortest length (see [**7**, § 2.3]). In particular, every arc in ζ_Q is perpendicular to ∂Q. It is not hard to show the following.

PROPOSITION 3.1. *Let $\mathcal{P}$ be an (ϵ_0, ϵ_1)-decomposition for σ and let $Q, A \in \mathcal{P}$ be, respectively, a hyperbolic and an annular component. Then, for any simple closed curve ζ, the following estimates hold:*

$$i(\zeta, \partial Q) \stackrel{*}{\succ} \big|\ell_\sigma(\zeta \cap Q) - \ell_\sigma(\zeta_Q)\big|, \tag{15}$$

$$i(\zeta, \gamma) \stackrel{*}{\succ} \left|\ell_\sigma(\zeta \cap A) - \left[2\log\frac{\epsilon_0}{\ell_\sigma(\gamma)} + \ell_\sigma(\gamma) \cdot |\mathrm{tw}_\sigma(\zeta, \gamma)|\right] i(\zeta, \gamma)\right|, \tag{16}$$

where γ is the core geodesic of A and $\mathrm{tw}_\sigma(\zeta, \gamma)$ is the twist of ζ around γ defined in [**7**, § 3]*; see also Section* 3.3.

In equation (16), the quantity $2\log[\epsilon_0/\ell_\sigma(\gamma)]$ is approximately the width of A; the right-most term of the right-hand side describes the sum of lengths of piecewise geodesic arcs homotopic to $\zeta \cap A$, relative to endpoints, each of which goes perpendicularly from one component of A to γ, wraps around γ a number of $|\mathrm{tw}_\sigma(\zeta,\gamma)|$ times (up to an error of 1), then goes out of the other end of A orthogonally. The idea is that most of the twisting that ζ does around γ takes place in A; see [**7**]. This is also the reason that equation (15) is true (for a proof, see [**2**]).

Since each component of ∂Q has a collar of some definite width, $\ell_\sigma(\zeta\cap Q) \overset{*}{\succ} i(\zeta,\partial Q)$ and $\ell_\sigma(\zeta_Q) \overset{*}{\succ} i(\zeta,\partial Q)$. Similarly, since γ has a collar of definite width, terms in the right-hand side of equation (16) are larger than a multiple of $i(\zeta,\gamma)$. Therefore, Proposition 3.1 implies the following.

Corollary 3.2. *Let Q, A and γ be as in Proposition* 3.1*. Then for any simple closed curve ζ on S, we have*

$$\ell_\sigma(\zeta\cap Q) \overset{*}{\asymp} \ell_\sigma(\zeta_Q), \tag{17}$$

$$\ell_\sigma(\zeta\cap A) \overset{*}{\asymp} \Big[2\log\frac{\epsilon_0}{\ell_\sigma(\gamma)} + \ell_\sigma(\gamma)\cdot|\mathrm{tw}_\sigma(\zeta,\gamma)|\Big] i(\zeta,\gamma). \tag{18}$$

3.3. *Metrics on annuli*

Let γ be a simple closed curve on S and let $\tilde S$ be the annular cover of S corresponding to γ. Since S admits a hyperbolic metric, $\tilde S$ has a well-defined boundary $\partial\tilde S$ at infinity. Let $\tilde\gamma$ be the lift of γ that is homotopic to the core curve of $\tilde S$. For $\epsilon > 0$, let $U_\epsilon(\gamma)$ be the space (equivalence classes) of hyperbolic metrics on $\tilde S$ such that the geodesic representative of $\tilde\gamma$ has length at most ϵ. Two metrics are considered equivalent in $U_\epsilon(\gamma)$ if they differ by an isotopy of $(\tilde S,\partial\tilde S)$ that fixes $\partial\tilde S$ pointwise.

Let $\mathcal{C}(\tilde S,\partial\tilde S)$ be the set of isotopy classes of non-trivial simple loops or arcs in $\tilde S$ with endpoints in $\partial\tilde S$, under isotopies that fix the endpoints. Here a loop is non-trivial if it is not homotopic to a point, and an arc is non-trivial if it is not homotopic into $\partial\tilde S$ by a homotopy fixing endpoints. For $\rho \in U_\epsilon(\gamma)$, let $N(\rho)$ be the annular neighborhood of the ρ-geodesic representative of $\tilde\gamma$ such that each component of $\partial N(\rho)$ has length ϵ_0. Although the length of the ρ-geodesic representative of an arc $\beta\in\mathcal{C}(\tilde S,\partial\tilde S)$ is obviously infinite, we abuse terminology and define the ρ-length $\ell_\rho(\beta)$ of β to be the length of the arc of intersection between the ρ-geodesic representative of β and $N(\rho)$. Observe that this definition extends consistently to the ρ-length of $\tilde\gamma$.

Define the distance between $\rho_1,\rho_2\in U_\epsilon(\gamma)$ to be

$$d_{L(\gamma)}(\rho_1,\rho_2) = \sup_{\beta\in\mathcal{C}(\tilde S,\partial\tilde S)} \left|\log\frac{\ell_{\rho_1}(\beta)}{\ell_{\rho_2}(\beta)}\right|.$$

Clearly $d_{L(\gamma)}(\rho_1,\rho_2)$ is symmetric and is zero if and only if $\rho_1=\rho_2$. To see that the triangle inequality holds, observe that

$$\left|\log\frac{\ell_{\rho_1}(\beta)}{\ell_{\rho_2}(\beta)}\right| + \left|\log\frac{\ell_{\rho_2}(\beta)}{\ell_{\rho_3}(\beta)}\right| \geqslant \log\frac{\ell_{\rho_1}(\beta)}{\ell_{\rho_2}(\beta)} + \log\frac{\ell_{\rho_2}(\beta)}{\ell_{\rho_3}(\beta)} = \log\frac{\ell_{\rho_1}(\beta)}{\ell_{\rho_3}(\beta)},$$

$$\left|\log\frac{\ell_{\rho_1}(\beta)}{\ell_{\rho_2}(\beta)}\right| + \left|\log\frac{\ell_{\rho_2}(\beta)}{\ell_{\rho_3}(\beta)}\right| \geqslant \log\frac{\ell_{\rho_2}(\beta)}{\ell_{\rho_1}(\beta)} + \log\frac{\ell_{\rho_3}(\beta)}{\ell_{\rho_2}(\beta)} = \log\frac{\ell_{\rho_3}(\beta)}{\ell_{\rho_1}(\beta)}.$$

Define the twist $\mathrm{tw}_\rho(\beta,\tilde\gamma)$ of β around $\tilde\gamma$ as follows (see also [**7**, § 3]). First, it is necessary to fix an orientation of $\tilde\gamma$. Consider the universal cover of $(\tilde S,\rho)$ in $\mathbb{H}^2$ and the lifts $\hat\gamma$, $\hat\beta$ of $\tilde\gamma,\beta$, respectively (see Figure 1). Let β_{L} and β_{R} be the endpoints of $\hat\beta$ that lie on the left and right of $\hat\gamma$, respectively. Let p_{L} and p_{R} be, respectively, the orthogonal projections of β_{L} and β_{R} to $\hat\gamma$.

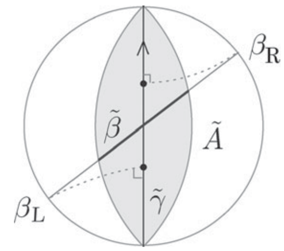

FIGURE 1. *Defining the twist* $\mathrm{tw}_\rho(\beta,\tilde\gamma)$.

Then the twist is defined as

$$\mathrm{tw}_\rho(\beta,\tilde\gamma) = \pm\frac{d_{\mathbb{H}^2}(p_{\rm L},p_{\rm R})}{\ell_\rho(\tilde\gamma)},$$

where the sign is $+$ if the direction from $p_{\rm L}$ to $p_{\rm R}$ coincides with the orientation of $\hat\gamma$ and $-$ if it is opposite. This is basically the same definition as the definition given for $\mathrm{tw}_\sigma(\zeta,\gamma)$ in [7, § 3], where σ is a hyperbolic metric on S, γ is a simple closed curve, and ζ is a transverse curve.

After fixing a simple arc $\omega\in\mathcal{C}(\tilde S,\partial\tilde S)$, we can define the twist parameter $\mathrm{tw}_\rho(\tilde S)$ of $(\tilde S,\rho)$ by setting $\mathrm{tw}_\rho(\tilde S)=\mathrm{tw}_\rho(\omega,\tilde\gamma)$. We have the following.

LEMMA 3.3. *Let $\rho,\rho_1,\rho_2\in U_\epsilon(\gamma)$ and let $\beta\in\mathcal{C}(\tilde S,\partial\tilde S)$ be any arc. Then*

$$\Big|[\mathrm{tw}_{\rho_2}(\beta,\tilde\gamma)-\mathrm{tw}_{\rho_1}(\beta,\tilde\gamma)]-[\mathrm{tw}_{\rho_2}(\tilde S)-\mathrm{tw}_{\rho_1}(\tilde S)]\Big| \overset{+}{\asymp} 0$$

and

$$\ell_\rho(\beta)\overset{*}{\asymp} 2\log\left[\frac{1}{\ell_\rho(\tilde\gamma)}\right]+|\mathrm{tw}_\rho(\beta,\tilde\gamma)|\ell_\rho(\tilde\gamma). \tag{19}$$

Note that $\log[1/\ell_\rho(\tilde\gamma)]>1$ since $\ell_\rho(\tilde\gamma)<0.263<1/e$. The proof of the first statement is similar to the proof of [7, Lemma 3.5] and the second is similar to equation (18). Details are omitted.

Then $U_\epsilon(\gamma)$ can be parametrized by the length of $\tilde\gamma$ and the twist parameter. The map $\rho\mapsto(\mathrm{tw}_\rho(\tilde S),1/\ell_\rho(\tilde\gamma))$ is a homeomorphism identifying $U_\epsilon(\gamma)$ with a subset of the upper half-plane

$$U_\epsilon(\gamma)=\left\{\ (x,y)\in\mathbb{R}^2\mid y\geqslant\frac{1}{\epsilon}\ \right\}.$$

We can formulate the distance $d_{L(\gamma)}$ on $U_\epsilon(\gamma)$ in terms of these coordinates as follows. Let $\rho_1,\rho_2\in U_\epsilon(\gamma)$ and let $t_i=\mathrm{tw}_{\rho_i}(\tilde S)$, $\ell_i=\ell_{\rho_i}(\tilde\gamma)$ for $i=1,2$.

LEMMA 3.4. *Assume that $\ell_1\leqslant\ell_2$. Then the following hold.*

(i) *If $|t_1-t_2|\,\ell_1\leqslant\log[1/\ell_1]$, then*

$$d_{L(\gamma)}(\rho_1,\rho_2)\overset{+}{\asymp}\log\frac{\ell_2}{\ell_1}.$$

(ii) *If $|t_1-t_2|\,\ell_1>\log[1/\ell_1]$, then*

$$d_{L(\gamma)}(\rho_1,\rho_2)\overset{+}{\asymp}\log\frac{|t_1-t_2|\,\ell_2}{\log[1/\ell_1]}=\log\frac{\ell_2}{\ell_1}+\log\frac{|t_1-t_2|\,\ell_1}{\log[1/\ell_1]}.$$

We remark that in comparison, the hyperbolic distance between $z_1=(t_1,1/\ell_1)$ and $z_2=(t_2,1/\ell_2)$ in the upper half-plane can be estimated as follows. Assume that $\ell_1\leqslant\ell_2$.

(i) If $|t_1 - t_2|\,\ell_1 \leqslant 1$, then

$$d_{\mathbb{H}^2}(z_1, z_2) \overset{+}{\asymp} \log \frac{\ell_2}{\ell_1}.$$

(ii) If $|t_1 - t_2|\,\ell_1 > 1$, then

$$d_{\mathbb{H}^2}(z_1, z_2) \overset{+}{\asymp} \log \frac{\ell_2}{\ell_1} + 2\log[|t_1 - t_2|\ell_1].$$

Proof of Lemma 3.4. For any arc $\beta \in \mathcal{C}(\tilde{S}, \partial\tilde{S})$, the second part of Lemma 3.3 implies that

$$\begin{aligned}\frac{\ell_{\rho_2}(\beta)}{\ell_{\rho_1}(\beta)} &\overset{*}{\asymp} \frac{2\log[1/\ell_2] + |\mathrm{tw}_{\rho_2}(\beta, \tilde{\gamma})|\ell_2}{2\log[1/\ell_1] + |\mathrm{tw}_{\rho_1}(\beta, \tilde{\gamma})|\ell_1} \\ &\overset{*}{\asymp} \frac{\log[1/\ell_2] + |\mathrm{tw}_{\rho_2}(\beta, \tilde{\gamma})|\ell_2}{\log[1/\ell_1] + |\mathrm{tw}_{\rho_1}(\beta, \tilde{\gamma})|\ell_1}.\end{aligned} \tag{20}$$

Combined with the first part of Lemma 3.3, we obtain the following:

$$\sup_{\beta \in \mathcal{C}(\tilde{S}, \partial\tilde{S})} \frac{\ell_{\rho_2}(\beta)}{\ell_{\rho_1}(\beta)} \overset{*}{\asymp} \max\left\{\frac{\ell_2}{\ell_1}, \frac{\log[1/\ell_2] + |t_2 - t_1|\ell_2}{\log[1/\ell_1]}\right\}. \tag{21}$$

To see this, note that Lemma 3.3 implies that for a sequence of arcs β_n with $|\mathrm{tw}_{\rho_1}(\beta_n, \tilde{\gamma})| \to \infty$, we have $|\mathrm{tw}_{\rho_2}(\beta_n, \tilde{\gamma})/\mathrm{tw}_{\rho_1}(\beta_n, \tilde{\gamma})| \to 1$, so that the limit of (20) for this sequence of arcs gives ℓ_2/ℓ_1. At the other extreme, when $\mathrm{tw}_{\rho_1}(\beta, \tilde{\gamma}) = 0$, we see that $|\mathrm{tw}_{\rho_2}(\beta, \tilde{\gamma})| \overset{+}{\asymp} |t_2 - t_1|$, and so we get the term on the right in equation (21).

To simplify the notation, let

$$R_1 = \frac{\log[1/\ell_2] + |t_2 - t_1|\,\ell_2}{\log[1/\ell_1]}, \quad R_2 = \frac{\log[1/\ell_1] + |t_2 - t_1|\,\ell_1}{\log[1/\ell_2]},$$

and

$$R = \frac{|t_2 - t_1|\,\ell_2}{\log[1/\ell_1]}.$$

The assumption that $\ell_1 \leqslant \ell_2$ implies that $R < R_1 \leqslant R + 1$. Moreover, since $\ell_1 \leqslant \ell_2 < \epsilon_1 < c_0 = 0.2629\ldots$, we see that $\log[1/\ell_1]/\log[1/\ell_2] \leqslant \ell_2/\ell_1$, and so $R_2 < \ell_2/\ell_1 + R$. Therefore

$$d_{L(\gamma)}(\rho_1, \rho_2) \overset{+}{\asymp} \log\max\left\{R + 1, \frac{\ell_2}{\ell_1}\right\}.$$

If $|t_1 - t_2|\,\ell_1 \leqslant \log[1/\ell_1]$, then $R \leqslant \ell_2/\ell_1$, and so

$$d_{L(\gamma)}(\rho_1, \rho_2) \overset{+}{\asymp} \log\left[\ell_2/\ell_1\right].$$

If $|t_1 - t_2|\,\ell_1 > \log[1/\ell_1]$, then $R > \ell_2/\ell_1$, and hence

$$d_{L(\gamma)}(\rho_1, \rho_2) \overset{+}{\asymp} \log(R + 1) \overset{+}{\asymp} \log R.$$

□

3.4. *Product region theorem.*

Let $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ be a collection of disjoint, homotopically distinct simple closed curves on S. Choose a pants system $\bar{\Gamma}$ that contains Γ and define a Fenchel–Nielsen coordinate system associated to $\bar{\Gamma}$, as explained in [7, § 3]. Let $s_\sigma(\gamma_i)$ denote the Fenchel–Nielsen twist coordinate of γ_i. Let $\tilde{S}_i$ be the annular cover of S corresponding to γ_i, let $\tilde{\gamma}_i$ be the lift of γ_i to $\tilde{S}_i$, and let $U_i = U_{\epsilon_1}(\gamma_i)$. For $\sigma \in \mathrm{Thin}_{\epsilon_1}(S, \Gamma)$, let $\Pi_{\gamma_i}(\sigma) \in U_i$ be the metric ρ such that the twist $\mathrm{tw}_\rho(\tilde{S}_i)$ equals $s_\sigma(\gamma_i)$ and such that $\ell_\rho(\tilde{\gamma}_i) = \ell_\sigma(\gamma_i)$. Each $\sigma \in \mathrm{Thin}_{\epsilon_1}(S, \Gamma)$ also defines a metric $\Pi_{S\setminus\Gamma}(\sigma)$ in $\mathcal{T}(S \setminus \Gamma)$, obtained by pinching the geodesic representatives of $\gamma_1, \ldots, \gamma_k$, but otherwise leaving the metric unchanged, that is, by retaining the same Fenchel–Nielsen

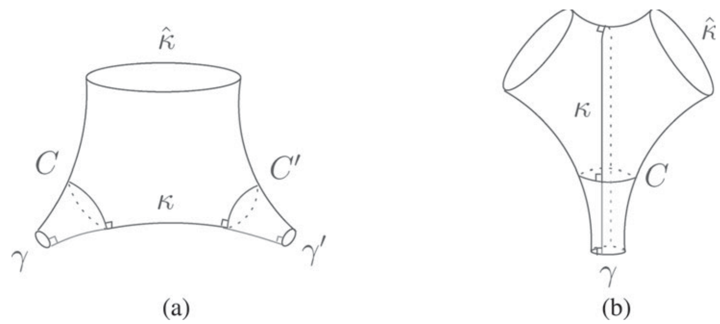

FIGURE 2. *Construction of $\hat{\kappa}$.*

coordinates. Thus we define a homeomorphism

$$\Pi : \mathrm{Thin}_{\epsilon_1}(S, \Gamma) \longrightarrow \mathcal{T}(S \setminus \Gamma) \times U_1 \times \ldots \times U_k.$$

Endow $\mathcal{T}(S \setminus \Gamma) \times U_1 \times \ldots \times U_k$ with the sup-metric

$$d_{L_\Gamma} = \sup\{d_{L(S\setminus\Gamma)}, d_{L(\gamma_1)}, \ldots, d_{L(\gamma_k)}\}.$$

THEOREM 3.5 (Product regions for the Lipschitz metric). *For any $\sigma, \tau \in \mathrm{Thin}_{\epsilon_1}(S, \Gamma)$, we have*

$$d_L(\sigma, \tau) \stackrel{+}{\asymp} d_{L_\Gamma}(\Pi(\sigma), \Pi(\tau)).$$

The important step of the proof is Proposition 3.7 below.

3.5. *Replacing an arc with a loop*

Let Q be a hyperbolic component of an (ϵ_0, ϵ_1)-decomposition which is not homeomorphic to a pair of pants. Next, we describe a procedure to replace an arc in ζ_Q with a non-trivial, non-peripheral simple closed curve in Q that has comparable length.

Let κ be a simple geodesic arc in Q, the endpoints of which lie in ∂Q and which is perpendicular to ∂Q. If the two endpoints of κ lie in distinct components C, C' of ∂Q, then the boundary of a regular neighborhood of $\kappa \cup C \cup C'$ in Q consists of a single curve η. Define $\hat{\kappa}$ to be the geodesic representative of η in S. Note that since Q is not a pair of pants, it follows that η is non-peripheral in Q, and in particular, $\hat{\kappa}$ is contained in Q (see Figure 2a).

If both endpoints of κ lie in a single component C of ∂Q, then the boundary of a regular neighborhood of $\kappa \cup C$ has two components (see Figure 2b). In this case, define $\hat{\kappa}$ to be the curve of greater length between the geodesic representatives in S of the two components (if one of the curves is peripheral, $\hat{\kappa}$ is the geodesic representative of the non-peripheral component). Note that $\hat{\kappa}$ is non-peripheral in Q and, in particular, it is contained in Q. Also note that unlike the preceding case, the choice of $\hat{\kappa}$ depends on the geometry of the surface.

LEMMA 3.6. *Suppose that Q is a hyperbolic component of an (ϵ_0, ϵ_1)-decomposition of σ which is not homeomorphic to a pair of pants. Let κ be an arc in Q perpendicular to ∂Q and let $\hat{\kappa}$ be the associated simple closed curve constructed above. If $\ell_\sigma(\hat{\kappa}) > c_0$ for the Margulis constant c_0, then*

$$\ell_\sigma(\kappa) \stackrel{*}{\asymp} \ell_\sigma(\hat{\kappa}).$$

Proof. Let C and C' denote the components of ∂Q that contain the endpoints of κ, where we take $C = C'$ if the endpoints lie on the same component. Let γ and γ' denote the geodesic

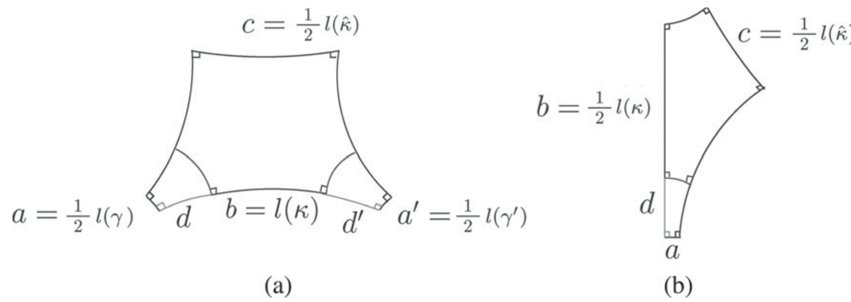

FIGURE 3. *Hexagon and pentagon of P.*

representatives of C and C' in S. By hypothesis, γ and γ' have embedded collars in S, with boundary components each having length ϵ_0. Cut the collars in half along γ, γ' and let $\overline{Q}$ be the surface obtained by attaching the half collars around γ and γ' to Q, along C and C', respectively. (In the case that $C \neq C'$ but $\gamma = \gamma'$ in S, we attach a half-collar around γ to each of C and C'.) Since κ intersects ∂Q perpendicularly, it has a natural extension to a (smooth) geodesic arc $\overline{\kappa}$ with endpoints in $\partial\overline{Q}$ and perpendicular to $\partial\overline{Q}$, as depicted in Figure 2.

First, consider the case when $C \neq C'$. Let P be the pair of pants with boundary components γ, γ', $\hat\kappa$ and consider one of the right-angled hexagons of P, as in Figure 3a.

Let $a = l(\gamma)/2, a' = l(\gamma')/2$ and let d and d' be the widths of the half-collars around γ and γ', respectively. Let $b = l(\kappa)$ and $c = l(\hat\kappa)/2$. By the formula for right-angled hexagons, we have

$$\cosh c + \cosh a \cosh a' = \sinh a \sinh a' \cosh(b + d + d'). \tag{22}$$

Since $a, a' < \epsilon_1/2$ and since ϵ_1 is smaller than the Margulis constant, we see that $\sinh a < 2a$ and $\sinh a' < 2a'$. Also, by a straightforward calculation in $\mathbb{H}^2$, we have $\epsilon_0/2 = a \cosh d = a' \cosh d'$. Therefore, the right-hand side of equation (22) satisfies

$$\sinh a \sinh a' \cosh(b + d + d') > a \cdot a' \, \frac{e^{b+d+d'}}{2} > a \cdot a' \, \frac{\cosh d \cosh d'}{2} \, e^b = \frac{\epsilon_0^2 \, e^b}{8},$$
$$\sinh a \sinh a' \cosh(b + d + d') < 4 \, a \cdot a' \cdot e^{b+d+d'} < 16 \, a \cdot a' \cdot \cosh d \cosh d' \, e^b = 4\epsilon_0^2 \, e^b.$$

On the other hand, since $a, a' < \epsilon_1/2 < \epsilon_0/4 < c_0/4$ and $c > c_0/2$, we have

$$\cosh a \cosh a' < \cosh(a + a') < \cosh \frac{c_0}{2} < \cosh c.$$

Therefore, equation (22) combined with the three equations above gives

$$\frac{\epsilon_0^2 e^b}{16} < \cosh c < 4\epsilon_0^2 \, e^b.$$

Hence

$$|\, c - b \,| = \left| \frac{l(\hat\kappa)}{2} - l(\kappa) \right| < 2 \log \frac{1}{\epsilon_0} + k$$

for some universal constant $k (= \log 8)$. Thus, if $l(\kappa)$ is sufficiently large, then the additive error can be absorbed into multiplicative constants to conclude that $l(\hat\kappa) \overset{*}{\asymp} l(\kappa)$. If $l(\kappa)$ is not sufficiently large, then $l(\hat\kappa) \overset{*}{\asymp} l(\kappa)$ holds almost tautologically, because $l(\hat\kappa)$ is bounded above by $2l(\kappa) + 2\epsilon_0$ and is bounded below, by assumption.

Next consider the case where $C = C'$. Let P be the geodesic pair of pants in S filled by $\overline{\kappa} \cup \gamma$. The arc $\overline{\kappa}$ divides the two right-angled hexagons of P into four right-angled pentagons. It is easy to see that the two pentagons that have edges originally contained in $\hat\kappa$ are isometric to each other. Let X be either one of them, as in Figure 3b. Let $b = l(\kappa)/2$, $c = l(\hat\kappa)/2$ and let

d be the width of the half-collar around γ. Let a be the length of the edge of X coming from γ. Now, by the formula for right-angled pentagons, we have

$$\cosh c = \sinh(b+d)\sinh a.$$

It is clear that $a \leqslant l(\gamma)/2$, and by applying the pentagon formula to the pentagon which together with X makes up a hexagon of P, we see that our choice of $\hat{\kappa}$ implies that $a \geqslant l(\gamma)/4$. Furthermore, as before we have $l(\gamma)\cdot\cosh d = \epsilon_0$ and since $l(\gamma) \leqslant \epsilon_1$, the assumption that $\epsilon_0/\epsilon_1 > 2$ is sufficient to guarantee that d is large enough that $e^{b+d}/4 < \sinh(b+d)$ holds. And, as above, ϵ_1 is small enough that $a < \sinh a < 2a$. Therefore, we have

$$\cosh c = \sinh(b+d)\sinh a > \frac{e^b\, e^d}{4} a > e^b \frac{\cosh d}{4}\cdot\frac{l(\gamma)}{4} = \frac{e^b\epsilon_0}{16},$$
$$\cosh c = \sinh(b+d)\sinh a < e^b\, e^d a < e^b\cdot 2\cosh d\cdot l(\gamma)/2 = e^b\epsilon_0.$$

Hence

$$|\, c - b\,| = \left|\, \frac{l(\hat{\kappa})}{2} - \frac{l(\kappa)}{2}\,\right| < \log\frac{1}{\epsilon_0} + k$$

for some universal constant $k(=\log 16)$. Thus we conclude as before that $l(\hat{\kappa}) \stackrel{*}{\asymp} l(\kappa)$. □

We remark that in the second case above, had we not chosen $\hat{\kappa}$ to be the longer of the two components of $\partial P - \gamma$, then the lemma would not be true. This can easily be seen by considering the construction in reverse as follows. Take a closed curve α in Q of moderate length and a very long arc β with one endpoint on α and the other on a component C of ∂Q. Construct a new arc κ with both endpoints on C by replacing β with two copies of itself very close together, and by connecting their two endpoints on α by the longer arc along α. It is easy to see that the pair of pants filled by $\kappa \cup C$ has α as a boundary component, yet $l(\alpha)/l(\kappa)$ can be made arbitrarily small.

3.6. *Proof of the product region theorem for the Lipschitz metric*

For any surface Σ, let $\mathcal{C}(\Sigma)$ be the set of homotopy classes of non-peripheral, non-trivial simple closed curves in Σ. Suppose that A is an annulus in an (ϵ_0,ϵ_1)-decomposition of $\sigma \in \mathcal{T}(S)$. Let γ be the core curve of A and let $\tilde{\sigma}$ be the lift of σ to $\tilde{S}$, where as before, $\tilde{S}$ is the cover of S corresponding to γ. The $\tilde{\sigma}$-length of an arc $\beta \in \mathcal{C}(\tilde{S},\partial\tilde{S})$ is as defined in Section 3.3. Note that if $\zeta \in \mathcal{C}(S)$ and $\tilde{\zeta} \subset \mathcal{C}(\tilde{S},\partial\tilde{S})$ are the lifts of ζ to $\tilde{S}$, then

$$\ell_{\tilde{\sigma}}(\tilde{\zeta}) = \ell_\sigma(\zeta \cap A),$$

where $\ell_\sigma(\zeta\cap A)$ is the σ-length of the intersection of the σ-geodesic representative of ζ with A.

To keep track of the dependence of $\tilde{S}$ on A, we will write $\mathcal{C}(A,\partial A)$ instead of $\mathcal{C}(\tilde{S},\partial\tilde{S})$ and for convenience, write $\ell_\sigma(\beta)$ instead of $\ell_{\tilde{\sigma}}(\beta)$. We are now ready to prove Proposition 3.7.

PROPOSITION 3.7. *Suppose that $\mathcal{P}$ is an (ϵ_0,ϵ_1)-decomposition for both $\sigma,\tau \in \mathcal{T}(S)$. Then*

$$\sup_{\zeta\in\mathcal{C}(S)} \frac{\ell_\tau(\zeta)}{\ell_\sigma(\zeta)} \stackrel{*}{\asymp} \max_{Q,A\in\mathcal{P}} \left\{ \sup_{\alpha\in\mathcal{C}(Q)} \frac{\ell_\tau(\alpha)}{\ell_\sigma(\alpha)},\ \sup_{\beta\in\mathcal{C}(A,\partial A)} \frac{\ell_\tau(\beta)}{\ell_\sigma(\beta)} \right\}. \tag{23}$$

Moreover, when taking the maximum, we may assume that Q is never a pair of pants.

Proof. By Corollary 3.2, for any curve $\zeta \in \mathcal{C}(S)$ and any $\rho \in \mathcal{T}(S)$ which has $\mathcal{P}$ as a partial (ϵ_0,ϵ_1)-decomposition, we have

$$\ell_\rho(\zeta) \stackrel{*}{\asymp} \sum_{Q,A\in\mathcal{P}} [\,\ell_\rho(\zeta_Q) + \ell_\rho(\zeta\cap A)\,].$$

Applying this to σ, τ gives

$$\begin{aligned}\frac{\ell_\tau(\zeta)}{\ell_\sigma(\zeta)} &\stackrel{*}{\asymp} \frac{\sum_{Q,A\in\mathcal{P}}[\,\ell_\tau(\zeta_Q)+\ell_\tau(\zeta\cap A)\,]}{\sum_{Q,A\in\mathcal{P}}[\,\ell_\sigma(\zeta_Q)+\ell_\sigma(\zeta\cap A)\,]}\\ &\leqslant \max_{Q,A\in\mathcal{P}}\left\{\frac{\ell_\tau(\zeta_Q)}{\ell_\sigma(\zeta_Q)}, \frac{\ell_\tau(\zeta\cap A)}{\ell_\sigma(\zeta\cap A)}\right\}.\end{aligned} \tag{24}$$

Fix Q and write $\zeta_Q = \sum_i m_i\kappa_i + \sum_j n_j\lambda_j$, where κ_i are arcs with endpoints on ∂Q and λ_j are non-peripheral simple closed curves contained in Q. Then

$$\begin{aligned}\frac{\ell_\tau(\zeta_Q)}{\ell_\sigma(\zeta_Q)} &\stackrel{*}{\asymp} \frac{\sum_i m_i\,\ell_\tau(\kappa_i)+\sum_j n_j\,\ell_\tau(\lambda_j)}{\sum_i m_i\,\ell_\sigma(\kappa_i)+\sum_j n_j\,\ell_\sigma(\lambda_j)}\\ &\leqslant \max_{i,j}\left\{\frac{\ell_\tau(\kappa_i)}{\ell_\sigma(\kappa_i)}, \frac{\ell_\tau(\lambda_j)}{\ell_\sigma(\lambda_j)}\right\}.\end{aligned} \tag{25}$$

The idea is to show that for every i,

$$\frac{\ell_\tau(\kappa_i)}{\ell_\sigma(\kappa_i)} \stackrel{*}{\prec} \sup_{\alpha\in\mathcal{C}(Q)} \frac{\ell_\tau(\alpha)}{\ell_\sigma(\alpha)} \tag{26}$$

by replacing $\kappa = \kappa_i$ with the associated simple closed curve $\hat\kappa = \hat\kappa_i$ in Q, as described above. In the case that Q is a pair of pants, it is not hard to see that there are multiplicative constants depending only on ϵ_0 such that $\ell(\kappa) \stackrel{*}{\asymp} i(\kappa, \partial Q)$ and so

$$\frac{\ell_\tau(\kappa)}{\ell_\sigma(\kappa)} \stackrel{*}{\asymp} \frac{i(\kappa,\partial Q)}{i(\kappa,\partial Q)} \stackrel{*}{\asymp} 1.$$

Therefore, it is sufficient to prove equation (26) on assuming that Q is not a pair of pants, so that we may apply Lemma 3.6.

Recall that when the two endpoints of κ lie in the same component of ∂Q, the choice of $\hat\kappa$ depends on the geometry of the surface. Let $\hat\kappa(\tau)$ and $\hat\kappa(\sigma)$ denote the curves associated to κ for the two metrics τ and σ, respectively. Note that by definition of $\hat\kappa$,

$$\ell_\sigma(\hat\kappa(\tau)) \leqslant \ell_\sigma(\hat\kappa(\sigma)).$$

Now, if $\ell_\tau(\hat\kappa(\tau)) > c_0$, then applying Lemma 3.6 and using the fact that $\ell(\hat\kappa) \leqslant 2\ell(\kappa) + 2\epsilon_0$ always holds, we have

$$\frac{\ell_\tau(\kappa)}{\ell_\sigma(\kappa)} \stackrel{*}{\asymp} \frac{\ell_\tau(\hat\kappa(\tau))}{\ell_\sigma(\kappa)} \stackrel{*}{\prec} \frac{\ell_\tau(\hat\kappa(\tau))}{\ell_\sigma(\hat\kappa(\sigma))} \leqslant \frac{\ell_\tau(\hat\kappa(\tau))}{\ell_\sigma(\hat\kappa(\tau))} \leqslant \sup_{\alpha\in\mathcal{C}(Q)} \frac{\ell_\tau(\alpha)}{\ell_\sigma(\alpha)}.$$

If $\ell_\tau(\hat\kappa(\tau)) \leqslant c_0$, then in the τ-metric, the three boundary curves of the geodesic pair of pants P spanned by $\overline{\kappa}$ and the two components of $\partial\overline{Q}$ that contain the endpoints of $\overline{\kappa}$ (see Lemma 3.6) all have length shorter than c_0. Using the formulae for right-angled pentagons and hexagons as in the proof of Lemma 3.6, it is easy to show that this implies that $\ell_\tau(\kappa)$ is bounded above. Furthermore, since κ meets ∂Q, and ∂Q has an embedded regular neighborhood of some definite width depending on ϵ_0, it follows that $\ell_\sigma(\kappa)$ is bounded below. Hence

$$\frac{\ell_\tau(\kappa)}{\ell_\sigma(\kappa)} \stackrel{*}{\prec} \frac{1}{\ell_\sigma(\kappa)} \stackrel{*}{\prec} 1.$$

Since the ratio $\ell_\tau(\kappa)/\ell_\sigma(\kappa)$ is bounded above, equation (26) is tautologically satisfied in this case. Thus equation (26) is proved.

Combining this with equations (24) and (25), we now have

$$\frac{\ell_\tau(\zeta)}{\ell_\sigma(\zeta)} \stackrel{*}{\prec} \max_{Q,A\in\mathcal{P}}\left\{\sup_{\alpha\in\mathcal{C}(Q)}\frac{\ell_\tau(\alpha)}{\ell_\sigma(\alpha)}, \sup_{\beta\in\,\mathcal{C}(A,\,\partial A)}\frac{\ell_\tau(\beta)}{\ell_\sigma(\beta)}\right\}.$$

Therefore, the supremum of the left-hand side, taken over all $\zeta \in \mathcal{C}(S)$, is bounded by the quantity on the right-hand side.

Finally, since $\mathcal{C}(Q) \subset \mathcal{C}(S)$, it is clear that for every $Q \in \mathcal{P}$,

$$\sup_{\zeta \in \mathcal{C}(S)} \frac{\ell_\tau(\zeta)}{\ell_\sigma(\zeta)} \geqslant \max_{Q \in \mathcal{P}} \left\{ \sup_{\alpha \in \mathcal{C}(Q)} \frac{\ell_\tau(\alpha)}{\ell_\sigma(\alpha)} \right\}.$$

To complete the proof, we will show that there is a simple closed curve ζ such that

$$\frac{\ell_\tau(\zeta)}{\ell_\sigma(\zeta)} \overset{*}{\succ} \sup_{\beta \in \mathcal{C}(A,\, \partial A)} \frac{\ell_\tau(\beta)}{\ell_\sigma(\beta)}. \tag{27}$$

If the supremum on the right is realized by the core curve γ of A, the statement is obviously true. Hence, assume that the supremum is realized by an arc β. It follows from [**7**, Lemmas 3.2 and 3.3] that given any $t \in \mathbb{R}$, there is a simple closed curve δ on S, the twist $\mathrm{tw}_\sigma(\delta, \gamma)$ of which equals t, up to an additive error that is uniformly bounded. Moreover, it was shown that δ consists of one or two arcs traversing A, together with one or two arcs in $S \setminus A$, the lengths of which are uniformly bounded above with respect to σ. Applying this to our situation where $t = \mathrm{tw}_{\tilde\sigma}(\beta, \tilde\gamma)$, we obtain a simple closed curve ζ, the twist of which satisfies $\mathrm{tw}_\sigma(\zeta, \gamma) \overset{+}{\asymp} \mathrm{tw}_{\tilde\sigma}(\beta, \tilde\gamma)$. Thus, combined with equations (18) and (19), its length satisfies

$$\begin{aligned}
\ell_\sigma(\zeta) &= \ell_\sigma(\zeta \cap S \setminus A) + \ell_\sigma(\zeta \cap A) \overset{*}{\asymp} \ell_\sigma(\zeta \cap A) \\
&\overset{*}{\asymp} \log \frac{1}{\ell_\sigma(\gamma)} + |\mathrm{tw}_\sigma(\zeta, \gamma)| \ell_\sigma(\gamma) \\
&\overset{*}{\asymp} \log \frac{1}{\ell_\sigma(\gamma)} + |\mathrm{tw}_{\tilde\sigma}(\beta, \tilde\gamma)| \ell_\sigma(\gamma) \overset{*}{\asymp} \ell_\sigma(\beta).
\end{aligned} \tag{28}$$

On the other hand, with respect to τ, we obtain

$$\ell_\tau(\zeta) > \ell_\tau(\zeta \cap A) \overset{*}{\asymp} \log \frac{1}{\ell_\tau(\gamma)} + |\mathrm{tw}_\tau(\zeta, \gamma)| \ell_\tau(\gamma).$$

Now, for any two arcs β_1 and β_2 in $\mathcal{C}(A, \partial A)$, it is not hard to see that the difference $\mathrm{tw}_{\tilde\rho}(\beta_1, \tilde\gamma) - \mathrm{tw}_{\tilde\rho}(\beta_2, \tilde\gamma)$ is, up to an additive error that is uniformly bounded, a topological quantity that is independent of $\tilde\rho = \tilde\sigma, \tilde\tau$ (namely, the algebraic intersection number of β_1, β_2; see proofs of [**7**, Lemmas 3.2 and 3.5]). Observe also that if $\tilde\zeta$ is a lift of ζ that intersects $\tilde\gamma$, then $\mathrm{tw}_{\tilde\rho}(\tilde\zeta, \tilde\gamma) \overset{+}{\asymp} \mathrm{tw}_\rho(\zeta, \gamma)$. It follows that

$$\mathrm{tw}_\tau(\zeta, \gamma) - \mathrm{tw}_{\tilde\tau}(\beta, \tilde\gamma) \overset{+}{\asymp} \mathrm{tw}_\sigma(\zeta, \gamma) - \mathrm{tw}_{\tilde\sigma}(\beta, \tilde\gamma) \overset{+}{\asymp} 0.$$

Therefore, we obtain

$$\begin{aligned}
\ell_\tau(\zeta) &\overset{*}{\succ} \log \frac{1}{\ell_\tau(\gamma)} + |\mathrm{tw}_\tau(\zeta, \gamma)| \ell_\tau(\gamma) \\
&\overset{*}{\asymp} \log \frac{1}{\ell_\tau(\gamma)} + |\mathrm{tw}_{\tilde\tau}(\beta, \tilde\gamma)| \ell_\tau(\gamma) \overset{*}{\asymp} \ell_\tau(\beta).
\end{aligned} \tag{29}$$

Inequality (27) now follows from equations (28) and (29), thus completing the proof. □

We conclude this section with the proof of Theorem 3.5.

Proof of Theorem 3.5. By Proposition 3.7, we have

$$d_{\mathrm{L}}(\sigma, \tau) \overset{+}{\asymp} \log \max_{Q, A \in \mathcal{P}} \left\{ \sup_{\alpha \in \mathcal{C}(Q)} \frac{\ell_\tau(\alpha)}{\ell_\sigma(\alpha)}, \sup_{\alpha \in \mathcal{C}(Q)} \frac{\ell_\sigma(\alpha)}{\ell_\tau(\alpha)}, \sup_{\beta \in \mathcal{C}(A, \partial A)} \frac{\ell_\tau(\beta)}{\ell_\sigma(\beta)}, \sup_{\beta \in \mathcal{C}(A, \partial A)} \frac{\ell_\sigma(\beta)}{\ell_\tau(\beta)} \right\}.$$

Therefore, to complete the proof, it would be sufficient to show that

$$\sup_{\alpha \in \mathcal{C}(Q)} \frac{\ell_\tau(\alpha)}{\ell_\sigma(\alpha)} \stackrel{*}{\asymp} \sup_{\alpha \in \mathcal{C}(Q)} \frac{\ell_{\Pi_{S\setminus\Gamma}(\tau)}(\alpha)}{\ell_{\Pi_{S\setminus\Gamma}(\sigma)}(\alpha)}$$

and that

$$\sup_{\beta \in \mathcal{C}(A,\partial A)} \frac{\ell_\tau(\beta)}{\ell_\sigma(\beta)} \stackrel{*}{\asymp} \sup_{\beta \in \mathcal{C}(A,\partial A)} \frac{\ell_{\Pi_\gamma(\tau)}(\beta)}{\ell_{\Pi_\gamma(\sigma)}(\beta)}.$$

Regarding the first estimate, it has already been shown in [7] that for $\rho \in \mathrm{Thin}_\epsilon(S,\Gamma)$, the space (Q,ρ) embeds K-quasiconformally (in fact, bi-Lipschitz), with uniform K, in $(Q, \pi_{S\setminus\Gamma}(\rho))$. Thus, the lengths of curves in the two spaces are comparable and the first estimate follows.

Now consider the second estimate. To simplify the notation, let $\ell_1 = \ell_{\Pi_\gamma(\sigma)}(\tilde\gamma), \ell_2 = \ell_{\Pi_\gamma(\tau)}(\tilde\gamma)$ and let $t_1 = \mathrm{tw}_{\Pi_\gamma(\sigma)}(\tilde S), t_2 = \mathrm{tw}_{\Pi_\gamma(\tau)}(\tilde S)$. Then by equation (21) we obtain

$$\sup_{\beta \in \mathcal{C}(A,\partial A)} \frac{\ell_{\Pi_\gamma(\tau)}(\beta)}{\ell_{\Pi_\gamma(\sigma)}(\beta)} \stackrel{*}{\asymp} \max\left\{\frac{\ell_2}{\ell_1}, \frac{\log[1/\ell_2] + |t_2 - t_1|\,\ell_2}{\log[1/\ell_1]}\right\}.$$

On the other hand, for any arc $\beta \in \mathcal{C}(A,\partial A)$, analogously to Lemma 3.3, we have

$$|[\mathrm{tw}_{\tilde\tau}(\beta,\tilde\gamma) - \mathrm{tw}_{\tilde\sigma}(\beta,\tilde\gamma)] - [s_\tau(\gamma) - s_\sigma(\gamma)]| \stackrel{+}{\asymp} 0,$$

where $s_\sigma(\gamma)$ and $s_\tau(\gamma)$ are, respectively, the Fenchel–Nielsen twist coordinates of σ and τ associated to γ (see [7, Lemma 3.5]). Thus, by the same reasoning used to derive equation (21) we obtain

$$\sup_{\beta \in \mathcal{C}(A,\partial A)} \frac{\ell_\tau(\beta)}{\ell_\sigma(\beta)} \stackrel{*}{\asymp} \max\left\{\frac{\ell_2}{\ell_1}, \frac{\log[1/\ell_2] + |s_\tau(\gamma) - s_\sigma(\gamma)|\,\ell_2}{\log[1/\ell_1]}\right\}.$$

By the definition of Π_γ we have $t_1 = s_\sigma(\gamma)$ and $t_2 = s_\tau(\gamma)$ and thus the second estimate is proved. □

4. *Comparison on a thin region*

We now prove Theorem A of Section 1, which illustrates the discrepancy between the Lipschitz and Teichmüller distances stated in Section 1.

Proof of Theorem A. Let σ_n be a hyperbolic metric on S such that there is exactly one short curve γ of length $\ell_{\sigma_n}(\gamma) = \epsilon_n$ and let $\tau_n = D_\gamma^{T_n}(\sigma_n)$ be the metric obtained from σ_n by T_n Dehn twists around γ. In this case, $\ell_{\sigma_n}(\gamma) = \ell_{\tau_n}(\gamma) = \epsilon_n$. Set $\epsilon_n = e^{-P_n}$, $T_n = e^{P_n + q_n}$ and choose the sequences of positive integers P_n, q_n so that

$$P_n \to \infty, \quad q_n \to \infty, \quad \text{and} \quad \frac{e^{q_n}}{P_n} \to 0 \quad \text{as} \quad n \to \infty.$$

On the one hand, it follows from Theorem 1.1 and the discussion following Lemma 3.4 that

$$d_{\mathcal{T}}(\sigma_n, \tau_n) \stackrel{+}{\asymp} \log[T_n \epsilon_n] = q_n \to \infty.$$

It follows from Proposition 3.1 that for a simple closed curve ζ in S, we have

$$\frac{\ell_{\tau_n}(\zeta)}{\ell_{\sigma_n}(\zeta)} = \frac{\ell_{\tau_n}(\zeta_Q) + \big[2\log[\epsilon_0/\epsilon_n] + \epsilon_n \cdot |\mathrm{tw}_{\tau_n}(\zeta,\gamma)| + \mathrm{O}(1)\big] \cdot i(\zeta,\gamma)}{\ell_{\sigma_n}(\zeta_Q) + \big[2\log[\epsilon_0/\epsilon_n] + \epsilon_n \cdot |\mathrm{tw}_{\sigma_n}(\zeta,\gamma)| + \mathrm{O}(1)\big] \cdot i(\zeta,\gamma)},$$

since in this situation, $i(\zeta,\partial Q) = 2i(\zeta,\gamma)$, where $\mathrm{O}(1)$ represents an error that is independent of ζ, σ_n, τ_n and that is bounded in absolute value by some uniform constant. Since σ_n, τ_n

coincide outside A, we have $\ell_{\tau_n}(\zeta_Q) = \ell_{\sigma_n}(\zeta_Q)$. Therefore

$$\sup_\zeta \frac{\ell_{\tau_n}(\zeta)}{\ell_{\sigma_n}(\zeta)} \leqslant \max\left\{1,\ \sup_\zeta \frac{2\log[\epsilon_0/\epsilon_n] + \epsilon_n \cdot |\mathrm{tw}_{\tau_n}(\zeta,\gamma)| + \mathrm{O}(1)}{2\log[\epsilon_0/\epsilon_n] + \epsilon_n \cdot |\mathrm{tw}_{\sigma_n}(\zeta,\gamma)| + \mathrm{O}(1)}\right\}$$

and by the same reasoning used to deduce equation (21), the supremum on the right-hand side is equal to

$$\frac{2\log[1/\epsilon_n] + \epsilon_n \cdot T_n + \mathrm{O}(1)}{2\log[1/\epsilon_n] + \mathrm{O}(1)} = \frac{2P_n + e^{q_n} + \mathrm{O}(1)}{2P_n + \mathrm{O}(1)}.$$

Thus, we have

$$\lim_{n\to\infty} d_{\mathrm{L}}(\sigma_n, \tau_n) = \lim_{n\to\infty} \log \frac{2P_n + e^{q_n} + \mathrm{O}(1)}{2P_n + \mathrm{O}(1)} = 0. \qquad \square$$

So far, we have seen that if $\sigma, \tau \in \mathcal{T}(S)$ are both in the thick part then $d_{\mathrm{L}}(\sigma,\tau) \asymp d_{\mathcal{T}}(\sigma,\tau)$, but that if σ, τ have a short curve in common, then the two distances are no longer comparable. The following proposition shows that, in some sense, this is the only way for the distances to diverge.

Proposition 4.1. *If $\sigma, \tau \in \mathcal{T}(S)$ have no short curves in common, then $d_{\mathrm{L}}(\sigma,\tau) \asymp d_{\mathcal{T}}(\sigma,\tau)$.*

Proof. Let Γ_σ be the set of curves the length of which is less than ϵ_1 at σ, and let $\bar\sigma$ be the point in the thick part of $\mathcal{T}(S)$ obtained from σ by increasing the length of each curve in Γ_σ to ϵ_1, but otherwise leaving the metric unchanged. More precisely, this can be achieved by choosing a pants system of S that contains Γ_σ and altering the associated Fenchel–Nielsen length coordinates as desired. We define $\bar\tau$ analogously by increasing the length of every short curve of τ to ϵ_1. It follows from Theorem 3.5 and Lemma 3.4 that

$$d_{\mathrm{L}}(\sigma,\bar\sigma) \asymp \log \max_{\alpha\in\Gamma_\sigma} \left\{\frac{\ell_{\bar\sigma}(\alpha)}{\ell_\sigma(\alpha)}\right\}, \quad d_{\mathrm{L}}(\tau,\bar\tau) \asymp \log \max_{\alpha\in\Gamma_\tau} \left\{\frac{\ell_{\bar\tau}(\alpha)}{\ell_\tau(\alpha)}\right\}.$$

Since curves that are short in σ are not short in τ and vice versa, the above equation implies that

$$d_{\mathrm{L}}(\sigma,\bar\sigma) \prec d_{\mathrm{L}}(\sigma,\tau) \quad \text{and} \quad d_{\mathrm{L}}(\tau,\bar\tau) \prec d_{\mathrm{L}}(\sigma,\tau). \tag{30}$$

By the triangle inequality, we also have

$$\begin{aligned} d_{\mathrm{L}}(\sigma,\tau) &\geqslant d_{\mathrm{L}}(\bar\sigma,\bar\tau) - d_{\mathrm{L}}(\sigma,\bar\sigma) - d_{\mathrm{L}}(\bar\tau,\tau), \\ d_{\mathrm{L}}(\sigma,\tau) &\leqslant d_{\mathrm{L}}(\bar\sigma,\bar\tau) + d_{\mathrm{L}}(\sigma,\bar\sigma) + d_{\mathrm{L}}(\bar\tau,\tau). \end{aligned} \tag{31}$$

Combining equations (30) and (31), we obtain

$$d_{\mathrm{L}}(\sigma,\tau) \asymp d_{\mathrm{L}}(\bar\sigma,\sigma) + d_{\mathrm{L}}(\bar\sigma,\bar\tau) + d_{\mathrm{L}}(\tau,\bar\tau). \tag{32}$$

Analogously, it follows from Theorem 1.1, the discussion following Lemma 3.4, and equation (3) that

$$d_{\mathcal{T}}(\sigma,\bar\sigma) \prec d_{\mathcal{T}}(\sigma,\tau) \quad \text{and} \quad d_{\mathcal{T}}(\tau,\bar\tau) \prec d_{\mathcal{T}}(\sigma,\tau)$$

and combining these with the triangle inequality again, we obtain

$$d_{\mathcal{T}}(\sigma,\tau) \asymp d_{\mathcal{T}}(\bar\sigma,\sigma) + d_{\mathcal{T}}(\bar\sigma,\bar\tau) + d_{\mathcal{T}}(\tau,\bar\tau). \tag{33}$$

Now, by Theorems 1.1 and 3.5 and Lemma 3.4, we have

$$d_{\mathrm{L}}(\sigma,\bar\sigma) \asymp d_{\mathcal{T}}(\sigma,\bar\sigma) \quad \text{and} \quad d_{\mathrm{L}}(\tau,\bar\tau) \asymp d_{\mathcal{T}}(\tau,\bar\tau)$$

and by Theorem B we have

$$d_{\mathrm{L}}(\bar{\sigma}, \bar{\tau}) \asymp d_{\mathcal{T}}(\bar{\sigma}, \bar{\tau}).$$

Thus, it follows from equations (32) and (33) that $d_{\mathrm{L}}(\sigma, \tau) \asymp d_{\mathcal{T}}(\sigma, \tau)$, as claimed. □

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References

1. A. Belkhirat, A. Papadopoulos and M. Troyanov, 'Thurston's weak metric on the Teichmüller space of the torus', *Trans. Amer. Math. Soc.* 357 (2005) 3311–3324.
2. Y. Choi, K. Rafi and C. Series, 'Lines of minima and Teichmüller geodesics' Preprint, 2006, arXiv:math.GT/0605135.
3. S. Kerckhoff, 'The asymptotic geometry of Teichmüller space', *Topology* 19 (1980) 23–41.
4. Z. Li, 'Length spectrums of Riemann surfaces and the Teichmüller metric', *Bull. London Math. Soc.* 35 (2003), 247–254.
5. H. Masur and Y. Minsky, 'Geometry of the complex of curves. II: hierarchical structure', *Geom. Funct. Anal.* 10 (2000) 902–974.
6. Y. Minsky, 'Teichmüller geodesics and ends of hyperbolic 3-manifolds', *Topology* 32 (1993) 625–647.
7. Y. Minsky, 'Extremal length estimates and product regions in Teichmüller space', *Duke Math. J.* 83 (1996) 249–286.
8. K. Rafi, 'A characterization of short curves of a Teichmüller geodesic', *Geom. Topol.* 9 (2005) 179–202.
9. K. Rafi, 'A combinatorial model for the Teichmüller metric', *Geom. Funct. Anal.* 17 (2007) 936–959.
10. K. Rafi, 'A thick–thin decomposition for quadratic differentials', *Math. Res. Lett.* 14 (2007) 333–341.
11. T. Sorvali, 'On Teichmüller spaces of Tori', *Ann. Acad. Sci. Fenn. Ser. A I Math.* 1 (1975), 7–11.
12. W. Thurston, 'Minimal stretch maps between hyperbolic surfaces', Preprint, 1998, arXiv:math.GT/9801039.
13. S. Wolpert, 'The length spectra as moduli for compact Riemann surfaces', *Ann. of Math.* 109 (1979) 323–351.
14. A. Yamada, 'On Marden's universal constant of Fuchsian groups', *Kodai Math. J.* 4 (1981) 266–277.

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