Here are some practice problems in number theory. They are, very roughly, in increasing order of difficulty.

1. (a) Show that $n^7 - n$ is divisible by 42 for every positive integer n.

   (b) Show that every prime not equal to 2 or 5 divides infinitely many of the numbers 1, 11, 111, 1111, etc.

2. Show that if $p > 3$ is a prime, then $p^2 \equiv 1 \pmod{24}$.

3. How many zeros are at the end of 1000!?

4. If p and $p^2 + 2$ are primes, show that $p^3 + 2$ is prime.

5. Show that $\gcd(2^a - 1, 2^b - 1) = 2^{\gcd(a,b)} - 1$ for positive integers a, b.

6. Suppose that a, b, c are distinct integers and that $p(x)$ is a polynomial with integer coefficients. Show that it is not possible to have $p(a) = b$, $p(b) = c$, $p(c) = a$.

7. A triangular number is a positive integer of the form $n(n+1)/2$. Show that m is a sum of two triangular numbers iff $4m + 1$ is a sum of two squares. (A-1, Putnam 1975)

8. For positive integers n define $d(n) = n - m^2$, where m is the greatest integer with $m^2 \leq n$. Given a positive integer b_0, define a sequence b_i by taking $b_{k+1} = b_k + d(b_k)$. For what b_0 do we have b_i constant for sufficiently large i? (B-1, Putnam 1991)

9. d, e and f each have nine digits when written in base 10. Each of the nine numbers formed from d by replacing one of its digits by the corresponding digit of e is divisible by 7. Similarly, each of the nine numbers formed from e by replacing one of its digits by the corresponding digit of f is divisible by 7. Show that each of the nine differences between corresponding digits of d and f is divisible by 7. (A-3, Putnam 1993)

10. Define the sequence of decimal integers a_n as follows: $a_1 = 0$; $a_2 = 1$; a_{n+2} is obtained by writing the digits of a_{n+1} immediately followed by those of a_n. When is a_n a multiple of 11? (A-4, Putnam 1998)

11. Suppose $n > 1$ is an integer. Show that $n^4 + 4^n$ is not prime.

12. (a) Let α and β be irrational numbers such that $1/\alpha + 1/\beta = 1$. Then the sequences $f(n) = \lfloor \alpha n \rfloor$ and $g(n) = \lfloor \beta n \rfloor$, $n = 1, 2, 3, \ldots$ are disjoint and their union is the set of positive integers. (A classic due to Beatty; variations of this appear again and again.)

    (b) Show the following converse: if α, β are positive reals such that the sequences $f(n) = \lfloor \alpha n \rfloor$ and $g(n) = \lfloor \beta n \rfloor$, $n = 1, 2, 3, \ldots$ are disjoint and their union is the set of positive integers, then α, β are irrational and $1/\alpha + 1/\beta = 1$.

13. If p is a prime number greater than 3 and $k = \lfloor 2p/3 \rfloor$, prove that the sum

$$\binom{p}{1} + \binom{p}{2} + \dots \binom{p}{k}$$

of binomial coefficients is divisible by p^2. (A-5, Putnam 1996)

14. Find all positive integers a, b, m, n with m relatively prime to n such that $(a^2 + b^2)^m = (ab)^n$. (A-3, Putnam 1992)

15. Suppose the positive integers x, y satisfy $2x^2 + x = 3y^2 + y$. Show that $x - y$, $2x + 2y + 1$, $3x + 3y + 1$ are all perfect squares.

16. Find all solutions of $x^{n+1} - (x+1)^n = 2001$ in positive integers x and n. (A-5, Putnam 2001)