

# Integration

web version:

<http://www.math.toronto.edu/~drorbn/classes/0304/157AnalysisI/Integration/Integration.html>

The setting:  $f$  bounded on  $[a, b]$ ,  $P : a = t_0 < t_1 < \dots < t_n = b$  a partition of  $[a, b]$ ,  $m_i = \inf_{[t_{i-1}, t_i]} f(x)$ ,  $M_i = \sup_{[t_{i-1}, t_i]} f(x)$ ,  $L(f, P) = \sum_{i=1}^n m_i(t_i - t_{i-1})$ ,  $U(f, P) = \sum_{i=1}^n M_i(t_i - t_{i-1})$ ,  $L(f) = \sup_P L(f, P)$ ,  $U(f) = \inf_P U(f, P)$ . Finally, if  $U(f) = L(f)$  we say that “ $f$  is integrable on  $[a, b]$ ” and set  $\int_a^b f = \int_a^b f(x)dx = U(f) = L(f)$ .

**Theorem 1.** For any two partitions  $P_{1,2}$ ,  $L(f, P_1) \leq U(f, P_2)$ .

**Theorem 2.**  $f$  is integrable iff for every  $\epsilon > 0$  there is a partition  $P$  such that  $U(f, P) - L(f, P) < \epsilon$ .

**Theorem 3.** If  $f$  is continuous on  $[a, b]$  then  $f$  is integrable on  $[a, b]$ .

**Theorem 4.** If  $a < c < b$  then  $\int_a^b f = \int_a^c f + \int_c^b f$  (in particular, the rhs makes sense iff the lhs does).

**Theorem 5.** If  $f$  and  $g$  are integrable on  $[a, b]$  then so is  $f + g$ , and  $\int_a^b f + g = \int_a^b f + \int_a^b g$ .

**Theorem 6.** If  $f$  is integrable on  $[a, b]$  and  $c$  is a constant, then  $cf$  is integrable on  $[a, b]$  and  $\int_a^b cf = c \int_a^b f$ .

**Theorem 7<sup>a</sup>.** If  $f \leq g$  on  $[a, b]$  and both are integrable on  $[a, b]$ , then  $\int_a^b f \leq \int_a^b g$ .

**Theorem 7.** If  $m \leq f(x) \leq M$  on  $[a, b]$  and  $f$  is integrable on  $[a, b]$  then  $m(b - a) \leq \int_a^b f \leq M(b - a)$ .

**Theorem 8.** If  $f$  is integrable on  $[a, b]$  and  $F$  is defined on  $[a, b]$  by  $F(x) = \int_a^x f$ , then  $F$  is continuous on  $[a, b]$ .

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This class' fundamental existential dilemma / schizophrenia:

- We want to develop a mathematical world view in which *everything* is intuitive.
- At the same time, we want to *prove* everything with no appeal whatsoever to the bad word “intuition”.

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White unicorns roam the earth.