THURSTON GEODESICS: NO BACKTRACKING AND ACTIVE INTERVALS

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Abstract. We develop the notion of the *active interval* for a subsurface along a geodesic in the Thurston metric on Teichmüller space of a surface S. That is, for any geodesic in the Thurston metric and any subsurface R of S, we find an interval of times where the length of the boundary of R is uniformly bounded and the restriction of the geodesic to the subsurface R resembles a geodesic in the Teichmüller space of R. In particular, the set of short curves in R during the active interval represents a reparametrized quasi-geodesic in the curve graph of R (no backtracking) and the amount of movement in the curve graph of R outside of the active interval is uniformly bounded which justifies the name *active interval*. These intervals provide an analogue of the active intervals introduced by the third author in the setting of Teichmüller space equipped with the Teichmüller metric.

1. Introduction

This is the third paper in the series [LRT12, LRT15] where the Thurston metric on Teichmüller space and the behavior of its geodesics is studied. Let S be a surface of finite hyperbolic type and let $\mathcal{T}(S)$ be the Teichmüller space of S equipped with the Thurston metric d_{Th} (see Section 2.8 for definition). It is well known that there may not be a unique Thurston geodesic connecting two points $X, Y \in \mathcal{T}(S)$ (see [Thu98] or [LRT12, Theorem 1.1]). In general, for a subsurface $R \subseteq S$, the projection of a geodesic segment $[X, Y]$ to the Teichmüller space of R can be essentially any path in $\mathcal{T}(R)$. Heuristically, this can be understood as follows: let X, Y be points in $\mathcal{T}(S)$ and let R, R' be disjoint subsurfaces such that the length of ∂R is short in both X and Y, the restriction of the hyperbolic structures of X and Y to R is the same but the restriction of X and Y to R' is very different. Then any geodesic segment from X to Y has to move efficiently in R' but it has the freedom to modify the hyperbolic structure inside R in different ways before returning back to the original structure (see [LRT15, Section 6] for a detailed examples of such phenomena). In other words, a copy of $\mathcal{T}(R) \times \mathcal{T}(R')$ equipped with the L^∞–metric embeds nearly isometrically in $\mathcal{T}(S)$ (see [CR07, Theorem 3.5]) which causes non-uniqueness of geodesics similar to that of $\mathbf{R}^2$ equipped with the L^∞–metric.

In this paper we will show that, in a coarse sense, this is essentially the only phenomenon responsible for the non uniqueness of geodesics. We now make this precise.

For a point $X \in \mathcal{T}(S)$, let μ_X be the *short marking* of X (see Section 2.6 for the definition). For a subsurface $R \subseteq S$, let $\mathcal{C}(R)$ be the curve graph of the subsurface R. Now define

$$\Upsilon_R : \mathcal{T}(S) \to \mathcal{C}(R)$$

to be the map which assigns to a point X the projection of μ_X on X to $\mathcal{C}(R)$ (see Definition 2.2). We call Υ_R the shadow of X to the curve complex of R.

Let $\mathbf{g}\colon I \to \mathcal{T}(S)$ be a geodesic segment from X to Y and let $\lambda_{\mathbf{g}}$ be the maximally stretched lamination from X to Y introduced by Thurston [Thu98] (see §2 for definition and discussion). We often denote $\mathbf{g}(t)$ simply by X_t. The following theorem summarizes the main results of this paper which are analogues of some of the results of Rafi in [Raf05, Raf07, Raf14] about the behavior of Teichmüller geodesics. Also some similar results about the behavior of Weil-Petersson geodesics in [BMM10, BMM11, Mod15, Mod16, MM21].

Theorem 1.1. *Let $\mathbf{g}\colon [a,b] \to \mathcal{T}(S)$ be a geodesic in the Thurston metric with maximally stretched lamination $\lambda_{\mathbf{g}}$. There are positive constants ρ and K such that, for any non-annular subsurface $R \subseteq S$ that intersects $\lambda_{\mathbf{g}}$, there is a possibly empty interval of times $J_R = [c,d] \subset [a,b]$, with the following properties:*

(1) For any $t \in J_R$, we have $\ell_t(\partial R) \le \rho$.

(2) If $s,t \in I$ are on the same side of J_R ($s,t < c$ or $s,t > d$), then

$$d_R(X_r, X_t) \le K.$$

If J_R is empty, this holds for every $s,t \in I$.

(3) The shadow of $\mathbf{g}$ to R, $\Upsilon_R \circ \mathbf{g}\big|_{J_R} : J_R \to \mathcal{C}(R)$, is a reparametrized quasi-geodesic in $\mathcal{C}(R)$.

The key assumption in the above theorem is that R intersects $\lambda_{\mathbf{g}}$. This restricts the freedom of changing the structure of subsurface R arbitrarily. In particular, if $\lambda_{\mathbf{g}}$ is a filling lamination (meaning it intersects every essential simple closes curve on S), the shadow of a Thurston geodesic to every surface is a reparametrized quasi-geodesic. At the end of the paper, we will show that the assumption that $\lambda_{\mathbf{g}}$ is filling can be replaces a assumption that $\lambda_{\mathbf{g}}$ is nearly filling (see Definition 5.12) which should be interpreted to mean, from the point of view of the hyperbolic metric X, $\lambda_{\mathbf{g}}$ looks like a filling lamination.

Theorem 1.2. *Let* $\mathbf{g}\colon [a,b] \to \mathcal{T}(S)$ *be a Thurston geodesic. If* $\lambda_{\mathbf{g}}$ *is nearly filling on* X_a*, then for any subsurface* $R \subseteq S$*, the set* $\Upsilon_R(\mathbf{g}([a,b]))$ *is a reparametrized quasi-geodesic in* $\mathcal{C}(R)$*.*

The main technical tool of this paper is to further develop the notion of a horizontal curve along a Thurston geodesic. This notion was first introduced in [LRT15], in analogy with the existing picture for the behavior of a curve along a Teichmüller geodesic, as developed in [MM99, Raf07, Raf14]. In essence, a curve α is horizontal if it spends a significant portion of its length fellow traveling $\lambda_{\mathbf{g}}$. It was shown in [LRT15] that horizontal curves stay horizontal (in fact get more horizontal) along a Thurston geodesic and the length of the portion of α that fellow travels $\lambda_{\mathbf{g}}$ grows fast. We generalize this to a curve inside a subsurface (Theorem 4.4). We then establish a trichotomy (Proposition 5.1) stating that, at every time t and for every subsurface R either ∂R is horizontal in X_t, ∂R is short at X_t or ∂R is vertical meaning that the projection of X_t and $\lambda_{\mathbf{g}}$ to $\mathcal{C}(R)$ are close. This implies that the only time the shadow of $\mathbf{g}$ to R can change is when ∂R is short.

Notations. Here we list some notation and conventions that are used in this paper. We will call a constant C *universal* if it depends only on the topological type of a surface. Given two quantities a and b, we will write

$$a \stackrel{*}{\prec} b$$

if there is a universal constant C such that $a \leq Cb$.

Similarly, we write

$$a \stackrel{+}{\prec} b$$

if $a \leq b + C$ for a universal constant C.

We write $a \stackrel{*}{\asymp} b$ if $a \stackrel{*}{\prec} b$ and $b \stackrel{*}{\prec} a$. Likewise, $a \stackrel{+}{\asymp} b$ if $a \stackrel{+}{\prec} b$ and $b \stackrel{+}{\prec} a$.

We define the logarithmic cut-off function $\operatorname{Log} : \mathbb{R}^+ \to [1, \infty)$ as

$$\operatorname{Log} x := \max\Big\{\ln x, 1\Big\}, \tag{1.1}$$

where $\ln x$ is the natural logarithm of x.

For the convenient of the reader, we also list the named constants that are used throughout the paper.

- ϵ_B = Bers constant
- ϵ_w = weakly-horizontal threshold constant
- ϵ_h = horizontal threshold constant
- ρ = boundary of R threshold constant
- s_0 = Theorem 4.4

- n_0 = horizontal intersection constant
- L_0 = horizontal fellow traveling constant
- w_ϵ = .5 width of the standard collar about a curve of ϵ length.
- b_ϵ = boundary length of the standard collar about a curve of ϵ length
- L_ϵ = Lemma 3.4
- A = Lemma 2.6
- B = Lemma 5.4
- $C(n, L)$ = Lemma 4.14
- M = Lemma 3.3

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2. Background

2.1. **Metric spaces and coarse maps.** Let $C > 0$, and let $\mathcal{X}$ and $\mathcal{Y}$ be two metric spaces. A multivalued map $\phi : \mathcal{X} \to \mathcal{Y}$ is called a C*–coarse map* if for every $x \in \mathcal{X}$,

$$\operatorname{diam}_{\mathcal{Y}} \big(\phi(x)\big) \leq C.$$

Moreover, for $K \geq 1$, we will say it is K*–Lipschitz* if in addition, for all $x, x' \in \mathcal{X}$,

$$\operatorname{diam}_{\mathcal{Y}} \big(\phi(x) \cup \phi(x')\big) \leq K d_{\mathcal{X}}(x, x').$$

Now, let $\mathcal{Y}$ be a subspace of $\mathcal{X}$ equipped with the induced metric. We say a K–coarse map $\Pi\colon \mathcal{X} \to \mathcal{Y}$ is a K*–retraction* if, for all $y \in \mathcal{Y}$,

$$\operatorname{diam}_{\mathcal{X}} \big(\Pi(y) \cup \{y\}\big) \leq K.$$

By a *path* in $\mathcal{X}$ we will mean a coarse map $\phi\colon [a, b] \to \mathcal{X}$. A path $\phi\colon [a, b] \to \mathcal{X}$ is a K–quasi-geodesic if for all $a \leq s \leq t \leq b$,

$$\frac{1}{K}(t - s) - K \leq \operatorname{diam}_X \big(\phi(s) \cup \phi(t)\big) \leq K(t - s) + K.$$

We will say ϕ is a *reparametrized* K–quasi-geodesic if there is an increasing function $h\colon [0, n] \to [a, b]$ such that $\phi \circ h$ is a K–quasi-geodesic. When $\mathcal{X}$ is δ–hyperbolic, then a path ϕ is a reparametrized K–quasi-geodesic for some K if an only if there exists K' such that for all $r, s, t \in [a, b]$, where $r \leq s \leq t$, we have

$$\operatorname{diam}_{\mathcal{X}}(\phi(r) \cup \phi(s)) + \operatorname{diam}_{\mathcal{X}}(\phi(s) \cup \phi(t)) \leq K' \operatorname{diam}_{\mathcal{X}}(\phi(r) \cup \phi(t)) + K'.$$

The following theorem is a standard fact in the coarse geometry; see e.g. the proof of [LRT15, Theorem 5.7].

Theorem 2.1. *Let $K \geq 1$ and suppose that ϕ is a path in $\mathcal{X}$ and that there exists a (coarse) K–Lipschitz retraction map $\Pi \colon \mathcal{X} \to Im(\phi)$. Then ϕ is a reparametrized K'–quasi-geodesic for a $K' \geq 1$ that depends only on K.*

In this section we will discuss arcs, curves and subsurfaces on a compact, connected, oriented surface S of genus g with possibly b punctures or boundary components. The *complexity* of S is defined by $\xi(S) := 3g - 3 + b$.

A *curve* on S is a closed connected 1–manifold on S defined up to free homotopy. We always assume that a curve is essential i.e. non-trivial and non-peripheral. We say two curves are disjoint if the curves have disjoint representatives. A *multicurve* is a union of pairwise disjoint curves. An *arc* on S is a properly embedded simple 1-manifold with boundary where the endpoints lie on ∂S defined up to homotopy relative to ∂S. We always assume that an arc is not boundary parallel. A *subsurface* $R \subseteq S$ is a compact, connected 2–dimensional submanifold of S with boundary such that each boundary component of R is either an essential curve or a boundary component of S defined up to homotopy relative to ∂S. In particular, the map $\pi_1(R) \to \pi_1(S)$ induced by inclusion is injective.

Two curves or arcs α, β *intersect* if the curves are not disjoint. We denote by $\mathrm{i}(\alpha, \beta)$ the minimal number of intersections between representatives of α and β. A curve α *essentially intersects* a subsurface R if α does not have a representative that is disjoint from R; equivalently, α is either a curve in R or it intersects a component of ∂R. Finally, two subsurfaces intersect if they do not have disjoint representatives.

2.2. **Pants decompositions and markings.** A *pants decomposition* on S is a multicurve whose complementary regions are all 3-holed spheres (or equivalently a maximal set of pairwise disjoint curves on S); the number of curves in a pants decomposition is equal to $\xi(S)$. A *marking* μ on S is a pants decomposition $P = \{\alpha_i\}$ together with a set $\overline{P} = \{\overline{\alpha}_i\}$ of curves such that, for each i, $\overline{\alpha}_i$ intersects α_i minimally, and $\overline{\alpha}_i$ does not intersect α_j for all $j \neq i$; $\overline{\alpha}_i$ is called the *transverse curve* to α. We will usually think of a marking μ as the set of the curves in $P \cup \overline{P}$.

2.3. **Geodesic laminations.** For the rest of this paper, we will assume that S is a surface with $\xi(S) \geq 1$, and by a hyperbolic metric on S we will mean a complete finite-area hyperbolic metric on the interior of S.

Now fix a hyperbolic metric on S. A *geodesic lamination* λ on S is a closed subset of S that is a disjoint union of simple complete geodesics. These geodesics are called the *leaves* of λ. The space of geodesic laminations, equipped with Hausdorff topology, is compact. Moreover, any two different hyperbolic metrics on S determine homeomorphic spaces of geodesic laminations, so the space of geodesic laminations $\mathcal{GL}(S)$ depends only on the topology of S. (see [PH92], [Thu98] [Bon01] for details).

2.4. **Curve graphs and subsurface projection.** Let $R \subseteq S$ be a subsurface. The *arc and curve graph* $\mathcal{C}(R)$ of R is defined as follows: If R is non-annular, then vertices of $\mathcal{C}(R)$ are the arcs and curves on R and two vertices are connected by an edge if they do not intersect. When R is an annulus, the definition of $\mathcal{C}(R)$ is a bit more involved. In this situation, the vertices of the complex are the homotopy classes of arcs that connect the boundaries of the natural compactification of the annular cover of S corresponding to R, and edges correspond to arcs with disjoint interiors.

Assigning length one to each edge turns $\mathcal{C}(R)$ into a metric graph which is Gromov hyperbolic [MM99]. We represent this metric by $d_R(\cdot, \cdot)$.

In the following we define the *subsurface projection map* and *subsurface projection coefficients* that play a crucial role in description of our results in the paper.

Definition 2.2. The subsurface projection map

$$\pi_R : \mathcal{GL}(S) \to \mathcal{P}(\mathcal{C}(R)) \cup \{\infty\}$$

is defined as follows. Fix a hyperbolic metric on S so that R is represented by a unique convex subsurface with geodesic boundary and any geodesic lamination λ is uniquely represented by geodesics in the metric so that all intersections between these representatives and the subsurface R are essential. Then

- If λ does not intersect R, define $\pi_R(\lambda) = \emptyset$.
- If $\lambda \cap R$ contains a curve or an arc, then define $\pi_R(\lambda)$ to be the set of all curves and arcs in $\lambda \cap R$ (as before up to homotopy).
- If $\lambda \cap R$ has no compact segment, and there is an element $\alpha \in \mathcal{C}(R)$ disjoint from $\lambda \cap R$, then define $\pi_R(\lambda)$ to be the set of all curves and arcs in $\mathcal{C}(R)$ disjoint from λ.
- Finally, if $\lambda \cap R$ has no compact segment and every element of $\mathcal{C}(R)$ intersects λ, then define $\pi_R(\lambda) = \infty$.

It is easy to see that $\text{diam}_R(\pi_R(\lambda)) \leq 2$ when $\pi_R(\lambda) \notin \{\emptyset, \infty\}$, so π_R is coarsely well-defined. Note that the above definition is independent of the choice of the hyperbolic metric. Note that a curve or lamination λ intersects a subsurface R essentially if $\pi_R(\lambda) \neq \emptyset$.

The notion of subsurface projection generalizes to markings on a surface as follows. When R is a non-annular subsurface, $\pi_R(\mu)$ is the projection of any curve in the base of μ that intersects R. When R is an annulus with core curve γ that in not contained in the base of μ, $\pi_R(\mu)$ is the projection of any curve in the base that intersects γ. But, when γ is contained in the base of the marking, we define $\pi_R(\mu)$ to be the projection of the transversal curve $\bar{\gamma} \in \mu$ to γ.

Let p, q be any pair of geodesic laminations or markings that intersect R essentially. If $\pi_R(p), \pi_R(q) \subset \mathcal{C}(R)$, we define the subsurface coefficient by:

$$d_R(p,q) := \min\Big\{ d_R(\alpha,\alpha') : \alpha \in \pi_R(p) \text{ and } \alpha' \in \pi_R(q) \Big\}. \tag{2.1}$$

If either p or q are mapped to ∞ by π_R, we define

$$d_R(p,q) = +\infty.$$

The subsurface projection coefficients satisfy the triangle inequality with an additive error, that is for any three geodesic laminations or markings p, q, r that intersect R we have

$$d_R(p,q) \overset{+}{\prec} d_R(p,r) + d_R(r,q) \tag{2.2}$$

where the additive error can be taken to be 2.

We also define the *intersection number* of any two laminations or curves λ and λ' that intersect R to be

$$\mathrm{i}_R(\lambda,\lambda') = \min\Big\{ \mathrm{i}_R(\alpha,\alpha') : \alpha \in \pi_R(\lambda),\ \alpha' \in \pi_R(\lambda') \Big\} \tag{2.3}$$

in the case when both $\pi_R(\lambda)$ and $\pi_R(\lambda')$ are contained in $\mathcal{C}(R)$, and $+\infty$ otherwise. Using a surgery argument [MM99, §2] we may see that

$$d_R(\alpha,\beta) \overset{*}{\prec} \operatorname{Log} \mathrm{i}_R(\alpha,\beta). \tag{2.4}$$

where Log is the cut-off logarithmic function defined in (1.1).

2.5. **The standard collar.** Let X be a hyperbolic metric on S and α a curve on X. Then α has a unique geodesic representative and we will denote its length by $\ell_X(\alpha)$.

For a multicurve σ let

$$\ell_X(\sigma) = \max_{\alpha\in\sigma} \ell_X(\alpha). \tag{2.5}$$

A multicurve σ is ϵ*–short* on X if $\ell_X(\sigma) \le \epsilon$. A subsurface R in X is ϵ*–thick* if $\ell_X(\partial R) \le \epsilon$ and every curve in R has length at least ϵ.

Lemma 2.3 (Collar Lemma[Bus10, §4.1])**.** *For any hyperbolic metric X on S, if α is a geodesic curve with $\ell_X(\alpha) = \epsilon$, then the regular neighborhood $U_X(\gamma) \subseteq X$ of α with width $2w_\epsilon$ where*

$$w_\epsilon = \sinh^{-1}\left(\frac{1}{\sinh(\epsilon/2)}\right), \tag{2.6}$$

is an embedded annulus, and each boundary of $U_X(\alpha)$ has length

$$b_\epsilon = \epsilon \coth(\epsilon/2). \tag{2.7}$$

If β is a geodesic curve disjoint from α, then $U_X(\beta)$ and $U_X(\alpha)$ are disjoint.

Definition 2.4. We will call $U_X(\gamma)$ the *standard collar* about γ on X. A boundary curve of a regular neighborhood of a closed geodesic is called a *hypercycle*.

Remark 2.5. It is easy to see that when $\epsilon < 1$, w_ϵ is $\log \frac{1}{\epsilon}$ up to an additive error of at most $\log 3$. Also, b_ϵ is an increasing function with $\lim_{\epsilon\to 0} b_\epsilon = 2$.

2.6. **Short marking.** Recall that the *Bers constant* is a constant ϵ_B that depends only on the complexity of S [Bus10, §5], such that any hyperbolic metric X on S admits a pants decomposition $\mathcal{P}$ where $\ell_X(\partial\mathcal{P}) \leq \epsilon_B$. By increasing ϵ_B if necessary, we may assume that $\epsilon_B \geq 1$ has the property that whenever $\ell_X(\alpha) \geq 1$, then the shortest transverse curve to α is ϵ_B–short.

A pants decomposition $\mathcal{P}$ with ϵ_B-short boundary will be called a *short pants decomposition*. A *short marking* μ_X on X is a marking consisting of a short pants decomposition as the base and that the transversal curve to each base curve is chosen to be the shortest possible on X.

For a subsurface $R \subseteq S$ we write $d_R(X, \cdot)$ for $d_R(\mu_X, \cdot)$.

2.7. **Twisting and length.** In the case when R is an annulus with core curve γ, we use the notation $d_\gamma(X, \cdot)$ instead of $d_R(\mu_X, \cdot)$. For a lamination λ, we will call $d_\gamma(X, \lambda)$ the *relative twisting* of λ about γ on X.

The following lemma provides us with some useful estimates for the length of an arc inside the collar neighborhood of a curve γ and the twisting of the arc about γ on X.

Lemma 2.6. *There is a universal constant $A > 0$ such that the following statements hold. Let X be a hyperbolic metric on S admitting a curve γ of $\ell_X(\gamma) = 1$. Then for any simple geodesic α intersecting γ, the length $L(\alpha)$ of the longest arc in $\alpha \cap U_X(\gamma)$ has*

$$d_\gamma(X, \alpha) - A \leq L(\alpha) \leq d_\gamma(X, \alpha) + A.$$

Proof. It follows from [LRT15, Lemma 3.1] that

$$\Big| L(\alpha) - \mathrm{i}(\tau_\gamma, \alpha_\gamma)\ell_X(\gamma) \Big| \overset{+}{\prec} w_1$$

where $\tau_\gamma \subset U_X(\gamma)$ is a geodesic arc crossing $U_X(\gamma)$ and perpendicular to γ, α_γ is a component of $\alpha \cap U_X(\gamma)$ and w_1 is half of the width of the standard collar of γ. Since $\mathrm{i}(\alpha_\gamma, \tau_\gamma) \overset{+}{\asymp} d_\gamma(\alpha, X)$, $\ell_X(\gamma) = 1$ and w_1 is a fixed number, the lemma follows. ∎

2.8. **Thurston metric.** In this subsection we recall the definition of a metric on Teichmüller space introduced by Thurston [Thu98].

Recall that the Teichmüller space of S, denoted by $\mathcal{T}(S)$, is the space of hyperbolic metrics on S modulo the action of homeomorphisms of S that are isotopic to the identity. The space $\mathcal{T}(S)$ is homeomorphic to an open ball of dimension $2\xi(S)$ and has been the subject of study in particular in conjunction with the mapping class groups of surfaces.

Given $X, Y \in \mathcal{T}(S)$, let $L(X,Y) = \inf_f L_f$, where $f\colon X \to Y$ is a Lipschitz homeomorphism from X to Y isotopic to the identity on S, and L_f is the Lipschitz constant of f. A map $f\colon X \to Y$ is called *optimal* if $L_f = L(X,Y)$. Thusrston [Thu98, Theorem 8.5] showed that such a map exists and defined *Thurston metric* on $\mathcal{T}(S)$ by

$$d_{\mathrm{Th}}(X,Y) := \log L(X,Y). \tag{2.8}$$

Thurston also introduced a geodesic lamination $\lambda(X,Y)$ called *maximal stretch lamination* where for any segment $\sigma \subseteq \lambda(X,Y)$ with finite length we have that $\mathrm{length}(f(\sigma)) = L \cdot \mathrm{length}(\sigma)$. In the following theorem we collect some of the important properties of Thurston metric.

Theorem 2.7 (Thurston metric)**.** [Thu98, Theorem 8.2, Theorem 8.5]

The Thurston metric $d(\cdot,\cdot)$ is an asymmetric complete geodesic metric on $\mathcal{T}(S)$. Moreover, for all $X, Y \in \mathcal{T}(S)$,

(1) $L(X,Y) = \sup_{\alpha \in \mathcal{C}(S)} \frac{\ell_Y(\alpha)}{\ell_X(\alpha)}$.

(2) *There exists an optimal map from X to Y and every optimal map preserves the leaves of $\lambda(X,Y)$, stretching the length of each leaf by a factor of $e^{d_{Th}(X,Y)}$.*

(3) *For any Thurston geodesic $\mathbf{g}\colon I \to \mathcal{T}(S)$, $I \subseteq \mathbf{R}$, there is a geodesic lamination $\lambda_{\mathbf{g}}$, maximal with respect to inclusion, such that for all $t > s$, $\lambda_{\mathbf{g}} \subseteq \lambda(\mathbf{g}(s), \mathbf{g}(t))$. Moreover, if $I = [a,b]$ is a finite interval, then $\lambda_{\mathbf{g}} = \lambda(\mathbf{g}(a), \mathbf{g}(b))$.*

2.9. **Shadow map.** Let $R \subseteq S$ be a subsurface and let $\Upsilon_R\colon \mathcal{T}(S) \to \mathcal{C}(R)$ be the coarse map defined by $\Upsilon_R(X) = \pi_R(\mu_X)$, where μ_X is a short marking on X. The final result of this section is that Υ_R is a Lipschitz map.

Lemma 2.8. *For any subsurface $R \subseteq S$, the coarse map Υ_R is K–Lipschitz for some $K \geq 1$ that depends only on S.*

Proof. Suppose $d_{\mathrm{Th}}(X,Y) = 1$ and α and β are ϵ_B–short curves on X and Y respectively both intersecting R. Since the length of a curve grows at most exponentially in the distance from X to Y, we have $\ell_Y(\alpha) \leq e\epsilon_B$. Therefore, $\mathrm{i}_R(\alpha,\beta) = O(1)$, which implies $d_R(X,Y) = O(1)$. □

3. Tools from hyperbolic geometry

In this section we gather some useful properties of the hyperbolic plane and hyperbolic surfaces. Some of the results are elementary and are known in spirit, but to our knowledge the statements do not directly follow from what is known in literature.

Lemma 3.1. *For any $0 < \epsilon_1 < \epsilon_2$, set $L = \log 2\left(\frac{\sinh \epsilon_2}{\sinh \epsilon_1}\right)$. Let λ and α be two geodesics in $\mathbf{H}$ and let p_0, p_1, p_2 be 3 points on λ appearing in that order. Assume that,*

$$d_{\mathbf{H}}(p_0,p_1) \geq L, \qquad d_{\mathbf{H}}(p_1,p_2) \geq L, \qquad d_{\mathbf{H}}(p_0,\alpha) \leq \epsilon_2, \qquad d_{\mathbf{H}}(p_2,\alpha) \leq \epsilon_2.$$

Then

$$d_{\mathbf{H}}(p_1,\alpha) \leq \epsilon_1.$$

Proof. Let a and b be points in $\mathbf{H}$ on the same side of λ and such that segments $[a,p_0]$ and $[b,p_2]$ are both perpendicular to $[a,b]$ and $d_{\mathbf{H}}(p_0,a) = d_{\mathbf{H}}(p_2,b) = \epsilon_2$. Let $c \in \lambda$ be the closest point to $[a,b]$, clearly it is also the midpoint of $[p_0,p_2]$. We have $d_{\mathbf{H}}(p_0,c) = \frac{1}{2} d_{\mathbf{H}}(p_0,p_2) \geq L$ and hence

$$\sinh d_{\mathbf{H}}(c,[a,b]) = \frac{\sinh d_{\mathbf{H}}(a,p_0)}{\cosh d_{\mathbf{H}}(p_0,c)} < \frac{\sinh \epsilon_2}{\frac{1}{2}e^L} = \sinh \epsilon_1.$$

This implies that some point $p' \in [p_0,c]$ is distance ϵ_1 from $[a,b]$. We have

$$\frac{\sinh \epsilon_2}{\sinh \epsilon_1} = \frac{\cosh d_{\mathbf{H}}(p_0,c)}{\cosh d_{\mathbf{H}}(p',c)} \geq \frac{1}{2} e^{d_{\mathbf{H}}(p_0,c) - d_{\mathbf{H}}(p',c)}$$

and so

$$d_{\mathbf{H}}(p',c) \geq d_{\mathbf{H}}(p_0,c) - L.$$

Similarly, let $p'' \in [c,p_2]$ which is distance ϵ_1 from $[a,b]$. By the same computation

$$d_{\mathbf{H}}(p'',c) \geq d_{\mathbf{H}}(p_2,c) - L.$$

This means in particular that $d_{\mathbf{H}}(p_1,[a,b]) \leq \epsilon_1$.

Let $[a',b']$ be the image of $[a,b]$ under the reflection in λ. Recall that $d_{\mathbf{H}}(p_0,\alpha) \leq \epsilon_2$ and $d_{\mathbf{H}}(p_2,\alpha) \leq \epsilon_2$. Hence α crosses the polygon $abp_2b'a'p_0$ at $[p_0,a]$ or $[p_0,a']$ and $[p_2,b]$ or $[p_2,b']$. We conclude that

$$d_{\mathbf{H}}(p_1,\alpha) \leq d_{\mathbf{H}}(p_1,[a,b]) = d_{\mathbf{H}}(p_1,[a',b']) \leq \epsilon_1. \quad \square$$

Lemma 3.2. *Let α and λ be disjoint geodesics in $\mathbf{H}$. Let $a, a' \in \alpha$ be the endpoints of the orthogonal projection of λ to α. Then for any hyperbolic isometry ϕ with axis α and translation length $\ell < d_{\mathbf{H}}(a, a')$, the geodesics $\phi(\lambda)$ and λ intersect.*

Proof. Let ϕ be a hyperbolic isometry with axis α and translation length $\ell < d_{\mathbf{H}}(a, a')$. Since λ is disjoint from α, the endpoints of λ and $\phi(\lambda)$ are all on the same side of α. The orthogonal projection of $\phi(\lambda)$ to α has endpoints $\phi(a)$ and $\phi(a')$. The condition on ℓ implies that exactly one of these points is between a and a'. Hence the endpoints of λ are separated by one endpoint of $\phi(\lambda)$, so λ and $\phi(\lambda)$ intersect. □

Lemma 3.3. *There exists a universal constant $M > 0$ such that the following holds. Let X be a hyperbolic metric on S. Let γ be a curve with $\ell_X(\gamma) = \epsilon \leq \epsilon_B$, and τ be a shortest curve intersecting γ. Then*

$$2w_\epsilon \leq \ell_X(\tau) \leq 4w_\epsilon + 2M.$$

Proof. Let $\mathcal{P}$ be a short pants decomposition on X. If $\ell_X(\gamma) = \epsilon \leq \epsilon_B$, then either γ intersects a curve in $\mathcal{P}$ or $\gamma \in \mathcal{P}$. In the former case, we have immediately that $\ell_X(\tau) \leq \epsilon_B$. In the latter case, γ is contained in a subsurface $R \subseteq S$ which is either a one-holed torus or a four-holed sphere with boundary curves in $\mathcal{P} \smallsetminus \gamma$.

The diameter of the complement of half collar neighborhoods of the boundary curves of R is uniformly bounded above by some $M > 0$ independent of X. Thus there is a curve τ that intersects γ once or twice whose length is at least $2w_\epsilon$ and at most $4w_\epsilon + 2M$. □

Lemma 3.4. *For any $\epsilon \in (0, \epsilon_B)$, there exists $L_\epsilon > 0$ such that the following statement holds. Let X be a hyperbolic metric on S and let R be an ϵ–thick subsurface. Let α be a simple closed geodesic contained in R. Then any segment of α of length at least L_ϵ must intersect some curve in a short marking μ_R on R.*

Proof. Let $Y \subseteq R$ be a pair of pants obtained by cutting R along the pants curves in μ_R. For each $\gamma \subset \partial Y$, let $U_Y(\gamma) = U_X(\gamma) \cap Y$, where $U_X(\gamma)$ is the standard collar neighborhood of γ. Each boundary component of Y is either a pants curve in μ_R or a boundary curve of R. Since R is ϵ–thick, the collar width of every pants curve in μ_R is at most w_ϵ. For each $\gamma \subset \partial Y$ that is not in ∂R, let τ_γ be a component of $\overline{\gamma} \cap U_Y(\gamma)$, where $\overline{\gamma}$ is the dual curve to γ in μ_R. Now consider the truncated pants

$$\overline{Y} = Y \setminus \bigcup_{\gamma \subset \partial Y} U_Y(\gamma).$$

Let α be a simple geodesic curve in R and let ω be a segment of α contained in Y. Note that the segment ω can enter each standard collar neighborhood at most once. Thus, ω can be divided into 3 pieces:

$$\omega = \omega_1 \cup \overline{\omega} \cup \omega_2,$$

where $\overline{\omega} = \omega \cap \overline{Y}$, and for each $i = 1, 2$, there is a boundary curve γ_i of Y such that $\omega_i = \omega \cap U_Y(\gamma_i)$. Since $\alpha \subseteq R$, ω cannot enter the standard collar of any curve in ∂R, so each γ_i is a pants curve of μ_R. By [LRT15, Lemma 3.1], we have

$$\ell_X(\overline{\omega}) = O(1) \quad \text{and} \quad \Big| \ell_X(\omega_i) - \mathrm{i}(\tau_{\bar{\gamma}_i}, \omega)\ell_X(\gamma_i) \Big| \overset{+}{\prec} w_\epsilon. \tag{3.1}$$

Now, if ω is long enough and it does not intersect a pants curve of μ_R, then ω is contained in a pair of pants Y. By Equation 3.1 then $\mathrm{i}(\omega, \tau_{\bar{\gamma}_i}) \geq 1$ for some $i = 1, 2$ and ω must intersect $\bar{\gamma}_i$ which is a curve of μ_R. □

Lemma 3.5. *Let X be a hyperbolic metric on S. Given a constant b and let U be an annulus bounded by hypercycles with boundary lengths at most b. Suppose $\tau \subset U$ and $\eta \subset U$ are two arcs connecting the boundaries of U, where τ is a geodesic arc. Then*

$$\ell_X(\tau) \leq \ell_X(\eta) + (\mathrm{i}(\tau, \eta) + 1)b.$$

If, in addition, τ and η intersect at least twice and the distance between any two consecutive intersection points along τ is at least $D > 3b$, then

$$\ell_X(\tau) \leq \frac{\ell_X(\eta)}{(1 - \frac{3b}{D})}.$$

Proof. Lifting to the universal cover, we can construct a path with the same endpoints as τ which is a concatenation of a segment of a boundary component of U of length at most b, followed by η and a segment of the other boundary component of U of length at most $\mathrm{i}(\eta, \tau)b$. Hence

$$\ell_X(\tau) \leq \ell_X(\eta) + (\mathrm{i}(\tau, \eta) + 1)b \tag{3.2}$$

which is the first statement of the lemma.

Suppose now that the distance between any two consecutive intersection points of τ and η along τ is at least D. We have $\ell_X(\tau) \geq D$ and $\mathrm{i}(\tau, \eta) - 1 \leq \frac{\ell_X(\tau)}{D}$. Incorporating this in 3.2 gives

$$\ell_X(\tau) \leq \ell_X(\eta) + \left(\frac{\ell_X(\tau)}{D} + 2 \right) b \leq \ell_X(\eta) + \frac{3b}{D}\ell_X(\tau),$$

which implies the second statement of the lemma. □

We proceed with the following lemma which is a modification of [LRT15, Lemma 3.2] about the length of a curve inside a pair of pants.

Lemma 3.6. *For any $\ell > 0$ there exists $D_0 = D_0(\ell)$ such that the following holds. Let X be a hyperbolic metric on S and $\mathcal{P}$ a pair of pants in X with boundary lengths at most ℓ. Let γ be a simple closed curve (not necessarily a geodesic) that intersects $\partial\mathcal{P}$ and let ω be a simple geodesic arc contained in $\mathcal{P}$ with endpoints on γ and has at least 2 intersections with γ in the interior. Write ω as a concatenation of ω_1, ω_2 and ω_3 where ω_i's are segments of ω with endpoints on γ. Suppose that the distance along ω between any two consecutive intersection points with γ is at least D_0. Then for at least one $i \in \{1, 2, 3\}$ we have*

$$\ell_X(\omega_i) \le 6\,\ell_X(\gamma).$$

Proof. The proof is similar to that of [LRT15, Lemma 3.2], but requires more care when it comes to additive and multiplicative errors.

Let α_i, $i = 1, 2, 3$, be the boundary components of $\mathcal{P}$ and $U_i \subseteq \mathcal{P}$ the standard collar of α_i in $\mathcal{P}$. By (2.7), the length of the non-geodesic boundary of U_i is bounded above by $b = \ell \coth(\ell/2)$. Let $M > 0$ be the length of the longest simple geodesic arc in $\mathcal{P}$ that is disjoint from U_i's. Note that M depends only on ℓ.

Let $D_0 \ge \max\{3M, 6b\}$. We claim the following holds.

Claim 3.7. There are $i, j \in \{1, 2, 3\}$ such that the following holds. There is an annulus $U \subset U_j$ bounded by hypercycles, a subarc τ of ω_i contained in U with

$$\ell_X(\tau) \ge \frac{1}{3}\ell_X(\omega_i),$$

a subarc $\eta \subset U$ of γ that connects the boundary components of U, so that one of the endpoints of τ is on η.

Proof. It follows from [LRT15, Lemma 3.1] that ω_2 consists of (possibly empty) an arc in some U_j followed by an arc in $\mathcal{P} \setminus \cup_i U_i$ followed by an arc in some U_k. Hence by the assumption on D_0, either the first or the last arc has length at least $\frac{1}{3}\ell_X(\omega_2)$; we call this arc ζ. Let j be such that $\zeta \subseteq U_j$. The arc ζ shares an endpoint p with some $\omega_k \subseteq U_j$ for $k = 1$ or $k = 3$. Now we have a division of U_j into two annuli with disjoint interiors, one containing ζ and the other containing ω_k. Since γ passes through p, it crosses at least one of the annuli. We denote the annulus by U and the subarc (ω_k or ζ) of ω that is contained in U by τ. Clearly the length of τ is at least a third of the length of the arc ω_i that contains it and the claim follows. □

We now finish the proof of the lemma. Let τ, η, ω_i and U be as in the claim. The length of either of the boundary components of U is at most b. If τ and η intersect at least twice, we apply Lemma 3.5 and $D_0 \ge 6b$ to get

$$\ell_X(\omega_i) \le 3\ell_X(\tau) \le \frac{3\ell_X(\eta)}{(1 - \frac{3b}{D_0})} \le \frac{3\ell_X(\gamma)}{(1 - \frac{3b}{D_0})} \le 6\,\ell_X(\gamma).$$

If τ and η intersect once, then $\ell_X(\tau) \leq \ell_X(\eta) + b$. Further, since each ω_i is at least D_0 long, $\ell_X(\tau) \geq \frac{1}{3}D_0 \geq 2b$ and so $\ell_X(\tau) \leq 2\ell_X(\tau)$ which implies

$$\ell_X(\omega_i) \leq 6\,\ell_X(\gamma).$$

□

We now introduce another useful tool, which is the notion of a *thin quadrilateral* with long sides lying along the boundary of some proper subsurface $R \subset S$. In a lemma below, we will show the existence of such quadrilaterals when the boundary of R is sufficiently long.

Definition 3.8 (Quadrilateral)**.** Let $R \subset S$ be a proper subsurface. By a *quadrilateral* Q *in* R we mean the image of a continuous map $f\colon [0,1] \times [0,1] \to R$, where f is an embedding on $(0,1) \times (0,1)$, the edges $[0,1] \times \{0\}$ and $[0,1] \times \{1\}$ are mapped to ∂R, and all other points are mapped to the interior of R. We will call the images of $[0,1] \times \{1\}$ and $[0,1] \times \{0\}$ the *top* and *bottom* edges, respectively, and $\{0\} \times [0,1]$ and $\{1\} \times [0,1]$ the *side edges* of Q. Also, note that top and bottom edges of the quadrilateral may not be distinct. By *width* of Q we will mean the maximal length over the top and bottom edges. We say Q is δ*–thin* if the side lengths are at most δ and Q is L*–wide* if its width is at least L.

Lemma 3.9. *Given* $\delta \in (0,1)$*, let* $L = \frac{2\pi|\chi(S)|}{\delta}$ *and* $K = \frac{|\chi(S)|}{\sinh^2(\delta/2)}$*. Let* X *be a hyperbolic metric on* S *and* $R \subset S$ *a proper non-annular subsurface with* $\ell_X(\partial R) > 2KL$*. Then there is a non-empty collection of pairwise disjoint* δ*–thin quadrilaterals in* R *with total width at least* $\ell_X(\partial R)/4$*, and at least one among them is* L*–wide. Moreover, we can ensure this collection consists of at most* K *quadrilaterals.*

Proof. Let $\widetilde{X} \cong \mathbf{H}$ be the universal cover of X. Let α be a curve in ∂R with $\ell_X(\alpha) = \ell_X(\partial R)$, then let U be the set of points $p \in \alpha$ such that for any lift $\widetilde{p}$ of p to $\widetilde{X}$, $B_\delta(\widetilde{p})$, the ball of radius δ centered at $\widetilde{p}$, meets exactly one component of the lifts of ∂R, i.e. the one that contains $\widetilde{p}$. In this case, $B_\delta(p) \cap R$ is an embedded half-disk in R. By definition, for each arc ω in $\alpha \smallsetminus U$, the δ–collar neighborhood of ω is either a cylinder in R and hence it is a δ–thin quadrilateral in R where the top and bottom edges are the same, or it contains an embedded δ–thin quadrilateral in R. Moreover, the width of the quadrilateral containing ω is at least $\ell_X(\omega)$. Now consider the collection $\mathcal{Q}$ of δ–thin quadrilaterals containing an arc of $\alpha \smallsetminus U$.

First note that the set U has at most K connected components. This follows from the fact that each connected component of U has a δ–collar neighborhood inside R that contains at least one half-disk, and thus it contributes at least $2\pi\sinh^2(\delta/2)$ (the area of half-disk) to the area of R. But the total area of R is at most $2\pi|\chi(S)|$ (area of the hyperbolic metric X), so U has at most K components. This also implies that $\mathcal{Q}$ consists of at most K quadrilaterals. Moreover, using a similar area argument, we see that each connected component of U has length at most L; because the δ–collar neighborhood of $\omega \subset U$ in R has area at least $\ell_X(\omega)\delta$ which is bounded by $2\pi|\chi(S)|$. Putting these observations and the assumption of the lemma about the length of α together we get the lower bound

$$\ell_X(\alpha) - KL > \ell_X(\alpha)/2,$$

for the total length of components of $\alpha \setminus U$. Since U has at most K components, so does $\alpha \setminus U$. With each component of U having length at most L, at least one component of $\alpha \setminus U$ has length at least $\ell_X(\alpha)/2K > L$. This implies that there is at least one δ–thin and L–wide quadrilateral in $\mathcal{Q}$. Finally, since the total length of all the arcs in $\alpha \setminus U$ is at least $\ell_X(\alpha)/2$ and each quadrilateral in $\mathcal{Q}$ contains at most 2 arcs in $\alpha \setminus U$, the total width of all the quadrilaterals is at least $\ell_X(\alpha)/4$. This finishes the proof of the lemma. □

We now prove a proposition that allows us to estimate the relative twisting of a curve in a subsurface R of S with respect to a hyperbolic metric on S.

Proposition 3.10. *Let X be a hyperbolic metric on S. Let $R \subseteq S$ be a non-annular subsurface and γ a curve intersecting R. Then for any component $\tau \subseteq \gamma \cap R$, we have*

$$d_R(\gamma, X) \overset{*}{\prec} \operatorname{Log} \frac{\ell_X(\tau)}{\ell_X(\partial R)},$$

where the error in the bound above depends only on S.

Proof. Let β be an ϵ_B–short curve intersecting R, and let $\tau \subseteq \gamma \cap R$ and $\tau' \subseteq \beta \cap R$ be curves or arcs with endpoints on the boundary of R.

For any $\delta \in (0,1)$, recall the constants K and L from Lemma 3.9. Now fix a small δ so that $8\delta^2 \leq 2\pi|\chi(S)|$ and $L - 2\delta > \epsilon_B$. We consider the following two cases.

First suppose that $\ell_X(\partial R) \leq 2KL$. Recall that w_{ϵ_B} denotes the width of the standard collar neighborhood about a curve of length ϵ_B. Since β is ϵ_B–short, the distance between any two intersection points with β along τ is at least $2w_{\epsilon_B}$, so we immediately have that

$$\mathrm{i}_R(\tau, \tau') \leq \frac{\ell_X(\tau)}{2w_{\epsilon_B}} + 1 \leq \frac{KL}{w_{\epsilon_B}} \frac{\ell_X(\tau)}{\ell_X(\partial R)} + 1.$$

If the quantity $\frac{KL}{w_{\epsilon_B}} \frac{\ell_X(\tau)}{\ell_X(\partial R)}$ is less than 1, then τ and τ' intersect at most once. If it is bigger than 1, we can write

$$\mathrm{i}_R(\tau, \tau') \leq 2\frac{KL}{w_{\epsilon_B}} \frac{\ell_X(\tau)}{\ell_X(\partial R)}$$

The constant $2\frac{KL}{w_{\epsilon_B}}$ is independent of X and the inequality (2.4) completes the proof in the case of $\ell_X(\partial R) \leq 2KL$.

Now assume that $\ell_X(\partial R) > 2KL$. Let $Q_1,\ldots,Q_m$, $m \le K$, be the collection of δ–thin quadrilaterals in R as guaranteed by Lemma 3.9. Set w_i to be the width of Q_i and assume that $w_1 > L$. Note that the side edges of all the Q_i's are pairwise disjoint. Moreover, any geodesic that crosses Q_i from side to side must be at least $w_i - 2\delta$ long. Thus τ', which is an arc of an ϵ_B–short curve, cannot cross Q_1 from side to side, so $d_R(\tau', \eta_1) = 1$, where η_1 is a side edge of Q_1 missed by τ'. Now let n_i be the number of times that τ crosses Q_i from side to side for $i = 1, \ldots, m$, and let $n = \min_{i=1,\ldots,m} n_i$. Then observe that

$$d_R(\gamma, X) \overset{*}{\prec} \operatorname{Log} n, \tag{3.3}$$

holds. To see this, note that if $n = 0$, then $d_R(\gamma, X) \le 1$ and we already have the above inequality. So assume that $n \ge 1$ and let η be the side edge that $\operatorname{i}_R(\tau, \eta) = n$, then by the triangle inequality 2.2 and 2.4 we have

$$\begin{aligned} d_R(\gamma, X) &\overset{+}{\prec} d_R(\tau, \eta) + d_R(\eta, \eta_1) + d_R(\eta_1, \tau') \\ &\le d_R(\tau, \eta) + 2 \\ &\overset{*}{\prec} \operatorname{Log} \operatorname{i}_R(\tau, \eta) = \operatorname{Log} n, \end{aligned}$$

again giving us the inequality (3.3).

We know that τ crosses each Q_i from side to side at least n times, and picks up a length of at least $w_i - 2\delta$ each time. Moreover, by Lemma 3.9 we have that $\sum_{i=1}^m w_i > \ell_X(\partial R)/4$, so

$$n \le \frac{\ell_X(\tau)}{\ell_X(\partial R)/4 - 2m\delta} \le \frac{\ell_X(\tau)}{\ell_X(\partial R)/4 - 2K\delta}.$$

Also by our assumption, $8\delta^2 \le 2\pi|\chi(S)|$ which implies that $2K\delta \le \frac{1}{4}KL$, and hence by the lower bound for the length of ∂R we have

$$\frac{\ell_X(\partial R)}{4} - 2K\delta \ge \frac{\ell_X(\partial R)}{8}.$$

From the above two inequalities we deduce that $\operatorname{Log} n \overset{*}{\prec} \operatorname{Log} \frac{\ell_X(\tau)}{\ell_X(\partial R)}$ which is the desired inequality. □

Lemma 3.11. *Let λ and λ' be a pair of disjoint non-asymptotic geodesics in $\mathbf{H}$. Let $f\colon \mathbf{H} \to \mathbf{H}$ be a K–Lipschitz map, $K \ge 1$, such that $f(\lambda)$ and $f(\lambda')$ are geodesics, and f stretches distances along λ by K. Let $q \in f(\lambda)$ and $q' \in f(\lambda')$ be the endpoints of the common perpendicular between $f(\lambda)$ and $f(\lambda')$. Then for any points $x, y \in \lambda$, if $d_{\mathbf{H}}(f(x), q) \ge d_{\mathbf{H}}(f(y), q)$, then for any $x' \in \lambda'$, we have*

$$\sinh d_{\mathbf{H}}(q, q') \cosh \frac{K d_{\mathbf{H}}(x, y)}{2} \le \sinh K d_{\mathbf{H}}(x, x').$$

Proof. Since $d_{\mathbf{H}}(f(x), q) \geq d_{\mathbf{H}}(f(y), q)$ and f stretches distances along λ by K, we have

$$d_{\mathbf{H}}(f(x), q) \geq \frac{d_{\mathbf{H}}(f(x), f(y))}{2} = \frac{K d_{\mathbf{H}}(x, y)}{2}. \tag{3.4}$$

Also it suffices to assume that

$$d_{\mathbf{H}}(f(x), f(x')) = d_{\mathbf{H}}(f(x), f(\lambda'))$$

and hence that the four points $f(x), q, q'$ and $f(x')$ form a Lambert quadrilateral (a quadrilateral with three right angles). We then have

$$\sinh d_{\mathbf{H}}(q, q') \cosh d_{\mathbf{H}}(f(x), q) = \sinh d_{\mathbf{H}}(f(x), f(x')). \tag{3.5}$$

Using the fact that f is K–Lipschitz, we obtain

$$d_{\mathbf{H}}(f(x), f(x')) \leq K d_{\mathbf{H}}(x, x').$$

The result now follows by plugging (3.4) and the above into the left and right sides of (3.5). □

Lemma 3.12. *Let γ and λ be a pair of intersecting geodesics in $\mathbf{H}$. Given $\epsilon > 0$, let $U(\gamma)$ be the w_ϵ–regular neighborhood of γ and let L be the length of the arc $\lambda \cap U(\gamma)$. Let ϕ be a hyperbolic isometry with axis γ and translation length ϵ. Let then $\lambda' = \phi(\lambda)$ and $v = d_{\mathbf{H}}(\lambda, \lambda')$. Then we have*

$$\sinh(L/2) = \frac{1}{\sinh(v/2)}.$$

Proof. We will use the Poincaré disc model for $\mathbf{H}$. See Figure 1 for this proof. Let p be the point of intersection between γ and λ, and let $p' = \phi(p)$. Let $q \in \lambda$ and $q' \in \lambda'$ be the points such that the arc $[q, q']$ in $\mathbf{H}$ is perpendicular to λ and λ'. Without a loss of generality, assume the midpoint of $[q, q']$ is the origin o of $\mathbf{H}$. Any isometry taking λ to λ' must have axis passing through o and thus o is also the midpoint of $[p, p']$. Let x be the point of intersection between λ and a boundary component of $U(\gamma)$. Rotation by an angle of π about p preserves $U(\gamma)$ and λ, and sends x to the other intersection of λ and $\partial U(\gamma)$. We than have that $d_{\mathbf{H}}(x, p) = L/2$, since Let $y \in \gamma$ be the foot of the perpendicular from x to γ. The triangles $\triangle(opq)$ and $\triangle(xpy)$ are right triangles with hypotenuse $[o, p]$ and $[x, p]$ respectively, and $\angle opq = \angle xpy$. Thus, by hyperbolic trigonometry of right triangles, we have

$$\frac{\sinh d_{\mathbf{H}}(o, p)}{\sinh d_{\mathbf{H}}(o, q)} = \frac{\sinh d_{\mathbf{H}}(x, p)}{\sinh d_{\mathbf{H}}(x, y)}.$$

The formula follows since

$$d_{\mathbf{H}}(o, p) = \epsilon/2, \quad d_{\mathbf{H}}(o, q) = v/2, \quad d_{\mathbf{H}}(x, y) = w_\epsilon, \quad d_{\mathbf{H}}(x, p) = L/2. \qquad \square$$

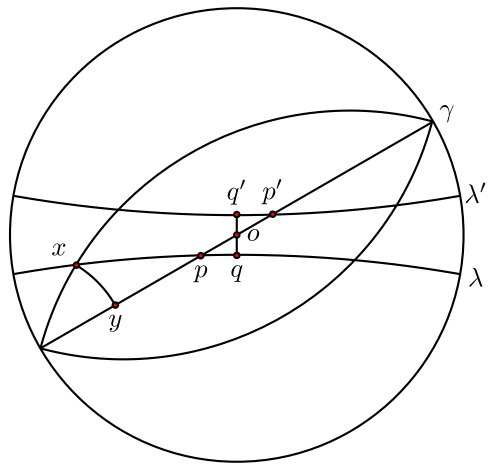

FIGURE 1. Figure for the proof of Lemma 3.12.

Proposition 3.13. *There exists $\epsilon_0 > 0$ such that the following statement holds. Let $\mathbf{g} : I \to \mathcal{T}(S)$ be a Thurston geodesic. Suppose a curve γ intersects $\lambda_{\mathbf{g}}$ and $\ell_s(\gamma) \leq \epsilon_0$ for some $s \in I$. Then for all $t > s$ with $\ell_t(\gamma) \geq 1$, we have:*

$$\mathcal{L}_t \geq \frac{1}{2} e^{t-s} \mathcal{L}_s,$$

where $\mathcal{L}_t$ is the X_t–length of longest arc of $\lambda_{\mathbf{g}}$ in the standard collar of γ.

Proof. Given ϵ, recall that w_ϵ denotes the width of the standard collar neighborhood of a curve of length ϵ and b_ϵ the length of the boundary of the standard collar neighborhood. Let $b = \sup_{\epsilon \in (0,1]} b_\epsilon = b_1$. Explicitly, by (2.7) we have

$$b = \coth \frac{1}{2}.$$

Let $\epsilon_0 \in (0, 1)$ be sufficiently small so that for any positive $\epsilon \leq \epsilon_0$, we have $w_\epsilon > 2b$. Such ϵ_0 exists since the function w_ϵ is decreasing and goes to $+\infty$ at 0.

For any $s \in I$, let $\widetilde{\gamma}_s$ be a lift of γ to $\widetilde{X}_s$. Let ϕ_s be the hyperbolic isometry with axis $\widetilde{\gamma}_s$ and translation length $\ell_s(\gamma)$.

Now fix s and set $\epsilon = \ell_s(\gamma)$ for some $\epsilon \in (0, \epsilon_0)$. Let U_s be the w_ϵ–regular neighborhood of $\widetilde{\gamma}_s$. Let λ be a lift of a leaf of $\lambda_{\mathbf{g}}$ that crosses $\widetilde{\gamma}_s$ such that $\mathcal{L}_s$ is the length of $\lambda \cap U_s$. Note that $\mathcal{L}_s \geq 2w_\epsilon > 4b$. Let x, y be the endpoints of $\lambda \cap U_s$. Set $\lambda' = \phi_s(\lambda)$.

Suppose $\ell_t(\gamma) = \eta \geq 1$ for some $t > s$. Let $\widetilde{f} : \widetilde{X}_s \to \widetilde{X}_t$ be the lift of an optimal map from $X_s \to X_t$. By composing with Möbius transformation if necessary, we may assume $\widetilde{f}\phi_s = \phi_t \widetilde{f}$. Let U be the w_η–regular neighborhood of $\widetilde{\gamma}_t$ and let $\mathcal{L}$ be the length of $\widetilde{f}(\lambda) \cap U$. Note that $\mathcal{L}_t \geq \mathcal{L}$, so it's enough to prove the statement of the proposition for $\mathcal{L}$. Let $q \in \widetilde{f}(\lambda)$ and $q' \in \widetilde{f}(\lambda')$ be the endpoints of the common perpendicular of $\widetilde{f}(\lambda)$ and $\widetilde{f}(\lambda')$. By Lemma 3.12 and monotonicity of sinh,

$$\sinh \frac{\mathcal{L}}{2} \geq \frac{1}{\sinh(d_{\mathbf{H}}(q, q'))}.$$

We now proceed to give an upper bound for $\sinh(d_{\mathbf{H}}(q, q'))$.

Without the loss of generality, we can assume that $d_{\mathbf{H}}(\widetilde{f}(x), q) \geq d_{\mathbf{H}}(\widetilde{f}(y), q)$. Let $x' = \phi_s(x)$. Since $\ell_s(\gamma) < 1$, the length along the boundary component of U_s from x to x' is less than b, so $d_{\mathbf{H}}(x, x') \leq b$. This, together with Lemma 3.11, yields

$$\sinh d_{\mathbf{H}}(q, q') \leq \frac{\sinh e^{t-s} d_{\mathbf{H}}(x, x')}{\cosh(e^{t-s}\mathcal{L}_s/2)} \leq \frac{\sinh e^{t-s} b}{\cosh(e^{t-s}\mathcal{L}_s/2)} \leq \frac{1}{e^{e^{t-s}(\frac{\mathcal{L}_s}{2} - b)}}$$

Since $\mathcal{L}_s > 4b$ and $\sinh^{-1} x \geq \log x$ for $x \geq 1$, we obtain

$$\mathcal{L} \geq 2 \sinh^{-1} e^{e^{t-s}(\frac{\mathcal{L}_s}{2} - b)} \geq 2e^{t-s} \left(\frac{\mathcal{L}_s}{2} - b \right) \geq \frac{1}{2} \mathcal{L}_s e^{t-s}. \qquad \square$$

4. A notion of being horizontal

Let $\mathbf{g} : I \to \mathcal{T}(S)$ be a Thurston geodesic and let $\lambda_{\mathbf{g}}$ be the maximally stretched lamination. The main goal of this section is to develop a notion of a curve being horizontal along $\mathbf{g}$ in such a way that as soon as α is sufficiently horizontal, it remains horizontal and its length grows essentially exponentially. There are two stages, weakly horizontal and strongly horizontal.

This latter notion requires a rather technical definition, but roughly speaking, it means that curve α fellow travels $\lambda_{\mathbf{g}}$ both geometrically and topologically for a long time. We will prove that strongly horizontal curves stay strongly horizontal and grow exponentially in length. Then, a curve α is weakly horizontal if it stays parallel with a leaf of $\lambda_{\mathbf{g}}$ through a long collar neighborhood of a very short curve. We will show that a weakly horizontal curve quickly becomes strongly horizontal.

4.1. Strongly horizontal.

We will start with the definition of strongly horizontal. This follows closely to the definition of horizontal as introduced in [LRT15], but with an adjustment of the constants involved. The constant $\epsilon_h \geq \epsilon_B$ in the following definition will be determined in Lemma 4.3.

Definition 4.1 (Strongly Horizontal)**.** Let $\mathbf{g}\colon I \to \mathcal{T}(S)$ be a Thurston geodesic and denote $X_t = \mathbf{g}(t)$. Given $n \in \mathbb{N}$ and $L > 0$, we say a curve α is (n, L)–*horizontal* on $X = X_t$ if there exists an ϵ_h–short curve γ on X and a leaf λ in $\lambda_{\mathbf{g}}$ such that the following statements hold:

(H1) In the universal cover $\widetilde{X} \cong \mathbf{H}$, there is a collection $\{\widetilde{\gamma}_1, \dots \widetilde{\gamma}_n\}$ of lifts of the curve γ and a lift $\widetilde{\lambda}$ of λ intersecting each $\widetilde{\gamma}_i$ at a point p_i on $\widetilde{l}$ (the p_i's are indexed by the order of their appearances along $\widetilde{\lambda}$) such that $d_{\mathbf{H}}(p_i, p_{i+1}) \geq L$ for $i = 1, \dots, n-1$.

(H2) There is a lift $\widetilde{\alpha}$ of α such that $\widetilde{\alpha}$ intersects $\widetilde{\gamma}_i$ at a point $q_i \in \mathbf{H}$ with $d_{\mathbf{H}}(p_i, q_i) \leq \epsilon_h$ for each $i = 1, \dots, n$.

We will call γ as above an *anchor curve* for α, and $\widetilde{\alpha}$ an (n, L)*–horizontal lift* of α. Moreover, we call the segment $[q_1, q_n] \subseteq \widetilde{\alpha}$ the *horizontal segment* of $\widetilde{\alpha}$, and the projection of $[q_1, q_n]$ to X the horizontal segment of α.

We recall the following proposition from [LRT15].

Proposition 4.2. [LRT15, Proposition 4.6] *There is a constant $L_0 > 0$ such that the following statement holds. Let X be a hyperbolic metric on S, γ an ϵ_B–short curve on X, λ is a complete simple geodesic on X. Then for any $n \in \mathbb{N}$ and $L \geq L_0$, if there are n lifts $\{\widetilde{\gamma}_i\}$ of γ and a lift $\widetilde{\lambda}$ of λ that satisfying condition (H1) of Definition 4.1, and α is a curve on X with a lift $\widetilde{\alpha}$ that lies ϵ_B–close to the segment $[p_1, p_n]$ in $\widetilde{\lambda}$, then for all $i = 3, \dots n-2$, $\widetilde{\alpha}$ intersects $\widetilde{\gamma}_i$ at a point q_i with $d_{\mathbf{H}}(p_i, q_i) \leq \epsilon_B$.*

Recall that we have fixed once and for all the Bers constant ϵ_B, which fixes the associated collar width w_{ϵ_B} and the length b_{ϵ_B} of the boundary of the collar neighborhood of a curve of length ϵ_B. In the following lemma we set several constants to quantify the notions of horizontal curve that we require in the rest of the paper.

Lemma 4.3 (Constants)**.** *Let A be the constant of Lemma 2.6 and M the constant from Lemma 3.3. Then there are positive constants ϵ_w, ϵ_h, L_0 and $n_0 \in \mathbb{N}$ such that the following conditions are satisfied simultaneously.*

(1) $\epsilon_w < \min\{1, 2w_{\epsilon_B}\}$ *and satisfies Proposition 3.13,*

(2) $\epsilon_h = \epsilon_B + 4w_{\epsilon_w} + 2M$

(3) $L_0 = 10\epsilon_h$.

(4) L_0 *satisfies Proposition 4.2.*

(5) $L_0 \geq w_{\epsilon_B} + 6b_{\epsilon_B}$

(6) $\frac{1}{\epsilon_w} w_{\epsilon_w} \geq n_0 \lceil L_0 + 1 \rceil + 5 + A$.

(7) $\sinh(1 + w_{\epsilon_B}) \sinh \frac{\epsilon_B}{2} \leq \exp(\frac{1}{6} L_0 - 1)$.

(8) $n_0 \geq 8$.

Proof. First note that for all ϵ_w sufficiently small (1) holds. Then we may define ϵ_h and L_0 as in (2) and (3). Since L_0 gets arbitrary large by choice of ϵ_w small enough, (4) and (5) hold for ϵ_w small enough. Also (6) holds for ϵ_w small enough, because the right hand side of the inequality in (6) is a linear function of w_{ϵ_w} and $\frac{1}{\epsilon_w} w_{\epsilon_w}$ is larger than that for all ϵ_w small enough. Finally (7) holds because the right hand side of (7) could be made arbitrary large for ϵ_w small enough. □

From now on we fix constants $n_0, \epsilon_w, \epsilon_h$ and L_0 that satisfy the conditions of Lemma 4.3. The following theorem provides us with a control on the growth of the quantifiers of a strongly-horizontal curve moving forward along a Thurston geodesic. It is a generalization of Theorem 4.2 in [LRT15, §4] to the situation that the curve is inside a subsurface and the maximal stretch lamination intersects the subsurface.

Theorem 4.4. *There exists $s_0 \geq 0$ such that the following statement holds. Let $\mathbf{g} : I \to \mathcal{T}(S)$ be a Thurston geodesic, and let $R \subseteq S$ be a non-annular subsurface that intersects $\lambda_{\mathbf{g}}$ essentially. Suppose that a curve $\alpha \in \mathcal{C}(R)$ is (n_s, L_s)–horizontal at $s \in I$ where $n_s \geq n_0$ and $L_s \geq L_0$. Then,*

(1) For any $t \geq s + s_0$, the curve α is (n_t, L_t)–horizontal on X_t where

$$n_t \overset{*}{\succ} n_s \quad \text{and} \quad L_t \geq L_s.$$

(2) Moreover, if for some $d > 1$ we have $d_R(X_s, X_t) \geq d$, then

$$\log \frac{n_t}{n_s} \overset{*}{\succ} d \quad \text{and} \quad L_t n_t \overset{*}{\succ} e^{t-s} L_s n_s.$$

Proof. The proof of both statements is very similar to that of [LRT15, Theorem 4.2], so we only sketch it here.

Let $\widetilde{\alpha}, \widetilde{\lambda}, p_i, q_i, \gamma$ and $\widetilde{\gamma}_i$ be as in Definition 4.1. Let $f : X_s \to X_t$ be an e^{t-s}–Lipschitz map. Let $\widetilde{f} : \widetilde{X}_s \to \widetilde{X}_t$ be a lift of f that preserves $\widetilde{\lambda}$ and that fixes pointwise its endpoints. The first step is to show that the geodesic representative of $\widetilde{f}(\widetilde{\alpha})$ is distance at most 1 to $\widetilde{\lambda}$ from $\widetilde{f}(p_3)$ to $\widetilde{f}(p_{n_s-2})$.

We will apply Proposition 3.5 in [LRT15, §4], and to do so we need that, if exists, the closest to $\widetilde{\alpha}$ point c on $\widetilde{\lambda}$ be at least $2\epsilon_h$ away from both p_1 and p_{n_s}. If this is not the case, note that at most one of the points p_i is within $2\epsilon_h$ of c, so we lose at most one point. Depending on that, label $l = 1$ or 2 and $r = n_s$ or $n_s - 1$.

By Proposition 3.5 in [LRT15, §4] there are translates $\widetilde{\lambda}_l$ and $\widetilde{\lambda}_r$ of $\widetilde{\lambda}$ by hyperbolic isometries with axes $\widetilde{\gamma}_l$ and $\widetilde{\gamma}_r$ so that the endpoints α_l and α_r of $\widetilde{\alpha}$ are sandwiched between those of $\widetilde{\lambda}$ and $\widetilde{\lambda}_l$ and of $\widetilde{\lambda}$ and $\widetilde{\lambda}_r$ respectively, and so that p_l and p_r are at most $4\epsilon_h + 3$ from respectively $\widetilde{\lambda}_l$ and $\widetilde{\lambda}_r$.

We now apply $\widetilde{f}$. Let $\widetilde{\alpha}'$ be the geodesic representative of $\widetilde{f}(\widetilde{\alpha})$. Its endpoint $\widetilde{f}(\alpha_l)$ is between $\widetilde{\lambda}$ and $\widetilde{f}(\widetilde{\lambda}_l)$, and $\widetilde{f}(\alpha_r)$ is between $\widetilde{\lambda}$ and $\widetilde{f}(\widetilde{\lambda}_r)$. Since $\widetilde{f}(\lambda_l)$ is within distance $e^{t-s}(4\epsilon_h+3)$ of $\widetilde{f}(p_l)$, and $\widetilde{f}(\lambda_r)$ is within distance $e^{t-s}(4\epsilon_h+3)$ of $\widetilde{f}(p_r)$, we have that $\widetilde{\alpha}'$ stays within distance $e^{t-s}(4\epsilon_h+3)$ of $\widetilde{\lambda}$ from $\widetilde{f}(p_l)$ to $\widetilde{f}(p_r)$. It now follows from Lemma 3.1 that $\widetilde{\alpha}'$ is 1– close to $\widetilde{\lambda}$ from $\widetilde{f}(p_{l+1})$ to $\widetilde{f}(p_{r-1})$.

The next step is to show that any segment $[\widetilde{f}(p_i), \widetilde{f}(p_{i+3})]$ for $i = l, \cdots, r-3$, intersects a lift of an ϵ_B– short curve. Let $\widetilde{\omega}$ be any such segment and let ω be its projection by $\pi : \widetilde{X}_t \to X_t$.

If $\pi : \widetilde{\omega} \to \omega$ is not injective, then $\widetilde{\lambda}$ is a lift of a closed curve λ which on X_s has to satisfy $\ell_s(\lambda) \geq w_{\epsilon_h}$ and hence $\ell_t(\lambda) \geq e^{s_0} w_{\epsilon_h}$. Assuming that $s_0 > \log \frac{\epsilon_B}{w_{\epsilon_h}}$ guarantees that λ intersects an ϵ_B–short curve, and hence that $\widetilde{\omega}$ intersects a lift of the ϵ_B-short curve.

If $\pi : \widetilde{\omega} \to \omega$ is injective, then we prove that it intersects an ϵ_B –short curve as in Claim 4.8 of [LRT15, §4], except that instead of [LRT15, Lemma 3.2] we apply Lemma 3.6, and we require that $s_0 \geq \log \frac{D_0}{w_{\epsilon_h}}$ where $D_0 = D_0(\epsilon_h)$ is the constant from Lemma 3.6, and check that $L_0 > 6\epsilon_h$ by condition (3) of Lemma 4.3.

Hence for some ϵ_B–short curve (call it β) and some $n_t \stackrel{*}{\succ} n_s$, the segment $[\widetilde{f}(p_l), \widetilde{f}(p_r)]$ intersects at least n_t lifts of β, such that any two of the intersection points are at least $L_t = L_s e^{t-s}$ apart. Applying Proposition 4.2 we conclude α is (n_t, L_t)–horizontal which proves part (1) of the theorem.

We now sketch the proof of part (2). Suppose that $d_R(X_s, X_t) \geq d > 1$. Let $\mathcal{P}$ be an ϵ_B–short pants decomposition of X_t. Note that for small values of d part (2) follows from part (1), hence we assume that d is large, which implies in particular that γ, the anchor curve of α at X_s as chosen as in Definition 4.1, intersects every curve in $\mathcal{P}$ that enters R. Let

$$m = \min_{\beta \in \mathcal{P}} \Big\{ \mathrm{i}_R(\beta, \gamma) \ : \ \beta \cap R \neq \emptyset \Big\}.$$

From Equation 2.4 we have

$$d \stackrel{*}{\prec} \log m, \tag{4.1}$$

also note that m satisfies

$$m \leq \frac{\ell_t(\gamma)}{w_{\epsilon_B}} \leq \frac{e^{t-s}\epsilon_h}{w_{\epsilon_B}}. \tag{4.2}$$

Cut the segment $[\widetilde{f}(p_{l+1}, \widetilde{f}(p_{r-1})]$ into $\max\{\lfloor \frac{m(r-l-2)w_{\epsilon_B}}{\epsilon_h} \rfloor, 1\}$ pieces of equal length. Let $\widetilde{\omega}$ be one of the pieces and let ω be its projection to X_t. We will show that ω intersects a curve in $\mathcal{P}$.

If $\pi : \widetilde{\omega} \to \omega$ is not injective, then ω is a closed leaf of λ. Since $s_0 > \log \frac{\epsilon_B}{w_{\epsilon_h}}$, $\ell_t(\omega) > \epsilon_B$ and therefore ω intersects some curve in $\mathcal{P}$.

Suppose now that $\pi : \widetilde{\omega} \to \omega$ is injective. Since the segment $[\widetilde{f}(p_{l+1}), \widetilde{f}(p_{r-1})]$ has length at least $L_0(r-l-2)e^{t-s}$ we have

$$\ell_t(\omega) \geq \frac{e^{t-s}\epsilon_h}{m} \frac{L_s}{w_{\epsilon_B}}. \tag{4.3}$$

Assume now for contradiction that ω misses all the curves in $\mathcal{P}$. Then it is contained in a pair of pants P with ϵ_B–short boundary components.

By [LRT15, Lemma 3.1] there is a boundary component β of P such that an arc τ of ω is contained in $U(\beta)$ and $\ell_t(\tau) \geq \frac{1}{3}\ell_t(\omega)$.

Claim 4.5. The curve β intersects R.

Proof. Since the curve α is contained in R, it suffices to show that α intersects β or that α and β are the same curve. In fact it suffices to show that there is a point on α that is distance less than $w_{\ell_t(\beta)}$ from β, that is, that α enters the standard collar of β at X_t. Recall that $\widetilde{\alpha}'$ is 1–close to $\widetilde{\lambda}$ from $\widetilde{f}(p_{l+1})$ to $\widetilde{f}(p_{r-1})$. Let $\tilde{\tau} \subseteq \widetilde{\omega}$ be the lift of τ, and $\widetilde{\beta}$ the appropriate lift of β. Let $[a_1, a_2] \subseteq \widetilde{\alpha}'$ be the longest segment in the 1–neighborhood of $\tilde{\tau}$. Then

$$d_{\mathbf{H}}(a_1, a_2) \geq \tfrac{1}{3}L_0 - 2 \text{ and } d_{\mathbf{H}}(a_i, \tilde{\beta}) \leq 1 + w_{\ell_t(\beta)}.$$

Now by Lemma 3.1 applied to $\epsilon_1 = w_{\ell_t(\beta)}$ and $\epsilon_2 = 1 + w_{\ell_t(\beta)}$, the midpoint p of $[a_1, a_2]$ is distance less than $w_{\ell_t(\beta)}$ away from β as long as

$$d_{\mathbf{H}}(a_1, a_2) \geq 2 \log 2 \frac{\sinh(1 + w_{\ell_t(\beta)})}{\sinh(w_{\ell_t(\beta)})}. \tag{4.4}$$

Since the function $\log 2 \frac{\sinh(1+x)}{\sinh(x)}$ is decreasing and $w_{\ell_t(\beta)} \geq w_{\epsilon_B}$, we have, by the choice of L_0 in (7) of Lemma 4.3, that the inequality 4.4 holds.

Hence p is within distance $w_{\ell_t(\beta)}$ of $\tilde{\beta}$, which implies that α intersects or coincides with β, and so β intersects R. □

The claim and the definition of m imply that γ intersects β at least m times. Let $\gamma' = f(\gamma)$ be the image of the geodesic γ. Then γ' also intersects β at least m times, and there are at least m disjoint arcs in γ' that connect both boundary components of $U(\beta)$. Let η be a shortest such arc. Clearly $\ell_t(\eta) \leq \frac{e^{t-s}\epsilon_h}{m}$.

If $\mathrm{i}(\tau,\eta)\geq 2$, the distance between two consecutive intersections along τ is at least $D=e^{t-s}w_{\epsilon_h}$. By requiring that $s_0>\log(6\frac{b_{\epsilon_B}}{w_{\epsilon_h}})$, we guarantee $D>6b_{\epsilon_B}$. Now by Lemma 3.5

$$\ell_t(\tau)\leq\frac{\ell_t(\eta)}{1-\frac{3b_{\epsilon_B}}{D}}<\frac{2\epsilon_h e^{t-s}}{m} \tag{4.5}$$

which implies

$$\ell_t(\omega)<\frac{6\epsilon_h e^{t-s}}{m}.$$

When τ and η intersect at most once, then by the second inequality of Lemma 3.5,

$$\ell_t(\tau)\leq\ell_t(\eta)+2b_{\epsilon_B}$$

So we obtain

$$\ell_t(\omega)\leq\frac{3\epsilon_h e^{t-s}}{m}+6b_{\epsilon_B}\leq\frac{\epsilon_h e^{t-s}}{m}\left(\frac{3w_{\epsilon_B}+6b_{\epsilon_B}}{w_{\epsilon_B}}\right)$$

By Lemma 4.3 both upper bounds on $\ell_t(\omega)$ contradict the Equation 4.3. Hence ω intersects a curve in $\mathcal{P}$.

Recall that we have cut $[\widetilde{f}(p_{l+1},\widetilde{f}(p_{r-1}))]$ into $\max\left\{\left\lfloor\frac{m(r-l-2)w_{\epsilon_B}}{\epsilon_h}\right\rfloor,1\right\}$ segments, and that the length of each segment is at least L_s. We also showed that each segment intersects a geodesic that projects to some ϵ_B–short curve. By choosing every other segment if necessary we guarantee that the distance between the intersection points is at least L_s. Then for some $\beta\in\mathcal{P}$ the number n_t of segments that intersect a lift of β satisfies $n_t\overset{*}{\succ}mn_s$. By Equation 4.1, $\log m\overset{*}{\succ}d$, thus applying Proposition 4.2 finishes the proof of the second part of the theorem. □

The following proposition gives some fundamental examples of horizontal curves on a hyperbolic metric X on S: a curve of length at least 1 with $\lambda_{\mathbf{g}}$ twisting a lot about it, and a curve that together with a leaf of $\lambda_{\mathbf{g}}$ twists a lot about some curve of length at least 1.

Proposition 4.6. *Let $\mathbf{g}:I\to\mathcal{T}(S)$ be a Thurston geodesic. Let γ be a curve, τ be the shortest curve on X_t intersecting γ, and λ be a leaf of $\lambda_{\mathbf{g}}$. Suppose $\ell_t(\gamma)\geq 1$ and $d_\gamma(\tau,\lambda)\geq n\left\lceil\frac{L_0}{\ell_t(\gamma)}+1\right\rceil+5$ for some $n\in\mathbb{N}$. Then γ and any curve α with $d_\gamma(\alpha,\lambda)\leq 3$ is (n,L_0)–horizontal on X_t with anchor curve τ.*

Proof. If $\ell_X(\gamma)>\epsilon_B$, then it intersects an ϵ_B–short curve by definition of ϵ_B. If γ itself is ϵ_B–short, applying Lemma 3.3 we have that the curve τ satisfies $\ell_t(\tau)\leq 4w_1+2M$. In both cases by the choice of ϵ_h in (2) of Lemma 4.3 we have $\ell_t(\tau)\leq\epsilon_h$.

Denote the standard collar neighborhood of γ by U, and choose arcs of τ, λ and α inside U, denoting them τ_γ, λ_γ and α_γ respectively. Let η be an arc perpendicular to γ crossing U and disjoint from τ_γ. Such an arc exists since τ is chosen to be the shortest possible curve that intersects γ.

Now the assumption on the annular coefficients and the triangle inequality imply

$$d_\gamma(\tau, \alpha) > n\Big\lceil L_0/\ell_t(\gamma) + 1\Big\rceil,$$

thus τ_γ intersects both α_γ and λ_γ at least $n\Big\lceil L_0/\ell_t(\gamma) + 1\Big\rceil$ times.

Choosing a lift $\widetilde{\gamma}$ of γ to $\tilde{X}_t = \mathbf{H}$, let $\widetilde{U}$ be the standard collar neighborhood of $\widetilde{\gamma}$. $\widetilde{U}$ contains infinitely many lifts of τ_γ and η. We can choose lifts $\widetilde{\alpha}$ and $\widetilde{\lambda}$ of α_γ and λ_γ in $\widetilde{U}$ so that they intersect together the same $n\lceil L_0/\ell_t(\gamma) + 1\rceil$ lifts of τ_γ. The length of τ is at most ϵ_h, so the distance between the intersection points of $\widetilde{\lambda}$ and $\widetilde{\alpha}$ with a lift of τ_γ is bounded by ϵ_h.

Since the distance between any two consecutive lifts of η is $\ell_t(\gamma)$ and any lift of τ_γ is strictly sandwiched between two consecutive lifts of η, taking every $\lceil \frac{L_0}{\ell_t(\gamma)} + 1\rceil$ lift of τ_γ insures that the distance between intersections of $\widetilde{\lambda}$ with these lifts is at least

$$\left\lceil \frac{L_0}{\ell_t(\gamma)} \right\rceil \ell_t(\gamma) \geq L_0.$$

Then $\widetilde{\alpha}$ also intersects these n lifts of τ_γ and we can conclude that α is (n, L_0)–horizontal. As for γ, it is (n, L_0)–horizontal since its lift $\widetilde{\gamma}$ intersects all the lifts of τ_γ that $\widetilde{\lambda}$ intersects and the distance between the intersection points is also bounded by ϵ_h. □

In the following, we will show that if a curve is (n, L)–horizontal at a point on a Thurston geodesic then its length essentially grows exponentially along the geodesic. Let us now recall the notion of horizontal and vertical components of a simple closed curve that intersects a leaf of $\lambda_{\mathbf{g}}$, that were introduced in [DLRT16].

Definition 4.7. Let $X \in \mathcal{T}(S)$ and let α be a simple closed geodesic on X. If α intersects a leaf λ of $\lambda_{\mathbf{g}}$, define V to be a shortest arc with endpoints on λ that, together with an arc H of λ, form a curve homotopic to α. Thus V and H meet orthogonally and α passes through the midpoints of both of these arcs. If α is a leaf of $\lambda_{\mathbf{g}}$, then we set $H = \alpha$ and let V be the empty set.

Define $h_X(\alpha)$ and $v_X(\alpha)$ to be the lengths of H and V respectively.

Proposition 4.8. *Suppose α is (n_s, L_s)–horizontal for $s \in I$ with $L_s \geq L_0$, and $n_s \geq n_0$. Then for any $t \geq s$,*

$$\ell_t(\alpha) \geq e^{t-s} \min\left\{\frac{1}{2}\ell_s(\alpha), \frac{L_s}{10}\right\}.$$

In particular,

$$\ell_t(\alpha) \geq w_{\epsilon_h} e^{t-s}.$$

Proof. Let α be an (n_s, L_s)–horizontal curve at X_s and $\widetilde{\alpha}_s$ a horizontal lift of α to $\widetilde{X}_s \cong \mathbf{H}$. That is, there is a lift $\widetilde{l}$ of a leaf of $\lambda_{\mathbf{g}}$, n_s–lifts $\{\widetilde{\gamma}_i\}$ of an ϵ_h–short curve γ, such that $d_{\mathbf{H}}(p_i, q_i) \leq \epsilon_h$ and $d_{\mathbf{H}}(p_i, p_{i+1}) \geq L_s$, where p_i and q_i are respectively the intersection of $\widetilde{\gamma}_i$ with $\widetilde{\lambda}$ and $\widetilde{\alpha}_s$.

Let $t \geq s$, and let $f_t \colon X_s \to X_t$ be an e^{t-s}–Lipschitz map, and $\widetilde{f}_t \colon \widetilde{X}_s \to \widetilde{X}_t$ be a lift of f_t. Recall that $\widetilde{f}_t$ is chosen to preserve $\widetilde{\lambda}$, and it stretches it by a factor of e^{t-s}. Let ϕ_s be the hyperbolic isometry preserving $\widetilde{\alpha}_s$ with translation length $\ell_s(\alpha)$, oriented in the direction of $\overrightarrow{q_1 q_{n_s}}$, and let ϕ_t be be such that $\phi_t \widetilde{f}_t = \widetilde{f}_t \phi_s$. That is, the axis of ϕ_t is the geodesic representative $\widetilde{\alpha}_t$ of $\widetilde{f}(\widetilde{\alpha}_s)$. In the proof of Theorem 4.4 it was shown that $\widetilde{\alpha}_t$ stays ϵ_h–close to $\widetilde{\lambda}$ from $\widetilde{f}_t(p_l)$ to $\widetilde{f}_t(p_r)$ where $r - l \geq n_s - 3$. Let $[b_t, b_t'] \subset \widetilde{\lambda}$ be the largest segment with $d_{\mathbf{H}}(b_t, \widetilde{\alpha}_t) = d_{\mathbf{H}}(b_t', \widetilde{\alpha}_t) = \epsilon_h$. Denote a_t and a_t' the feet of the perpendiculars from b_t and b_t' to $\widetilde{\alpha}_t$. We have

$$d_{\mathbf{H}}(a_t, a_t') \geq d_{\mathbf{H}}(b_t, b_t') - 2\epsilon_h \geq e^{t-s} L_s(n_s - 4) - 2\epsilon_h \geq e^{t-s} L_s(n_s - 5) \tag{4.6}$$

Let k_t be the largest number such that $\phi_t^{k_t}(a_t) \in [a_t, a_t']$. That is, $k_t = \left\lfloor \frac{d_{\mathbf{H}}(a_t, a_t')}{\ell_t(\alpha)} \right\rfloor$. There are several cases to consider.

First suppose that $k_t < 5$. Then by 4.6,

$$5\ell_t(\alpha) \geq d_{\mathbf{H}}(a_t, a_t') \geq e^{t-s} L_s(n_s - 5)$$

which implies

$$\ell_t(\alpha) \geq \frac{L_s e^{t-s}}{5}. \tag{4.7}$$

Now suppose that $k_t \geq 5$. Then by Lemma 3.2, either λ and α intersect, or α is a closed leaf of $\lambda_{\mathbf{g}}$. In the later case the length of α grows exponentially. In the former case, we will estimate the horizontal and vertical components of α with respect to the leaf λ on X_t. We will write h_t and v_t instead of $h_{X_t}(\alpha)$ and $v_{X_t}(\alpha)$.

By a basic hyperbolic geometry computation,

$$\frac{\sinh(v_s/2)}{\sinh(\ell_s(\alpha)/2)} \leq \frac{\sinh(\epsilon_h)}{\sinh(d_{\mathbf{H}}(a_s, a_s')/2)} \leq 2e^{\epsilon_h - \frac{1}{2} d_{\mathbf{H}}(a_s, a_s')}. \tag{4.8}$$

We have

$$\sinh(v_s/2) \le 2\sinh(\ell_s(\alpha)/2)e^{\epsilon_h - \frac{1}{2}d_{\mathbf{H}}(a_s,a'_s)}.$$

If $\ell_s(\alpha) \le d_{\mathbf{H}}(a_s, a'_s)$, that is, $k_s \ge 1$, then differentiating when $\ell_s(\alpha) \le 2$ and when $\ell_s(\alpha) > 2$ and using 4.6 and the fact that $L_s \ge L_0 = 9\epsilon_h$, $\epsilon_h \ge 1$ and $n_s \ge 8$ we get the following generous estimate

$$v_s \le \ell_s(\alpha)/8$$

which implies

$$h_s(\alpha) - v_s(\alpha) \ge \frac{1}{2}\ell_s(\alpha). \tag{4.9}$$

By Lemma 3.4 in [DLRT16] and definition of h_t we have $\ell_t(\alpha) \ge h_t \ge e^{t-s}(h_s - v_s)$ and so together with 4.9 this implies

$$\ell_t(\alpha) \ge \frac{1}{2}e^{t-s}\ell_s(\alpha).$$

The last case to consider is when $k_t \ge 5$ so α and λ intersect, but $k_s = 0$. That is, the length of α on X_s is bigger than the length of the segment which fellow travels λ, and on X_t the leaf λ wraps about α.

Since the endpoints of $\widetilde{\alpha}_t$ vary continuously with t, the function $d_{\mathbf{H}}(a_t, a'_t)/\ell_t(\alpha)$ is continuous, and therefore there is some $s' \in (s,t)$ such that $1 \le k_{s'} < 5$. Then, as in 4.7, we have $\ell_{s'}(\alpha) \ge \frac{1}{5}L_s e^{s'-s}$. Further, on $X_{s'}$ the horizontal and vertical components $h_{s'}$ and $v_{s'}$ of α are estimated as h_s and v_s above and so

$$h_{s'} - v_{s'} \ge \frac{1}{2}\ell_{s'}(\alpha).$$

Applying again Lemma 3.4 of [DLRT16] gives

$$\ell_t(\alpha) \ge \frac{1}{2}e^{t-s'}\ell_{s'}(\alpha) \ge \frac{L_s}{10}e^{t-s}.$$

Hence we have shown that

$$\ell_t(\alpha) \ge e^{t-s}\min\left\{\frac{1}{2}\ell_s(\alpha), \frac{1}{10}L_s\right\}.$$

The last inequality in the statement follows from the first one and the fact that $\ell_s(\alpha)$ is at least $2w_{\epsilon_h}$ since α intersects an ϵ_h– short curve, and that $w_{\epsilon_h} \le \epsilon_h \le L_s/10$. □

4.2. **Weakly horizontal.** We now define the notion of weakly horizontal.

Definition 4.9 (Weakly Horizontal)**.** Let $\mathbf{g}\colon I \to \mathcal{T}(S)$ be a Thurston geodesic. We say a curve α is *weakly horizontal* on $X_t = \mathbf{g}(t)$ if there is an ϵ_w–short curve γ on X_t intersecting α and $\lambda_{\mathbf{g}}$ such that inside the standard collar $U_X(\gamma)$ of γ, there is an arc of $\alpha \cap U_X(\gamma)$ and an arc of $\lambda \cap U_X(\gamma)$ that are disjoint.

For convenience, we will also call γ an *anchor curve* for α. Note that when α is weakly horizontal, then $d_\gamma(\alpha, \lambda_{\mathbf{g}}) \leq 3$.

We also call any ϵ_w–short curve that intersects $\lambda_{\mathbf{g}}$ a *pre-horizontal* curve. Note that the anchor curve of a weakly-horizontal curve is pre-horizontal.

The following theorem tells us that a weakly horizontal curve and its anchor curve both become strongly horizontal as soon as the length of the anchor curve grows.

Theorem 4.10. *Let $\mathbf{g} : I \to \mathcal{T}(S)$ be a Thurston geodesic. Suppose α is weakly horizontal on X_s with anchor curve γ. Let $t > s$ be the first time that $\ell_t(\gamma) = 1$. Then α and γ are both (n_0, L_0)–horizontal on X_t.*

Proof. Let λ be a leaf of $\lambda_{\mathbf{g}}$ crossing γ. For any $u \in [s, t]$ denote $U_u(\gamma) := U_{X_u}(\gamma)$ the standard collar neighborhood of γ in X_u, and let $\mathcal{L}_u$ be the X_u–length of a longest arc of $\lambda_{\mathbf{g}} \cap U_u(\gamma)$.

Let ϵ be the length $\ell_s(\gamma)$ and recall that by the definition of weakly horizontal $\epsilon \leq \epsilon_w$. By Proposition 3.13, we have $\mathcal{L}_t \geq \frac{1}{2} e^{t-s} \mathcal{L}_s$. Also since $\ell_t(\gamma) = 1$ and $\ell_s(\gamma) = \epsilon$, from Theorem 2.7 (2) we have that $e^{t-s} \geq \frac{1}{\epsilon}$.

Further, the length $\mathcal{L}_s$ is at least the width of $U_s(\gamma)$, so $\mathcal{L}_s \geq 2w_\epsilon$. Thus we obtain

$$\mathcal{L}_t \geq \frac{1}{\epsilon} w_\epsilon.$$

Again since $\ell_t(\gamma) = 1$, letting τ be a shortest curve on X_t transverse to γ from Lemma 2.6 we have that

$$d_\gamma(\tau, \lambda) \geq \mathcal{L}_t - A.$$

Putting the above two inequalities together and using the fact that $\epsilon \leq \epsilon_w$ and the property of ϵ_w from part (6) of Lemma 4.3, we obtain

$$d_\gamma(\tau, \lambda) \geq \frac{1}{\epsilon} w_\epsilon - A \geq n_0 \left\lceil L_0 + 1 \right\rceil + 5.$$

The above inequality, the fact that $d_\gamma(\alpha, \lambda) \leq 3$ and Proposition 4.6 imply that both γ and α are (n_0, L_0)–horizontal on X_t, as desired. □

Definition 4.11 (Horizontal)**.** We say a curve α is *horizontal* on X if it is either weakly horizontal or (n_0, L_0)–horizontal on X. Similarly, we will say a multicurve σ is horizontal if each component of σ is horizontal.

The following proposition gives us a uniform upper bound for the diameter of the shadow of any Thurston geodesic to the curve graph of a subsurface if the boundary of the subsurface is horizontal on the initial point of the segment.

Proposition 4.12. *Let* $\mathbf{g} : [a, b] \to \mathcal{T}(S)$ *be a Thurston geodesic and let* $R \subseteq S$ *be a subsurface. If* ∂R *is horizontal on* X_a*, then* $\operatorname{diam}_R(\mathbf{g}([a, b])) = O(1)$.

Proof. If the boundary ∂R is horizontal at X_a, then it is either weakly horizontal or (n_0, L_0)–horizontal. Then we choose a time $s \in [a, b]$ as follows:

- If $\alpha \in \partial R$ is weakly horizontal on X_a with anchor curve γ, then let $s \in [a, b]$ be, if exists, the first time such that $\ell_s(\gamma) = 1$. If no such s exists, let $s = b$.
- If α is (n_0, L_0)–horizontal on X_a, let $s = a$.

Now note that along the interval $[a, s]$ the curve γ remains 1–short so

$$\operatorname{diam}_R(\mathbf{g}([a, s])) = O(1).$$

So we only need to show that $\operatorname{diam}_R(\mathbf{g}([s, b])) = O(1)$; we can assume $b - s \geq s_0$, otherwise we are done. First note that when s is chosen as the first bullet by Theorem 4.10, α is (n_0, L_0)–horizontal on X_s and when s is chosen as the second bullet α is obviously (n_0, L_0)–horizontal on X_s. So in any case α is obviously (n_0, L_0)–horizontal on X_s. Now let γ be an ϵ_B–short curve on X_s crossing R. By Proposition 4.8, for all t that $t - s \geq s_0$, $\ell_t(\alpha) \geq e^{t-s} w_{\epsilon_h}$, and $\ell_t(\gamma) \leq e^{t-s} \epsilon_B$. Thus, by Proposition 3.10,

$$d_R(X_s, X_t) \leq d_R(\gamma, X_t) \prec \log \frac{\ell_b(\gamma)}{\ell_t(\alpha)} \leq \log \frac{\epsilon_B}{w_{\epsilon_h}}.$$

This is true for all t with $t - s \geq s_0$, so $\operatorname{diam}_R(\mathbf{g}([s, b])) = O(1)$ as desired. □

4.3. **Corridors.** In this section, we recall a tool for constructing horizontal curves on hyperbolic surfaces assuming certain conditions on intersection numbers between curves. This is the notion of corridors on hyperbolic surfaces as was introduced in [LRT15, §5.1].

Definition 4.13 (Corridor)**.** Let $X \in \mathcal{T}(S)$ and let (τ, ω) be an ordered pair of simple geodesic segments. An (n, L)–corridor generated by (τ, ω) is the image Q of a map $f : [0, a] \times [0, b] \to X$ such that

- f is an embedding on $(0, a) \times (0, b)$.
- Edges $[0, a] \times \{0\}$ and $[0, a] \times \{b\}$ are mapped to a segment of ω and the interior of Q does not meet ω.
- There is a partition $0 = t_0 < \ldots < t_{n+1} = a$ such that each $\{t_i\} \times [0, b]$ is mapped to a segment of τ, for all $i = 0, \ldots, n + 1$.
- Segments $f([t_i, t_{i+1}]) \times \{0\})$ and $f([t_i, t_{i+1}] \times \{b\})$ have lengths at least L.

We will call $f(\{0\} \times [0, b])$ and $f(\{a\} \times [0, b])$ the *vertical* sides of Q, and $f([0, a] \times \{0\})$ and $f([0, a] \times \{b\})$ the *horizontal* sides of Q.

The following lemma guarantees the existence of a corridor generated by two curves given that the curves intersect many times inside a subsurface.

Lemma 4.14. *Given $n \in \mathbb{N}$ and $L > 0$, let*

$$C(n, L) := (6|\chi(S)| + 1)n \left\lceil \frac{L}{2w_{\epsilon_h}} \right\rceil + 3|\chi(S)| + 1.$$

Then given $X \in \mathcal{T}(S)$ and $R \subseteq S$ subsurface, let γ be an ϵ_h–short curve intersecting R and let τ be a component of $\gamma \cap R$. If ω is a simple geodesic segment in R with $\mathrm{i}_R(\tau, \omega) \geq C(n, L)$, then there exists an (n, L)–corridor Q in R generated by (τ, ω). Moreover, there exists a curve $\alpha \in \mathcal{C}(R)$ such that $\mathrm{i}_R(\tau, \alpha) \leq C(n, L)$. Moreover, if β is any curve or arc that does not cross the interior of Q from vertical to vertical sides, then $\mathrm{i}_R(\beta, \alpha) \leq \mathrm{i}_R(\beta, \omega) + 1$.

Proof. With the exception of the last statement, the proof of the lemma is identical to that of [LRT15, Lemma 5.4]. The only adjustment is the replacement of the constant $\delta_B = 2w_{\epsilon_B}$ by $2w_{\epsilon_h}$. Since τ and ω both lie in R, the corridor they generate will also lie in R. Finally, the existence of the curve α can be found in the proof of [LRT15, Proposition 5.5]. □

The following lemma will be useful later to determine when a curve is horizontal. The proof is immediate by lifting the picture to the universal cover.

Lemma 4.15. *Suppose $\mathbf{g} \colon I \to \mathcal{T}(S)$ is a Thurston geodesic. For $t \in I$, suppose there is an (n_0, L_0)–corridor Q in X_t whose vertical sides belong to an ϵ_h–short curve γ. Then the following statements hold.*

- *If a leaf of $\lambda_{\mathbf{g}}$ crosses the interior of Q from vertical to vertical sides, and the horizontal sides of Q belong to a curve α, then α is (n_0, L_0)–horizontal on X_t with anchor curve γ.*
- *If the horizontal sides of Q belong to a leaf of $\lambda_{\mathbf{g}}$ or a horizontal segment of a curve, then any curve that crosses the interior of Q from vertical to vertical sides is (n_0, L_0)–horizontal on X_t with anchor curve γ.*

5. Active intervals

Let $\mathbf{g} \colon I \to \mathcal{T}(S)$ be a Thurston geodesic and let $R \subseteq S$ be a subsurface. In this section, we introduce the notion of *an active interval* $J_R \subseteq I$ of R along $\mathbf{g}$ with the property that for all $t \in J_R$, the length of ∂R is uniformly bounded (independently of $\mathbf{g}, t$ and R) and the restriction of $X_t = \mathbf{g}(t)$ to R resembles a geodesic in the Teichmüller space of R. In particular, we will show that, similar to the main result of [LRT15], $\mathbf{g}$ restricted to J_R projects to a reparametrized quasi-geodesic in $\mathcal{C}(R)$. Furthermore, outside of J_R, $\mathbf{g}$ does not make much progress in $\mathcal{C}(R)$.

These statements will be made precise within this section. Theorem 1.1 in the introduction which is the main result of the paper is a combination of Theorem 5.7 and Corollary 5.10 in this section.

Let ϵ_h, ϵ_w, n_0 and L_0 be the constants that we have fixed in the previous section (Lemma 4.3). Recall that ∂R is *horizontal* on X_t if a component of ∂R is either weakly or (n_0, L_0)–horizontal on X_t (see Definition 4.1). We first show the following trichotomy about ∂R along Thurston geodesics.

Proposition 5.1. *There exists a constant $\rho > 0$ such that, for any Thurston geodesic $\mathbf{g} : I \to \mathcal{T}(S)$, any subsurface $R \subseteq S$ intersecting $\lambda_{\mathbf{g}}$ and any $t \in I$, one of the following three statements holds:*

(T1) $\ell_t(\partial R) = \max_{\alpha \in \partial R} \ell_t(\alpha) \leq \rho$.

(T2) ∂R *is horizontal on* X_t.

(T3) $d_R(X_t, \lambda_{\mathbf{g}}) \leq 1$.

Proof. To start the proof recall that by Lemma 3.4, there exists a constant $L_{\epsilon_w} > 0$ such that any geodesic of length L_{ϵ_w} on an ϵ_w–thick subsurface $Y \subseteq S$ must intersect the short marking μ_Y on Y. Define the number

$$L = (4n_0\, \xi(S) + 2) \max \{L_{\epsilon_w}, L_0\}.$$

Also define the numbers

$$\delta = \frac{2\pi|\chi(S)|}{L}, \qquad K = \frac{|\chi(S)|}{\sinh^2(\delta/2)}, \qquad \rho = 2KL.$$

Now let λ be a leaf of $\lambda_{\mathbf{g}}$ intersecting R essentially. Suppose first that a component $\alpha \subset \partial R$ intersects an ϵ_w–short curve γ. If λ does not intersect γ, then $d_R(X_t, \lambda_{\mathbf{g}}) \leq d_R(\gamma, \lambda) \leq 1$, so (T3) holds. Now suppose that λ intersects γ. If α is not weakly horizontal, then inside the standard collar neighborhood $U(\gamma)$ of γ, every arc of $\alpha \cap U(\gamma)$ must intersect every arc of $\lambda \cap U(\gamma)$. In this case, we can find an arc of $\omega \subseteq \lambda \cap R$, such that ω has at least one endpoint on an arc of $\alpha \cap U(\gamma)$, and then homotope ω (relative to ∂R) to be disjoint from γ. This shows $d_R(X_t, \lambda_{\mathbf{g}}) \leq 1$ so again (T3) holds.

So we may assume that ∂R does not intersect any ϵ_w–short curve. Let α be a component of ∂R with $\ell_t(\alpha) > \rho$. Then we can find an ϵ_w–thick subsurface Y containing α such that either $\partial Y = \emptyset$ or $\ell_t(\partial Y) \leq \epsilon_w$. A short marking μ_Y on Y is a subset of the short marking on X; in particular, every curve of μ_Y is ϵ_h–short. Since $\ell_t(\alpha) > \rho$, by Lemma 3.9, there exists an L–wide quadrilateral Q in R, whose top edge is a segment of α and bottom edge belongs to ∂R. By subdividing Q, we can find a collection of quadrilaterals $Q_1, Q_2, \ldots, Q_n \subseteq R$ with the following properties:

- $n = 4n_0\,\xi(S) + 2$
- $Q = \bigcup_{i=1}^{n} Q_i$, and the indices are arranged so that the right side of Q_i is the left side of Q_{i+1} for all $i = 1, \ldots, n-1$.
- The top edge of Q_i is a segment of α with length at least $\max\{L_{\epsilon_w}, L_0\}$.

By the third property above, for each i, there exists a curve γ_i in μ_Y that intersects the top edge of Q_i. Since all curves in μ_Y are ϵ_h short and $L_0 > \epsilon_h$, if the curve γ_i, for $i = 2, \ldots, n-1$, intersects the top edge of Q, then it must cross the quadrilateral Q_i, and exit the bottom edge of Q_j for some $j \in \{i-1, i, i+1\}$. That is, $\gamma_i \cap Q$ has a component δ_i which goes from the top edge of Q_i to the bottom edge of Q_j, $j \in \{i-1, i, i+1\}$. Then, since $n = 4n_0\,\xi(S)+2$ and there are at most $2|\xi(S)|$ curves in μ_Y, at least $2n_0$ of the δ_i's belong to the same curve γ in μ_Y. By skipping every other component if necessary, we can find n_0 arcs in Q belonging to γ with pairwise distance at least L_0. If λ does not cross Q from side to side, then there is an arc of $\lambda \cap R$ that misses an arc of some $\gamma_i \in \mu_Y$ with end point on ∂R, thus $d_R(X_t, \lambda_{\mathbf{g}}) \leq 1$, which means that (T3) holds. On the other hand, if λ crosses Q from side to side, then α is (n_0, L_0)–horizontal with anchor curve γ and hence (T2) holds. This finishes the proof of the trichotomy. □

Let $\rho > 0$ be the constant from Proposition 5.1.

In preparation for the next section, we now prove two technical propositions about when curves are horizontal in a subsurface R.

Proposition 5.2. *There exists an integer $N_0 \geq n_0$ such that the following statement holds. Let $\mathbf{g}\colon I \to \mathcal{T}(S)$ be a Thurston geodesic, $R \subseteq S$ a subsurface, and $X = \mathbf{g}(t)$ a point with $\ell_X(\partial R) \leq \rho$. If a curve $\alpha \in \mathcal{C}(R)$ is (n, L)–horizontal on X, where $n \geq N_0$, $L \geq L_0$, then any curve $\beta \in \mathcal{C}(R)$ that intersects α at most twice is either (n_0, L_0)–horizontal on X or $d_R(X, \beta) = O(1)$.*

Proof. Let $C = C(n_0, L_0)$ be the constant from Lemma 4.14 and let $N_0 = \frac{\epsilon_h}{2w_\rho} C$. Assume that α is (n, L)–horizontal on X, where $n \geq N_0$ and $L \geq L_0$.

Let γ be an anchor curve for α. Since α lies in R, $\gamma \cap R$ is non-empty. By definition, γ is ϵ_h–short, so $\mathrm{i}_R(X, \gamma) = O(1)$. Let $\widetilde{\alpha}$ be an (n, L)–horizontal lift of α. Let $q_1, \ldots, q_n$ be the points along $\widetilde{\alpha}$ given by the definition of (n, L)–horizontal. Let $\widetilde{\omega}$ be the innermost segment of $[q_1, q_n)$ that intersects exactly N_0–lifts of $\widetilde{\gamma}$ and let ω be the projection of $\widetilde{\omega}$ to X. If $\omega = \alpha$, then $\mathrm{i}_R(\gamma, \alpha) \leq N_0$. In this case, by the triangle inequality and (2.4), we obtain

$$\begin{aligned} d_R(X, \beta) &\overset{+}{\prec} d_R(X, \gamma) + d_R(\gamma, \alpha) + d_R(\alpha, \beta) \\ &\overset{*}{\prec} \operatorname{Log} \mathrm{i}_R(X, \gamma) + \operatorname{Log} \mathrm{i}_R(\gamma, \alpha) + d_R(\alpha, \beta) \overset{*}{\prec} \operatorname{Log} N_0, \end{aligned}$$

which is one of the conclusions of the proposition.

Now suppose ω is a proper arc of α. By construction, $\mathrm{i}_R(\gamma, \omega) = N_0$. Moreover, $\ell_X(\partial R) \le \rho$, so any arc in $\gamma \cap R$ is at least $2w_\rho$ long. Thus, $\gamma \cap R$ has at most $\epsilon_h/2w_\rho$ arcs, and hence there is at least one arc τ with $\mathrm{i}_R(\tau, \omega) \ge C$. By Lemma 4.14, τ and ω generates a (n_0, L_0)–corridor Q in R. If β crosses the interior of Q from vertical to vertical side, then β is (n_0, L_0)–horizontal with anchor curve γ by Lemma 4.15. Otherwise, let $\alpha' \in \mathcal{C}(R)$ be the curve guaranteed by Lemma 4.14. The curve α' intersects β at most twice and $\mathrm{i}_R(\tau, \alpha') \le \mathrm{i}_R(\tau, \omega) \le N_0$. Then by the triangle inequality and (2.4) we obtain

$$d_R(X_s, \beta) \overset{+}{\prec} d_R(X_s, \alpha') + d_R(\alpha', \beta) \overset{*}{\prec} \operatorname{Log} \mathrm{i}_R(\tau, \alpha') \le \log N_0. \qquad \square$$

The following proposition gives us a pre-horizontal or strongly horizontal curve of bounded length in a subsurface R along a Thurston geodesic, provided that ∂R is sufficiently short and not horizontal. It is a generalization of [LRT15, Proposition 4.5] to subsurfaces.

Proposition 5.3. *There exists a constant $N_1 \in \mathbb{N}$ such that the following statement holds. Let $\mathbf{g}\colon I \to \mathcal{T}(S)$ be a Thurston geodesic and let $X = \mathbf{g}(t)$ be a point such that $\ell_X(\partial R) \le \rho$ and ∂R is not horizontal on X. If $\mathrm{i}_R(X, \lambda_{\mathbf{g}}) \ge N_1$, then there exists a curve $\alpha \in \mathcal{C}(R)$ of uniformly bounded length which is either pre-horizontal or (n_0, L_0)–horizontal on X.*

Proof. Let $K \ge 1$ be the additive error from Proposition 4.2 that corresponds to the maximum of the fellow traveling distances in the statements of Lemma 5.8 and Lemma 5.9 of [LRT15]. Then, let $N = n_0 \left\lceil \frac{L_0}{2w_{\epsilon_B}} \right\rceil + K$ and $N_1 = 5N$. We also fix a small constant $\epsilon_\rho \le \epsilon_w$ so that $w_{\epsilon_\rho} > \rho$. Note that, by our choice ϵ_ρ, any ϵ_ρ–short curve of X is disjoint from ∂R.

If there is an ϵ_ρ–short curve in R that intersects $\lambda_{\mathbf{g}}$, then by definition this curve is pre-horizontal curve and we are done. Thus, we may proceed with the case that no such curve exists.

Let P be the set of ϵ_B–short curves on X, and suppose there exists a leaf λ of $\pi_R(\lambda_{\mathbf{g}})$ that intersects some element of $\pi_R(P)$ at least N_1 times. Take a segment $\overline{\lambda} \subseteq \lambda$ and an arc or curve $\tau \in \pi_R(P)$ such that $\overline{\lambda}$ has endpoints on τ, $\mathrm{i}_R(\tau, \overline{\lambda}) = N_1$, and $\mathrm{i}_R(\tau, \overline{\lambda})$ is maximal among all other elements in $\pi_R(P)$. Note that $\overline{\lambda}$ does not intersect any ϵ_ρ–short curves, so its length is uniformly bounded. If $\overline{\lambda}$ closes up to a curve, then we have found our (n_0, L_0)–horizontal curve. Otherwise, we will find such a curve by applying a surgery between τ and $\overline{\lambda}$. This procedure is involved and will depend on the intersection pattern between τ and $\overline{\lambda}$. The conclusion will follow if either of the following three cases occur.

Case (1): There is a subsegment ω of $\overline{\lambda}$ that hits τ on opposite sides with $\mathrm{i}_R(\tau,\omega) \geq N$, and the endpoints of ω can be joined by a segment of τ that is disjoint from the interior of ω.

In this case, let α be the curve obtained from the concatenation of ω and a segment of τ disjoint from the interior of ω. Then α is (n_0, L_0)–horizontal with anchor curve γ. To see this, note that by Lemma 5.8 of [LRT15] $\widetilde{\alpha}$ a lift of α stays in a bounded neighborhood of $\widetilde{\omega}$ and $\widetilde{\eta}$ lifts of ω and η. Then since γ and ω intersect at least N_1 times ($\tau \subseteq \gamma$) and $N_1 > N$ at least N lifts of γ intersect $\widetilde{\omega}$. Moreover, by the choice of N, skipping some of the lifts of γ, we may choose at least $n_0 + K$ lifts of γ so that the distance between the intersection points between any two consecutive lifts and $\widetilde{\omega}$ is at least L_0. But then by Proposition 4.2 there are at least n_0 lifts of γ that intersect both $\widetilde{\omega}$ and $\widetilde{\alpha}$. This shows that α is (n_0, L_0)–horizontal (see the definition of horizontal).

Case (2): There is a subsegment ω of $\overline{\lambda}$ and a closed curve β disjoint from ω, such that $\mathrm{i}_R(\tau,\omega) \geq N$ and $\ell_X(\beta) \geq 2w_{\epsilon_B}$, and the endpoints of ω are close to the same point on β. Furthermore, each endpoint of ω can be joined to a nearby point on β by a segment of τ that is disjoint from β and the interior of ω.

In this case let α be the curve obtained by closing up ω with β and one or two subsegments of τ. By Lemma 5.9 of [LRT15] $\widetilde{\omega}$ a lift of ω stays in a bounded neighborhood of $\widetilde{\alpha}^*$ a lift of α^* (the geodesic representative of α) and two lifts of β. Moreover, since ω and γ intersect at least $5N$ times, the lifts of γ intersect either $\widetilde{\alpha}^*$ or a lift of β at least N times. To see this, note that the lifts of γ do not intersect the lifts of subsegments of τ ($\tau \subseteq \gamma$), so the lift of ω and two lifts of β are intersected by the $5N$ lifts of γ.

In the former situation applying Proposition 4.2 and by the choice of N we can see that α is (n_0, L_0)–horizontal with anchor curve γ. In the later situation β is (n_0, L_0)–horizontal by Proposition 4.2 with anchor curve γ. In fact, in this case ω is twisting around β.

Case (3): There is a subsegment ω of $\overline{\lambda}$ and two closed curves β and β' disjoint from ω, such that $\mathrm{i}_R(\tau,\omega) \geq N$, $\ell_X(\beta) \geq 2w_{\epsilon_B}$ and $\ell_X(\beta') \geq 2w_{\epsilon_B}$, and the two endpoints of ω are close to β and β'. Furthermore, τ has a segment joining one endpoint of ω to β and another segment joining the other endpoint of ω to β', such that both segments are disjoint from β, β' and the interior of ω.

In this case, let α be the curve obtained by gluing two copies of ω, β, β' and a few subsegments of τ. By Lemma 5.9 of [LRT15] $\widetilde{\omega}$ a lift of ω stays in a bounded neighborhood of $\widetilde{\alpha}^*$ a lift of α^*, $\widetilde{\beta}$ a lift of β and $\widetilde{\beta}'$ a lift of β'. Then similar to case (2) at least N lifts of γ intersects $\widetilde{\alpha}$, $\widetilde{\beta}$ or $\widetilde{\beta}'$. Thus again appealing to Proposition 4.2 and by the choice of N either α, β or β' is (n_0, L_0)–horizontal.

We now show that at least one of the above three cases happens.

Let p and q be the endpoints of $\overline{\lambda}$. Consider the intersection points between $\overline{\lambda}$ and τ ordered along τ. If each p and q is adjacent to two intersection points, then this proof is contained in [LRT15, Proposition 5.10]. In the following, we will consider the case that at least one of p or q is not adjacent to two intersection points. Note that when this happens, τ must be an arc in R and any component of ∂R crossing τ has length at least $2w_{\epsilon_B}$.

First suppose that neither p nor q is adjacent to two intersection points (see Figure 2). In this case, the intersection points along τ line up from p to q. Let β and β' be the two (possibly non-distinct) components of ∂R containing the endpoints of τ. By relabeling if necessary, assume β is closest to p and β' is closest to q. Let $\alpha \in \mathcal{C}(R)$ be the curve obtained by gluing two copies of $\overline{\lambda}$, β, β', and the subsegments of τ from p to β and from q to β'. Similar to Case (3), either α, β, or β' is (n_0, L_0)–horizontal. However, since by assumption ∂R is not horizontal, it must be that α is the desired (n_0, L_0)–horizontal curve.

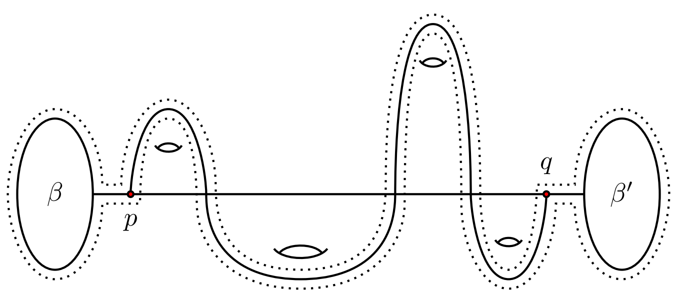

FIGURE 2. Neither p nor q is adjacent to two intersection points of $\overline{\lambda}$ and τ.

Now assume exactly one of p or q is adjacent to two intersection points. Without the loss of generality, assume p is adjacent to p_1 and p_2. In this case, the intersection points along τ ends with q at one end. After relabeling if necessary, assume $\overline{\lambda}$ passes from p to p_1 and then to p_2. (We include the possibility that $p_2 = q$.) Let ω_1 be the subsegment of $\overline{\lambda}$ from p to p_1 and ω_2 be the subsegment from p_1 to p_2. Among the 3 segments ω_1, ω_2, and $\omega_1 \cup \omega_2$, there is at least one that hits τ on opposite sides. Pick the one that has maximal intersection number with τ and call it η. Let β be the closed curve obtained from closing up η with a segment of τ. Since β stays close to η by Lemma 5.8 of [LRT15]] and was constructed from a segment with endpoints on τ, $\ell_X(\beta) \geq 2w_{\epsilon_B}$. If $\mathrm{i}_R(\tau, \eta) \geq N$, then we are in case (1) with $\alpha = \beta$. If $\mathrm{i}_R(\tau, \eta) < N$, then there is a segment ω of $\overline{\lambda}$ such that $\mathrm{i}_R(\tau, \omega) \geq N$, ω shares one endpoint with η, and the other endpoint of ω is either p or q. If the other endpoint of ω is p, then let $\omega = \omega_1$. In this case, there are segments of τ connecting the endpoints of ω to β which are disjoint from the interiors of β and ω, so we can do a surgery as in case (2) to obtain the desired curve α. Finally, if q is an endpoint of ω, then let β' be the component of ∂R connected to q by a segment in τ which is disjoint from the interior of ω and β'. There is also a segment of τ connecting the other endpoint of ω with β which is disjoint from their interior. Now we are in case (3), where the curve α is obtained by gluing two copies of $\overline{\lambda}$, β, β', and these subsegments of τ. In this case, since ∂R is not horizontal, either α or β is (n_0, L_0)–horizontal. In either case, we have found our (n_0, L_0)–horizontal curve in R.

Finally, the (n_0, L_0)–horizontal curve α that we have constructed is a concatenation of a segment of $\overline{\lambda}$ (possibly traversed twice), at most 4 segments of τ, and at most 2 components of ∂R. All of these arcs and curves involved have uniformly bounded length, whence so does α. This finishes the proof of the proposition. □

5.1. **Active interval.** In this subsection we introduce the notion of an active interval which is central to this paper. Then we will prove some results about the behavior of Thurston geodesics over active intervals.

Lemma 5.4 (Constant B)**.** *Let $C = C(n_0, L_0)$ be the constant from Lemma 4.14, s_0 the constant from Theorem 4.4, and $s_1 = \log \frac{\rho}{2w_{\epsilon_h}}$. Then there is a constant $B > 3$ such that.*

- *For all $R \subseteq S$ and $\alpha, \beta \in \mathcal{C}(R)$, if $\mathrm{i}_R(\alpha, \beta) \le C$, then $d_R(\alpha, \beta) \le B$.*
- *For all $X, Y \in \mathcal{T}(S)$ and all $R \subseteq S$, if $d(X, Y) \le \max\{s_0, s_1\}$, then $d_R(X, Y) \le B$.*
- *If α is ρ–short on X, then $d_S(X, \alpha) \le B$.*

Proof. First note that by (2.4) $d_R(\alpha, \beta) \overset{*}{\prec} \operatorname{Log} \mathrm{i}_R(\alpha, \beta)$, thus the first bullet holds for a B that is larger than a multiple of $\operatorname{Log} C$.

For the second bullet point, recall that the shadow map $\mathcal{T}(S) \to \mathcal{C}(R)$ is coarsely-Lipschitz with Lipschitz constant that only depends on S. Thus such B exists for a fixed s_1.

Finally, since $d(X, \alpha) \overset{*}{\prec} \operatorname{Log} \mathrm{i}(\beta, \alpha)$ where β is an ϵ_B–short curve on X and $\mathrm{i}(\beta, \alpha) \le \frac{\rho}{2w_{\epsilon_B}}$, so the last bullet point holds when B is larger than a multiple of $\operatorname{Log} \frac{\rho}{2w_{\epsilon_B}}$. □

For the rest of the paper, we will fix a constant $B > 6$ as in the above lemma, and recall the constant ρ of Proposition 5.1. We establish the following statement.

Lemma 5.5. *Given a Thurston geodesic $\mathbf{g} : [a, b] \to \mathcal{T}(S)$ and a subsurface $R \subseteq S$ that intersects $\lambda_{\mathbf{g}}$. For $s \in [a, b]$, if $d_R(X_s, \lambda_{\mathbf{g}}) \ge 2B$, then for all $t \ge s$, $d_R(X_t, \lambda_{\mathbf{g}}) > 1$.*

Proof. Let P_t be the set of ϵ_B–short curves on X_t. Assume that $d_R(X_t, \lambda_{\mathbf{g}}) \leq 1$. Then by the triangle inequality (2.2) and that $B > 3$ we deduce that

$$d_R(X_s, X_t) \geq d_R(X_t, \lambda_{\mathbf{g}}) - d_R(\lambda_{\mathbf{g}}, X_s) - 2 \geq 2B - 3 > B. \tag{5.1}$$

Thus, there are $\tau \in \pi_R(P_s)$, $\omega \in \pi_R(P_t)$, and $\lambda \in \pi_R(\lambda_{\mathbf{g}})$ such that $\mathrm{i}_R(\tau, \omega) \geq C$, and λ is disjoint from ω. By Lemma 4.14, τ and ω generate an (n_0, L_0)–corridor Q, whose horizontal sides are disjoint from λ. Thus, either λ is disjoint from Q or it crosses the interior of Q from vertical to vertical sides. In the latter case, the curve $\gamma \in P_t$ containing ω is (n_0, L_0)–horizontal on X_s by Lemma 4.15. Since γ is ϵ_B–short on X_t, by Proposition 4.8, $t - s \leq \log \max\{s_0, s_1\}$, but this implies $d_R(X_s, X_t) \leq B$, contradicting Equation 5.1. On the other hand, if λ is disjoint from Q, then let α be the curve guaranteed by Lemma 4.14, which has the property that $\mathrm{i}_R(\tau, \alpha) \leq C$ and $\mathrm{i}_R(\alpha, \lambda) \leq 1$. By the triangle inequality, this yields

$$d_R(X_s, \lambda_{\mathbf{g}}) \leq d_R(\tau, \alpha) + d_R(\alpha, \lambda) + 2 \leq B + 3 < 2B,$$

contradicting the assumption. This shows $d_R(X_t, \lambda_{\mathbf{g}}) > 1$. □

We are now ready to give the definition of the *active interval* of a subsurface $R \subsetneq S$ along a Thurston geodesic $\mathbf{g}$.

Definition 5.6 (Active Interval)**.** Let $\mathbf{g}\colon [a, b] \to \mathcal{T}(S)$ be a Thurston geodesic and let $R \subseteq S$ be a subsurface that intersects $\lambda_{\mathbf{g}}$. We define the *active interval* $J_R = [c, d] \subseteq [a, b]$ of R along $\mathbf{g}$ as follows. The right endpoint of the interval is

$$d := \inf \Big\{ t \in [a, b] \colon \partial R \text{ is horizontal on } X_t \Big\},$$

and if ∂R is never horizontal along $[a, b]$, then we set $d = b$. The left endpoint of the interval is

$$c := \inf \Big\{ t \in [a, d] \colon d_R(X_t, \lambda_{\mathbf{g}}) \geq 2B \Big\},$$

and if $d_R(X_t, \lambda_{\mathbf{g}}) < 2B$ for all $t \in [a, d]$, then we set $c = d$.

Theorem 5.7. *Given a Thurston geodesic $\mathbf{g} : [a, b] \to \mathcal{T}(S)$ and a subsurface $R \subseteq S$ that intersects $\lambda_{\mathbf{g}}$, the active interval $[c, d] \subseteq [a, b]$ of R satisfies the following properties.*

(i) $\mathrm{diam}_R(\mathbf{g}([a, c]))$ *and* $\mathrm{diam}_R(\mathbf{g}([d, b]))$ *are uniformly bounded. In particular, if $d_R(X_a, X_b)$ is sufficiently large, then $[c, d]$ is a non-trivial interval.*

(ii) *If $[c, d]$ is a non-trivial interval, then $\ell_t(\partial R) \leq \rho$ for all $t \in [c, d]$.*

Proof. For all $s, t \in [a, c]$, by the choice of c and the triangle inequality (2.2) we have

$$d_R(X_s, X_t) \overset{+}{\prec} d_R(X_s, \lambda_{\mathbf{g}}) + d_R(\lambda_{\mathbf{g}}, X_t) \leq 6B,$$

so $\mathrm{diam}_R(\mathbf{g}([a,c]) = O(1)$. Moreover, if $d \leq b$, then ∂R is horizontal on X_d by definition. Then by Proposition 4.12 we have $\mathrm{diam}_R(\mathbf{g}([d,b]) = O(1)$, thus (i) is proved.

For property (ii), if $c < d$, then by our choice of c, $d_R(X_c, \lambda_{\mathbf{g}}) \geq 2B$. By Lemma 5.5 $d_R(X_t, \lambda_{\mathbf{g}}) > 1$, for all $t \in [c,d]$. Thus, by Proposition 5.1, either ∂R is horizontal on X_t or $\ell_t(\partial R) \leq \rho$. But by our choice of d, we must have $\ell_t(\partial R) \leq \rho$ for all $t \in [c,d)$. The latter bound extends to $t = d$ by continuity. □

5.2. **No backtracking along active intervals.** We now define the *balanced time of a curve in a subsurface* $R \subseteq S$ which is a time in the active interval of R which helps us to define a contraction map from $\mathcal{C}(R)$ to the shadow of a Thurston geodesic in $\mathcal{C}(R)$.

Definition 5.8 (Balanced time of a curve)**.** Let $\mathbf{g}\colon I \to \mathcal{T}(S)$ be a Thurston geodesic and let R be a subsurface intersecting $\lambda_{\mathbf{g}}$. Also, let $J_R = [c,d]$ be the active interval of R along $\mathbf{g}$. Then, for a curve $\alpha \in \mathcal{C}(R)$, the *balanced time* of α in $[c,d]$ is the time

$$t_\alpha := \inf\Big\{t \in [c,d] \colon \alpha \text{ is } (n_0, L_0)\text{–horizontal on } X_t\Big\}.$$

But, if α is never (n_0, L_0)–horizontal along $[c,d]$, then set $t_\alpha = d$.

Now we define a coarse map

$$\Pi\colon \mathcal{C}(R) \to \Upsilon_R(\mathbf{g}([c,d]))$$

as follows. For any curve $\alpha \in \mathcal{C}(R)$, let $\Pi(\alpha) = \Upsilon_R(\mathbf{g}(t_\alpha))$, where t_α is the balanced time of α along $\mathbf{g}$. When τ is an arc in R, then set $\Pi(\tau) = \Pi(\alpha)$, where α is any disjoint curve from τ in $\mathcal{C}(R)$.

Our main theorem of this section is the following.

Theorem 5.9. *Let $\mathbf{g}\colon [a,b] \to \mathcal{T}(S)$ be a Thurston geodesic and let $[c,d] \subseteq [a,b]$ be the active interval of a subsurface R along $\mathbf{g}$. Then $\Pi\colon \mathcal{C}(R) \to \Upsilon_R(\mathbf{g}([c,d]))$ is a K–Lipschitz retraction, where K depends only on S. More precisely,*

- Π *is Lipschitz: Given two curves $\alpha, \beta \in \mathcal{C}(R)$ that intersect at most twice,*
  $$d_R(X_{t_\alpha}, X_{t_\beta}) \leq K.$$
  In particular, Π is coarsely well-defined, i.e. when τ is an arc, then $\Pi(\tau)$ is independent of the choice of a curve in $\mathcal{C}(R)$ disjoint from τ.
- Π *is a retraction: For any $t \in [c,d]$ and any curve $\alpha \in \mathcal{C}(R)$ disjoint from any element of $\pi_R(X_t)$, we have $d_R(X_t, X_{t_\alpha}) \leq K$.*

Proof. All statements hold trivially if $c = d$, so we will assume $c < d$. Then by part (ii) of Theorem 5.7, $\ell_t(\partial R) \le \rho$ for all $t \in [c, d]$.

We first show that the map Π is Lipschitz. Let $\alpha, \beta \in \mathcal{C}(R)$ be two curves with $d_R(\alpha, \beta) = 1$. Without a loss of generality, we may assume that $t_\alpha < t_\beta$. Let N_0 be the constant of Proposition 5.2 and let $s \in [t_\alpha, d]$ be the first time that α is (N_0, L_0)–horizontal; if no such time exists then set $s = d$. Then since α is (n_0, L_0)–horizontal at X_t from the second part of Theorem 4.4 we can deduce that $\operatorname{diam}_R(\mathbf{g}([t_\alpha, s])) \stackrel{+}{\prec} \log(\frac{N_0}{n_0})$. So if $t_\beta \in [t_\alpha, s]$, then we are done. If $t_\beta > s$, then it is enough to show that $d_R(X_s, X_{t_\beta}) = O(1)$. By definition of balanced time, β is not (n_0, L_0)–horizontal along $[s, t_\beta)$. Now for any $t \in [s, t_\beta)$, if α remains (N_0, L_0)–horizontal on X_t , then since the length of ∂R is at most ρ on X_s and X_t and α is (N_0, L_0)–horizontal on X_s and X_t, we may apply Proposition 5.2 to get

$$d_R(X_s, X_t) \stackrel{+}{\prec} d_R(X_s, \beta) + d_R(\beta, X_t) = O(1).$$

But, if α is not (N_0, L_0)–horizontal on X_t, then again by the second part of Theorem 4.4, $t - s \stackrel{+}{\prec} 0$. In either case, we have our desired bound for $d_R(X_t, X_s)$.

Now we show Π is a retraction. Let N_1 be the constant of Proposition 5.3. For any $t \in [c, d]$, let P_t be the set of ϵ_B–short curves on X_t. Note that there is a curve α of bounded length and disjoint from an element of $\pi_R(P_t)$; if $\pi_R(P_t)$ contains a curve, then let α be the curve. Otherwise, take an arc $\tau \in \pi_R(P_t)$ and do a surgery with ∂R to get a curve α disjoint from τ of length at most $2(\rho + \epsilon_B)$. Since Π is Lipschitz, to show that Π is a retraction it suffices to show that $d_R(X_t, X_{t_\alpha}) = O(1)$.

If $t_\alpha < t$, then in particular $t_\alpha < d$. By definition of the balanced time, α is (n_0, L_0)–horizontal on X_{t_α}, so α crosses an ϵ_h–short curve on X_{t_α} and hence $\ell_{t_\alpha}(\alpha) \ge 2w_{\epsilon_h}$. Then, since α has length at most $2(\rho + \epsilon_B)$ on X_t, by Proposition 4.8, $t - t_\alpha \le \log \frac{\rho + \epsilon_B}{w_{\epsilon_h}}$. This shows that $d_R(X_t, X_{t_\alpha}) = O(1)$ when $t_\alpha < t$.

Now assume that $t_\alpha > t$. Let $s \in [t, d]$ be the first time that $\mathrm{i}_R(X_s, \lambda_{\mathbf{g}}) \ge N_1$; set $s = d$ if no such time exists. It is immediate from (2.4) that $\operatorname{diam}_R(\mathbf{g}([t, s])) \stackrel{*}{\prec} \operatorname{Log} N_1$, so we would be done if $t_\alpha \in [t, s]$. Thus, assume that $t_\alpha \in (s, d]$. By Proposition 5.3, there exists a bounded length curve $\beta \in \mathcal{C}(R)$ which is either pre-horizontal or (n_0, L_0)–horizontal on X_s. In either case, $d_R(X_{t_\beta}, X_s) = O(1)$. Indeed, if β is pre-horizontal, then let $s_\beta > s$ be the first moment when $\ell_{s_\beta}(\beta) = 1$; set $s_\beta = d$ if no such time exists. Of course, $\operatorname{diam}_R(\mathbf{g}[s, s_\beta]) = O(1)$. But $t_\beta \in [s, s_\beta]$ by Proposition 4.8 and Theorem 4.10, so $d_R(X_{t_\beta}, X_s) = O(1)$. On the other hand, if β is (n_0, L_0)–horizontal, then its length grows essentially exponentially from its balanced time $t_\beta < s$ onward by Proposition 4.8. The conclusion now follows from the fact that $\ell_s(\beta)$ is bounded. To proceed, we note that $d_R(X_s, \beta) = O(1)$, and recall that α is 1–close to $\pi_R(P_t)$ and $\operatorname{diam}_R(\mathbf{g}([t, s]))$ is bounded. Thus, by the triangle inequality,

$$d_R(\alpha,\beta) \overset{+}{\prec} d_R(\alpha,X_t)+d_R(X_t,X_s)+d_R(X_s,\beta)=O(1).$$

As Π is Lipschitz by the first part of the Theorem, the inequality above implies that

$$d_R(X_{t_\alpha},X_{t_\beta})=O(1).$$

A final application of the triangle inequality now yields

$$d_R(X_{t_\alpha},X_t)\le d_R(X_{t_\alpha},X_{t_\beta})+d_R(X_{t_\beta},X_s)+d_R(X_s,X_t)=O(1). \qquad \square$$

In light of Theorem 2.1, we obtain the following corollary.

Corollary 5.10. *Let $\mathbf{g}\colon [a,b]\to\mathcal{T}(S)$ and let $[c,d]\subseteq[a,b]$ be the active interval of a subsurface R along $\mathbf{g}$. Then $\Upsilon_R(\mathbf{g}([c,d])$, the shadow of $\mathbf{g}|_{[c,d]}$ to $\mathcal{C}(R)$, is a reparametrized quasi-geodesic in $\mathcal{C}(R)$.*

Remark 5.11. Let $\mathbf{g}\colon [a,b]\to\mathcal{T}(S)$ be a Thurston geodesic, and recall that for any curve $\alpha\in\mathcal{C}(R)$, our definition of the balanced time t_α is the first time that α is (n_0,L_0)–horizontal in the active interval $[c,d]$ of R along $\mathbf{g}$. However we can also define t'_α to be the first time α is (n_0,L_0)–horizontal in $[a,b]$ (see also [LRT15, §5.1]). The two times are equivalent from the point of view of coarse geometry of $\mathcal{C}(R)$. More precisely, we claim $d_R(X_{t_\alpha},X_{t'_\alpha})$ is uniformly bounded. To see this, note that by definition, if $t_\alpha\neq t'_\alpha$, then either $t'_\alpha<c$ or $t'_\alpha>d$. If $t'_\alpha<c$ and $t_\alpha=c$, then the conclusion follows from Theorem 5.7(i). If $t'_\alpha<c$ and $t_\alpha>c$, then the conclusion follows by Theorem 4.4. Finally, if $t'_\alpha>d$, then $t_\alpha=d$ by definition. In this case, the conclusion again follows by Theorem 5.7(i). Hence, from the point of view of the coarse geometry of $\mathcal{C}(R)$, there is not any difference between the two definitions of the balanced time.

Finally, note that by Theorem 5.7, $\operatorname{diam}_R(\mathbf{g}([a,c]))$ and $\operatorname{diam}_R(\mathbf{g}([d,b]))$ are bounded which imply that $\Upsilon_R(\mathbf{g}([a,b]))$ is also a reparametrized quasi-geodesic in $\mathcal{C}(R)$.

5.3. **Nearly filling.** In this final section after introducing the notion of a *nearly filling lamination*, we show that when the maximal stretch lamination of a Thurston geodesic segment is nearly filling at the initial point of the segment, then the shadow of the geodesic is a reparametrized quasi-geodesic in the curve complex of any subsurface that intersects the stretch lamination.

Definition 5.12 (Nearly filling)**.** Let $\mathbf{g}\colon I\to\mathcal{T}(S)$ be a Thurston geodesic, then we say that $\lambda_{\mathbf{g}}$ is *nearly filling* on $X_t=\mathbf{g}(t)$ if the inequality $d_S(X_t,\lambda_{\mathbf{g}})\ge 4B$ holds.

Theorem 5.13. *Let* $\mathbf{g}\colon [a,b] \to \mathcal{T}(S)$ *be a Thurston geodesic and suppose that* $\lambda_{\mathbf{g}}$ *is nearly filling on* X_a*. Then for all* $t \geq a$*, any curve* α *disjoint from* $\lambda_{\mathbf{g}}$ *is horizontal on* X_t*.*

Proof. Since α is disjoint from $\lambda_{\mathbf{g}}$, $R = S \setminus \alpha$ contains $\lambda_{\mathbf{g}}$. The inclusion map $\iota : \mathcal{C}(R) \to \mathcal{C}(S)$ is a contraction, so $d_R(X_a, \lambda_{\mathbf{g}}) \geq 4B$. Thus, for all $t \geq a$, by Lemma 5.5, $d_R(X_t, \lambda_{\mathbf{g}}) > 1$. We assume that α is not horizontal on X_t to derive a contradiction. This assumption leads to $\ell_t(\alpha) \leq \rho$ by Proposition 5.1, and also that $d_S(X_t, \alpha) \leq B$, by definition of B as in Lemma 5.4.

Consider the active interval for R along $\mathbf{g}$, which must be of the form $[a,d]$ for some $d \leq b$. We now analyze the two cases that either $[a,d]$ is a trivial or non-trivial interval.

If $a < d$, then $\ell_a(\alpha) \leq \rho$ by Theorem 5.7. In this case, $d_S(X_a, \alpha) \leq B$ by the definition of B. But then

$$d_S(X_a, \lambda_{\mathbf{g}}) \leq d_S(X_a, \alpha) + d_S(\alpha, \lambda_{\mathbf{g}}) \leq B + 1 < 2B,$$

contradicting the assumption.

If $a = d$, then α is horizontal on X_a. Let $s \in [a,b]$ be the first time such that α is (n_0, L_0)–horizontal on X_s; and set $s = b$ if no such s exists. It is not possible for $t > s + s_1$, where $s_1 \geq \log \frac{\rho}{w_{\epsilon_h}}$. This is because on X_s, α must intersect some ϵ_h–short curve, so $\ell_s(\alpha) \geq 2w_{\epsilon_h}$. But then $\ell_t(\alpha) > \rho$ by Proposition 4.8, which is not possible. On the other hand, $\operatorname{diam}_S(\mathbf{g}([a,s])) \leq 1$ and $\operatorname{diam}_S(\mathbf{g}([s, s+s_1]) \leq B$. So if $t \in [a, s+s_1]$, then

$$\begin{aligned} d_S(X_a, \lambda_{\mathbf{g}}) &\leq d_S(X_a, X_t) + d_S(X_t, \alpha) + d_S(\alpha, \lambda_{\mathbf{g}}) + 4 \\ &\leq (B+1) + B + 1 + 4 \\ &= 2B + 6 < 4B. \end{aligned}$$

This again contradicts the assumption. The proof that α must be horizontal on X_t is now complete. ◻

Now we restate Theorem 1.2 from the introduction and prove in as a corollary of our results.

Corollary 5.14. *Let* $\mathbf{g}\colon [a,b] \to \mathcal{T}(S)$ *be a Thurston geodesic. If* $\lambda_{\mathbf{g}}$ *is nearly filling on* X_a*, then for any subsurface* $R \subseteq S$*, the set* $\Upsilon_R(\mathbf{g}([a,b]))$ *is a reparametrized quasi-geodesic in* $\mathcal{C}(R)$*.*

Proof. If $\lambda_{\mathbf{g}}$ intersects R, then the conclusion follows from Corollary 5.10. If $\lambda_{\mathbf{g}}$ does not intersect R, then it is disjoint from ∂R. By Theorem 5.13, ∂R is horizontal and the conclusion follows because $\operatorname{diam}_R(\mathbf{g}[a,b])$ is bounded by Proposition 4.12. ◻

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