Math 157 Analysis I — Solution of Term Exam 3

web version: http://www.math.toronto.edu/~drorbn/classes/0304/157AnalysisI/TE3/Solution.html

Problem 1. In a very condensed form, the definition of integration is as follows: For f bounded on $[a,b]$ and $P : a = t_0 < t_1 < \cdots < t_n = b$ a partition of $[a,b]$ set $m_i = \inf_{[t_{i-1},t_i]} f(x)$, $M_i = \sup_{[t_{i-1},t_i]} f(x)$, $L(f,P) = \sum_{i=1}^n m_i(t_i - t_{i-1})$ and $U(f,P) = \sum_{i=1}^n M_i(t_i - t_{i-1})$. Then set $L(f) = \sup_P L(f,P)$ and $U(f) = \inf_P U(f,P)$. Finally, if $U(f) = L(f)$ we say that "f is integrable on $[a,b]$" and set $\int_a^b f = \int_a^b f(x)dx = U(f) = L(f)$.

From this definition alone, without using *anything* proven in class about integration, prove that the function f given below is integrable on $[-1,1]$ and compute its integral $\int_{-1}^1 f$:

$$f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0. \end{cases}$$

Solution. (Graded by Cristian Ivanescu) Let $P : -1 = t_0 < t_1 < \cdots < t_n = 1$ be an arbitrary partition of $[-1,1]$. Then for any i the infimum $m_i = \inf_{[t_{i-1},t_i]} f(x)$ is 0 and so $L(f,P) = \sum_{i=1}^n m_i(t_i - t_{i-1}) = 0$. Thus $L(f) = \sup_P L(f,P) = 0$. At the same time, for any i the supremum $M_i = \sup_{[t_{i-1},t_i]} f(x)$ is ≥ 0, and hence $U(f,P) = \sum_{i=1}^n M_i(t_i - t_{i-1}) \geq 0$ and so $U(f) = \inf_P U(f,P) \geq 0$. Now let $0 < \epsilon < 1$ be given and let P_ϵ be the partition $-1 = t_0 < t_1 = -\frac{\epsilon}{2} < \frac{\epsilon}{2} = t_2 < 1 = t_3$. Then $M_1 = M_3 = 0$ while $M_2 = 1$ and so $U(f,P_\epsilon) = 0(1-\frac{\epsilon}{2}) + 1(\frac{\epsilon}{2} + \frac{\epsilon}{2}) + 0(1-\frac{\epsilon}{2}) = \epsilon$. Thus $U(f) = \inf_P U(f,P) \leq \epsilon$. But this is true for any $0 < \epsilon < 1$ and we already know that $U(f) \geq 0$. So it must be that $U(f) = 0$. Thus $U(f) = L(f) = 0$ and hence f is integrable on $[-1,1]$ and its integral is $\int_{-1}^1 f = U(f) = L(f) = 0$.

Problem 2. Prove that the function

$$g(x) := \int_0^x \frac{dt}{1+t^2} + \int_0^{1/x} \frac{dt}{1+t^2}$$

is a constant function.

Solution. (Graded by Julian C.-N. Hung) Differentiate g using the first fundamental theorem of calculus. The first summand yields $\frac{1}{1+x^2}$. The second summand is the first summand pre-composed with the function $x \mapsto 1/x$. So by the chain rule, the derivative of the second summand is $\frac{1}{1+(1/x)^2}(1/x)' = -\frac{1}{1+(1/x)^2}(1/x^2) = -\frac{1}{1+x^2}$. g' is the sum of these two terms, $g' = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$. Hence g is a constant.

Problem 3. In class we have proven that a twice-differentiable function f satisfying the equation $f'' = -f$ is determined by $f(0)$ and $f'(0)$. Use this fact and the known formulas for the derivatives of $\cos x$ and $\sin x$ to derive a formula for $\cos(\alpha+\beta)$ in terms of $\cos\alpha$, $\cos\beta$, $\sin\alpha$ and $\sin\beta$.

Solution. (Graded by Julian C.-N. Hung) Let β be a constant and consider the functions $f_1(\alpha) = \cos(\alpha+\beta)$ and $f_2(\alpha) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$. Then $f_1'' = (\cos(\alpha+\beta))'' = (-\sin(\alpha+\beta))' = -\cos(\alpha+\beta) = -f_1$ and $f_2'' = (\cos\alpha\cos\beta - \sin\alpha\sin\beta)'' = (-\sin\alpha\cos\beta -$

$\cos\alpha\sin\beta)' = -\cos\alpha\cos\beta + \sin\alpha\sin\beta = -f_2$ so both f_1 and f_2 satisfy $f'' = -f$. We also have $f_1(0) = \cos\beta = \cos 0\cos\beta - \sin 0\sin\beta = f_2(0)$ and $f_1'(0) = -\sin\beta = (-\sin 0)\cos\beta - \cos 0\sin\beta = f_2'(0)$. So by what we have proven in class $f_1 = f_2$ or $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

Problem 4. The function F is defined by $F(x) := x^x$.

1. Compute $F'(x)$ for all $x > 0$.
2. Explain why $F(x)$ has a differentiable inverse for $x > \frac{1}{e}$.
3. Let S be the inverse function of F (with the domain of F considered to be $(\frac{1}{e}, \infty)$). Compute $S'(x)$ and simplify your result as much as you can. Your end result may still contain $S(x)$ in it, but not S', F or F'.

Solution. (Graded by Vicentiu Tipu)

1. $F'(x) = \left(e^{x\log x}\right)' = e^{x\log x}\left(\log x + x\frac{1}{x}\right) = x^x(1+\log x) = F(x)(1+\log x).$
2. For $x > \frac{1}{e}$ we have that $\log x > \log\frac{1}{e} = -1$ and hence $1 + \log x > 0$ and $F'(x) > 0$. So F is increasing on $(\frac{1}{e}, \infty)$. It is also differentiable on that interval, so by a theorem proven in class, it has a differentiable inverse.
3. $S'(x) = \dfrac{1}{F'(S(x))} = \dfrac{1}{F(S(x))(1+\log S(x))} = \dfrac{1}{x(1+\log S(x))}.$

The results. 82 students took the exam; the average grade was 69.3, the median was 78 and the standard deviation was 26.76.