UNIQUE EQUILIBRIA AND SUBSTITUTION EFFECTS IN A STOCHASTIC MODEL OF THE MARRIAGE MARKET

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Abstract. Choo-Siow (2006) propose a behavioral model for the marriage market which incorporates random identically distributed noise into the preferences of each of the participants — à la McFadden — to fit virtually any observed distribution of matches.

We prove existence and uniqueness of the resulting equilibrium marriage distribution, and find a representation of it in closed form. This allows us to derive smooth dependence of this distribution on exogenous preference and population parameters, and establish sign, symmetry, and size of the various substitution effects, making comparative statics possible. For example, we show that an increase in the population of men of any given type in this model leads to an increase in unmarried men of each type, and a decrease in unmarried women of each type. We show that an increase in the number of men of a given type increases the equilibrium transfer paid by such men to their spouses, and also increases the percentage of men of that type who choose to remain unmarried. While the above trends may not seem surprising, the verification of such properties helps to substantiate the validity of the model. Moreover, we make unexpected predictions which could be tested: namely, the percentage change of type i unmarrieds with respect to fluctuations in the total number of type j men or women turns out to form a symmetric positive-definite matrix $r_{ij} = r_{ji}$ in this model, and thus to satisfy bounds such as $|r_{ij}| \leq (r_{ii}r_{jj})^{1/2}$.

Our existence result follows from a variational principle and a simple estimate, rather than a fixed point theorem. Fixed point approaches to the existence part of our result have been explored by others [6] [8] [12], but are much more complicated and yield neither uniqueness, nor comparative statics, nor an explicit representation of the solution.

1. Introduction

In 'Who Marries Whom and Why?' [7], Choo and Siow propose a behavioral model of the marriage market which is flexible enough to fit any observed data. As in McFadden's random utility model [18], agents are assumed to have deterministic preferences with respect to observable characteristics and stochastic preferences respect to unobservable

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characteristics, which spreads the preferences of agents on one side of the marriage market over the entire type-distribution of agents on the other and yields non-assortative matching. Along with its variants [14] [5], the Choo-Siow model has found empirical applications which range from studying the effects of abortion legislation on the US marriage market [7] to the effects of famine on the marriage market in China [3]. Although empirical researchers are actively using this model [6] [21], its analytical properties — which we shall seek to determine — are not well-understood.

Even under simplifying assumptions concerning the form of the randomness, it is not evident whether all choices of deterministic preferences lead to existence of a distribution of marriages which clears the market [6] [12] [8]. Our first result reconfirms that it does. Moreover, we show this equilibrium is unique, and give an explicit formula for the resulting marriage distribution, essentially solving the model completely. This allows us to derive smooth dependence of the resulting equilibrium state on the specified population and preference parameters, and establish sign, symmetry and bounds on the response of the predicted distribution of marriages to changes in each parameter. Although existence of an equilibrium was also discussed by Choo, Seitz and Siow [6] and Fox [12] (and by Dagsvik [8] for a related model), our proof relies on the reformulation of the problem as a variational minimization, hence is much simpler than the fixed point argument they suggest. Moreover, as mentioned above, it yields the solution in closed form. Our uniqueness result is the first concerning this model [5], and is based on convexity (in appropriate variables) of the new variational principle formulated in Section §4. Section §2 details the model, while §3 specifies our results. Section §4 establishes the existence and uniqueness of equilibria, while §5 is devoted to comparative statics. The remainder of this introduction addresses related literature, provides motivation for the Choo-Siow model, and discusses our results. Further comments concerning the derivation of the model may be found in Appendix A.

The random component of agent preferences is a salient feature of the Choo-Siow model. Due to this randomness, the equilibrium marriage distribution predicted by the model will not be positive assortative, even when the observed attributes of the agents are one-dimensional. This is consistent with empirical data. Even in experiments where the agents are parameterized by ordered types, such as age, observed marital data will almost never be genuinely assortative. For example, in any given population it is unlikely to be true that the age of the youngest woman married to a 34 year old man always exceeds that of the oldest woman married to a 33 year old. Similarly, one always finds matches in large populations that pair high with low qualities as measured by any given ordered observable characteristic (e.g. income, years of education, etc.). Thus a strictly assortative framework fails to explain the presence of, for example, the existence of PhD graduates married to high-school drop-outs.

The classic transferable utility model of the marriage market, introduced by Gary Becker [1], in principle predicts how agents will marry given exogenous preference parameters. However, it has seldom been estimated. There are two main obstacles in estimating a model of the marriage market. First, equilibrium transfers in modern marriages (except in the case of dowries) are not observed. Hence any behavioural model that requires their presence in data is not identifiable. Second, real-world agents are described by discrete, multi-dimensional, possibly unordered, types. But the classic Becker model predicts positive assortative matching under the assumption that agent type is one dimensional, continuous, ordered, and that preferences are super-modular. This positive assortative matching is limiting, but does ensure that his model predicts a unique marital distribution.

The Choo-Siow model eliminates the structural assumptions of the classic model. First, it is not necessary to observe transfers in order to determine the equilibrium marriage distribution generated by the model. In fact, we provide an explicit formula for the equilibrium marriage distribution in terms of the derivative of the Legendre transform of a known function. Second, the model places no a priori structure on the nature or number of types that agents (men and women) can have. This allows consideration of a wide range of attributes, like race, religion, level of income, and educational achievements.

In this more realistic framework, with its lack of structure for the agents' deterministic preferences and types, the issue of whether there exists an equilibrium marital distribution, and if so whether it is unique, becomes a question of fundamental theoretical and econometric significance. The theoretical importance arises from the fact that uniqueness of equilibria in two-sided matching problems is usually not better than a generic property, except perhaps in certain convex programming settings like [10] [11], which include continuous Monge-Kantorovich matching [15] [4]. Further, the randomness considered in the model below is the commonly used Gumbel extreme value (logit) type, thus any result that describes properties of the equilibrating matches has potentially wider applicability. The econometric importance arises from the fact that models of the marriage market are useful to econometricians only insofar as they make unique predictions of a martial distribution, given exogenous preferences. From a practical point of view having closed form solutions which permit comparative statics may be even more crucial.

2. The Choo-Siow Marriage Matching Model

Our presentation emphasizes the stochastic heterogeneity that differentiates the Choo-Siow model from classical models. The competitive framework, which uses transfers of utility from spouses to equilibrate the market, is explored in detail in Choo-Siow [7] but treated here only at the end of §5.3. It should be noted at the outset that the methods developed here also apply to other non-transferable utility models present in the literature. For example, Dagsvik [8] develops a model of the marriage market which uses an assignment algorithm (deferred acceptance) rather than utility transfers to sort matches, but his equilibrium conditions are functionally similar to ours.

2.1. **Setting.** What is exogenous in this model are the observed types of men and of women, the numbers of men and women of each type in the population, and the total gains π_{ij} of marriage between a man of observed type i and a woman of observed type j, relative to both partners remaining unmarried. The quantity π_{ij} will not reappear until (7). On the other hand, individual agents have a utility functions that depend on both an endogenous deterministic component that captures systematic utility, and an exogenous random one that models heterogeneity within the population of each given type. Thus the utility accrued by a man of type i and specific identity g who marries a woman of type j is assumed to be:

$$V^m_{ijg} = \eta^m_{ij} + \sigma\epsilon_{ijg}; \tag{1}$$

the case $j = 0$ represents the utility of remaining unmarried. The deterministic component is η^m_{ij}; its endogeneity can be interpreted to reflect the possibility of interspousal transfer, as in Choo-Siow [7] and §5.3 below. It is set in equilibrium, and depends explicitly on the type of the man and the type of the woman, and implicitly on market conditions, i.e. on the relative abundance or scarcity of men and women of each different type. The random term ϵ_{ijg} depends additionally on the specific identity of the man, but not on the specific identity of the woman. Hence a specific thirty-five year old man may have stronger than typical (with respect to his age group) attraction for fifty-year old women. But this attraction does not depend on whether, for example, the older woman has an especially strong attraction to younger men (assuming this latter characteristic is unobservable in the data and hence not reflected in j).

The random term is assumed to have the Gumbel extreme value distribution described in Appendix A. This distribution was introduced to the economics literature by McFadden [18]. Finally, the real number σ is a scaling parameter which measures the degree of randomness; its reciprocal can be interpreted as the signal to noise ratio. It is equal to unity in the original Choo-Siow model. For illustrative purposes, we will have occasion to allow σ to vary and in doing so embed the Choo-Siow model in a one parameter family of models that differ by the degree of randomness present in them.

Unlike in deterministic matching models, agents of a particular type do not have a uniform preferred match. This because their preferences depend on the random variable ϵ_{ijg}. Using the Gumbel structure, the probability that a man of type i prefers a woman of type j among all other possible marital choices $k \in \{0, ..., J\}$ is given by

$$\Pr(\text{Man of type } ig \text{ prefers a woman of type } j) = \frac{\exp(\frac{\eta^m_{ij}}{\sigma})}{\sum_{k=0}^{J} \exp(\frac{\eta^m_{ik}}{\sigma})}; \tag{2}$$

(see Appendix A for a derivation). This probability distribution is endogenous, because it depends on the various η^m_{ik}. Note that it does not depend on the specific identity g of the man of type i, since the noise is identically distributed for each different g. Yet it is possible already to see how the equilibrium marriage output will differ markedly from a deterministic one. Whereas in the deterministic case all members of a given type typically have the same preferred match, here the preferred matches of type i men are smeared across all female types according to the distribution defined by (2). The mean and spread of the smearing are determined by the endogenous values η^m_{ij} and by σ, respectively.

Consider $\sigma \in [0, \infty]$. The case $\sigma = 1$ corresponds to the Choo-Siow model where some smearing is present. The case $\sigma = 0$ corresponds to a deterministic matching model, for which there is no smearing. Indeed, as $\sigma \to 0$, the largest exponentials dominate all others, and the probability that a man of type i prefers a woman of type j converges to 0 or $\frac{1}{\#\arg\max\{\eta^m_{ij} | 0 \le j \le J\}}$ depending on whether or not η^m_{ij} weakly dominates all other preference parameters η_{ik}^m. Conversely, as $\sigma \to \infty$, the stochastic term dominates the utility function, and the resulting probability distribution converges to the uniform distribution. In this case, there is maximal smearing, as preferences are completely random, constrained only by availability of prospective partners to marry.

Female preferences are also smeared, and the equilibrium marriage distribution is determined when η_{ij}^m and η_{ij}^f are such that the number of desired marriages of each type is the same on both sides of the market.

We now elaborate on the Choo-Siow model. We henceforth fix $\sigma = 1$; since preferences are relative, this normalization can always be attained by rescaling all of the preferences in the model. Our presentation of the model is notationally different than in [7] and better suited to our subsequent arguments.

2.2. **The Choo-Siow model.** Suppose we wish to predict the number of marriages between men and women of different types. The number of men of type i is denoted m_i. The number of marriages of type i men to type j women is denoted μ_{ij}. The number of men of type i and women of type j who choose to remain unmarried is denoted by μ_{i0} and μ_{0j} respectively. If each man marries his preferred woman, the equality

$$\text{Pr(Man of type } ig \text{ prefers a woman of type } j) = \frac{\mu_{ij}}{m_i} \tag{3}$$

will be valid, or at least as the population size becomes large, the right hand side of the equality converges to the left hand side by the law of large numbers, or the maximum likelihood theorem.

Using equations (2)–(3) to compute the ratio of the probability that a man of type i prefers a woman of type j to the probability that he prefers to remain unmarried, we arrive at the following formula:

$$\mu_{ij}^m = \frac{e^{\eta_{ij}^m}}{e^{\eta_{i0}^m}} \mu_{i0}^m. \tag{4}$$

These $I \times J$ equations are in fact quasi-demand equations, because they indicate the number of type μ_{ij} marriages that men of type i would like to participate in. Viewing the female market cohort as the supply side, there are analogous supply equations. Letting the utility acquired by a woman of type j and specific identity h who marries a man of type i be

$$V_{ijh}^f = \eta_{ij}^f + \epsilon_{ijh}, \tag{5}$$

the above analysis produces $I \times J$ supply equations of the form:

$$\mu_{ij}^f = \frac{e^{\eta_{ij}^f}}{e^{\eta_{0j}^f}} \mu_{0j}^f. \tag{6}$$

The equilibrium output in the Choo-Siow model is a specification of μ_{ij} for all $0 \leq i \leq I$, and all $0 \leq j \leq J$. This output is obtained by requiring that supply balance demand: $\mu_{ij}^f = \mu_{ij}^m$. Under this market-clearing hypothesis, we have the following equation. The endogenous parts of the η_{ij}^m, η_{ij}^f, η_{i0}^m, η_{0j}^f are eliminated upon adding them to arrive at the definition of an exogenous[1] aggregated gains variable π_{ij} associated to each observable type of marriage:

$$\pi_{ij} := \frac{\eta^m_{ij} + \eta^f_{ij} - \eta^m_{i0} - \eta^f_{0j}}{2}. \tag{7}$$

Using the market-clearing hypothesis, we may re-write the equilibrium condition $\mu^m_{ij} = \mu^f_{ij} =: \mu_{ij}$ in terms of the exogenous variable π_{ij} as follows:

$$\frac{\mu_{ij}}{\sqrt{\mu_{i0}\mu_{0j}}} = e^{\pi_{ij}}. \tag{8}$$

Finally, letting $\Pi_{ij} = e^{\pi_{ij}}$, the equilibrium output is given by

$$\frac{\mu_{ij}}{\sqrt{\mu_{i0}\mu_{0j}}} = \Pi_{ij}. \tag{9}$$

The equilibrium conditions expressed in equation (9) are implicit. They give necessary conditions for real numbers μ_{ij} to be an output of the Choo-Siow model. However, they are not sufficient; a secondary set of necessary conditions, population constraints, must also be satisfied. Let there be I types of men, and J types of women. The number of men of type i is denoted m_i, and the number of women of type j is denoted f_j. The vector whose i^{th} component is m_i, and whose $(I+j)^{th}$ component is f_j, is denoted by ν. Called the *population vector*, it has $(I+J)$ components and may also be denoted by $[\, m \mid f \,]$. Let μ_{ij} be the number of marriages of type i men to type j women. Let μ_{i0} be the number of unmarried men of type i and μ_{0j} be the number of unmarried women of type j. A specification of μ_{ij},μ_{i0},μ_{0j} for all i and j is called a *marital distribution.* The following population constraints must be satisfied by all marital distributions, and are a consequence of the definitions:

$$\mu_{i0} + \sum_{j=1}^{J} \mu_{ij} = m_i, \tag{10}$$

$$\mu_{0j} + \sum_{i=1}^{I} \mu_{ij} = f_j, \tag{11}$$

$$\mu_{ij} \geq 0. \tag{12}$$

Two questions naturally arise. First, given an exogenous matrix Π (with positive entries) will there always be a specification of μ_{ij} that satisfies (9)–(12)? Second, if there is a satisfying marital distribution, is it unique? We call these questions the *Choo-Siow inverse problem.* The problem is important for several reasons:

First, the implicit conditions present in equation (9) are the equilibrium outcome of a competitive market. There are not so many realistic environments with finitely many agent

[1]See [7] or §5.3 for an explanation in terms of spousal transfers.

types and many commodities which are known to generate unique competitive equilibria — except possibly generically. While there are generic uniqueness results for matching problems that can be reduced to convex programing problems such as Monge-Kantorovich matching, e.g. [15] [4] [10] [11], the stochastic heterogeneity prevents the equilibrium in our model from being formulated as such. Instead, stochasticity effectively removes the need for a genericity assumption.

Second, an affirmative explicit solution to the *Choo-Siow Inverse Problem* makes the Choo-Siow model useful in econometric analysis. The matrix Π is exogenous and unobserved in data, but can be point-estimated from an observed marriage distribution. An economic or social shock will affect the systematic utilities that agents of various types incur by marrying agents of various others, and will therefore alter the value of Π. This effect can be approximated to form an updated matrix of aggregated systematic parameters $\Pi^{'}$. Existence and uniqueness guarantee that there will be exactly one marriage distribution that results from the shock, making the model predictive. In the same vein, demographers are often interested in predicting how marriage distributions will change due to changing demographics, i.e. changes in the population vector ν. Our closed form solution makes it is possible to compute the sign and in some cases the magnitude of such changes explicitly.

Finally, if the *Choo-Siow inverse problem* has a unique solution, the estimated parameters π_{ij} are an alternative characterization of the observed marriage distribution. The recharacterization is useful because the parameters of the Choo-Siow model have a behavioral interpretation, and are not merely observed data.

2.3. **Summary of progress.** A related local uniqueness question was resolved by Choo and Siow in [7]. However the issue of global uniqueness was left open, and posed as an open problem in a subsequent working paper by Siow [22]. We resolve this question positively by introducing a variational principal and a change of variables which allows us to exploit convexity. The question of existence of (μ_{ij}) for all $\Pi = (e^{\pi_{ij}})$ was addressed in a working paper of Choo, Seitz and Siow [6] by appealing to the Tarski fixed point theorem; see also the related results of Fox [12] and Dagsvik [8]. However the proofs there are long and obscure — at least to us — whereas the variational proof in the present paper is simple and direct and follows from continuity and compactness by way of an elementary estimate. Moreover, it leads to an explicit representation of the solution. This allows us to rigorously confirm various desirable and intuitive features of Choo-Siow matching, whose presence or absence might in principle be used as a test to refute the validity of various alternative matching models. Among other results, we show for example that an increase in the number of men of a given type increases the equilibrium transfer paid by such men to their spouses, while also increasing the percentage of such men who choose to remain unmarried. See Theorem 2 below for related statements and more surprising conclusions.

3. Precise Statement of Results

In the preceding remarks, the *Choo-Siow inverse problem* was phrased in terms of finding existence and uniqueness of equilibrium μ given exogenous data Π and ν. As the name suggests, it is also useful to think of this problem as one of inverting a function. From this point of view, even though Π is exogenous, we may prefer instead to consider Π as the image of a marriage distribution under a transformation that we seek to invert. We say that a marriage distribution μ generates the gains matrix Π if (9) holds.

Remark 1 (Incomplete participation)**.** *From the market equilibrium point of view, the fact that the left hand-side of* (9) *becomes infinite when μ_{i0} or μ_{0j} is equal to zero is unproblematic. It means that for no finite value of the exogenous Π is sufficient to induce all the representatives of some type to marry. However from the inverse problem point of view, it is necessary to stipulate that μ_{i0} and μ_{0j} be strictly positive.*

We can now precisely formulate the Choo-Siow Inverse Problem:

Problem (Choo-Siow inverse problem) *Given a gains matrix $\Pi = (\Pi_{ij})$ and a population vector $\nu = [\, m \mid f \,]$, does there exist a unique marital arrangement generating Π? In other words, assuming the entries Π_{ij} to be non-negative and m_i and f_j to be strictly positive, does exactly one matrix (μ_{ij}) with non-negative entries exist that satisfies* (9)–(12)*?*

3.1. **Preliminaries.** Let us begin with a reformulation of the problem; Siow attributes this reformulation to Angelo Melino. Let $\alpha_{ij}^2 = \mu_{ij}$. In this new notation, the gains matrix and population constraints (9)–(12) take the form:

$$\alpha_{i0}\alpha_{0j}\Pi_{ij} = \alpha_{ij}^2, \tag{13}$$

$$\alpha_{i0}^2 + \sum_{j=1}^{J} \alpha_{ij}^2 = m_i, \tag{14}$$

$$\alpha_{0j}^2 + \sum_{i=1}^{I} \alpha_{ij}^2 = f_j. \tag{15}$$

Borrowing terminology from quantum physics, we call any collection of α_{ij}'s which solve (13)–(15) *amplitudes* corresponding to Π. By substituting (13) into (14) and (15), we can eliminate all variables but those that correspond to unmarried men and women. Renaming the variables α_{i0} to β_i, and α_{0j} to β_{I+j}, we obtain a system of $(I+J)$ quadratic polynomials in the $(I+J)$ variables $\{\beta_k\}_{k=1}^{I+J}$:

$$\begin{aligned} \beta_i^2 + \sum_{j=1}^{J} \beta_i\beta_{I+j}\Pi_{ij} - \nu_i = 0, &\qquad 1 \le i \le I, \\ \beta_{I+j}^2 + \sum_{i=1}^{I} \beta_i\beta_{I+j}\Pi_{ij} - \nu_{I+j} = 0, &\qquad 1 \le j \le J. \end{aligned} \tag{16}$$

A solution to this system of equations is a vector of amplitudes β that has $(I+J)$ components. Abstractly, its components might be real, complex, or both. The Choo-Siow Inverse Problem is equivalent to showing that the polynomial system (16) has a unique solution with real positive amplitudes for all gains matrices Π and population vectors $\nu = [\, m \mid f \,]$

with positive components. Our proof is variational. We construct a functional $E(\beta)$ with the property that β is a *critical point* of E — meaning point where E has zero derivative — if and only if β satisfies equation (16). We then show that E has exactly one critical point in the positive *orthant* $(\mathbf{R}_+)^{I+J}$, and give a formula for this critical point using the Legendre transform of a related function.

The main result of this paper is the following theorem, which solves the Choo-Siow Inverse problem.

Theorem 1 (Existence and uniqueness of real positive solution)**.** *If all the entries of* $\Pi = (\Pi_{ij})$ *are non-negative, and those of* $\nu = [\,m \mid f\,]$ *are strictly positive, then precisely one solution* β *of* (16) *lies in the positive orthant of* $\mathbf{R}^{I+J}$.

Remark 2 (Explicit solution)**.** *As we shall see, the solution* $b := (\log \beta_1, \ldots, \log \beta_{I+J})$ *satisfies* $b = (DH)^{-1}(\nu) = DH^*(\nu)$ *where* $H(b)$ *and* $H^*(\nu)$ *are the smooth strictly convex dual functions defined by* (25)*–*(26)*.*

Remark 3 (Unpopulated types)**.** *In case* $m_i = 0$ *or* $f_j = 0$*, we simply reformulate the problem in fewer than* $I + J$ *variables, corresponding only to the populated types. This reformulation shows the conclusions of Theorem 1 extend also to population vectors* $\nu = [\,m \mid f\,]$ *whose entries are merely non-negative, instead of strictly positive.*

Since each matrix (μ_{ij}) with non-negative entries solving (9)–(12) corresponds to a solution β of (16) having positive amplitudes $\beta_i = \sqrt{\mu_{i0}}$ and $\beta_{I+j} = \sqrt{\mu_{0j}}$, this theorem gives the sought characterization of (μ_{ij}) by Π. Moreover, this characterization facilitates computing variations in the marital arrangements in response to changes in the data (Π, ν):

Theorem 2 (Comparative statics)**.** *Let the unique solution to the Choo-Siow inverse problem with exogenous data* Π *and* ν *be given by* $\beta(\Pi, \nu)$*. Then the percentage change of unmarrieds* β_k^2 *with respect to the population parameter* ν_ℓ *turns out to define a symmetric and positive definite matrix*

$$r_{k\ell} := \frac{1}{\beta_k^2} \frac{\partial \beta_k^2}{\partial \nu_\ell} =: 2[D_\nu \log(\beta)]_{k,\ell} \tag{17}$$

(sometimes denoted $D_\nu \log(\beta) > 0$*); here* $k, \ell \in \{1, \ldots, I + J\}$*. This positive definiteness implies, among other things, the expected monotonicity* $r_{kk} > 0$*, the unexpected symmetry* $r_{k\ell} = r_{\ell k}$*, and more subtle constraints relating these percentage rates of change and the corresponding substitution effects such as* $|r_{k\ell}| < \sqrt{r_{kk} r_{\ell\ell}}$*.*

Additionally we can account for the sign, and in some cases bound the magnitude, of each entry of the matrix $R = (r_{k\ell})$*. To avoid trivialities, assume no column or row of* Π *vanishes, so no observable type of individual is compelled to remain unmarried. Then,*

$$r_{k\ell} < 0, \tag{18}$$

if $k \in \{1 \ldots I\}$ *and* $\ell \in \{I + 1 \ldots I + J\}$ *(or vice versa). Second, if* $k, \ell \in \{1 \ldots I\}$*, then*

$$\frac{1}{2}(\beta_k^2 + \nu_k) r_{k\ell} > \delta_{k\ell} := \begin{cases} 0 & \text{if } k \neq \ell \\ 1 & \text{otherwise.} \end{cases} \tag{19}$$

Similarly, (19) *also holds if both* $k, \ell \in \{I + 1 \ldots I + J\}$*.*

These qualitative comparative statics have a simple interpretation. Increased supply of any type of man coaxes more women into marriage (due to increased competition among men leading to an increased equilibrium transfer — see subsection §5.3) and decreases the number of men who wish to marry. The last statement of the theorem says that this decrease is not merely due to the fact that there are more men. Rather, men of type $k \neq \ell$ who would have chosen marriage under the old regime choose to be unmarried after the shock.

We conclude our results with a corollary asserting monotonicity of utility transferred by men of type i, and of the percentage who choose to remain unmarried, as a function of their abundance in the population.

Corollary 3 (Utility transferred and non-participant fraction increase with abundance)**.** *For all $i \leq I$, $j \leq J$, and $k \leq I + J$, with the hypotheses and notation of Theorem 2,*

$$\frac{\partial}{\partial \nu_i}(\eta^f_{ij} - \eta^m_{ij}) \;>\; 0 \tag{20}$$

$$\text{and} \quad \frac{\partial}{\partial \nu_k}\Big(\frac{\beta_k^2}{\nu_k}\Big) \;>\; 0. \tag{21}$$

4. A New Variational Principle (Proof of Theorem 1)

4.1. **Variational method: existence of a solution.** Consider the function $E : \mathbf{R}^{I+J} \to \mathbf{R} \cup \{+\infty\}$, defined as follows:

$$E(\beta) := \frac{1}{2}\sum_{k=1}^{I+J} \beta_k^2 + \sum_{i=1}^{I}\sum_{j=1}^{J} \Pi_{ij}\beta_i\beta_{I+j} - \sum_{k=1}^{I} \nu_k \log|\beta_k|. \tag{22}$$

It diverges to $+\infty$ on the coordinate hyperplanes where the β_k vanish, but elsewhere is smooth.

We differentiate and observe that β is a critical point of E if and only if (16) holds. Notice strict positivity of the components of $\nu = [\, m \mid f \,]$ implies the corresponding component of a solution β to (16) is non-vanishing, hence no solutions occur on the coordinate hyperplanes which separate the different orthants. In words, the critical points of E are precisely those that satisfy the system of equations we wish to show has a unique real positive root. It therefore suffices to show that $E(\beta)$ has a unique real positive critical point; for then (16) admits exactly one real positive solution. Let us show at least one such solution exists, by showing $E(\beta)$ has at least one critical point: namely, its minimum in the positive orthant.

Claim 4 (Existence of a minimum)**.** *If all the entries of $\Pi = (\Pi_{ij})$ are non-negative, and those of $\nu = [\, m \mid f \,]$ are strictly positive, the function $E(\beta)$ on the positive orthant defined by* (22) *attains its minimum value.*

Proof. Since $E(\beta)$ is continuous, the claim will be established if we show the sublevel set $B_\lambda := \{\beta \in (\mathbf{R}_+)^{I+J} \mid E(\beta) \leq \lambda\}$ is compact for each $\lambda \in \mathbf{R}$. Non-negativity of Π_{ij}

combines with positivity of ν_k, β_k, and the inequality $\log \beta_k \leq \beta_k - 1$ to yield

$$E(\beta) \geq \sum_{k=1}^{I+J} \frac{1}{2}\beta_k^2 - \nu_k(\beta_k - 1) \tag{23}$$

$$= \frac{1}{2}\sum_{k=1}^{I+J}(\beta_k - \nu_k)^2 - (\nu_k - 1)^2 + 1. \tag{24}$$

It follows that B_λ is bounded away from infinity. Since $E(\beta)$ diverges to $+\infty$ on the coordinate hyperplanes, it follows that B_λ is also bounded away from the coordinate hyperplanes — hence compactly contained in the positive orthant. □

4.2. **Uniqueness, convexity, and Legendre transforms.** With this critical point characterization of the solution in mind, let us observe for $\beta \in \mathbf{R}^{I+J}$ in the positive orthant, defining $b_k := \log \beta_k$ implies $E(\beta) = H(b) - \langle \nu, b\rangle$, where

$$H(b) := \frac{1}{2}\sum_{k=1}^{I+J} e^{2b_k} + \sum_{i=1}^{I}\sum_{j=1}^{J} \Pi_{ij} e^{b_i + b_{I+j}} \tag{25}$$

and $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathbf{R}^{I+J}$. Since the change of variables $\beta_k \in \mathbf{R}_+ \longmapsto b_k = \log \beta_k \in \mathbf{R}$ is a diffeomorphism, it follows that critical points of $H(b) - \langle \nu, b\rangle$ in the whole space $\mathbf{R}^{I+J}$ are in one-to-one correspondence with critical points of $E(\beta)$ in the positive orthant.

On the other hand, $H(b)$ is manifestly convex, being a non-negative sum of convex exponential functions of the real variables b_k; in fact $\Pi_{ij} \geq 0$ shows the Hessian $D^2H(b)$ dominates what it would be in case $\Pi = 0$, namely the diagonal matrix with positive entries $diag[2e^{2b_1}, \ldots, 2e^{2b_{I+J}}]$ along its diagonal. Thus $H(b)$ is strictly convex throughout $\mathbf{R}^{I+J}$, and $E(\beta) = H(b) - \langle \nu, b\rangle$ can admit only one critical point β in the positive orthant — the minimizer whose existence we have already shown. The solution β to (16) which we seek therefore coincides with the unique point at which the maximum is attained.

This last fact means that b maximizes the right-hand side of the following equation:

$$\begin{aligned} H^*(\nu) &:= \sup_{b \in \mathbf{R}^{I+J}} \langle \nu, b\rangle - H(b) \\ &= \sup_{\beta \in (\mathbf{R}_+)^{I+J}} -E(\beta). \end{aligned} \tag{26}$$

The function H^* defined pointwise by the above equation is the Legendre transform or convex dual function of H; see Appendix B for details. It follows that the solution b satisfies $\nu = DH(b)$. Thus $b = DH^*(\nu)$ by the duality of H and H^*. This provides an explicit formula for b in terms of the derivative of H^*.

5. Comparative Statics (Proof of Theorem 2)

5.1. **Positive definiteness.** Our representation of the solution in terms of the Legendre transform of the convex function H can be used to obtain information about the derivatives of the solutions with respect to the population parameters ν.

Suppose we wish to know how the number of marriages $\mu_{ij} = \Pi_{ij}\beta_i\beta_{I+j}$ of each type (i,j) varies in response to slight changes in the population vector ν, assuming the gains matrix Π remains fixed. This is easily computed from the percentage rate of change $r_{k\ell}$ in the number β_k^2 of unmarrieds of each type, which is given in terms of the Hessian of either (25) or (26) by

$$r_{k\ell} := \frac{1}{\beta_k^2}\frac{\partial \beta_k^2}{\partial \nu_\ell} = 2D^2_{k\ell}H^*(\nu) = 2(D^2H|^{-1}_{(\log\beta_1,\ldots,\log\beta_{I+J})})_{k\ell}, \quad 1 \le k,\ell \le I+J. \tag{27}$$

To see that these equalities hold, observe that the solution β is the point where the maximum (26) is attained. The Legendre transform $H^*(\nu)$ of H defined by this maximum is manifestly convex, and its smoothness is well-known to follow from the positive-definiteness of $D^2H(b) > 0$ as in Lemma 9. Moreover $b = DH^*(DH(b))$, whence the maximum (26) is attained at $b = DH^*(\nu)$ and $D^2H(b)^{-1} = D^2H^*(DH(b)) = D^2H^*(\nu) > 0$. This positive definiteness implies the first half of the Theorem 2.

5.2. **Qualitative characterization of comparative statics.** To complete our qualitative description of the substitution effects in this section, we apply the following theorem from functional analysis to linear transformations $T : \mathbf{R}^n \longrightarrow \mathbf{R}^n$.

Theorem 5 (Neumann series for the resolvent of a linear contraction)**.** *Let $\|\bullet\|_{op}$ be the operator norm. Then if $\|T\|_{op} <1$, the operator $(1-T)^{-1}$ exists and is equal to $\sum_{n=0}^\infty T^n$.*

Next, we consider the matrix $D^2H(b)|_{(\log\beta_1,\ldots,\log\beta_{I+J})}$, and derive properties of its inverse, whose entries give the various values (27) of $r_{k\ell}/2$. Differentiating the known function $H(b)$ twice yields a positive-definite $(I+J)\times(I+J)$ matrix which can be factored into the form

$$2R^{-1} = D^2H|_{b=(\log\beta_1,\ldots,\log\beta_{I+J})} = \Delta\begin{pmatrix}\Delta_I & \Pi \\ \Pi^T & \Delta_J\end{pmatrix}\Delta \tag{28}$$

where $\Delta = \mathrm{diag}[e^{b_1},\ldots,e^{b_{I+J}}] = \mathrm{diag}[\beta]$, while Δ_I and Δ_J are $I\times I$ and $J\times J$ diagonal submatrices whose diagonal entries are all larger than two:

$$\begin{aligned}(\Delta_I)_{ii} &= 2 + \frac{1}{\beta_i^2}\sum_{j=1}^J \Pi_{ij}\beta_i\beta_{I+j} = 1 + \frac{\nu_i}{\beta_i^2},\\ (\Delta_J)_{jj} &= 2 + \frac{1}{\beta_{I+j}^2}\sum_{i=1}^I \Pi_{ij}\beta_i\beta_{I+j} = 1 + \frac{\nu_{I+j}}{\beta_{I+j}^2}.\end{aligned}$$

Here we have used the fact that the values β are critical points and therefore satisfy the first order conditions (16) to simplify these diagonal terms.

There are determinant and inverse formulae for block matrices which assert [16] that

$$\det\begin{pmatrix}\Delta_I & \Pi \\ \Pi^T & \Delta_J\end{pmatrix} = \det(\Delta_I)\det(\Delta_J)\det(1-{\Delta_I}^{-1}\Pi{\Delta_J}^{-1}\Pi^T), \tag{29}$$

and

$$\begin{pmatrix} \Delta_I & \Pi \\ \Pi^T & \Delta_J \end{pmatrix}^{-1} = \begin{pmatrix} (\Delta_I - \Pi{\Delta_J}^{-1}\Pi^T)^{-1} & -(\Delta_I - \Pi{\Delta_J}^{-1}\Pi^T)^{-1}\Pi{\Delta_J}^{-1} \\ -(\Delta_J - \Pi^T\Delta_I^{-1}\Pi)^{-1}\Pi^T{\Delta_I}^{-1} & (\Delta_J - \Pi^T\Delta_I^{-1}\Pi)^{-1} \end{pmatrix}. \tag{30}$$

The determinant (29) is positive by (28) and Theorem 1. We will now show that the eigenvalues of the matrix $A(s) = {\Delta_I}^{-1}s\Pi{\Delta_J}^{-1}s\Pi^T$, appearing in (29)–(30) are bounded above by 1 and below by -1 for all values of $s \in [0,1]$. This will have implications respecting the signs of the entries of (30), whose $(k,\ell)^{th}$ entry is in fact equal to $\beta_k\beta_\ell r_{k\ell}/2$ hence shares the sign of the change (18)–(19) which we desire to estimate. Namely, it will allow us to apply Theorem 5 to block entries such as $(\Delta_I - \Pi{\Delta_J}^{-1}\Pi^T)^{-1} = (1-A(1))^{-1}{\Delta_I}^{-1}$ in (30).

Let $\lambda^{max}(s)$ be the largest eigenvalue of $A(s)$. Then, the smallest eigenvalue of $(1-A(s))$ is equal to $(1-\lambda^{max}(s))$. We proceed by continuously deforming from $s=0$ to $s=1$: The eigenvalues of $(1-A(0))$ are equal to 1, as $A(0)$ is in fact equal to the zero matrix. Since $\det(1-A(s)) > 0$ for all $s \in [0,1]$, continuity of $\lambda^{max}(s)$ and the intermediate value theorem imply that $1-\lambda^{max}(s) > 0$ for all s, so that $\lambda^{max}(1) < 1$. Since no row of Π vanishes, $A(s)$ has positive entries whenever $s > 0$. The Perron-Frobenius theorem therefore implies that any negative eigenvalue λ of $A(1)$ is bounded by $|\lambda| < \lambda^{max}(1)$.

Since A has positive entries and $\|A\|_{op} < 1$, Theorem 5 indicates that the entries of $(1-A)^{-1}$ are all positive — exceeding one on the diagonal. But $\beta_k\beta_\ell r_{k\ell}/2$ coincides with the $(k,\ell)^{th}$ entry of $(1-A)^{-1}\text{diag}[\beta_1^2/(\beta_1^2+\nu_1), \ldots, \beta_I^2/(\beta_I^2+\nu_I)]$, giving the desired inequalities (19) whenever $k,\ell \in \{1,\ldots,I\}$. The signs of the remaining derivatives (18)–(19) may be verified by applying the same technique to the three other submatrices present in (30), thus completing the proof of Theorem 2.

5.3. **Transfer utilities and percentage unmarried.** Given a specification of Π and $\nu = [\,m_i \mid f_j\,]$, the Choo-Siow model predicts a unique vector $\beta = [\,\mu_{i0} \mid \mu_{0j}\,]$ of unmarrieds. Given a fixed Π and a fixed β, the full marriage distribution can then be uniquely recovered. It is therefore possible to view the β as a single valued (smooth) function of Π and ν.

5.3.1. *Varying the population vectors ν.* By Theorem 2, the signs of $r_{k\ell}$ are independent of Π and ν and depend only on whether $k \in \{1,\ldots,I\}$, or $k \in \{I+1,\ldots,I+J\}$, and likewise for ℓ. It is perhaps useful to visualize these comparative statics as the entries of the matrix $D\beta$ with $D_\ell\beta_k := \frac{\partial\beta_k}{\partial\nu_\ell}$. Then, $D\beta$ is a block matrix that is positive in its upper-left and lower-right blocks, and negative in its upper-right and lower-left blocks. Schematically, (30) yields

$$D\beta = \begin{pmatrix} + & - \\ - & + \end{pmatrix}. \tag{31}$$

Reverting back to the Choo-Siow notation for unmarrieds and population vectors, we have for all k and ℓ:

$$\frac{\partial\mu_{k0}}{\partial m_\ell} > 0, \quad \frac{\partial\mu_{k0}}{\partial f_\ell} < 0, \quad \frac{\partial\mu_{0k}}{\partial f_\ell} > 0, \quad \frac{\partial\mu_{0k}}{\partial m_\ell} < 0. \tag{32}$$

These basic comparative statics yield qualitative information about other more complex quantities of interest. As indicated following equation (7), the quantity $\eta^m_{ij} + \eta^f_{ij} - \eta^m_{i0} - \eta^f_{0j}$ is exogenous, whereas the first two individual summands are separately endogenous and determined within the model. In the original formulation of this model, present in [7], our endogenous payoff $\eta^m_{ij} = \tilde{\eta}^m_{ij} - \tau_{ij}$ is separated into a systematic return $\tilde{\eta}^m_{ij}$ presumed to be exogenous, and a utility transfer τ_{ij} from husband to wife, which is endogenous and set in equilibrium. Similarly, $\eta^f_{ij} = \tilde{\eta}^f_{ij} + \tau_{ij}$.

In equilibrium (6), both of the following equations hold:

$$\log(\mu_{ij}) - \log(\mu_{0j}) = \eta^f_{ij} - \eta^f_{0j} = \tilde{\eta}^f_{ij} + \tau_{ij} - \tilde{\eta}^f_{0j}, \tag{33}$$

$$\log(\mu_{ij}) - \log(\mu_{i0}) = \eta^m_{ij} - \eta^m_{i0} = \tilde{\eta}^m_{ij} - \tau_{ij} - \tilde{\eta}^m_{i0}; \tag{34}$$

there is no utility transferred by remaining unmarried. Subtracting one from the other, we see that:

$$\log(\frac{\mu_{i0}}{\mu_{0j}}) = 2\tau_{ij} + c_{ij}, \tag{35}$$

where $c_{ij} = (\tilde{\eta}^f_{ij} - \tilde{\eta}^f_{0j} - \tilde{\eta}^m_{ij} + \tilde{\eta}^m_{i0})$ is exogenous.

We denote the differentiation operator $\frac{\partial}{\partial \nu_k} f$ by $\dot{f}$ (suppressing the dependence on k). Differentiating $c_{ij} = (\eta^f_{ij} - 2\tau_{ij} - \tilde{\eta}^f_{0j} - \eta^m_{ij} + \tilde{\eta}^m_{i0})$ and (35) yields:

$$\frac{\partial}{\partial \nu_k}(\eta^f_{ij} - \eta^m_{ij}) = 2\dot{\tau_{ij}} = \frac{\dot{\mu_{i0}}}{\mu_{i0}} - \frac{\dot{\mu_{0j}}}{\mu_{0j}}.$$

The inequalities (32) now determine the sign of $\dot{\tau_{ij}}$, which depends on the differentiation variable ν_k. Since $\dot{\mu_{i0}}$ and $\dot{\mu_{0j}}$ have opposite signs, according to Theorem 2, we find

$$\frac{\partial \tau_{ij}}{\partial m_i} > 0, \tag{36}$$

which means the transfer of type i men to each type of spouse must increase in response to an isolated increase in the population of men of type i. This is expected because an increase in the number of type i men introduces additional competition for each type of women, due to the smearing present in the model. To decrease the number of type i men demanding marriage to a particular type of woman to a level that permits one-to-one matching requires an increase in the transfer to crowd out some men.

While in principle the men might re-distribute so that the proportion of married men remains the same, our next computation shows this is not the case. We consider the marital participation rate of type k individuals, or rather the non-participation rate $s_k(\nu) := \beta_k^2/\nu_k$, defined as the proportion of individuals who choose not to marry. Differentiation yields

$$\begin{aligned}\frac{\partial s_k}{\partial \nu_k} &= \frac{\beta_k^2}{\nu_k^2}\Big(\nu_k r_{kk} - 1\Big) \\ &> \frac{\beta_k^2}{\nu_k^2}\Big(\frac{\nu_k - \beta_k^2}{\nu_k + \beta_k^2}\Big),\end{aligned}$$

according to (19). But this is manifestly positive since the number β_k^2 of unmarrieds of type k cannot exceed the total number ν_k of type k individuals. This means, for example, that an increase in the total population of type k men increases the percentage of type k men who choose to remain unmarried, given a fixed population of women and men of other types (and assuming, as always, that the exogenous gains matrix Π remains fixed). It concludes the proof of Corollary 3.

5.3.2. *Varying the gains data* Π. The population vector ν is one variable of interest. However the function β also depends on the gains parameters Π_{ij}. The complete derivative $D_{(\nu,\Pi)}\beta = [D_\nu\beta \mid D_\Pi\beta]$ is an $(I+J)\times(IJ+I+J)$ matrix. As such there are linear dependencies among its rows and columns. Since the matrix $D_\nu\beta$ is invertible, its columns are linearly independent and form a basis of the column space. Hence, the remaining the columns of the complete derivative can be expressed using linear combinations of them. The implicit function theorem applied to this problem turns out to yield the following simple linear relationship:

$$\frac{\partial\beta_k}{\partial\Pi_{ij}} = -\beta_i\beta_{I+j}\left(\frac{\partial\beta_k}{\partial\nu_i} + \frac{\partial\beta_k}{\partial\nu_{I+j}}\right) \tag{37}$$

for all $i \in \{1,\ldots,I\}$, $j \in \{1,\ldots,J\}$, and $k \in \{1,\ldots,I+J\}$.

Equilibrium (16) coincides with vanishing of the function $F(\beta,\nu,\Pi) : \mathbf{R}^{(I+J)+(I+J)+(IJ)} \to \mathbf{R}^{I+J}$ defined by

$$F_i(\nu,\Pi) = \beta_i^2 + \sum_{j=1}^{J}\beta_i\beta_{I+j}\Pi_{ij} - \nu_i, \qquad 1 \le i \le I$$

$$F_j(\nu,\Pi) = \beta_{I+j}^2 + \sum_{i=1}^{I}\beta_i\beta_{I+j}\Pi_{ij} - \nu_{I+j}, \qquad 1 \le j \le J. \tag{38}$$

The implicit function theorem stipulates that if the derivative $D_\beta F|_{\beta_0,\nu_0,\Pi_0}$ is invertible, there is a small neighbourhood around (β_0,ν_0,Π_0) inside which for each (ν,Π) there is a unique β satisfying equation (16), and further that β depends smoothly on (ν,Π). The implicit function theorem also provides a formula for the derivative of the implicit function $\beta(\nu,\Pi)$. It is obtained by applying the chain-rule to $F(\beta(\nu,\Pi),\nu,\Pi)$:

$$[D_\nu\beta \mid D_\Pi\beta]_{\nu_0,\Pi_0} = -[D_\beta F]^{-1}[D_\nu F \mid D_\Pi F]_{\beta_0,\nu_0,\Pi_0}. \tag{39}$$

Since $\frac{\partial F_k}{\partial\nu_\ell} = -\delta_{k\ell}$, and $\frac{\partial F_\ell}{\partial\Pi_{ij}} = \beta_i\beta_{I+j}(\delta_{i\ell} + \delta_{I+j,\ell})$, the first part of the preceding formula yields $[D_\beta F]^{-1} = D_\nu\beta$, and the second part then implies (37). Theorem 2 shows $D_\nu\beta$ is invertible, so the hypotheses of the implicit function theorem are globally satisfied and our calculations are valid.

The equation (37) has an intuitive interpretation. An increase in the total systematic gains to an (i,j) marriage (produced, for example, by an isolated increase in the value of type j marriages to type i men, or an isolated decrease in the value of remaining unmarried) has the same effect as decreasing the supply of the men or women of the respective types by a proportionate amount, weighted by the geometric mean of the unmarried men and women of type i and j. Since Theorem 2 shows the the summands in (37) to have opposite signs, the sign of their sum will fluctuate according to market conditions.

Appendix A. Derivation of the preference probabilities

The random variable present in the definition of male and female utility is the Gumbel extreme value distribution, introduced to the economics literature by McFadden [18]:

Definition 6 (Gumbel distribution)**.** *A random variable ϵ is Gumbel if it has cumulative distribution function $F(\epsilon) = \exp(-\exp(-\epsilon))$.*

Here $Pr(\epsilon < x) = F(x)$ gives the probability that the realization of this random variable takes a value less $x \in \mathbf{R}$. The corresponding density function is $F'(x) = f(x) = \exp(-(x + \exp(-x))$. The mean of ϵ is the Euler-Mascheroni constant, which is approximately equal to $\gamma = 0.57\ldots$. Its variance is equal to $\frac{\pi^2}{6}$.

We now use this distribution to derive the discrete probability distribution (2).

Lemma 7. *Suppose $\sigma > 0$ and $\eta_{ij} \in \mathbf{R}$ are constants, while for each choice of $j = 0, \ldots, J$, the ϵ_{ijg} are independent identically distributed random variables with the Gumbel distribution. Then*

$$\Pr(\eta_{ij} + \epsilon_{ijg} = \max_{0 \le k \le J} \eta_{ik} + \epsilon_{ikg}) = \frac{\exp(\frac{\eta_{ij}}{\sigma})}{\sum_{k=0}^{J} \exp(\frac{\eta_{ik}}{\sigma})}. \tag{40}$$

Proof. It costs no generality to assume $\sigma = 1$. Then

$$P := Pr(\eta_{ij} + \epsilon_{ijg} \ge \eta_{ik} + \epsilon_{ikg} \forall k) = \int_{-\infty}^{\infty} d\epsilon\, F'(\epsilon)\, \Pi_{k \ne j} F(\eta_{ij} + \epsilon - \eta_{ik}). \tag{41}$$

This formula follows from Bayes' rule for conditional probability, and independence of the various random variables involved. Substituting in the explicit formula for the Gumbel distribution from Definition 6 yields

$$P = \int_{-\infty}^{\infty} d\epsilon \exp(-(\epsilon + \exp(-\epsilon))\, \Pi_{k \ne j} \exp(-\exp(\eta_{ik} - \eta_{ij} - \epsilon)). \tag{42}$$

We make a change of variables by setting $t = \exp(-\epsilon)$, so $d\epsilon{=}{-dt/t}$. Evaluating the integral in the new variables yields

$$\begin{aligned} P &= \int_0^{\infty} dt\, \exp(-t)\, \Pi_{k \ne j} \exp(-t \exp(\eta_{ik} - \eta_{ij})) \\ &= \int_0^{\infty} dt\, \exp(-t \sum_{k=0}^{J} \exp(\eta_{ik} - \eta_{ij})) \\ &= \frac{1}{\sum_{k=0}^{J} \exp(\eta_{ik} - \eta_{ij})} \\ &= \frac{\exp(\eta_{ij})}{\sum_{k=0}^{J} \exp(\eta_{ik})} \end{aligned}$$

as desired. □

Corollary 8 (Expected marital preferences by observed types)**.** *Suppose a man with observable type i and (unobservable) specific identity g derives utility $V^m_{ijg} = \eta^m_{ij} + \sigma\epsilon_{ijg}$ from being married to a woman of observable type j, independent of her specific identity. If $\sigma > 0$, $\eta^m_{ij} \in \mathbf{R}$ and ϵ_{ijg} are as in Lemma 7, then the probability he prefers a woman of type j to all other alternatives in $\{0, 1, \ldots, J\}$ is given by* (2)*.*

Remark 4 (The Boltzmann / Gibbs distribution from statistical physics)**.** *The probabilities which appear in* (2) *and* (40) *take the form of the* Boltzmann *or* Gibbs *distributions which govern large collections of identical particles in statistical physics, but with the deterministic component η^m_{ij} of the derived utility playing the role of the energy associated to marital state j, and the strength σ of the random component playing the role of the physical temperature. Pursuing this analogy further, since the observed preference probabilities coincide with the predicted ones* (3) *only for large populations, the Choo-Siow model can be viewed as an approximate theory which becomes exact asymptotically, as the population size increases but the fraction of individuals of each observed type remains fixed. Such connections are also discussed by Galichon and Salanié* [14]*.*

Appendix B. The Legendre Transform

Here some well-known results pertaining to convexity and the Legendre transform are recalled. Let $F : \mathbf{R}^n \to \mathbf{R}$ be a twice continuously differentiable function; F is *convex* if Hess(F) $:= D^2F \geq 0$, and strictly convex if the line segment connecting any two points on the graph of F lies above the graph. The *Legendre transform* or convex dual function to $F(p)$ is denoted $F^*(q)$ and defined pointwise by:

$$F^*(q) = \sup_{p \in \mathbf{R}^n} \{q \cdot p - F(p)\}. \tag{43}$$

Since the supremum of affine functions is convex, it is clear that $F^*(q)$ is a convex function. Additionally, the following duality result is true:

Lemma 9 (Legendre duality)**.** *Let $F \in C^2$ be strongly convex on $\mathbf{R}^n$, meaning* Hess$(F) > 0$*. Then F^* is also twice continuously differentiable. Further, if $q = DF(p)$, then p=$DF^*(q)$.*

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