11 Darboux's Theorem

References: Audin II.1.3; Guillemin-Sternberg §22.

Theorem 11.1 *(Darboux-Weinstein) Suppose ω is a symplectic form on a manifold M. Then for any $x \in M$ there is a neighbourhood U of x and a diffeomorphism $\phi : U \to \mathbb{R}^{2n}$ such that*

$$\phi^*(\sum_i dq_i \wedge dp_i) = \omega$$

for coordinates q_i, p_i on $\mathbb{R}^{2n}$.

Equivariant version:

Suppose $N \subset M$ is a submanifold and ω_0, ω_1 are two closed 2-forms on M for which $(\omega_0)|_N = (\omega_1)|_N$. Then there is a neighbourhood U of N and a diffeomorpism $f : U \to M$ such that

(a) $f(n) = n$ for all $n \in N$

(b) $f^*\omega_1 = \omega_0$

If G is a compact group acting on M with ω_0 and ω_1 invariant under its action, and if N is invariant under the action of G, then f can be chosen equivariant with respect to G.

Important example: Fix coordinates identifying a neighbourhood of 0 in T_mM with a neighbourhood of m in M. Let ω_0 be the symplectic form on T_mM (antisymmetric quadratic form on a vector space) and let ω_1 be the symplectic form on the manifold M.

Important example 2: If T acts symplectically on M, the fixed point set of T is a symplectic submanifold of M.

Remark: If G fixes the point m then the action of G on T_mM is linear. Equivariant Darboux says that a diffeomorphism ϕ can be found in a neighbourhood of m such that

(1)

$$\phi^*\omega_1 = \omega_0$$

(2) There is a coordinate system on a neighbourhood of m with respect to which ω_0 is the standard antisymmetric form on a symplectic vector space and the action of G is linear.

In other words, there exist Darboux coordinates with respect to which the action of G is linear.

Corollary 11.2 *If $F \subset M$ is a submanifold fixed by G then (by the tubular neighbourhood theorem) an open subset U of the normal bundle $\nu(F)$ of F in M containing the zero section of F embeds in M, and G acts linearly on the fibres of $\nu(F)$. So Darboux coordinates may be chosen near F for which the action of G is linear on the fibres of the normal bundle to F.*

Proof: (of equivariant Darboux-Weinstein):

This proof uses Moser's method. Consider

$$\omega_t = (1-t)\omega_0 + t\omega_1.$$

For all t, ω_t is closed. In a neighbourhood of N, we can find a 1-form ϕ such that $d\phi = \omega_0 - \omega_1$ (since $d(\omega - \omega_1) = 0$). If Y is a point then we can choose a contractible neighbourhood of N and the result is obvious. Otherwise we choose an equivariant family $\phi_t : M \to M$ such that

(a) ϕ_t fixes N

(b) $\phi_0 : M \to N$, $\phi_1 = \mathrm{id}$. If X is a tubular neighbourhood of N identified with the normal bundle $\nu(N)$, so $x = (y, \nu)$, $y \in N$, $\nu \in \nu(N)$ (the normal bundle to N) then $\phi_t(y, \nu) = t\nu$.

Then for any form σ on M,

$$\phi_1^*\sigma - \phi_0^*\sigma = \int_0^1 \frac{d}{dt}\left(\phi_t^*\sigma\right) dt$$

$$= \int_0^1 \phi_t^*(L_{\xi_t}\sigma) dt$$

$$= \int_0^1 \phi_t^*\left(di_{\xi_t}\sigma + i_{\xi_t} d\sigma\right) dt$$

$$= Id\sigma + dI\sigma$$

where we have defined a chain homotopy

$$I : \Omega^*(M) \to \Omega^{*-1}(M)$$

with

$$I\sigma = \int_0^1 \phi_t^*\left(i_{\xi_t}\sigma\right) dt.$$

Choosing $\sigma = \omega_0 - \omega_1$, we see that $d\sigma = 0$ so in some neighbourhood of N,

$$\sigma = d\beta, \qquad \beta = I\sigma$$

(hence $\beta|_Y = 0$).

Now $(\omega_t)_Y$ is nondegenerate for all $t \in [0,1]$. Hence this is also true on some suitably small neighbourhood of N. Then we can find a time dependent vector field η_t such that

$$i_{\eta_t}\omega_t = -\beta$$

(recall $d\beta = \omega_0 - \omega_1$).

(Note that β may be chosen to be invariant under G, hence so is the time dependent vector field η_t.)

Integrate the vector field η_t: this gives a family of local diffeomorphisms f_t with $f_0 = \mathrm{id}$ with

$$\frac{d}{dt} f_t(m) = \eta_t(f_t(m)).$$

Since the vector field η_t commutes with the action of G, the maps f_t are G-equivariant. Also $(\eta_t)|_Y = 0$ so $(f_t)|_Y = \mathrm{id}$. We then have

$$(f_1)^*\omega_1 - \omega_0 = \int_0^1 \frac{d}{dt}(f_t^*\omega_t)dt$$

$$= \int f_t^*(di_{\eta_t})\omega_t dt + \int f_t^*(\omega_0 - \omega_1)dt$$

(because $d\omega_t = 0$, one term in the Lie derivative vanishes)

$$= \int f_t^* d(-\beta)dt + \int f_t^*(\omega_0 - \omega_1)dt$$

$$= -\int f_t^*(\omega_0 - \omega_1)dt + \int f_t^*(\omega_0 - \omega_1)dt = 0$$

So f_1 is the desired equivariant diffeomorphism. $\square$