Math 157 Analysis I — Solution of Term Exam 4

web version:
http://www.math.toronto.edu/~drorbn/classes/0203/157AnalysisI/TermExam4/Solution.html

Problem 1. Is there a non-zero polynomial $p(x)$ defined on the interval $[0,\pi]$ and with values in the interval $[0,\frac{1}{2})$ so that it and all of its derivatives are integers at both the point 0 and the point π? In either case, prove your answer in detail. (Hint: How did we prove the irrationality of π?)

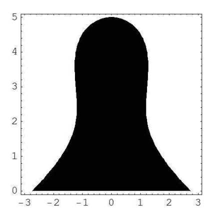
Solution. There isn't. Had there been one, we could reach a contradiction as in the proof of the irrationality of π. Indeed we would have that $0<\int_0^\pi p(x)\sin x\,dx<\frac{1}{2}\int_0^\pi \sin x\,dx=1$, hence the integral $I=\int_0^\pi p(x)\sin x\,dx$ is not an integer. But repeated integration by parts gives

$$I=\begin{pmatrix}\text{boundary}\\ \text{terms}\end{pmatrix}\pm\int_0^\pi p'(x)\cos x\,dx=\begin{pmatrix}\text{boundary}\\ \text{terms}\end{pmatrix}\pm\int_0^\pi p''(x)\sin x\,dx=\dots$$

$$=\begin{pmatrix}\text{boundary}\\ \text{terms}\end{pmatrix}\pm\int_0^\pi p^{(2n)}(x)\sin x\,dx.$$

The assumptions on $p^{(k)}(0)\in\mathbb{Z}$ and $p^{(k)}(\pi)\in\mathbb{Z}$ along with the fact that $\sin 0$, $\sin\pi$, $\cos 0$ and $\cos\pi$ are all integers imply that the boundary terms are all integers. If n is large enough, $p^{(2n)}=0$ and hence the remaining integral is 0. So I is an integer, and that's a contradiction.

Problem 2. Compute the volume V of the "Black Pawn" on the right — the volume of the solid obtained by revolving the solutions of the inequalities $4x^2\leq y+3-(y-3)^3$ and $y\geq 0$ about the y axis (its vertical axis of symmetry). (Check that $5+3-(5-3)^3=0$ and hence the height of the pawn is 5).



Solution. This is the area of the rotation solid with radius $r(y)=\frac{1}{2}\sqrt{y+3-(y-3)^3}$ bounded by $y=0$ and $y=5$. Thus

$$V=\pi\int_0^5 r(y)^2dy=\frac{\pi}{4}\int_0^5(y+3-(y-3)^3)dy$$

$$=\frac{\pi}{4}\left(\frac{y^2}{2}+3y-\frac{(y-3)^4}{4}\right)\Bigg|_0^5=\frac{175\pi}{16}.$$

Problem 3.

1. Compute the degree n Taylor polynomial P_n of the function $f(x)=\frac{1}{1-x}$ around the point 0.

2. Write a formula for the remainder $f-P_n$ in terms of the derivative $f^{(n+1)}$ evaluated at some point $t\in[0,x]$.

3. Show that at least for very small values of x, $f(x)=\lim_{n\to\infty}P_n(x)$.

Solution.

1. $f' = \frac{1}{(1-x)^2}$, $f'' = \frac{2}{(1-x)^3}$, $f''' = \frac{2\cdot 3}{(1-x)^4}$ and so it can be shown by induction that $f^{(k)} = \frac{k!}{(1-x)^{k+1}}$. Thus $f^{(k)}(0) = k!$ and hence $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$.

2. Cauchy's formula for the remainder is $R_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(t)}{(n+1)!} x^{n+1} = \frac{x^{n+1}}{(1-t)^{n+2}} = \frac{1}{1-t}\left(\frac{x}{1-t}\right)^{n+1}$ for some $t \in [0, x]$.

3. If $|x| < \frac{1}{2}$ then $|t| < \frac{1}{2}$ and $|1-t| > \frac{1}{2}$ and hence $\left|\frac{x}{1-t}\right| < 2|x| < 1$ and thus $|R_n(x)| < \frac{1}{1-t}(2|x|)^{n+1} \to 0$. Therefore $f(x) - P_n(x) \to 0$, as required.

Problem 4.

1. Prove that if $\lim_{n\to\infty} a_n = l$ and the function f is continuous at l, then $\lim_{n\to\infty} f(a_n) = f(l)$

2. Let $b > 1$ be a number, and define a sequence a_n via the relations $a_1 = 1$ and $a_{n+1} = \frac{1}{2}(a_n + b/a_n)$ for $n \geq 1$. Assuming that this sequence is convergent to a positive limit, determine what this limit is.

Solution.

1. See the "easy" part of Theorem 1 of Spivak's Chapter 22.

2. Assume $\lim a_n = l > 0$. Then $l = \lim a_{n+1} = \lim \frac{1}{2}(a_n + b/a_n)$. Using the first part of this question on the function $x \mapsto \frac{1}{2}(x + b/x)$, which is continuous at l, we find that $\lim \frac{1}{2}(a_n + b/a_n) = \frac{1}{2}(l + b/l)$. Hence l satisfies $l = \frac{1}{2}(l + b/l)$. Dividing by l we get $1 = \frac{1}{2} + \frac{b}{2l^2}$ which is $1 = \frac{b}{l^2}$ which along with $l > 0$ implies that $l = \sqrt{b}$.

Problem 5. Do the following series converge? Explain briefly why or why not:

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n+1}$.

   Solution. $\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$ hence by the vanishing test the series cannot converge.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n\sqrt{n}+1}$.

   Solution. $\frac{\sqrt{n}}{n\sqrt{n}+1} > \frac{\sqrt{n}}{2n\sqrt{n}} = \frac{1}{2n}$. The latter is a multiple of the harmonic series which doesn't converge, hence the original series doesn't converge either.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n!}$.

   Solution. Ignoring the first two terms of the series, which don't change convergence anyway,

$$\frac{n^2}{n!} = \frac{n^2}{n(n-1)(n-2)!} < \frac{n^2}{2n^2(n-2)!} = \frac{1}{2(n-2)!}.$$

The latter sequence is summable as we have shown in class, hence the original series is convergent.

4. $\displaystyle\sum_{n=1}^{\infty} \frac{\log n}{n^2}$.

   Solution. The function $f(x) = \sqrt{x} - \log x$ is positive at $x = 1$ and simple differentiation shows that $f'(x) > 0$ for $x \geq 1$, hence it is increasing, and hence it is positive for all $x \geq 1$. Thus $\frac{\log n}{n^2} < \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$ which is summable as was shown in class.

5. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \log n}$.

   Solution. That's a tough one. Here's a solution inspired by the solution to Problem 20 of Spivak's Chapter 23, which by itself is inspired by the proof of the divergence of the harmonic series:

$$\sum_{n=2}^{2^K} \frac{1}{n \log n} = \sum_{k=1}^{K} \left(\sum_{n:\ 2^{k-1} < n \leq 2^k} \frac{1}{n \log n} \right) = \#.$$

   If we replace each of the inner sums here by the number of terms in it times the smallest of those, which is the last of those, it only becomes smaller. Hence

$$\# > \sum_{k=1}^{K} 2^{k-1} \frac{1}{2^k \log 2^k} = \sum_{k=1}^{K} \frac{2^{k-1}}{2^k k \log 2} = \frac{2}{\log 2} \sum_{k=1}^{K} \frac{1}{k}.$$

   The latter are partial sums of a divergent positive series, hence they approach infinity. Therefore the partial sums $\sum_{n=2}^{2^K} \frac{1}{n \log n}$ approach infinity and our series is divergent.

The results. 75 students took the exam; the average grade is 47.4, the median is 46 and the standard deviation is 23.55.