Math 157 Analysis I — Solution of Term Exam 4

web version: http://www.math.toronto.edu/~drorbn/classes/0304/157AnalysisI/TE4/Solution.html

Problem and Solution 1. (Graded by Vicentiu Tipu) Compute the following definite and indefinite integrals in elementary terms:

1. $\displaystyle\int \frac{du}{u(1-u)} = \int \left(\frac{1}{u} + \frac{1}{1-u}\right) du = \log u - \log(1-u) = \log \frac{u}{1-u}$.
2. With $u = e^x$ we have $du = e^x dx$ and hence $dx = du/e^x = du/u$. Thus
$$\int \frac{dx}{1-e^x} = \int \frac{du}{u(1-u)} = \log \frac{u}{1-u} = \log \frac{e^x}{1-e^x}.$$
3. With $u = x^2$ we have $du = 2xdx$ and and when $x = 0, 1$, also $u = 0, 1$. Thus
$$\int_0^1 2x(1+x^2)^7 dx = \int_0^1 (1+u)^7 du = \left.\frac{(1+u)^8}{8}\right|_0^1 = \frac{2^8 - 1^8}{8} = \frac{255}{8}.$$
4. Using integration by parts with $f = x$ (thus $f' = 1$) and $g' = e^x$ (thus say $g = e^x$ as well), we get
$$\int_0^1 xe^x dx = xe^x|_0^1 - \int_0^1 1e^x dx = (xe^x - e^x)|_0^1 = 1e^1 - e^1 - (0e^0 - e^0) = e^0 = 1.$$

Problem 2. The "unit ball" B in $\mathbb{R}^3$ is the result of revolving the domain $0 \le y \le \sqrt{1-x^2}$ (for $-1 \le x \le 1$) around the x axis.

1. State the general "Cosmopolitan Integral" formula for the volume of a body obtained by revolving a domain bounded under the graph of a function f around the x axis.
2. Compute the volume of B.
3. State the general "Cosmopolitan Integral" formula for the surface area of a body obtained by revolving a domain bounded under the graph of a function f around the x axis.
4. Compute the surface area of B.

Solution. (Graded by Cristian Ivanescu)

1. The volume V of a body obtained by revolving a domain bounded under the graph of a function f around the x axis, between the vertical lines $x = a$ and $x = b$, is $V = \pi \displaystyle\int_a^b f^2(x) dx$.

2. Taking $f(x) = \sqrt{1-x^2}$ in the above formula we get

$$V = \pi \int_{-1}^{1} \sqrt{1-x^2}^2 dx = \pi \int_{-1}^{1} (1-x^2) dx = \pi \left(x - \frac{x^3}{3} \right) \Big|_{-1}^{1} = \frac{4}{3}\pi.$$

3. The area S of the surface obtained by revolving the graph of a function f around the x axis, between $x = a$ and $x = b$, is $S = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}$.

4. Taking $f(x) = \sqrt{1-x^2}$ and thus $f' = -\frac{x}{\sqrt{1-x^2}}$ in the above formula we get

$$S = 2\pi \int_{-1}^{1} \sqrt{1-x^2}\sqrt{1 + \frac{x^2}{1-x^2}} dx = 2\pi \int_{-1}^{1} dx = 4\pi.$$

Problem 3. Let α be a real number which is not a positive integer or 0, let $f(x) = (1+x)^\alpha$ and let n be a positive integer.

1. Compute the Taylor polynomial $P_{n,0,f}$ of degree n for f around 0.
2. Write the corresponding remainder term using one of the formulas discussed in class.
3. Determine (with proof) if there is an interval around 0 on which $f(x) = \lim_{n\to\infty} P_{n,0,f}(x)$.

Solution. (Graded by Julian C.-N. Hung)

1. $f' = \alpha(1+x)^{\alpha-1}$, $f'' = \alpha(\alpha-1)(1+x)^{\alpha-2}$, $f''' = \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3}$ and in general, $f^{(k)} = \alpha(\alpha-1)\cdots(\alpha-k+1)(1+x)^{\alpha-k}$. Thus the Taylor coefficients are $a_k = \frac{f^{(k)}(0)}{k!} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$ and so

$$P_{n,0,f}(x) = \sum_{k=0}^{n} a_k x^k = \sum_{k=0}^{n} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)x^k}{k!}.$$

2. Our first remainder formula says that there is some t between 0 and x so that (with $a = 0$)

$$R_{n,0,f}(x) := f(x) - P_{n,0,f}(x) = \frac{f^{(n+1)}(t)}{(n+1)!}(x-a)^{n+1}$$

$$= \frac{\alpha(\alpha-1)\cdots(\alpha-n)(1+t)^{\alpha-n-1}}{(n+1)!}x^{n+1} = \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{(n+1)!}(1+t)^\alpha \left(\frac{x}{1+t}\right)^{n+1}.$$

3. The remainder formula above is a product of three factors. The first in itself is a product of $n+1$ factors of the form $\frac{\alpha+1-k}{k}$ for $k = 1, \ldots, n+1$. If $k > |\alpha+1|$ then $\left|\frac{\alpha+1-k}{k}\right| < 2$, so the first factor is bounded by a constant times 2^n. For any given x the second factor is bounded by $(1+|x|)^\alpha$ which is independent of n, and if $|x| < \frac{1}{4}$ then $|t| < \frac{1}{4}$ and so $|\frac{x}{1+t}| < \frac{1/4}{3/4} = \frac{1}{3}$ and so the third factor in the remainder formula is bounded by $1/3^{n+1}$. Multiplying the three bounds we find that for $|x| < \frac{1}{4}$ the remainder is bounded by a

constant times $(2/3)^n$, and this goes to 0 when n goes to ∞. So at least on the interval $(-\frac{1}{4}, \frac{1}{4})$ the remainder goes to 0 and hence $f(x) = \lim\limits_{n\to\infty} P_{n,0,f}(x)$.

(Further analysis show that convergence occurs for all $x \in (-1, 1)$, but this doesn't concern us here).

Problem 4. Let $a_{n,m}$ be a "sequence of sequences" (an assignment of a real number $a_{n,m}$ to every pair (n, m) of positive integers) and assume that l_n is a sequence so that for every n we have $\lim_{m\to\infty} a_{n,m} = l_n$. Further assume that $\lim_{n\to\infty} l_n = l$.

$$\begin{array}{cccccc} a_{1,1} & a_{1,2} & a_{1,3} & \dots & \longrightarrow & l_1 \\ a_{2,1} & a_{2,2} & a_{2,3} & \dots & \longrightarrow & l_2 \\ a_{3,1} & a_{3,2} & a_{3,3} & \dots & \longrightarrow & l_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ & & & & & \downarrow \\ & & & & & l \end{array}$$

1. Show that for every positive integer n there is a positive integer m_n so that $|a_{n,m_n} - l_n| < |l_n - l| + 1/n$.

2. Show that $\lim\limits_{n\to\infty} a_{n,m_n} = l$.

3. (5 points bonus, no partial credit) Is it always true that also $\lim\limits_{n\to\infty} a_{n,n} = l$?

Solution. (Graded by Vicentiu Tipu)

1. For any fixed n we have that $\epsilon := |l_n - l| + 1/n > 0$ so by the convergence $\lim_{m\to\infty} a_{n,m} = l_n$ we can find an m for which $|a_{n,m} - l_n| < \epsilon$. Rename m to be m_n and you are done.

2. $|a_{n,m_n} - l| \le |a_{n,m_n} - l_n| + |l_n - l| < 2|l_n - l| + 1/n \underset{n\to\infty}{\longrightarrow} 0$.

3. Take $a_{n,m}$ to be the "identity matrix": $a_{n,m} = 0$ if $n \neq m$ though $a_{n,n} = 1$ for all n. But then for any fixed n the sequence $a_{n,m}$ (regarded with m varying) is eventually the constant 0, so $l_n = \lim_{m\to\infty} a_{n,m} = 0$ and so $l = \lim_{n\to\infty} l_n = 0$. But $\lim_{n\to\infty} a_{n,n} = \lim_{n\to\infty} 1 = 1$.

Problem 5.

1. Compute the first 5 partial sums of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

2. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

Solution. (Graded by Cristian Ivanescu)

1. $s_1 = \frac{1}{2}$, $s_2 = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$, $s_3 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{2}{3} + \frac{1}{12} = \frac{3}{4}$, $s_4 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{3}{4} + \frac{1}{20} = \frac{4}{5}$ and $s_5 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} = \frac{4}{5} + \frac{1}{30} = \frac{5}{6}$.

2. Following this trend we guess that $s_N := \sum_{n=1}^{N} \frac{1}{n(n+1)} = \frac{N}{N+1}$. This we prove by induction. There is no need to check low N cases — we've already done that. So all that remains is

$$s_N = s_{N-1} + \frac{1}{N(N+1)} = \frac{N-1}{N} + \frac{1}{N(N+1)} = \frac{N}{N+1}.$$

But now,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{N\to\infty} s_N = \lim_{N\to\infty} \frac{N}{N+1} = 1.$$

Alternative Solution. Note that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. Then use telescopic summation to find that

$$s_N := \sum_{n=1}^{N} \frac{1}{n(n+1)} = \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1 - \frac{1}{N+1}.$$

Now continue as at the end of the previous solution.

The results. 79 students took the exam; the average grade was 62.99, the median was 68 and the standard deviation was 23.32. The average is thus noticeably below the averages for the first three term exams, but still higher than last year's average. I don't know if the same factors from last year applied this time as well; but for what it's worth, see last year's handout "What Went Wrong with Term Exam 4?".

An unrelated computation.

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drorbn@coxeter:~/classes/157AnalysisI:1 math
Mathematica 4.1 for IBM AIX
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 -- Motif graphics initialized --

In[1]:= D[ArcTan[x], {x, 10}]

                    9               7               5               3
        -185794560 x    371589120 x    243855360 x    58060800 x    3628800 x
Out[1]= ------------- + ------------ - ------------ + ----------- - ---------
                2 10            2 9             2 8            2 7          2 6
         (1 + x )        (1 + x )        (1 + x )       (1 + x )     (1 + x )
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